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EXISTENCE AND CONTINUATION PROPERTIES OF SOLUTIONS

OF A NON-LINEAR VOLTERRA INTEGRAL EQUATION

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EXISTENCE AND CONTINUATION PROPERTIES OF SOLUTIONS

OF A NON-LINEAR VOLTERRA INTEGRAL EQUATION

1. Introduction. The subject of this paper is an n-dimensional non-linear integral equation of Volterra type,

$$\text{(E)} \qquad x(t) = f(t) + \int_0^t g(t,s,x(s))ds, \quad t \in [0,\infty).$$

The prime consideration is the study of properties of solutions of equation (E). In particular, there are derived sufficient conditions for this equation to possess a solution on a subinterval $I_o = [0,T)$ of $[0,\infty)$, and sufficient conditions for the extensibility of such a solution to an interval I containing I_o.

Section 2 presents some known results for linear Volterra integral equations, which are used to establish a "Gronwall type" inequality for integral equations. A device introduced by Tonelli [8] is employed in Section 3 to find approximate solutions of (E). Using these approximate solutions as the main tool, there are then established certain existence theorems for (E). Also within Section 3 there are derived sufficient conditions to insure that if x is a solution of (E) on a subinterval I_o of I then x can be extended to be a solution of (E) on I. In Section 4 it is shown that whenever (E) possesses a unique solution, a certain type of stability holds for the set of solutions of a corresponding equation involving "nearby data". Section 5 is devoted to establishing a sufficient condition for a solution $x(t)$ of (E) on its maximal interval of existence $[0,T)$ to possess the property that $\|x(t)\| \to +\infty$ as $t \to T^-$.

Matrix notation is used throughout; in particular, matrices of one column are termed vectors. Lower class letters are used for vectors, and capital letters for all other matrices. The real n-dimensional space is denoted by $\mathbb{R}^n$, and for $x = (x_\alpha) \in \mathbb{R}^n$ the norm $\|x\|$ is given by $(x_1^2 + \cdots + x_n^2)^{1/2}$. For simplicity $\mathbb{R}$ is used for $\mathbb{R}^1$; also $\mathbb{R}^+$ is used to denote $\{t : t \in \mathbb{R},\ t \geq 0\}$. For an $m \times n$ matrix A, the matrix norm $\|A\|$ is defined as the supremum of $\|Ax\|$ on the unit ball $\{x : \|x\| \leq 1\}$ of $\mathbb{R}^n$.

For a general $m \times n$ matrix $M = [M_{\alpha\beta}]$ the symbol $M \cdot\geq\cdot 0$ signifies that the elements of M are real, and $M_{\alpha\beta} \geq 0$ for $\alpha = 1,\cdots,m$ and $\beta = 1,\cdots,n$. The symbol $M \cdot\leq\cdot 0$ denotes the condition $-M \cdot\geq\cdot 0$; also, $M \cdot\geq\cdot N$ is used to signify $M - N \cdot\geq\cdot 0$. The symbol 0 is used indiscriminately for the zero matrix of any dimensions. For $x \in \mathbb{R}^n$ the symbol $|x|$ is employed for the vector $(|x_\alpha|)$ $\alpha = 1,\cdots,n$.

For $T \in (0,\infty)$ the set $\{(t,s) : 0 \leq t \leq T,\ 0 \leq s \leq T\}$ is denoted by Q_T. A matrix function $M(t,s)$ is said to be an element of $L^2(Q_T)$ if $M(t,s)$ is measurable in the Lebesgue sense on Q_T, and $M(t,s)$ is quadratically integrable in the sense of Lebesgue on the square Q_T; the corresponding norm of M as an element of $L^2(Q_T)$ is defined as

$$\|M\|_{2;T} = \left(\int_0^T \int_0^T \|M(t,s)\|^2 \, dtds \right)^{1/2}.$$

Since a vector function $p(t,s)$ is a one column matrix function, the above also gives the meaning for $p \in L^2(Q_T)$.

For any interval I contained in $\mathbb{R}^+$, and any set $B \subset \mathbb{R}^n$, we denote by $C(I;B)$ the class of all functions $f : I \to \mathbb{R}^n$ such that f is continuous on I and $f(t) \in B$ for all $t \in I$; in particular, $C(I;\mathbb{R}^n)$ is abbreviated to simply $C(I)$. Similar to the notations introduced above for matrix functions on Q_T, the symbol $L^2(I)$ denotes the set of all functions $f:I \to \mathbb{R}^n$ such that f is Lebesgue measurable and the integral $\int_I \|f(s)\|^2 ds$ exists in the Lebesgue sense.

If M and N are matrix functions defined on a common domain $\mathcal{D}$, and such that M is equal to N almost everywhere on $\mathcal{D}$, then we write simply M = N. Correspondingly, if $M \cdot\leq\cdot N$ almost everywhere on $\mathcal{D}$, then we write simply $M \cdot\leq\cdot N$ on $\mathcal{D}$.

A matrix function is called continuous, integrable, measurable, etc., when each element of the matrix possesses the specified property.

2. Preliminary results on linear Volterra integral equations. We shall consider now the linear vector equation

$$x(t) = h(t) + \int_0^t K(t,s)x(s)ds, \qquad t \geq 0, \tag{L}$$

under the following hypotheses.

(H1) *The function h is an element of $L^2[0,T]$ for arbitrary $T \in [0,\infty)$.*

(H2) *The matrix function K is an element of $L^2(Q_T)$ for each $T > 0$, and $K(t,s) = 0$ for $s > t$.*

As is well known from the Fubini theorem, (see, for example, Taylor [7; Theorem 7-1, p. 328; Problem 2, p. 333]), upon possibly re-defining K on a set of two-dimensional measure zero one has the following point properties;

(i) *for each value of* t, K(t,s) *is a measurable function of* s *on* $\mathbb{R}^+$ *such that*

(2.1) $$k^2(t) = \int_0^t \|K(t,s)\|^2 ds < \infty, \qquad t \in [0,\infty);$$

(ii) *for each value of* s, K(t,s) *is a measurable function of* t *on* $\mathbb{R}^+$ *such that*

(2.2) $$b^2(s;T) = \int_s^T \|K(t,s)\|^2 dt < \infty, \qquad s \in [0,T].$$

A function K satisfying (H2) is called an L^2 Volterra kernel on $\mathbb{R}^+ \times \mathbb{R}^+$, but for convenience we shall simply say that K is an L^2 kernel.

For an L^2 kernel K and $T \in [0,\infty)$, we define an operator A on $L^2[0,T]$ in the following way. For $x \in L^2[0,T]$ let $y = Ax$, where

(2.3) $$y(t) = \int_0^t K(t,s)x(s)ds, \qquad t \in [0,T].$$

The function y is in $L^2[0,T]$, and the integral operator $A:L^2[0,T] \to L^2[0,T]$ is compact. For proofs of these statements concerning the operator A, see Taylor [6; p. 167, p. 277].

For a given L^2 kernel K we define the iterated kernels R_i as follows:

(2.4) $$R_1(t,s) = K(t,s),$$
$$R_{i+1}(t,s) = \int_s^t K(t,u)R_i(u,s)du, \qquad (i = 1,2,\cdots).$$

Also, we shall let $c(t,s) = \int_s^t k^2(u)du$ for $0 \leq t \leq T$, and $c(t) = \int_0^t k^2(u)du = c(t,0)$, where k is defined by (2.1).

LEMMA 2.1. *If* K *satisfies* (H2) *then the iterated kernels* $R_i(t,s)$, $i = 1,2,\cdots$, *defined by* (2.4) *satisfy* (H2).

The proof of Lemma 2.1 follows directly from applying the Schwarz inequality and using mathematical induction.

A sequence $\{R_n\}$ of L^2 matrix kernels will be said to be "relatively uniformly convergent" to R if there is a non-negative real valued L^2 kernel p(t,s) such that for $T \in (0,\infty)$ and $\varepsilon > 0$ there is a positive integer $n_o(\varepsilon,T)$ for which

$$\|R_n(t,s) - R(t,s)\| \leq \varepsilon\, p(t,s), \text{ whenever } n \geq n_o(\varepsilon,T),\ (t,s) \in Q_T.$$

The limit matrix function R is again an L^2 kernel. An infinite series of L^2 kernels is said to be relatively uniformly convergent if the sequence formed by its partial sums is relatively uniformly convergent. Finally, an infinite series $\sum_{n=1}^{\infty} R_n(t,s)$ of L^2 kernels is said to be "relatively uniformly absolutely convergent" if the series $\sum_{n=1}^{\infty} \|R(t,s)\|$ is relatively uniformly convergent. The concept of relatively uniform convergence was introduced by E. H. Moore [3].

The following theorem gives the principal properties of relatively uniform convergence pertinent for the proofs of results listed in Theorems (2.2) - (2.5). For a proof of this theorem the reader is referred to Smithies [5; p. 24].

THEOREM 2.1. (i) _If_ $R_n(t,s) \to R(t,s)$, _as_ $n \to \infty$, _as a relatively uniformly convergent sequence of_ L^2 _kernels, and_ x _is an_ $L^2([0,T])$ _function for_ $T \in (0,\infty)$, _then_

$$\int_0^T R_n(t,s)x(s)ds \to \int_0^T R(t,s)x(s)ds, \quad \text{as} \quad n \to \infty,$$

relatively uniformly.

(ii) _If_

$$\sum_{n=1}^{\infty} R_n(t,s) = R(t,s), \tag{2.5}$$

the left-hand side being a relatively uniformly absolutely convergent series of L^2 *kernels, and* $x(t)$ *is an* $L^2([0,T])$ *function for* $T \in (0,\infty)$, *then*

$$\sum_{n=1}^{\infty} \int_0^T R_n(t,s)x(s)ds = \int_0^T R(t,s)x(s)ds,$$

the series being relatively uniformly absolutely convergent.

