DISCRETE VARIABLE STRUCTURE CONTROL FOR UNCERTAIN LINEAR MULTIVARIABLE SYSTEMS

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PREFACE

The methodology of variable structure with sliding mode is proven to be very successful in controlling uncertain continuous-time dynamical systems. When the system is sampled or purely discrete, the invariance property of sliding mode, which is originally a continuous-time concept, no longer hold and the reaching condition has to be modified to allow a pseudo-sliding mode. Moreover, the state dependency of parametric uncertainties makes the satisfaction of reaching condition considerably more difficult especially in multivariable systems. These difficulties have offered challenges that attracted a great deal of research interests. This thesis presents theoretical results on the discrete variable structure control of uncertain linear multivariable systems using the concepts of sliding mode and switching sector. It considers both the state and output feedback cases for systems with additive uncertainties and the state feedback case for systems with parametric uncertainties. The thesis also presents a sliding surface design procedure for single-input systems based on the version of discrete variable structure control developed by Dr. Eduardo Misawa.

Here, I would like to express my gratitude to Dr. Eduardo Misawa, my thesis adviser, who has always been encouraging me and supporting me throughout the course of this research. I am also grateful to my committee members, Dr. Gary Young and Dr. Prabhakar Pagilla, for their help and guidance.

To my parents Chooi Leng Tang and Chai Kim Ng, my sincere thanks for their love and support during all these years of my education. Also, my thanks to Ms. Yeun Fun Yau for her patience during my study in the United States. Finally, I wish to thanks my lab-mates, Brian, May-Win, and Hanz for proofreading the thesis.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

1.1 Background and Motivations

The problem of controlling uncertain continuous-time dynamical systems has been the subject of research activity for many years (Gutman [19], Utkin [42], Corless and Leitmann [8], Slotine [36]). By virtue of increasing availability of low cost digital computers, for the past decade, a tremendous amount of work has been devoted to the problem of stabilizing uncertain discrete-time systems with bounded uncertainties of deterministic nature. Deterministic, state space approach based work can be roughly classified into three categories. First, inspired by the success of variable structure control with sliding mode for uncertain continuous-time systems (Utkin [43], DeCarlo *et al.* [11], Zinober [54]), many investigators attempted to extend this notion to sampled-data or discrete-time systems (Sira-Ramirez [35], Su *et al.* [40], Koshkouei and Zinober [22], Misawa [27]). Second, a number of investigators retained the idea of variable structure by introducing the use of *switching sector*, also called sliding sector or switching region, in place of sliding surface (Furuta [15], Yu [51], Pan and Furuta [31], Wang *et al.* [48]). Third, a number of investigators employed the direct method of Lyapunov in synthesizing stabilizing controllers (Corless and Manela [7], Magaña and Żak [24]).

In sampled-data systems, the control signal space shrinks from $(\mathcal{L}_2[0, \Delta t])^m$ to $\mathbf{R}^m$, where $\mathcal{L}_2$, Δt, and m are the Lebesgue space, the sampling period, and the number of control inputs respectively (Su *et al.* [38]). Thus, the controller is inherently less capable than the continuous one and sliding mode is hardly achieved under nonideal conditions. If Δt is *sufficiently small*, the sampling effect is insignificant and the state can be kept sufficiently close to the sliding surface with a continuous-time sliding mode control law. Unfortunately, no general result is available to date on the upper bound of Δt so that the sampling effect can be "safely" ignored. Although it is shown by Su *et al.* [38] that the state can be maintained in the vicinity of the sliding surface up to at least $O(\Delta t^q)$ for some positive q depending on the situations, this does not imply that the system is BIBO stable because the state may still go unbounded along the vicinity of the sliding surface. A study conducted by Yu [51] based on the Lyapunov exponents method (Grantham and Athalye [18]) shows that a *sufficiently small* Δt may still cause chaotic behavior. It is therefore necessary to analyze and design discrete variable structure systems in a complete discrete-time framework.

As a result, several reaching conditions specifically tailored for the existence of pseudo-sliding mode are proposed. From a geometric viewpoint, currently there are two kinds of pseudo-sliding mode. For systems with additive uncertainties, i.e. uncertainties that can be bounded by constants, pseudo-sliding mode usually refers to a subset in the state space with uniform thickness, possibly a *boundary layer*. Meanwhile, for systems with parametric uncertainties, i.e. uncertainties with state dependent bounds, pseudo-sliding mode refers to a switching sector whose thickness grows with the magnitude of the states. Switching sector is adopted mainly because the effect of parametric uncertainties is more severe as the states are located farther from the origin. This does not allow the use of boundary layer with uniform thickness. The terminology of switching "sector" actually originates from second order systems. In general, its shape depends on the particular control law and the number of sectors may depend on the number of inputs as well.

Conventionally, sliding mode is defined as the intersection of m sliding hyperplanes (Utkin and Young [45]), which results in perfect invariance to matched uncertainties for continuous-time systems. When it comes to discrete-time, additive uncertainties can easily destroy the discrete sliding mode and one has to seek help from other means, possibly through linear control design strategies, to attenuate the effect of uncertainties. However, of the existing discrete variable structure control schemes for linear multivariable systems, only $n - m$ eigenvalues of the dynamics on the sliding surface can be freely assigned (Su *et al.* [38], Chan [5], Koshkouei and Zinober [22]). This imposes a certain level of difficulties in the design of sliding surface. Furthermore, most existing work considers the case where full state feedback is available. To date, very few work exists for discrete variable structure systems with observers.

1.2 Scope and Contributions

The scope of this research covers the discrete control of uncertain linear multivariable systems using the theory of variable structure with sliding mode and with switching sector. For systems with additive uncertainties, both the state and output feedback cases are studied, as opposed to systems with parametric uncertainties, where the state feedback case is studied. Sliding surface design problem is also considered. Highlighted below are the major contributions of this research.

- The development of a state feedback discrete variable structure control technique for linear multivariable systems with additive uncertainties. In contrast to existing schemes, it utilizes one sliding hyperplane regardless of the number of inputs. This attribute enhances the design freedoms of tracking error dynamics inside the boundary layer while preserving the same robustness properties. It allows the use of well-established linear control design strategies under a minor eigenvalue constraint.

- The extension of the technique to incorporate a prediction observer with uncertainty estimation to make the controller practically reliazable. The resulting observer-based controller guarantees the attractiveness and invariance of the estimated boundary layer, which is dynamic and parallel to the sliding hyperplane. Linear control design strategies can then be employed under the eigenvalue constraint.

- The development of a state feedback discrete variable structure control technique for linear multivariable systems with parametric uncertainties using the concept of switching sector. Potential stability problems with existing schemes are avoided. Global uniform asymptotic stability may be achieved despite the noninvariance of switching sector.

- The development of an optimal sliding surface design procedure for single-input system based on the LQR technique which allows the prespecification of the desired real eigenvalue.

1.3 Thesis Outline and Notations

This thesis is organized into eight chapters. Chapter 1 briefly overviews the problem of discrete variable structure control of uncertain dynamical systems and summarizes the accomplishment of this research. Chapter 2 reviews the existing literature on discrete variable structure control and on sliding surface design.

In Chapter 3, a state feedback discrete variable structure control law for linear multivariable systems with additive uncertainties is derived. Its convergence properties and related design issues are discussed. Extensions of the results to the output feedback case are given in Chapter 4, where the properties of the observer-based controller with the use of prediction observer with uncertainty estimation are investigated.

Chapter 5 presents a state feedback discrete variable structure control law using switching sector for linear multivariable systems with parametric uncertainties. Chapter 6 presents an optimal sliding surface design procedure based on LQR technique for single-input systems.

Examples used to illustrate the effectiveness of the proposed controllers and sliding surface design procedure are given in Chapter 7. Finally, Chapter 8 lists the concluding remarks of this work and suggestions for future research.

Throughout the thesis, the following conventions are adopted for the vector and matrix norms unless otherwise specified: Let $x \in \mathbf{R}^n$ and $A \in \mathbf{R}^{m \times n}$. Then the Euclidean 2-norm of x is denoted by $\|x\|$ and the induced 2-norm of A is denoted by $\|A\|$. In addition, $[a_{ij}]$ denotes a matrix with a_{ij} as its ith row and jth column element.

Chapter 2

Literature Review

2.1 Discrete Variable Structure Control with Sliding Mode

Early papers on discrete-time sliding mode control examined the *reaching or hitting conditions*. Milosavljevic [25] has pointed out that the sampling process limits the existence of ideal sliding mode in the digital implementation of sliding mode controllers. In light of this, definitions of quasi-sliding mode, pseudo-sliding mode, discrete sliding mode, and convergent discrete sliding mode have been suggested and the conditions for the existence of such modes have been investigated. In particular, Milosavljevic [25] proposed the idea of quasi-sliding mode and presented a reaching condition modified from the continuous-time reaching condition. However, it is shown in Sira-Ramirez [35] and Yu [51] [52] that this condition guarantees only the states to approach and/or to cross the sliding surface, which may allow an unstable sliding mode. Sapturk *et al.* [34] suggested a reaching condition that is widely used in current DVSC research. Koshkouei and Zinober [22] clarified the concept of discrete-time sliding mode and presented several new sufficient conditions for the existence of discrete-time sliding mode.

As a part of this research, versions of discrete-time sliding mode control (DSMC) or discrete variable structure control (DVSC) schemes for linear systems based on the sliding mode concept have been proposed. Most of these schemes considered *single-input systems*. Chan [4] proposed a DSMC strategy that ensures sliding mode to be achieved exponentially fast to keep the system robust. Sira-Ramirez [35] investigated the behavior of the nonlinear DVSC in quasi-sliding mode. Paden and Tomizuka [30] proposed a DVSC technique with parabolic sliding surface for position control of second order systems with parametric uncertainties. Chan [5] proposed a linear feedback control law, developed using the delta operator, to achieve discrete sliding mode or quasi-sliding mode for systems with no uncertainties. Next, he developed an adaptive DSMC for linear systems in state-space form [6]. Gao *et al.* [16] developed a version of DVSC for system with no uncertainties. Koshkouei and Zinober [21] proposed a DSMC for system with additive uncertainties. Fradkov and Furuta [12] analyzed the behavior of a number of control schemes under both deterministic and stochastic disturbances. Using the concept of time delay control, Corradini and Orlando [9] developed a DVSC for systems with both matched parametric and additive uncertainties. They assumed that the rate of change of the matched uncertainties is considerably slower than the sampling rate. Misawa [27] proposed a DSMC for nonlinear systems with unmatched uncertainties and uncertain control vector. The linear case of the above DSMC is treated in Misawa [28], where tracking control under additive uncertainties is considered. To explicitly account for computational delay inherent in any digital implementation, the prediction observer-based DSMC is proposed by Misawa [26].

A number of DVSC schemes considered *multiple-input systems*. Kaynak and Denker [20] developed a DVSC for a class of nonlinear systems. Su *et al.* [40] proposed a nonlinear DSMC for matched uncertainties. The linear case was treated in Su *et al.* [38], where it was shown that the states can be maintained in the vicinity of the sliding surface up to at least $O(\Delta t^2)$. They later proposed the use of pre-filtering and post-filtering settings to eliminate chattering while ensuring the robustness of DSMC [39]. Koshkouei and Zinober [22] proposed a DVSC for systems with additive uncertainties. They showed that if sliding mode can occur, the systems behavior will be governed by $n-m$ eigenvalues. Utkin [44] presented several DSMC design for linear systems, infinite-dimensional systems, and systems with delays. Chan [5] proposed DSMC which is robust against slowly varying perturbations while Fujisaki *et al.* [14] proposed a DSMC for systems with no uncertainties. Yu *et al.* [50] proposed a DVSC with an adaptive discrete reaching law and a periodic convergence law for systems with no uncertainties.

On the other hand, a number of investigators applied the discrete sliding mode concept to systems represented by *input-output models*. Pieper and Goheen [32] developed a DSMC for systems described by ARMA models while Suzuki and Furuta [41] studied the hyperplane of the discrete variable structure systems by using only input and output signals.

2.2 Discrete Variable Structure Control with Switching Sector

DVSC using the concept of *switching sector* for system with parametric uncertainties has been the subject of a number of investigators. The earliest work is by Furuta [15]. He designed a DSMC for *single-input systems* and provide a condition on the stability of the equivalent dynamics on the hyperplane. Wang and Wu [47] considered the work of Furuta and presented a simpler control law. They used the concept of equilibrium point of the diagonalized system to determine the switching region which results in explicit reduction of chattering. Yu [51] analyzed some of the inherent properties peculiar to DVSC and developed a DVSC that enables the elimination of chattering as well as divergence from the switching hyperplane. Pan and Furuta [31] presented a robust DVSC and presented a robust stability criterion for system inside the switching sector. Wang *et al.* [48] proposed a DVSC and introduced the concept of locating the equilibrium point of the nominal subsystem on each hyperplane parallel to the sliding hyperplane outside the switching sector. In a later paper, they generalized the result to *multiple-input systems* [46]. Lee and Wang [23] proposed a DVSC for model-following systems with no uncertainties.

2.3 Sliding Surface Design

Several sliding surface design techniques for DSMC or DVSC have been proposed. For *single-input systems*, Richter *et al.* [33] solved the eigenvalue assignment problem for the equivalent dynamics inside the boundary layer and proposed the use of LQ technique in the design of sliding surface. Pan and Furuta [31] proposed the use of LQR technique in the design of sliding sector. For *multiple-input systems*, Spurgeon [37] proposed an advanced hyperplane design methodology based on the Lyapunov approach.

Chapter 3

Additive Uncertainties: The State Feedback Case

In this chapter, a state feedback discrete variable structure control technique for linear multivariable systems with additive uncertainties, a generalization of the result for single-input systems by Misawa [28], is presented. It is shown that the boundary layer under the control law is attractive and invariant. Discussion on the eigenvalue constraint in variable structure systems, followed by comparison on the use of one hyperplane and multiple hyperplanes are then made. It is shown that the tracking error dynamics inside the linear region, namely the boundary layer, can be matched with any dynamics having the state feedback form under the eigenvalue constraint. This chapter ends with discussion on design issues and conclusion on stability.

Consider the following discrete-time linear multivariable system which may be obtained by discretizing its continuous-time equivalent with sampling period Δt:

$$x(k+1) = Ax(k) + Bu(k) + D_o w_o(k) \tag{3.1}$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^m$ is the input vector, and $w_o \in \mathbf{R}^{q_o}$ is the additive uncertainty vector. A, B, and D_o are perfectly known constant matrices with appropriate dimensions with $B = [B_1 \mid B_2 \mid \cdots \mid B_m]$, $B_i \in \mathbf{R}^n$, and the matching condition $\text{rank}([B\ D_o]) = \text{rank}(B)$ not necessarily satisfied, i.e. the additive uncertainties may occur not only on the control channels. The objective is to find a suitable control input $u(k)$ so that $x(k)$ will track a known desired trajectory $x_d(k)$.

Assumption 3.1. The system (3.1) satisfies the following conditions: $\text{rank}(B) = m$, the pair (A, B) is controllable, and $w_o(k)$ is bounded.

Assumption 3.2. The desired trajectory $x_d(k+1)$ is "consistently generated" by a model-based x_d-generator (Misawa [28]) using the nominal system

$$x_d(k+1) = Ax_d(k) + Bu_d(k)$$

where $u_d(k)$ is the hypothetical input.

3.1 Attractiveness and Invariance of Boundary Layer

Definition 3.1. Let the tracking error $\tilde{x} \in \mathbf{R}^n$, the sliding function $s \in \mathbf{R}$, the sliding surface $\mathcal{S} \subset \mathbf{R}^n$, the boundary layer $\mathcal{B} \subset \mathbf{R}^n$, the saturation function, and the signum function be respectively defined as

$$\begin{aligned}
\tilde{x}(k) &= x_d(k) - x(k) \\
s(k) &= G\tilde{x}(k) \\
\mathcal{S} &= \{\tilde{x} : s = G\tilde{x} = 0\} \\
\mathcal{B} &= \{\tilde{x} : |s| = |G\tilde{x}| \leq \phi \ , \ \phi > 0\} \\
\text{sat}\left(\frac{s}{\phi}\right) &= \begin{cases} \text{sgn}(s) & , \quad |s| > \phi \\ s/\phi & , \quad |s| \leq \phi \end{cases} \\
\text{sgn}(s) &= \begin{cases} +1 & , \quad s > 0 \\ 0 & , \quad s = 0 \\ -1 & , \quad s < 0 \end{cases}
\end{aligned}$$

where ϕ is the boundary layer thickness.

Assumption 3.3. The row vector G is determined such that $GB_i \neq 0$, $1 \leq i \leq m$ and $\|G\| = 1$.

Theorem 3.1 (Attractiveness and Invariance of $\mathcal{B}$ with State Feedback). *Consider the system (3.1) and let Assumptions 3.1, 3.2, and 3.3 hold. If the control law is chosen as*

$$\begin{aligned}
u(k) &= (B^TB)^{-1}B^T(x_d(k+1) - Ax_d(k)) \\
&\quad + \Gamma^{-1}M(x_d(k) - x(k)) + \Gamma^{-1}K\,\text{sat}\left(\frac{s(k)}{\phi}\right) && (3.2) \\
\Gamma &= \text{diag}(GB_1, GB_2, \ldots, GB_m)\,, \quad \Gamma \in \mathbf{R}^{m\times m} && (3.3) \\
M &= [\mu_1\,|\,\mu_2\,|\cdots|\,\mu_m]^T\,, \quad M \in \mathbf{R}^{m\times n}\,, \quad \mu_i \in \mathbf{R}^n && (3.4) \\
\sum_{i=1}^{m}\mu_i^T &= G(A - I) && (3.5) \\
K &= [K_1\ K_2\ \cdots\ K_m]^T\,, \quad K \in \mathbf{R}^m\,, \quad K_i \in \mathbf{R} && (3.6) \\
\sum_{i=1}^{m}K_i &= K_\Sigma = \gamma + 2\Delta t\epsilon\,, \quad \epsilon > 0 && (3.7) \\
\phi &\geq \gamma + \Delta t\epsilon && (3.8) \\
\gamma &\geq |GD_ow_o(k)| && (3.9)
\end{aligned}$$

where K_i's are the sliding gains and γ is the bound on uncertainties, then $\mathcal{B}$ is attractive and invariant, i.e. there exists a k_s such that $\tilde{x}(k) \in \mathcal{B}$ for all $k \geq k_s$. In particular, $s(k)$ asymptotically approaches $\mathcal{S}$ if $w_o(k) = 0$.

Proof. Consider a Lyapunov function candidate $V(k) = s^2(k)$. $|s(k)|$ decreases monotonically if the following inequality holds:

$$V(k+1) < V(k) \Rightarrow [\Delta s(k) + 2s(k)]\Delta s(k) < 0\,, \quad \forall s(k) \neq 0 \tag{3.10}$$

where $\Delta s(k) = s(k+1) - s(k)$. Using Eqs. (3.2), (3.5), and (3.7), $\Delta s(k)$ is obtained as

$$\begin{aligned}
\Delta s(k) &= Gx_d(k+1) - GAx(k) - GB(B^TB)^{-1}B^T(x_d(k+1) - Ax_d(k)) \\
&\quad - \sum_{i=1}^{m}\mu_i^T(x_d(k) - x(k)) - \sum_{i=1}^{m}K_i\,\text{sat}(s(k)/\phi) - GD_ow_o(k) - G(x_d(k) - x(k)) \\
&= G[I - B(B^TB)^{-1}B^T](x_d(k+1) - Ax_d(k)) - K_\Sigma\,\text{sat}(s(k)/\phi) - GD_ow_o(k)
\end{aligned}$$

It follows from Assumption 3.2 that the first term in the above expression vanishes, i.e.

$$[I - B(B^TB)^{-1}B^T](x_d(k+1) - Ax_d(k)) = [I - B(B^TB)^{-1}B^T]Bu_d(k) = 0 \tag{3.11}$$

Therefore,

$$\Delta s(k) = -K_\Sigma\,\text{sat}(s(k)/\phi) - GD_ow_o(k) \tag{3.12}$$

and the left-hand side of inequality (3.10) can be written as

$$[-K_\Sigma \operatorname{sat}(s(k)/\phi) - GD_o w_o(k) + 2s(k)][-K_\Sigma \operatorname{sat}(s(k)/\phi) - GD_o w_o(k)] \tag{3.13}$$

For $s(k) > \phi$, let $s(k) = \phi + \xi_1(k)$ where $\xi_1(k) > 0$. Also, in view of Eqs. (3.8) and (3.9), let $\phi = \gamma + \Delta t\epsilon + \xi_2$ and $\xi_3(k) = \gamma - GD_o w_o(k)$ where $\xi_2 \geq 0$ and $0 \leq \xi_3(k) \leq 2\gamma$. Expression (3.13) then becomes

$$\begin{aligned} & [-\gamma - 2\Delta t\epsilon - GD_o w_o(k) + 2\gamma + 2\Delta t\epsilon + 2\xi_2 + 2\xi_1(k)][-\gamma - 2\Delta t\epsilon - GD_o w_o(k)] \\ \Rightarrow \quad & \underbrace{[\xi_3(k) + 2\xi_2 + 2\xi_1(k)]}_{positive}\underbrace{[\xi_3(k) - 2\gamma - 2\Delta t\epsilon]}_{negative} \end{aligned}$$

which implies that inequality (3.10) is satisfied. Similarly, for $s(k) < -\phi$, let $s(k) = -\phi - \xi_1(k)$ where $\xi_1(k) > 0$. Along with the definitions of ξ_2 and $\xi_3(k)$, expression (3.13) becomes

$$\begin{aligned} & [\gamma + 2\Delta t\epsilon - GD_o w_o(k) - 2\gamma - 2\Delta t\epsilon - 2\xi_2 - 2\xi_1(k)][\gamma + 2\Delta t\epsilon - GD_o w_o(k)] \\ \Rightarrow \quad & \underbrace{[\xi_3(k) - 2\gamma - 2\xi_2 - 2\xi_1(k)]}_{negative}\underbrace{[\xi_3(k) + 2\Delta t\epsilon]}_{positive} \end{aligned}$$

which implies that inequality (3.10) is again satisfied. This concludes that $\mathcal{B}$ is attractive, i.e. for all $\tilde{x}(k) \notin \mathcal{B}$, there exists a $k_s > k$ such that $\tilde{x}(k_s) \in \mathcal{B}$. Next, for $|s(k)| \leq \phi$, let $s(k) = \xi_4(k)\phi$ where $-1 \leq \xi_4(k) \leq 1$. Then, from Eq. (3.12) and definition of ξ_2 one has

$$\begin{aligned} s(k+1) &= (1 - K_\Sigma/\phi)s(k) - GD_o w_o(k) \\ &= \left(\frac{-\Delta t\epsilon + \xi_2}{\gamma + \Delta t\epsilon + \xi_2}\right)\xi_4(k)(\gamma + \Delta t\epsilon + \xi_2) - GD_o w_o(k) \\ \Rightarrow |s(k+1)| &\leq |-\Delta t\epsilon\xi_4(k)| + |\xi_2\xi_4(k)| + |GD_o w_o(k)| \leq \Delta t\epsilon + \xi_2 + \gamma = \phi \end{aligned}$$

Hence, $\mathcal{B}$ is invariant, i.e. if $\tilde{x}(k_s) \in \mathcal{B}$, then $\tilde{x}(k) \in \mathcal{B}$ for all $k > k_s$. Finally, it is obvious from Eqs. (3.7) and (3.8) that $-1 < (1 - K_\Sigma/\phi) < 1$ which implies that $s(k)$ is a stable first order filter that asymptotically approaches $\mathcal{S}$ if $w_o(k) = 0$. □

Corollary 3.1. *With the control law stated in Theorem 3.1, the closed-loop tracking error dynamics can be classified as follows: for $s(k)$ outside $\mathcal{B}$,*

$$\tilde{x}(k+1) = A_s\tilde{x}(k) - B\Gamma^{-1}K \operatorname{sgn}(G\tilde{x}(k)) - D_o w_o(k) \; ; \tag{3.14}$$

for $s(k)$ inside $\mathcal{B}$,

$$\tilde{x}(k+1) = A_{eq}\tilde{x}(k) - D_o w_o(k) \; ; \tag{3.15}$$

and for $s(k)$ on $\mathcal{S}$ with $w_o(k) = 0$,

$$\tilde{x}(k+1) = A_s\tilde{x}(k) \; ; \tag{3.16}$$

where

$$A_s = A - B\Gamma^{-1}M \tag{3.17}$$

$$A_{eq} = A - B\Gamma^{-1}(M + \phi^{-1}KG) \tag{3.18}$$

Thus, A_s represents the linear portion of the dynamics for $s(k)$ outside $\mathcal{B}$ as well as the dynamics in sliding mode while A_{eq} represents the dynamics for $s(k)$ inside $\mathcal{B}$.

