ELASTIC WEIGHTS FOR TRUSSES BY THE STRING POLYGON METHOD

By

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PREFACE

The writer wishes to express indebtedness and gratitude to the following persons:

To Professor Jan J. Tuma for introducing him to the String Polygon Method and its application to the "conjugate cell," also for the many helpful suggestions made during this period of study.

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In addition, gratitude is due Mrs. W. Arlene Starwalt, who typed the final manuscript.

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NOMENCLATURE

a, h . . . . Dimensions

d_{ij} . . . . Length of Truss Member ij

i, j, k, l, m . . . General Subscripts Denoting Functions of Members Between Joints

p . . . . Subscript Denoting Functions of Loads

r . . . . "Arm" Matrix

x, y, z . . . . . . Reference Axes (Rectangular Coordinates), Projected Lengths

x', y' . . . . Auxiliary Reference Axes

A, B, C, D . . . Notation for Cells

D_H . . . . Determinate of the Arm Matrix for the General Cell H

D'_{Hx}, D'_{Hy} . . . . Deformation Constants for the General Cell H in x, y Directions

E . . . . Modulus of Elasticity

F . . . . Notation of Forces for Summations

L . . . . Length

N_{ij} . . . . Normal Force in Member ij

P_i . . . . External Load at Joint i

R_{ix}, R_{iy} . . . . . Reaction of Support at Joint i in the x, y Direction

$\overline{}$ . . . . Quantities Involved on Conjugate Structure i.e.

$\overline{F}$ . . . . Notation of Force for Summation on Conjugate Structure

$\overline{M}$ . . . . Notation of Moment on Conjugate Structure

$\overline{P}_i$ . . . . Total Joint Elastic Weight, Load on Conjugate Structure for Joint i

$\overline{P}_{iH}$ . . . . . . Joint Elastic Weight for Joint i of Cell H

$\overline{R}_{ix}$, $\overline{R}_{iy}$ . . . . Reaction of Support at Joint i on the Conjugate Structure in the x, y Direction

α, β . . . . . . Direction Constants

$\Delta_{ijx'}$, $\Delta_{ijy'}$ . . Axial Deformation of Member ij in the x', y' Direction

δ . . . . Deflection of Joint

θ_{ij} . . . . Angles

λ . . . . Axial Flexibilities of Member ij

Σ . . . . Summation

$[\omega_N]$ . . . . . . Direction Matrix for Normal Forces

$[\omega_P]$ . . . . . . Direction Matrix for Loads

[ ] . . . . . . Brackets, Matrix Notation

CHAPTER I

INTRODUCTION

1-1. Statement of the Problem.

The purpose of this thesis is the extension of the string polygon method to the calculation of deformation of coplanar trusses. The analytical expressions for the angle changes of truss panels are derived in general form and represented by force vectors acting normal to the plane of the structure. The axial deformation of each member is shown as a moment vector acting on the corresponding member. Each cell of the truss is considered as a closed polygon and later on two or more cells are combined into a new conjugate structure. The investigation is limited to a coplanar statically determinate system.

1-2. Historical Background.

The idea of representing the deformation of trusses by means of fictitious loads designated as elastic weights was introduced by Otto Mohr (1, 2). In using elastic weights, Mohr classified these fictitious loads as top and bottom bar elastic weights and diagonal bar elastic weights and assigned to the angle changes a special value due to each of these deformations. Then he applied these elastic weights on a conjugate beam and used this beam in computation of deformation similar to the manner he had done for beams. The calculation of elastic weights due to the top and bottom member deformation is relatively easy, but the calculation of elastic weight due to the deformation of diagonal members is more involved.

Later on H. Muler-Breslau developed an alternate method of elastic loads which became well known as the bar-chain method (3). The disadvantage being that displacements of only a few joints may be obtained for any given bar-chain calculation.

In the American literature the application of elastic weights to the calculation of deformation of trusses has been discussed in connection with new ideas by Scordelis and Smith (4) and Lee and Patel (5).

Scordelis and Smith used elasto-static equations to obtain angular and elastic weights and introduced a conjugate structure of the bar-chain type. Lee and Patel extended the bar-chain method suggested by Muler-Breslau to the analysis of indeterminate truss structures.

Algebraic formulas for the elastic weights of coplanar trusses have been developed by Tuma (6). The disadvantage here is their algebraic complexity and the difficulty arising in locating the point of application.

The development of the string polygon method opened new possibilities for the analysis of trusses and led to the investigation presented in this thesis. The historical background was recorded by Tuma and Oden (7) and reference is given to their work.

In formulation of the stereo-static and elasto-static matrices, experience was drawn from the papers published by Martin (8) and Chen (9).

1-3. Assumptions.

In this study the usual assumptions of the analysis of trusses are observed. These assumptions are as follows:

1. Members are made of materials which are homogeneous, isotropic, and continuous.
2. Materials are elastic and follow Hooke's Law.
3. Modulus of Elasticity is the same in tension as in compression.
4. Geometry is not greatly changed by application of loads.
5. Joints are frictionless hinges where forces and moments can be neglected.
6. All loads are applied at the joints or are carried over to joints.
7. The supports are resting on unyielding foundations.

1-4. Sign Convention.

The following sign convention is used:

(a) All coordinates are positive if measured in the positive direction of the coordinate axes.

(b) All direction angles are positive if measured counterclockwise from the positive direction of the coordinate axes X .

(c) All force and moment vectors are positive if acting in the positive direction of the coordinate axes.

(d) All deformations represented as force or moment vectors are positive if acting in the positive direction of the coordinate axes.

CHAPTER II

STEREO-STATICS

2-1. Geometry.

In this study, each bar of the truss will be considered to be lying in the X - Y plane. The general subscripts i, j, k, · · · will denote the bar between joints i and j , j and k or k and i · · · . The bar ij is shown in Figure 2-1a.

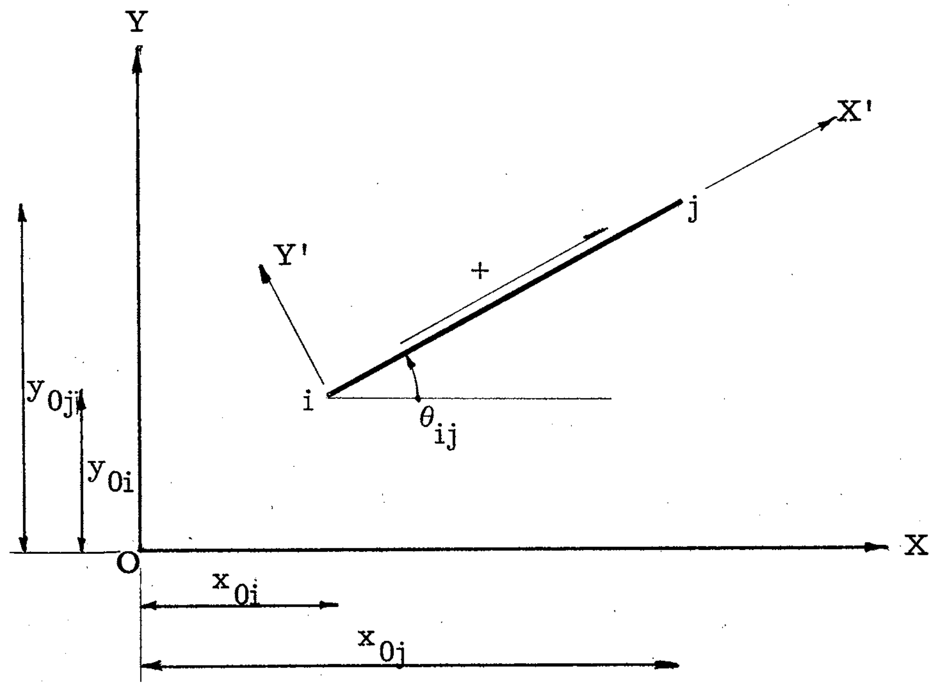


Fig. 2-1a

Geometry of Bar ij - $\theta_{ij} = 0° \to 90°$

A set of axes coincident and perpendicular to the neutral axis of the bar are called X' and Y' , respectively. The angle between X' and X measured in a counterclockwise direction is θ_{ij} . Figures 2-1a, -1b and -1c show different orientations of bar ij .

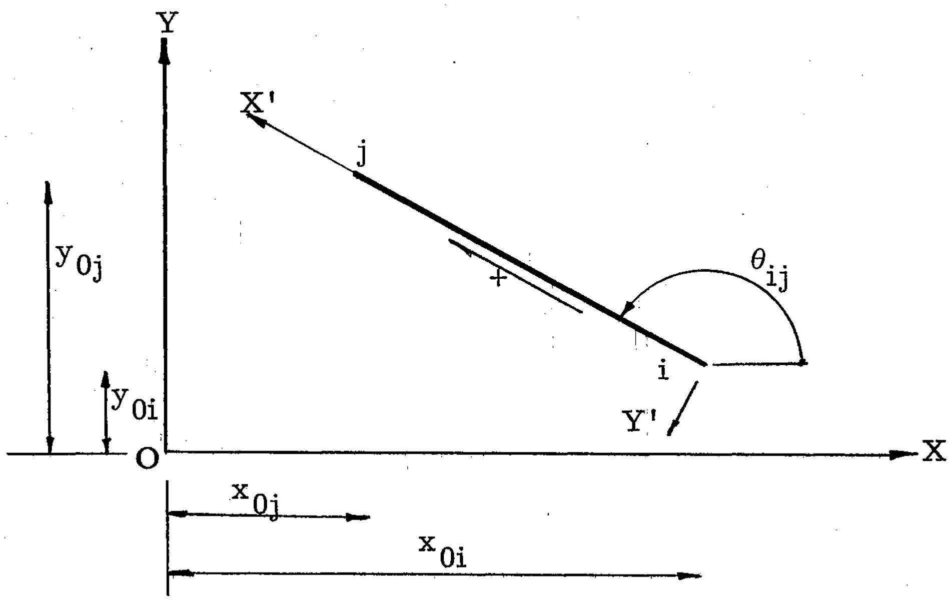


Fig. 2-1b

Geometry of Bar ij - $\theta_{ij} = 90° \rightarrow 180°$

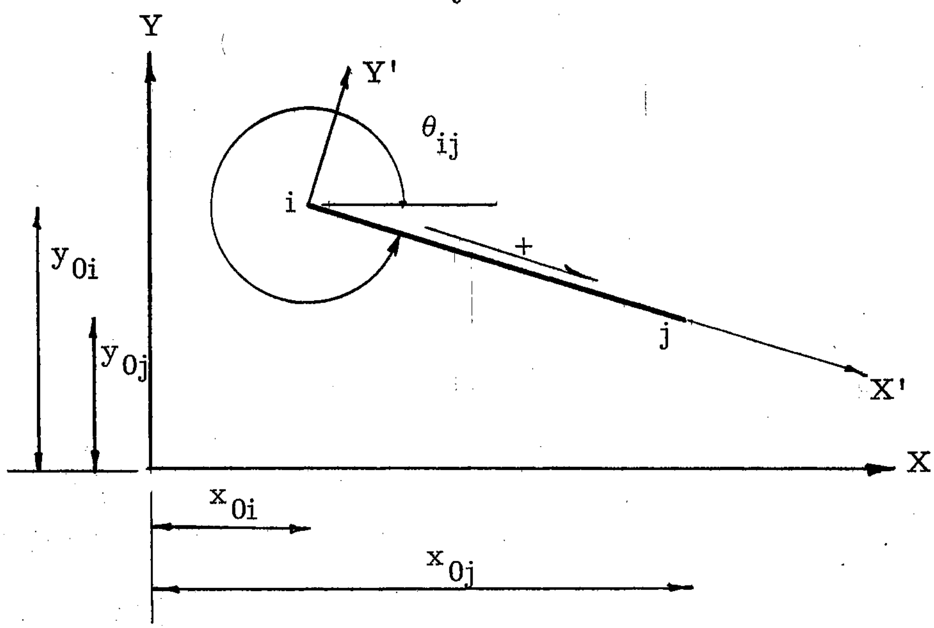


Fig. 2-1c

Geometry of Bar ij - $\theta_{ij} = 180° \rightarrow 360°$

The positive direction of bar ij is the ij-direction with the origin i at the end i . The coordinates x' , y' and the axes X' , Y' are denoted as coordinates and axes of the bar. On the other hand, the coordinates x , y and the axes X , Y are designated as coordinates and axes of the whole structural system.