Proofs for the following lemma and theorem are found in Smithies [5; pp. 32-35].

LEMMA 2.2. *If* K *is an* L^2 *kernel, then for* $T \in (0,\infty)$ *we have*

$$\|R_{n+1}(t,s)\| \le (\|K\|_{2;T}^{n-1}/[(n-1)!]^{1/2})k(t)b(s;T), (t,s) \in Q_T, \; n = 1,2,\cdots.$$

THEOREM 2.2. *If* K *is an* L^2 *kernel the series* $\sum_{n=1}^{\infty} R_n$ *is relatively uniformly absolutely convergent.*

The function $R = \sum_{n=1}^{\infty} R_n$ is called the "resolvent kernel" corresponding to the kernel K, and possesses the following well-known properties, (see, for example, Miller [2; Chapter IV], Smithies [5; Chapter II], or Tricomi [9; Chapter I]).

THEOREM 2.3. *If* K *satisfies* (H2) *then for* $T > 0$ *the function* $R(t,s) = \sum_{i=1}^{\infty} R_i(t,s)$, *where* R_i *is defined by* (2.4), *is an element of* $L^2(Q_T)$ *and satisfies both equation*

$$R(t,s) = K(t,s) + \int_s^t R(t,u)K(u,s)du \tag{R}$$

and

(R′) $$R(t,s) = K(t,s) + \int_s^t K(t,u)R(u,s)du$$

for almost all (t,s) *in the region* $\mathbb{R}^+ \times \mathbb{R}^+$. *Moreover, the function* R *satisfies the inequality*

$$\|R(t,s)\| \leq \|K(t,s)\| + k(t)b(s;T)\sum_{i=2}^{\infty} [(c(T))^i/i!]^{1/2}, \text{ for } (t,s) \in Q_T.$$

The matrix equations (R), (R′) are called the "resolvent equations" associated with the kernel K. The basic property of the resolvent kernel is given in the following theorem.

THEOREM 2.4. *If hypotheses* (H1), (H2) *are satisfied, and* x *satisfies equation* (L) *on the interval* [0,T], *then*

(2.6) $$x(t) = h(t) + \int_0^t R(t,u)h(u)du, \quad \text{for} \quad t \in [0,T].$$

THEOREM 2.5. *Suppose that in addition to hypotheses* (H1) *and* (H2), *we have*

(H3) $$K(t,s) \cdot\geq\cdot 0, \quad \text{for} \quad (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Then for T > 0, *and* $x \in L^2[0,T]$ *satisfying*

(2.7) $$x(t) \cdot\leq\cdot h(t) + \int_0^t K(t,s)x(s)ds, \quad \text{for} \quad t \in [0,T],$$

we have

(2.8) $$x(t) \cdot\leq\cdot h(t) + \int_0^t R(t,s)h(s)ds, \quad \text{for} \quad t \in [0,T],$$

where R *is the resolvent kernel defined by* (2.5).

If K satisfies (H2) and (H3), then the resolvent kernel R defined by (2.5) is an element of $L^2(Q_T)$ for arbitrary $T \in (0,\infty)$, and $R(t,s) \cdot\geq\cdot 0$ for $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$. Consequently, if $T \in (0,\infty)$ and $h \in L^2[0,T]$, then for an $x \in L^2[0,T]$ satisfying (2.7) we have that there exists a vector function $r \in L^2[0,T]$ with $r(t) \cdot\geq\cdot 0$ and

$$x(t) = h(t) - r(t) + \int_0^t K(t,s)x(s)ds, \quad \text{for} \quad t \in [0,T].$$

Then by Theorem 2.4 it follows that

$$x(t) = h(t) + \int_0^t R(t,u)h(u)du - \{r(t) + \int_0^t R(t,u)r(u)du\},$$

and as $r(t) \cdot\geq\cdot 0$ on $[0,T]$, and $R(t,s) \cdot\geq\cdot 0$ for $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$, we have that x satisfies (2.8).

The inequality of Theorem 2.5 is a "Gronwall type" inequality. For a similar inequality under somewhat different hypothesis, see Chu and Metcalf [1].

3. Existence theorems for non-linear Volterra integral equations. We shall now state some hypotheses concerning the vector functions f and g occurring in the non-linear Volterra integral equation (E).

(H4) The function f is continuous on $\mathbb{R}^+$.

(H5) The function g is defined for all $(t,s,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n$, $g(t,s,x) = 0$ whenever $s > t$ and $x \in \mathbb{R}^n$; moreover, $g(t,s,x)$ is measurable in s on $[0,t]$ for each $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, and $g(t,s,x)$ is continuous in x for each fixed pair $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$.

(H6) There exists an $n \times n$ matrix function M and an n-dimensional vector function p satisfying the following conditions.

(i) $M(t,s) \cdot\geq\cdot 0$, $p(t,s) \cdot\geq\cdot 0$, for $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$, and $M(t,s) = 0$, $p(t,s) = 0$ whenever $s > t$.

(ii) $M \in L^2(Q_T)$ and $p \in L^2(Q_T)$, for each $T \in (0,\infty)$.

(iii) For $(t,s,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n$ we have that

(3.1) $$|g(t,s,x)| \le M(t,s)|x| + p(t,s).$$

(H7) If J is a compact subinterval of $\mathbb{R}^+$, and B is a compact set in $\mathbb{R}^n$, then the function $w(t;x) = \int_J g(t,s,x(s))ds$ is continuous in t on J, uniformly with respect to $x \in C(J;B)$.

(H8) For $T \in (0,\infty)$, and B a compact set in $\mathbb{R}^n$, the function $v(t,u;x) = \int_0^u g(t,s,x(s))ds$ is continuous in (t,u) on Q_T, uniformly for $x \in C([0,T];B)$.

(H9) For the functions M and p of (H6) we have that:

(i) there exists a $k_2(T;M) < \infty$ such that

(3.2) $$\int_0^t \|M(t,s)\|^2 ds \le k_2(T;M), \quad \text{for} \quad t \in [0,T];$$

(ii) there exists a $k(T;p) < \infty$ such that

(3.3) $$\int_0^t \|p(t,s)\| ds \le k(T;p), \quad \text{for} \quad t \in [0,T].$$

If hypothesis (H8) is satisfied then it follows that hypothesis (H7) is also satisfied. Hypotheses (H7), (H8) hold if certain corresponding conditions with "norm" inside the integral sign hold; for example, hypothesis (H7) is satisfied if the following condition holds for the function g.

(H7′) For J a compact subinterval of $\mathbb{R}^+$, B a bounded set in $\mathbb{R}^n$, and $t_0 \in \mathbb{R}^+$, we have that

$$\sup\{\int_J \|g(t,s,x(s)) - g(t_o,s,x(s))\| ds : x \in C(J;B)\} \to 0 \text{ as } t \to t_o.$$

In Miller [2; Chapter II], hypothesis (H7′) is employed to prove existence theorems for solutions of equation (E).

Let $T \in (0,\infty)$ be given, and $k_2(T;M)$ be the constant of (H9) such that (3.2) holds on $[0,T]$. In view of the Schwarz inequality, it follows that

$$\int_o^t \|M(t,s)\| ds \leq (Tk_2(T;M))^{1/2}, \text{ for } t \in [0,T].$$

Thus as a consequence of (H9), there exists a $k_1(T;M) < \infty$ such that

$$\int_o^t \|M(t,s)\| ds \leq k_1(T;M), \text{ for } t \in [0,T].$$

Now consider the equation (E) where the vector functions f and g are such that hypotheses (H4), (H5), and (H6) are satisfied. By definition a "solution" of (E) on an interval I, of the form $[0,T]$ or $[0,T)$, is a function which is continuous on I and satisfies (E) on this interval.

For equation (E) all existence theorems involve some sort of limiting process to pass from an "approximate" solution to an actual solution of this equation. For I of the form $[0,T]$ or $[0,T)$, a function $y:I \to \mathbb{R}^n$ is said to be an ε-approximate solution of (E) on I if y is continuous on this interval, $g(t,s,y(s))$ is integrable in s on $[0,t]$ for $t \in I$, and

$$\|y(t) - f(t) - \int_o^t g(t,s,y(s))ds\| < \varepsilon, \text{ for } t \in I. \tag{3.4}$$

Clearly a solution of (E) on an interval I is also an ε-approximate solution of (E) on this interval for every positive real number ε.

THEOREM 3.1. *Suppose that hypotheses* (H4) - (H7) *and* (H9) *are satisfied. If* $\{\varepsilon_m\}$ *is a sequence of positive constants converging to* zero, and $\{y^{(m)}(t)\}$ is a corresponding sequence of ε_m-approximate solutions satisfying (3.4) for $\varepsilon = \varepsilon_m$ on the interval $[0,T]$, then there exists a subsequence $\{y^{(m_k)}(t)\}$, $(m_1 < m_2 < \cdots)$, which converges uniformly on $[0,T]$ to a solution y of the equation (E).

Define the vector functions $r^{(m)} \in C([0,T];\mathbb{R}^n)$ as

$$(3.5)\qquad r^{(m)}(t) = y^{(m)}(t) - f(t) - \int_0^t g(t,s,y^{(m)}(s))ds, \quad t \in [0,T],$$

for $m = 1,2,\cdots$. Equality (3.5), together with the assumption that $y^{(m)}$ is an ε_m-approximate solution of equation (E) on $[0,T]$, implies that $\|r^{(m)}(t)\| < \varepsilon_m$ for all $t \in [0,T]$; in particular, the sequence $\{r^{(m)}(t)\}$ converges to zero uniformly on $[0,T]$.