Proof. By straightforward verification:

$$\begin{aligned}\tilde{x}(k+1) &= [I - B(B^TB)^{-1}B^T](x_d(k+1) - Ax_d(k)) + \left[A - B\Gamma^{-1}M\right]\tilde{x}(k) \\ &\quad - B\Gamma^{-1}K\,\text{sat}(G\tilde{x}(k)/\phi) - D_o w_o(k)\end{aligned}$$

The first term in the above expression vanishes in view of Eq. (3.11) and the results follow immediately from the definition of sat(·) function. □

Remark 3.1. In the single-input case, $m = 1$, $\mu_1 = G(A - I)$, $K_1 = K_\Sigma$, and $B_1 = B$. Also, B^T in the first term of the control law (3.2) can be replaced by G, since

$$\begin{aligned}&(B^TB)^{-1}B^T(x_d(k+1) - Ax_d(k)) \\ =\;& (B^TB)^{-1}B^T(Bu_d(k)) = u_d(k) = (GB)^{-1}G(x_d(k+1) - Ax_d(k))\end{aligned} \tag{3.19}$$

Thus, the control law (3.2) reduces to

$$u(k) = (GB)^{-1}\left[G\left((I - A)x(k) + x_d(k+1) - x_d(k)\right) + K_\Sigma\,\text{sat}\left(\frac{s(k)}{\phi}\right)\right] \tag{3.20}$$

which is the same as that of Misawa [28]. Moreover, the tracking error dynamics stated in Corollary 3.1 for $s(k)$ on $\mathcal{S}$ with $w_o(k) = 0$ and for $s(k)$ inside $\mathcal{B}$ reduce to

$$\tilde{x}(k+1) = \left[A - \frac{BG}{GB}(A - I)\right]\tilde{x}(k) \tag{3.21}$$

$$\tilde{x}(k+1) = \left[A - \frac{BG}{GB}\left(A - (1 - \frac{K_\Sigma}{\phi})I\right)\right]\tilde{x}(k) - D_o w_o(k) \tag{3.22}$$

respectively which are the same as that of Furuta [15] and Richter *et al.* [33].

Remark 3.2. It can be seen from Eq. (3.19) that the first term in the control law (3.2) is actually the hypothetical input $u_d(k)$. This term is used instead because it is usually more convenient to work with signals x_d rather than u_d.

3.2 The Eigenvalue Constraint

The attractiveness and invariance of $\mathcal{B}$ is not sufficient for the stability of the variable structure systems. In addition, the equivalent matrix A_{eq} should be determined so that the tracking error dynamics inside $\mathcal{B}$ is asymptotically stable in the absence of uncertainties. Since A_{eq} defined in Eq. (3.18) has the form of $A - BF$, well-established linear control design strategies such as pole placement, LQR, H_∞ etc. can be applied to determine the feedback gain matrix F. As shown below, this is possible under the *eigenvalue constraint*.

Definition 3.2. In variable structure systems, the *eigenvalue constraint* is the restriction that the matrix representing the linear dynamics on the sliding surface $\mathcal{S}$ or inside the boundary layer $\mathcal{B}$ cannot have all eigenvalues being complex.

This fact is best illustrated by considering a continuous-time second-order system described by

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_c(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_c(t)$$

Suppose the sliding surface, boundary layer thickness, and the variable structure control law are respectively designed as

$$\begin{aligned} s_c(t) &= G_c x_c(t) = [1\ 1]\, x_c(t) = 0 \\ \phi_c &= 0.1 \\ u_c(t) &= \begin{cases} -(G_c B_c)^{-1} G_c A_c x_c(t) - \operatorname{sgn}(s_c(t)) & , \quad |s_c(t)| > \phi_c \\ -[1\ 0]\, x_c(t) & , \quad |s_c(t)| \le \phi_c \end{cases} \end{aligned}$$

so that the linear dynamics inside the boundary layer is governed by a pair of complex poles at $-1 \pm j$. The phase plane plot for different initial conditions is shown in Figure 3.1. It can be seen that the attractiveness of the boundary layer and the nature of a stable focus cause

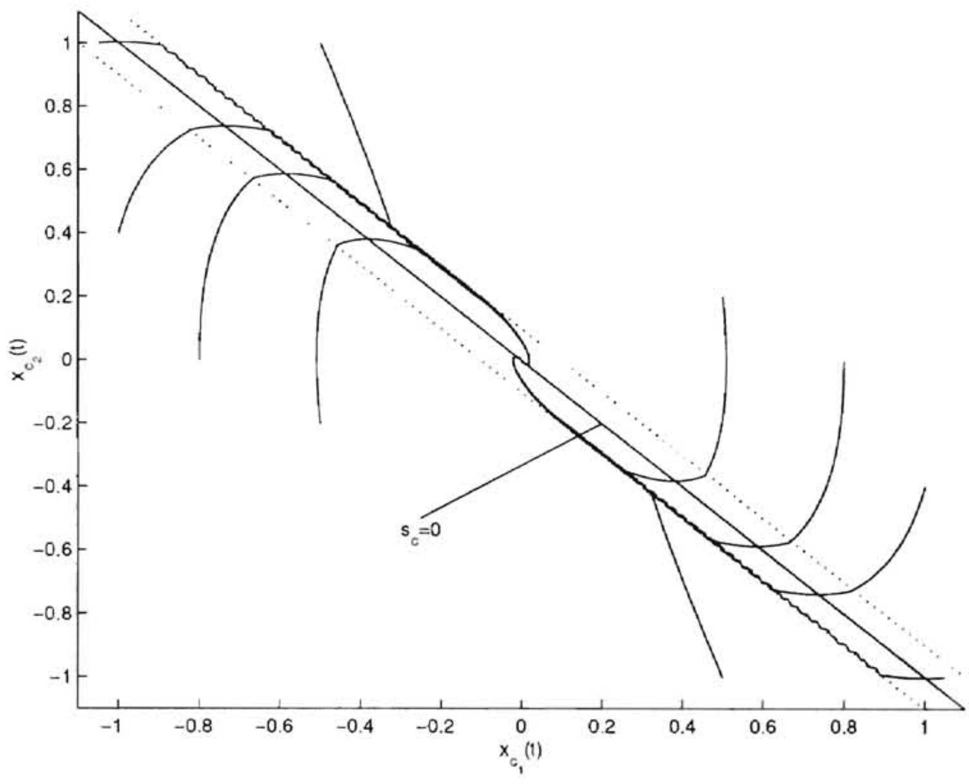

Figure 3.1: Phase plane plot of a continuous-time second-order variable structure system with a pair of complex poles inside the boundary layer

the trajectories to "slide" along the edges of the boundary layer and then approach the origin spirally. This undesirable effect indicates that making the sliding function decay and placing a pair of complex poles for the dynamics inside the boundary layer are conflicting objectives. This notion can be simply extended to higher, even order systems with all eigenvalues being complex. Also, this *eigenvalue constraint* arises despite the thickness of the boundary layer since spiral behavior in linear systems is global. Moreover, it is independent of how the control is derived, but an intrinsic nature of variable structure control with linear dynamics on the sliding surface $\mathcal{S}$ or inside the boundary layer $\mathcal{B}$.

3.3 One Hyperplane versus Multiple Hyperplanes

The sliding regime in multivariable systems is traditionally defined as the intersection of m hyperplanes and depending on the approaches, either each of the hyperplanes or just the intersection is made attractive. This convention is also followed in existing discrete variable structure control schemes (Su *et al.* [38], Chan [5], Koshkouei and Zinober [22]), which allows only $n-m$ eigenvalues to be freely assigned, with the remaining m eigenvalues at zero. This eigenvalue assignment constraint is usually tolerable for continuous-time systems because perfect invariance to matched uncertainties is obtained as a trade-off. Since perfect invariance no longer holds for discrete-time systems, unless if $m = n$ where the sliding regime is trivially defined as the state space origin then achieving a "degraded" sliding mode means that the tracking error is small. Otherwise, there is no significant advantage in confining the error within the *neighborhood* of an $n-m$ dimensional sliding regime as opposed to that of an $n-1$ dimensional one using one hyperplane. Besides, for approach that makes each of the hyperplanes attractive, the system usually suffers from "jagged" motion in the reaching phase because the trajectory must reach the first hyperplane and then slide along that hyperplane until it reaches the second hyperplane and so on. Furthermore, with only $n-m$ eigenvalues freely assignable, many of the linear control design strategies are applicable under severe restrictions.

In view of these, the proposed technique attempts to drive the error trajectory and restrict it to stay within the neighborhood of an $n-1$ dimensional subspace, the highest dimension possible for variable structure systems with sliding mode. Since only one hyperplane is attractive, "jagged" motion in the reaching phase is reduced. As is shown next, all the eigenvalues of A_{eq} can be freely assigned with at least one being real so that the *eigenvalue constraint* is satisfied. Hence, the design constraint is kept at a minimum and major elements in linear multivariable feedback systems design such as quadratic optimization, disturbance rejection, controller bandwidth, etc. can be easily captured using the well-developed time and frequency domain tools.

3.4 Model Matching in the Linear Region

Lemma 3.1. *If (A, B) is controllable, then for almost any $F \in \mathbf{R}^{m\times n}$, all the eigenvalues of $A - BF$ are distinct.*

Proof. See Zhou *et al.* [53] pg. 62 for a list of sources of proof. □

Theorem 3.2 (Model Matching via A_{eq}). *Consider the matrices A and B of the system (3.1). Let Assumptions 3.1, 3.2, and 3.3 hold. Let $F \in \mathbf{R}^{m\times n}$ be chosen such that $A - BF$ has all its eigenvalues distinct and inside the unit circle and has at least one real eigenvalue at λ. Then, $A_{eq} = A - BF$ if and only if G, K, ϕ, and M satisfy*

$$M + \phi^{-1}KG = \Gamma F \tag{3.23}$$

$$1 - \frac{K_\Sigma}{\phi} = \lambda \tag{3.24}$$

$$G = \ker\left((A - BF - \lambda I)^T\right)^T \tag{3.25}$$

Proof. Let $F = [\,F_1 \mid F_2 \mid \cdots \mid F_m\,]^T$, $F_i \in \mathbf{R}^n$. It follows from Eq. (3.18) that $A_{eq} = A - BF$ if and only if Eq. (3.23) or its equivalent, given by

$$\mu_i^T + \frac{K_i}{\phi} G = G B_i F_i^T\,, \quad 1 \le i \le m \tag{3.26}$$

is satisfied. Taking the sum of both sides of Eq. (3.26) from $i = 1$ to m gives

$$\sum_{i=1}^{m} \mu_i^T + \sum_{i=1}^{m} (K_i/\phi) G = \sum_{i=1}^{m} G B_i F_i^T$$

Since Eqs. (3.5) and (3.7) must hold, the above expression becomes

$$G(A - I) + (K_\Sigma/\phi) G = GBF$$

$$\Rightarrow \quad [A - BF - (1 - K_\Sigma/\phi) I]^T G^T = 0$$

which implies that a nontrivial solution G exists if and only if Eq. (3.24) if satisfied. It is obvious that the solution is given by Eq. (3.25). □

To illustrate the use of Theorem 3.2, one may proceed as follows. First, select F using any linear control design strategy with the restriction that F yields at least a real eigenvalue at λ. Next, determine G from Eq. (3.25) and make sure Assumption 3.3 is satisfied. Then, obtain γ from Eq. (3.9) using the knowledge of the bounds on uncertainties $w_o(k)$. Since K_Σ, ϕ, and ϵ depend on Eqs. (3.7), (3.8), and (3.24), fixing either parameter fixes the others. Some guidelines on how ϵ and γ affect the convergence rate and steady state value of $s(k)$ are given in Misawa [28]. Finally, M can be computed using Eq. (3.23).

It can be seen from Eq. (3.7) that for a given K_Σ, the choice of sliding gains K_i, $1 \le i \le m$ is not unique. This suggests that after the matching of A_{eq} with $A - BF$ is completed, these design freedoms can be further utilized, e.g. in avoiding controller saturation and in the design of nonlinear dynamics outside $\mathcal{B}$ given in Eq. (3.14). Unlike linear control laws, this technique allows the use of lower gain outside $\mathcal{B}$ through the sat$(\cdot)$ function which signifies a smaller chance of saturation and hence a larger operating region. However, formal treatment of this issue is beyond the scope of current research and is not pursued here.

The following corollary reveals the relation between A_{eq} and A_s, which may be viewed as a generalization of the relation between the results by Furuta [15] and Richter *et al.* [33] to multivariable systems.

Corollary 3.2. *Let* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be the distinct eigenvalues of* A_{eq} *and let* $\lambda_n = 1 - (K_\Sigma/\phi)$*. Then, the eigenvalues of* A_s *are* $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, 1\}$*.*

Proof. Let x_i and y_i be respectively the right and left eigenvector of A_{eq} associated with eigenvalue λ_i. Premultiply A_{eq} by G gives

$$GA_{eq} = GA - \sum_{i=1}^{m} \mu_i^T - G\sum_{i=1}^{m}(K_i/\phi) = G(1 - K_\Sigma/\phi)$$

which implies that G is the left eigenvector associated with the eigenvalue $(1 - K_\Sigma/\phi)$, i.e. $y_n^* = G$. Similarly, premultiply A_s by G gives

$$GA_s = GA - \sum_{i=1}^{m} \mu_i^T = G$$

which implies that A_s has an eigenvalue at 1 and G is the corresponding left eigenvector. Since A_{eq} and A_s are related by

$$A_s = A_{eq} + \phi^{-1}B\Gamma^{-1}KG\ ,$$

using the spectral decomposition of A_{eq} leads to

$$\begin{aligned}
A_s &= \sum_{i=1}^{n} \lambda_i x_i y_i^* + \phi^{-1}B\Gamma^{-1}KG \\
&= \sum_{i=1}^{n-1} \lambda_i x_i y_i^* + (1 - K_\Sigma/\phi)x_n G + \phi^{-1}B\Gamma^{-1}KG \\
&= \sum_{i=1}^{n-1} \lambda_i x_i y_i^* + [(1 - K_\Sigma/\phi)x_n + \phi^{-1}B\Gamma^{-1}K]y_n^*
\end{aligned}$$

Thus, both A_{eq} and A_s have $(n-1)$ eigenvalues in common, namely $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}\}$. □

Corollary 3.3. *Consider the system (3.1) and let Assumptions 3.1, 3.2, and 3.3 hold. Suppose the control law given in Theorem 3.1 is used and suppose* F *and the corresponding* G, K, ϕ, *and* M *are chosen to meet the conditions stated in Theorem 3.2. Then, the tracking error dynamics is asymptotically stable if* $w_o(k) = 0$ *and BIBO stable otherwise.*

Proof. This is obvious in view of Theorems 3.1 and 3.2. □

Chapter 4

Additive Uncertainties: The Output Feedback Case

The controller proposed in Chapter 3 cannot be implemented in situations when full state feedback is not available, which is usually the case in practice. This chapter considers the output feedback case where a prediction observer along with uncertainty estimation is first presented. Then, the discussions will be centered around the convergence properties and design issues related to the resulting observer-based controller.

Consider the following discrete-time linear multivariable system which may be obtained by discretizing its continuous-time equivalent with sampling period Δt:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + D_o w_o(k) \\ y(k) &= Cx(k) + v(k) \end{aligned} \tag{4.1}$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^m$ is the input vector, $y \in \mathbf{R}^p$ is the output vector, $w_o \in \mathbf{R}^{q_o}$ is the additive uncertainty vector, and $v \in \mathbf{R}^p$ is the measurement noise vector. Let w_o be decomposed into the matched and unmatched portion as follows:

$$D_o w_o(k) = BDw(k) + B_\perp D_\perp w_\perp(k) \tag{4.2}$$

where $w \in \mathbf{R}^q$ is the matched uncertainty vector, $w_\perp \in \mathbf{R}^{q_\perp}$ is the unmatched uncertainty vector, $q_o = q + q_\perp$, and the dynamics of the matched portion be described by

$$w(k+1) = w(k) + r(k) \tag{4.3}$$

where $r \in \mathbf{R}^q$ is the vector indicating its variation. A, B, $B_\perp$, C, D_o, D, and $D_\perp$ are perfectly known constant matrices with appropriate dimensions with $B = [\,B_1 \,|\, B_2 \,|\, \cdots \,|\, B_m\,]$, $B_i \in \mathbf{R}^n$, and the column space of $B_\perp \in \mathbf{R}^{n\times(n-m)}$ being the orthogonal complement of that of B. The objective is to find a suitable control input $u(k)$ so that $x(k)$ will track a known desired trajectory $x_d(k)$.

Assumption 4.1. The system (4.1) satisfies the following: $\text{rank}(B) = m$, $\text{rank}(C) = p$, the pair (A, B) is controllable, the pair (A, C) is observable, and $w_o(k)$ and $v(k)$ are bounded.

To facilitate the presentation that follows, introduce the augmented state vector $\eta = [x^T \; w^T]^T \in \mathbf{R}^{n+q}$. Along with Eqs. (4.1), (4.2), and (4.3), the augmented system can be written as

$$\begin{aligned}
\begin{bmatrix} x(k+1) \\ w(k+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} A & BD \\ 0 & I \end{bmatrix}}_{A_a} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_a} u(k) + \underbrace{\begin{bmatrix} B_\perp D_\perp w_\perp(k) \\ r(k) \end{bmatrix}}_{\zeta(k)} \\
y(k) &= \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C_a} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + v(k)
\end{aligned}$$

or compactly as

$$\begin{aligned}
\eta(k+1) &= A_a \eta(k) + B_a u(k) + \zeta(k) \\
y(k) &= C_a \eta(k) + v(k)
\end{aligned} \tag{4.4}$$

where $\zeta \in \mathbf{R}^{n+q}$ is the vector of uncertainties to this augmented system.

4.1 Prediction Observer with Uncertainty Estimation

Definition 4.1. Let $w_{\perp i}$, r_i, and v_i be the ith entry of $w_\perp$, r, and v respectively. Let $\hat{\eta}(k) = [\hat{x}^T(k) \; \hat{w}^T(k)]^T$ be the estimate of $\eta(k)$ and $\tilde{\eta}(k) = [\tilde{x}_e^T(k) \; \tilde{w}^T(k)]^T$ be the estimation error, i.e. $\tilde{\eta}(k) = \eta(k) - \hat{\eta}(k)$.

Since parametric uncertainties are assumed absent, state observers that provide more flexibility over static output feedback can be easily constructed. Estimation of the matched uncertainty vector $w(k)$ can also be performed as long as the augmented system (4.4) remains at least detectable. A simple lemma on the necessary and sufficient conditions for the observability of the pair (A_a, C_a) is given as follows.

Lemma 4.1. *Let Assumption 4.1 holds. The pair (A_a, C_a) is observable if and only if*

$$W := \begin{bmatrix} A - I & BD \\ C & 0 \end{bmatrix} \in \mathbf{R}^{(n+p)\times(n+q)}$$

has full column rank. Moreover, it is observable only if $p \geq q$ and $\mathrm{rank}(D) = q$.