The length of the bar ij is d_{ij} and the length of the same bar measured from j is d_{ji} . The components of d_{ij} in terms of the direction parameters

$$\alpha_{ij} = \cos\theta_{ij} \qquad \alpha_{ji} = \cos\theta_{ji}$$
$$\beta_{ij} = \sin\theta_{ij} \qquad \beta_{ji} = \sin\theta_{ji} \qquad (2\text{-}1)$$

are

$$x_{ij} = d_{ij}\,\alpha_{ij} \qquad x_{ji} = d_{ji}\,\alpha_{ji}$$
$$y_{ij} = d_{ij}\,\beta_{ij} \qquad y_{ji} = d_{ji}\,\beta_{ji} \quad . \qquad (2\text{-}2)$$

The same components in terms of the coordinates x and y are

$$x_{ij} = x_{0j} - x_{0i} \qquad x_{ji} = x_{i0} - x_{j0}$$
$$y_{ij} = y_{0j} - y_{0i} \qquad y_{ji} = y_{i0} - y_{j0} \quad . \qquad (2\text{-}3)$$

The relationships between the parameters α_{ij} , α_{ji} and β_{ij} , β_{ji} are from the trigonometry (Fig. 2-2).

$$\alpha_{ij} = -\alpha_{ji} \qquad \beta_{ij} = -\beta_{ji} \quad . \qquad (2\text{-}4)$$

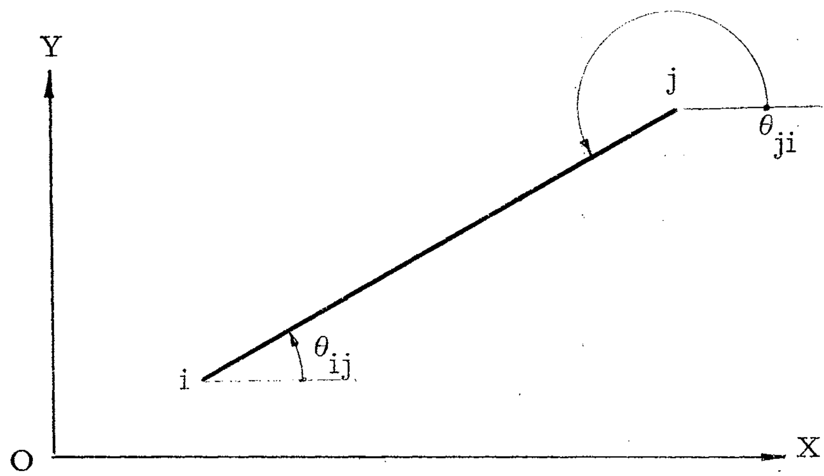


Fig. 2-2

Direction Relationship

2-2. Stereo-Static Equilibrium.

The only force existing in a bar ij is the axial force N_{ij} (Fig. 2-3).

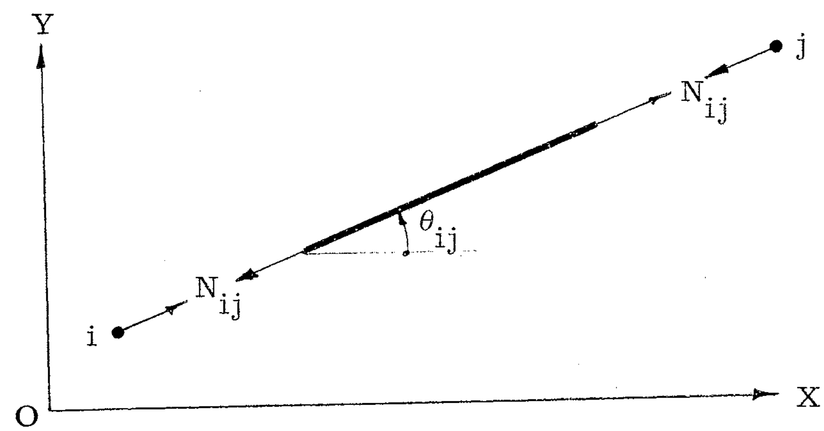


Fig. 2-3

Free Body of Bar ij

From Eq. (2-2) components of this force (Fig. 2-4) are

$$N_{ijx} = N_{ij}\,\alpha_{ij}$$
$$N_{ijy} = N_{ij}\,\beta_{ij}\ . \qquad (2\text{-}5)$$

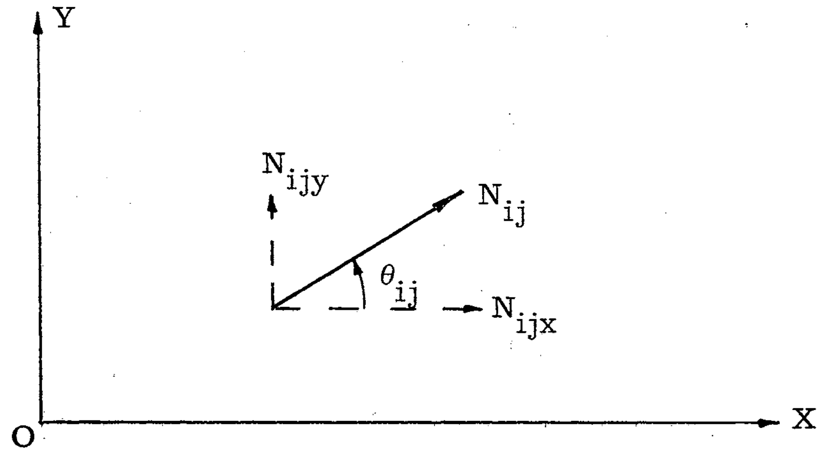


Fig. 2-4

Axial Force N_{ij} and Its Components

Joint i of a general truss is considered and isolated as a free body shown in Fig. 2-5, two types of forces will occur:

(a) Loads

(b) Axial forces.

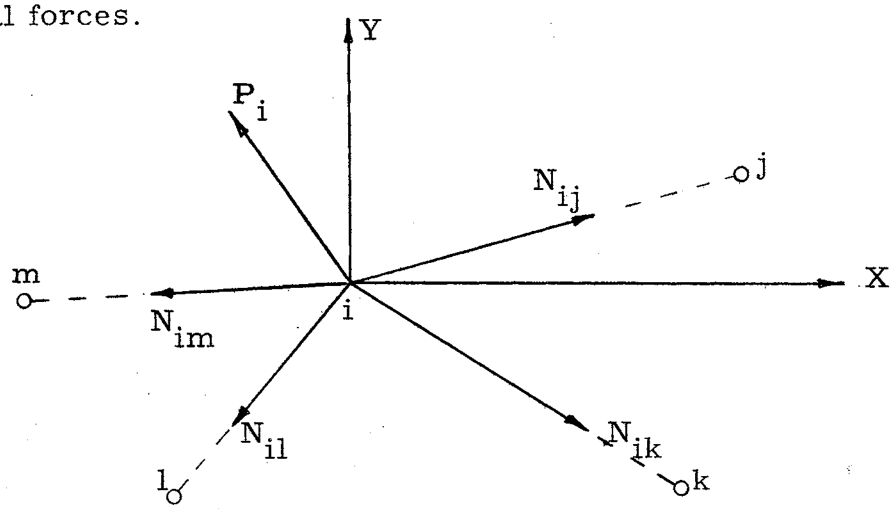


Fig. 2-5

Free Body of Joint i

The components of these forces in terms of parameters α's and β's give two conditions of joint equilibrium.

$$\sum_i F_x = 0 \tag{2-6a}$$

$$N_{ij}\,\alpha_{ij} + N_{ik}\,\alpha_{ik} + N_{il}\,\alpha_{il} + N_{im}\,\alpha_{im} + P_i\,\alpha_{ip} = 0$$

and

$$\sum_i F_y = 0 \tag{2-6b}$$

$$N_{ij}\,\beta_{ij} + N_{ik}\,\beta_{ik} + N_{il}\,\beta_{il} + N_{im}\,\beta_{im} + P_i\,\beta_{ip} = 0$$

where α_{ip} and β_{ip} are $\cos\theta_{ip}$ and $\sin\theta_{ip}$ respectively, and P_i is the resultant of all external forces applied at joint i .

2.3. Stereo-Static Equilibrium Matrix.

Eq's. (2-6a, b) may be written for each joint of the truss and there will be twice as many equilibrium equations as joints. The system of joint equilibrium equations is called hereafter the stereo-static equilibrium matrix. This matrix in full form is:

$$\begin{bmatrix} \Sigma F_{ix} \\ \Sigma F_{jx} \\ \Sigma F_{kx} \\ \vdots \\ \Sigma F_{iy} \\ \Sigma F_{jy} \\ \Sigma F_{ky} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{2-7}$$

and in symbolic form is:

$$\left[\Sigma F\right] = \left[0\right]$$

The stereo-static equilibrium matrix written for a particular truss shown in Fig. 2-6 is recorded in full form in Table 2-1.

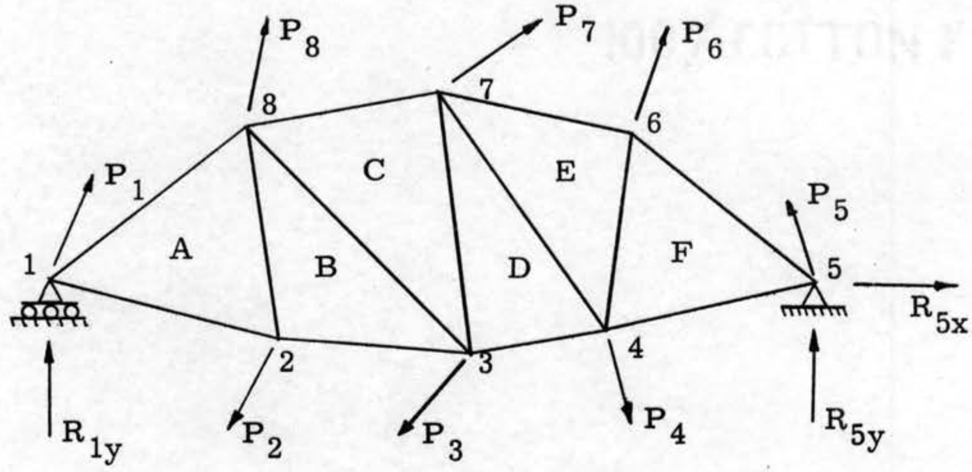


Fig. 2-6

General Simple Truss

From Table 2-1 it becomes evident that the stereo-static equilibrium matrix (Eq. 2-7) consists of two matrix products

$$\left[\omega_N\right]\left[N\right] + \left[\omega_P\right]\left[P\right] = 0 \qquad (2\text{-}8)$$

Inspection of this set of equations shows that there are thirteen unknown internal forces and three unknown reactions. Also, there are two equations for each of the eight joints. With sixteen unknowns and sixteen equations, this system may be solved and the unknowns evaluated.

