THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

THE POLYNOMIALS GENERATED BY $f(t) \exp(p(x)u(t))$

A THESIS

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

FRANK JOSEPH PALAS

Norman, Oklahoma

1955

THE POLYNOMIALS GENERATED BY f(t) exp (p(x)u(t))

A THESIS

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

BY W. N. Huff

John B. Giever

Albert A. Grau

Arthur Bernhart

R. G. Fowler

ACKNOWLEDGMENT

I wish to express my sincere thanks and appreciation to Associate Professor William Nathan Huff for his valuable criticisms and suggestions during the preparation of this thesis, and to the members of the thesis committee for their helpful comments.

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
II.	A DIFFERENTIAL EQUATION	4
III.	K-SETS OF POLYNOMIALS	11
IV.	THE TYPES OF k-SETS	22
V.	FURTHER PROPERTIES OF k-SETS	33
BIBLIOGRAPHY		49

THE POLYNOMIALS GENERATED BY $f(t)\exp(p(x)u(t))$

CHAPTER I

INTRODUCTION

One of the first studies of polynomial sets was made by Appell [1] [1] who characterized a class of sets by demanding that they satisfy the relation $D[P_n(x)] = P_{n-1}(x)$, where $D \equiv d/dx$. These sets may be defined equivalently by the generating function $g(x,t) = f(t)\exp(xt)$ where $f(t)$ is a power series of the form $\sum_{i=0}^{\infty} b_i t^i$. The expressions $(x-a)^n/n!$ which appear in the Taylor's series expansion of a function serve as a simple example of an Appell set.

Sheffer in 1935 [6] developed a differential equation for the Appell sets and later [7] studied a generalized form of these sets which he called sets of type zero. The latter are characterized by a generating function of the form $g(x,t) = f(t)\exp(xu(t))$, where $u(t)$ is a power series of the form $\sum_{i=1}^{\infty} a_i t^i$. Meixner [5] and also Sheffer [7] were interested in finding whether there are any orthogonal sets among the sets of type zero other than the sets of Hermite and Laguerre.

[1] Numbers in square brackets refer to bibliography.

More recently Huff [2] treated the sets generated by $g(x,t) = \varphi(t)f(xt)$ where both $\varphi(t)$ and $f(xt)$ are power series in their respective variables. He found among other things the differential equation and conditions for orthogonality for these sets.

In the present discussion we develop the differential equation in explicit form for Sheffer's sets of type zero and give several special cases. In Chapter III we consider the class of sets generated by $g(x,t) = f(t) \exp(p(x)u(t))$, where $p(x)$ is a polynomial of degree k. The differential equation and a recurrence relation are obtained.

Several properties of these sets are studied in Chapter IV. In particular they are classified by studying the form of the recurrence relations they satisfy. Also conditions an arbitrary k-set must satisfy in order to be characterized by the function $g(x,t) = f(t)\exp(p(x)u(t)$ are investigated.

In Chapter V some further properties of k-sets and several special cases are presented, in particular the k-set corresponding to the finite operator $L(x,D) = D^k$ and the 2-set obtained by demanding $p(x)$ be of degree 2 and $u(t)$ be t itself. It is found this latter 2-set is necessarily of infinite A-type. The basic 2-set, that is when $g(x,t) = \exp(p(x)t)$, is found to satisfy a differential equation of infinite order. The chapter is concluded with the development of a k-set which satisfies a Rodrigues' formula.

It should be pointed out that a great deal of the material of Chapters III and IV generalizes Sheffer's article on polynomial sets of type zero [7].

Before proceeding it may be well to state the condition that a set be of A-type k, a designation to which we shall refer in the sequel. We follow Sheffer [7].

Definition: A polynomial set is a sequence of polynomials $\{P_n(x)\}$ such that $P_n(x)$ has exactly degree n, n = 0, 1, ...

Theorem: Let $\{P_n(x)\}$ be a polynomial set. There is a unique operator $L(x,D)$ for which

(1) $$L[P_n(x)] \equiv \sum_{n=1}^{\infty} L_n(x)D^n [P_n(x)] = P_{n-1}(x), \quad D = d/dx,$$

where $L_n(x)$ is a polynomial of degree at most n-1.

Definition: If the maximum of the degrees of the polynomials $L_n(x)$ in (1) is exactly k then the set $\{P_n(x)\}$ is said to be of A-type k. If the degrees of the $L_n(x)$ are unbounded then the set is said to be of infinite A-type.

The simplest and probably most interesting case is the A-type zero, that is, the $L_n(x)$ are constants for all n.

CHAPTER II

A DIFFERENTIAL EQUATION

The class of polynomials sets $\{Y_n(x)\}$ generated by the function

$$g(x,t) = f(t)e^{x\,u(t)} = \sum_{n=0}^{\infty} Y_n(x)t^n,$$

where $f(t) = \sum_{i=0}^{\infty} b_i t^i$ and $u(t) = \sum_{i=1}^{\infty} a_i t^i$, $a_1 \neq 0$, has been shown to be of A-type zero by Sheffer [7]. We shall find a recurrence relation and the differential equation for this class of sets.

The above notation will be used throughout as consistently as possible.

<u>Proposition 1:</u> The set $\{Y_n(x)\}$ satisfies the recurrence relation

$$nY_n(x) = \sum_{i=1}^{n} (d_i + ia_i x)Y_{n-i}(x), \quad n = 1,2, \cdots,$$

in which we define constants d_i so that $\sum_{i=1}^{\infty} d_i t^i = f'(t)/f(t)$.

<u>Proof:</u> Consider

$$\frac{\partial}{\partial t} g(x,t) = xu'(t)g(x,t) + \frac{f'(t)}{f(t)}g(x,t),$$

then

$$\sum_{n=1}^{\infty} nY_n(x)t^{n-1} = \sum_{i=1}^{\infty} (d_i + ia_i x)t^{i-1} \sum_{j=0}^{\infty} Y_j(x)t^j,$$
$$= \sum_{n=1}^{\infty} \sum_{i=1}^{n} (d_i + ia_i x)Y_{n-i}(x)t^{n-1}.$$

Equating the coefficients of t^{n-1} we have the result.

In order to get the differential equation in explicit form we need the following.

<u>Lemma 1</u>: The set $\{Y_n(x)\}$ satisfies the relation $Y_{n-k}(x) = a_1^{-k}\left[Y_n^{(k)}(x) - a_{k+1}^{(k)}\, Y_{n-k-1}(x) - a_{k+2}^{(k)} Y_{n-k-2}(x) - \cdots - a_n^{(k)} Y_0(x)\right.$, for $k = 1, 2, \cdots, n$.

<u>Proof</u>: From the generating function $g(x,t) = f(t)e^{x u(t)}$ we find $\frac{\partial}{\partial x} g(x,t) = u(t)g(x,t)$, from which can be written, by equating coefficients of t^n,

(2) $$Y'_n(x) = \sum_{i=1}^{n} a_i Y_{n-i}(x),\ n = 1, 2, \cdots.$$

Solving this for $Y_{n-1}(x)$ we have

(3) $$Y_{n-1}(x) = a\left[Y'_n(x) - \sum_{i=2}^{n} a_i Y_{n-i}(x)\right]$$

which is the required result for $k=1$. This result, of course, holds for $n - 2$ so that

(4) $$\begin{aligned} Y_{n-2}(x) &= a_1^{-1}\left[Y'_{n-1}(x) - \sum_{i=2}^{n-1} a_i Y_{n-1-i}(x)\right], \\ &= a_1^{-2}\left[Y''_n(x) - \sum_{i=1}^{2} a_i a_{3-i} Y_{n-3}(x) - \sum_{i=1}^{3} a_i a_{4-i} Y_{n-4}(x) - \cdots \right. \\ &\quad \left. - \sum_{i=1}^{n-1} a_i a_{n-i} Y_0(x)\right]. \end{aligned}$$

In the last step we used 2) for various indices and the derivative of 3). The coefficient of $Y_{n-m}(x)$ in (4) is of the form

$$\left(\sum_{i=1}^{\infty} a_i t^i\right)^2 = \sum_{m=2}^{\infty}\sum_{j=1}^{m-1} a_{m-j}\, a_j t^m = \sum_{m=2}^{\infty} a_m^{(2)} t^m,$$

and we follow Knopp [9] by defining $a_n^{(m)}$ so that $\left(\sum_{i=1}^{\infty} a_i t^i\right)^m = \sum_{n=m}^{\infty} a_n^{(m)} t^n$. Thus equation (4) is the required result for $k=2$. The general result is shown to hold for $k=1, 2, \cdots n$ by a somewhat involved finite induction.

The step $k=2$ has been included to indicate the formation of the coefficients.

By means of Proposition 1 and repeated application of Lemma 1 we find the following theorem, in which use is made of the well known symbol $\delta^{j_1\,,\,j_2\,,\,\cdots,\,j_i}_{m-i,\,m+1-i,\,\cdots,\,m-1}$ defined as 1(-1) for an even (odd) permutation of the j's.

Theorem 1: The set $\{Y_n(x)\}$ satisfies the differential equation

$$nY_n(x) = \sum_{i=1}^{n} (r_i + s_i x) Y_n^{(i)}(x),$$

where

$$r_m = a_1^{-\binom{m+2}{2}} \sum_{i=0}^{m-1} (-1)^i a_1^{\binom{m-i}{2}} Q_{mi}\, d_{mi},$$

$$s_1 = 1, \quad s_m = a_1^{-\binom{m+2}{2}} \sum_{i=0}^{m-2} (-1)^i (m-1-i)\, a_1^{\binom{m-i}{2}} a_{m-i}\, Q_{mi},$$

in which $Q_{mi} = \delta^{j_1\,,\,j_2\,,\,\cdots,\,j_i}_{m-i;\,m+1-i,\,\cdots,\,m-1}\, a^{(j_1)}_{m+1-i}\, a^{(j_2)}_{m+2-i} \cdots a^{(j_i)}_{m}$,

with the conventions $a_s^{(r)} = 0$ for $r > s$ and $Q_{om} = 1$.

Some Special Cases

(a) The Appell sets. In this case $u(t) = t$ so that $a_1 = 1$ and $a_i = 0$ for $i = 2, 3, \cdots$. From the definition of the $a_n^{(m)}$ we find $a_n^{(m)} = 0$ for $m \neq n$.

The recurrence relation (Proposition 1) becomes

$$nY_n(x) = (d_1 + x)Y_{n-1}(x) + d_2 Y_{n-2}(x) + \cdots d_n Y_0(x).$$

Since $r_m = d_m$, $m \geq 1$; and $s_m = 0$, $m \geq 2$ the differential equation can be written as

$$nY_n(x) = (d_1 + x)Y_n'(x) + d_2 Y_n''(x) + \cdots + d_n Y_n^{(n)}(x).$$

This is the usual differential equation for the Appell sets as obtained by Sheffer [6].

(b) The Hermite set. This is a particular set rather than a class of sets so that both functions f(t) and

$u(t)$ are specialized. Namely, we have, $f(t) = \exp(-t^2/2)$ $u(t) = t$.

Thus,

$$\sum_{i=1}^{\infty} d_i t^i = f'(t)/f(t) = -t,$$

which implies $d_2 = -1$, $d_i = 0$, for $i \neq 2$.

The recurrence relation is $nH_n(x) = xH_{n-1}(x) - H_{n-2}(x)$, and the Hermite differential equation is

$$nH_n(x) = xH_n'(x) - H_n''(x), \text{ (cf. [10]).}$$

(c) <u>The Laguerre set</u>. In this case we have $f(t) = (1-t)^{-1}$ and $u(t) = -t(1-t)^{-1} = -\sum_{i=1}^{\infty} t^i$. This implies $a_i = -1$ and $d_i = 1$ for $i = 1, 2, \cdots$.