Proof. It is well known (Zhou *et al.* [53]) that (A_a, C_a) is observable if and only if

$$W_o(\lambda) := \begin{bmatrix} A_a - \lambda I \\ C_a \end{bmatrix} = \begin{bmatrix} A - \lambda I & BD \\ 0 & (1-\lambda)I \\ C & 0 \end{bmatrix} \in \mathbf{R}^{(n+q+p)\times(n+q)}$$

has full column rank $\forall \lambda \in \mathbf{C}$. The hypothesis on the observability of (A, C) implies that $W_o(\lambda)$ has full column rank $\forall \lambda \in \mathbf{C}$ except possibly for $\lambda = 1$ where $\mathrm{rank}(W_o(1)) = \mathrm{rank}(W)$. Since $W \in \mathbf{R}^{(n+p)\times(n+q)}$, the condition $p \geq q$ is necessary for full column rank. Finally, it can be shown that $\mathrm{rank}(D) = q$ is also a necessary condition using Sylvester's inequality. □

Assumption 4.2. The pair (A_a, C_a) is observable and the observer gain matrix $H_a := [H_1^T \ H_2^T]^T$ with $H_1 \in \mathbf{R}^{n\times p}$ and $H_2 \in \mathbf{R}^{q\times p}$ is chosen such that $A_a - H_a C_a$ is Schur stable.

Lemma 4.2. *Consider the augmented system (4.4) and let Assumptions 4.1 and 4.2 hold. If the Luenberger prediction observer with uncertainty estimation*

$$\hat{\eta}(k+1) = A_a\hat{\eta}(k) + B_a u(k) + H_a(y(k) - C_a\hat{\eta}(k)) \tag{4.5}$$

is used, then the observer error dynamics

$$\tilde{\eta}(k+1) = (A_a - H_a C_a)\tilde{\eta}(k) - H_a v(k) + \zeta(k) \tag{4.6}$$

is asymptotically stable if $w_\perp(k) = 0$, $r(k) = 0$, and $v(k) = 0$ and BIBO stable otherwise.

Proof. This is a standard result. See, e.g., Franklin *et al.* [13] pg. 250. □

It is known that observers with uncertainty estimation often yield better observation when the sampling rate is considerably faster than the dynamics of the $w(k)$, i.e. when $r(k)$ is small. It also gives other advantages in spite of heavier computation burden, e.g. the bounds on $w(k)$ need not be known and the estimate $\hat{w}(k)$ can be fedforward to the controller to reduce the effect of $w(k)$.

Unlike current observer which estimates the state at the kth instant based on measurements up to and including the kth instant, the prediction observer estimates the state at the kth instant based on measurements up to and including the $(k-1)$th instant. As a result of this computational delay, it might not be as accurate as the current observer but as a trade-off, it allows the entire sampling period for computations.

The estimation of the unmatched uncertainties $w_{\perp}(k)$ as well as the use of current or reduced order observer are not pursued here although possible as long as the resulting system remains observable.

Assumption 4.3. The unknown signals $w_{\perp i}(k)$, $r_i(k)$, and $v_i(k)$ are bounded by known scalars $\delta_{w_{\perp i}}$, δ_{r_i}, and δ_{v_i} in magnitude and can be expressed as sums of sinusoidal signals with frequencies belong to the known sets $\Omega(w_{\perp i})$, $\Omega(r_i)$, and $\Omega(v_i)$ respectively. The sets $\Omega(\cdot) \subseteq (-\pi/\Delta t, \pi/\Delta t]$ may be discontinuous and may take into account the phenomenon of aliasing when components with frequencies exceeding the Nyquist frequency $\omega = \pi/\Delta t$ exist.

Lemma 4.3. *Let Assumptions 4.1, 4.2, and 4.3 hold and let $\beta \in \mathbf{R}^{1\times(n+q)}$ be an arbitrary row vector. Let $\Phi_{w_{\perp i}}(z,\beta)$, $\Phi_{r_i}(z,\beta)$, and $\Phi_{v_i}(z,\beta)$ represent the transfer functions from $w_{\perp i}$, r_i, and v_i to $\beta\tilde{\eta}$ respectively, where they are obtained from the stable transfer matrix*

$$\tilde{\eta}(z) = [zI - A_a + H_a C_a]^{-1}\left[\zeta(z) - H_a v(z)\right] \tag{4.7}$$

Then in steady-state,

$$\begin{aligned}|\beta\tilde{\eta}(k)| \;\le\; & \sum_{i=1}^{q_\perp} \sup_{\omega\in\Omega(w_{\perp i})} |\Phi_{w_{\perp i}}(e^{j\omega\Delta t},\beta)|\delta_{w_{\perp i}} + \sum_{i=1}^{q} \sup_{\omega\in\Omega(r_i)} |\Phi_{r_i}(e^{j\omega\Delta t},\beta)|\delta_{r_i} \\ & + \sum_{i=1}^{p} \sup_{\omega\in\Omega(v_i)} |\Phi_{v_i}(e^{j\omega\Delta t},\beta)|\delta_{v_i}\end{aligned} \tag{4.8}$$

Proof. This is a standard result. See, e.g., Franklin *et al.* [13] pg. 77. □

Remark 4.1. For white noise, e.g. if $v_1(k)$ is white, then $\Omega(v_1) = (-\pi/\Delta t, \pi/\Delta t]$ and

$$\sup_{\omega\in\Omega(v_1)} |\Phi_{v_1}(e^{j\omega\Delta t}, \beta)| = \|\Phi_{v_1}(z,\beta)\|_\infty = \mathcal{H}_\infty \text{ norm of } \Phi_{v_1}(z,\beta)$$

4.2 Attractiveness and Invariance of Estimated Boundary Layer

The following is introduced in addition to Definition 3.1.

Definition 4.2. Let the estimated sliding function $\hat{s} \in \mathbf{R}$, the estimated sliding surface $\hat{\mathcal{S}} \subset \mathbf{R}^n$, and the estimated boundary layer $\hat{\mathcal{B}} \subset \mathbf{R}^n$ be respectively defined as

$$\begin{aligned} \hat{s}(k) &= G(x_d(k) - \hat{x}(k)) \\ \hat{\mathcal{S}} &= \{\tilde{x} : \hat{s} = 0\} \\ \hat{\mathcal{B}} &= \{\tilde{x} : |\hat{s}| \le \phi\ ,\ \phi > 0\} \end{aligned}$$

Theorem 4.1 (Attractiveness and Invariance of $\hat{\mathcal{B}}$ with Prediction Observer). *Consider the system (4.1) and let Assumptions 3.2, 3.3, 4.1, 4.2, and 4.3 hold. If the prediction observer stated in Lemma 4.2 is used and if the control law is chosen as*

$$\begin{aligned} u(k) &= (B^TB)^{-1}B^T(x_d(k+1) - Ax_d(k)) + \Gamma^{-1}M(x_d(k) - \hat{x}(k)) \\ &\quad + \Gamma^{-1}K\,\text{sat}\left(\frac{\hat{s}(k)}{\phi}\right) - D\hat{w}(k) \end{aligned} \tag{4.9}$$

with Γ, M, μ_i, K, K_i, K_Σ, ϕ, and ϵ defined similarly as in Eqs. (3.3) through (3.8), and

$$\begin{aligned} \gamma &\ge \sum_{i=1}^{q_\perp} \sup_{\omega\in\Omega(w_{\perp i})} |\Phi_{w_{\perp i}}(e^{j\omega\Delta t}, GH_1C_a)|\delta_{w_{\perp i}} + \sum_{i=1}^{q} \sup_{\omega\in\Omega(r_i)} |\Phi_{r_i}(e^{j\omega\Delta t}, GH_1C_a)|\delta_{r_i} \\ &\quad + \sum_{i=1}^{p} \sup_{\omega\in\Omega(v_i)} |\Phi_{v_i}(e^{j\omega\Delta t}, GH_1C_a)|\delta_{v_i} + |GH_1v(k)|\ , \end{aligned} \tag{4.10}$$

then $\hat{\mathcal{B}}$ is attractive and invariant, i.e. there exists a k_s such that $\tilde{x}(k) \in \hat{\mathcal{B}}$ for all $k \ge k_s$. In particular, $\hat{s}(k)$ asymptotically approaches $\hat{\mathcal{S}}$ if $w_\perp(k) = 0$, $r(k) = 0$, and $v(k) = 0$.

Proof. Consider a Lyapunov function candidate $V(k) = \hat{s}^2(k)$. $|\hat{s}(k)|$ decreases monotonically if the following inequality holds:

$$V(k+1) < V(k) \Rightarrow [\Delta\hat{s}(k) + 2\hat{s}(k)]\Delta\hat{s}(k) < 0\ , \quad \forall \hat{s}(k) \ne 0 \tag{4.11}$$

where $\Delta\hat{s}(k) = \hat{s}(k+1) - \hat{s}(k)$. Using Eqs. (4.9), (3.5), and (3.7), $\Delta\hat{s}(k)$ is obtained as

$$\begin{aligned}\Delta\hat{s}(k) &= Gx_d(k+1) - GA\tilde{x}_e(k) - GA\hat{x}(k) - GB(B^TB)^{-1}B^T(x_d(k+1) - Ax_d(k)) \\ &\quad -\sum_{i=1}^{m}\mu_i^T(x_d(k) - \hat{x}(k)) - \sum_{i=1}^{m}K_i\,\mathrm{sat}(\hat{s}(k)/\phi) - GB_\perp D_\perp w_\perp(k) \\ &\quad -GBD(w(k) - \hat{w}(k)) + G\tilde{x}_e(k+1) - G(x_d(k) - \hat{x}(k)) \\ &= G[I - B(B^TB)^{-1}B^T](x_d(k+1) - Ax_d(k)) - K_\Sigma\,\mathrm{sat}(\hat{s}(k)/\phi) \\ &\quad +G[\tilde{x}_e(k+1) - A\tilde{x}_e(k) - BD\tilde{w}(k) - B_\perp D_\perp w_\perp(k)]\end{aligned}$$

It follows from Eq. (3.11) that the first term in the above expression vanishes. Also, it can be seen by inspecting Eq. (4.6) that the last term may be expressed as

$$G[\tilde{x}_e(k+1) - A\tilde{x}_e(k) - BD\tilde{w}(k) - B_\perp D_\perp w_\perp(k)] = -GH_1C_a\tilde{\eta}(k) - GH_1v(k)$$

which simplifies $\Delta\hat{s}(k)$ to

$$\Delta\hat{s}(k) = -K_\Sigma\,\mathrm{sat}(\hat{s}(k)/\phi) - GH_1C_a\tilde{\eta}(k) - GH_1v(k) \tag{4.12}$$

For $\hat{s}(k)$ outside $\hat{\mathcal{B}}$, Eq. (4.12) represents a marginally stable first-order system which implies that $\hat{s}(k)$ is bounded. The observer error dynamics (4.6) must finally come to a steady-state since it is independent of $\hat{s}(k)$. In steady-state, the bound on $GH_1C_a\tilde{\eta}(k)$ is given by inequality (4.8) with β replaced by GH_1C_a. If γ is chosen to satisfy inequality (4.10), then

$$\gamma \geq |GH_1C_a\tilde{\eta}(k)| + |GH_1v(k)| \geq |GH_1C_a\tilde{\eta}(k) + GH_1v(k)| \tag{4.13}$$

Since $\Delta\hat{s}(k)$ in Eq. (4.12) and γ in Eq. (4.13) are analogous to $\Delta s(k)$ in Eq. (3.12) and γ in Eq. (3.9) for the state feedback case, the remaining proof follows along the same lines of Theorem 3.1. □

Corollary 4.1. *With the control law stated in Theorem 4.1, the value of $s(k)$ as $k \to \infty$ is bounded by*

$$\begin{aligned}|s(k)| &< \phi + \sum_{i=1}^{q_\perp}\sup_{\omega\in\Omega(w_{\perp i})}|\Phi_{w_{\perp i}}(e^{j\omega\Delta t},[G\ 0])|\delta_{w_{\perp i}} + \sum_{i=1}^{q}\sup_{\omega\in\Omega(r_i)}|\Phi_{r_i}(e^{j\omega\Delta t},[G\ 0])|\delta_{r_i} \\ &\quad + \sum_{i=1}^{p}\sup_{\omega\in\Omega(v_i)}|\Phi_{v_i}(e^{j\omega\Delta t},[G\ 0])|\delta_{v_i}\end{aligned} \tag{4.14}$$

In particular, $s(k) \to 0$ as $k \to \infty$ if $w_\perp(k) = 0$, $r(k) = 0$, and $v(k) = 0$.

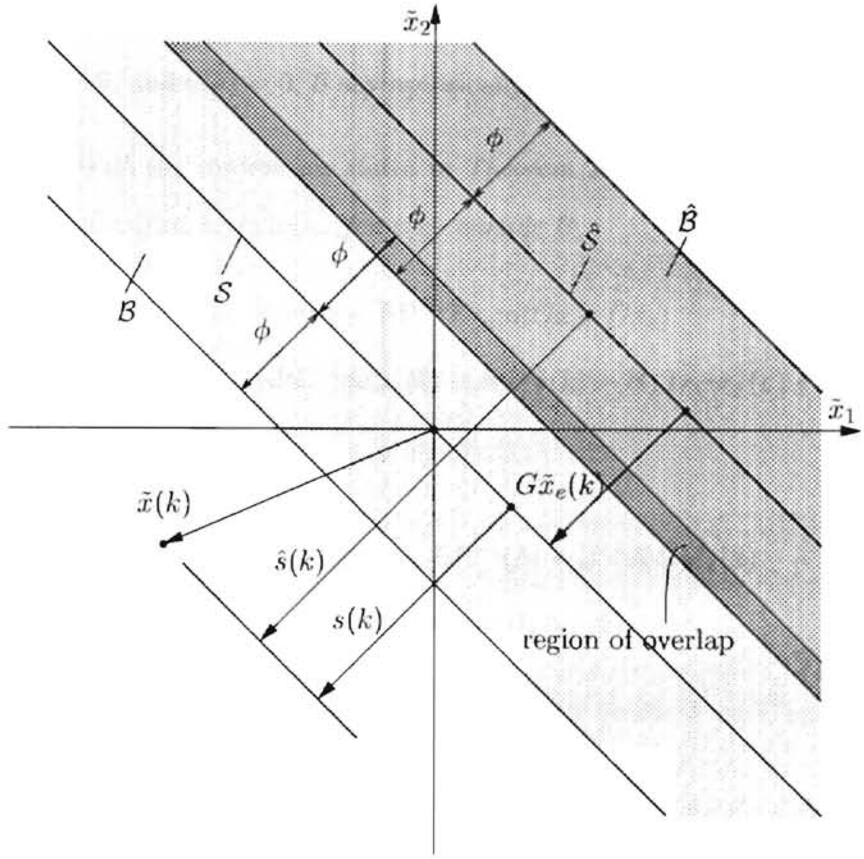

Figure 4.1: Phase plane plot showing the dynamic region of attraction $\hat{\mathcal{B}}$ and the region of overlap between $\hat{\mathcal{B}}$ and $\mathcal{B}$ at kth instant

Proof. Observe that

$$s(k) = \hat{s}(k) - G\tilde{x}_e(k) = \hat{s}(k) - [G\ 0]\,\tilde{\eta}(k) \tag{4.15}$$

It follows from Theorem 4.1 and Lemma 4.3 that as $k \to \infty$, $|\hat{s}(k)| < \phi$ and $|[G\ 0]\tilde{\eta}(k)|$ satisfies inequality (4.8) with $\beta = [G\ 0]$, which leads immediately to inequality (4.14). If $w_\perp(k) = 0$, $r(k) = 0$, and $v(k) = 0$, then $\hat{s}(k) \to 0$ from Theorem 4.1 and $\tilde{\eta}(k) \to 0$ from Lemma 4.2 inply that $s(k) \to 0$ as $k \to \infty$. □

The region of attraction $\hat{\mathcal{B}}$ under this observer-based controller is dynamic in the direction of G. This is in contrast to the state feedback case where the region of attraction $\mathcal{B}$ is static. Whether $\hat{\mathcal{B}}$ overlaps $\mathcal{B}$ depends statically on ϕ and dynamically on the term $G\tilde{x}_e(k)$ as evident from Eq. (4.15). Graphical interpretation of the dynamic region of attraction and the region of overlap for the case $n = 2$ is given in Figure 4.1. As can be expected, if $w_\perp(k) = 0$, $r(k) = 0$, and $v(k) = 0$, $\hat{\mathcal{B}}$ asymptotically approaches and overlaps $\mathcal{B}$ as $k \to \infty$.

Corollary 4.2. *With the control law stated in Theorem 4.1, the closed-loop tracking error dynamics can be classified as follows: for $\hat{s}(k)$ outside $\hat{\mathcal{B}}$.*

$$\begin{aligned}\tilde{x}(k+1) &= A_s\tilde{x}(k) - B\Gamma^{-1}K\,\mathrm{sgn}(G\tilde{x} + G\tilde{x}_e) \\ &\quad -B\Gamma^{-1}M\tilde{x}_e(k) - BD\tilde{w}(k) - B_\perp D_\perp w_\perp(k)\,; \qquad (4.16)\end{aligned}$$

and for $\hat{s}(k)$ inside $\hat{\mathcal{B}}$,

$$\begin{aligned}\tilde{x}(k+1) &= A_{eq}\tilde{x}(k) - B\Gamma^{-1}(M + \phi^{-1}KG)\tilde{x}_e(k) \\ &\quad -BD\tilde{w}(k) - B_\perp D_\perp w_\perp(k) \qquad (4.17)\end{aligned}$$

where A_s and A_{eq} are defined similarly as in the state feedback case under Eqs. (3.17) and (3.18) respectively.

Proof. Noticing that $x(k) = \tilde{x}_e(k) + \hat{x}(k) = x_d(k) - \tilde{x}(k)$ and by straightforward verification:

$$\begin{aligned}\tilde{x}(k+1) &= [I - B(B^TB)^{-1}B^T](x_d(k+1) - Ax_d(k)) + \left[A - B\Gamma^{-1}M\right]\tilde{x}(k) \\ &\quad -B\Gamma^{-1}K\,\mathrm{sat}(\hat{s}(k)/\phi) - B\Gamma^{-1}M\tilde{x}_e(k) - BD\tilde{w}(k) - B_\perp D_\perp w_\perp(k)\end{aligned}$$

The first term in the above expression vanishes in view of Eq. (3.11) and the results follow immediately from the definition of $\mathrm{sat}(\cdot)$ function. □

4.3 Model Matching in the Linear Region

Lemma 4.4. *For $\hat{s}(k)$ inside $\hat{\mathcal{B}}$, the separation principle (Franklin* et al. *[13]) holds for the overall error dynamics governed by Eqs. (4.6) and (4.17).*

Proof. Since the overall error dynamics inside $\hat{\mathcal{B}}$ has the form

$$\begin{bmatrix}\tilde{x}(k+1)\\ \tilde{\eta}(k+1)\end{bmatrix} = \begin{bmatrix}A_{eq} & \star\\ 0 & A_a - H_aC_a\end{bmatrix}\begin{bmatrix}\tilde{x}(k)\\ \tilde{\eta}(k)\end{bmatrix} + \begin{bmatrix}\star\\ \star\end{bmatrix}$$

□

The above lemma is a natural result since parametric uncertainties are assumed absent in the system (4.1). Hence, for $\hat{s}(k)$ inside $\hat{\mathcal{B}}$, Theorem 3.2 is applicable to match A_{eq} with any $A - BF$ obtained using linear control design strategies as long as the *eigenvalue constraint* discussed in Section 3.2 is satisfied.

Similar to the state feedback case, the choice of sliding gains K_i, $1 \leq i \leq m$ is not unique. Several suggestions on how to utilize these freedoms for the state feedback case are given in Section 3.4. Formal treatment of this issue for the output feedback case is beyond the scope of current research and is not pursued here.

Corollary 4.3. *Consider the system (4.1) and let Assumptions 3.2, 3.3, 4.1, 4.2, and 4.3 hold. Suppose the prediction observer with uncertainty estimation and the control law given in Lemma 4.2 and Theorem 4.1 are used and suppose F and the corresponding G, K, ϕ, and M are chosen to meet the conditions stated in Theorem 3.2. Then, the overall error dynamics is asymptotically stable if $w_{\perp}(k) = 0$, $r(k) = 0$, and $v(k) = 0$ and BIBO stable otherwise.*

Proof. This is obvious in view of Lemmas 4.2, 4.4 and Theorem 4.1. □

Chapter 5

Parametric Uncertainties: The State Feedback Case

This chapter studies the problem of state feedback stabilization of a class of discrete linear multivariable systems with parametric uncertainties using the theory of variable structure with switching sector. First, difficulties in ensuring stability of discrete variable structure systems with switching sector and potential stability problems with existing schemes are discussed. Then, a discrete variable structure control law with switching sector is designed as an attempt to overcome these difficulties. It is shown that global uniform asymptotic stability can be guaranteed despite the noninvariance of switching sector. Finally, properties of the resulting systems and admissible bounds on the uncertainties are compared to linear controllers.