From (3.5) we have that

$$(3.6)\qquad |y^{(m)}(t)| \cdot\leq\cdot |f(t)| + |r^{(m)}(t)| + \int_0^t |g(t,s,y^{(m)}(s))|ds, \text{ for } t \in [0,T],$$

and thus in view of (H6) it follows that

$$(3.7)\qquad |y^{(m)}(t)| \cdot\leq\cdot |f(t)| + |r^{(m)}(t)| + \int_0^t M(t,s)|y^{(m)}(s)|ds + \int_0^t p(t,s)ds.$$

Letting $h^{(m)}(t) = |f(t)| + |r^{(m)}(t)| + \int_0^t p(t,s)ds$, and $w^{(m)}(t) = |y^{(m)}(t)|$ for $t \in [0,T]$, (3.7) can be written as

$$(3.8)\qquad w^{(m)}(t) \cdot\leq\cdot h^{(m)}(t) + \int_0^t M(t,s)w^{(m)}(s)ds, \quad t \in [0,T].$$

Each function $h^{(m)}:[0,T] \to \mathbb{R}^n$ is an element of $L^2[0,T]$, and M is an element of $L^2(Q_T)$, while $w^{(m)} \in L^2[0,T]$. Consequently inequality (3.8), together with Theorem 2.5, implies that

$$(3.9) \qquad w^{(m)}(t) \cdot\leq\cdot h^{(m)}(t) + \int_0^t H(t,s)h^{(m)}(s)ds, \quad \text{for} \quad t \in [0,T],$$

where the matrix function H is the resolvent kernel corresponding to the kernel M. As was seen in Section 2, the function H is an element of $L^2(Q_T)$, and $H(t,s) \cdot\geq\cdot 0$ on $\mathbb{R}^+ \times \mathbb{R}^+$.

Concerning the functions $h^{(m)}$, we have that

$$\|h^{(m)}(t)\| \leq \|f(t)\| + \|r^{(m)}(t)\| + \int_0^t \|p(t,s)\|ds$$

$$\leq \sup\{\|f(t)\| : 0 \leq t \leq T\} + \varepsilon_m + \int_0^t \|p(t,s)\|ds, \text{ for all } t \in [0,T].$$

In particular, applying (3.3) of (H9), and noting that $\{\varepsilon_m\}$ converges to zero, it follows that $\{h^{(m)}(t)\}$ is uniformly bounded on the interval $[0,T]$. Let $B_o(T)$ be a real number such that $\|h^{(m)}(t)\| \leq B_o(T)$ for $t \in [0,T]$, and each $m = 1,2,\cdots$. From (3.9) it follows that

$$(3.10) \qquad \|y^{(m)}(t)\| = \|w^{(m)}(t)\| \leq \|h^{(m)}(t)\| + \int_0^t \|H(t,s)h(s)\|ds,$$

$$\leq B_o(T) + B_o(T)\int_0^t \|H(t,s)\|ds,$$

for $t \in [0,T]$, and each $m = 1,2,\cdots$.

Applying the moreover part of Theorem 2.3 to the resolvent kernel H, it follows that

$$(3.11) \qquad \|H(t,s)\| \leq \|M(t,s)\| + \left(\int_0^t \|M(t,u)\|^2du\right)^{1/2}\left(\int_s^T \|M(u,s)\|^2du\right)^{1/2}\alpha,$$

$$\text{for } (t,s) \in Q_T$$

where $\alpha = \sum_{i=2}^{\infty} ([\int_0^T \int_0^u \|M(u,s)\|^2 ds\, du]^i/i!)^{1/2}$, (see Miller [2; p. 197]).

Inequality (3.11), together with the Schwarz inequality, yield

$$\int_0^t \|H(t,s)\| ds \leq \int_0^t \|M(t,s)\| ds$$

$$+ \alpha\left(\int_0^t \int_0^t \|M(t,u)\|^2 du ds\right)^{1/2}\left(\int_0^t \int_s^T \|M(u,s)\|^2 du ds\right)^{1/2},$$

which implies

$$(3.12) \quad \int_0^t \|H(t,s)\| ds \leq \int_0^t \|M(t,s)\| ds + \alpha T^{1/2}\left(\int_0^t \|M(t,u)\|^2 du\right)^{1/2}\|M\|_{2;T},$$

for $t \in [0,T]$.

Inequality (3.12) and hypothesis (H9) imply the existence of a positive constant $B_H(T)$ such that $\int_0^t \|H(t,s)\| ds \leq B_H(T)$ for $t \in [0,T]$. From (3.10), (3.11) it then follows that

$$(3.13) \qquad \|y^{(m)}(t)\| \leq B_0(T) + B_0(T)B_H(T), \quad \text{for} \quad t \in [0,T].$$

In particular, the bound of (3.13) is independent of m. Since inequality (3.13) holds for each $t \in [0,T]$, it follows that the sequence $\{y^{(m)}\}$ is uniformly bounded on [0,T].

Now consider the sequence of functions $\{z^{(m)}\}$ defined by

$$(3.14) \qquad z^{(m)}(t) = f(t) + \int_0^t g(t,s,y^{(m)}(s))ds, \quad \text{for} \quad t \in [0,T].$$

For each $m = 1,2,\cdots$, we have that $z^{(m)} : [0,T] \to \mathbb{R}^n$. Also in view of (H6) it follows that for each $m = 1,2,\cdots$, we have

$$(3.15) \quad |z^{(m)}(t)| \leq |f(t)| + \int_0^t M(t,s)|y^{(m)}(s)|ds + \int_0^t p(t,s)ds,$$

for $t \in [0,T]$.

From (3.13), (3.15), (H4) and (H9) it follows that the sequence $\{z^m\}$ is uniformly bounded on [0,T].

If $t_1, t_2 \in [0,T]$ then

$$\int_0^{t_i} g(t_i,s,y^{(m)}(s))ds = \int_0^T g(t_i,s,y^{(m)}(s))ds, \quad \text{for } i = 1,2,$$

since $g(t,s,y^{(m)}(s)) = 0$ whenever $s > t$. We now have that for $m = 1,2,\cdots,$

$$(3.16)\quad \|z^{(m)}(t_1) - z^{(m)}(t_2)\| \le \|f(t_1) - f(t_2)\| + \left\|\int_0^T \{g(t_1,s,y^{(m)}(s)) - g(t_2,s,y^{(m)}(s))\}ds\right\|.$$

Inequalities (3.13), (3.16), together with hypotheses (H4), (H7), then imply that the sequence $\{z^{(m)}(t)\}$ is uniformly equicontinuous on the interval [0,T]. Hence by the Ascoli selection theorem, (see for example Reid [4; p. 527]), there is a continuous function y and a subsequence $\{z^{(m_k)}\}$, $(m_1 < m_2 < \cdots)$, of the sequence $\{z^{(m)}\}$ such that $\{z^{(m_k)}\}$ converges uniformly to y on [0,T].

For $t \in [0,T]$ it then follows that

$$\begin{aligned}\|y^{(m_k)}(t) - y(t)\| &= \left\|r^{(m_k)}(t) + f(t) + \int_0^t g(t,s,y^{(m_k)}(s))ds - y(t)\right\|,\\ &\le \|r^{(m_k)}(t)\| + \|z^{(m_k)}(t) - y(t)\|,\\ &\le \varepsilon_{m_k} + \|z^{(m_k)}(t) - y(t)\|, \quad t \in [0,T].\end{aligned}$$

As the sequence $\{\varepsilon_{m_k}\}$ converges to zero, and the sequence $\{z^{(m_k)}\}$ converges uniformly to y on [0,T], the sequence $\{y^{(m_k)}\}$ of ε_{m_k}-approximate solutions converges uniformly to y on [0,T]. From (3.13) it follows that there exists a compact set $B \subset \mathbb{R}^n$ such that each function $y^{(m_k)}(t)$ is contained in B for $t \in [0,T]$, and hence the limit function $y(t)$ is contained in B for $t \in [0,T]$; that is, $y \in C([0,T];B)$.

Clearly $y(0) = f(0)$, since $z^{(m_k)}(0) = f(0)$ for $k = 1,2,\cdots$, and $\{z^{(m_k)}(0)\}$ converges to $y(0)$. Now fix t_1 in the interval $[0,T]$, and fix s_1 in the interval $[0,t_1]$. From (H5) we have that $g(t,s,x)$ is continuous in x for any fixed pair $(t_1,s_1) \in Q_T$, and this, together with the uniform convergence of $\{y^{(m_k)}\}$ to y on $[0,t_1]$, implies that $\{g(t_1,s,y^{(m_k)}(s))\}$ converges to $g(t_1,s,y(s))$ for every $s \in [0,t_1]$. In view of hypotheses (H6), (H9), and inequality (3.13), we have that $g(t,s,y^{(m_k)}(s))$ is bounded by a function integrable in s on the interval $[0,t]$. As each of the functions $y^{(m_k)}$ is continuous on $[0,T]$, hypothesis (H6) implies that $g(t,s,y^{(m_k)}(s))$ is measurable in s on $[0,T]$, and hence using the Lebesgue dominated convergence theorem it follows that for each $t \in [0,T]$ we have

$$\int_0^t g(t,s,y^{(m_k)}(s))ds \longrightarrow \int_0^t g(t,s,y(s))ds, \text{ as } k \to \infty.$$

By (3.4) we have

$$z^{(m_k)}(t) = f(t) + \int_0^t g(t,s,y^{(m_k)}(s))ds, \quad \text{for} \quad t \in [0,T],$$

and hence upon taking the limit as $k \to \infty$ it follows that

$$y(t) = f(t) + \int_0^t g(t,s,y(s))ds, \quad \text{for} \quad t \in [0,T];$$

that is, the function y is a solution of equation (E) on $[0,T]$.