From the definition of $a_n^{(m)}$ we find

$$a_n^{(2)} = \sum_{i=1}^{n-1} a_{n-i} a_i = (-1)^2 (n-1),$$

$$a_n^{(3)} = \sum_{i=2}^{n-1} a_{n-i} a_i^{(2)} = (-1)^3 \sum_{i=2}^{n-1} (i-1) = (-1)^3 \binom{n-1}{2},$$

and in general,

(5) $$a_n^{(h)} = \sum_{i=h-1}^{n-1} a_{n-i} a_i^{(h-1)} = (-1)^h \sum_{i=h-1}^{n-1} \binom{i-1}{h-2} = (-1)^h \binom{n-1}{h-1}.$$

In order to evaluate r and s we need to examine the expression

$$\alpha_{i\,m} \equiv a_1^{-\binom{m+1}{2}} a_1^{\binom{m-i}{2}} \delta^{j_1, j_2, \cdots, j_i}_{m-i, m+1-i, \cdots, m-1} a_{m+1-i}^{(j_1)} \cdots a_m^{(j_i)} d_{m-i},$$

$$= (-1)^{\frac{i^2+i-2mi-2m}{2} + \sum_{j=1}^{i}(m-j)} \delta^{j_1, \cdots, j_i}_{m-i, \cdots, m-1} \binom{m-i}{j_1 - 1} \cdots \binom{m-1}{j_i - 1}.$$

The exponent of (-1) is $-m$ or equivalently m so that for $i = 1$ the expression can be written as

$$\alpha_{1m} = (-1)^m \delta^{j_1}_{m-1} \binom{m-1}{j_1 - 1} = (-1)^m \binom{m-1}{1}.$$

If we assume $\alpha_{\ell,m} = (-1)^m \binom{m-1}{\ell}$ for $\ell = 1, 2, \cdots, k$ in which $m > \ell$, then

$$\alpha_{k+1,m} = (-1)^m \delta^{j_1, \cdots, j_{k+1}}_{m-k-1, \cdots, m-1} \binom{m-k-1}{j_1 - 1} \cdots \binom{m-1}{j_{k+1} - 1},$$

can be written, by means of the induction hypothesis, as

$$\alpha_{k+1,m} = (-1)^m \Big\{ \binom{m-k-1}{m-k-2}\binom{m-1}{k} - \binom{m-k-1}{m-k-1}\Big[\binom{m-k}{m-k-2}\binom{m-1}{k-1}$$

$$- \binom{m-k}{m-k}\Big[\binom{m-k+1}{m-k-2}\binom{m-1}{k-2} - \binom{m-k+1}{m-k+1}\Big[\binom{m-k+2}{m-k-2}\binom{m-1}{k-3} - \cdots$$

$$- \binom{m-3}{m-3}\Big[\binom{m-2}{m-k-2}\binom{m-1}{1} - \binom{m-2}{m-2}\binom{m-1}{m-k-2}\Big]\cdots\Big]\Big\}$$

$$= (-1)^m \sum_{i=0}^{k} (-1)^i \binom{m-k-1+i}{m-k-2}\binom{m-1}{k-i} = (-1)^m \binom{m-1}{k+1} \sum_{i=0}^{k} (-1)^i \binom{k+1}{i+1}.$$

The sum in the preceding expression can be evaluated easily by renaming the dummy index and we find

$$\alpha_{k+1,m} = (-1)^{m+1} \binom{m-1}{k+1} \sum_{i=1}^{k+1} (-1)^i \binom{k+1}{i}.$$

Thus we have $\alpha_{k+1,m} = (-1)^m \binom{m-1}{k+1}$, which completes the induction.

This result enables us to write

$$r_m = (-1)^m \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} = -1, \text{ for } m = 1,$$
$$= 0, \text{ for } m \geqq 2,$$

and $s_1 = 1$,

$$s_m = (-1)^m \sum_{i=0}^{m-2} (-1)^i (m-1-i)(-1)\binom{m-1}{i} = -1, \text{ for } m = 2,$$
$$= 0, \text{ for } m \geqq 3.$$

In view of these results, the recurrence relation simplifies to

$$nL_n(x) = \sum_{i=1}^{n} (1 - ix)\, L_{n-i}(x),$$

and the differential equation reduces to

$$nL_n(x) = (x-1)\, L_n'(x) - xL_n''(x),$$

the Laguerre differential equation of second order (cf. [10]).

As the concluding item in this section, we shall find a condition that the class of sets $\{Y_n(x)\}$ of A-type zero coincide with the class of sets generated by

$$g(x,t) = \varphi(t)f(xt) = \sum_{n=0}^{\infty} h_n(x)\, t^n,$$

with $\varphi(t) = \sum_{i=0}^{\infty} b_i t^i$ and, $f(xt) = \sum_{n=0}^{\infty} c_n (xt)^n/n!$; $c_n \neq 0$.

Many interesting properties of the class of sets $\{h_n(x)\}$ have been discovered and investigated by Huff [2]. In addition, Huff and Rainville [3] found the condition that $\{h_n(x)\}$ be of A-type k can be expressed by stating that f(xt) must be a hypergeometric function of k denominator parameters and no numerator parameters.

<u>Proposition 2</u>: A necessary and sufficient condition that the sets $\{Y_n(x)\}$ and $\{h_n(x)\}$ coincide is that $u(t) = at$, $a \neq 0$.

<u>Proof</u>: We ask that the equation

$$\psi(t) \exp(\, xu(t)\,) = \varphi(t)\, f(xt)$$

be an identity. Letting $x = 0$ we see that $\psi(t)$ and $\varphi(t)$ are the same except for a constant factor. Absorbing this factor in f(xt) we have that

$$\exp(\, xu(t)\,) = f(xt)$$

must be an identity. Thus,

$$\sum_{n=0}^{\infty} \frac{c_n (xt)^n}{n!} = \sum_{j=0}^{\infty} \left(\frac{x\, u(t)}{j!}\right)^j = \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{n=j}^{\infty} a_n^{(j)} t^n$$

$$= 1 + \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{x^i}{i!} a_n^{(i)} t^n.$$

By equating coefficients, we have

$$\sum_{i=1}^{n} \frac{a_n^{(i)} x^i}{i!} = \frac{C_n x^n}{n!}$$

from which $a_n^{(n)} = a_1^n = C_n$, and $a_n^{(j)} = 0$, $j = 1, \cdots, n-1$, so that $u(t) = a_1 t$.

<u>Corollary</u>: Polynomial sets that are both $\{Y_n(x)\}$ and $\{h_n(x)\}$ are Appell sets.

CHAPTER III

K-SETS OF POLYNOMIALS

In this chapter we first define a k-set of polynomials and then develop a linear operator which is applicable to a k-set of polynomials. Moreover, this operator corresponds to the k-set, that is, it takes each member of the k-set into the preceding member. Using the operator we find a recurrence relation, the differential equation, and several properties of the k-set $\{Y_{kn}(x)\}$.

Consider a generating function of the form

$$g(x,t) = f(t)\exp\,(p(x)u(t)), \tag{6}$$

where

$$f(t) = \sum_{i=0}^{\infty} b_i t^i \ , b_0 \neq 0,$$
$$u(t) = \sum_{i=1}^{\infty} a_i t^i \ , a_1 \neq 0,$$

and $p(x)$ is a polynomial of degree k such that $p(0)=0$.

<u>Definition</u>: A k-set of polynomials is a sequence of polynomials of degrees exactly mk, m = 0, 1, 2,···, designated in general as $\{P_{kn}(x)\}$.

To show that (6) generates a k-set, we expand as a series in t, that is,

$$e^{p(x)\,u(t)} = \sum_{j=0}^{\infty} \frac{p(x)^j}{j!} \sum_{i=j}^{\infty} a_i^{(j)} t^i,$$

$$= 1 + \sum_{m=1}^{\infty} \sum_{i=1}^{m} \frac{p^i(x)\, a_m^{(i)}}{i!} t^m,$$

$$= \sum_{m=0}^{\infty} U_{km}(x)\, t^m \text{ , where}$$

(7) $$U_{km}(x) = \sum_{i=1}^{m} p^i(x)\, a_m^{(i)}/i!.$$

Using this result we see the form of the general case is

$$g(x,t) = f(t)e^{p(x)\,u(t)} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} b_{n-m} U_{km}(x)\, t^n = \sum_{n=0}^{\infty} Y_{kn}(x)\, t^n,$$

where

(8) $$Y_{kn}(x) = \sum_{m=0}^{n} b_{n-m} U_{km}(x), \; n = 0, 1, \cdots.$$

Remark 1: $\{U_{kn}(x)\}$ and $\{Y_{kn}(x)\}$ consistently denote the k-sets (7) and (8) in the sequel, moreover $\{U_{kn}(x)\}$ is said to be the basic set relative to $\{Y_{kn}(x)\}$.

Sheffer [7] has the following, which we state as

Lemma 2: Let $L(x,D)$ be a linear operator applicable to the function x^n, $n = 0, 1, 2, \cdots$ (hence to all polynomials) and such that $L[x^n]$ is a polynomial of degree at most n. Then $L(x,D)$ has the form

(9) $$L(x,D) = \sum_{m=0}^{\infty} L_m(x) D^m, \; D \equiv d/dx,$$

valid for all polynomials, where $L_m(x)$ is a polynomial of degree at most m.

Of special interest to us is the case for which $L[x^n]$ is of degree n-k.

Lemma 3: In order that the operator (9) carry every polynomial of degree n ($n \geqq k$) into one of degree n-k, it is necessary and sufficient that $L(x,D)$ be of the form

(10) $$\sum_{m=k}^{\infty} L_m(x)D^m = \sum_{m=k}^{\infty}(l_{m0} + l_{m1}x + \cdots + l_{m,m-k}x^{m-k})\, D^m,$$

with the restriction

(11) $$\lambda_n \equiv l_{k0}\, n(n-1)\cdots(n-k+1) + l_{k+1,1}\, n(n-1)\cdots(n-k) + \cdots + l_{n,n-k}\, n! \neq 0,$$

for $n \geqq k$.

Proof: From Lemma 2 we have the operator $L(x,D)$, and demand that

$$\sum_{i=0}^{\infty}(l_{i0} + l_{i1}x + \cdots + l_{ii}x^i)\, D^i[x^n] = P_{n-k}(x),$$

where the polynomial $P_{n-k}(x)$ is zero for $n = 0, 1, \cdots, k-1$, and is of degree $n-k$ for $n \geqq k$. Rearranging the terms, we find

(12) $$\sum_{j=0}^{n}\left[l_{j0}\, n(n-1)\cdots(n-j+1) + l_{j+1,1}\, n(n-1)\cdots(n-j) + \cdots + l_{n,n-j}\, n!\right] x^{n-j} = P_{n-k}(x).$$

These relations are to be satisfied for $n = 0, 1, 2, \cdots$ so that we find it is necessary that

$$l_{i+j,i} = 0 \text{ for } j = 0, 1, \cdots, k-1 \text{ and } i = 0, 1, 2, \cdots.$$

In order to assure the degree of $P_{n-k}(x)$ is precisely $n-k$,

$$\lambda_n \equiv l_{k0}\, n(n-1)\cdots(n-k+1) + l_{k+1,1}\, n(n-1)\cdots(n-k) + \cdots + l_{n,n-k}\, n!$$

must not vanish.

The conditions are also sufficient since

$$L[x^r] = \sum_{n=k}^{\infty}(l_{n0} + l_{n1}x + \cdots + l_{n,n-k}x^{n-k})\, D^n[x^r]$$

is of degree $r-k$.