5.1 On Stability of Discrete Variable Structure Systems with Switching Sector

The underlying stability problem in existing schemes with switching sector is that the switching sector is attractive but not an invariant set in general. This is in contrast with the results in Chapter 3 for the case of additive uncertainties; where the boundary layer is an invariant set and to conclude stability, it is sufficient to show that the boundary layer is

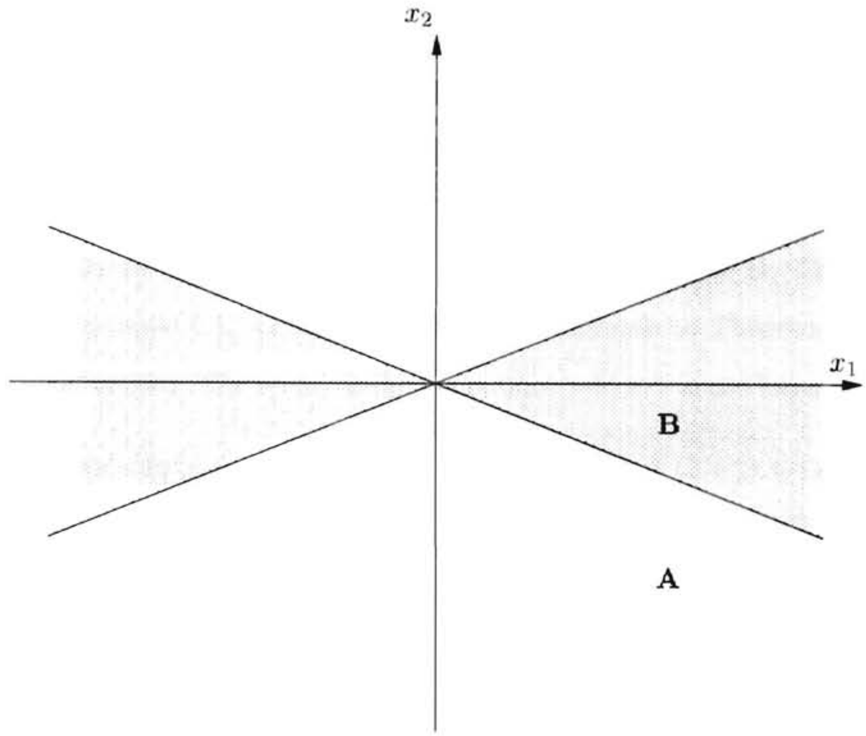

Figure 5.1: The switching sector (region **B**) and its surroundings (region **A**) of a second-order system

attractive and the dynamics inside the boundary layer is stable. Due to this reason, there are at least two potential stability problems with the use of switching sector. To illustrate this, let $x(k)$ denotes the state and consider the second-order case shown in Figure 5.1.

First, limit cycles may exist if these controllers are used. To show this, suppose $x(k)$ is initially in **A**. Since the switching sector is attractive, $x(k)$ then moves into **B**. However, the dynamics inside **B** is stable but **B** is not invariant, so $x(k)$ might go back into **A**. As $x(k)$ switches back to **A**, if it coincides with any point on its previous path in **A**, then it might repeat its history and limit cycles are encountered.

Second, the phenomenon of "unstable switchback" pointed out later by Wang *et al.* [46] may occur. Similar to the case of limit cycles, $x(k)$ will move from **A** into **B**. Since there is no additional condition imposed on how $x(k)$ should move from **A** into **B**, $\|x(k)\|$ might increase as it moves in. Next, consider a Lyapunov function candidate $V(k) = x^T(k)Px(k)$. An asymptotically stable linear dynamics inside **B** means that the difference $V(k+1) - V(k)$

is negative definite, which is equivalent to saying that if $x(k)$ is on the ellipsoid defined by $x^T(k)Px(k)$, then $x(k+1)$ is on a smaller ellipsoid defined by $x^T(k+1)Px(k+1)$. However, this does not imply that the $\|x(k)\|$ is decreasing. Since **B** is not invariant, $x(k)$ might go back into **A** with increasing $\|x(k)\|$. Therefore, it is possible that $\|x(k)\|$ increase continually as $x(k)$ switches back and forth between **A** into **B**. This phenomenon is illustrated in Example 5.1, which is also a counter example to Theorem 2 of an earlier paper by Wang *et al.* [48]. The notations and equations referred to are based on that paper.

Example 5.1. Consider a second order system described by $\bar{X}(k+1) = (\bar{A} + \Delta\bar{A})\bar{X}(k) + \bar{B}\bar{U}(k)$. Let $\bar{A} = \left[\begin{smallmatrix} 0.9 & 0.6 \\ 0 & 0.2 \end{smallmatrix}\right]$, $\bar{B} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, and $\bar{D} = [0.5\ 0.2]$ where $\Delta\bar{A} = \bar{B}\bar{D}$. Since $\bar{B}$ has the form of $\left[\begin{smallmatrix} 0 \\ B_2 \end{smallmatrix}\right]$, a change of coordinates is not necessary and therefore, $\bar{X} = X$, $\bar{A} = A$, $\bar{B} = B$, $\bar{U} = U$, $B_2 = 1$, and $\bar{D} = D$. Select the vector that defines the sliding hyperplane to be $C = [-0.7071\ \ -0.7071]$ where $C_1 = C_2 = -0.7071$ so that the nominal system on the hyperplane, namely $X(k+1) = (A + B(C - CA)/(CB))X(k) = \left[\begin{smallmatrix} 0.9 & 0.6 \\ 0.1 & 0.4 \end{smallmatrix}\right]$ has eigenvalues at 0.3 and 1. Next, L_1, L_2, and $\left|\frac{2}{C_2B_2(E_{n-1}|L_1|+|L_2|)}\right|$ are computed to be -1.2122, -0.2020, and 2 respectively. Also, select $\delta = 1.2$ and $\bar{d} = 0.6$ so that inequality (4), (12) are satisfied. Finally, select $k_d = -0.72$, 0, or 0.72 depending on Eqs. (11a), (11b), and (11c). The simulation result for initial condition $X(0) = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ is shown in Figure 5.2. It can be seen from Figure 5.2 that the "unstable switchback" phenomenon takes place.

The two potential stability problems discussed above are not considered in the papers proposed by Furuta [15] and Pan and Furuta [31]. In a later paper by Wang *et al.* [46], they tried to avoid the "unstable switchback" phenomenon. However, the critical statement $\|X_1(h+1)\|_2 < \|X_1(h)\|_2$ in the proof of Theorem 1 in that paper is incorrect. In the following claim, the notations and equations referred to are based on that paper. The nominal system on the hyperplane $S_{\hat{h}_i}$, represented by equation A.8, is stable in which the nominal state moves towards the nominal equilibrium point $\hat{X}_{ep}^{\hat{h}_1}$ at each time k. However, the 2-norm of the unperturbed components of X, i.e. $\|X_1(k)\|$ may not be decreasing because $\|A_{11} - A_{12}C_2^{-1}C_1\|_2$ may be larger than 1 although $A_{11} - A_{12}C_2^{-1}C_1$ is asymptotically stable. Thus, there is no guarantee that $\|X_1(h+1)\|_2 < \|X_1(h)\|_2$.

Based on the above discussions, one can see that the stability proof for systems with

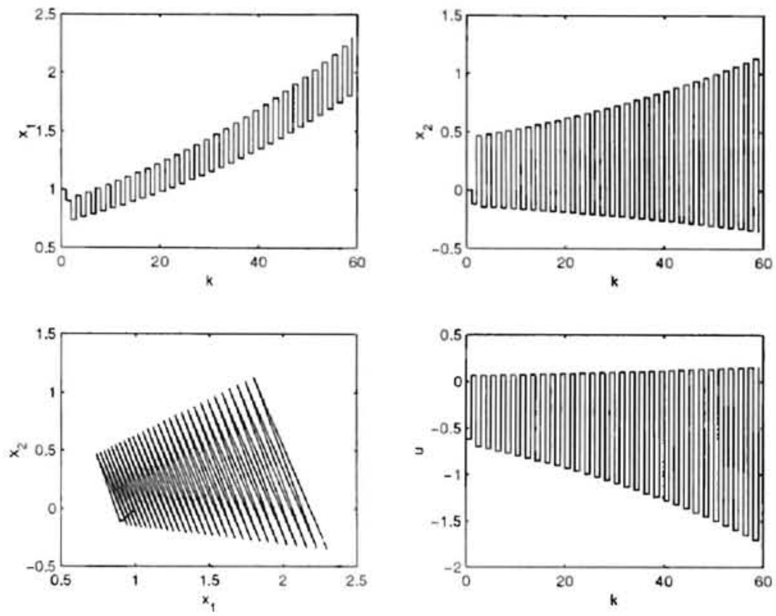

Figure 5.2: Phenomenon of "unstable switchback" in Example 5.1

switching sector is considerably more involved when the state switches between two regions. If the system has multiple-inputs, the problem might become more complicated. In the next section, a discrete variable structure control law with switching sector is designed as an attempt to solve the problems mentioned above.

5.2 Robust Stabilization with Switching Sector

Consider the following discrete-time linear multivariable system which may be obtained by discretizing its continuous-time equivalent with sampling period Δt:

$$x(k+1) = (A + \Delta A(k))x(k) + Bu(k) \tag{5.1}$$

where $x = [x_1\ x_2\ \cdots\ x_n]^T \in \mathbf{R}^n$ is the state vector, $u = [u_1\ u_2\ \cdots\ u_m]^T \in \mathbf{R}^m$ is the input vector, A and B are perfectly known constant matrices with appropriate dimensions, and $\Delta A(k)$ is the time-varying parametric uncertainty matrix satisfying the matching condition $\text{rank}([B\ \Delta A(k)]) = \text{rank}(B)$. The objective is to find a suitable control input $u(k)$ so that $x(k)$ will go asymptotically to zero in the presence of unknown $\Delta A(k)$.

Assumption 5.1. The system (5.1) satisfies the following: $\text{rank}(B) = m$, the pair (A, B) is controllable, and $\Delta A(k)$ is bounded.

Lemma 5.1. *Let Assumption 5.1 holds. Let $n_1, n_2, \ldots, n_m$ be the Kronecker invariant of the system (5.1) in which $\sum_{i=1}^{m} n_i = n$ and define $h_1, h_2, \ldots, h_m$ as $h_j = \sum_{i=1}^{j} n_i$. Then there exists a nonsingular matrix $T \in \mathbf{R}^{n\times n}$ such that with a change in coordinates $x = T\bar{x}$, the system (5.1) can be transformed into the controllable canonical form*

$$\bar{x}(k+1) = (\bar{A} + \Delta\bar{A}(k))\bar{x}(k) + \bar{B}u(k) \tag{5.2}$$

where

$$\bar{A} = T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \cdots & \bar{A}_{1m} \\ \bar{A}_{21} & \bar{A}_{22} & \cdots & \bar{A}_{2m} \\ \vdots & \vdots & & \vdots \\ \bar{A}_{m1} & \bar{A}_{m2} & \cdots & \bar{A}_{mm} \end{bmatrix}, \ \Delta\bar{A}(k) = T^{-1}\Delta A(k)T, \ \bar{B} = T^{-1}B = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \vdots \\ \bar{B}_m \end{bmatrix},$$

$$\bar{A}_{ij} = \begin{bmatrix} 0 & \delta_{ij} & 0 & \ldots & 0 \\ 0 & 0 & \delta_{ij} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \delta_{ij} \\ a_{i(h_j-n_j+1)} & a_{i(h_j-n_j+2)} & \cdots & \cdots & a_{ih_j} \end{bmatrix} \in \mathbf{R}^{n_i\times n_j},$$

δ_{ij} is the Kronecker delta function, and

$$\bar{B}_i = \begin{bmatrix} 0 & \cdots & 0 & \ldots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \ldots & 0 & \cdots & 0 \\ \underbrace{0 \ \cdots \ 0}_{(i-1)\ zeros} & & & 1 & b_{i(i+1)} & \cdots & b_{im} \end{bmatrix} \in \mathbf{R}^{n_i\times m}$$

Moreover, there exists a nonsingular matrix $R \in \mathbf{R}^{m\times m}$ given by

$$R = \begin{bmatrix} 1 & b_{12} & b_{13} & \ldots & b_{1m} \\ 0 & 1 & b_{23} & \ldots & b_{2m} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & b_{(m-1)m} \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}^{-1}$$

such that

$$\bar{B}R = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_m \end{bmatrix} , \quad P_i = \begin{bmatrix} 0 & \cdots & 0 & \dots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \dots & 0 & \cdots & 0 \\ \underbrace{0 \; \cdots \; 0}_{(i-1)\ zeros} & & & 1 & \underbrace{0 \; \cdots \; 0}_{(m-i)\ zeros} & & \end{bmatrix} \in \mathbf{R}^{n_i \times m}$$

Proof. This is a standard result. See, e.g., Ogata [29] pg. 704. □

Definition 5.1. Let $\hat{A} := [a_{ij}] \in \mathbf{R}^{m\times n}$, $\Delta\hat{A}(k) := [\Delta a_{ij}(k)] \in \mathbf{R}^{m\times n}$, $F := [f_{ij}] \in \mathbf{R}^{m\times n}$, $K := [K_{ij}] \in \mathbf{R}^{m\times n}$, and $\Delta\hat{A}(k)$ be related to $\Delta\bar{A}(k)$ by

$$\Delta\bar{A}(k) = \bar{B}R\Delta\hat{A}(k) \tag{5.3}$$

Definition 5.2. Let the switching sector corresponding to the ith input, $\mathcal{W}_i \subset \mathbf{R}^n$, be defined as

$$\mathcal{W}_i = \left\{\bar{x} : |\bar{x}_{h_i}| \leq \textstyle\sum_{j=1}^{n} K_{ij}|\bar{x}_j|\right\}$$

and the overall switching sector $\mathcal{W} \subset \mathbf{R}^n$ be defined as $\mathcal{W} = \bigcap_{i=1}^{m} \mathcal{W}_i$.

Theorem 5.1. *Consider the system described by (5.1) and let Assumption 5.1 holds. If the control law is chosen as*

$$u(k) = -RF\bar{x}(k) + Ru_d(k) \; , \quad u_d = [u_{d_1} \; u_{d_2} \; \cdots \; u_{d_m}]^T \in \mathbf{R}^m \tag{5.4}$$

$$u_{d_i}(k) = \begin{cases} \bar{x}_{h_i}(k) - \mathrm{sgn}(\bar{x}_{h_i}(k)) \sum_{j=1}^{n} K_{ij}|\bar{x}_j(k)| & , \; |\bar{x}_{h_i}(k)| > \sum_{j=1}^{n} K_{ij}|\bar{x}_j(k)| \\ 0 & , \; |\bar{x}_{h_i}(k)| \leq \sum_{j=1}^{n} K_{ij}|\bar{x}_j(k)| \end{cases} \tag{5.5}$$

and if $\Delta\hat{A}(k)$, F, and K satisfy

$$|a_{ij} + \Delta a_{ij}(k) - f_{ij}| < K_{ij} \; , \quad 1 \leq i \leq m \; , \quad 1 \leq j \leq n \tag{5.6}$$

$$\|K\|_\infty \leq 1 \tag{5.7}$$

then the following hold:

1. *For $1 \leq i \leq m$, for all $\bar{x}(k) \notin \mathcal{W}_i$ there exists a $k_i > k$ such that $\bar{x}(k_i) \in \mathcal{W}_i$.*

2. *For $1 \leq i \leq m$, for all $\bar{x}(k) \in \mathcal{W}_i$, $|\bar{x}_{h_i}(k+1)| < \|\bar{x}(k)\|_\infty$.*

3. $\|\bar{x}(k+1)\|_\infty \leq \|\bar{x}(k)\|_\infty$.

4. The system (5.2) is globally uniformly asymptotically stable, i.e. $\bar{x}(k) \to 0$ *as* $k \to \infty$.

Proof. (*1.*) Substituting Eqs. (5.3) and (5.4) into Eq. (5.2) gives

$$\bar{x}(k+1) = (\bar{A} + \bar{B}R\Delta\hat{A}(k) - \bar{B}RF)\bar{x}(k) + \bar{B}Ru_d(k) \tag{5.8}$$

It follows from Lemma 5.1 that the system (5.8) can be expressed as

$$\begin{aligned}
\bar{x}_{h_i-n_i+1}(k+1) &= \bar{x}_{h_i-n_i+2}(k) \\
\bar{x}_{h_i-n_i+2}(k+1) &= \bar{x}_{h_i-n_i+3}(k) \\
&\vdots \\
\bar{x}_{h_i}(k+1) &= u_{d_i}(k) + \sum_{j=1}^{n}(a_{ij} + \Delta a_{ij}(k) - f_{ij})\bar{x}_j(k)
\end{aligned} \tag{5.9}$$

for $1 \leq i \leq m$. Consider the Lyapunov function candidates $V_i(k) = \bar{x}^2_{h_i}(k)$ for $1 \leq i \leq m$. $|\bar{x}_{h_i}(k)|$ decreases monotonically if the following inequality holds:

$$V_i(k+1) < V_i(k) \Rightarrow [\Delta\bar{x}_{h_i}(k) + 2\bar{x}_{h_i}(k)]\Delta\bar{x}_{h_i}(k) < 0\,, \quad \forall \bar{x}_{h_i}(k) \neq 0 \tag{5.10}$$

where $\Delta\bar{x}_{h_i}(k) = \bar{x}_{h_i}(k+1) - \bar{x}_{h_i}(k)$. For $\bar{x}_{h_i}(k) > 0$, using Eqs. (5.5) and (5.9), $\Delta\bar{x}_{h_i}(k)$ can be written as

$$\Delta\bar{x}_{h_i}(k) = -\sum_{j=1}^{n} K_{ij}|\bar{x}_j(k)| + \sum_{i=1}^{m}(a_{ij} + \Delta a_{ij}(k) - f_{ij})\bar{x}_j(k)$$

It can be seen from inequality (5.6) that $\Delta\bar{x}_{h_i}(k)$ is bounded by

$$-2\sum_{j=1}^{n} K_{ij}|\bar{x}_j(k)| < \Delta\bar{x}_{h_i}(k) < 0$$

This implies that inequality (5.10) is satisfied for

$$2\bar{x}_{h_i}(k) > -\Delta\bar{x}_{h_i}(k) \Rightarrow \bar{x}_{h_i}(k) > \sum_{j=1}^{n} K_{ij}|\bar{x}_j(k)|$$

Similarly, for $\bar{x}_{h_i}(k) < 0$, it can be shown that inequality (5.10) is satisfied for

$$2\bar{x}_{h_i}(k) < -\Delta\bar{x}_{h_i}(k) \Rightarrow \bar{x}_{h_i}(k) < -\sum_{j=1}^{n} K_{ij}|\bar{x}_j(k)|$$

Thus, for all $\bar{x}(k) \notin \mathcal{W}_i$, the inequality

$$|\bar{x}_{h_i}(k+1)| < |\bar{x}_{h_i}(k)| \tag{5.11}$$

is satisfied until $\bar{x}(k_i) \in \mathcal{W}_i$ for some $k_i > k$.

(*2.*) Observe that for all $\bar{x}(k) \in \mathcal{W}_i$, one has $u_{d_i}(k) = 0$ from Eq. (5.5). Thus, Eq. (5.9) reduces to

$$\bar{x}_{h_i}(k+1) = \sum_{j=1}^{n}(a_{ij} + \Delta a_{ij}(k) - f_{ij})\bar{x}_j(k)$$

It then follows from inequalities (5.6) and (5.7) that $\bar{x}_{h_i}(k+1)$ is bounded by

$$\begin{aligned} |\bar{x}_{h_i}(k+1)| &\le \sum_{j=1}^{n}|(a_{ij} + \Delta a_{ij}(k) - f_{ij})\bar{x}_j(k)| \le \sum_{j=1}^{n}|a_{ij} + \Delta a_{ij}(k) - f_{ij}||\bar{x}_j(k)| \\ &< \sum_{j=1}^{n} K_{ij}|\bar{x}_j(k)| \le \|K\|_\infty \|\bar{x}(k)\|_\infty \le \|\bar{x}(k)\|_\infty \end{aligned} \tag{5.12}$$

(*3.*) Without loss of generality, suppose $\bar{x}(k) \notin \mathcal{W}_i$ for $1 \le i \le m'$, and $\bar{x}(k) \in \mathcal{W}_i$ for $m'+1 \le i \le m$ where $0 \le m' \le m$. Since $|\bar{x}_{h_i}(k+1)| < |\bar{x}_{h_i}(k)|$ for $1 \le i \le m'$ and $|\bar{x}_{h_i}(k+1)| < \|\bar{x}(k)\|_\infty$ for $m'+1 \le i \le m$ in view of (5.11) and (5.12), it is apparent that the nonstrict inequality $\|\bar{x}(k+1)\|_\infty \le \|\bar{x}(k)\|_\infty$ holds.

(*4.*) Let $n_{\max} = \max(n_1, n_2, \ldots, n_m)$. It follows from (*3.*) that

$$\|[\bar{x}_{h_1}(k+1)\ \bar{x}_{h_2}(k+1)\ \cdots\ \bar{x}_{h_m}(k+1)]^T\|_\infty < \|\bar{x}(k)\|_\infty$$

This relation also holds for

$$\|[\bar{x}_{h_1}(k+k')\ \bar{x}_{h_2}(k+k')\ \cdots\ \bar{x}_{h_m}(k+k')]^T\|_\infty < \|\bar{x}(k)\|_\infty$$

where $k' > 1$. In addition, it can be seen from Eq. (5.9) that $\bar{x}_i(k+1) = \bar{x}_{i+1}(k)$ for $i \notin \{h_1, h_2, \ldots, h_m\}$. Thus, after at most $n_{\max}$ sampling periods, the element of $\bar{x}(k)$ which contributes to the value of $\|\bar{x}(k)\|_\infty$ must have left the stack. This implies that the strict inequality

$$\|\bar{x}(k+n_{\max})\|_\infty < \|\bar{x}(k)\|_\infty$$

holds whenever $\bar{x}(k) \ne 0$ which in turn implies that $\bar{x}(k) \to 0$ as $k \to \infty$.

This completes the proof. □

Corollary 5.1. *Suppose $\bar{x}(k) \in \mathcal{W}_i$. Then it is possible that $\bar{x}(k+1) \notin \mathcal{W}_i$.*

Proof. It is sufficient to consider the case of $n = 2$ and $m = 1$. Let $\bar{A} = \left[\begin{smallmatrix} 0 & 1 \\ 0.1 & 0.1 \end{smallmatrix}\right]$, $\bar{B} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, and $\Delta\hat{A} = [0\ 0]$. Thus, $n_1 = h_1 = n = 2$ and $R = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Choose $F = [0\ 0]$ and $K = [0.2\ 0.2]$ so that inequalities (5.6) and (5.7) in Theorem 5.1 are satisfied. Suppose $\bar{x}(k) = \left[\begin{smallmatrix} 10 \\ 1 \end{smallmatrix}\right]$. One has $\bar{x}(k) \in \mathcal{W}_1$ since $|\bar{x}_{h_1}(k)| = 1 < \sum_{j=1}^{2} K_{1j}|\bar{x}_j(k)| = 2.2$. Thus, $u_{d_1}(k) = 0$ and $\bar{x}(k+1) = \left[\begin{smallmatrix} 1 \\ 1.1 \end{smallmatrix}\right]$. Since $|\bar{x}_{h_1}(k+1)| = 1.1 > \sum_{j=1}^{2} K_{1j}|\bar{x}_j(k+1)| = 0.42$, one has $\bar{x}(k+1) \notin \mathcal{W}_1$. □

It is shown in the above corollary that $\mathcal{W}_i$ is not an invariant set. Nevertheless, it is proven in Theorem 5.1 that the phenomenons of limit cycles and "unstable switchback" cannot occur—the resulting system is globally uniformly asymptotically stable. The implication of this result is that the switching sector $\mathcal{W}_i$ should be attractive but does not have to be invariant as long as additional stability requirements can be imposed. The stability requirement used here when deriving the control law is to make $\|\bar{x}\|_\infty$ decrease, although this may not be necessary.