THEOREM 3.2. <u>Given</u> $T > 0$, <u>suppose that hypotheses</u> (H4) - (H9) <u>are satisfied</u>. <u>Then we have the following results</u>.

(i) <u>For each real number</u> $\varepsilon > 0$ <u>there is a corresponding</u> ε-<u>approximate solution of</u> (E) <u>satisfying</u> (3.4) <u>on</u> $[0,T]$.

(ii) <u>There exists a function</u> y <u>which is a solution of</u> (E) <u>on</u> $[0,T]$.

In view of Theorem 3.1, conclusion (ii) is a ready consequence of conclusion (i). Indeed, if conclusion (i) is valid and for a sequence $\{\varepsilon_m\}$ of positive constants converging to zero a corresponding approximate solution satisfying (3.4) on [0,T] with $\varepsilon = \varepsilon_m$ is denoted by $y = y^{(m)}(t)$, then from Theorem 3.1 it follows that there is a subsequence of these approximate solutions that converges uniformly on [0,T] to a solution y of (E).

For $0 < \delta < T$ define the function $y(t;\delta) : [0,T] \to \mathbb{R}^n$ as

$$(3.17) \qquad y(t;\delta) = f(t) + \int_0^{\zeta(t;\delta)} g(t,s,y(s;\delta))ds,$$

where the function $\zeta(t;\delta) : [0,T] \to \mathbb{R}^+$ is defined by

$$(3.18) \qquad \zeta(t;\delta) = 0 \text{ for } t \in [0,\delta], \quad \zeta(t;\delta) = t - \delta \text{ for } t \in [\delta,T].$$

Fix $\delta \in (0,T)$, and let r be a natural number such that $(r-1)\delta < T \le r\delta$. For $j = 1,\cdots,r-1$, the knowledge of $y(t;\delta)$ on the interval $[0,(j-1)\delta]$ provides the value of $y(t;\delta)$ on the larger interval $[0,j\delta]$, and the knowledge of $y(t;\delta)$ on the interval $[0,(r-1)\delta]$ provides the value of $y(t;\delta)$ on the entire interval [0,T]. From (3.17) and (3.18) we have that $y(t;\delta) = f(t)$ on $[0,\delta]$, and hence it follows that $y(t;\delta)$ is a well defined function on [0,T].

Clearly $y(t;\delta)$ is continuous on the interval $[0,\delta]$ in view of hypothesis (H4). From (H4), (H8) it then follows that $y(t;\delta)$ is continuous on the interval $[\delta,2\delta]$. Continuing this argument for each of the compact subintervals $[j\delta,(j+1)\delta]$, $j = 1,\cdots r-2$, and $[(r-1)\delta,T]$, we establish that the function $y(t;\delta)$ is continuous on each of these intervals. Thus it follows that $y(t;\delta)$ is a continuous function on [0,T].

Now for the function $y(t;\delta)$ of (3.17) we have that

$$|y(t;\delta)| \leq |f(t)| + \int_0^{\zeta(t;\delta)} |g(t,s,y(s;\delta))|ds,$$

$$\leq |f(t)| + \int_0^t |g(t,s,y(s;\delta))|ds, \text{ for } t \in [0,T],$$

where the second inequality holds since $\zeta(t;\delta) \leq t$ for $t \in [0,T]$. Thus it follows by hypothesis (H6) that

$$(3.19) \qquad |y(t;\delta)| \leq |f(t)| + \int_0^t M(t,s)|y(s;\delta)|ds + \int_0^t p(t,s)ds, \text{ for } t \in [0,T].$$

Inequality (3.19) implies that the functions $y(t;\delta)$ of (3.17) are uniformly bounded on $[0,T]$, independent of δ on $(0,T)$. Indeed, this uniform bound may be established by an argument similar to the one used to establish a uniform bound for the vector functions $y^{(m)}$ of Theorem 3.1. Let B denote a compact set in $\mathbb{R}^n$ such that $y(t;\delta) \in B$ for all $t \in [0,T]$ and all $\delta \in (0,T)$.

Let $\varepsilon > 0$ be given. From (H8) we have that there exists a real number $\rho = \rho(\varepsilon;T;B) > 0$ such that if $0 < \delta < \rho$ then

$$(3.20) \qquad \left\| y(t;\delta) - f(t) - \int_0^t g(t,s,y(s;\delta))ds \right\| < \varepsilon, \quad \text{for} \quad t \in [0,T];$$

that is, for $\delta \in (0,\rho)$ the function $y(t;\delta)$ is an ε-approximate solution of (E) on $[0,T]$.

The device used in the above determination of approximate solutions was introduced by Tonelli [8] in the proof of existence theorems for functional equations of Volterra type.

Theorem 3.2 establishes the existence of a solution for equation (E) that is defined on the interval $[0,T]$. Now there may be other solutions of (E) that exist on some subinterval, say $[0,T_1]$ or $[0,T_1)$ of the interval $[0,T]$, and for any one such solution there is the possibility that it may be impossible to extend its interval of definition to be the whole interval $[0,T]$. In Theorem 3.3 below we prove that under the hypotheses of Theorem 3.2 such an extension of the interval of definition is always possible.

THEOREM 3.3. Suppose that hypotheses (H4) - (H9) are satisfied and that $T \in (0,\infty)$ is given. If ϕ is a solution of equation (E) on the interval $[0,T_1)$, where $0 < T_1 < T$, then there exists on $[0,T]$ a solution y of (E) such that $y(t) \equiv \phi(t)$ for $t \in [0,T_1)$.

Hypothesis (H6), together with the result of Theorem 2.5, implies that the solution ϕ is bounded on $[0,T_1)$. Let δ be any real number satisfying $0 < \delta < T_1$, and define the function $y(t;\delta)$ for $t \in [0,T]$ as

$$y(t;\delta) = \phi(t), \quad \text{for} \quad t \in [0,T_1-\delta],$$

$$y(t;\delta) = f(t) + \int_0^{T_1-\delta} g(t,s,\phi(s))ds, \text{ for } t \in [T_1-\delta,T_1],$$

$$y(t;\delta) = f(t) + \int_0^{t-\delta} g(t,s,y(s;\delta))ds, \text{ for } t \in [T_1,T].$$

Then we have

$$(3.21) \qquad y(t;\delta) = f(t) + \int_0^{\eta(t;\delta)} g(t,s,y(s;\delta))ds, \text{ for } t \in [0,T],$$

where the real valued function $\eta(t;\delta)$ is defined as $\eta(t;\delta) = t$ on $[0,T_1-\delta]$, $\eta(t;\delta) = T_1-\delta$ on $[T_1-\delta,T_1]$, $\eta(t;\delta) = t-\delta$ on $[T_1,T]$. Since $0 \le \eta(t;\delta) \le t$ on $[0,T]$ and $|\eta(t;\delta) - t| \le \delta$ on $[0,T]$, as in the proof of the corresponding results for the vector function (3.17) it follows that $y(t;\delta)$ is a continuous function on $[0,T]$, and $y(t;\delta)$ is bounded on $[0,T]$ independent of $\delta \in (0,T_1)$. Let B denote a compact set in $\mathbb{R}^n$ such that $y(t;\delta) \in B$ for all $t \in [0,T]$ and all $\delta \in (0,T_1)$.

Hypothesis H(8) then implies that for arbitrary $\varepsilon > 0$ there exists a real number $\rho(\varepsilon;T;B) > 0$ such that if $0 < \delta < \min\{T_1,\rho(\varepsilon;T;B)\}$ then

$$\left\| y(t;\delta) - f(t) - \int_0^t g(t,s,y(s;\delta))ds \right\| < \varepsilon, \quad \text{for} \quad t \in [0,T].$$

Let $\{\varepsilon_m\}$ be a sequence of positive constants converging to zero, and choose a sequence $\{\delta_m\}$ which converges to zero, and such that $0 < \delta_m < \min\{T_1,\rho(\varepsilon_m;T;B)\}$ for $m = 1,2,\cdots$. For each m the function $y(t;\delta_m)$ is an ε_m-approximate solution of equation (E) on $[0,T]$, and thus as a consequence of Theorem 3.1 there is a subsequence $\{y(t;\delta_{m_k})\}$, $(m_1 < m_2 < \cdots)$, which converges uniformly on $[0,T]$ to a solution y of (E). As $y(t;\delta_{m_k}) = \phi(t)$ on $[0,T_1-\delta_{m_k}]$ and $\{\delta_{m_k}\}$ converges to zero, it follows that the limit function y is identical to the function ϕ on $[0,T_1)$. It is to be noted that if ϕ is a solution of (E) on the closed interval $[0,T_1]$, then by the continuity of the limit function y and the left-hand continuity of ϕ at T_1, we have that $y(T_1) = \phi(T_1)$, and therefore y is a solution of (E) on $[0,T]$ and $y(t) = \phi(t)$ for $t \in [0,T_1]$.

The following hypothesis for the function g of (E) is employed in the next theorem.