The operator may be further simplified for application to a k-set of polynomials since n takes values $n = mk$, $m = 0, 1, 2, \cdots$.

<u>Lemma 4</u>: The operator L(x,D) of the form

$$(13)\quad \sum_{m=1}^{\infty} L_{km}(x) D^{km} = \sum_{m=1}^{\infty} (l_{km,0} + l_{km,1}x + \cdots + l_{km,k(m-1)} x^{k(m-1)}) D^{km}$$

defined on any k-set $\{P_{kn}(x)\}$ carries $P_{kn}(x)$ into $P_{k(n-1)}(x)$ for each n provided

$$\lambda_{km} \equiv l_{k0}\, km(km-1)\cdots(km-k+1) + l_{k2,k}\, km(km-1)\cdots(km-2k+1) + \cdots + l_{km,k(m-1)}(km)! \neq 0.$$

<u>Proof</u>: Let $P_{kn}(x) = \sum_{i=0}^{kn} c^i_{kn} x^i$ with $c^{kn}_{kn} \neq 0$.
Using Lemma 3 we have

$$(14)\quad \sum_{m=k}^{\infty} L_m(x) D^m [P_{kn}(x)] = P_{k(n-1)}(x) \;,\; n = 1, 2, \cdots,$$

or more explicitly

$$\sum_{m=k}^{\infty} (l_{m0} + l_{m1}x + \cdots + l_{m,m-k} x^{m-k}) \sum_{i=m}^{kn} \frac{i!}{(i-m)!} c^i_{kn} x^{i-m} = \sum_{j=0}^{k(n-1)} c^j_{k(n-1)} x^j.$$

If we let $L_m(x)$ be identically zero for $m \neq nk$, $n = 1, 2, \cdots$, the remaining $L_{kn}(x)$ are sufficient, moreover they are uniquely determined. To insure $P_{k(n-1)}(x)$ (on the right of (14)) is of degree exactly (n-1)k, we must require that

$$\lambda_{km} \equiv l_{k0}\, km(km-1)\cdots(km-k+1) + l_{k2,k}\, km(km-1)\cdots(km-2k+1) + \cdots + l_{km,k(m-1)}(km)!$$

must not vanish.

<u>Theorem 2</u>: The basic k-set $\{U_{kn}(x)\}$ satisfies the relation

$$(15)\quad L[U_{kn}(x)] = U_{k(n-1)}(x),$$

with $L(x, D) = \sum_{n=1}^{\infty} (l_{kn,0} + l_{kn,1}x + \cdots + l_{kn,k(n-1)} x^{k(n-1)}) D^{kn}$,
where the $l_{nk,i}$ are uniquely determined by

$$(16)\quad l_{k0}\, u^k_{k1}\, k! = 1 \quad , \text{ and}$$

$$\sum_{h=1}^{r} \sum_{\substack{i+j=m \\ i=0,1,\cdots,k(r-h) \\ j=0,1,\cdots,k(h-1)}} l_{kh,j} \frac{(kh+i)!}{i!} u_{kr}^{kh+i} = u_{k(r-1)}^{m} \; ; \; m = 0,1,2,\cdots,k(r-1), \; r \geq 2.$$

<u>Proof:</u> The first part of the theorem follows from Lemma 4, and the rest can be seen as follows. From equation (7) we have

$$\mathcal{U}_{k0}(x) = 1 = u_0^0 \,, \quad u_{kr}^0 = 0,$$

$$\mathcal{U}_{kr}(x) = \sum_{i=1}^{r} p^i(x)\, a_r^{(i)}/i! = \sum_{i=1}^{kr} u_{kr}^i x^i, \; r = 1,2,\cdots,$$

which serve to define the constants u_{kr}^i. Writing equation (15) for r = 1, we have

(17) $$l_{k0} D^k \Big[\sum_{i=1}^{k} u_{k1}^i x^i\Big] = l_{k0} u_1^k k! = u_0^0 = 1.$$

For the r^{th} ($r \geq 2$) case, we have

$$l_{k0} \sum_{i=0}^{k(r-1)} \frac{(k+i)!}{i!} u_{kr}^{k+i} x^i + \sum_{j=0}^{k} l_{k2,j} x^j \sum_{i=0}^{k(r-2)} \frac{(2k+i)!}{i!} u_{kr}^{2k+i} x^i + \cdots$$
$$+ \sum_{i=0}^{k(r-1)} l_{kr,j} x^j (kr)!\, u_{kr}^{kr} = \sum_{m=1}^{k(r-1)} u_{k(r-1)}^m x^m,$$

in which the powers of x may be combined to give

$$l_{k0} \sum_{m=0}^{k(r-1)} \frac{(k+m)!}{m!} u_{kr}^{k+m} x^m + \sum_{\substack{i+j=m \\ i=0,1,\cdots,k(r-2) \\ j=0,1,\cdots,k}} l_{k2,j} \frac{(2k+i)!}{i!} u_{kr}^{2k+i} x^m + \cdots$$
$$+ \sum_{m=0}^{k(r-1)} l_{kr,m} x^m (kr)!\, u_{kr}^{kr} = \sum_{m=0}^{k(r-1)} u_{k(r-1)}^m x^m.$$

Equating the coefficients of x^m, ($m = 0, k, \cdots, (r-1)k$) we find the result, along with equation (17), is the required condition (16).

<u>Corollary:</u> If L(x,D) is the operator corresponding

to the basic set $\{U_{kn}(x)\}$, then $L(x,D)$ also corresponds to $\{Y_{kn}(x)\}$.

<u>Proof</u>: From equation (8), and the fact that $L(x,D)$ is a linear operator, we have

$$(18) \qquad L\left[Y_{kn}(x)\right] = \sum_{m=1}^{n} b_{n-m} U_{k(m-1)}(x) = Y_{k(n-1)}(x),$$

in which it is necessary to rename the index $(m = i + 1)$ to conform with equation (8).

Although the preceding theorem is stated in terms of the basic set $\{U_{kn}(x)\}$, it actually applies to any k-set which has the property $P_{k0}(x) = 1$.

Noting the result of the corollary, equation (18), we proceed to define

$$(19) \qquad L^{m}(x,D) \quad L\left[L^{m-1}(x,D)\right],$$

which has the property,

$$L\left[Y_{kn}(x)\right] = Y_{k(n-m)}(x).$$

It is convenient to express the polynomial involved in the generating function for $\{Y_{kn}(x)\}$ as $p(x) = p_1 x + p_2 x^2 + \cdots + p_k x^k$.

<u>Theorem 3</u>: The k-set $\{Y_{kn}(x)\}$ satisfies the functional equation

$$(20) \qquad \mathcal{L}\left[y(x)\right] \equiv \sum_{j=1}^{\infty} \sum_{i=1}^{k} d_{ij}\, x^{i}\, L^{j}\left[y(x)\right] = \lambda y(x),$$

where $\lambda = n$ for $y(x) = Y_{kn}(x)$. The constants are defined by

$$(21) \qquad f'(t)/f(t) = \sum_{n=0}^{\infty} d_{0,n+1}\, t^{n},$$

and

(22) $$p_j\, u'(t) = \sum_{n=0}^{\infty} d_{j,n+1}\, t^n, \quad j = 1, 2, \cdots, k.$$

Proof: Consider the right side of (20) for $\lambda = n$ and $y(x) = Y_{kn}(x)$ multiplied by t^n and summed on n. The result is

(23) $$\sum_{n=1}^{\infty} n Y_{kn}(x)\, t^n = t \frac{\partial}{\partial t} \sum_{n=0}^{\infty} Y_{kn}(x)\, t^n = t\frac{\partial}{\partial t}\left[f(t)\, e^{p(x)\,u(t)}\right],$$

(24) $$= t\, f(t)\, e^{p(x)\,u(t)}\left[f'(t)/f(t) + (p_1 x + p_2 x^2 + \cdots + p_k x^k)\, u'(t)\right],$$

in which we used the defining equation (6) for the k-set $\{Y_{kn}(x)\}$.

We now turn to the left side of (20), multiply by t^n and sum on n. This gives us

(25) $$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} (d_{0j} + d_{1j}x + \cdots + d_{kj}x^k)\, L^j\left[Y_{kn}(x)\right] t^n,$$

$$= \sum_{j=1}^{\infty} (d_{0j} + d_{1j}x + \cdots + d_{kj}x^k)\, t^j \sum_{i=0}^{\infty} Y_{ki}(x)\, t^i,$$

noting that $Y_{-kj}(x) \equiv 0$. By changing appropriate indices this may be written as

$$\left[\sum_{n=0}^{\infty} (d_{0,n+1} + d_{1,n+1}x + \cdots + d_{k,n+1}x^k)\, t^n\right] t\, f(t)\, e^{p(x)\,u(t)}.$$

Using relations (21) and (22), we have

$$\left[f'(t)/f(t) + (p_1 x + p_2 x^2 + \cdots + p_k x^k)\, u'(t)\right] t\, f(t)\, e^{p(x)\,u(t)},$$

which is precisely the right side of (24). Hence the expressions (23) and (25) are formally equal and so in turn are the coefficients of t^n, n = 1, 2, •••, which gives us the result (20).

Corollary: $\{Y_{kn}(x)\}$ satisfies the recurrence relation

(26) $$\sum_{j=1}^{n} (d_{0j} + d_{1j}x + \cdots + d_{kj}x^k) Y_{k(n-j)}(x) = nY_{kn}(x).$$

It may be of some interest to note that the preceding results simplify to the usual relations for the classical polynomial sets.

(a) Appell set. This set is generated by

$$g(x,t) = f(t)\exp(xt) = \sum_{n=0}^{\infty} P_n(x)\, t^n.$$

The basic set is $U_{1m}(x) = x^m/m!$ so that the operator $L(x,D) = D$, and $p_1u'(t) = 1 = d_{11}$, $d_{1j} = 0$ for $j \neq 1$. Hence, the differential equation is

$$(d_{01} + x)D[y] + d_{02}D^2[y] + d_{03}D^3[y] + \cdots = \lambda y,$$

where $\lambda = n$ for $y = P_n(x)$.

The recursion relation (26) becomes

$$nP_n(x) = (d_{01} + x)P_{n-1}(x) + d_{02}P_{n-2}(x) + d_{03}P_{n-3}(x) + \cdots + d_{0n}P_0(x).$$

(b) Hermite set. This set is generated by

$$g(x,t) = \exp(-t^2/2)\exp(xt) = \sum_{n=0}^{\infty} H_n(x)t^n.$$

Using the results in (a), we see in addition that $f'(t)/f(t) = -t = \sum_{n=0}^{\infty} d_{0,n+1}t^n$, so that the differential equation is

$$d_{11}xH'_n(x) + d_{02}H''_n(x) = xH'_n(x) - H''_n(x) = nH_n(x),$$

and the recursion formula is

$$nH_n(x) = xH_{n-1}(x) - H_{n-2}(x).$$

(c) Laguerre set. The consideration of this set is somewhat more complicated than the similar discussion for the Appell sets since the constants $l_{kn,i}$ in the linear operator must be determined by means of the relations (16) as stated in Theorem 2.