Since the control law (5.4) has a linear state feedback term $-RF\bar{x}(k)$, it is of interest to compare the admissible bounds on the uncertainties with linear control. In fact, if $\bar{x}(k) \in \mathcal{W} = \bigcap_{i=1}^{m} \mathcal{W}_i$, the system is essentially linear with closed-loop system matrix $\bar{A} - \bar{B}RF$ because all the $u_{d_i}(k)$'s are identically zero. The following lemma is useful for upcoming discussion.

Lemma 5.2. *Let X be a block partitioned matrix with*

$$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1m} \\ X_{21} & X_{22} & \cdots & X_{2m} \\ \vdots & \vdots & & \vdots \\ X_{m1} & X_{m2} & \cdots & X_{mm} \end{bmatrix}$$

and let each X_{ij} be an appropriately dimensioned matrix. Then for any induced matrix *p-norm*

$$\|X\|_p \leq \left\| \begin{bmatrix} \|X_{11}\|_p & \|X_{12}\|_p & \cdots & \|X_{1m}\|_p \\ \|X_{21}\|_p & \|X_{22}\|_p & \cdots & \|X_{2m}\|_p \\ \vdots & \vdots & & \vdots \\ \|X_{m1}\|_p & \|X_{m2}\|_p & \cdots & \|X_{mm}\|_p \end{bmatrix} \right\|$$

Proof. This is a standard result. See, e.g., Zhou *et al.* [53] pg. 30. □

It can be seen from Eq. (5.9) that if $F = \hat{A}$, then the nominal system undergoes a deadbeat response for $\bar{x}(k) \in \mathcal{W}$ and inequalities (5.6) and (5.7) reduce to

$$\|\Delta\hat{A}\|_\infty < 1 \tag{5.13}$$

which is the maximum robustness this control law can achieve based on inequalities (5.6) and (5.7). Since system (5.2) is in controllable canonical form, the row sums of the closed-loop system matrix is always one except possibly for rows $h_1, h_2, \ldots, h_n$. It then follows from Lemma 5.2 that inequality (5.13) is also the maximum tolerable perturbations if a linear control with all poles assigned at the origin were used, i.e. let $F = \hat{A}$ and remove the discontinuous $u_d(k)$ from the control law (5.4).

Furthermore, it is known that the condition (5.13) is sufficient but not necessary because the spectral radius of any matrix is less than or equal to any of its induced-norm (Zhou *et al.* [53] pg. 30). This suggests that the admissible bounds on uncertainties is quite conservative.

Hence, one might wonder what is the need of having a variable structure control law with switching sector as opposed to a linear one. As is discussed earlier in Section 5.1, the basic motivation of the this work is to show that stability of discrete variable structure systems with switching sector that is not an invariant set can be achieved but requires more careful analysis. Additionally, it lays foundation for future research because from a variable structure point of view, switching region can take on other shapes and not necessarily having the shape of a sector.

Chapter 6

Optimal Sliding Surface Design for Single-Input Systems

Section 2.3 has reviewed several methodologies available in the design of sliding surface. To create more sophisticated tools, this chapter investigates the use of the LQR technique in sliding surface design for the version of discrete variable structure control proposed by Misawa [28]. Use of the LQR technique in sliding sector design has been proposed by Pan and Furuta [31] along with their version of discrete variable structure control. It is remarked in the paper that if the optimal closed-loop eigenvalues are strictly complex, solution for the optimal sliding sector cannot be obtained and it is necessary to reselect the weighting matrix—a consequence of the *eigenvalue constraint* discussed in Section 3.2. However, the existence of weighting matrix under a prespecified real eigenvalue, which is called the *inverse optimal problem* in this chapter, is not addressed in the paper. This chapter first solves the *inverse optimal problem* constructively by showing that a feasible weighting matrix always exists for almost any prespecified real eigenvalue. It is then shown that finding a feasible weighting matrix closest to the desired one is a constrained optimization problem that can be solved using the least squares-convex programming approach. This chapter ends by giving an automated design procedure.

6.1 LQR Technique

It is shown in Misawa [28], Richter *et al.* [33], and Remark 3.1 that for single-input systems, the tracking error dynamics inside the boundary layer is governed by the equivalent matrix

$$A_{eq} = A - \frac{BG}{GB}(A - \alpha I) \tag{6.1}$$

where $\alpha = 1-(K_\Sigma/\phi)$ is one of the real, stable eigenvalues of A_{eq} according to Theorem 3.2. It follows from Eq. (6.1) that the tracking error dynamics can be represented as

$$\tilde{x}(k+1) = A\tilde{x}(k) + B\tilde{u}(k) \tag{6.2}$$

where $\tilde{u}(k) = -\frac{G}{GB}(A - \alpha I)\,\tilde{x}(k)$. The LQR performance index for the system (6.2) is given by

$$J = \sum_{k=k_s}^{\infty} \tilde{x}^T(k)Q\tilde{x}(k) + R\tilde{u}^2(k)$$

where k_s is the instant at which the error trajectory enters the boundary layer and $Q = Q^T \geq 0$, $R > 0$ are the weighting terms. It is known that the static state feedback control law $\tilde{u}(k) = -F\tilde{x}(k)$ minimizes J if the feedback gain matrix is chosen as

$$F = \left[R + B^T PB\right]^{-1} B^T PA$$

where $P = P^T > 0$ is the stabilizing solution to the discrete-time algebraic Ricatti equation

$$A^T PA - P - A^T PB\left[R + B^T PB\right]^{-1} B^T PA + Q = 0$$

The symmetric root locus can also be plotted with respect to varying scalar R using the LQR characteristic equation

$$1 + \frac{1}{R}G^T(z^{-1})G(z) = 0 \tag{6.3}$$

where $G(z) = C(zI - A)^{-1}B$ and C is the *fictitious* output depending on Q, i.e. $Q = C^T C$.

There are two possible alternatives in satisfying the *eigenvalue constraint* and making system (6.2) behave like an LQR regulator:

1. Specify a desired Q and plot the symmetric root locus using Eq. (6.3). The set of possible α is given by the segments of root loci that lie on the real axis inside the unit

circle. Difficulty arises since the existence of such segments is not known beforehand. If they exist (always the case if the system is of odd order), α may be selected in accordance with R. Otherwise the process is repeated. Since α depends on factors such as K_Σ, ϕ, and γ, it is usually fixed *a priori* and has less degree of freedom than Q. The desired Q should not only give such segments, but should include that α as well. Thus, a substantial amount of guess work is generally needed which make this approach infavorable.

2. Specify a desired α depending on K_Σ, ϕ, and γ and then constrain the weighting terms Q and R. Even though this approach does not have the disadvantages associated with Alternative 1, its feasibility should be further explored. The theoretical aspects of this approach—the treatment of the so-called *inverse optimal problem*—is presented next.

6.2 The Inverse Optimal Problem

Alternative 2 allows α to be arbitrarily specified regardless of the system (6.2). As far as optimality is concerned, the following question should be answered: *Given the system (6.2), is a particular choice of α optimal with respect to the LQR technique; that is, can a $Q \geq 0$ and a $0 < R < \infty$ be found such that they give real closed-loop eigenvalue(s) at α?* This is referred to as the *inverse optimal problem*. Two issues motivate the seek for a solution to this problem:

- Some classes of systems may not possess any real, optimal closed-loop pole.
- Optimal closed-loop poles may not be placed on some real segments inside the unit circle for a particular system.

Thus, an insightful solution is required so that a reliable design procedure capable of avoiding these pitfalls can be developed. To begin with, consider the following definition.

Definition 6.1. Given the system (6.2), $\alpha \in (-1, 1)$ is *optimal with respect to the LQR technique* if there exist $Q = C^T C \geq 0$ and $0 < R < \infty$ such that the LQR characteristic equation

$$1+\frac{1}{R}B^T\left[z^{-1}I-A^T\right]^{-1}Q\left[zI-A\right]^{-1}B=0 \tag{6.4}$$

gives root(s) at $z=\alpha$. Otherwise, α is *not* optimal with respect to the LQR technique.

Assumption 6.1. The pair (A,B) is controllable.

For convenience, let $\varphi(\alpha)\in\mathbf{R}^n$ be defined as

$$\varphi(\alpha)=\left[\alpha I-A\right]^{-1}B$$

The LQR characteristic equation (6.4) can then be written as

$$\varphi^T(\alpha^{-1})Q\varphi(\alpha)=-R \tag{6.5}$$

where α replaces z in the argument.

Lemma 6.1. *Suppose $n\geq 2$, $\alpha\notin\{\lambda_i\}$, $\alpha^{-1}\notin\{\lambda_i\}$, and $\alpha\neq 0$ where $\{\lambda_i, i=1,\ldots,n\}$ denotes the eigenvalues of A. Then, $\varphi(\alpha)$ and $\varphi(\alpha^{-1})$ are nonzero column vectors and can never be collinear.*

Proof. Since both $\left[\alpha I-A\right]^{-1}$ and $\left[\alpha^{-1}I-A\right]^{-1}$ are nonsingular and a necessary condition for controllability is that B is nonzero, it is obvious that both $\varphi(\alpha)$ and $\varphi(\alpha^{-1})$ are nonzero. Next, let $X(\alpha)=\left[\alpha I-A\right]\left[\alpha^{-1}I-A\right]$ be the matrix representing the linear transformation $\mathcal{T}:\mathbf{R}^n\to\mathbf{R}^n$. The image vectors $X(\alpha)\varphi(\alpha)$ and $X(\alpha)\varphi(\alpha^{-1})$ are found to be

$$\begin{aligned}
X(\alpha)\varphi(\alpha) &= \left[\alpha I-A\right]\Big[\alpha^{-1}\left[\alpha I-A\right]^{-1}-A\left[\alpha I-A\right]^{-1} \\
&\qquad +\alpha\left[\alpha I-A\right]^{-1}-\alpha\left[\alpha I-A\right]^{-1}\Big]B \\
&= \left[\alpha I-A\right]\Big[(\alpha^{-1}-\alpha)\left[\alpha I-A\right]^{-1}+I\Big]B \\
&= \alpha^{-1}B-AB \qquad (6.6) \\
X(\alpha)\varphi(\alpha^{-1}) &= \alpha B-AB \qquad (6.7)
\end{aligned}$$

Since $n\geq 2$ and a necessary condition for controllability is that B and AB are linearly independent vectors, it is obvious from Eqs. (6.6) and (6.7) that $X(\alpha)\varphi(\alpha)$ and $X(\alpha)\varphi(\alpha^{-1})$ are collinear if and only if $\alpha=\pm 1$. This cannot be achieved because $\alpha\in(-1,1)$ by Definition 6.1. Since the mapping $\mathcal{T}$ is one-to-one and onto, this in turn implies that $\varphi(\alpha)$ and $\varphi(\alpha^{-1})$ can never be collinear. □

Lemma 6.2. *Suppose $n \geq 2$ and let α or α^{-1} be an eigenvalue of A. Then α is not optimal with respect to the LQR technique.*

Proof. The LQR characteristic equation (6.5) can be expressed as

$$\varphi^T(\alpha^{-1})Q\varphi(\alpha) = \frac{B^T \operatorname{adj}(\alpha^{-1}I - A^T)\, Q \operatorname{adj}(\alpha I - A)\, B}{\det(\alpha^{-1}I - A^T)\det(\alpha I - A)} = -R$$

If A does not have any real eigenvalue, there is nothing to prove. So, let $\lambda \in \mathbf{R}$ be the eigenvalue of A that α or α^{-1} is approaching. Since either one of the two determinants above must vanish as $\alpha \to \lambda$ or $\alpha^{-1} \to \lambda$, i.e. $\lim_{\alpha \to \lambda} \varphi^T(\alpha^{-1})Q\varphi(\alpha) = \infty$ or $\lim_{\alpha^{-1} \to \lambda} \varphi^T(\alpha^{-1})Q\varphi(\alpha) = \infty$, it is immediate that $R \to \pm\infty$ violates the condition in Definition 6.1. This concludes the proof. □

Lemma 6.3. *Suppose $n \geq 2$ and let $\alpha = 0$. Then α is not optimal with respect to the LQR technique.*

Proof. The LQR characteristic equation (6.5) can be expressed as

$$\frac{b_p\alpha^p + b_{p-1}\alpha^{p-1} + b_{p-2}\alpha^{p-2} + \cdots + b_{p-2}\alpha^{-p+2} + b_{p-1}\alpha^{-p+1} + b_p\alpha^{-p}}{\alpha^n + a_{n-1}\alpha^{n-1} + a_{n-2}\alpha^{n-2} + \cdots + a_{n-2}\alpha^{-n+2} + a_{n-1}\alpha^{-n+1} + \alpha^{-n}} = -R$$

where $n > p$ since $G(z)$ defined in Eq. (6.3) is strictly proper. Multiply both numerator and denominator by α^n and take the limit as $\alpha \to 0$ gives

$$\lim_{\alpha \to 0} \frac{b_p\alpha^{n+p} + b_{p-1}\alpha^{n+p-1} + \cdots + b_{p-1}\alpha^{n-p+1} + b_p\alpha^{n-p}}{\alpha^{2n} + a_{n-1}\alpha^{2n-1} + \cdots + a_{n-1}\alpha + 1} = 0$$

It is immediate that $R \to 0$ as $\alpha \to 0$ violates the condition in Definition 6.1. This concludes the proof. □

Theorem 6.1. *For $n \geq 2$, α is optimal with respect to the LQR technique if and only if $\alpha \notin \{\lambda_i\}$, $\alpha^{-1} \notin \{\lambda_i\}$, and $\alpha \neq 0$ where $\{\lambda_i, i = 1, \ldots, n\}$ denotes the eigenvalues of A.*

Proof. It is known (Curtis [10] pg. 274) that there exists an orthogonal matrix $V = [v_1 \,|\, v_2 \,|\, \cdots \,|\, v_n]$ which orthogonally diagonalizes Q, i.e.

$$V^T QV = \operatorname{diag}(q_1, q_2, \cdots, q_n)$$

where $\{q_i, i = 1, \ldots, n\}$ and $\{v_i, i = 1, \ldots, n\}$ are the sets of real nonnegative (with at least one positive) eigenvalues and corresponding eigenvectors of Q respectively. Thus, the LQR characteristic equation (6.5) may be written as

$$\varphi^T(\alpha^{-1}) \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \varphi(\alpha) = \sum_{i=1}^{n} q_i \varphi^T(\alpha^{-1}) v_i v_i^T \varphi(\alpha) = -R \quad (6.8)$$

Moreover, since $\varphi(\alpha)$ and $\varphi(\alpha^{-1})$ are nonzero and noncollinear vectors in view of Lemma 6.1, it is known that there exists a hyperplane $\{x : y^T x = 0,\ x, y \in \mathbf{R}^n, y \neq 0\}$ separating them, i.e. the inner products $y^T\varphi(\alpha)$ and $y^T\varphi(\alpha^{-1})$ will have opposite sign. So, by choosing $q_1 > 0$, $q_i = 0$ for $2 \leq i \leq n$, and $v_1 = y$, the left-hand side of Eq. (6.8) reduces to

$$q_1 \varphi^T(\alpha^{-1}) y y^T \varphi(\alpha) < 0$$

which corresponds to some $R > 0$ and sufficiency is verified. The necessity then follows directly from Lemmas 6.2 and 6.3. □

Corollary 6.1. *For $n \geq 2$, almost any $\alpha \in (-1, 1)$ is optimal with respect to the LQR technique.*

Proof. Since the system (6.2) is finite-dimensional, Theorem 6.1 implies that α cannot be equal to $n + 1$ isolated points, i.e. the origin and the n closed-loop eigenvalues or their reciprocals inside the unit circle. □

Theorem 6.2. *For $n = 1$, α is optimal with respect to the LQR technique if and only if*

$$\begin{cases} 0 < \alpha < \min(A^{-1}, A)\,, & \forall A > 0 \\ \max(A^{-1}, A) < \alpha < 0\,, & \forall A < 0 \end{cases} \quad (6.9)$$

If $A = 0$, any α is not optimal.

Proof. For $n = 1$, A, B, Q, and R are scalars. The LQR characteristic equation (6.4) becomes

$$1 + A^2 - (\alpha^{-1} + \alpha)A = -\frac{B^2 Q}{R} \quad (6.10)$$

Since Q and R are positive and finite, the right-hand side of Eq. (6.10) is bounded, i.e.

$$-\infty < -\frac{B^2 Q}{R} < 0$$

Clearly this inequality is satisfied if and only if

$$\begin{cases} \infty > \alpha^{-1} + \alpha > A^{-1} + A\,, & \forall A > 0 \\ -\infty < \alpha^{-1} + \alpha < A^{-1} + A\,, & \forall A < 0 \end{cases}$$

which leads to inequality (6.9). Finally, it can be seen from (6.9) that as A tends to zero the region of allowable α vanishes. □

In the case of $n = 1$, there may exist segments and even the whole real axis inside the unit circle for which α is not optimal. However, it is unnecessary to proceed any further because sliding surface for first order system is trivially the origin of the error state space. Hence, all discussions that follow will exclude this case.

In the case of $n \geq 2$, Corollary 6.1 says that the available freedom for α is promising in which only a small finite number of *isolated points* are not allowed. Thus, the two issues brought up earlier in this section are proved to be false. It can therefore be concluded that Alternative 2 is conceivable—one can always specify the more important α since it affects both the sliding gain and boundary layer thickness and then constrain Q and R.

6.3 Least Squares-Convex Programming Approach

Unlike the approach taken by Pan and Furuta [31], which is quite similar to Alternative 1 where the existence of real eigenvalue is not guaranteed, the present approach allows both the desired α and desired weighting matrix Q_d to be specified *simultaneously* while letting $R = 1$. However, in most cases this combination of α, Q_d, and R do not satisfy the LQR characteristic equation (6.4). Since α is usually fixed *a priori*, one way to satisfy Eq. (6.4) is by replacing Q_d with a feasible Q that is "closest" to Q_d, satisfy Eq. (6.4), and is positive semi-definite. This section is intended to solve this constrained optimization problem by the least-squares approach if possible; or otherwise formulate it as a convex optimization problem that is amenable to computer solution.

Without loss of generality, let R be normalized to unity and rewrite Eq. (6.5) as

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_i(\alpha^{-1})\varphi_j(\alpha)Q_{ij} = -1 \tag{6.11}$$

where $\varphi(\alpha^{-1}) = [\varphi_i(\alpha^{-1})]$, $\varphi(\alpha) = [\varphi_i(\alpha)]$, and $Q = [Q_{ij}]$.

Definition 6.2. Given a symmetric matrix $X = [X_{ij}] \in \mathbf{R}^{n\times n}$, let $\mathcal{V} : \mathbf{R}^{n\times n} \to \mathbf{R}^h$ be a linear transformation defined as

$$\mathcal{V}(X) = [X_{11}\cdots X_{n1} \,|\, X_{22}\cdots X_{n2} \,|\, X_{33}\cdots X_{n3} \,|\, \cdots \,|\, X_{nn}]^T$$

where $h = \sum_{i=1}^{n} i$. That is, $\mathcal{V}(X)$ is a column vector formed by stacking up the columns of the lower triangular part of X. Also, let the inverse transformation $\mathcal{V}^{-1} : \mathbf{R}^h \to \mathbf{R}^{n\times n}$ be defined as

$$\mathcal{V}^{-1}(\mathcal{V}(X)) = X$$

Definition 6.3. Given that the kth entry of $\mathcal{V}(Q)$ contains Q_{ij}, let $\beta = [\beta_i] \in \mathbf{R}^h$ be defined as

$$\beta_k = \begin{cases} \varphi_i(\alpha^{-1})\varphi_i(\alpha) & , \quad i = j \\ \varphi_i(\alpha^{-1})\varphi_j(\alpha) + \varphi_j(\alpha^{-1})\varphi_i(\alpha) & , \quad i \neq j \end{cases} \quad , \quad 1 \leq k \leq h$$

By Definitions 6.2 and 6.3, the LQR characteristic equation (6.11) can be written as

$$\beta^T\mathcal{V}(Q) = -1 \tag{6.12}$$

The solution to this underdetermined equation is given by the sum of the nonhomogeneous (minimum norm) and homogeneous solutions, namely

$$\mathcal{V}(Q) = -\beta(\beta^T\beta)^{-1} + N\xi \tag{6.13}$$

where $N \in \mathbf{R}^{h\times(h-1)}$ is an orthonormal basis for the $h-1$ dimensional null space of β^T denoted as $\mathcal{N}(\beta^T)$, and $\xi \in \mathbf{R}^{h-1}$ is arbitrary.