(H10) <u>There is a</u> $T' > 0$ <u>and</u> $\eta > 0$ <u>for which there exists a vector function</u> $p_o : Q_{T'} \to \mathbb{R}^n$ <u>such that</u>:

(i) $p_o \in L^2(Q_{T'})$, $p_o(t,s) \cdot\geq\cdot 0$ <u>on</u> $Q_{T'}$, <u>and</u> $p_o(t,s) = 0$ <u>whenever</u> $s > t$;

(ii) $|g(t,s,x)| \le p_o(t,s)$ for $(t,s,x) \in \Delta$, where

$$\Delta = \{(t,s,x) : (t,s) \in Q_{T'}, \|x - f(s)\| \le \eta\};$$

(iii) there exists a value $k(T',p_o) < \infty$ such that

$$\int_0^t \|p_o(t,s)\|ds < k(T',p_o), \text{ for } t \in [0,T'].$$

As a consequence of Theorem 3.2 we have the following local existence theorem.

THEOREM 3.4. If hypotheses (H4), (H5), (H8) and (H10) are satisfied then there exists a number $\beta > 0$ and a function y such that y is a solution of equation (E) on the interval $[0,\beta]$ satisfying $\|y(t) - f(t)\| < \eta$. Moreover, if ϕ is a solution of (E) on a subinterval $[0,\beta_1)$, where $0 < \beta_1 < \beta$, then there exists on $[0,\beta]$ a solution y of (E) such that $y(t) \equiv \phi(t)$ on $[0,\beta_1)$, and $\|y(t) - f(t)\| < \eta$ on $[0,\beta]$.

Let T' be the positive constant of hypothesis (H10), and define the vector function h on $R^+ \times R^+ \times R^n$ as:

$$h(t,s,x) = g(t,s,x), \text{ for } (t,s,x) \in \Delta; \tag{3.22}$$

and

$$h(t,s,x) = g(t,s,f(s) + \eta(x - f(s))/\|x - f(s)\|), \tag{3.23}$$

$$\text{for all } (t,s,x) \in \Delta^o = \{(t,s,x) : (t,s) \in Q_{T'}, \|x - f(s)\| \ge \eta\};$$

$$h(t,s,x) = 0, \text{ elsewhere.}$$

For $(t,s,x) \in \{(t,s,x) : (t,s) \in Q_T, \|x - f(s)\| = \eta\}$, we have that $g(t,s,x) = g(t,s,f(s) + \eta(x - f(s))/\|x - f(s)\|)$, and for all $(t,s,x) \in \Delta^o$ it follows that

$$\|f(s) - \{f(s) + \eta(x - f(s))/\|x - f(s)\|\}\| = \eta(\|x - f(s)\|/\|x - f(s)\|) = \eta.$$

In particular, the function h is well defined on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n$ and satisfies

(3.24) $\quad |h(t,s,x)| \leq p_o(t,s) \quad \text{for} \quad (t,s,x) \in Q_{T'} \times \mathbb{R}^n.$

Inequality (3.24) implies that the function h satisfies (H6), where M is the zero matrix and $p = p_o$ for $(t,s) \in Q_{T'}$, $p \equiv 0$ elsewhere in $\mathbb{R}^+ \times \mathbb{R}^+$. Clearly hypotheses (H4) and (H9) are satisfied for the functions f and $M \equiv 0$, $p = p_o$, respectively. The only non-trivial point in showing that the function h defined above satisfies (H5) is the measurability of h(t,s,x) in the variable s on the interval [0,t] for each $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, (see Reid [4; Problem 3, p. 99]).

For $T > 0$ and B a compact set in $\mathbb{R}^n$, we have that there exists a $k(B) < \infty$ such that if $x \in C([0,T];B)$ then $\|x(t)\| \leq k(B)$ for $t \in [0,T]$. Let $r = \max\{k(B), \sup[\|f(t)\| : t \in [0,T]] + \eta\}$, and denote by B_r the closed ball centered at the origin having radius r. For each function $x \in C([0,T];B)$ define $\hat{x}$ as follows for $s \in [0,T]$: $\hat{x}(s) = x(s)$ if $\|x(s) - f(s)\| \leq \eta$; $\hat{x}(s) = f(s) + \eta(x(s) - f(s))/\|x(s) - f(s)\|$ if $\|x(s) - f(s)\| \geq \eta$. Then the function $\hat{x}$ is an element of $C([0,T];B_r)$, and $g(t,s,\hat{x}(s)) = h(t,s,x(s))$ for $s \in [0,T]$.

Let $\varepsilon > 0$ be given. For $T > 0$ and the compact set B_r, hypothesis (H8) implies that there exists a $\rho(\varepsilon;T;B_r) > 0$ such that if $(t_o,u_o),(t_1,u_1) \in Q_T$ and $\|(t_o,u_o) - (t_1,u_1)\| < \rho(\varepsilon;T;B_r)$, then it follows that

$$\left\|\int_o^{u_o} g(t_o,s,x(s))ds - \int_o^{u_1} g(t_1,s,x(s))ds\right\| < \varepsilon, \text{ for all } x \in C([0,T];B_r).$$

Consequently it follows that if $u_o' = \min\{u_o,T'\}$ and $u_1' = \min\{u_1,T'\}$, then

$$\left\|\int_0^{u_o} h(t_o,s,x(s))ds - \int_0^{u_1} h(t_1,s,x(s))ds\right\|$$

$$= \left\|\int_0^{u_o'} g(t_o,s,\hat{x}(s))ds - \int_0^{u_1'} g(t_1,s,\hat{x}(s))ds\right\| < \varepsilon,$$

for all $x \in C([0,T];B)$ whenever $(t_o,u_o),(t_1,u_1) \in Q_T$ and satisfy $\|(t_o,u_o) - (t_1,u_1)\| < \rho = \rho(\varepsilon;T;B_r)$. That is, the function h satisfies (H8), and thus h also satisfies (H7).

As a consequence of Theorem 3.2 we have that the equation

$$(E') \qquad x(t) = f(t) + \int_0^t h(t,s,x(s))ds$$

has a solution y on each interval $[0,T]$ for $T > 0$, and, in particular the interval $[0,T']$.

Let $d = \eta + \sup\{\|f(t)\| : t \in [0,T']\} + k(T',p_o)$ where $k(T',p_o)$ is the positive constant of (H10), and let B_d denote the closed ball in $\mathbb{R}^n$ centered at the origin and having radius d. Hypothesis (H7) implies that there exists a $\rho(\eta;T';B_d)$ such that if $0 < \beta < \min\{T',\rho(\eta;T';B_d)\}$ and $z \in C([0,T'];B_d)$, then

$$(3.25) \qquad \left\|\int_0^t h(t,s,z(s))ds\right\| = \left\|\int_0^{T'} \{h(t,s,z(s)) - h(0,s,z(s))\}ds\right\|$$

$$< \eta, \quad \text{for} \quad t \in [0,\beta].$$

Since y is a solution of (E′) on $[0,T']$, and (3.24) holds, we have that

$$|y(t)| \le |f(t)| + \int_0^t p_o(t,s)ds, \quad \text{for} \quad t \in [0,T'],$$

which implies that $\|y(t)\| < d$ for $t \in [0,T']$. Thus y is an element of $C([0,T'];B_d)$ and (3.25) holds with $z = y$. From (3.25), we have that

$\|y(s) - f(s)\| < \eta$ on $[0,\beta]$, and thus $h(t,s,y(s)) = g(t,s,y(s))$ on the set $\{(t,s,y(s)) : (t,s) \in Q_\beta, \|y(s) - f(s)\| \leq \eta\}$. That is,

$$(3.26) \qquad y(t) = f(t) + \int_0^t h(t,s,y(s))ds = f(t) + \int_0^t g(t,s,y(s))ds,$$

$$\text{for } t \in [0,\beta],$$

and hence y is a solution of equation (E) on $[0,\beta]$. Application of the result of Theorem 3.3 to the equation (E′), and the fact that the resulting vector function $y(t)$ has the property that $\{(t,s,y(s)) : (t,s) \in Q_\beta\}$ is in Δ, then implies the final conclusion of the theorem.

If g satisfies a Lipschitz condition with respect to x, then the local solution of equation (E) is unique, (see, for example, Miller [2; Chapter II, Section 1]).

4. Properties of solutions. In this section we shall continue the consideration of the non-linear Volterra equation (E).

LEMMA 4.1. *Suppose that in addition to hypotheses* (H4) - (H10) *the following three conditions are satisfied:*

(i) $\{\delta_m\}$, $m = 1,2,\cdots$, *is a sequence of non-negative constants converging to zero;*

(ii) $\{f^{(m)}\}$ *is a sequence of continuous vector functions converging uniformly on the interval* $[0,T]$ *to the function* f *of equation* (E);

(iii) $\{y^{(m)}\}$ *is a sequence of continuous vector functions defined on* $[0,T]$ *and satisfying*

$$\lim_{m\to\infty} \|y^{(m)}(t) - f^{(m)}(t) - \int_{\delta_m}^t g(t,s,y^{(m)}(s))ds\| = 0, \text{ uniformly on } [0,T].$$

Then there exists a subsequence $\{y^{(m_k)}\}$, $(m_1 < m_2 < \cdots)$, *that converges*

<u>uniformly on [0,T] to a solution</u> y <u>of equation</u> (E).