The generating function is

$$g(x,t) = (1-t)^{-1} \exp\left(-xt(1-t)^{-1} \right) = \sum_{n=0}^{\infty} L_n(x)t^n,$$

and the basic Laguerre polynomials (7) are

$$\mathcal{U}_m(x) = \sum_{i=1}^{m} \frac{p^i(x)\, a_m^{(i)}}{i\,!} = \sum_{i=1}^{m} \frac{(-x)^i}{i\,!}\binom{m-1}{i-1} = \sum_{i=1}^{m} u_m^i x^i,$$

where the evaluation of $a_m^{(i)}$ is from (5) and the u_m^i are found to be

(27) $$u_m^i = \frac{(-1)^i}{i\,!}\binom{m-1}{i-1} \text{ with } U_0(x) = u_0^0 = 1.$$

In order to evaluate the $l_{kn,i}$, we have from (16) for $r = 1$, $m = 0$ that $l_{10} = -1$. Assume $l_{j0} = -1$ for $j = 1, 2, \cdots, r-1$ then using (27), we find

$$\sum_{k=1}^{r-1} (-1)\, k!\, \frac{(-1)^k}{k!}\binom{r-1}{k-1} + l_{r0}\, r!\, \frac{(-1)^r}{r!}\binom{r-1}{r-1} = 0,$$

from which it can be seen that

(28) $$l_{r0} = -1 \text{ for } r = 1, 2, \cdots.$$

When $m = 1$ and $r = 2$ equation (16) gives $l_{21} = 0$. By an induction similar to the above in which use is made of (28) we find that $l_{r1} = 0$ for $r = 2, 3, \cdots$. In fact, we can show by an induction somewhat like the preceding that $l_{rj} = 0$, $r > j \geqq 1$.

The remaining constants involved in the differential equation can be determined as follows. From (21) we have

$$\sum_{i=0}^{\infty} d_{0,i+1} t^i = f'(t)/f(t) = (1-t)^{-1} = \sum_{i=0}^{\infty} t^i,$$

so that $d_{0,n+1} = 1$, for $n = 0, 1, 2, \cdots$. In addition, from (22) we have

$$p_1 u'(t) = -\left(\sum_{i=1}^{\infty} t^i\right)' = -\sum_{i=0}^{\infty}(i+1)t^i = \sum_{i=0}^{\infty} d_{1,i+1}\, t^i,$$

from which it is seen that $d_{1,n+1} = -(n+1)$, $n = 0, 1, 2, \cdots$.

Using the results thus far obtained we may write the differential equation (20) as

$$\sum_{j=1}^{\infty}(d_{0j} + d_{1j}\, x)L^j\,[y] = \lambda y,$$

or

$$\sum_{j=1}^{\infty}(1 - jx)\, L^j\,[y] = \lambda y,$$

where $L = \sum_{n=1}^{\infty}(-1)D^n$. This explicit form of the operator gives the incidental result

$$L'_n(x) + L''_n(x) + \cdots + L^{(n)}_n(x) = -L_{n-1}(x).$$

Also we have

$$L^j = \left[\sum_{n=1}^{\infty}(-1)D^n\right]^j = \sum_{n=j}^{\infty}(-1)^j\binom{n-1}{j-1}D^n,$$

in which use was made of relation (5). Thus the differential equation takes the form

$$\sum_{j=1}^{\infty}(1 - jx)\sum_{n=j}^{\infty}(-1)^j\binom{n-1}{j-1}D^n\,[y] = \lambda y,$$

which may be rearranged as

$$\sum_{m=0}^{\infty}\sum_{i=0}^{m}\left[1 - (i+1)x\right](-1)^{i+1}\binom{m}{i}D^{i+1}[y] = \lambda y.$$

Splitting the sum on i into three sums corresponding to the three terms in the bracket, we see that the first and third sums are zero for $m \geqq 1$, while the second sum is zero for

$m \geqq 2$. Hence we arrive at the Laguerre differential equation

$$(-1+x)D[y] - xD^2[y] = \lambda y,$$

which may be written more familiarly as

$$(x-1)L_n'(x) - xL_n''(X) = nL_n(x).$$

CHAPTER IV

THE TYPES OF k-SETS

In analogy to Sheffer's classification of polynomial sets as B and C type [7], we shall define a classification for k-sets of polynomials, and various characterizations will be found for the k-set $\{Y_{kn}(x)\}$ and for sets of B_k and C_k type. We shall define B_k and C_k type below.

Lemma 5: To each k-set $\{P_{kn}(x)\}$ corresponds a unique sequence of polynomials $\{Q_{kn}(x)\}$, with $Q_{kn}(x)$ of degree not exceeding nk, such that

(29) $nP_{kn}(x) = Q_{k1}(x)P_{k(n-1)}(x)+Q_{k2}(x)P_{k(n-2)}(x) + \cdots+Q_{kn}(x)P_{k0}(x)$,

$n = 1, 2,\cdots$.

Proof: Set $n = 1, 2, \cdots$ successively and it is seen that the $Q_{kn}(x)$ are uniquely determined. However, the $Q_{kn}(x)$ do not determine the k-set $\{P_{kn}(x)\}$ although, as will be seen, they characterize the k-set in several ways.

Definition: A k-set $\{P_{kn}(x)\}$ is of C_k-type l if the maximum degree of the $Q_{kn}(x)$ in (29) is $l+1$. If the degrees of the polynomials $Q_{kn}(x)$ are unbounded, the k-set $\{P_{kn}(x)\}$ is of infinite C_k-type.

Proposition 3: $\{Y_{kn}(x)\}$ is of C_k-type k-1.

Proof: The statement can be seen from equation (26).

Lemma 6: To each k-set $\{P_{kn}(x)\}$ corresponds a unique sequence $\{T_{kn-1}(x)\}$, with $T_{kn-1}(x)$ a polynomial of degree not exceeding nk-1 such that

(30) $P'_{kn}(x) = T_{k-1}(x)P_{k(n-1)}(x) + T_{k2-1}(x)P_{k(n-2)}(x) + \cdots + T_{kn-1}(x)P_{k0}(x);$ $n = 1, 2, \cdots.$

Proof: If we set $n = 1, 2, \cdots$ successively, the $T_{kn-1}(x)$ are uniquely determined.

Definition: A k-set $\{P_{kn}(x)\}$ is of B_k-type ℓ if the maximum degree of the $T_{ki-1}(x)$ in (30) is ℓ.

Proposition 4: $\{Y_{kn}(x)\}$ is of B_k-type k-1.

Proof: From the generating function for the k-set $\{Y_{kn}(x)\}$ (see (6)), we find that the partial derivative with respect to x can be written

$$\sum_{n=1}^{\infty} Y'_{kn}(x)\, t^n = p'(x)\, u(t)\, g(x,t)$$
$$= p'(x) \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} a_{n-j}\, Y_{kj}(x)\, t^n.$$

Equating the coefficients of t^n we have

(31) $$Y'_{kn}(x) = p'(x) \sum_{j=0}^{n-1} a_{n-j}\, Y_{kj}(x)$$

It should be noticed that when k = 1, the B_1-type ℓ and C_1-type ℓ classes coincide with Sheffer's B and C type ℓ respectively.

We now turn our attention to conditions that a k-set must satisfy in order to be of class B_k-type ℓ.

Lemma 7: The relation (30),

$$P'_{kn}(x) = T_{k-1}(x)P_{k(n-1)}(x) + T_{k2-1}(x)P_{k(n-2)}(x) + \cdots + T_{kn-1}(x)P_{k0}(x)$$

is equivalent to

(32) $$\frac{\partial}{\partial x} g(x,t) = T(x,t)g(x,t),$$

where

$$g(x,t) = \sum_{n=0}^{\infty} P_{kn}(x)t^n,$$

$$T(x,t) = \sum_{n=1}^{\infty} T_{kn-1}(x)t^n.$$

Proof: Suppose (30) is given, then

$$\frac{\partial}{\partial x} g(x,t) = \sum_{n=1}^{\infty} P'_{kn}(x)t^n = \sum_{n=1}^{\infty} \sum_{j=1}^{n} T_{kj-1}(x)\, P_{k(n-j)}(x)t^n$$
$$= \sum_{i=1}^{\infty} T_{ki-1}(x)t^i \quad \sum_{j=0}^{\infty} P_{kj}(x)t^j.$$

Conversely, if (32) is given then (30) follows by equating coefficients of t^n.

We have as the solution of (32) that

(33) $$g(x,t) = g(0,t) \exp \int_0^x T(x,t)dx.$$

The integrand of this expression can be written

$$T(x,t) = \sum_{m=1}^{\infty} T_{km-1}(x)t^m = \sum_{m=1}^{\infty} \sum_{i=0}^{km-1} t_{mi} x^i t^m.$$

We rearrange this in powers of x as

$$T(x,t) = \sum_{j=0}^{k-1} \sum_{i=1}^{\infty} t_{ij} t^i x^j + \sum_{j=k}^{2k-1} \sum_{i=2}^{\infty} t_{ij} t^i x^j + \cdots$$
$$+ \sum_{j=k(m-1)}^{km-1} \sum_{i=km}^{\infty} t_{ij} t^i x^j + \cdots,$$

so that the integral, after changing indices, is

$$\int_0^x T(x,t)\, dx = \sum_{j=1}^{k} \sum_{i=1}^{\infty} h_{ij} t^i x^j + \sum_{j=k+1}^{2k} \sum_{i=2}^{\infty} h_{ij} t^i x^j + \cdots,$$

where

$$h_{ij} = t_{ij\,j-1}/j.$$

In order that $\{P_{kn}(x)\}$ be of B_k-type ℓ, $T(x,t)$ must be a polynomial of degree ℓ in x, and after integration it must be of degree $\ell+1$. Thus we have the following.

Proposition 5: A necessary and sufficient condition that a k-set $\{P_{kn}(x)\}$ be of B_k-type ℓ is that the generating function for the k-set is of the form

$$g(x,t) = f(t)\exp\left(\sum_{m=1}^{r}\sum_{i=k(m-1)+1}^{km} H_{mi}(t)x^i + \sum_{i=kr+1}^{\ell+1} H_{r+1,i}(t)x^i\right),$$

where r is determined by $rk < \ell \leqq (r+1)k$ and

$$H_{mi}(t) = h_{mi}t^m + h_{m+1,i}t^{m+1} + \cdots .$$ [1]

Note that for $\ell = k-1$ and $H_{1i}(t) = p_i u(t)$, $i = 1, 2, \cdots, k$, we have

$$g(x,t) = f(t)\exp\left(\sum_{i=1}^{k} p_i x^i u(t)\right) = f(t)\exp(p(x)u(t)),$$

which is the generating function for the k-set $\{Y_{kn}(x)\}$.

Lemma 8: The relations

(29) $nP_{kn}(x) = Q_{k1}(x)P_{k(n-1)}(x) + Q_{k2}(x)P_{k(n-2)}(x) + \cdots + Q_{kn}(x)P_{k0}(x)$

for $n = 1, 2, \cdots$ are equivalent to

(34)
$$\frac{\partial}{\partial t}g(x,t) = Q(x,t)g(x,t),$$
$$g(x,t) = \sum_{n=0}^{\infty} P_{kn}(x)t^n$$
$$Q(x,t) = \sum_{i=1}^{\infty} Q_{ki}(x)t^i .$$

Proof: Suppose (29) is given, then

---

[1] We should note the restrictions $h_{1k} \neq 0$ to insure a k-set and $H_{r+1,\ell+1}(t) \neq 0$ to insure exactly B -type ℓ. In addition, we see from the restriction $h_{1k} \neq 0$ that $\ell \geqq k-1$. Thus the k-set $\{Y_{kn}(x)\}$ (see Proposition 4) is the simplest k-set in the B_k-type classification.

$$\frac{\partial}{\partial t} g(x,t) = \sum_{n=1}^{\infty} nP_{kn}(x)t^{n-1} = \sum_{n=1}^{\infty}\sum_{i=1}^{n} Q_{ki}(x)P_{k(n-i)}(x)t^{n-1}$$
$$= \sum_{i=1}^{\infty} Q_{ki}(x)t^{i-1} \sum_{j=0}^{\infty} P_{kj}(x)t^{j}.$$

Conversely, if (34) is given then (29) follows by equating the coefficients of t^n.