The truss selected (Fig. 2-6) is consistent with the equation

$$m + 3 = 2j \qquad (2\text{-}9)$$

$$
\underbrace{\begin{bmatrix}
\alpha_{12} &&&&&&& \alpha_{18} & \alpha_{1y} &&&&&&& \\
\alpha_{21} & \alpha_{23} &&&&&&&& \alpha_{28} &&&&&& \\
& \alpha_{32} & \alpha_{34} &&&&&&&& \alpha_{37} &&& \alpha_{38} && \\
&& \alpha_{43} & \alpha_{45} &&&&&&&& \alpha_{46} &&& \alpha_{47} & \\
&&& \alpha_{54} & \alpha_{56} &&&&&&&& \alpha_{5y} &&& \alpha_{5x} \\
&&&& \alpha_{65} & \alpha_{67} &&&&&& \alpha_{64} &&& \alpha_{74} & \\
&&&&& \alpha_{76} & \alpha_{78} &&&& \alpha_{73} &&& \alpha_{83} && \\
&&&&&& \alpha_{87} & \alpha_{81} && \alpha_{82} &&&&&& \\
\beta_{12} &&&&&&& \beta_{18} & \beta_{1y} &&&&&&& \\
\beta_{21} & \beta_{23} &&&&&&&& \beta_{28} &&&&&& \\
& \beta_{32} & \beta_{34} &&&&&&&& \beta_{37} &&& \beta_{38} && \\
&& \beta_{43} & \beta_{45} &&&&&&&& \beta_{46} &&& \beta_{47} & \\
&&& \beta_{54} & \beta_{56} &&&&&&&& \beta_{5y} &&& \beta_{5x} \\
&&&& \beta_{65} & \beta_{67} &&&&&& \beta_{64} &&& \beta_{74} & \\
&&&&& \beta_{76} & \beta_{78} &&&& \beta_{73} &&& \beta_{83} && \\
&&&&&& \beta_{87} & \beta_{81} && \beta_{82} &&&&&&
\end{bmatrix}}_{[\omega_N]}
\underbrace{\begin{bmatrix} N_{12} \\ N_{23} \\ N_{34} \\ N_{45} \\ N_{56} \\ N_{67} \\ N_{78} \\ N_{81} \\ R_{1y} \\ N_{28} \\ N_{37} \\ N_{46} \\ R_{5y} \\ N_{83} \\ N_{74} \\ R_{5x} \end{bmatrix}}_{[N]}
+
\underbrace{\begin{bmatrix}
\alpha_1 &&&&&&&&&&&&&&& \\
& \alpha_2 &&&&&&&&&&&&&& \\
&& \alpha_3 &&&&&&&&&&&&& \\
&&& \alpha_4 &&&&&&&&&&&& \\
&&&& \alpha_5 &&&&&&&&&&& \\
&&&&& \alpha_6 &&&&&&&&&& \\
&&&&&& \alpha_7 &&&&&&&&& \\
&&&&&&& \alpha_8 &&&&&&&& \\
&&&&&&&& \beta_1 &&&&&&& \\
&&&&&&&&& \beta_2 &&&&&& \\
&&&&&&&&&& \beta_3 &&&&& \\
&&&&&&&&&&& \beta_4 &&&& \\
&&&&&&&&&&&& \beta_5 &&& \\
&&&&&&&&&&&&& \beta_6 && \\
&&&&&&&&&&&&&& \beta_7 & \\
&&&&&&&&&&&&&&& \beta_8
\end{bmatrix}}_{[\omega_P]}
\underbrace{\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{bmatrix}}_{[P]}
= 0
$$

TABLE 2-1

STEREO-STATIC EQUILIBRIUM MATRIX - GENERAL SIMPLE TRUSS where:

m = number of members in the system

j = number of joints in the system .

Using matrix algebra and general matrix notations, Eq. (2-8) yields

$$\left[N \right] = - \left[\omega_N \right]^{-1} \left\{ \left[\omega_P \right] \left[P \right] \right\} \tag{2-10}$$

where

$\left[N \right]$ = Column matrix $(2j \times 1)$ giving the 13 internal forces and 3 reactions.

$\left[\omega_N \right]$ = Square direction matrix $(2j \times 2j)$ for internal forces and reactions.

$\left[P \right]$ = Column matrix $(2j \times 1)$ giving all external forces.

$\left[\omega_P \right]$ = Diagonal direction matrix $(2j \times 2j)$ for external forces.

j = Number of joints in the truss system (8 in this case).

The matrix product

$$\left[\omega_N \right]^{-1} \left\{ \left[\omega_P \right] \left[P \right] \right\}$$

is a column matrix composed of the values of each internal force and reaction.

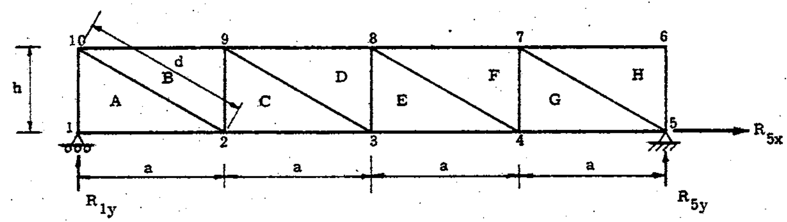
The application of the stereo-static equilibrium matrix is not restricted by the size of the truss system, but rather by the computers capacity to invert the direction matrix.

2.4. Special Cases.

The direction matrix portion of the application of the stereostatic equilibrium matrix to special truss sections is given in Table 2-2, Table 2-3, and Table 2-4.

$C_1 = \frac{a}{d}$

$C_2 = \frac{h}{d}$



10 joints
17 members
3 reactions

GENERAL DIRECTION MATRIX

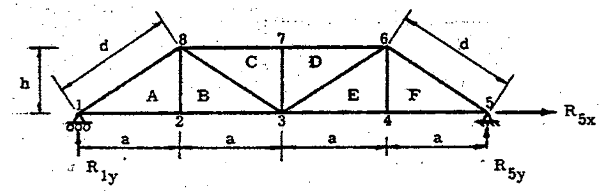
$$
\left[\begin{array}{cccccccccccccccccccc}
\alpha_{1\,2} &&&&&&&&& \alpha_{1\,10} & \alpha_{1\,y} &&&&&&&&& \\
\alpha_{2\,1} & \alpha_{2\,3} &&&&&&&&&& \alpha_{2\,9} &&&& \alpha_{2\,10} &&&& \\
& \alpha_{3\,2} & \alpha_{3\,4} &&&&&&&&&& \alpha_{3\,8} &&&& \alpha_{3\,9} &&& \\
&& \alpha_{4\,3} & \alpha_{4\,5} &&&&&&&&&& \alpha_{4\,7} &&&& \alpha_{4\,8} && \\
&&& \alpha_{5\,4} & \alpha_{5\,6} &&&&&&&&&& \alpha_{5\,4} &&&& \alpha_{5\,7} & \alpha_{5\,x} \\
&&&& \alpha_{6\,5} & \alpha_{6\,7} &&&&&&&&&&&&&& \\
&&&&& \alpha_{7\,6} & \alpha_{7\,8} &&&&&&& \alpha_{7\,4} &&&&& \alpha_{7\,5} & \\
&&&&&& \alpha_{8\,7} & \alpha_{8\,9} &&&&& \alpha_{8\,3} &&&&& \alpha_{8\,4} && \\
&&&&&&& \alpha_{9\,8} & \alpha_{9\,10} &&& \alpha_{9\,2} &&&&& \alpha_{9\,3} &&& \\
&&&&&&&& \alpha_{10\,9} & \alpha_{10\,1} &&&&&& \alpha_{10\,2} &&&& \\
\beta_{1\,2} &&&&&&&&& \beta_{1\,10} & \beta_{1\,y} &&&&&&&&& \\
\beta_{2\,1} & \beta_{2\,3} &&&&&&&&&& \beta_{2\,9} &&&& \beta_{2\,10} &&&& \\
& \beta_{3\,2} & \beta_{3\,4} &&&&&&&&&& \beta_{3\,8} &&&& \beta_{3\,9} &&& \\
&& \beta_{4\,3} & \beta_{4\,5} &&&&&&&&&& \beta_{4\,7} &&&& \beta_{4\,8} && \\
&&& \beta_{5\,4} & \beta_{5\,6} &&&&&&&&&& \beta_{5\,4} &&&& \beta_{5\,7} & \beta_{5\,x} \\
&&&& \beta_{6\,5} & \beta_{6\,7} &&&&&&&&&&&&&& \\
&&&&& \beta_{7\,6} & \beta_{7\,8} &&&&&&& \beta_{7\,4} &&&&& \beta_{7\,5} & \\
&&&&&& \beta_{8\,7} & \beta_{8\,9} &&&&& \beta_{8\,3} &&&&& \beta_{8\,4} && \\
&&&&&&& \beta_{9\,8} & \beta_{9\,10} &&& \beta_{9\,2} &&&&& \beta_{9\,3} &&& \\
&&&&&&&& \beta_{10\,9} & \beta_{10\,1} &&&&&& \beta_{10\,2} &&&&
\end{array}\right]
$$

SIMPLIFIED DIRECTION MATRIX

$$
\left[\begin{array}{cccccccccccccccccccc}
1 &&&&&&&&& 0 & 0 &&&&&&&&& \\
-1 & 1 &&&&&&&&&& 0 &&&& -C_1 &&&& \\
& -1 & 1 &&&&&&&&&& 0 &&&& -C_1 &&& \\
&& -1 & 1 &&&&&&&&&& 0 &&&& -C_1 && \\
&&& -1 & 0 &&&&&&&&&& 0 &&&& -C_1 & 1 \\
&&&& 0 & -1 &&&&&&&&&&&&&& \\
&&&&& 1 & -1 &&&&&&& 0 &&&&& +C_1 & \\
&&&&&& 1 & -1 &&&&& 0 &&&&& +C_1 && \\
&&&&&&& 1 & -1 &&& 0 &&&&& +C_1 &&& \\
&&&&&&&& 1 & 0 &&&&&& +C_1 &&&& \\
0 &&&&&&&&& 1 & -1 &&&&&&&&& \\
0 & 0 &&&&&&&&&& 1 &&&& +C_2 &&&& \\
& 0 & 0 &&&&&&&&&& 1 &&&& +C_2 &&& \\
&& 0 & 0 &&&&&&&&&& 1 &&&& +C_2 && \\
&&& 0 & 1 &&&&&&&&&& -1 &&&& +C_2 & 0 \\
&&&& -1 & 0 &&&&&&&&&&&&& -C_2 & \\
&&&&& 0 & 0 &&&&&&& -1 &&&& -C_2 && \\
&&&&&& 0 & 0 &&&&& -1 &&&& -C_2 &&& \\
&&&&&&& 0 & 0 &&& -1 &&&& -C_2 &&&& \\
&&&&&&&& 0 & -1 &&&&&&&&&&
\end{array}\right]
$$

TABLE 2-2

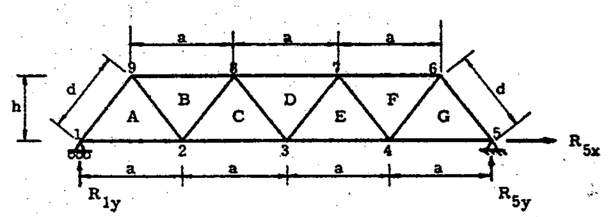
DIRECTION MATRIX



$C_1 = \frac{a}{d} = C_2$ $\qquad$ $C_3 = \frac{h}{d} = C_4$

	12	23	34	45	56	78	81	1y	28	37	46	5y	38	36	5x
1	1						$+C_2$	0							
2	-1	1							0						
3		-1	1							0			$-C_1$	$+C_2$	
4			-1	1							0				
5				-1	$+C_1$							0			1
6					$-C_1$	-1					0			$-C_2$	
7						1	-1			0					
8							1	$-C_2$	0				$+C_1$		
1	$+C_4$							$+C_4$	-1						
2	0	0								1					
3		0	0								1		$+C_3$	$+C_4$	
4			0	0								1			
5				0	$+C_3$								-1		0
6					$-C_3$	0						-1			
7						0	0				-1			$-C_4$	
8							0	$-C_4$		-1			$-C_3$		

TABLE 2-3



$C_1 = \frac{a}{d} = C_2$ $\qquad$ $C_3 = \frac{h}{d} = C_4$

	12	23	34	45	56	67	78	89	91	1y	5y	29	28	38	37	47	46	5x
1	1								$+C_2$	0								
2	-1	1										$-C_1$	$+C_2$					
3		-1	1											$-C_1$	$+C_2$			
4			-1	1												$-C_1$	$+C_2$	
5				-1	$+C_1$						0							1
6					$-C_1$	-1											$-C_2$	
7						1	-1								$-C_2$	$+C_1$		
8							1	-1					$-C_2$	$+C_1$				
9								1	$-C_2$			$+C_1$						
1	0								$+C_4$	-1								
2	0	0										$+C_3$	$+C_4$					
3		0	0											$+C_3$	$+C_4$			
4			0	0												$+C_3$	$+C_4$	
5				0	$+C_3$						-1							0
6					$-C_3$	0											$-C_4$	
7						0	0								$-C_4$	$-C_3$		
8							0	0					$-C_4$	$-C_3$				
9								0	$-C_4$			$-C_3$						

TABLE 2-4

SIMPLIFIED DIRECTION MATRIX

CHAPTER III

ELASTO-STATICS

3-1. Geometry.