Condition (ii) implies that the sequence $\{f^{(m)}\}$ is uniformly bounded and equicontinuous on [0,T]. From condition (iii) it follows that there exists an integer k such that if $m \geq k$ then

$$\|y^{(m)}(t) - f^{(m)}(t) - \int_{\delta_m}^{t} g(t,s,y^{(m)}(s))ds\| < 1$$

and hence

$$(4.1) \qquad |y^{(m)}(t)| \cdot\leq\cdot |f^{(m)}(t)| + \int_{\delta_m}^{t} |g(t,s,y^{(m)}(s))|ds + (1),$$

for $t \in [0,T]$,

where (1) denotes the vector with all components equal to 1. From (4.1) and hypothesis (H6) it follows that for $m \geq k$ we have

$$(4.2) \quad |y^{(m)}(t)| \cdot\leq\cdot |f^{(m)}(t)| + \int_{o}^{t} M(t,s)|y^{(m)}(s)|ds + \int_{o}^{t} p(t,s)ds + (1)$$

for all $t \in [0,T]$. Using an argument similar to the one employed for the corresponding vector functions $y(t;\delta)$ in the proof of Theorem 3.2, and remembering that $\{f^{(m)}(t)\}$ is uniformly bounded on [0,T], inequality (4.2) together with the result of Theorem 2.5 imply that the sequence $\{y^{(m)}\}_{m=k}^{\infty}$ is uniformly bounded on [0,T]. Since each of the functions $y^{(m)}$, $m = 1,2,\cdots,k-1$, is continuous on [0,T] it then follows that $\{y^{(m)}\}$, $m = 1,2,\cdots$, is uniformly bounded on [0,T].

Now for $m = 1,2,\cdots$ and $t_1,t_2 \in [0,T]$, we have the following inequality

$$(4.3) \qquad \|y^{(m)}(t_1) - y^{(m)}(t_2)\| \leq A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = \|y^{(m)}(t_1) - f^{(m)}(t_1) - \int_{\delta_m}^{t_1} g(t_1,s,y^{(m)}(s))ds\|,$$

$$A_2 = \|f^{(m)}(t_2) + \int_{\delta_m}^{t_2} g(t_2,s,y^{(m)}(s))ds - y^{(m)}(t_2)\|,$$

$$A_3 = \|f^{(m)}(t_1) - f^{(m)}(t_2)\|,$$

$$A_4 = \|\int_{\delta_m}^{t_1} g(t_1,s,y^{(m)}(s))ds - \int_{\delta_m}^{t_2} g(t_2,s,y^{(m)}(s))ds\|.$$

Condition (iii) implies that uniformly with respect to t the quantities A_1, A_2 are small for large m. As the sequence $\{f^{(m)}\}$ is uniformly equicontinuous on [0,T], A_3 is arbitrarily small whenever $|t_1 - t_2|$ is sufficiently small. Hypothesis (H8) implies that A_4 can be made small by restricting $|t_1 - t_2|$ to be sufficiently small. Thus the sequence $\{y^{(m)}\}$, $m = 1,2,\cdots$, is uniformly equicontinuous on [0,T].

Since $\{y^{(m)}\}$ is uniformly bounded and uniformly equicontinuous on [0,T], the Ascoli selection theorem yields a subsequence, which will also be indexed by m, that converges uniformly on [0,T] to a continuous vector function y. For $t \in (0,T]$, pick m sufficiently large so that $\delta_m \leq t$. It then follows that

$$(4.4) \qquad \|y(t) - f(t) - \int_0^t g(t,s,y(s))ds\| < D_{1m} + D_{2m} + D_{3m},$$

where

$$D_{1m} = \|y(t) - y^{(m)}(t)\| + \|f^{(m)}(t) - f(t)\|,$$

$$D_{2m} = \|y^{(m)}(t) - f^{(m)}(t) - \int_{\delta_m}^t g(t,s,y^{(m)}(s))ds\|,$$

$$D_{3m} = \|\int_{\delta_m}^t g(t,s,y^{(m)}(s))ds - \int_0^t g(t,s,y(s))ds\|.$$

The sequences $\{y^{(m)}\}$ and $\{f^{(m)}\}$ converge uniformly on [0,T] to y and f, respectively, and hence $D_{1m} \to 0$ as $m \to \infty$. Condition (iii) implies that $D_{2m} \to 0$ as $m \to \infty$. Concerning D_{3m}, it follows that

$$D_{3m} \leq \left\|\int_0^t \left(g(t,s,y^{(m)}(s)) - g(t,s,y(s))\right)ds\right\| + \left\|\int_0^{\delta_m} g(t,s,y^{(m)}(s))ds\right\|,$$

and hence the Lebesgue dominated convergence theorem along with (H8) implies that $D_{3m} \to 0$ as $m \to \infty$. Condition (iii) implies that $\{y^{(m)}(0)\}$ converges to f(0), so that y(0) = f(0), and from inequality (4.4) it follows that for arbitrary $\varepsilon > 0$ we have

$$\left\|y(t) - f(t) - \int_0^t g(t,s,y(s))ds\right\| < \varepsilon \text{ for } t \in (0,T].$$

That is, y is a solution of equation (E) on [0,T].

THEOREM 4.1. *Suppose that hypotheses* (H4) - (H10) *are satisfied, and that* $\phi(t)$ *is the unique solution of equation* (E) *on the interval* [0,T]. *Then for arbitrary* $\beta > 0$ *there are positive constants* $\varepsilon_1(\beta)$, $\varepsilon_2(\beta)$ *and* $\varepsilon_3(\beta)$ *such that if*

(a) $\tau \in [0,T)$ *and* $\tau < \varepsilon_1(\beta)$,

(b) h *is a continuous function defined on* [0,T] *such that*

$$\|h(t) - f(t)\| < \varepsilon_2(\beta) \text{ for } t \in [0,T],$$

and

(c) y *is a continuous function on* $[\tau,T]$ *satisfying*

$$\left\|y(t) - h(t) - \int_\tau^t g(t,s,y(s))ds\right\| < \varepsilon_3(\beta) \text{ for } t \in [\tau,T],$$

then we have that $\|y(t) - \phi(t)\| < \beta$ *on* $[\tau,T]$.

The theorem will be established by an indirect argument. If the conclusion of the theorem is not valid then there is at least one positive value $\beta = \beta_o$ for which there are no corresponding values $\varepsilon_1(\beta)$, $\varepsilon_2(\beta)$, $\varepsilon_3(\beta)$ satisfying the conclusion of the theorem. In particular, the conclusion does not hold for $\varepsilon_1(\beta) = \varepsilon_2(\beta) = \varepsilon_3(\beta) = 1/m$, $m = 1,2,\cdots$. That is, for each positive integer m there exists a value $\tau_m \in [0,1/m]$ and continuous functions $h^{(m)}$, $y^{(m)}$ satisfying

$$(4.5) \qquad \|h^{(m)}(t) - f(t)\| < 1/m, \quad \text{for} \quad t \in [0,T],$$

$$(4.6) \quad \|y^{(m)}(t) - h^{(m)}(t) - \int_{\tau_m}^{t} g(t,s,y^{(m)}(s))ds\| < 1/m, \text{ for } t \in [\tau_m,T],$$

while there is a point $t_m \in [\tau_m,T]$ such that $\|y^{(m)}(t_m) - \phi(t_m)\| \geq \beta_o$.

In particular, the sequence $\{\tau_m\}$ converges to zero, and from (4.5) it follows that the sequence $\{h^{(m)}\}$ converges uniformly on $[0,T]$ to the function f. We shall extend the domain of each $y^{(m)}$ to the interval $[0,T]$ by defining $y^{(m)}(t) = h^{(m)}(t) + y(\tau_m) - h^{(m)}(\tau_m)$ for $t \in [0,\tau_m]$, and thus extended the sequence $\{y^{(m)}\}$ satisfies condition (iii) of Lemma 4.1. From Lemma 4.1 it follows that there exists a subsequence $\{y^{(m_k)}\}$ which converges uniformly on $[0,T]$ to a solution y of equation (E). Now by hypothesis ϕ is the unique solution of (E) on $[0,T]$, so that $y(t) \equiv \phi(t)$ on $[0,T]$, and the uniform convergence on $[0,T]$ of the subsequence $\{y^{(m_k)}(t)\}$ to $\phi(t)$ contradicts the above condition that for each m there is a value t_m on $[\tau_m,T]$ such that $\|y^{(m)}(t_m) - \phi(t_m)\| \geq \beta_o$.

Contained in the above theorem is the following result.

COROLLARY 4.1. *Under the hypotheses of Theorem 4.1, a continuous function y satisfying the equation*

$$y(t) = h(t) + \int_{\tau}^{t} g(t,s,y(s))ds, \quad \textit{for} \quad t \in [\tau,T],$$

where τ *and* h *satisfy conditions* (a) *and* (b), *respectively, is such that* $\|y(t) - \phi(t)\| < \beta$ *for* $t \in [\tau,T]$.

COROLLARY 4.2. *Under the hypotheses of Theorem* 4.1, *if condition* (c) *is replaced by*

(c′) y *is a continuous function on* [0,T] *satisfying*

$$\|y(t) - h(t) - \int_{\tau}^{t} g(t,s,y(s))ds\| < \varepsilon_3(\beta), \text{ for } t \in [0,T],$$

then it follows that $\|y(t) - \phi(t)\| < \beta$ *on the whole interval* [0,T].

The proof of Corollary 4.2 differs from that of Theorem 4.1 only by the fact that we need not extend the domains of the functions $y^{(m)}$.

COROLLARY 4.3. *Under the hypotheses of Theorem* 4.1, *a solution* y *of the equation*

$$y(t) = h(t) + \int_{0}^{t} g(t,s,y(s))ds, \quad t \in [0,T],$$

where h *is a continuous function satisfying* $\|h(t) - f(t)\| < \varepsilon_2(\beta)$ *on* [0,T], *is such that* $\|y(t) - \phi(t)\| < \beta$ *for* $t \in [0,T]$.

The conclusion follows from Corollary 4.2, since y(t) satisfies condition (c′) of Corollary 4.2 with $\tau = 0$.