By means of Lemma 8, we are able to find a relation which shows the equivalence of the B_k-type ℓ and C_k-type ℓ classifications. Solving equation (34) we have

(35) $\quad g(x,t) = c \exp \int_0^t Q(x,t)dt, \; c = g(x,0).$

Since an arbitrary k-set $\{P_{kn}(x)\}$ satisfies both equations (33) and (35) we see the two expressions for the generating function must be equal, that is

(36) $$g(t) \exp \int_0^x T(x,t)dx = c \exp \int_0^t Q(x,t)dt.$$

If we find the logarithm of both sides of (36) and then take the partial derivative with respect to x, the result is

(37) $$T(x,t) = \frac{\partial}{\partial x} \int_0^t Q(x,t)dt.$$

<u>Theorem 4</u>: A k-set $\{P_{kn}(x)\}$ is B_k-type ℓ if and only if it is C_k-type ℓ.

<u>Proof</u>: If the k-set is C_k-type ℓ, then $Q(x,t)$ is a polynomial of degree $\ell+1$ in x, and hence by (37) $T(x,t)$ is of degree ℓ so that the k-set is B_k-type ℓ.

Although the two classifications of k-sets are equivalent, it is of some interest to see the condition that a k-set is of C_k-type ℓ. For convenience we restate equation

(35) $\quad g(x,t) = c \exp \int_0^t Q(x,t)dt, \; c = g(x,0).$

In order that $g(x,t)$ generate a k-set, it is necessary that one of the following statements hold.

(a) The $Q_{ki}(x)$ are of degree ik for $i = 1, 2, \cdots$. This implies infinite C_k-type.

(b) The $Q_{ki}(x)$ are of degree ik for $ik \leq \ell+1$ and the maximum of the degrees of $Q_{ki}(x)$ for $ik > \ell+1$ is $\ell+1$. This implies C_k-type ℓ, $(\ell+1 \geq k)$.

We incorporate the second condition in the statement

$$Q(x,t) = \sum_{i=1}^{\infty} Q_{ki}(x)t^{i-1}$$
$$= \sum_{i=1}^{r} \sum_{j=0}^{ki} q_{ji} x^j t^{i-1} + \sum_{i=r+1}^{\infty} \sum_{j=0}^{\ell+1} q_{ji} x^j t^{i-1},$$

where r is determined so that $rk \leq \ell+1<(r+1)k$. Interchanging the summations the expression can be written

$$(38) \quad Q(x,t) = \sum_{j=0}^{\ell+1} x^j \sum_{i=s}^{\infty} q_{ji} t^{i-1}, \quad \begin{cases} j=0 & ; \; s=1, \\ 1 \leq j \leq rk & ; (s-1)k+1 < j \leq sk, \\ j > rk & ; \; s = r+1, \end{cases}$$

with the same condition on r, i.e. $rk \leq \ell+1<(r+1)k$. Upon integrating the relation (38), we have

<u>Proposition 6</u>: An arbitrary k-set $\{P_{kn}(x)\}$ is of finite C_k-type ℓ if and only if the generating function is of the form

$$g(x,t) = c \exp \left(\sum_{j=0}^{\ell+1} x^j G_j(t) \right) \; ; \; c = g(x,0),$$

and

$$G_j(t) = \sum_{i=s}^{\infty} q_{ji} t^i/i, \quad \begin{cases} j=0 & ; \; s=1, \\ 1 \leq j \leq rk & ; (s-1)k+1 < j \leq sk, \\ j > rk & ; \; s = r+1, \end{cases}$$

where r is determined by $rk \leq \ell+1 < (r+1)k$.

Example 1: If $\ell = k-1$ then $s = 1$, and if

$$G_j(t) = p_j \sum_{i=1}^{\infty} a_i t^i, \; j = 1, 2, \cdots, k;$$

then

$$g(x,t) = c \exp \sum_{j=0}^{k} x^j G_j(t),$$

$$= c\, e^{G_0(t)} e^{p(x)u(t)} = f(t)\, e^{p(x)u(t)},$$

which is the generating function for the k-set $\{Y_{kn}(x)\}$. We have used the following.

Remark 2: If $G(t)$ is a power series then $\exp G(t)$ can be written as a power series $f(t)$ where $f(0) > 0$.

Example 2: Suppose $k = 1$, then

$$G_j(t) = \sum_{i=s}^{\infty} q_{ji}/i \; t^i \quad \text{where for} \quad \begin{cases} j = 0 & ; \quad s = 1, \\ 1 \leq j \leq k = \ell+1 & ; \quad s = j. \end{cases}$$

Thus the generating function for a C_1-type ℓ set is of the form

$$g(x,t) = c \exp \left(\sum_{i=1}^{\infty} \frac{g_{0i}}{i} t^i\right) \exp \left(\sum_{j=1}^{\ell+1} x^j \sum_{i=j}^{\infty} \frac{g_{ji}}{i} t^i\right),$$
$$= f(t) \exp \left(\sum_{j=1}^{\ell+1} x^j G_j(t)\right),$$

in which we make use of the preceding remark. This result coincides with Theorem 5.3 of Sheffer [7].

Example 3: If in addition to $k = 1$ we suppose $\ell = 0$, then

$$g(x,t) = f(t) \exp (xG(t)),$$

which is a characterizing relation for sets of type zero.

According to Lemmas 5 and 6, we found that to each k-set there correspond unique sequences of polynomials

$\{Q_{kn}(x)\}$ and $\{T_{kn-1}(x)\}$ such that relations (29) and (30) respectively are satisfied. As we noted there, these sequences do not determine the k-set uniquely. However they do characterize the k-set in the following ways.

<u>Proposition 7</u>: An arbitrary k-set $\{P_{kn}(x)\}$ is a $\{Y_{kn}(x)\}$ k-set if and only if the relation

$$P'_{kn}(x) = \sum_{j=1}^{n} a_j p'(x) P_{k(n-j)}(x),$$

is satisfied, that is

$$T_{kj-1}(x) = a_j p'(x) \text{ in } (30).^1$$

<u>Proof</u>: For the k-set $\{Y_{kn}(x)\}$ we have by (31),

$$Y'_{kn}(x) = \sum_{j=1}^{n} p'(x) a_j Y_{k(n-j)}(x).$$

Conversely, if

$$T_{kj-1}(x) = a_j p'(x),$$

then

$$T(x,t) = p'(x) \sum_{j=1}^{\infty} a_j t^j = p'(x) u(t).$$

From equation (33) we write

$$g(x,t) = g(t) \exp \int_0^x T(x,t) dx,$$

$$= g(t) \exp (p(x) u(t)),$$

so that the k-set is $\{Y_{kn}(x)\}$.

We now find a particularly simple characterization of a $\{Y_{kn}(x)\}$ k-set.

---

[1]As usual p(x) is a polynomial of degree k such that p(0) = 0.

<u>Proposition 8</u>: An arbitrary k-set $\{P_{kn}(x)\}$ is a $\{Y_{kn}(x)\}$ k-set if and only if each of the polynomials of the sequence $\{Q_{kj}(x)\}$ in relation (29) is of degree k.

<u>Proof</u>: For the k-set $\{Y_{kn}(x)\}$ we have from (26) that

$$Q_{kj}(x) = d_{0j} + d_{1j}x + d_{2j}x^2 + \cdots + d_{kj}x^k.$$

Conversely if

$$Q_{kj}(x) = \sum_{i=0}^{k} q_{ji}x^i,$$

then

$$Q(x,t) = \sum_{j=1}^{\infty}\sum_{i=0}^{k} q_{ji}x^i t^{j-1},$$
$$= \sum_{j=1}^{\infty}(q_{j0} + ja_j p(x))t^{j-1},$$

for some values of a_1, a_2, $\cdots$, From equation (35) we have

$$g(x,t) = c \exp \int_0^t \sum_{j=1}^{\infty}(q_{j0} + ja_j p(x))t^{j-1}\,dt,$$

$$= c \exp \sum_{j=1}^{\infty} \frac{q_{j0}}{j} t^j \exp(p(x)u(t)),$$

$$= f(t) \exp(p(x)u(t)),$$

in which we use Remark 2.

The relation,

(30) $nP_{kn}(x) = Q_{k1}(x)P_{k(n-1)}(x) + Q_{k2}(x)\,P_{k(n-2)}(x) + \cdots + Q_{kn}(x)\,P_{k0}(x),$

may be used to obtain the following result.

<u>Proposition 9</u>: Every k-set $\{P_{kn}(x)\}$ satisfies the functional equation

$$\sum_{m=1}^{\infty} Q_{km}(x)L^m[y(x)] = \lambda y(x),$$

with $\lambda = n$ for $y(x) = P_{kn}(x)$, and $\{Q_{kn}(x)\}$ the unique sequence of polynomials determined by (30).

Proof: From Lemma 4 we have the operator $L(x,D)$, which by the convention expressed by equation (19) has the property

$$L^m\left[P_n(x)\right] = P_{n-m}(x), \quad m = 1, 2, \cdots, n.$$

This property in conjunction with equation (30) gives the result.

As the concluding item (Proposition 2) of section 2 we found the condition that the class of polynomial sets generated by

$$g(x,t) = \varphi(t)f(xt) = \sum_{n=0}^{\infty} h_n(x)t^n$$

with

$$\varphi(t) = \sum_{i=0}^{\infty} b_i t^i, \quad b_0 \neq 0,$$
$$f(xt) = \sum_{n=0}^{\infty} c_n (xt)^n/n!, \quad c_n \neq 0,$$

were of A-type zero, that is, $g(x,t) = \varphi(t) \exp(x\,u(t))$. We shall carry this discussion a little further. If we insist that $c_n \neq 0$ for all $n = 0, 1, 2, \cdots$ then the only sets in the class of sets $\{h_n(x)\}$ that are also among the k-sets $\{Y_{kn}(x)\}$ are the Appell sets. However if this condition is removed, we may state

Proposition 10: The k-sets $\{Y_{kn}(x)\}$ coincide with the sets $\{h_n(x)\}$ if and only if

$$g(x,t) = \varphi(t) \exp(ax^k t^k).$$

Proof: The k-set $\{Y_{kn}(x)\}$ is generated by $g(x,t) = \psi(t) \exp(p(x)u(t))$ with $p(x)$ of degree k (exactly) and $p(0) = 0$. The sets coincide whenever

$$\psi(t)\exp(p(x)u(t)) = \varphi(t)f(xt)$$

is an identity. Setting $x = 0$ we see $\psi(t)$ and $\varphi(t)$ differ only by a constant factor. Absorbing this constant in $f(xt)$ we state that $f(xt) = \exp\,(p(x)u(t))$ must be an identity, that is

$$\sum_{i=0}^{\infty} c_i (xt)^i/i! = \sum_{i=0}^{\infty} \frac{p^i(x)}{i} u^i(t) = \sum_{i=0}^{\infty} \frac{p^i(x)}{i} \sum_{j=1}^{\infty} a_j^{(i)} t^j,$$

$$= 1 + p(x)\, a_1^{(1)} t + \cdots + \Big[\sum_{j=1}^{n} \frac{p^j(x)}{j} a_n^{(j)}\Big] t^n \cdots .$$

Since the power of x involved in the bracket is n alone and $p(x)$ is degree k, we have $a_{ik}^{(i)} \neq 0$ for $i = 1, 2, \cdots$ and $a_n^{(i)} = 0$ for $n \neq ik$. This also implies $a_{ik}^{(i)} = (a_k)^i$, so that $u(t) = a_k t^k$.