The simple truss shown in Fig. 2-6 is considered, and the axial forces and reactions are assumed to be known as a result of Eq. (2-10). Each cell is separated as shown in Fig. 3-1. One of these cells is shown in Fig. 3-2, and its geometry is discussed in the following paragraph. For the sake of generality, the joints of the cell are designated by i , j , k and the cell is denoted by the symbol H .

From Eq's. (2-3) the components of the bars ij , jk , and ki are:

$$x_{ij} = x_{0j} - x_{0i} \qquad y_{ij} = y_{0j} - y_{0i}$$
$$x_{jk} = x_{0k} - x_{0j} \qquad y_{jk} = y_{0k} - y_{0j} \qquad (3\text{-}1)$$
$$x_{ki} = x_{0i} - x_{0k} \qquad y_{ki} = y_{0i} - y_{0k} \;.$$

3-2. Elasto-Static Equilibrium.

Due to the deformation of truss members, changes will occur in the angle of intersection of the members making up each cell. These "angle changes" may be represented by joint elastic weights. Once more the objective of this study is to find expressions for these elastic weights.

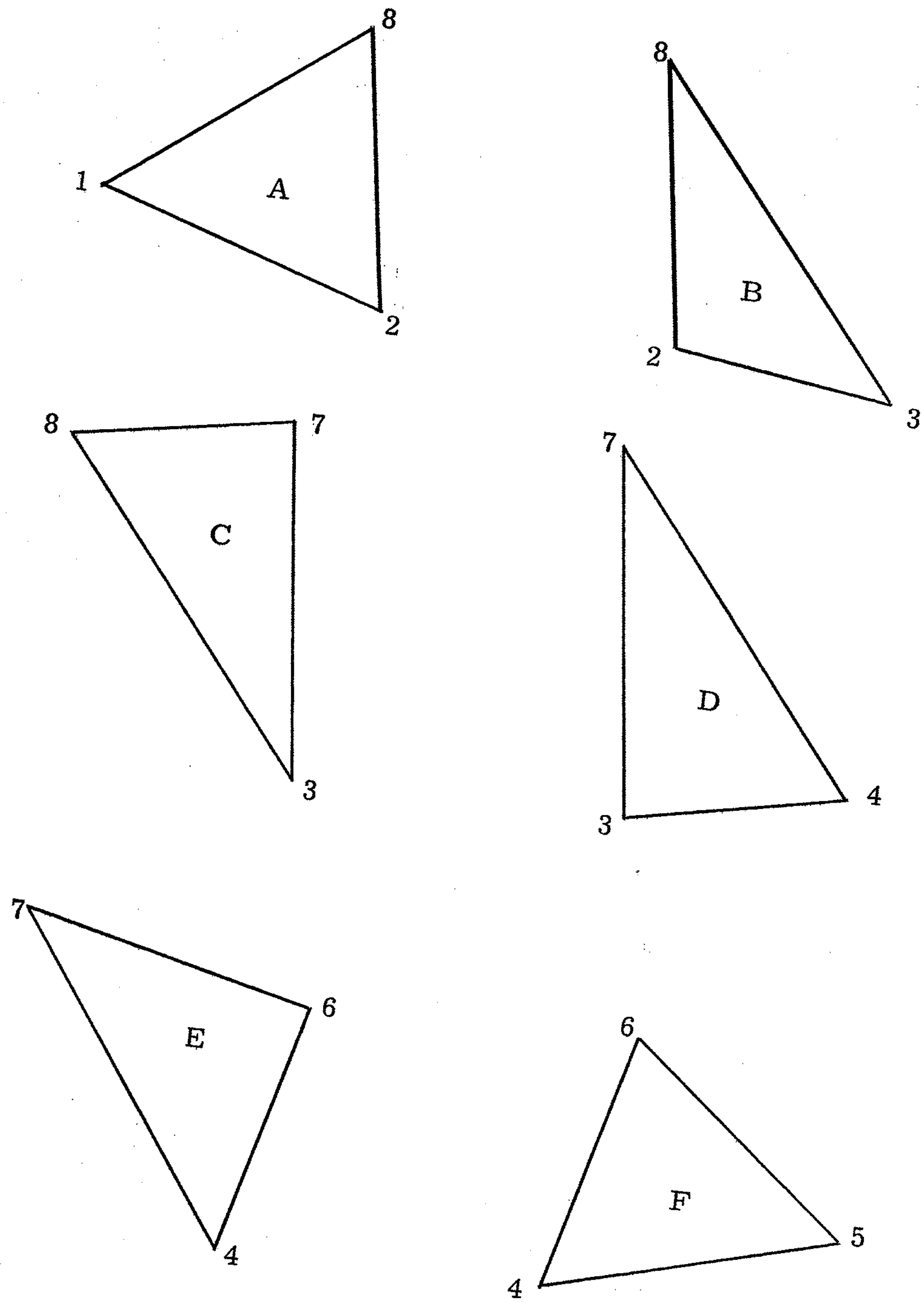


Fig. 3-1

Isolated Cells

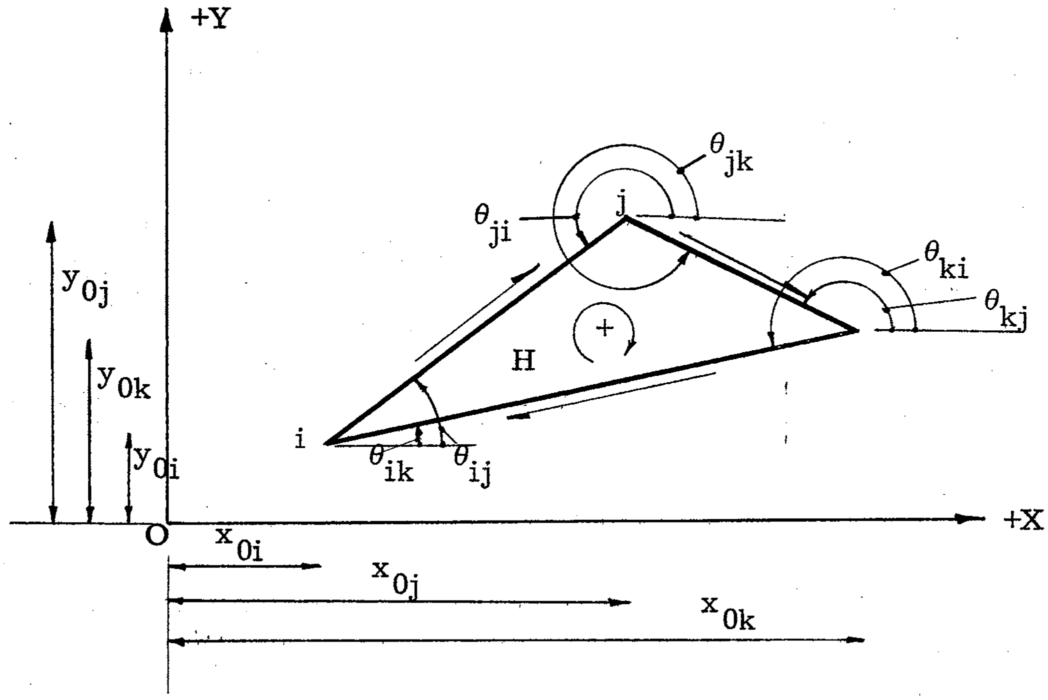


Fig. 3-2

Cell H of Any Truss

Stating Hooke's Law in the form:

$$\Delta = \frac{NL}{AE} \tag{3-2}$$

or

$$\Delta = N\lambda \tag{3-3}$$

where A is the cross-sectional area of the bar; E is the modulus of elasticity; Δ is the total change in length of the member; N is the normal force in the member (Eq. 2-10); and λ is the axial flexibility of the member. Applying this equation to the truss in Fig. 2-6 and stating it in matrix form, Eq. (3-3) becomes

$$\left[\Delta\right] = \left[\lambda\right]\left[N\right] \tag{3-4}$$

where

$[\Delta]$ = Column matrix $(2j \times 1)$ giving the change in length of each member in the truss.

$[N]$ = Column matrix $(2j \times 1)$ giving the normal force in each member.

$[\lambda]$ = Diagonal flexibility matrix $(2j \times 2j)$ giving the axial flexibility of each member.

Representing the conjugate structure as a closed string polygon and knowing the axial elongation of each member, the conjugate cell may be loaded as shown in Fig. 3-3. The moment vectors (double headed vectors) on the conjugate cell are the axial elongation of the the members of the real cell, and the force vectors are the joint elastic weights due to deformation of the members of cell "H" .

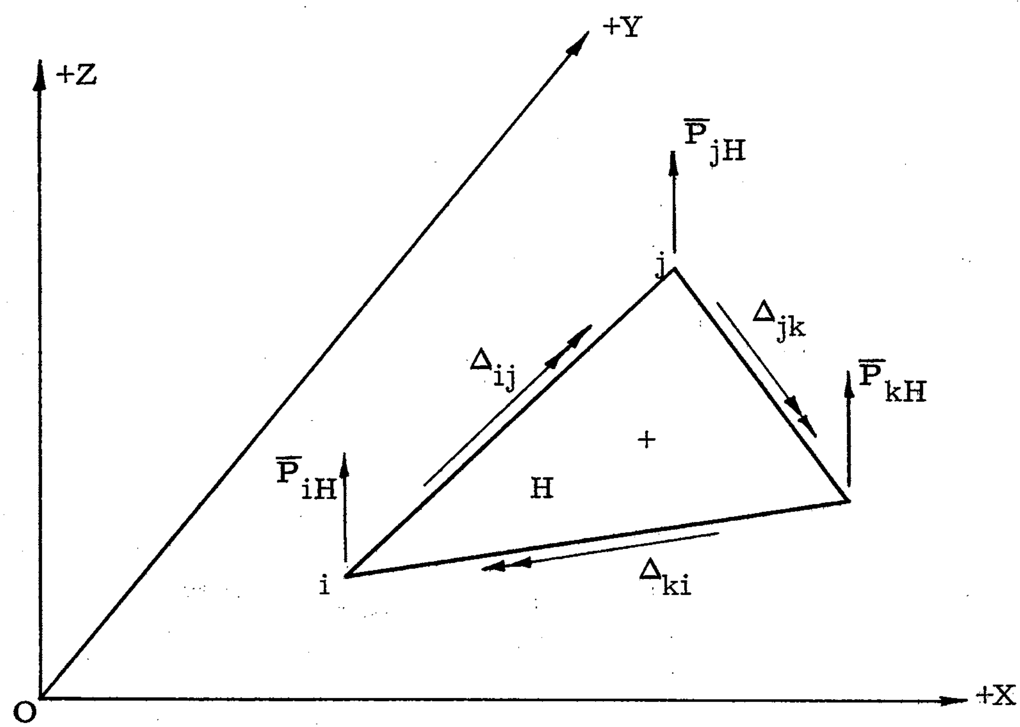


Fig. 3-3

Conjugate Cell H

Denoting the hollow circle "O" as a vector perpendicular to the X-Y plane in the direction of the positive Z-axis, this conjugate cell may be shown in the X-Y plane (Fig. 3-4).