By definition, an open region $\mathcal{R}$ of the (t,s,x)-space shall be an open connected set in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, and throughout the remainder of this section we shall restrict attention to an open region which contains a point of the form (0,0,x) for some $x \in \mathbb{R}^n$. For an open region $\mathcal{R}$ we now state the following hypothesis concerning the vector function g.

(H11) *There exists a function* $\hat{p}:\mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ *such that*

(i) $\hat{p}(t,s) \cdot\geq\cdot 0$, *and* $\hat{p}(t,s) = 0$ *whenever* $s > t$;

(ii) for every $T \in (0,\infty)$, $\hat{p} \in L^2(Q_T)$ and there exists a value $k(T;\hat{p}) < \infty$ such that for every $t \in [0,T]$ we have

$$\int_0^t \|\hat{p}(t,s)\| ds < k(T;\hat{p}),$$

(iii) $|g(t,s,x)| \le \hat{p}(t,s)$ for $(t,s,x) \in \mathcal{R}$.

From the definition of an open region $\mathcal{R}$ we see that $\mathcal{R}$ contains some points of the form (t,s,x) where t and s are negative; consequently, we shall extend the domain of g by defining $g(t,s,x) = g(0,0,x)$ if either t or s is negative, and define $f(t) = f(0)$ if t is negative.

THEOREM 4.2. Suppose that $\mathcal{R}$ is an open region in the (t,s,x)-space and that hypotheses (H4), (H5), (H8), (H11) are satisfied, while $\phi(t)$ is the unique solution of equation (E) on $[0,T]$ with $\{(t,s,\phi(s)) : 0 \le s \le t \le T\}$ in $\mathcal{R}$. If β is a positive constant such that the set

$$S_\beta = \{(t,s,x) : 0 \le s \le t \le T, \|x - \phi(s)\| \le \beta\}$$

is in $\mathcal{R}$, then there exist positive constants $\delta_1(\beta)$, $\delta_2(\beta)$ such that if $0 < \delta < \delta_2(\beta)$ and $0 \le \tau < \delta_1(\beta)$, then the function $y = y(t,\delta)$ of (3.17) satisfies $\|y(t;\delta) - \phi(t)\| < \beta$ on $[0,T]$.

Let the vector function $h_o(t,s,x)$ be defined on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n$ as follows

$$h_o(t,s,x) = g(t,s,x) \text{ on } S_\beta,$$

$$h_o(t,s,x) = g(t,s,\phi(s) + \beta(x - \phi(s))/\|x - \phi(s)\|)$$

$$\text{if } (t,s) \in Q_T \text{ and } \|x - \phi(s)\| > \beta,$$

$$h_o(t,s,x) = 0, \text{ otherwise.}$$

Using an argument similar to the one employed in the proof of Theorem 3.4

for the corresponding vector function $h(t,s,x)$ of (3.21), (3.22), it follows that hypotheses (H4) - (H9) are all satisfied for the functions f and h_o. For a given $\beta > 0$ we shall denote by ε_1, ε_2 and ε_3 the quantities of Theorem 4.1 determined for the equation

$$y(t) = f(t) + \int_o^t h_o(t,s,y(s))ds \quad t \in [0,T]. \tag{4.7}$$

Since $h_o(t,s,\phi(s)) = g(t,s,\phi(s))$ for $0 \le s \le t \le T$, we have that ϕ is a solution of (4.7).

Denote by $y_o(t;\delta)$, $0 < \delta < T$, the vector functions determined by (3.17) with $g(t,s,y)$ replaced by $h_o(t,s,y)$. It now follows that there exists a compact set B such that $y_o(t;\delta) \in C([0,T];B)$ for each $\delta \in (0,T)$. Moreover, hypothesis (H8) implies that there exists a $\rho = \rho(\varepsilon_3;T;B)$ such that if $0 < u < \rho$ then for $0 \le t \le T$ we have

$$\left\| \int_o^u h_o(t,s,y_o(s))ds \right\| < \varepsilon_3/2, \text{ uniformly for } y_o \in C([0,T];B). \tag{4.8}$$

Now let $\delta_1(\beta) = \min\{\varepsilon_1,\rho\}$ and $0 < \delta_2(\beta) < \rho$. In view of (4.8) and (3.20), it follows that if $0 < \tau < \delta_1(\beta)$ and $0 < \delta < \delta_2(\beta)$ then

$$\left\| y_o(t;\delta) - f(t) - \int_\tau^t h_o(t,s,y_o(s;\delta))ds \right\| < \varepsilon_3, \text{ for } t \in [0,T].$$

Then Corollary 4.2 yields the result that $\|y_o(t;\delta) - \phi(t)\| < \beta$ on $[0,T]$, and hence the set $\{(t,s,y_o(s;\delta)) : 0 \le s \le t \le T\}$ lies in S_β and $y_o(t;\delta) = y(t;\delta)$ on $[0,T]$. Consequently for $\delta_1(\beta)$ and $\delta_2(\beta)$ the values determined above, we have that Theorem 4.2 holds.

5. Continuation of solutions. Suppose that hypothesis (H11) is satisfied for the open region $\mathcal{R}$ of (t,s,x)-space, and ϕ is a solution of equation (E) on an interval I of the form $[0,T]$ or $[0,T)$, with

$\{(t,s,\phi(s)) : 0 \le s \le t,\ t \in I\}$ in $\mathcal{R}$. The interval I is said to be a maximal interval of existence of ϕ if there does not exist a solution $y(t)$, $t \in I_o$, of (E) with $\{(t,s,y(s)) : 0 \le s \le t,\ t \in I_o\}$ in $\mathcal{R}$, where $I \subset I_o$, $y(t) = \phi(t)$ for $t \in I$, and there is a $t_o \in I_o$ such that $t < t_o$ for every $t \in I$.

If a half open interval $[0,T)$ is an interval of existence of a vector function $y(t)$ with $\{(t,s,y(s)) : 0 \le s \le t,\ t \in [0,T)\}$ in $\mathcal{R}$, then $y(t)$ is said to tend to the boundary of $\mathcal{R}$ as $t \to T^-$ if either $T = +\infty$, or $T < +\infty$ and for S an arbitrary compact subset of $\mathcal{R}$ there is a corresponding $T_o < T$ such that if $t \in (T_o,T)$ then $(t,t,y(t)) \notin S$. An alternate specification of this latter condition is that there is no point (T,T,η) in $\mathcal{R}$ which is a limit point of the set $\{(t,s,y(s)) : 0 \le s \le t,\ t \in [0,T)\}$.

The following hypothesis is employed in the next theorem.

(H8-$\mathcal{R}$) _For the open region_ $\mathcal{R}$ _of the_ (t,s,x)-_space, we have that for_ $T \in (0,\infty)$ _and_ B _a compact set in_ R^n _the function_

$$v(t,u;x) = \int_0^u g(t,s,x(s))ds$$

is continuous in (t,u), _uniformly for_ $x \in C([0,T];B)$ _satisfying_ $(t,u,x(u)) \in \mathcal{R}$ _for_ $0 \le u \le t \le T$.

THEOREM 5.1. _Suppose that_ $\mathcal{R}$ _is an open region of the_ (t,s,x)-_space, hypotheses_ (H4), (H5), (H8-$\mathcal{R}$) _and_ (H11) _are satisfied, and_ $y_o(t)$, $t \in [0,T_o)$, _is a solution of_ (E) _with_ $\{(t,s,y_o(s)) : 0 \le s \le t < T_o\}$ _in_ $\mathcal{R}$. _Then there exists a solution_ $y(t)$, $t \in [0,T)$, _of_ (E) _which is an extension of_ $y_o(t)$ _and such that_ $[0,T)$ _is a maximal interval of existence for_ $y(t)$; _moreover,_ $y(t)$ _tends to the boundary of_ $\mathcal{R}$ _as_ $t \to T^-$.

Let the rational real numbers greater than T_o be ordered as a sequence $\{T_m\}$, $(m = 1,2,\cdots)$. If $[0,T_o)$ is not a maximal interval of existence of the solution $y^o(t)$, then there exists a smallest integer m_1 such that there is a solution $y^{(1)}(t)$, $t \in [0,T_{m_1})$, of (E) which is an extension of $y^{(o)}(t)$. Now suppose that integers m_j, $(j = 1,2,\cdots,k)$, have been determined, where $m_1 < m_2 < \cdots < m_k$, and there are solutions $y^{(j)}(t)$, $t \in [0,T_{m_j}]$, of (E) such that $y^{(j)}(t)$ is an extension of $y^{(j-1)}(t)$ for $j = 1,2,\cdots,k$. If $[0,T_{m_k})$ is not a maximal interval of existence of $y^{(k)}(t)$, let m_{k+1} be the smallest integer such that $T_{m_{k+1}} > T_{m_k}$, and there is a solution $y^{(k+1)}(t)$, $t \in [0,T_{m_{k+1}})$, which is an extension of $y^{(k)}(t)$. Proceeding in this fashion, there is obtained a finite or denumerably infinite sequence of solutions $y^{(j)}(t)$, $t \in [0,T_{m_j})$. If $I = [0,T)$ is the union of the intervals $[0,T_{m_j})$, then for $t_o \in I$ and k such that $t_o \in [0,T_{m_k})$, the condition $y(t_o) = y^{(k)}(t_o)$ unambiguously defines a solution $y(t)$ of (E) on $[0,T)$ with $\{(t,s,y(s)) : 0 \leq s \leq t < T\}$ in $\mathcal{R}$. Moreover, the manner of choice of the T_{m_j} implies that $[0,T)$ is a maximal interval of existence of $y(t)$.