Definition 6.4. Let $\mathcal{P}, \mathcal{Q} \subset \mathbf{R}^h$ be defined as

$$\begin{aligned} \mathcal{P} &= \{x : x \in \mathbf{R}^h, \mathcal{V}^{-1}(x) \geq 0\} \\ \mathcal{Q} &= \{x : x \in \mathbf{R}^h, x = -\beta(\beta^T\beta)^{-1} + N\xi \ , \ \forall\xi \in \mathbf{R}^{h-1}\} \end{aligned}$$

It is clear that a Q is feasible if and only if $\mathcal{V}(Q) \in \mathcal{P} \cap \mathcal{Q}$, i.e. it is positive semi-definite and satisfies the LQR characteristic equation. Furthermore, to maximally recover the desired weighting, the feasible Q should stay as "close" to Q_d as possible. A judicious choice will be to minimize the $\|\mathcal{V}(Q_d - Q)\|$, which is actually the sum of squares of the h lower triangular entries in matrix $Q_d - Q$. Along with these definitions and motivations, the constrained optimization problem can be formally stated as follows.

Problem 6.1. *Given α and Q_d where α is optimal with respect to the LQR technique and $\mathcal{V}(Q_d) \in \mathcal{P}$, find a Q that minimizes $\|\mathcal{V}(Q_d - Q)\|$ subject to $\mathcal{V}(Q) \in \mathcal{P} \cap \mathcal{Q}$.*

It has been shown in Theorem 6.1 there exists a feasible solution to Problem 6.1, i.e. $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$. In fact, the feasible solution can be constructed through the following proposition. Before this is presented, a lemma is introduced which is necessary for the proof.

Lemma 6.4. *Suppose $n \geq 2$ and α is optimal with respect to the LQR technique. Let $M = \varphi(\alpha^{-1})\varphi^T(\alpha) + \varphi(\alpha)\varphi^T(\alpha^{-1})$, $\lambda_M = \varphi^T(\alpha)\varphi(\alpha^{-1}) - \|\varphi(\alpha)\|\,\|\varphi(\alpha^{-1})\|$, and $v_M = \|\varphi(\alpha)\|\,\varphi(\alpha^{-1}) - \|\varphi(\alpha^{-1})\|\,\varphi(\alpha)$. Then, λ_M and v_M are the negative eigenvalue and corresponding eigenvector of M respectively.*

Proof. It suffices to verify that $Mv_M = \lambda_M v_M$ holds:

$$\begin{aligned}
Mv_M &= \varphi(\alpha^{-1})\varphi^T(\alpha)\,\|\varphi(\alpha)\|\,\varphi(\alpha^{-1}) - \varphi(\alpha^{-1})\varphi^T(\alpha)\,\|\varphi(\alpha^{-1})\|\,\varphi(\alpha) \\
&\quad +\varphi(\alpha)\varphi^T(\alpha^{-1})\,\|\varphi(\alpha)\|\,\varphi(\alpha^{-1}) - \varphi(\alpha)\varphi^T(\alpha^{-1})\,\|\varphi(\alpha^{-1})\|\,\varphi(\alpha) \\
&= \varphi^T(\alpha)\varphi(\alpha^{-1})\,\|\varphi(\alpha)\|\,\varphi(\alpha^{-1}) - \|\varphi(\alpha)\|^2\,\|\varphi(\alpha^{-1})\|\,\varphi(\alpha^{-1}) \\
&\quad + \|\varphi(\alpha^{-1})\|^2\,\|\varphi(\alpha)\|\,\varphi(\alpha) - \varphi^T(\alpha)\varphi(\alpha^{-1})\,\|\varphi(\alpha^{-1})\|\,\varphi(\alpha) \\
&= \left[\varphi^T(\alpha)\varphi(\alpha^{-1}) - \|\varphi(\alpha)\|\,\|\varphi(\alpha^{-1})\|\right]\left[\|\varphi(\alpha)\|\,\varphi(\alpha^{-1}) - \|\varphi(\alpha^{-1})\|\,\varphi(\alpha)\right] \\
&= \lambda_M v_M
\end{aligned}$$

Moreover, it is easy to see from Cauchy-Schwarz inequality and Lemma 6.1 that $\lambda_M < 0$. □

Proposition 6.1. *Let $\{q_i, i = 1, \ldots, n\}$ and $\{v_i, i = 1, \ldots, n\}$ be the sets of eigenvalues and corresponding orthonormal eigenvectors of Q_0, respectively. If $q_1 = -2/\lambda_M$, $q_2 = \ldots = q_n = 0$, $v_1 = v_M/\|v_M\|$, then $\mathcal{V}(Q_0) \in \mathcal{P} \cap \mathcal{Q}$, i.e. Q_0 is a feasible solution to Problem 6.1.*

Proof. Since $q_1 > 0$, it is obvious that $\mathcal{V}(Q_0) \in \mathcal{P}$. Furthermore, with $R = 1$, the LQR characteristic equation (6.8) can be rewritten as

$$\sum_{i=1}^{n} q_i v_i^T \left[\varphi(\alpha^{-1})\varphi^T(\alpha) + \varphi(\alpha)\varphi^T(\alpha^{-1})\right] v_i = \sum_{i=1}^{n} q_i v_i^T M v_i = -2$$

If $q_1 = -2/\lambda_M$, $q_2 = \ldots = q_n = 0$, $v_1 = v_M/\|v_M\|$, it follows from Lemma 6.4 that the above equation is satisfied. Thus, $\mathcal{V}(Q_0) \in \mathcal{P} \cap \mathcal{Q}$ and this completes the proof. □

The following theorem characterizes the result from the least-squares approach.

Theorem 6.3. *The unique least-squares solution*

$$\xi = \left[N^T N\right]^{-1} N^T \left(\mathcal{V}(Q_d) + \beta(\beta^T\beta)^{-1}\right) =: \xi_\ell \tag{6.14}$$

which leads to

$$\mathcal{V}(Q) = -\beta(\beta^T\beta)^{-1} + N\left[N^T N\right]^{-1} N^T \left(\mathcal{V}(Q_d) + \beta(\beta^T\beta)^{-1}\right) =: \mathcal{V}(Q_\ell) \tag{6.15}$$

minimizes $\|\mathcal{V}(Q_d - Q)\|$. *The resulting* Q_ℓ *is a solution to Problem 6.1 if and only if* $\mathcal{V}(Q_\ell) \in \mathcal{P}$. *Moreover,*

$$\mathcal{V}(Q_d - Q_\ell) \perp \mathcal{N}(\beta^T)$$

Proof. Using Eq. (6.13), the least-squares problem can be cast into the standard form:

$$\mathcal{V}(Q_d - Q) = \left(\mathcal{V}(Q_d) + \beta(\beta^T\beta)^{-1}\right) - N\xi$$

It is well known (Brogan [2]) that $\|\mathcal{V}(Q_d - Q)\|$ is uniquely minimized by choosing ξ as in Eq. (6.14), which by substitution into Eq. (6.13) leads to Eq. (6.15). Since the least-squares solution guarantees that $\mathcal{V}(Q_\ell) \in \mathcal{Q}$ but not $\mathcal{V}(Q_\ell) \in \mathcal{P}$, Q_ℓ is a solution to Problem 6.1 if and only if $\mathcal{V}(Q_\ell)$ happens to be an element in $\mathcal{P}$ as well. Finally, to show that $\mathcal{V}(Q_d - Q_\ell)$ is orthogonal to the subspace $\mathcal{N}(\beta^T)$, consider the decomposition

$$\mathbf{R}^h = \mathcal{N}(\beta^T) \oplus \mathcal{N}(\beta^T)^\perp$$

where $\mathcal{N}(\beta^T)^\perp$ denotes the orthogonal complement of $\mathcal{N}(\beta^T)$. Suppose $\mathcal{V}(Q_d - Q_\ell)$ is decomposed as

$$\mathcal{V}(Q_d - Q_\ell) = e_1 + e_2\,, \quad e_1 \in \mathcal{N}(\beta^T)\,, \quad e_2 \in \mathcal{N}(\beta^T)^\perp$$

Since $N\xi \in \mathcal{N}(\beta^T)$, the choice of ξ cannot affect e_2. The least-squares solution is the one for which $e_1 = 0$ which implies that $\mathcal{V}(Q_d - Q_\ell) = e_2 \in \mathcal{N}(\beta^T)^\perp$. This completes the proof. □

In the case where the least-squares approach fails, i.e. $\mathcal{V}(Q_\ell) \notin \mathcal{P}$, Problem 6.1 can still be solved via convex programming. In what follows, $\mathcal{V}(Q_\ell) \notin \mathcal{P}$ can be assumed so that the formulation of a convex optimization problem is necessary. To set up the problem, consider the following lemma and propositions.

Lemma 6.5. *Let $\{x_i, i = 1, \dots, n\}$, $\{y_i, i = 1, \dots, n\}$, and $\{z_i, i = 1, \dots, n\}$ be the eigenvalues of symmetric matrices X, Y, and Z, respectively, where all three sets are arranged in non-increasing order. If $Z = X + Y$, then*

$$\max(x_i + y_n, x_n + y_i) \leq z_i \leq \min(x_i + y_1, x_1 + y_i) , \quad 1 \leq i \leq n$$

Proof. This is a standard result. See, e.g., Wilkinson [49] pg. 101. □

Proposition 6.2. *$\mathcal{P} \cap \mathcal{Q}$ is a convex set.*

Proof. It follows from Definition 6.4 and Lemma 6.5 that $\mathcal{P}$ is a convex set since for all X, $Y \in \mathbf{R}^{n\times n}$ with $X, Y \geq 0$, one has $\mathcal{V}(X) \in \mathcal{P}$, $\mathcal{V}(Y) \in \mathcal{P}$, and

$$\mathcal{V}(\lambda X + (1-\lambda)Y) = \lambda\mathcal{V}(X) + (1-\lambda)\mathcal{V}(Y) \in \mathcal{P} \ , \ 0 \leq \lambda \leq 1$$

Furthermore, it is obvious from Definition 6.4 that $\mathcal{Q}$ forms a hyperplane parallel to $\mathcal{N}(\beta^T)$. The fact that every hyperplane in $\mathbf{R}^h$ is a convex set and the intersection of two convex sets is also convex concludes the proof (Cameron [3] pg. 4). □

The drawback associated with solving Problem 6.1 directly via convex programming is that the numerical search is carried out in a h dimensional space with constraint $\mathcal{V}(Q) \in \mathcal{P} \cap \mathcal{Q}$. It is possible, however, to confine the search in a $h-1$ dimensional space with constraint $\mathcal{V}(Q) \in \mathcal{P}$. This is accomplished by reconsidering the result from least-squares approach and the following proposition.

Proposition 6.3. *Let Q_* be obtained from Eq. (6.13) with $\xi = \xi_*$ and $\mathcal{V}(Q_*) \in \mathcal{P} \cap \mathcal{Q}$. Suppose that N is an orthonormal basis for the subspace $\mathcal{N}(\beta^T)$. The following are equivalent:*

1. $\|\xi_\ell - \xi_*\|$ *is a minimum.*

2. $\|\mathcal{V}(Q_\ell - Q_*)\|$ *is a minimum.*

3. $\|\mathcal{V}(Q_d - Q_*)\|$ *is a minimum.*

4. Q_* *is a solution to Problem 6.1.*

Proof. **(*1.*⇔*2.*)** Using Eq. (6.13), it is straightforward to verify that $\|\mathcal{V}(Q_\ell - Q_*)\| = \|\xi_\ell - \xi_*\|$.

(*2.*⇒*3.*) Since $\mathcal{Q}$ forms a hyperplane parallel to $\mathcal{N}(\beta^T)$, it is obvious that for all Q with $\mathcal{V}(Q) \in \mathcal{Q}$, $\mathcal{V}(Q_\ell - Q) \in \mathcal{N}(\beta^T)$. In addition, it follows from Theorem 6.3 that $\mathcal{V}(Q_d - Q_\ell) \perp \mathcal{N}(\beta^T)$. Using the above two facts and the Pythagorean theorem, one has

$$\|\mathcal{V}(Q_d - Q)\|^2 = \|\mathcal{V}(Q_d - Q_\ell)\|^2 + \|\mathcal{V}(Q_\ell - Q)\|^2 \qquad (6.16)$$

If $\|\mathcal{V}(Q_\ell - Q_*)\|$ is a minimum not necessarily unique, i.e.

$$\|\mathcal{V}(Q_\ell - Q_*)\| \leq \|\mathcal{V}(Q_\ell - Q)\|$$

for all Q with $\mathcal{V}(Q) \in \mathcal{P} \cap \mathcal{Q}$, then the following inequality can be obtained from Eq. (6.16):

$$\begin{aligned} \|\mathcal{V}(Q_d - Q)\|^2 &= \|\mathcal{V}(Q_d - Q_\ell)\|^2 + \|\mathcal{V}(Q_\ell - Q)\|^2 \\ &\geq \|\mathcal{V}(Q_d - Q_\ell)\|^2 + \|\mathcal{V}(Q_\ell - Q_*)\|^2 \geq \|\mathcal{V}(Q_d - Q_*)\|^2 \end{aligned}$$

Thus, $\|\mathcal{V}(Q_d - Q_*)\|$ is a minimum as well. Conversely, if $\|\mathcal{V}(Q_\ell - Q_*)\|$ is a not a minimum, i.e. there exists Q_{**} with $\mathcal{V}(Q_{**}) \in \mathcal{P} \cap \mathcal{Q}$ such that

$$\|\mathcal{V}(Q_\ell - Q_{**})\| < \|\mathcal{V}(Q_\ell - Q_*)\| ,$$

then the following inequality can be obtained from Eq. (6.16):

$$\begin{aligned} \|\mathcal{V}(Q_d - Q_{**})\|^2 &= \|\mathcal{V}(Q_d - Q_\ell)\|^2 + \|\mathcal{V}(Q_\ell - Q_{**})\|^2 \\ &< \|\mathcal{V}(Q_d - Q_\ell)\|^2 + \|\mathcal{V}(Q_\ell - Q_*)\|^2 < \|\mathcal{V}(Q_d - Q_*)\|^2 \end{aligned}$$

Thus, $\|\mathcal{V}(Q_d - Q_*)\|$ is not a minimum as well.

(*3.*⇒*2.*) The proof follows the same lines as the one above.

(*3.*⇔*4.*) This follows easily from the problem statement.

□

An immediate consequence of Proposition 6.3 is that $\|\xi_\ell - \xi\|$ can be used as the objective function instead of $\|\mathcal{V}(Q_d - Q)\|$, which yields the advantage of confining the search in a $h-1$ dimensional hyperplane formed by the elements of $\mathcal{Q}$. This also implies that the constraint $\mathcal{V}(Q) \in \mathcal{Q}$ is satisfied naturally. The modified problem can be restated as follows.

Problem 6.2. *Given α and Q_d where α is optimal with respect to the LQR technique and $\mathcal{V}(Q_d) \in \mathcal{P}$, find a ξ that minimizes $\|\xi_\ell - \xi\|$ subject to $\mathcal{V}(Q) \in \mathcal{P}$.*

The objective function and the constraint in Problem 6.2 are convex in view of triangular inequality and Proposition 6.2. It is well known that this convex programming problem has a global solution point. A rich collection of algorithms are available for this problem, e.g. the ellipsoid algorithm (Boyd *et al.* [1]), the interior-point methods (Boyd *et al.* [1]), and the sequential quadratic programming methods [17]. The initial guess for these algorithms may be obtained from Proposition 6.1.

Remark 6.1. The constraint in Problem 6.2 can be cast into the form of linear matrix inequality (LMI). To show this, let $\xi = [\xi_i]$ and $N = [\eta_1 \,|\, \eta_2 \,|\, \cdots \,|\, \eta_{h-1}]$ where the η_i's form an orthonormal basis set in $\mathcal{N}(\beta^T)$. Then, applying $\mathcal{V}^{-1}$ to Eq. (6.13) and writing $Q = Q(\xi)$ yields

$$Q(\xi) = -(\beta^T\beta)^{-1}\mathcal{V}^{-1}(\beta) + \sum_{i=1}^{h-1} \xi_i \mathcal{V}^{-1}(\eta_i)$$

with constraint $Q(\xi) \geq 0$, which is exactly a nonstrict LMI.

6.4 Design Procedure

The proposed LQR design procedure consists of the following steps:

1. **Compute α.** α is computed from prespecified K_Σ, ϕ, and γ using Eqs. (3.7), (3.8), and (3.9). The value of α should not violate Theorem 6.1.

2. **Specify Q_d and set $R = 1$.** The desired weighting matrix Q_d should be symmetric and positive semi-definite.

3. **Run MATLAB file `dvsclqr1.m` with input data A, B, Q_d, and α.** The following tasks are performed:

   (a) Compute a symmetric positive semi-definite matrix Q that will give root at α and is "closest" to Q_d in the least-squares sense using the least squares-convex programming approach.

   (b) Run `dlqr.m` to solve for the optimal feedback gain matrix F.

   (c) Compute G using Eq. (3.25).

   (d) Make sure $GB \neq 0$ and all of the n entries of G are nonzero. A warning message will be printed if any of which is not satisfied.

   (e) Return Q, F, and G as output data.

4. **(Optional) Fine-Tune Q and/or α.** The following routes on fine-tuning are suggested:

   (a) Manually adjust Q and/or α using Eq. (6.12). Whether the modified Q and/or modified α is positive semi-definite and/or satisfy Theorem 6.1 should be checked, respectively.

   (b) Change Q and/or α accordingly via the symmetric root locus using `dsrlocus.m`.

Remark 6.2. In addition to fine-tuning purposes, the manual adjustment approach is suitable when the system order is low and when Q is to be diagonal, in which case the positive semi-definiteness of Q can be easily observed.

Remark 6.3. The MATLAB files used to carry out the above operations, i.e. `dvsclqr1.m`, `dvsclqr2.m`, `dsrlocus.m`, and `vecsym.m` can be found in Appendix 8.

Chapter 7

Application Examples

7.1 Control of a Mechanical System

7.1.1 Additive Uncertainties: The State Feedback Case

Example 7.1. Consider the mechanical system shown in Figure 7.1. Let $m = 1$, $c = 2$, $b = 3$, state vector $x = [q_1\ \dot{q}_1\ q_2\ \dot{q}_2\ q_3\ \dot{q}_3]^T$, input vector $u = [u_1\ u_2]^T$, and additive uncertainty or disturbance vector $f = [f_1\ f_2\ f_3]^T$. The continuous-time state space representation is given by

$$\dot{x}(t) = A_c x(t) + B_c u(t) + D_c f(t) \tag{7.1}$$

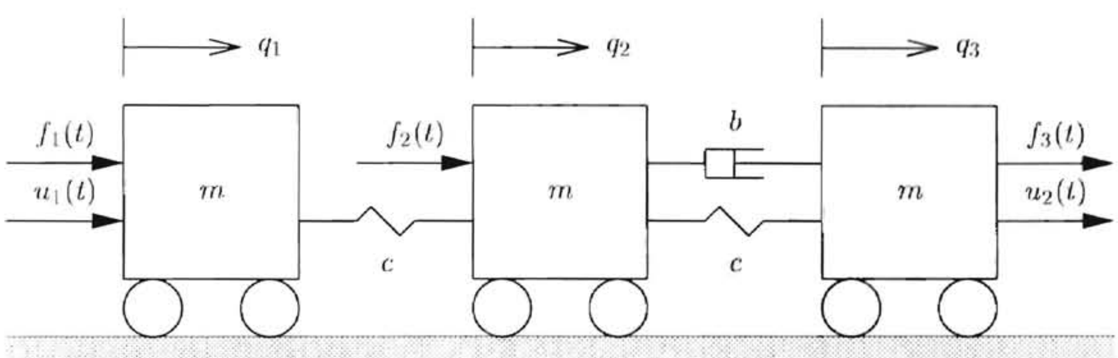

Figure 7.1: The mechanical system

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -4 & -3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 3 & -2 & -3 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad D_c = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The discrete-time equivalent of the system (7.1), obtained by applying u through a zero-order hold with $\Delta t = 0.2$, is given by Eq. (3.1) where $A = e^{A_c \Delta t}$, $B = \int_0^{\Delta t} e^{A_c \lambda} d\lambda B_c$, $D_o = \int_0^{\Delta t} e^{A_c \lambda} d\lambda D_c$, i.e.

$$A = \begin{bmatrix} 0.9605 & 0.1974 & 0.0393 & 0.0023 & 0.0002 & 0.0003 \\ -0.3901 & 0.9605 & 0.3861 & 0.0333 & 0.0040 & 0.0062 \\ 0.0333 & 0.0023 & 0.9394 & 0.1547 & 0.0273 & 0.0430 \\ 0.3048 & 0.0333 & -0.5282 & 0.6042 & 0.2234 & 0.3625 \\ 0.0062 & 0.0003 & 0.0213 & 0.0430 & 0.9725 & 0.1567 \\ 0.0853 & 0.0062 & 0.1421 & 0.3625 & -0.2274 & 0.6313 \end{bmatrix}, \tag{7.2}$$

$$B = \begin{bmatrix} 0.0199 & 0.0000 \\ 0.1974 & 0.0003 \\ 0.0001 & 0.0031 \\ 0.0023 & 0.0430 \\ 0.0000 & 0.0169 \\ 0.0003 & 0.1567 \end{bmatrix}, \quad D_o = \begin{bmatrix} 0.0199 & 0.0001 & 0.0000 \\ 0.1974 & 0.0023 & 0.0003 \\ 0.0001 & 0.0168 & 0.0031 \\ 0.0023 & 0.1547 & 0.0430 \\ 0.0000 & 0.0031 & 0.0169 \\ 0.0003 & 0.0430 & 0.1567 \end{bmatrix};$$

and $w_o = [w_{o_1} \; w_{o_2} \; w_{o_3}]^T$ is assumed to satisfy $D_o w_o(k) = \int_0^{\Delta t} e^{A_c \lambda} D_c f((k+1)\Delta t - \lambda) d\lambda$ for all k. Suppose $w_o(k) = [0 \;\; \sin(3.5\pi k \Delta t) \;\; 0]^T$, i.e. the disturbance is a scalar and is unmatched.