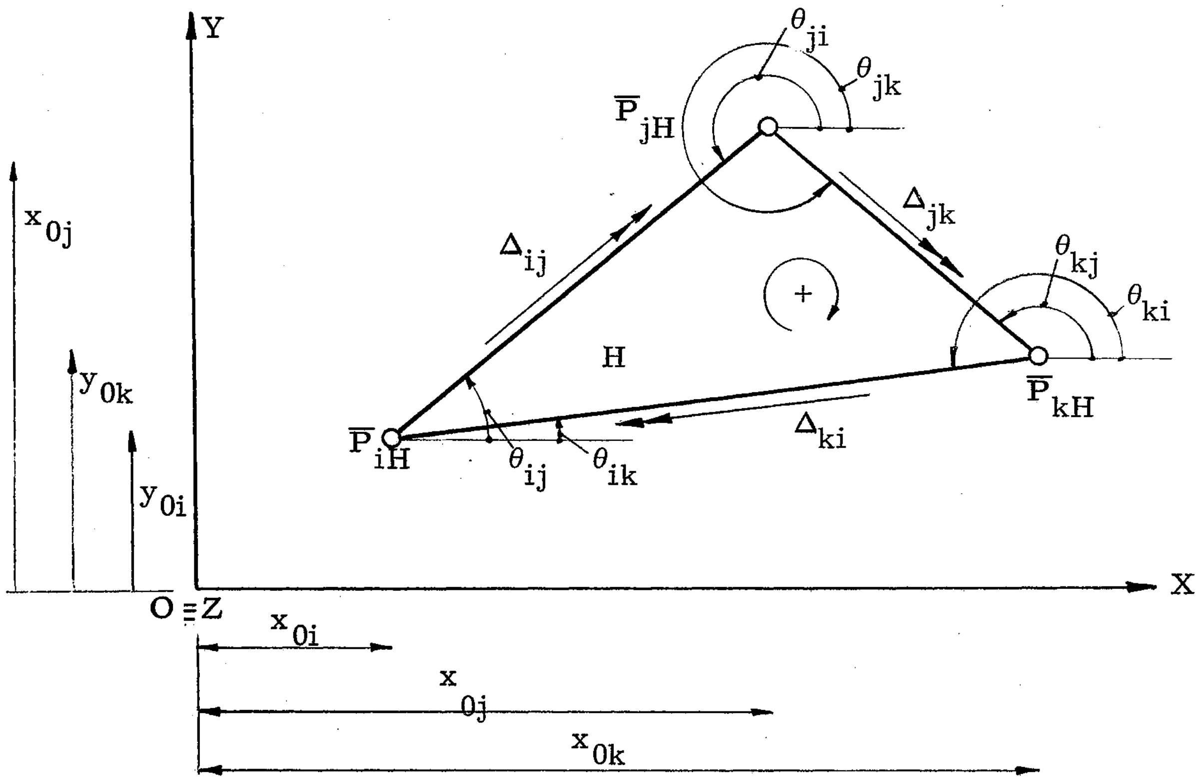


Fig. 3-4

Conjugate Cell H

Elasto-static equilibrium of this conjugate cell will give the unknown joint elastic weights. Since this is a space problem of second order, i.e., a structure lying in a plane and loaded out of plane, the elasto-static equilibrium equations are:

$$\Sigma\overline{M}_x = 0 \tag{3-5a}$$

$$\alpha_{ij}\,\Delta_{ij} + \alpha_{jk}\,\Delta_{jk} + \alpha_{ki}\,\Delta_{ki} + \overline{P}_{iH}\,y_{0i} + \overline{P}_{jH}\,y_{0j} + \overline{P}_{kH}\,y_{0k} = 0$$

$$\Sigma \overline{M}_y = 0 \tag{3-5b}$$

$$\beta_{ij} \Delta_{ij} + \beta_{jk} \Delta_{jk} + \beta_{ki} \Delta_{ki} - \overline{P}_{iH} x_{0i} - \overline{P}_{jH} x_{0j} - \overline{P}_{kH} x_{0k} = 0$$

$$\Sigma \overline{F}_z = 0 \tag{3-5c}$$

$$\overline{P}_{iH} + \overline{P}_{jH} + \overline{P}_{kH} = 0$$

where

$\overline{P}_{iH}$ = Joint elastic weight (angle change) for joint i of cell H

$\overline{P}_{jH}$ = Joint elastic weight (angle change) for joint j of cell H

$\overline{P}_{kH}$ = Joint elastic weight (angle change) for joint k of cell H

and the other terms were defined earlier.

3-3. Elasto-Static Equilibrium Matrix

In matrix form, Eq's. (3-5) become:

$$\begin{bmatrix} \alpha_{ij} & \alpha_{jk} & \alpha_{ki} \\ \beta_{ij} & \beta_{jk} & \beta_{ki} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{ij} \\ \Delta_{jk} \\ \Delta_{ki} \end{bmatrix} + \begin{bmatrix} y_{0i} & y_{0j} & y_{0k} \\ x_{0i} & x_{0j} & x_{0k} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \overline{P}_{iH} \\ \overline{P}_{jH} \\ \overline{P}_{kH} \end{bmatrix} = 0 \tag{3-6}$$

or in matrix shorthand and denoting the matrices in the same order

$$\left[\omega_H\right]\left[\Delta_H\right] + \left[r_{0H}\right]\left[\overline{P}_H\right] = 0 \quad . \tag{3-7}$$

Again using the standard matrix notations:

$$\left[\overline{P}_H\right] = - \left[r_{0H}\right]^{-1} \left\{\left[\omega_H\right]\left[\Delta_H\right]\right\} \quad . \tag{3-8}$$

This matrix equation gives the joint elastic weights for cell H shown in Figs. 3-3 and 3-4. Inspection of these figures and the arm matrix $\left[r_{0H}\right]$ which is shown again here

$$\left[r_{0H}\right] = \begin{bmatrix} +y_{0i} & +y_{0j} & +y_{0k} \\ -x_{0i} & -x_{0j} & -x_{0k} \\ 1 & 1 & 1 \end{bmatrix}$$

shows that this arm matrix may be simplified if the origin is selected to coincide with one of the joints, say joint i. The simplification is, of course, a result of the fact that $x_{ii} = 0$ and $y_{ii} = 0$ (Fig. 3-5). Elasto-static equilibrium of this conjugate cell gives:

$$\left[\overline{\Sigma}\right] = \left[0\right]$$

where

$$\left[\overline{\Sigma}\right] = \begin{bmatrix} \Sigma\overline{M}_x \\ \Sigma\overline{M}_y \\ \Sigma\overline{F}_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\left[\overline{P}_H\right] = -\left[r_{iH}\right]^{-1}\left\{\left[\omega_H\right]\left[\Delta_H\right]\right\} \qquad (3\text{-}9a)$$

where now

$$\left[r_{iH}\right] = \begin{bmatrix} 0 & +y_{ij} & +y_{ik} \\ 0 & -x_{ij} & -x_{ik} \\ 1 & 1 & 1 \end{bmatrix} \qquad (3\text{-}9b)$$

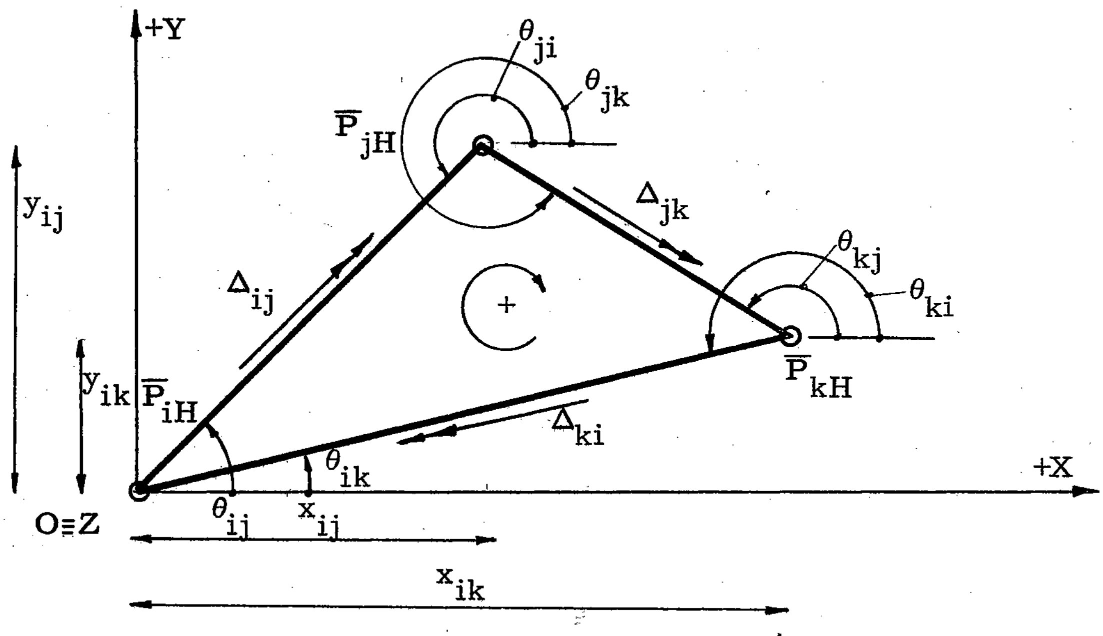


Fig. 3-5

Cell H With Origin at Joint i

and

$$\left[r_{iH} \right]^{-1} = \frac{\begin{bmatrix} -(x_{ij} - x_{ik}) & -(y_{ij} - y_{ik}) & D_{iH} \\ -x_{ik} & -y_{ik} & 0 \\ +x_{ij} & +y_{ij} & 0 \end{bmatrix}}{D_{iH}}$$

$$D_{iH} = x_{ij}\, y_{ik} - x_{ik}\, y_{ij} \tag{3-9c}$$

= Determinate of arm matrix for cell H (called hereafter D_H) .

Substituting from Eq.(3-1)

$$\left[r_H \right]^{-1} = \frac{\begin{bmatrix} x_{jk} & y_{jk} & D_H \\ x_{ki} & y_{ki} & 0 \\ x_{ij} & y_{ij} & 0 \end{bmatrix}}{D_H} \quad . \tag{3-10}$$

3-4. Total Joint Elastic Weights.