According to a theorem of Huff [2; 3.4] his set $\{h_n(x)\}$ is of B-type ℓ if and only if $f(xt) = \exp H(xt)$ where $H(xt)$ is a polynomial in xt of degree $\ell+1$. Thus $\{Y_{kn}(x)\}$ which has generating function $g(x,t) = f(t) \exp\,(p(x)u(t))$, $p(x)$ of degree k could satisfy this condition only if $p(x) = x^k$, $u(t) = at^k$ or $\{Y_{kn}(x)\}$ is of B-type k-1. From Proposition 4 $\{Y_{kn}(x)\}$ is of B_k-type k-1 so that our result (Proposition 10) is consistent with Huff's theorem.

CHAPTER V

FURTHER PROPERTIES OF k-SETS

The simplest form of the generating function for the k-set $\{Y_{kn}(x)\}$ is

$$(39) \qquad g_s(x,t) = \exp(x^k t) = \sum_{n=0}^{\infty} \frac{x^{kn}}{n!} t^n .$$

From Theorem 2 we have the existence of an operator L(x,D) corresponding to this set with the constants defined by equations (16), that is

$$(16)' \qquad \ell_{k1,1} u^k_{k1}\, k! = 1,$$

$$\sum_{h=1}^{r} \sum_{i+j=m} \ell_{kh,j} \frac{(kh+i)!}{i!} u^{kh+i}_{kr} = u^m_{k(r-1)}, \quad m = 0, 1, \cdots, k(r-1).$$

But $\quad u^i_{km} = 1/m!$ for $\quad i = km,$

$\qquad\qquad = 0 \quad$ for $\quad i = 0, 1, \cdots, km-1,$

so that in (16)' we have $rk-k = m = i+j = (hk+i)-k$ which implies $j = hk-k$ and $i = rk-hk$. Thus the relation (16)' simplifies to

$$(40) \qquad \sum_{h=1}^{r} \ell_{kh} \binom{kr}{kh} (kh)! = r, \quad r = 1, 2, \cdots,$$

and the operator reduces to

$$(41) \qquad L_s(x,D) = \sum_{h=1}^{\infty} \ell_{kh}\, x^{k(h-1)} D^{kh},$$

in which $\ell_{kh,k(h-1)} \equiv \ell_{kh}$.

We shall now investigate conditions under which an arbitrary k-set $\{P_{kn}(x)\}$ corresponds to the operator $L_s(x,D)$. Clearly a sufficient condition is that there exist a sequence of constants $\{r_i\}$ such that

$$P_{kn}(x) = \sum_{i=0}^{n} r_i \frac{x^{k(n-i)}}{(n-i)!} .$$

This condition is equivalent to considering the generating function

$$g(x,t) = f(t)\exp(x^k t)$$

(42)
$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} b_i \frac{x^{k(n-i)}}{(n-i)!} t^n,$$

instead of (39). The following statement will show that there are k-sets which correspond to the operator $L_s(x,D)$ other than those noted in (42).

<u>Proposition 11</u>: An arbitrary k-set $\{P_{kn}(x)\}$ corresponds to the operator $L_s(x,D)$ defined by (41) if and only if there exists a sequence of polynomials $\{r_{00},\ r_{0i}+r_{1ni}x + \cdots + r_{k-1,ni}x^{k-1}\}$ such that

$$P_{kn}(x) = r_{00}\frac{x^{kn}}{n!} + \sum_{i=1}^{n}(r_{0i} + r_{1ni}x + \cdots + r_{k-1,ni}x^{k-1}) \frac{x^{k(n-i)}}{(n-i)!},$$

where

$$r_{sni} = \prod_{m=1}^{n-i} \gamma_{sm} = (n-i)!\, \kappa_{si},$$

with $\gamma_{sm} = \sum_{j=1}^{m} \ell_{kj}\binom{km+s}{kj}(kj)!$, $s = 1, 2, \cdots k-1$.

<u>Proof</u>: For any k-set $\{P_{kn}(x)\}$ there exists a sequence of polynomials $\{r_{ono}, r_{oni} + r_{1ni}x + \cdots + r_{k-1,ni}x^{k-1}\}$ such that

(43) $P_{kn}(x) = r_{ono}\frac{x^{kn}}{n!} + \sum_{i=1}^{n}(r_{oni} + r_{1ni}x + \cdots + r_{k-1,ni}x^{k-1})\frac{x^{k(n-i)}}{(n-i)!}.$

We demand that $L_s[P_{kn}(x)] = P_{k(n-1)}(x)$, that is

$$\sum_{i=0}^{n} r_{oni} L_s\left[\frac{x^{k(n-i)}}{(n-i)!}\right] + \sum_{i=1}^{n} r_{1ni} L_s\left[\frac{x^{k(n-i)+1}}{(n-i)!}\right]$$

$$+ \cdots + \sum_{i=1}^{n} r_{k-1,ni} L_s\left[\frac{x^{k(n-i)+k-1}}{(n-i)!}\right],$$

$$= \sum_{i=0}^{n-1} r_{oni}\frac{x^{k(n-1-i)}}{(n-1-i)!} + \sum_{i=1}^{n-1} r_{1ni}\sum_{h=1}^{n-i} l_{kh}x^{k(h-1)}D\left[\frac{x^{k(n-i)+1}}{(n-i)!}\right]$$

$$+ \cdots + \sum_{i=1}^{n-1} r_{k-1,ni}\sum_{h=1}^{n-i} l_{kh}x^{k(h-1)}D\left[\frac{x^{k(n+1-i)-1}}{(n-i)!}\right].$$

This expression must be expanded and set equal to $P_{k(n-1)}(x)$ using (43). The resulting expressions are identically equal whenever the coefficients of corresponding powers of x are equal. Equating coefficients of $x^{k(n-1-i)}$, we have $r_{oni} = r_{o,n-1,i}$ for n = 1, 2, $\cdots$, so that the second index is superfluous. In effect, we find $r_{oni} = r_{oi}$, i = 0, 1, $\cdots$; $n \geqq 1$. Equating the coefficients of $x^{k(n-1-i)+s}$, $1 \leqq s \leqq k-1$, we have

(44) $r_{sni}\sum_{h=1}^{n-i} l_{kh}(kn-ki+s)\cdots(kn-ki-kh+s+1) = (n-i)\, r_{s,n-1,i}$.

For notational convenience, we introduce constants

(45) $$\gamma_{sm} = \sum_{h=1}^{m} \ell_{kh} \binom{km+s}{kh} (kh)! \quad , \quad 0 \leq s \leq k-1,$$

so that (44) is written as $r_{sni}\, \gamma_{s,n-i} = (n-i)\, r_{s,n-1,i}$ for $n \geq i$. By an iterative process the coefficients can be written as

$$r_{si} \equiv r_{sii} = r_{s,i+1,i}\, \gamma_{s1} = \frac{1}{2!} r_{s,i+2,i}\, \gamma_{s2} \gamma_{s1} = \cdots = \frac{r_{sni}}{(n-i)!} \prod_{m=1}^{n-i} \gamma_{sm} = \cdots,$$

which completes the proof.

It is interesting to note from (45) that $\gamma_{om} = m$ are the defining equations for the sequence $\{\ell_{kh}\}$ (cf. (40)).

Another result analogous to the preceding proposition has to do with finite operator $L(x,D) = D^k$.

Proposition 12: The operator $L(x,D) = D^k$ corresponding to the k-set $\left\{\frac{x^{kn}}{(kn)!}\right\}$ corresponds to an arbitrary k-set $\{P_{kn}(x)\}$ if and only if there is a sequence of polynomials $\{r_{oi} + r_{1ni}x + \cdots + r_{k-1,ni}x^{k-1}\}$ such that

$$P_{kn}(x) = r_{oo}\frac{x^{kn}}{(kn)!} + \sum_{i=1}^{n} (r_{oi} + r_{1ni}x + \cdots + r_{k-1,ni}x^{k-1}) \frac{x^{kn-ki}}{(kn-ki)!} ,$$

where

$$r_{jni} \binom{kn-ki+j}{j} = r_{ji} , \quad j = 1, 2, \cdots, k-1; \ n \geq i = 1, 2, \cdots.$$

Proof: For any k-set $\{P_{kn}(x)\}$ there is a sequence of polynomials such that

$$P_{kn}(x) = r_{ono}\frac{x^{kn}}{(kn)!} + \sum_{i=1}^{n} (r_{oni} + r_{1ni}x + \cdots + r_{k-1,ni}x^{k-1}) \frac{x^{kn-ki}}{(kn-ki)!} .$$

We demand that $D^k[P_{kn}(x)] = P_{k(n-1)}(x)$ and find after expanding and equating coefficients of like powers of x that

$$r_{oni} = r_{o,n-1,i} = r_{oi} \quad \text{for } n-1 \geq i = 1, 2, \cdots,$$

and

$$r_{1ni} \frac{(kn-ki+1)}{k(n-1)-ki+1} = r_{1,n-1,i} , \quad n \geq i.$$

By an iterative process we find this may be written

$$r_{1ni}(nk-ik+1) = r_{1ii} \equiv r_{1i}.$$

In a similar manner, the coefficients of $x^{k(n-i-1)+j}$, $j = 1, 2, \cdots, k-1$, are

$$r_{jni}\frac{(kn-ki+j)(kn-ki+j-1)\cdots(kn-ki+j-k+1)}{(kn-ki)!} = \frac{r_{j,n-1,i}}{(kn-k-ki)!}$$

for $n-1 \geqq i$, so that by iteration we are able to state the conditions as

$$r_{jni}\binom{kn-ki+j}{j} = r_{jii} \equiv r_{ji}\ , \ j = 1, 2, \cdots, k-1;\ n \geqq i = 1,2,\cdots.$$

The k-set $\left\{\frac{x^{kn}}{(kn)!}\right\}$ mentioned in Proposition 12 is not a $\{Y_{kn}(x)\}$ k-set since the generating function (6) for the $\{Y_{kn}(x)\}$ k-set cannot be modified to obtain this particular k-set. In order to verify this statement we shall derive the generating function for $\left\{\frac{x^{kn}}{(kn)!}\right\}$. The linear operator corresponding to this k-set is $L(x,D) = D^k$, so that we need a function $u(x)$ such that

(46) $$D^k[u(x)] = u(x).$$

The auxiliary equation is

$$\xi^k - 1 = 0,$$

which is satisfied by the k^{th} roots of unity. Let α be a primitive k^{th} root of unity. Then the general solution of (46) is

(47) $$u(x) = c_0 e^x + c_1 e^{\alpha x} + \cdots + c_{k-1} e^{\alpha^{k-1} x}.$$

We alter (47) to obtain the generating function

$$g(x,t) = 1/k \left[e^{xt^{1/k}} + e^{\alpha x t^{1/k}} + \cdots + e^{\alpha^{k-1} x t^{1/k}} \right].$$

<u>Proposition 13</u>: The function

$$g(x,t) = 1/k \sum_{m=0}^{k-1} \exp (\alpha^m x t^{1/k}),$$

where α is a primitive k^{th} root of unity, is the generating function for the k-set $\left\{ \frac{x^{kn}}{(kn)!} \right\}$.