The control objective is to make the tracking error small with q_2 being emphasized the most. Consider using the LQR technique to design the equivalent matrix A_{eq}, where the weighting matrices are chosen as $Q = \text{diag}(10, 0, 1000, 0, 10, 0)$ and $R = \text{diag}(1, 1)$. The feedback gain matrix F is obtained as

$$F = \begin{bmatrix} 1.7992 & 2.0044 & 0.6058 & 0.4914 & -0.4175 & 0.3546 \\ 3.8262 & 0.6742 & 12.2644 & 5.9504 & 2.5449 & 4.0341 \end{bmatrix}$$

The optimal closed-loop eigenvalues are found to be 0.3145, 0.8427, $0.4955 \pm 0.3799j$, and $0.7591 \pm 0.2776j$, which satisfy the *eigenvalue constraint*. It follows from Theorem 3.2 that A_{eq} can be matched with $A - BF$ by first letting $1-(K_\Sigma/\phi) = 0.3145$. Then, G is computed using Eq. (3.25) to be

$$G = \begin{bmatrix} -0.1394 & -0.0055 & -0.9432 & -0.1389 & -0.1600 & -0.2146 \end{bmatrix}$$

Next, the bound on uncertainties is selected as $\gamma = |GD_o w_o(k)| = 0.0471$, which satisfies inequality (3.9) because $|w_{o_2}(k)| \leq 1$. Since fixing either K_Σ, ϕ, or ϵ fixes the others; if $K = [0.001\ 0.1]^T$, then it follows from Eqs. (3.7), (3.8), and (3.24) that $K_\Sigma = 0.101$, $\epsilon = 0.1348$, and $\phi = 0.1473$. Finally, M is obtained using Eq. (3.23). Since w_{o_2} has a frequency at $3.5\pi \cong 11$ rad/sec, good disturbance attenuation can be expected in view of the discrete singular value plot of the transfer function from w_{o_2} to q_2 in Figure 7.2. A simulation with initial state $x(0) = [1\ 2\ 3\ 4\ 5\ 6]^T$ and desired trajectory $x_d(k) = [1\ 0\ 4\ 0\ 7\ 0]^T$ is carried out and the result is shown in Figure 7.3. It is seen that the $\mathcal{B}$ is attractive and invariant, and good disturbance rejection is attained.

Example 7.2. Reconsider Example 7.1 but now with initial state $x(0) = [3\ 0\ 4\ 0\ 5\ 0]^T$, desired trajectory $x_d(k) = [1\ 0\ 2\ 0\ 3\ 0]^T$, and $w_o(k) = [0\ 0\ 0]^T$, i.e. no uncertainties. It is seen from Figure 7.4 that under this ideal condition discrete sliding mode is achieved and the tracking error dynamics is asymptotically stable.

Example 7.3. Reconsider Example 7.2 but now the saturation term in the control law (3.2) is replaced by a linear term, i.e. sat $\left(\frac{s(k)}{\phi}\right)$ is replaced by $\frac{s(k)}{\phi}$ so that (3.2) becomes linear. With all parameters remain the same as those in Example 7.2, the simulation result for this linear controller is shown in Figure 7.5. It is seen that the linear controller provides a much faster response in the expense of more control effort than the variable structure controller. However, the result is unrealistic because the masses ran into each other. These

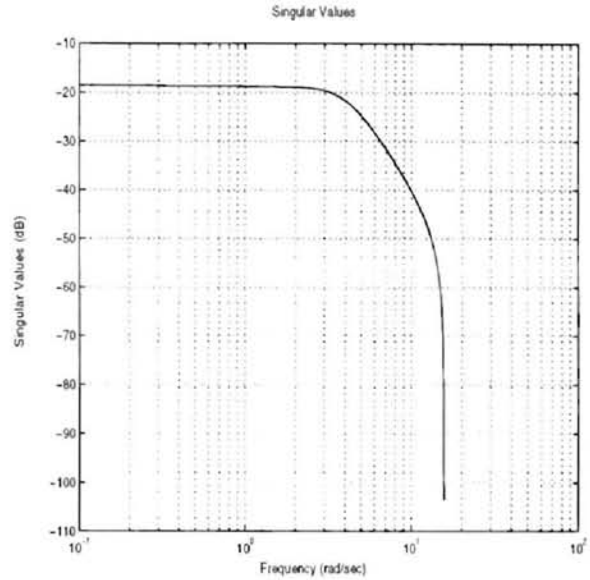

Figure 7.2: Discrete singular value plot of transfer function from disturbance w_{o_2} to position q_2 in Example 7.1

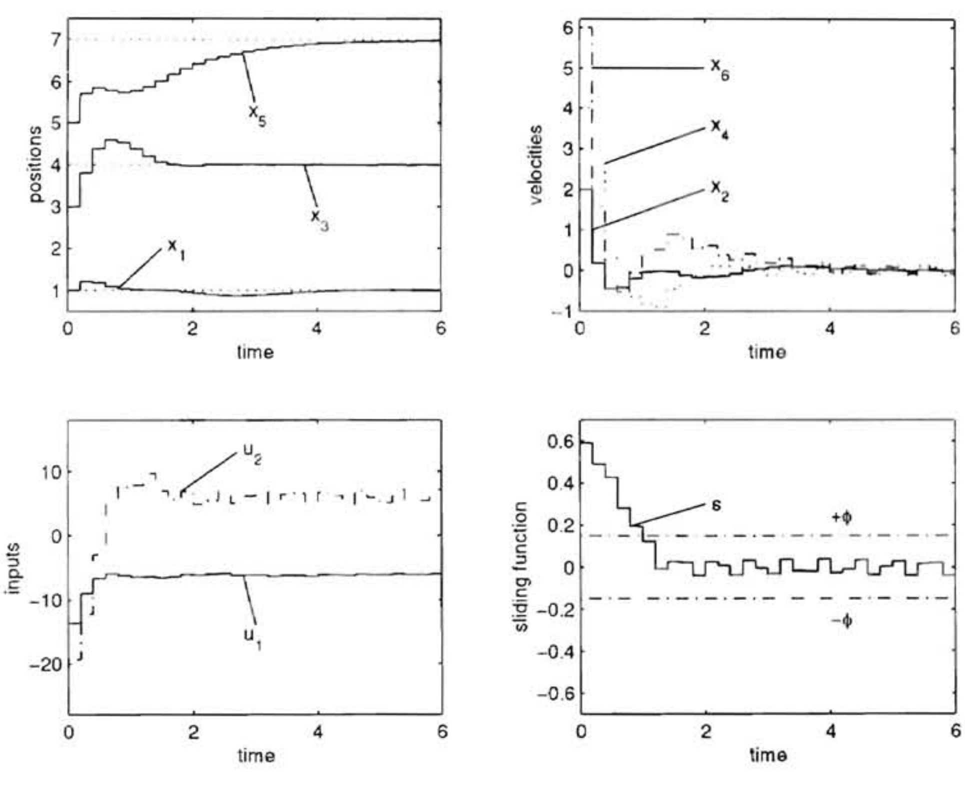

Figure 7.3: Time response of the mechanical system in Example 7.1

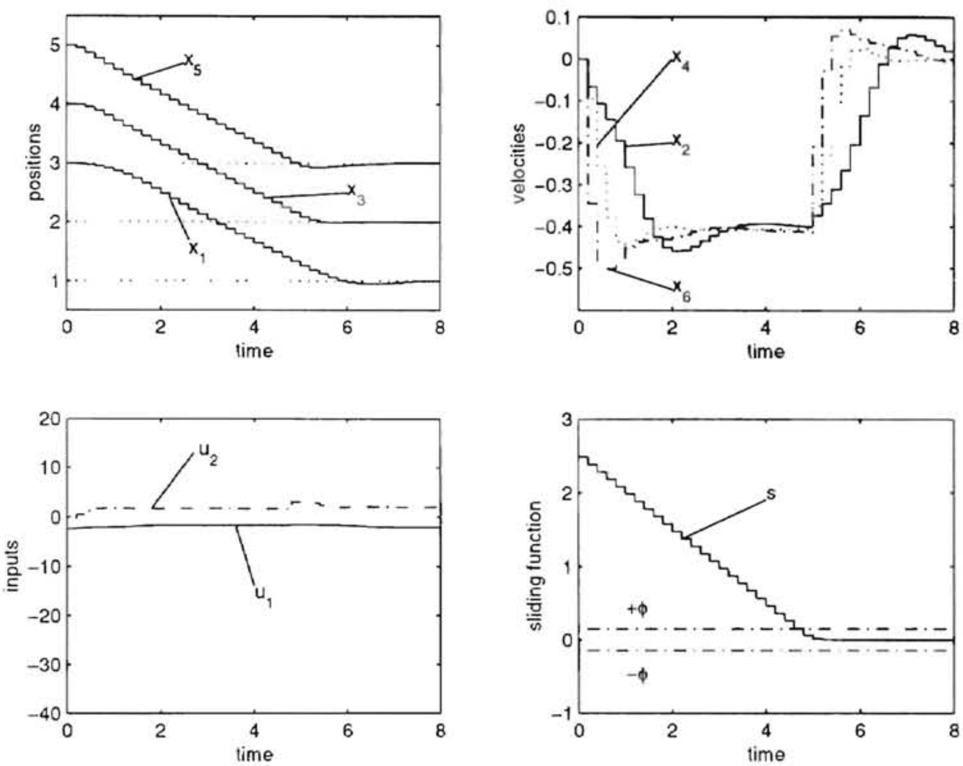

Figure 7.4: Time response of the mechanical system in Example 7.2

collisions are avoided by the variable structure controller in Example 7.2 due to the use of lower gain outside $\mathcal{B}$.

7.1.2 Additive Uncertainties: The Output Feedback Case

Example 7.4. Reconsider Example 7.1 but now use the prediction observer with uncertainty estimation. Suppose the output equation is given by

$$y(k) = Cx(k) + v(k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(k) + v(k)$$

Let w_o be decomposed into matched and unmatched portion as in Eq. (4.2) where D_o is given in Eq. (7.2), D is a 2×2 identity matrix, $B_\perp D_\perp$ is the second column of D_o, $w = [w_{o_1}\ w_{o_3}]^T$, and $w_\perp = [w_{\perp 1}] = w_{o_2}$. Let the dynamics of the matched portion be described by Eq. (4.3) with $r = [r_1\ r_2]^T$. Suppose $w_{\perp 1}(k) = \sin(3.5\pi k\Delta t)$, $r(k) = [\sin(2\pi k\Delta t)\ \cos(2\pi k\Delta t)]^T$, and $v(k) = [0\ 0]^T$, i.e. no measurement noise is present. Consider Assumption 4.3 and suppose

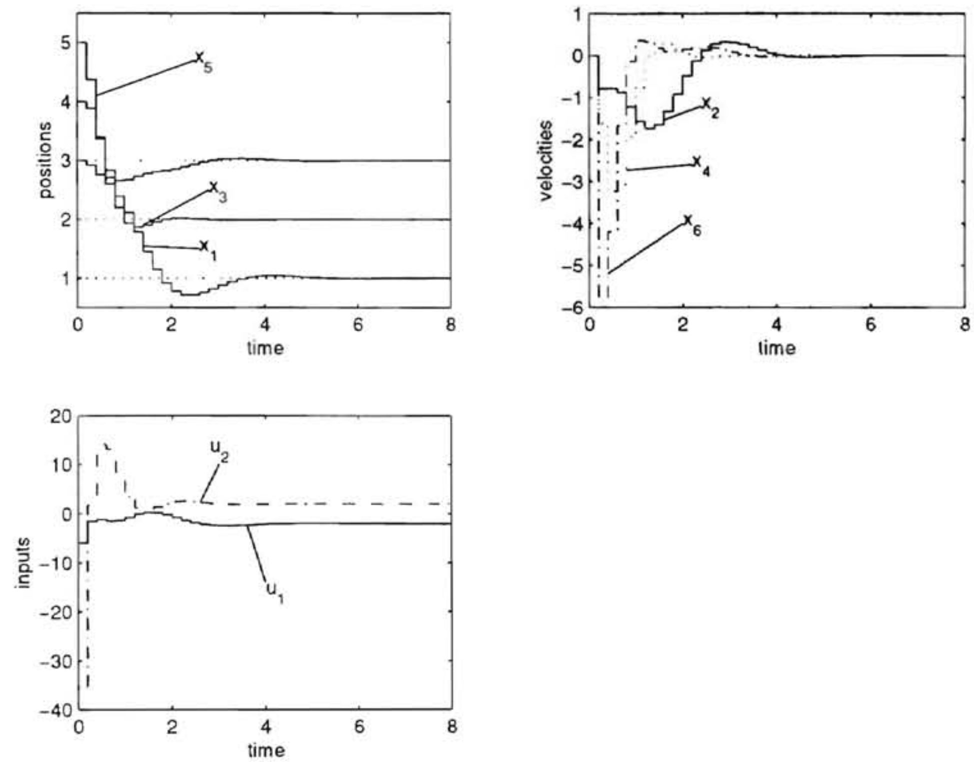

Figure 7.5: Time response of the mechanical system in Example 7.3

the following are known:

$$\delta_{w_{\perp 1}} = \delta_{r_1} = \delta_{r_2} = 1\,, \quad \Omega(w_{\perp 1}) = \Omega(r_1) = \Omega(r_2) = \{\omega : \pi \leq \omega \leq \pi/\Delta t\} \qquad (7.3)$$

The control objective is to make the tracking error small with q_2 being emphasized the most. Consider using the LQR technique again to design the equivalent matrix A_{eq}, where the weighting matrices Q and R are the same as those in Example 7.1. Consequently, the feedback gain matrix F and the optimal closed-loop eigenvalues are the same as those in Example 7.1. Next, let $1 - (K_\Sigma/\phi) = 0.3145$ so that G is also the same as that in Example 7.1.

The augmented system with state vector $\eta = [x^T\ w^T]^T$ is described by Eq. (4.4). Let the observer gain matrix $H_a = [H_1^T\ H_2^T]^T$ with $H_1 \in \mathbf{R}^{6\times 2}$ and $H_2 \in \mathbf{R}^{2\times 2}$ be chosen as

$$H_a = \begin{bmatrix} 1.3387 & 2.3847 & 0.1261 & -0.3368 & -0.1880 & -0.6985 & 1.9958 & -1.6711 \\ 0.2138 & 1.1364 & 0.8448 & 1.0869 & 1.0297 & 1.2479 & 1.0352 & 0.6550 \end{bmatrix}^T$$

so that it yields observer poles at 0.3, 0.4, 0.5, 0.6, 0.65, 0.7, 0.75, and 0.8. To find a γ

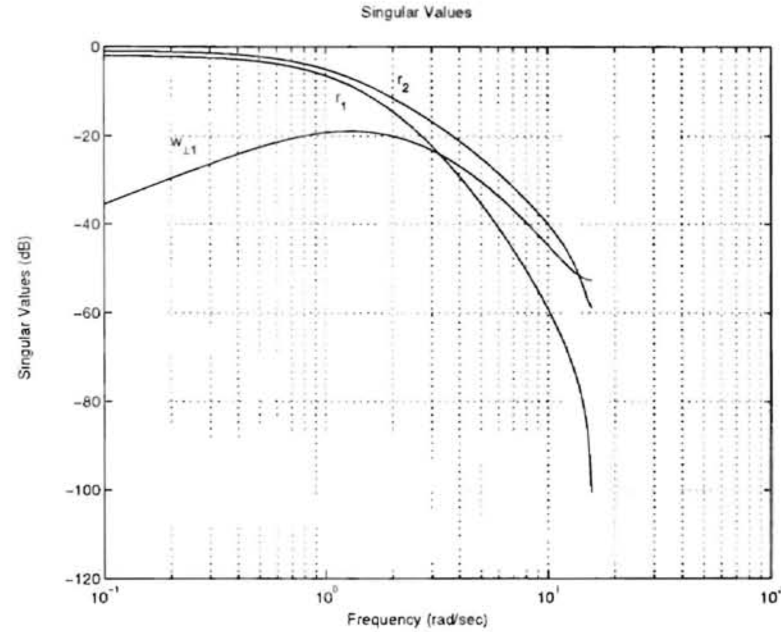

Figure 7.6: Discrete singular value plot of the transfer functions from r_1, r_2, and $w_{\perp 1}$ to $GH_1C_a\tilde{\eta}$ in Example 7.4

that satisfy inequality (4.10), first notice that the transfer functions from $w_{\perp 1}$, r_1, and r_2 to $GH_1C_a\tilde{\eta}$ are respectively given by

$$\begin{aligned}\Phi_{w_{\perp 1}}(z, GH_1C_a) &= GH_1C_a[zI - A_a + H_aC_a]^{-1}\begin{bmatrix} B_\perp D_\perp \\ 0 \end{bmatrix} \\ \Phi_{r_1}(z, GH_1C_a) &= GH_1C_a[zI - A_a + H_aC_a]^{-1}e_7 \\ \Phi_{r_2}(z, GH_1C_a) &= GH_1C_a[zI - A_a + H_aC_a]^{-1}e_8\end{aligned}$$

where e_i is a column vector with 1 in its ith entry and 0 elsewhere. Next, it is seen from Eq. (7.3) and from the discrete singular value plots of these transfer functions in Figure 7.6 that

$$\begin{aligned}\sup_{\omega\in\Omega(w_{\perp 1})} |\Phi_{w_{\perp 1}}(e^{j\omega\Delta t}, GH_1C_a)| &\cong -24\text{dB} = 0.063 \\ \sup_{\omega\in\Omega(r_1)} |\Phi_{r_1}(e^{j\omega\Delta t}, GH_1C_a)| &\cong -22\text{dB} = 0.079 \\ \sup_{\omega\in\Omega(r_2)} |\Phi_{r_2}(e^{j\omega\Delta t}, GH_1C_a)| &\cong -16\text{dB} = 0.158\end{aligned}$$

Inequality (4.10) then becomes

$$\gamma \geq (0.063)(1) + (0.079)(1) + (0.158)(1) = 0.3$$

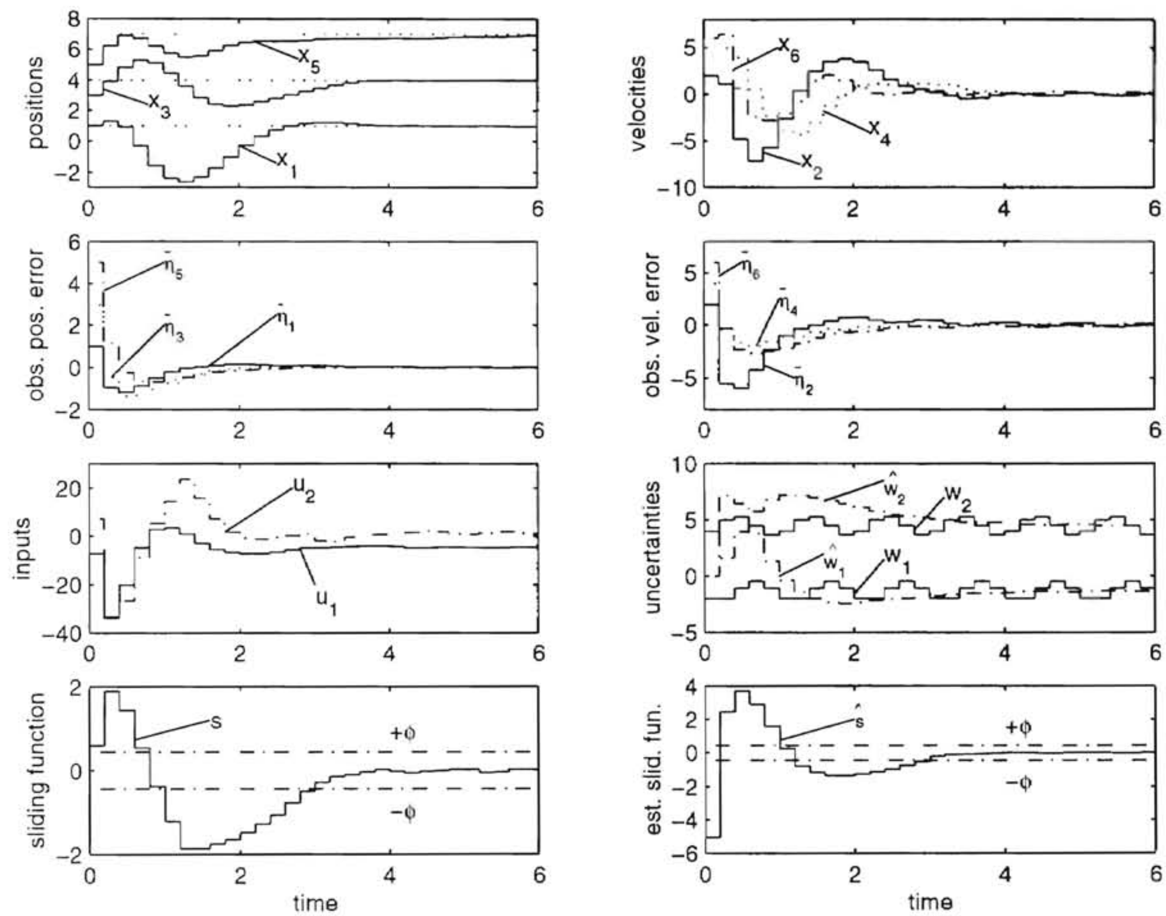

Figure 7.7: Time response of the mechanical system in Example 7.4

which implies that it is sufficient to let $\gamma = 0.3$. Since fixing either K_Σ, ϕ, or ϵ fixes the others; if $K = [0.001\ \ 0.3]^T$, then it follows from Eqs. (3.7), (3.8), and (3.24) that $K_\Sigma = 0.301$, $\epsilon = 2.5 \times 10^{-3}$, and $\phi = 0.4391$. Finally, M is obtained using Eq. (3.23). A simulation with initial state $x(0) = [1\ 2\ 3\ 4\ 5\ 6]^T$, desired trajectory $x_d(k) = [1\ 0\ 4\ 0\ 7\ 0]^T$, initial disturbance $w(0) = [-2\ \ 4]^T$, and observer initial state $\hat{\eta}(0) = 0$ is carried out and the result is shown in Figure 7.7. It is seen that the $\hat{\mathcal{B}}$ is attractive and invariant after the observer error dynamics has reached the steady-state, at approximately 3 seconds. Meanwhile, its performance is not as superior as in Example 7.1 where full state feedback is available. Furthermore, it is seen that the uncertainty estimation is effective and the system is kept BIBO stable.