Writing out in matrix longhand and using Eq. (3-10), Eq. (3-9) gives the analytical expressions for the joint elastic weights of cell H . These joint elastic weights are expressed in terms of the coordinates of their location and the axial elongation of the members of the cell.

$$\begin{bmatrix} \overline{P}_{iH} \\ \overline{P}_{jH} \\ \overline{P}_{kH} \end{bmatrix} = - \frac{1}{D_H} \begin{bmatrix} x_{jk} & y_{jk} & D_H \\ x_{ki} & y_{ki} & 0 \\ x_{ij} & y_{ij} & 0 \end{bmatrix} \begin{bmatrix} \alpha_{ij} & \alpha_{jk} & \alpha_{ki} \\ \beta_{ij} & \beta_{jk} & \beta_{ki} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{ij} \\ \Delta_{jk} \\ \Delta_{ki} \end{bmatrix} \tag{3-11}$$

Performing the matrix multiplication and writing the joint elastic weights for the cell, the joint elastic weight for say joint i becomes:

$$\begin{aligned} \overline{P}_{iH} = & - \left[x_{jk}\, \alpha_{ij} + y_{jk}\, \beta_{ij} \right] \frac{\Delta_{ij}}{D_H} \\ & - \left[x_{jk}\, \alpha_{jk} + y_{jk}\, \beta_{jk} \right] \frac{\Delta_{jk}}{D_H} \\ & - \left[x_{jk}\, \alpha_{ki} + y_{jk}\, \beta_{ki} \right] \frac{\Delta_{ki}}{D_H} \quad . \end{aligned} \tag{3-12}$$

Rearranging the terms

$$\overline{P}_{iH} = -\frac{x_{jk}}{D_H}\left[\Delta_{ij}\,\alpha_{ij} + \Delta_{jk}\,\alpha_{jk} + \Delta_{ki}\,\alpha_{ki}\right]$$

$$-\frac{y_{jk}}{D_H}\left[\Delta_{ij}\,\beta_{ij} + \Delta_{jk}\,\beta_{jk} + \Delta_{ki}\,\beta_{ki}\right] . \qquad (3\text{-}13)$$

It should be noted that the expressions in the brackets represent the horizontal and vertical deformations of the respective bars.

$$\Sigma\,\Delta_{Hx} = \Delta_{ijx} + \Delta_{jkx} + \Delta_{kix}$$

$$= \text{Summation of } \Delta\text{'s in the x-direction of cell H}$$

$$\Sigma\,\Delta_{Hy} = \Delta_{ijy} + \Delta_{jky} + \Delta_{kiy} \qquad (3\text{-}14)$$

$$= \text{Summation of } \Delta\text{'s in the y-direction of cell H}$$

with this notation

$$\overline{P}_{iH} = -\frac{x_{jk}}{D_H}\,\Sigma\,\Delta_{Hx} - \frac{y_{jk}}{D_H}\,\Sigma\,\Delta_{Hy} . \qquad (3\text{-}15a)$$

Denoting

$$D'_{Hx} = -\frac{\Sigma\,\Delta_{Hx}}{D_H} \qquad (3\text{-}15b)$$

$$D'_{Hy} = -\frac{\Sigma\,\Delta_{Hy}}{D_H} \qquad (3\text{-}15c)$$

the joint elastic weight for joint i of cell H becomes

$$\overline{P}_{iH} = x_{jk}\,D'_{Hx} + y_{jk}\,D'_{Hy} . \qquad (3\text{-}16a)$$

Similar operations with the balance of Eq. (3-11) and

$$\overline{P}_{jH} = x_{ki}\,D'_{Hx} + y_{ki}\,D'_{Hy} \qquad (3\text{-}16b)$$

$$\overline{P}_{kH} = x_{ij}\,D'_{Hx} + y_{ij}\,D'_{Hy} . \qquad (3\text{-}16c)$$

The expressions for the joint elastic weights of cell H given by Eq's. (3-16a, b, c) are perfectly general and can be used for the calculation of joint elastic weights for any cell of the simple truss shown in Fig. 2-6. Because each joint elastic weight represents the angle change of the cell at a particular point, the sum of joint elastic weights of two or more adjacent panels may be denoted as the total joint elastic weight and interpreted as the total angle change at that particular joint. This statement may be interpreted graphically and analytically. If three adjacent cells shown in Fig. 3-6 are considered, the joint

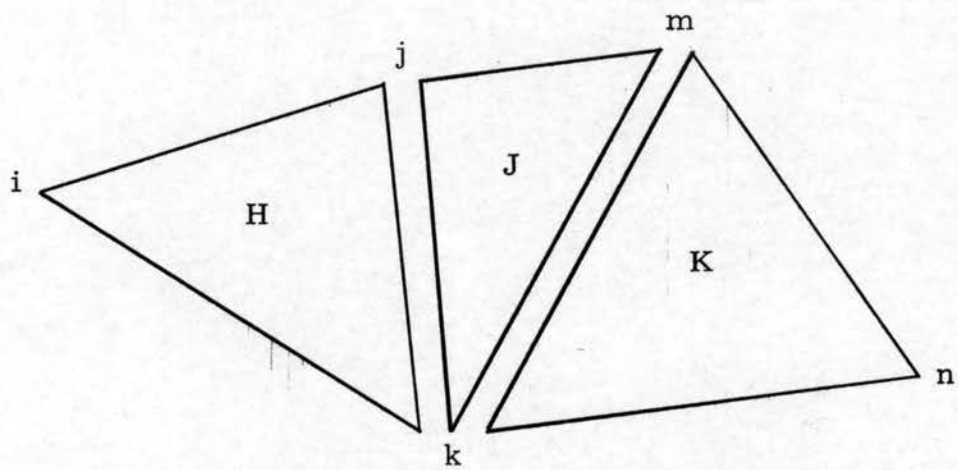


Fig. 3-6

Three Adjacent Cells H , J , K

elastic weight of each cell at joint k in terms of Eq's. (3-16) is

$$\begin{aligned} \overline{P}_{kH} &= x_{ij} D'_{Hx} + y_{ij} D'_{Hy} \\ \overline{P}_{kJ} &= x_{jm} D'_{Jx} + y_{jm} D'_{Jy} \\ \overline{P}_{kK} &= x_{mn} D'_{Kx} + y_{mn} D'_{Ky} \quad . \end{aligned} \tag{3-17}$$

They all represent the angle change at k and the total angle change at k is

$$\begin{aligned} \overline{P}_{k} &= \Sigma \, \overline{P}_{k} \\ &= \overline{P}_{kH} + \overline{P}_{kJ} + \overline{P}_{kK} \quad . \end{aligned} \tag{3-18}$$

Because each joint elastic weight is a normal vector to the plane X-Y at k the total joint elastic weight at k is the sum of these vectors.

3-5. Truss String Polygon.

The deformation of each cell has been represented by a triangular conjugate structure acted on by conjugate moments (axial deformations) and conjugate forces (angular deformations). The relationship between deformation and geometry of the structure has been explained by a conjugate analogy in which the conjugate forces have been denoted as elastic weights and the conjugate moments as elastic moments. It has been shown that the conjugate forces and moments are forming a system of elasto-static equilibrium (Eq's. 3-5a, b, c). Because each conjugate cell is in a state of elasto-static equilibrium, the whole structure or any part of it must be in a state of elasto-static equilibrium. Consequently, we can now combine two or any number of cells (by joining them together) into a conjugate system which takes the shape of a conjugate polygon. Two particular features of this conjugate polygon become apparent. The conjugate moments on adjacent lines will cancel each other and the conjugate moments on the circumference will remain in action. Second, the joint elastic weights at adjacent points of application will sum and form the total joint elastic weight.

These statements may now be illustrated by a few typical examples. If the deformation of the truss shown in Fig. 2-6 is represented by conjugate cells A , B , C , D , E , F (Fig. 3-7), six isolated conjugate systems in a state of elasto-static equilibrium are available. From these six conjugate cells by successive addition a large number

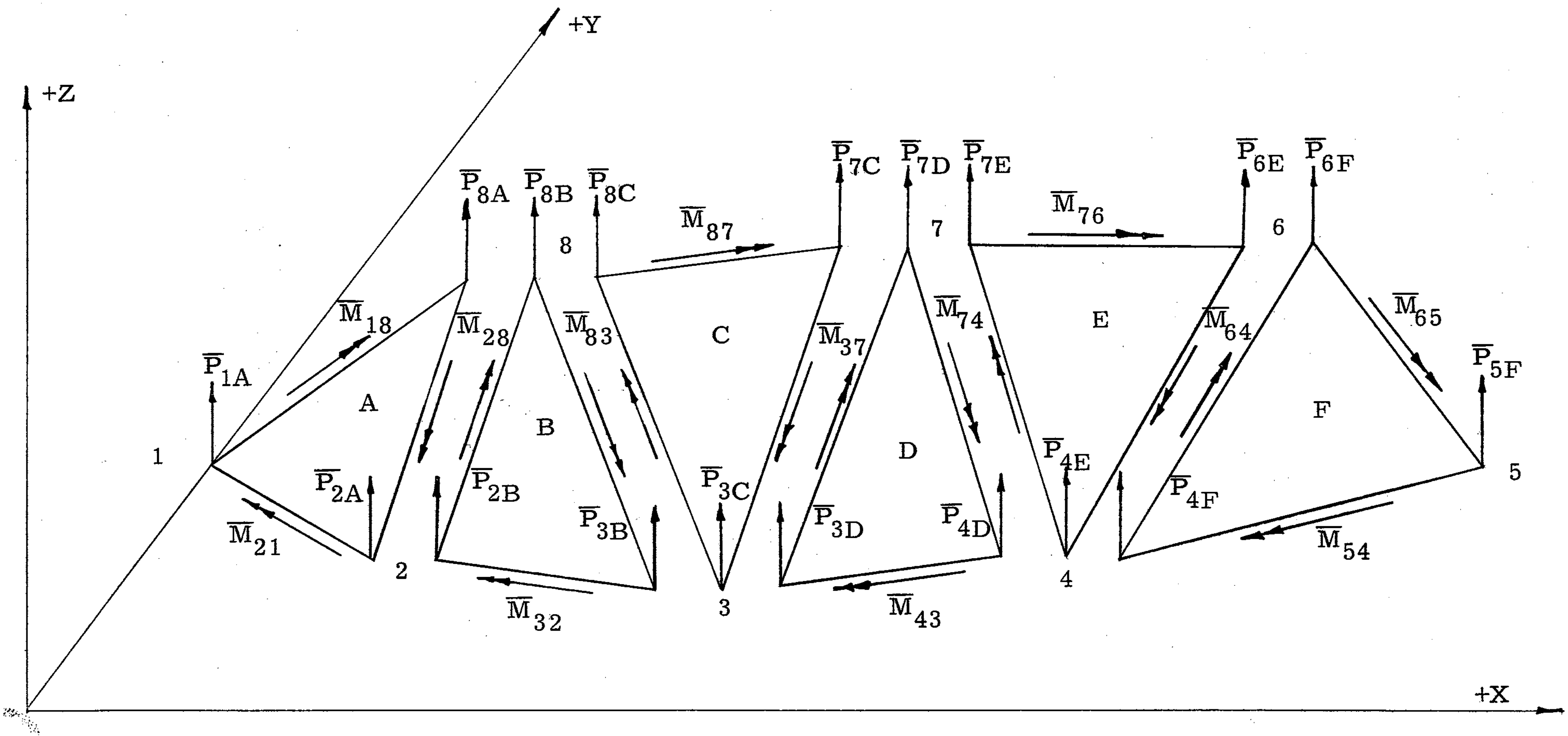


Fig. 3-7

Conjugate Cells

of composite structures can be produced. For example, if cells A and B are combined as shown in Fig. 3-8a the conjugate system 1832 is derived with conjugate moments on line 2-8 eliminated. If A , B , and C are combined, the elimination of conjugate moments will be done on lines 2-8 and 3-8. This process may be repeated for the combination of cells A , B , C , D and A , B , C , D , E and A , B , C , D , E , F . This last formation is the total conjugate system (Fig. 3-8e). The initial form of this system is a string polygon which can be denoted as a geometric string polygon. After the deformation takes place, the geometric string polygon becomes the deformation polygon (all joints will take new positions, all sides will elongate or contract). Because the sides of the deformation polygon remain straight, the term "String Polygon" can be introduced again as in previous investigations (Tuma, 7; Oden, 10; Wu, 11).

It was said before that this new system represented as a string polygon with elastic weights at joints and elastic moments on the circumference is in a state of elasto-static equilibrium and consequently is itself in equilibrium (no reaction required). From the experience with the string polygon discussed by Tuma (7), Oden (10), and Wu (11), it becomes apparent that the slope of the real structure becomes the shear of the conjugate structure, and the deformation of the real structure becomes the moment on the conjugate structure. These theorems can be utilized for calculation of displacements of the real structure or redundants in indeterminant trusses.