Now if $T < +\infty$ it will be shown that the assumption that $y(t)$ does not tend to the boundary of $\mathcal{R}$ as $t \to T^-$ leads to a contradiction. Let the point $(T,T,x_o) \in \mathcal{R}$ be a limit point of the set $\{(t,s,y(s)) : 0 \leq s \leq t < T\}$.

Hypothesis (H11) implies that there exists a natural number k such that $\|y(t)\| \leq k$ for all $t \in [0,T)$; that is, for $t \in [0,T)$ we have that $y(t)$ is in the closed ball B_k centered at the origin and having radius k.

Let $\{t_m\}$ be an increasing sequence of positive numbers converging to T, such that $y(t_m) \to x_o$ as $m \to \infty$. For each $m = 1,2,\cdots$ define the function $y^{(m)}(t)$ as

(5.1) $\quad y^{(m)}(t) = y(t)$, for $t \in [0,t_m]$; $y^{(m)}(t) = y(t_m)$, for $t \in [t_m,T]$.

For each $m = 1,2,\cdots$ the function $y^{(m)}$ is an element of $C([0,T];B_k)$. Since (T,T,x_o) is a point in $\mathcal{R}$, there exist constants $\varepsilon' > 0$ and $\beta' > 0$ such that the set $S_{\beta'}(T) = \{(t,s,x) : T-\varepsilon' \leq s \leq t \leq T, \|x-x_o\| \leq \beta'\}$ is in $\mathcal{R}$. Let $\hat{k}$ denote a positive integer such that if $m \geq \hat{k}$ then $T-\varepsilon' < t_m < T$ and $\|y^{(m)}(s) - x_o\| \leq \beta'$ for $t_m \leq s \leq T$. It then follows that for $m \geq \hat{k}$ the set $\{(t,s,y^{(m)}(s)) : t_m \leq s \leq t \leq T\}$ is in $\mathcal{R}$.

As a consequence of (5.1), if $m > n \geq \hat{k}$ we have that

$$(5.2) \quad \begin{aligned} \|y(t_m) - y(t_n)\| &\leq \|f(t_m) - f(t_n)\| \\ &\quad + \left\|\int_o^{t_m} g(t_m,s,y(s))ds - \int_o^{t_n} g(t_n,s,y(s))ds\right\|, \\ &\leq \|f(t_m) - f(t_n)\| \\ &\quad + \left\|\int_o^{t_m} g(t_m,s,y^{(m)}(s))ds - \int_o^{t_n} g(t_n,s,y^{(m)}(s))ds\right\|, \\ &\leq \|f(t_m) - f(t_n)\| \\ &\quad + \left\|\int_o^{T} \{g(t_m,s,y^{(m)}(s)) - g(t_n,s,y^{(m)}(s))\}ds\right\|, \end{aligned}$$

where the last inequality is a direct consequence of the preceding since $g(t,s,x) = 0$ for $s > t$. As hypotheses (H4) and (H8-$\mathcal{R}$) are satisfied, inequality (5.2) implies that $\|y(t_m) - y(t_n)\| \to 0$ as $m,n \to \infty$, and in view of the arbitrariness of the increasing sequence $\{t_m\}$ converging to T, the limit of $y(t)$ as $t \to T^-$ exists. When we extend y to be a continuous function on the closed interval $[0,T]$ by defining $y(T)$ to be the limit of $y(t)$ as $t \to T^-$, the function y thus defined on $[0,T]$ is an element of $C([0,T];B_k)$. Hypothesis (H8-$\mathcal{R}$) then implies that $y(t)$ satisfies equation (E) for $t = T$, and thus y is a solution of (E) on the interval $[0,T]$. From the assumption that (T,T,x_o) is a limit point of $\{(t,s,y(s)) : 0 \le s \le t \le T\}$ it follows that $y(T) = x_o$, and thus $(T,T,y(T))$ is in $\mathcal{R}$.

Since $y \in C([0,T];B_k)$ and $\{(t,s,y(s)) : 0 \le s \le t \le T\}$ is in $\mathcal{R}$, there exists a $\beta > 0$ such that the set $S_\beta(T) = \{(t,s,x) : 0 \le s \le t \le T, \|x - y(s)\| \le \beta\}$ is in $\mathcal{R}$. Define the function $\hat{y}: R^+ \to R^n$ as $\hat{y}(t) = y(t)$, for $0 \le t \le T$; $\hat{y}(t) = y(T)$, for $t \in [T,\infty)$. Now we have that there is an $\varepsilon_o > 0$ such that the set $S_\beta(T + \varepsilon_o) = \{(t,s,x) : 0 \le s \le t \le T + \varepsilon_o, \|x - \hat{y}(s)\| \le \beta\}$ is in $\mathcal{R}$.

Let $\{\delta_m\}$ $m = 1,2,\cdots$ be a decreasing sequence of positive numbers converging to zero and satisfying $\delta_m < T$ for $m = 1,2,\cdots$, and define the function $h: R^+ \times R^+ \times R^n$ as

(5.3) $$h(t,s,x) = g(t,s,x), \text{ for } (t,s,x) \in S_\beta(T+\varepsilon_o);$$

$$h(t,s,x) = g(t,s,\hat{y}(s) + \beta(x-\hat{y}(s))/\|x-\hat{y}(s)\|)$$

$$\text{for } 0 \le s \le t \le T+\varepsilon_o, \ \|x-\hat{y}(s)\| \ge \beta$$

$$h(t,s,x) = 0, \text{ otherwise.}$$

As was seen for the corresponding vector function of (3.22), (3.23), the function h satisfies (H5), (H7) and (H8), and hypothesis (H11) implies that h also satisfies (H6) with $M(t,s) \equiv 0$ and $p(t,s) = \hat{p}(t,s)$.

Now for each δ_m, $m = 1,2,\cdots$, we define

$$(5.4) \qquad y(t;\delta_m) = y(t), \quad \text{for} \quad t \in [0,T-\delta_m],$$

$$y(t;\delta_m) = f(t) + \int_0^{\eta(t;\delta_m)} h(t,s,y(s;\delta_m))ds, \quad t \in [0,T+\varepsilon_o]$$

where the real valued function $\eta(t;\delta_m)$, $(m = 1,2,\cdots)$, is defined by $\eta(t;\delta_m) = t$, for $t \in [0,T-\delta_m]$, $\eta(t,\delta_m) = T - \delta_m$, for $t \in [T-\delta_m,T]$, and $\eta(t;\delta_m) = t-\delta_m$, for $t \in [T,T+\varepsilon_o]$.

Since $\{(t,s,y(s)) : 0 \le s \le t \le T\}$ is in $\mathcal{R}$, we have that $h(t,s,y(s)) = g(t,s,y(s))$ for $0 \le s \le t \le T$, and thus $y(t)$ is a solution on $[0,T]$ of the equation

$$(5.5) \qquad y(t) = f(t) + \int_0^t h(t,s,y(s))ds.$$

As a consequence of Theorem 3.3 it follows that there is a subsequence $\{\delta_{m_k}\}$, $(m_1 < m_2 < \cdots)$, of $\{\delta_m\}$ such that $\{y(t;\delta_{m_k})\}$ converges uniformly on $[0,T+\varepsilon_o]$ to a solution y_o of the equation (5.5); moreover, $y_o(t) = y(t)$ for $t \in [0,T]$.

The function $y_o(t)$ is continuous on $[0,T+\varepsilon_o]$, and thus there exists an ε_1 satisfying $0 < \varepsilon_1 < \varepsilon_o$, and such that $\|y_o(t) - y_o(T)\| \le \beta/2$ for $t \in [T,T+\varepsilon_1]$.

For the function $y(t;\delta_{m_k})$, consider the following inequality for $t \in [T,T+\varepsilon_1]$,

$$(5.6) \qquad \|y(t;\delta_{m_k}) - y_o(T)\| \le \|y(t;\delta_{m_k}) - y_o(t)\| + \|y_o(t) - y_o(T)\|.$$

From inequality (5.6) it follows that there exists a positive integer N such that if $k \ge N$ and $t \in [T,T+\varepsilon_1]$ then

$$\|y(t;\delta_{m_k}) - y_o(T)\| = \|y(t;\delta_{m_k}) - \hat{y}(t)\| < \beta.$$

From (5.3), (5.4) it now follows that $h(t,s,y(s;\delta_{m_k})) = g(t,s,y(s;\delta_{m_k}))$ for $k \geq N$, $0 \leq s \leq t \leq T + \varepsilon_1$. That is, while $\{(t,s,y_o(s)) : 0 \leq s \leq t \leq T + \varepsilon_1\}$ is in $S_\beta(T + \varepsilon_o)$ the vector function $y_o(t)$ is a solution of (E) on $[0,T + \varepsilon_1]$ and $y_o(t) = y(t)$ on the interval $[0,T]$. This contradicts $[0,T)$ being a maximal interval of existence of $y(t)$, and thus $y(t)$ must tend to the boundary of $\mathcal{R}$ as $t \to T^-$.

The following result is clearly contained in the above theorem.

Corollary 5.1. _If hypotheses_ (H4) - (H9) _are satisfied, and_ $x(t)$ _is a solution of equation_ (E) _on a half open interval_ $[0,T_1)$, _where_ T_1 _is finite, then_ x _can be extended as a solution of_ (E) _to an interval_ $[0,T_o]$ _where_ $T_o > T_1$.

From Theorem 5.1 it follows that if $\mathcal{R}$ is the whole (t,s,x)-space and a finite T is such that [0,T) is the maximal interval of existence for a solution y of (E), then $\|y(t)\| \to \infty$ as $t \to T^-$.

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