Example 7.5. Reconsider Example 7.4 but now with $w_{\perp 1}(k) = 0$, $r(k) = [0\ \ 0]^T$, and

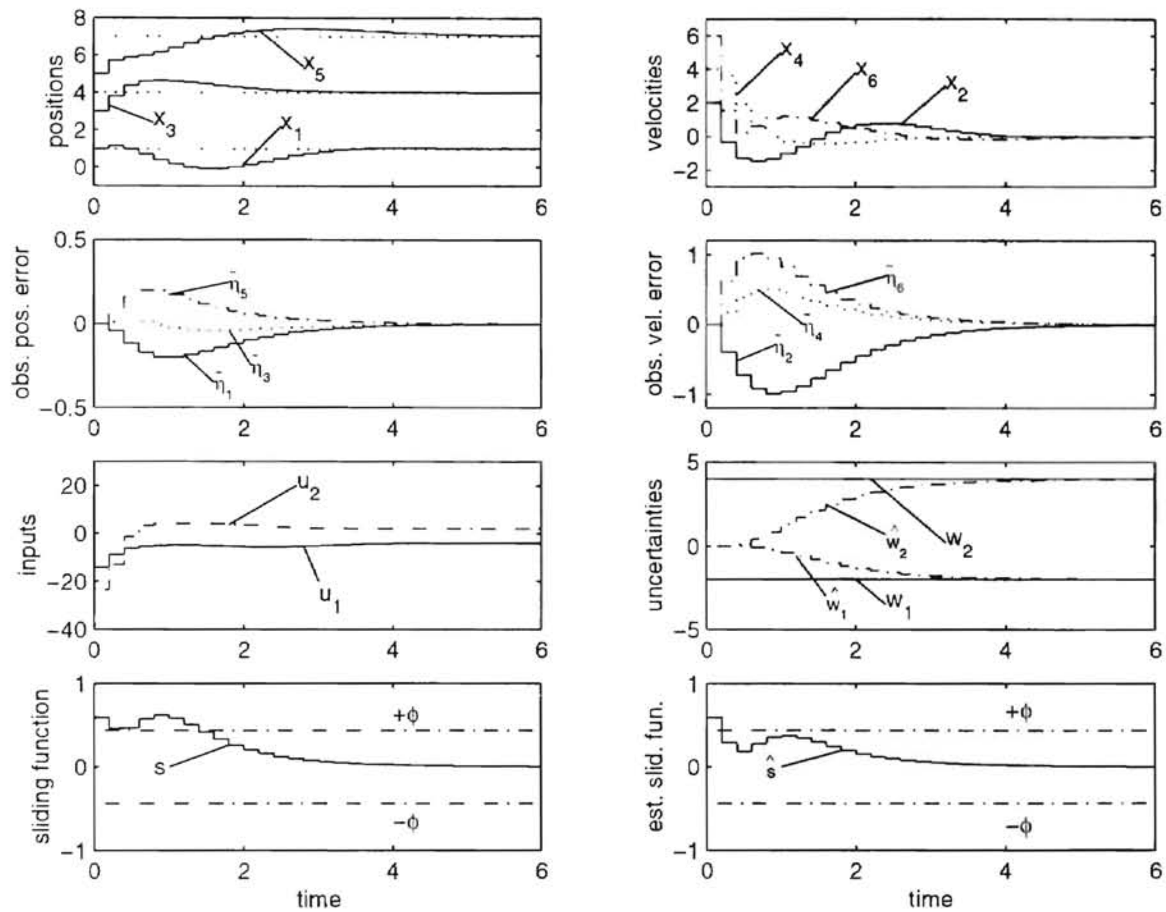

Figure 7.8: Time response of the mechanical system in Example 7.5

$v(k) = [0\ 0]^T$, i.e. no unmatched disturbance, constant matched disturbances, and no measurement noise. Also, let the observer initial state $\hat{\eta}(0) = [x(0)^T\ 0\ 0]^T$, i.e. no mismatched initial condition in the system state. With all parameters remain the same as those in Example 7.4, the simulation result is shown in Figure 7.8. It is seen that under this ideal condition discrete sliding mode is achieved and both the tracking and observer error dynamics are asymptotically stable.

7.2 Control of a Pressurized Flow Box

7.2.1 Parametric Uncertainties: The State Feedback Case

Example 7.6. Consider the pressurized flow box system (Franklin *et al.* [13], p.g. 788) described by

$$\begin{bmatrix} \dot{H}(t) \\ \dot{h}(t) \\ \dot{u}_a(t) \end{bmatrix} = \begin{bmatrix} -0.2 & 0.1 & 1 \\ -0.05 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} H(t) \\ h(t) \\ u_a(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0.7 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_c(t) \\ u_s(t) \end{bmatrix} \tag{7.4}$$

The discrete-time equivalent of the system (7.4), obtained by applying $u_c(t)$ and $u_s(t)$ through a zero-order hold with $\Delta t = 0.2$, is given by Eq. (5.1) where

$$A = \begin{bmatrix} 0.9607 & 0.0196 & 0.1776 \\ -0.0098 & 0.9999 & -0.0009 \\ 0 & 0 & 0.8187 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0185 & 0.1974 \\ -0.0001 & 0.1390 \\ 0.1813 & 0 \end{bmatrix},$$

and $\Delta A(k)$ is the parametric uncertainty matrix satisfying $\text{rank}([B \ \Delta A(k)]) = \text{rank}(B)$. It can be shown that the Kronecker invariant for this system is $n_1 = 2$, $n_2 = 1$. Using Lemma 5.1, the system (5.1) is transformed into the controllable canonical form (5.2) via a change in coordinates $x = T\bar{x}$ where

$$T = \begin{bmatrix} 0.0168 & 0.0185 & 0.1986 \\ -0.0003 & -0.0001 & 0.1390 \\ -0.1767 & 0.1813 & 0.0116 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ -0.7979 & 1.7935 & -0.0107 \\ -0.0025 & 0 & 0.9858 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0 & 0 \\ 1 & -0.0642 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & -0.0642 \\ 0 & 1 \end{bmatrix}^{-1}$$

By Definition 5.1, one has

$$\hat{A} = [a_{ij}] = \begin{bmatrix} -0.7979 & 1.7935 & -0.0107 \\ -0.0025 & 0 & 0.9858 \end{bmatrix}$$

Let the desired eigenvalues be 0.2 and $0.1 \pm 0.3j$. The corresponding feedback gain matrix F is found to be

$$F = [f_{ij}] = \begin{bmatrix} -0.6978 & 1.5937 & 0.0066 \\ -0.0052 & 0.0085 & 0.7856 \end{bmatrix}$$

Suppose the time-varying parametric uncertainty matrix is given by

$$\Delta\hat{A}(k) = [\Delta a_{ij}(k)] = 0.2 \times \begin{bmatrix} -1 & \mathrm{sgn}(\cos(2k\Delta t)) & 1 \\ \mathrm{sgn}(\sin(3k\Delta t)) & -1 & -\mathrm{sgn}(\sin(4k\Delta t)) \end{bmatrix}$$

where $\max(\Delta a_{ij}(k)) = 0.2$. To satisfy inequality (5.6), it suffices to choose

$$K_{ij} = |a_{ij} - f_{ij}| + \max(\Delta a_{ij}(k)) + 0.01 \ , \quad 1 \le i \le m \ , \quad 1 \le j \le n$$

It then follows that

$$K = [K_{ij}] = \begin{bmatrix} 0.3101 & 0.4098 & 0.2273 \\ 0.2127 & 0.2185 & 0.4102 \end{bmatrix}$$

and $\|K\|_\infty = 0.9473 < 1$, which implies that inequality (5.7) is satisfied. A simulation with initial state $x(0) = [1 \ 2 \ 3]^T$ is carried out and the result is shown in Figure 7.9. It is seen that the system is globally uniformly asymptotically stable in the presence of the perturbation $\Delta\hat{A}(k)$. Also, the control effort is quite high because the poles are placed near to the origin.

7.3 Sliding Surface Design for a Double Integrator Plant

Example 7.7. Consider a double integrator plant described by

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_c(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_c(t)$$

The corresponding discrete-time system obtained using a zero-order hold with $\Delta t = 0.1$ is given by

$$x(k+1) = Ax(k) + Bu(k) = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix} u(k)$$

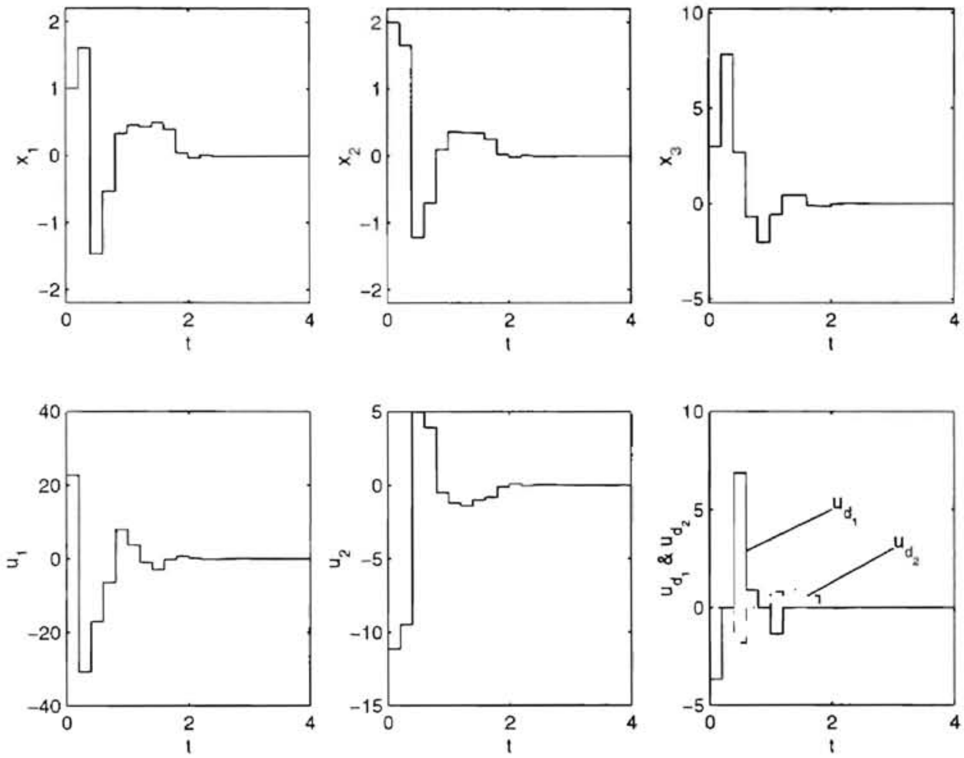

Figure 7.9: Time response of the pressurized flow box in Example 7.6

Let $\gamma = 0.05$ and $\epsilon = 0.1$. It follows from Eqs. (3.7) and (3.8) that $K_\Sigma = \gamma + 2\Delta t\epsilon = 0.07$, $\phi = \gamma + 2\Delta t\epsilon = 0.06$, and $\alpha = 1 - (K_\Sigma/\phi) = -0.1667$. Theorem 6.1 is satisfied since α is nonzero and not equal to any eigenvalue of A. Suppose the desired weighting matrices are chosen as $Q_d = \text{diag}(1 \times 10^6, 0)$ and $R = 1$. The subsequent step is performed in MATLAB using `dvsclqr1.m`:

```
>> Qd=[1e6 0;0 0];
>> [Q,F,G]=dvsclqr1(A,B,Qd,alpha)
Least-squares approach is used
Q =
  1.0e+005 *
   10.0000         0
         0    0.0046
F =
  128.1651   16.2440
G =
   -0.9992   -0.0410
>> Aeq=A-B*G/(G*B)*(A-alpha*eye(2));
```

```
>> eig(Aeq)
ans =
   -0.0986
   -0.1667 % this is alpha
>> eig(A-B*dlqr(A,B,Qd,1))
ans =
   -0.0557
   -0.3820 % if use Qd, both eigenvalues are not equal to alpha
```

Notice that in this example the least-squares approach is used to find the closest feasible Q. If Q_d is to be used, then α must be chosen as either -0.0557 or -0.3820.

Example 7.8. Reconsider the double integrator plant in Example 7.7. Suppose all the controller parameters remain unchanged except for Q_d where now $Q_d = \text{diag}(1 \times 10^5, 0)$. Again, the subsequent step is performed in MATLAB using `dvsclqr1.m`:

```
>> Qd=[1e5 0;0 0];
>> [Q,F,G]=dvsclqr1(A,B,Qd,alpha)
Convex programming approach is used
Q =
  1.0e+005 *
    6.4027    0.0000
    0.0000    0.0000
F =
  137.2667   16.5690
G =
   -0.9994   -0.0350
>> Aeq=A-B*G/(G*B)*(A-alpha*eye(2));
>> eig(Aeq)
ans =
   -0.1667 % this is alpha
   -0.1766
>> eig(A-B*dlqr(A,B,Qd,1))
ans =
   0.0639 + 0.2985i
   0.0639 - 0.2985i % if use Qd, both eigenvalues are strictly complex
```

Notice that in this example the convex programming approach is used to find the closest feasible Q. The desired Q_d can never be used because the closed-loop eigenvalues are strictly complex, which is not possible since α must be one of the real eigenvalues.

Chapter 8

Conclusions

A state feedback discrete variable structure control technique for linear multivariable systems with additive uncertainties is developed. It is shown that the boundary layer under the control law is attractive and invariant. Also, model matching in the linear region is possible as long as the *eigenvalue constraint* is satisfied. Furthermore, the benefits of using one hyperplane over multiple hyperplanes in discrete variable structure systems are discussed. The resulting system is found to be asymptotically stable if no uncertainties are present and BIBO stable otherwise.

For practical reasons, the use of a prediction observer with uncertainty estimation is proposed. It is shown that the estimated boundary layer is attractive and invariant after the observer has come to a steady-state and model matching in the linear region is possible as in the state feedback case. The resulting system is found to be asymptotically stable if there is no unmatched uncertainties, no noise, and the matched uncertainties are constant bias. Otherwise, the resulting system is BIBO stable.

For linear multivariable systems with parametric uncertainties, the concept of switching sector is used and a control law capable of avoiding the potential pitfalls associated with existing schemes is developed. It is shown that the switching sector should be attractive but does not have to be invariant as long as additional stability requirements can be imposed on the system. The resulting system is found to be globally uniformly asymptotically stable under certain conditions. However, the admissible bounds of uncertainties obtained are found to be conservative.

The use of LQR technique in sliding surface design for single-input systems is examined in details. It is shown that one can always specify the real eigenvalue related to the sliding gain and boundary layer thickness and then constrain the weighting matrix. The least squares-convex programming approach is then used to solve this constrained optimization problem. This leads to the development of an automated optimal sliding surface design procedure.

To illustrate the effectiveness of the proposed control techniques and sliding surface procedure, three examples are presented, namely the control of a mechanical system and a pressurized flow box as well as the sliding surface design for a double integrator plant.

Suggestions for future research include:

- Investigation of the possibility of extending the results on systems with additive uncertainties to the nonlinear case, perhaps with a nonlinear sliding surface $\mathcal{S} = \{x : s(x) = 0, s \in \mathbf{R}\}$.

- Investigation on the use of available freedoms among the sliding gains K_i's to avoid controller saturation and to improve the nonlinear behavior of the dynamics outside the boundary layer.

- Investigation of the possibility of extending the results on switching sector to the output feedback case, as well as the use of switching region not necessarily having the shape of a sector to improve the admissible bounds on uncertainties.

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Appendix: MATLAB Files

dvsclqr1.m

```
% LQR Technique in Sliding Surface Design using Least Squares-
% Convex Programming Approach
% by Choon Yik Tang 8/97
%
% [Q,F,G]=dvsclqr1(A,B,Qd,alp)
%
%   A,B=Single input plant in state space form
%   Qd=Symmetric positive semi-definite desired weighting matrix
%   alp=1-K/phi, valid range is -1<alp<1, alp~=0, alp~=eig(A),
%       and 1/alp~=eig(A)
%   Q=Symmetric positive semi-definite weighting matrix that gives
%     root at alp and is closest to Qd in the least-squares sense
%   F=Optimal feedback gain matrix corresponds to Q and R=1
%   G=Row vector defining the sliding surface
%
% Note: Which approach is used will be displayed
% Note: Warning message will be displayed if the dimension of G is
%       greater than 1 or G*B=0 or some entries in G is zero

function [Q,F,G]=dvsclqr1(A,B,Qd,alp)

global bet xil N
error(abcdchk(A,B));
if size(B,2)~=1
  error('Must be single input');
elseif any(eig(Qd)<-eps*norm(Qd,1))|(norm(Qd'-Qd,1)/norm(Qd,1)>eps)
  error('Qd must be symmetric and positive semi-definite')
elseif abs(alp)>=1 | alp==0 | any(alp==eig(A)) | any(1/alp==eig(A))
```

```
  error('Valid range is -1<alp<1,alp~=0,alp~=eig(A),1/alp~=eig(A)');
end
n=size(A,1);
phia=inv(alp*eye(n)-A)*B;
phiai=inv(1/alp*eye(n)-A)*B;
k=1;
for j=1:n
  for i=j:n
    if i==j
      bet(k,1)=phia(i)*phiai(i);
    else
      bet(k,1)=phia(i)*phiai(j)+phia(j)*phiai(i);
    end
    k=k+1;
  end
end
N=null(bet');
xil=inv(N'*N)*N'*(vecsym(Qd,1)+bet*inv(bet'*bet));
Ql=vecsym(-bet*inv(bet'*bet)+N*xil,-1);
if ~any(eig(Ql)<-eps*norm(Ql,1))
  Q=Ql;
  disp('Least-squares approach is used')
else
  M=phiai*phia'+phia*phiai';
  [wM,mM]=eig(M);
  [mn,k]=min(diag(mM));
  V=[wM(:,k),null(wM(:,k)')];
  Q0=V*diag([-2/mn,zeros(1,n-1)])*V';
  xi0=inv(N'*N)*N'*(vecsym(Q0,1)+bet*inv(bet'*bet));
  xi=constr('dvsclqr2',xi0);
  Q=vecsym(-bet*inv(bet'*bet)+N*xi,-1);
  disp('Convex programming approach is used')
end
F=dlqr(A,B,Q,1);
E=eig(A-B*F,'nobalance'); % avoid numerical error
[e,i]=min(abs(E-alp));
G=null((A-B*F-E(i)*eye(n))')';
if size(G,1)>1
  disp('Warning: The dimension of G is greater than 1')
```

```
elseif G*B==0 | any(abs(G)<eps)
  disp('Warning: G*B=0 or some entries in G is zero')
end
clear global
```

dvsclqr2.m

```
% LQR Technique in Sliding Surface Design using Least Squares-
% Convex Programming Approach
% by Choon Yik Tang 8/97
%
% dvsclqr2
%
% Note: To be called by internally dvsclqr1

function [f,g]=dvsclqr2(xi)
global bet xil N
f=(xil-xi)'*(xil-xi);
g=-eig(vecsym(-bet*inv(bet'*bet)+N*xi,-1));
```

dsrlocus.m

```
% Discrete-time LQR symmetric root locus
% by Choon Yik Tang 7/97
%
% dsrlocus(A,B,Q)
% dsrlocus(Gnum,Gden)
%
%   A,B=Single input plant in state space form
%   Q=C'*C=DLQR weighting matrix
%   C=plant/fictitious output matrix
%   Gnum/Gden=Single input plant/fictitious
%             transfer function=C*inv(zI-A)*B
%   Gnum,Gden=polynomial coefficients in descending powers of z

function dsrlocus(A,B,Q)
error(nargchk(2,3,nargin));
if nargin==3
  if size(B,2)>1
    error('Must be single input')
```

```
  end
  if any(eig(Q)<-eps*norm(Q,1))|(norm(Q'-Q,1)/norm(Q,1)>eps)
    error('Q must be symmetric and positive semi-definite')
  end
  error(abcdchk(A,B,Q));
  [Gnum1,Gden1]=ss2tf(A,B,Q,zeros(size(Q,1),1));
  [Gnum2i,Gden2i]=ss2tf(A,B,eye(size(A)),zeros(size(Q,1),1));
  Gnum2=fliplr(Gnum2i);
  Gden2=fliplr(Gden2i);
else
  [Gnum1,Gden1]=tfchk(A,B);
  Gnum1=[zeros(size(Gnum1,1),length(Gden1)-size(Gnum1,2)),Gnum1];
  Gnum2=fliplr(Gnum1);
  Gden2=fliplr(Gden1);
end
Num=zeros(1,2*size(Gnum1,2)-1);
for i=1:size(Gnum1,1)
  Num=Num+conv(Gnum1(i,:),Gnum2(i,:));
end
rlocus(Num,conv(Gden1,Gden2))
```

vecsym.m

```
% Vectorize a symmetric matrix and the reverse
% by Choon Yik Tang 8/97
%
% y=vecsym(x,ver)
%
%   ver=1:
%     x=symmetric matrix
%     y=column vector stacking columnwise the lower triangular
%       elements of x
%
%   ver=-1:
%     x=column vector stacking columnwise the lower triangular
%       elements of y
%     y=symmetric matrix

function y=vecsym(x,ver)
error(nargchk(2,2,nargin));
```

```
if ver==1
  if size(x,1)~=size(x,2)
    error('x must be square')
  elseif (norm(x'-x,1)/norm(x,1)>eps)
    error('x must be symmetric')
  end
  n=size(x,1);
  k=1;
  for i=1:n
    y(k:k+n-i,1)=x(i:n,i);
    k=k+n-i+1;
  end
else
  n=find(size(x,1)==cumsum(linspace(1,32,32)));
  if size(x,2)~=1 | n==[]
    error('x must be column vector and vectorizable')
  end
  k=1;
  for i=1:n
    y(i:n,i)=x(k:k+n-i);
    y(i,i+1:n)=x(k+1:k+n-i)';
    k=k+n-i+1;
  end
end
```

VITA

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