To illustrate the application of these principles, two cases will be considered. The conjugate structure of Fig. 3-8e is resolved into two free body sketches as shown in Fig. 3-9. The bent line 18765 is

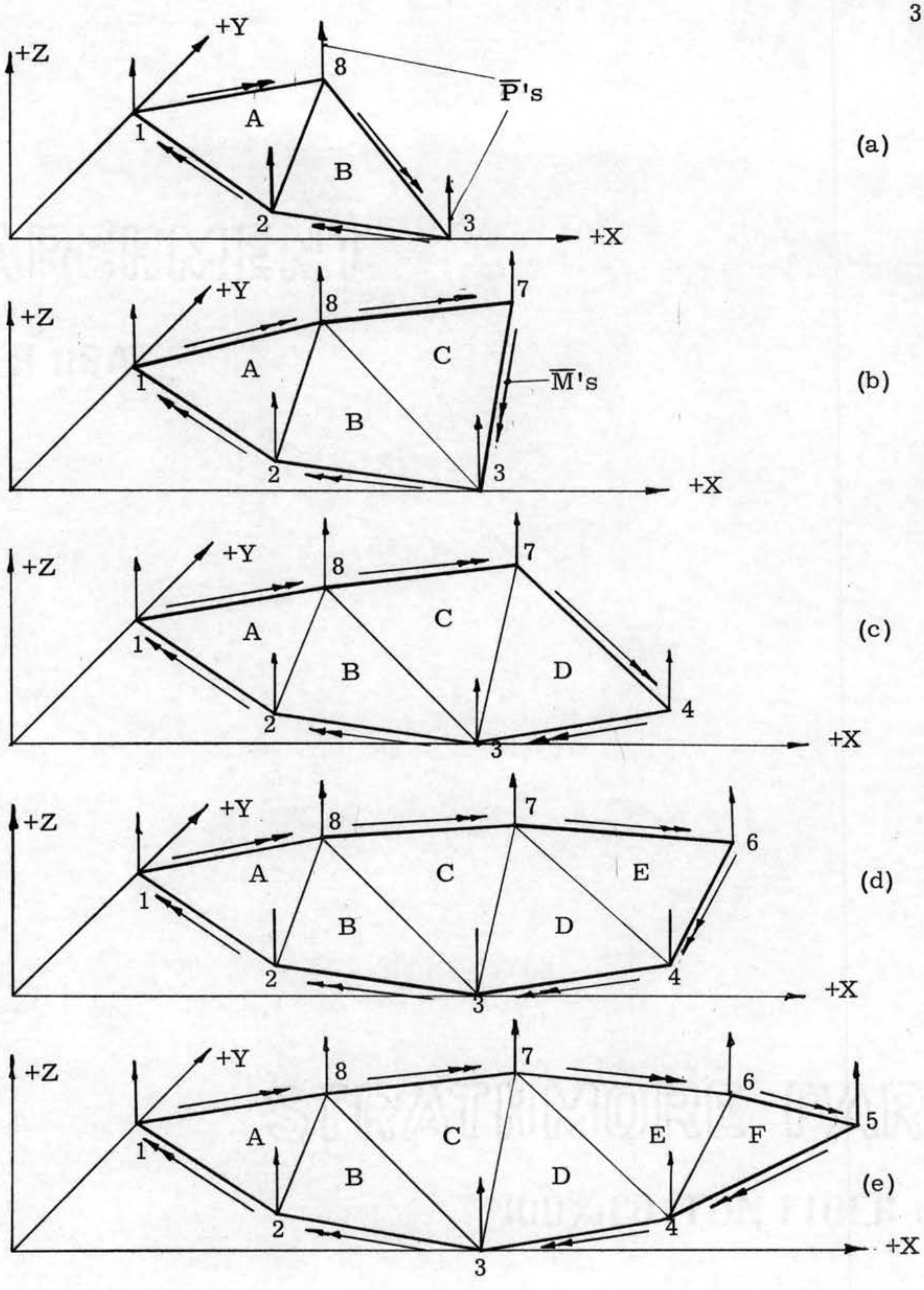


Fig. 3-8

Combination of Conjugate Cells

one free body and the bent line 12345 is the other.

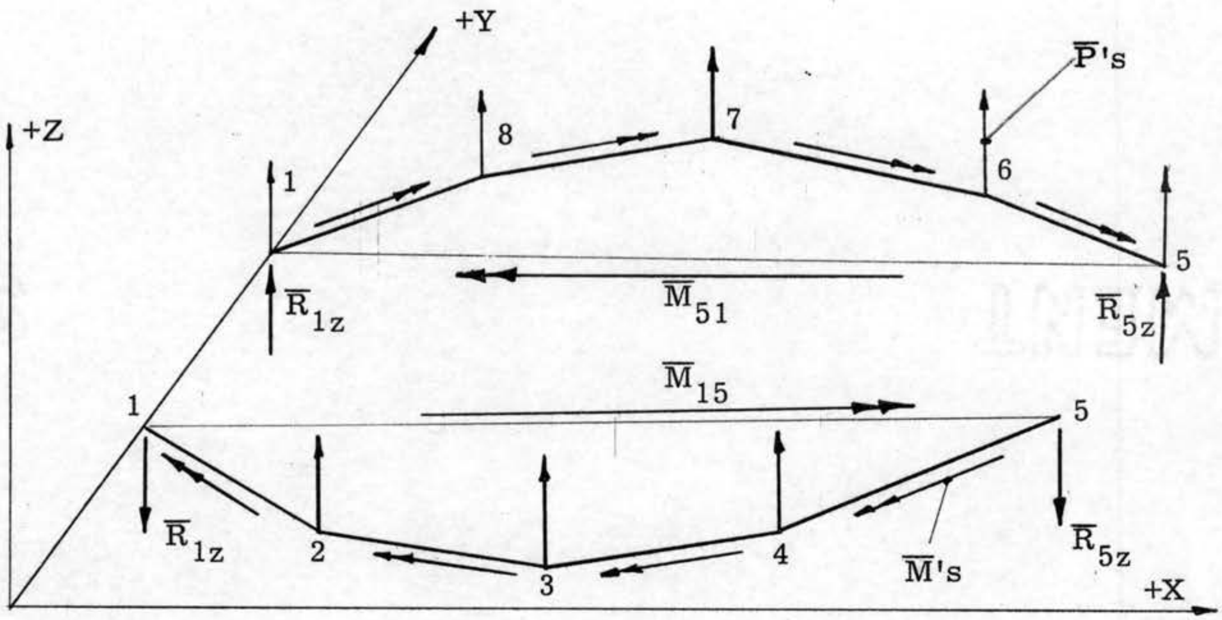


Fig. 3-9

Free Body of Conjugate Structure

The conjugate loads of both free body sketches are well defined by equations in preceding discussions. The cross-sectional elements on each branch are unknown quantities which, however, must satisfy elasto-static equilibrium on each branch and can be easily calculated from equations of elasto-static equilibrium and from either free body sketch. Because line 1-5 is a line connecting the points of supports, the cross-sectional elements are also giving the deformation of the real structure related to this base line.

The second case which is of even more importance is the resolution of the conjugate structure along a line which does not pass through points of support (Fig. 3-10). The bent lines 21876 and 23456 are the

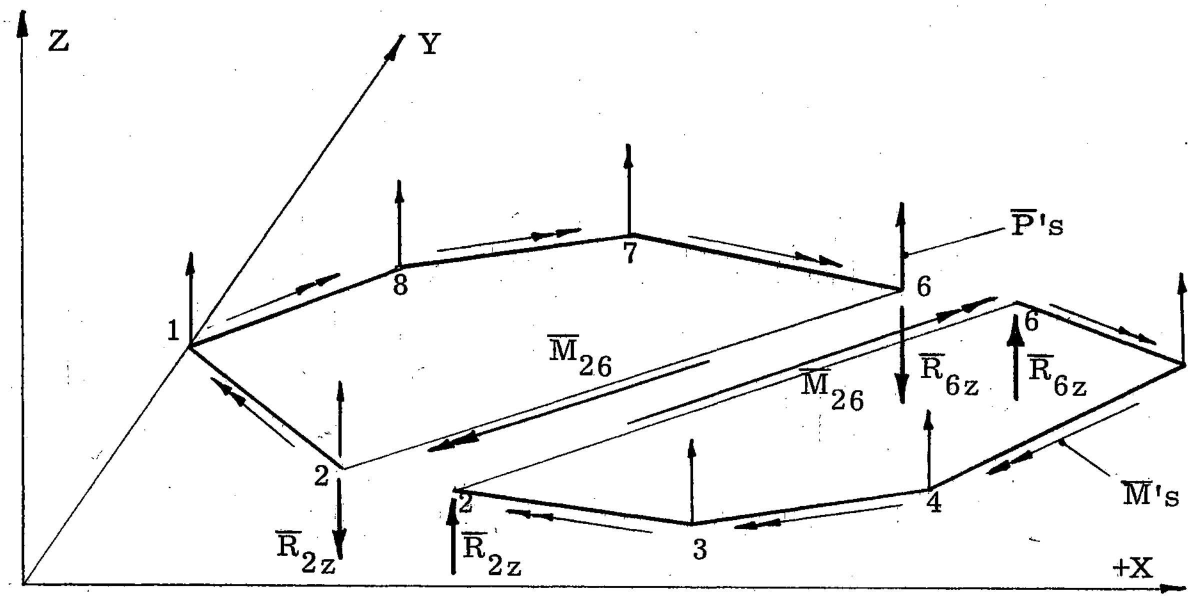


Fig. 3-10

Free Body of Conjugate Structure

two free body sketches. Again the cross-sectional elements are unknown quantities and satisfy elasto-static equilibrium on either branch. Because line 2-6 is a line connecting points that are not supported, i.e., the points may deflect, the cross-sectional elements are giving the relative deformation of the real structure with respect to this base line.

The application of stereo-static equilibrium, elasto-static equilibrium, and truss string polygon is shown in the last chapter of this thesis.

CHAPTER IV

APPLICATION AND CONCLUSION

4-1. Procedure of Analysis.

The general analysis for the calculation of elastic weights is divided into three parts: geometry, stereo-statics, and elasto-statics. The procedure is set forth in the following steps:

Part 1. Geometry.

(a) All joints are designated by Arabic numbers and each cell is denoted by a capital letter.

(b) The length, slope, and the component of the length of each member is calculated.

(c) Loads at each joint are resolved into one horizontal and one vertical component.

Part 2. Stereo-Statics.

(a) The axial forces in all members and the reactive forces at all points of support are designated by symbols "N" and "R" respectively and are introduced as unknowns.

(b) The stereo-static equilibrium conditions must be satisfied at each joint and written in terms of the stereo-static equilibrium matrix (Eq. 2-8 or Tables 2-1, 2, 3, 4).

(c) The stereo-static equilibrium matrix is solved by matrix inversion and for the unknown values of N and R (Eq. 2-10).

Part 3. Elasto-Statics.

(a) The axial deformation of all members is calculated by means of the Hooke law (Eq. 3-4).

(b) The determinant D of each cell is calculated with the origin of coordinates of joints of the cell at any one of the joints (Eq. 3-9c).

(c) The components of deformation of all members are calculated and summed for each cell (Eq. 3-14).

(d) The numerical constants D'_x and D'_y are computed for each cell (Eq. 3-15b, c) and substituted into Eq. (3-16a, b, c), from which the elastic weights are calculated. It should be noted that the expressions x_{ij} , y_{ij} ; x_{jk} , y_{jk} ; and x_{ki} , y_{ki} are geometric values with their corresponding signs.

(e) Once the joint elastic weights are known for all cells of the structure, the conjugate structure of the form of a string polygon is selected. This can be done in many different ways, but in any case it is desirable that the conjugate structure start at a point of zero linear displacement and finish at an arbitrary point with known or unknown deformation conditions.

(f) Once the conjugate structure is selected, the total joint elastic weights are calculated by means of Eq. (3-18) and these loads must be applied as force vectors normal to the plane of the structure at the respective joints. The axial deformations in a form of moment vectors must be applied along the sides of the polygon in the positive direction (Fig. 3-5).