<u>Proof</u>: The expansion of $g(x,t)$ can be written

$$g(x,t) = 1/k \left(\sum_{i=0}^{\infty} \left[1 + \alpha^i + \alpha^{2i} + \cdots + \alpha^{(k-1)i} \right] \frac{x^i t^{i/k}}{i!} \right),$$

where $i = nk$, $n = 0, 1, \cdots$ implies

$$\alpha^{ri} = (\alpha^k)^{rn} = 1; \; 0 \leqq r \leqq k-1,$$

and $i \neq nk$ implies $\alpha^i \neq 1$ since $i \equiv r \pmod{k}$, $1 \leqq r \leqq k-1$.

The sum of the geometric series

$$1 + \alpha^i + \cdots + \alpha^{(k-1)i} = \frac{1 - (\alpha^k)^i}{1 - \alpha^i} = 0, \text{ for } i \neq nk.$$

Hence we have the result,

$$g(x,t) = \sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!} t^n.$$

We note that in case $k = 2$, then $g(x,t)$ is the familiar $\cosh (x \sqrt{t})$.

<u>Remark</u>: The generating function for the k-set can also be expressed in terms of the generalized hypergeometric function of one numerator parameter and k denominator parameters. Using the usual definition (cf. [8]) we have after simplifying

$${}_1F_k \left(1; 1/k, 2/k, \cdots, k/k; \left(\frac{x}{k}\right)^k t\right) = \sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!} t^n,$$

which identifies the hypergeometric function with $g(x,t)$ of Proposition 13. In fact when $k = 2$, we notice that

$$\cosh x = {}_1F_2\,(1;\ \tfrac{1}{2},\ 1;\ x^2/4).$$

In order to make some remarks concerning the operator and the differential equation corresponding to the k-set $\{Y_{kn}(x)\}$ which are not readily apparent for the general case, we present the case $k = 2$ in some detail.

We consider the basic 2-set generated by

$$(48)\qquad g(x,t) = \exp\left[\,(ax+bx^2/2)t\,\right] = \sum_{n=0}^{\infty} U_{2n}(x)\ t^n,\ b \neq 0.$$

Two lemmas are needed.

<u>Lemma 9</u>: $\sum_{j=m}^{n-1}\binom{j}{m} = \binom{n}{m+1}$ for $n \geqq m+1$.

<u>Proof</u>: By induction.

<u>Lemma 10</u>: If $g(x,t)$ is given by (48), then

$$(49)\qquad D_x^n\left[\,g(x,t)\right] = \sum_{i=0}^{[n/2]} c_{in}\, b^i\, (a+bx)^{n-2i}\, t^{n-i} g(x,t),$$

where

$$c_{0n} = 1,$$

$$(50)\qquad c_{mn} = \frac{(2m-1)!}{2^{m-1}(m-1)!}\binom{n}{2m};\ n \geqq 2m > 0.$$

<u>Proof</u>: Equation (49) can be determined by an iterative procedure with the constants related as follows:

$$c_{1n} = \sum_{j=1}^{n-1} j = \binom{n}{2}\quad ;\ n = 2, 3, \cdots,$$

$$c_{2n} = \sum_{j=3}^{n-1} (j-2)c_{1j}\quad ;\ n = 4, 5, \cdots,$$

$$\cdots$$

$$c_{mn} = \sum_{j=2m-1}^{n-1} (j-2m+2)\ c_{m-1,j}\quad ;\ n \geqq 2m > 0.$$

We have (50) for $m = 1$, and assume it for $m = k$ and $n = 2k$, $2k+1, \cdots$. When $m = k+1$ and $n \geqq 2k+2$ we have

$$c_{k+1,n} = \sum_{j=2k+1}^{n-1} (j-2k)\, c_{kj} = \frac{(2k-1)!}{2^{k-1}(k-1)!} \sum_{j=2k+1}^{n-1} (j-2k)\binom{j}{2k},$$
$$= \frac{(2k+1)!}{2^k k!}\binom{n}{2k+2},$$

in which Lemma (9) was applied to obtain the last step.

There is a linear operator $L(x,D) = \sum_{m=1}^{\infty} L_{2m}(x)D^{2m}$; $D \equiv \frac{\partial}{\partial x}$, corresponding to the basic set $\{U_{2n}(x)\}$ of $\{Y_{2n}(x)\}$ and hence to $\{Y_{2n}(x)\}$ itself as is shown by Theorem 2 and its corollary, We have by linearity

$$L\left[g(x,t)\right] = \sum_{m=1}^{\infty} L_{2m}(x)D^{2m}\left[g(x,t)\right] = \sum_{n=1}^{\infty} U_{2(n-1)}(x)t^n$$
$$= \sum_{m=1}^{\infty} L_{2m}(x) \left[\sum_{i=0}^{m} c_{i,2m}\, b^i (a+bx)^{2m-2i} t^{2m-i}\right] g(x,t),$$

in which use was made of Lemma (10). By rearranging this expression in powers of t and then equating corresponding coefficients of t , r. 1, 2, •••, we are able to write

$$U_{2(k-1)}(x) = \sum_{m=1}^{k} L_{2m}(x) \sum_{j=s}^{m} c_{j,2m}\, b^j (a+bx)^{2m-2j} U_{2(k+j-2m)}(x),$$

where $s = \max\left[0,\ 2m-r\right]$ and $r = 1, 2, \cdots$. Each of these relations may be written more conveniently as

(51) $$U_{2m}(x) = \sum_{i=0}^{n} A_i(x)\ U_{2(n-i)}(x),$$

where

$$A_{2m+1}(x) = \sum_{j=0}^{m+1} L_{2,m+j+1}(x)\, c_{2j,2m+2j+2}\, b^{2j}(a+bx)^{2m-2j+2},$$

(52)

$$A_{2m}(x) = \sum_{j=0}^{m} L_{2,m+j+1}(x)\, c_{2j+1,2m+2j+2}\, b^{2j+1}(a+bx)^{2m-2j}, \quad m = 0, 1, \cdots.$$

Since the relations (51) must hold for $n = 0, 1, 2, \cdots$ $A_0(x) = 1$ and $A_i(x) \equiv 0$ for $i = 1, 2, \cdots$. By means of these values the equations (52) are seen to yield the explicit expressions for the polynomials in the operator $L(x,D)$. That is

(53) $$L_{2n}(x) = \sum_{i=0}^{2n-2} \ell_{2n,i}\, x^{i} = K_n(a+bx)^{2n-2}, \quad n = 1,\ 2,\ \cdots,$$

and the constants K are defined by recurrence as

$$K_1 b = 1,$$

$$K_{2m+1}\, c_{2m+1,4m+2}\, b^{2m+1} = -\sum_{j=0}^{m-1} K_{m+j+1}\, c_{2j+1,2m+2j+2}\, b^{2j+1}\ ; \quad m = 1,\ 2, \cdots,$$

$$K_{2m}\, c_{2m+2,4m+4}\, b^{2m+2} = -\sum_{j=0}^{m} K_{m+j+1}\, c_{2j,2m+2j+2}\, b^{2j}\ ; \quad m = 0,\ 1, \cdots,$$

from which it can be seen that the K_n are independent of the value of a and involve some power of b as a factor, specifically,

(54) $$K_n = (-1)^{n+1}\ b^{1-2n}(\text{function of } c_{ij}).$$

Since we insist $b \neq 0$, neither a nor b can affect the vanishing of the K_n.

Having the operator $L(x,D)$ in the form (53) we are able to determine whether $\{Y_{2n}(x)\}$ can be of finite A-type. In view of our previous remark (54), we may set $a = 0$ and $b = 2$ and the A-type will not be altered. Moreover this is a special case of the generating function given by (39) for which the operator is

(41)' $$L_s\ (x,D) = \sum_{h=1}^{\infty} \ell_{kh}\ x^{k(h-1)} D^{kh},$$

with the constants related by

(40)' $$\sum_{h=1}^{r} \ell_{kh} \binom{kr}{kh} (kh)! = r, \quad r = 1,\ 2,\ \cdots.$$

Proposition 14: The k-set $\{x^{kn}/(kn)!\}$, $k \geq 2$ generated by $g_5(x,t) = \exp(x^k t)$ is of infinite A-type.

Proof: Assume the k-set is of finite A-type r, that is the constants in the operator (41)' are such that $\ell_{kr} \neq 0$ and $\ell_{ki} = 0$ for $i \geq r+1$. This implies that (40)' can be written as

$$\sum_{h=1}^{r} \ell_{kh}\, kn(kn-1)\cdots(kn-kh+1) = n \quad \text{for } n \geq r,$$

or

$$\ell_{kr} = \frac{n}{kn(kn-1)\cdots(kn-kr+1)} - \sum_{h=1}^{r-1} \frac{\ell_{kh}}{(kn-kh)\cdots(kn-kr+1)} .$$

From this we see that

$$|\ell_{kr}| \leq \frac{1}{kn-1} + \sum_{h=1}^{r-1} \frac{|\ell_{kh}|}{kn-kr+1} .$$

Let $\epsilon > 0$ be arbitrary. There is an N such that $n > N$ implies $\max\left[\frac{1}{kn-1}, \frac{|\ell_{kh}|}{kn-kr+1}, h = 1,2,\cdots,r-1\right] < \frac{\epsilon}{r}$, so that $|\ell_{kr}| < \epsilon$ or equivalently $|\ell_{kr}| = 0$, which is contrary to the assumption.

In view of (53) and the remarks following (54), Proposition 14 allows us to state that the basic 2-set $\{U_{2n}(x)\}$ given by (48) is of infinite A-type. Moreover, as shown by the Corollary to Theorem 2, the basic set and the set $\{Y_{2n}(x)\}$ have the same operator. This completes the proof of the following statement.

Proposition 15: The 2-set $\{Y_{2n}(x)\}$ is of infinite A-type.

The k-set $\{Y_{kn}(x)\}$ satisfies a functional equation which is given by Theorem 3. For the basic 2-set $\{U_{2n}(x)\}$

this equation becomes

$$\mathcal{L}[y] = (d_{11}x + d_{21}x^2)L[y],$$
$$= (ax + bx^2/2)\sum_{m=1}^{\infty} L_{2m}(x)D^{2m}[y] = \lambda y,$$

where $\lambda = n$ whenever $y = U_{2n}(x)$. By means of (53) this can be written

(55) $$\mathcal{L}[y] = (ax + bx^2/2)\sum_{m=1}^{\infty} K_m(a+bx)^{2m-2} D^{2m}[y] = \lambda y.$$

Proposition 14 implies that there are arbitrarily large h such that $\ell_{2h} \neq 0$ and from (53) the corresponding $K_h \neq 0$, so that we have the following statement.

Proposition 16: The differential equation (55) for the basic 2-set $\{U_{2n}(x)\}$ is of infinite order.

A Rodrigues' Formula

The classical Laguerre polynomials, which are generated by

$$g(x,t) = (1-t)^{-1} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)t^n,$$

satisfy a Rodrigues' formula

$$L_n(x) = e^x D^n\left(\frac{x^n e^{-x}}{n!}\right), \quad D \equiv d/dx.$$

This formula may be used to show that the Laguerre polynomials form an orthogonal set. We first note

(56) $$\int_0^{\infty} e^{-x} x^s L_n(x)dx = 0, \quad s = 0, 1, \cdots, n-1;$$

so that orthogonality follows from the linearity of the integral.

In 1923 Humbert [4] generalized these relations to a particular 2-set. We shall show there is a $\{Y_{kn}(x)\}$ k-set which is determined by a Rodrigues' formula and satisfies a restricted orthogonality condition analogous to (56).

We consider the particular k-set generated by

(57) $$g(x,t) = (1-t)^{-1} \exp (x^k u(t)) = \sum_{n=0}^{\infty} T_{kn}(x), t^n,$$

where

$$u(t) = (1-t)^{-k} \sum_{j=1}^{k} (-1)^j \binom{k}{j} t^j = -\sum_{n=1}^{\infty} \binom{n+k-1}{k-1} t^n.^{1}$$

The recurrence relation (31) for the general $\{Y_{kn}(x)\}$ k-set becomes

(58) $$T'_{kn}(x) = -kx^{k-1} \sum_{j=0}^{n-1} \binom{n+k-1-j}{k-1} T_{kj}(x).$$

In order to proceed further, we develop a recurrence relation for the k-set $\{T_{kn}(x)\}$ which involves a fixed number of terms.

Lemma 11: $\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m = 0, \ n > m \geq 0.$

Proof: The well known proof for m = 0 may be extended by induction.

Lemma 12: $\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} \binom{a-i}{k} = 0$ for all a.

Proof: The statement follows as a consequence of Lemma 11.

Proposition 17: The k-set $\{T_{kn}(x)\}$ satisfies the

---

[1]The equivalence of the two forms of u(t) may be verified by means of Lemma 12.

relation

$$T'_{kn}(x) = \sum_{j=1}^{k} (-1)^{j+1}\binom{k}{j}\left[T'_{k(n-j)}(x) - kx^{k-1}\, T_{k(n-j)}(x)\right].$$

<u>Proof:</u> By means of the recurrence relation (58) we have

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}T'_{k(n-j)}(x) = -kx^{k-1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{i=1}^{n-j-1}\binom{n+k-1-j-i}{k-1}T_{ki}(x),$$

$$= -kx^{k-1}\Big\{\sum_{i=0}^{n-k-1}\Big[\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{n-1+k-i-j}{k-1}\Big]T_{ki}(x) + \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{2k-1-j}{k-1}T_{k(n-k)}(x)$$
$$+ \sum_{j=0}^{k-2}(-1)^j\binom{k}{j}\binom{2k-2-j}{k-1}T_{k(n-k+1)}(x)+\cdots+\binom{k}{0}\binom{k}{k-1}T_{k(n-1)}(x)\Big\},$$

in which the bracket is zero for each i by Lemma 12. In addition we find on applying Lemma 12 to each of the remaining sums that

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}T'_{k(n-j)}(x) = -kx^{k-1}\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}T_{k(n-j)}(x),$$

which proves the proposition.

The proof of the main result of this section, namely that the k-set $\{T_{kn}(x)\}$ satisfies a Rodrigues' formula, requires several facts which we state as lemmas.

<u>Lemma 13:</u> $D^n\left(\dfrac{x^{l+n}e^{-x^k}}{n!}\right) = \sum_{i=0}^{l}\binom{l}{i}x^{l-i}D^{n-i}\left(\dfrac{x^n e^{-x^k}}{(n-i)!}\right)$.

<u>Proof:</u> The statement is an application of the Leibniz' formula.

<u>Lemma 14:</u> $D^{n-m}\left(\dfrac{x^{n-m-1}e^{-x^k}}{(n-m-1)!}\right) = D^{n-m-1}\left(\dfrac{x^{n-m-2}e^{-x^k}}{(n-m-2)!}\right)$
$\qquad - k\sum_{i=m+1}^{k-1}\binom{k-m-2}{i-m-1}x^{k-1-i}D^{n-i}\left(\dfrac{x^n e^{-x^k}}{(n-i)!}\right).$

Proof: Perform one of the differentiations and apply Lemma 13.

Lemma 15: $\binom{k}{j+1} - \binom{k-1}{j} = \binom{k-1}{j+1}$, $0 \leq j \leq k-1$.

Proof: By induction.

Lemma 16: $\sum_{j=0}^{m}(-1)^j\binom{n}{j}\binom{n-j}{m-j} = 0$; $0 < m \leq n$.

Proof: The sum is equal to $\binom{n}{m}\sum_{j=0}^{m}(-1)^j\binom{m}{j}$.

Theorem 5: The k-set $\{T_{kn}(x)\}$ defined by (59) is determined by the formula

$$T_{kr}(x) = e^{x^k} D^r\left(\frac{x^r e^{-x^k}}{r!}\right), \quad r = 0, 1, \cdots.$$

Proof: The statement is true for $r = 0$; we assume the relation holds for $r = 0, 1, \cdots, n-1$, so that

$$T'_{kr}(x) = e^{x^k} D^{r+1}\left(\frac{x^r e^{-x^k}}{r!}\right) + kx^{k-1}T_{kr}(x), \quad r = 0, 1, \cdots, n-1.$$

From Proposition 17 we see that

$$T'_{kn}(x) = e^{x^k}\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}D^{n-j+1}\left(\frac{x^{n-j}e^{-x^k}}{(n-j)!}\right),$$

which can be written as

$$T'_{kn}(x) = e^{x^k}\Big\{D^{n+1}(x^n e^{-x^k}) + k\sum_{i=0}^{k-1}\binom{k-1}{i}x^{k-1-i}D^{n-i}\left(\frac{x^n e^{-x^k}}{(n-i)!}\right)$$
$$+ (k-1)D^n\left(\frac{x^{n-1}e^{-x^k}}{(n-1)!}\right) + \sum_{j=2}^{k}(-1)^{j+1}\binom{k}{j}D^{n-j+1}\left(\frac{x^{n-j}e^{-x^k}}{(n-j)!}\right)\Big\},$$

in which use was made of Lemma 14. This expression may be rearranged slightly to obtain

$$(59)\qquad T'_{kn}(x) = D\left[e^{x^k} D^n (x^n e^{-x^k})\right] + e^{x^k} A(x, D; n, k),$$

where

$$A = k\sum_{i=1}^{k-1} \binom{k-1}{1} x^{k-1-i} D^{n-i}\left(\frac{x^n e^{-x^k}}{(n-i)!}\right) + \binom{k-1}{1} D^n \left(\frac{x^{n-1} e^{-x^k}}{(n-1)!}\right)$$
$$+ \sum_{j=2}^{k} (-1)^{j+1} \binom{k}{j} D^{n-j+1}\left(\frac{x^{n-j} e^{-x^k}}{(n-j)!}\right).$$

We shall show that A(x, D; n, k) vanishes identically to complete the argument. Applying Lemma 14 to the single term we find

$$A = k\sum_{i=1}^{k-1}\left[\binom{k-1}{0}\binom{k-1}{1} - \binom{k-1}{1}\binom{k-2}{i-1}\right] x^{k-1-i} D^{n-i}\left(\frac{x^n e^{-x^k}}{(n-i)!}\right)$$
$$+(-1)\left[\binom{k}{2} - \binom{k-1}{1}\right] D^{n-1}\left(\frac{x^{n-2} e^{-x^k}}{(n-2)!}\right) + \sum_{j=3}^{k} (-1)^{j+1}\binom{k}{j} D^{n-j+1}\left(\frac{x^{n-j} e^{-x^k}}{(n-j)!}\right).$$

A(x, D; n, k) can be placed into a form similar to the preceding, that is, we shall show that

$$(60)\qquad A = k\sum_{i=m}^{k-1}\left[\sum_{j=0}^{m} (-1)^j \binom{k-1}{j}\binom{k-1-j}{i-j}\right] x^{k-1-i} D^{n-i}\left(\frac{x^n e^{-x^k}}{(n-i)!}\right)$$
$$+(-1)^m \binom{k-1}{m+1} D^{n-m}\left(\frac{x^{n-m-1} e^{-x^k}}{(n-m-1)!}\right) + \sum_{j=m+2}^{k} (-1)^{j+1}\binom{k}{j} D^{n-j+1}\left(\frac{x^{n-j} e^{-x^k}}{(n-j)!}\right),$$

holds for m = 1, 2,···. As a matter of fact we have the relation for m = 1 by means of Lemma 5. Assuming (60) is a valid for $m < k-1$ and noticing that the bracket vanishes for i = m by Lemma 16, and applying Lemma 14 to the single term we have

$$A = k\sum_{i=m+1}^{k-1}\left[\sum_{j=0}^{m+1} (-1)^j \binom{k-1}{j}\binom{k-1-j}{i-j}\right] x^{k-1-i} D^{n-i}\left(\frac{x^n e^{-x^k}}{(n-i)!}\right)$$
$$+(-1)\left[\binom{k}{m+2} - \binom{k-1}{m+1}\right] D^{n-m-1}\left(\frac{x^{n-m-2} e^{-x^k}}{(n-m-2)!}\right) + \sum_{j=m+3}^{k} (-1)^{j+1}\binom{k}{j} D^{n-j+1}\left(\frac{x^{n-j} e^{-x^k}}{(n-j)!}\right),$$

in which the results obtained from Lemma 14 are included in the existing sums. Application of Lemma 15 completes the induction.

We now evaluate $A(x, D; n, k)$ as expressed in (60) for $m = k - 2$ and find, after making use of Lemma 16, that $A(x, D; n, k)$ vanishes identically. In view of equation (59) we see that

$$T_{kn}(x) = e^{x^k} D^n \left(\frac{x^n e^{-x^k}}{n!}\right) + c_n.$$

However $c_n = 0$ since $T_{kn}(0) = 1$, which completes the proof of the theorem.

<u>Lemma 17</u>: $D^m(x^n e^{-x^k})\Big]_0^\infty = 0$, for $n > m \geq 0$.

<u>Proof</u>: As can be seen by means of the Leibniz' formula the expression has the form $Q(x)e^{-x^k}$ where $Q(x)$ is a polynomial such that $Q(0) = 0$.

<u>Proposition 18</u>: The k-set $\{T_{kn}(x)\}$ has the property

$$\int_0^\infty e^{-x^k} x^s T_{kn}(x)dx = 0, \; s = 0, 1, \cdots, n-1.$$

<u>Proof</u>: We substitute the Rodrigues' formula of Theorem 5, integrate by parts, and make use of Lemma 17 to verify this property.

In the case of the Laguerre polynomials $(k = 1)$ the property is equivalent to the property of orthogonality. When $k \geq 2$ this is no longer true and we are able to state only that the first $n - 1$ moments of $e^{-x^k} T_{kn}(x)$ vanish.

BIBLIOGRAPHY

Articles

1. Appell, P. "Sur une classe de polynômes," Annales Scientifiques de l'Ecole Normale Supérieure, vol. 9, 1880, pp. 119-144.

2. Huff, W. N. "The Type of the Polynomials Generated by $f(xt)\varphi(t)$," Duke Mathematical Journal, vol. 14, 1947, pp. 1091-1104.

3. Huff, W. N. and Rainville, E. D. "On the Sheffer A-type Polynomials Generated by $\varphi(t)f(xt)$," Proceedings of the American Mathematical Society, vol. 3, no. 2, 1952, pp. 296-299.

4. Humbert, P. "Sur certains polynômes orthogonaux," Comptes Rendus de l'Académie de Paris, vol. 176, 1923, pp. 1282-1284.

5. Meixner, J. "Orthogonale Polynomsystems mit einer besonderer Gestalt der erzeugenden Funktion," Journal of the London Mathematical Society, vol. 9, 1934, pp. 6-13.

6. Sheffer, I. M. "A Differential Equation for Appell Polynomials," Bulletin of the American Mathematical Society, vol. 41, 1935, pp. 914-928.

7. Sheffer, I. M. "Some Properties of Polynomials Sets of Type Zero," Duke Mathematical Journal, vol. 5, pp. 590-622.

Books

8. Bateman Manuscript Project, Higher Transcendental Functions, vol. 1, New York: McGraw-Hill Book Co., 1953.

9. Knopp, K. Theory and Applications of Infinite Series. 2d ed. London: Blackie and Sons, 1947.

10. Szego, G. Orthogonal Polynomials. New York: American Mathematical Society Colloquium Publication, vol. 23, 1939.