(g) The conjugate reactions of the conjugate structure are calculated by means of elasto-static equilibrium and the cross-sectional elements of the conjugate structure are obtained from the same equation.

(h) The real deformations are equal to their conjugate equivalents.

4-2. Example No. 1.

The procedure above is illustrated in the following problem. The Warren truss, loaded as shown in Fig. 4-1, is given. It is required to evaluate the vertical deflection of the lower chord. The cross-sectional area of the members is either known or assumed and is recorded on each member (Fig. 4-1). The truss is statically determinate (Eq. 2-9).

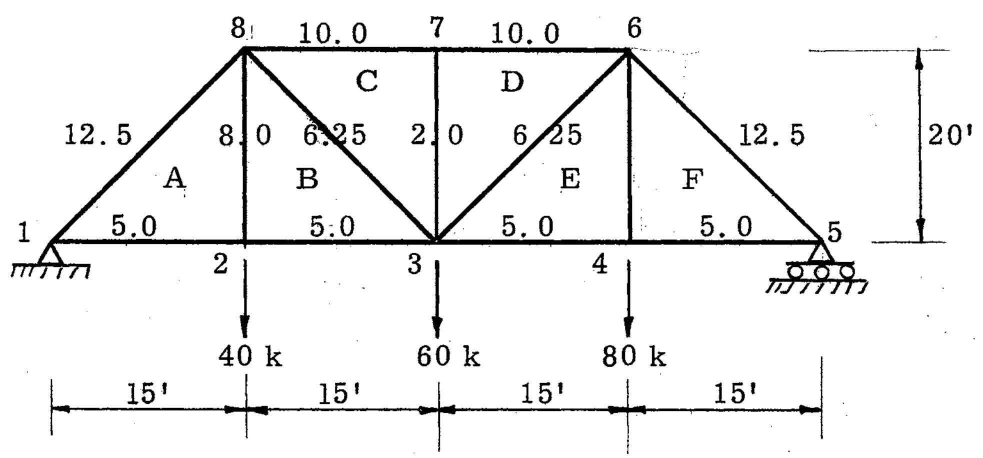


Fig. 4-1

Given Structure

The analysis is as follows:

Part 1. Geometry.

Joint and cell designation is shown in Fig. 4-1, and the properties of the members are shown in Table 4-1.

TABLE 4-1

PROPERTIES OF MEMBERS

Members	Length (ft)	Components (ft)	
		x	y
1 2	15	15	0
2 3	15	15	0
3 4	15	15	0
4 5	15	15	0
5 6	25	15	20
6 7	15	15	0
7 8	15	15	0
8 1	25	15	20
2 8	20	0	20
3 8	25	15	20
3 7	20	0	20
3 6	25	15	20
4 6	20	0	20

Part 2. Stereo-Statics.

The stereo-static equilibrium matrix (Table 2-3) becomes:

$$
\begin{bmatrix}
+1.0 & 0 & 0 & 0 & 0 & 0 & 0 & +0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.0 & +1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.0 & +1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.8 & +0.8 & 0 \\
0 & 0 & -1.0 & +1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.0 & -0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1.0 \\
0 & 0 & 0 & 0 & +0.8 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.8 & 0 \\
0 & 0 & 0 & 0 & 0 & +1.0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & +1.0 & -0.8 & 0 & 0 & 0 & 0 & 0 & +0.8 & 0 & 0 \\
+0.6 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1.0 & 0 & 0 & +0.6 & +0.6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1.0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & +0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.6 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & -0.6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6 & 0 & -1.0 & 0 & 0 & 0 & -0.6 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
N_{1\,2} \\ N_{2\,3} \\ N_{3\,4} \\ N_{4\,5} \\ N_{5\,6} \\ N_{6\,7} \\ N_{7\,8} \\ N_{8\,1} \\ R_{1\,y} \\ N_{2\,8} \\ N_{3\,7} \\ N_{4\,6} \\ R_{5\,y} \\ N_{3\,8} \\ N_{3\,6} \\ R_{5\,x}
\end{bmatrix}
+
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -40 \\ -60 \\ -80 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
= 0
$$

Normal Forces and Reactions

$N_{1\ 2} = +60$ kips

$N_{2\ 3} = +60$

$N_{3\ 4} = +75$

$N_{4\ 5} = +75$

$N_{5\ 6} = -125$

$N_{6\ 7} = -90$

$N_{7\ 8} = -90$

$N_{8\ 1} = -100$

$R_{1\ y} = +80$

$N_{2\ 8} = +40$

$N_{3\ 7} = 0$

$N_{4\ 6} = +80$

$R_{5\ y} = +100$

$N_{3\ 8} = +50$

$N_{3\ 6} = +25$

$R_{5\ x} = 0$

Part 3. Elasto-Statics

Axial Deformations

$E = 30 \times 10^6$ psi.

$$\Delta_{1\ 2} = +\ 72 \times 10^{-5} \text{ in}$$

$$\Delta_{2\ 3} = +\ 72$$

$$\Delta_{3\ 4} = +\ 90$$

$$\Delta_{4\ 5} = +\ 90$$

$$\Delta_{5\ 6} = -\ 100$$

$$\Delta_{6\ 7} = -\ 54$$

$$\Delta_{7\ 8} = -\ 54$$

$$\Delta_{8\ 1} = -\ 80$$

$$\Delta_{2\ 8} = +\ 40$$

$$\Delta_{3\ 7} = 0$$

$$\Delta_{4\ 6} = +\ 80$$

$$\Delta_{3\ 8} = +\ 80$$

$$\Delta_{3\ 6} = +\ 40$$

<u>Determinant D</u>

$$D_A = -\ 43200 \text{ in}^2$$

$$D_B = -\ 43200$$

$$D_C = -\ 43200$$

$$D_D = -\ 43200$$

$$D_E = -\ 43200$$

$$D_F = -\ 43200$$

The sum of components of deformation and the numerical constants D' are given in Table 4-2. The joint elastic weights are shown in Table 4-3.

TABLE 4-2

CELL DEFORMATION CONSTANTS

H	$\Sigma\ \Delta_{Hx}$ (in.)	$\Sigma\ \Delta_{Hy}$ (in.)	D'_{Hx} (in. $^{-1}$)	D'_{Hy} (in. $^{-1}$)
A	$- 120 \times 10^{-3}$	$- 104 \times 10^{-3}$	$+ 2.80 \times 10^{-6}$	$+ 2.40 \times 10^{-6}$
B	- 24	- 24	+ 0.55	+ 0.55
C	- 102	+ 64	+ 2.36	- 1.48
D	- 78	- 32	+ 1.81	+ 0.74
E	- 66	- 48	+ 1.53	+ 1.11
F	- 150	+ 160	+ 3.47	- 3.70

TABLE 4-3

JOINT ELASTIC WEIGHTS

i	A	B	C	D	E	F
1	- 48.2					
2	+ 89.8	- 2.8				
3		+ 11.1	+ 35.4	+ 35.4	- 22.2	
4					+ 45.2	+ 126.2
5						- 74.2
6				+ 14.9	- 22.9	
7			- 65.4	- 41.8		
8	- 41.7	- 8.3	+ 30.0			

(All values $\times 10^{-6}$)

The conjugate structure is shown in Fig. 4-2, and the total joint elastic weights in Table 4-4.

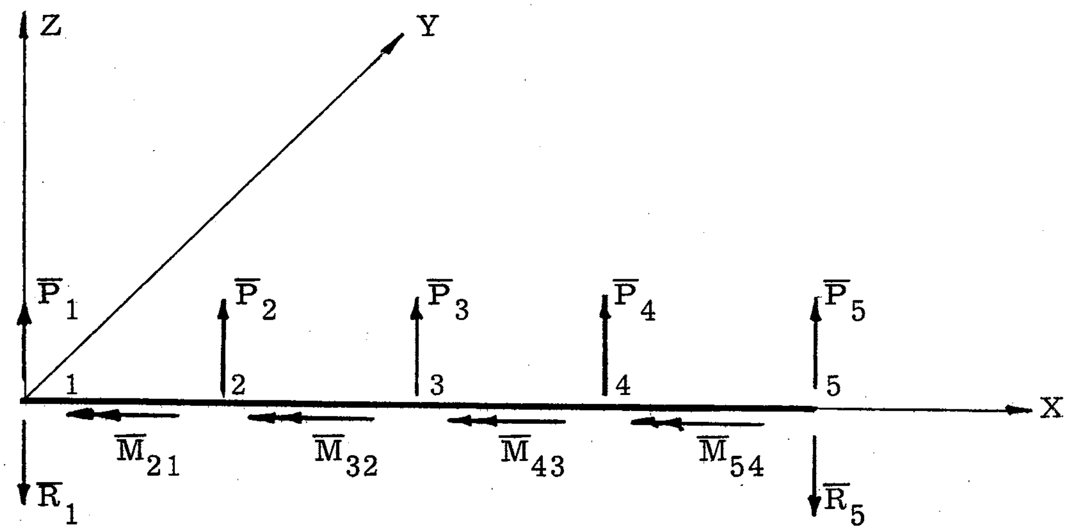


Fig. 4-2

Conjugate Structure with Elastic Loads

TABLE 4-4

TOTAL JOINT ELASTIC WEIGHTS

i	$\overline{P}_i$
1	-48.2×10^{-6}
2	+ 87.1
3	+ 51.4
4	+171.4
5	- 74.2
6	- 75.0
7	-107.4
8	- 20.0

Conjugate Reactions and Cross-Sectional Elements

$$\overline{R}_1 = 1029 \times 10^{-6}$$

$$\overline{R}_5 = 1222 \times 10^{-6}$$

$$\overline{M}_{2y} = -0.289 \text{ in.} = \delta_2$$

$$\overline{M}_{3y} = -0.390 \text{ in.} = \delta_3$$

$$\overline{M}_{4y} = -0.380 \text{ in.} = \delta_4$$

These deflections are shown in Fig. 4-3 as the moment diagram of the conjugate structure reduced to a conjugate bar and loaded by the total joint elastic weights.

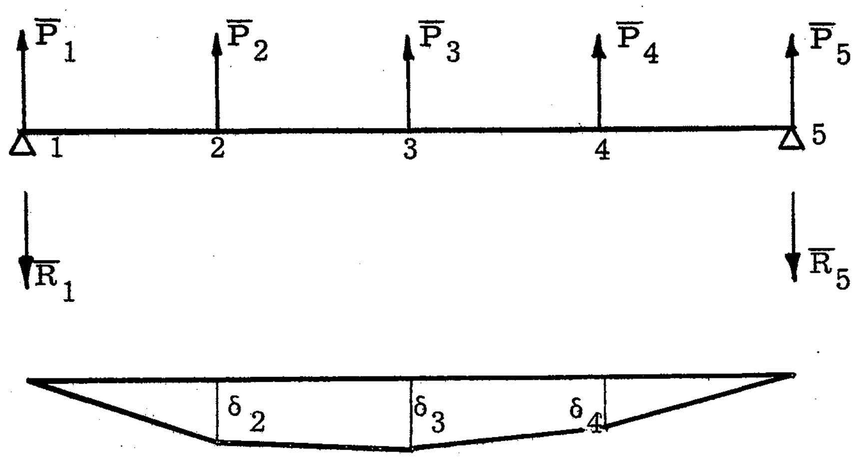


Fig. 4-3

Deflections of Lower Chord

The deflections are identical to those obtained by Wang (12).

4-3. Example No. 2.

The truss tower (Fig. 4-4) is given. It is required to find both the vertical and horizontal deflection and the slope at joint 7. The cross-sectional area of each member is recorded on the member in Fig. 4-4. The structure is statically determinate (Eq. 2-9). The analysis is as follows:

Part 1. Geometry

Joint and cell designation is shown in Fig. 4-4 and the properties of the members are given in Table 4-5.

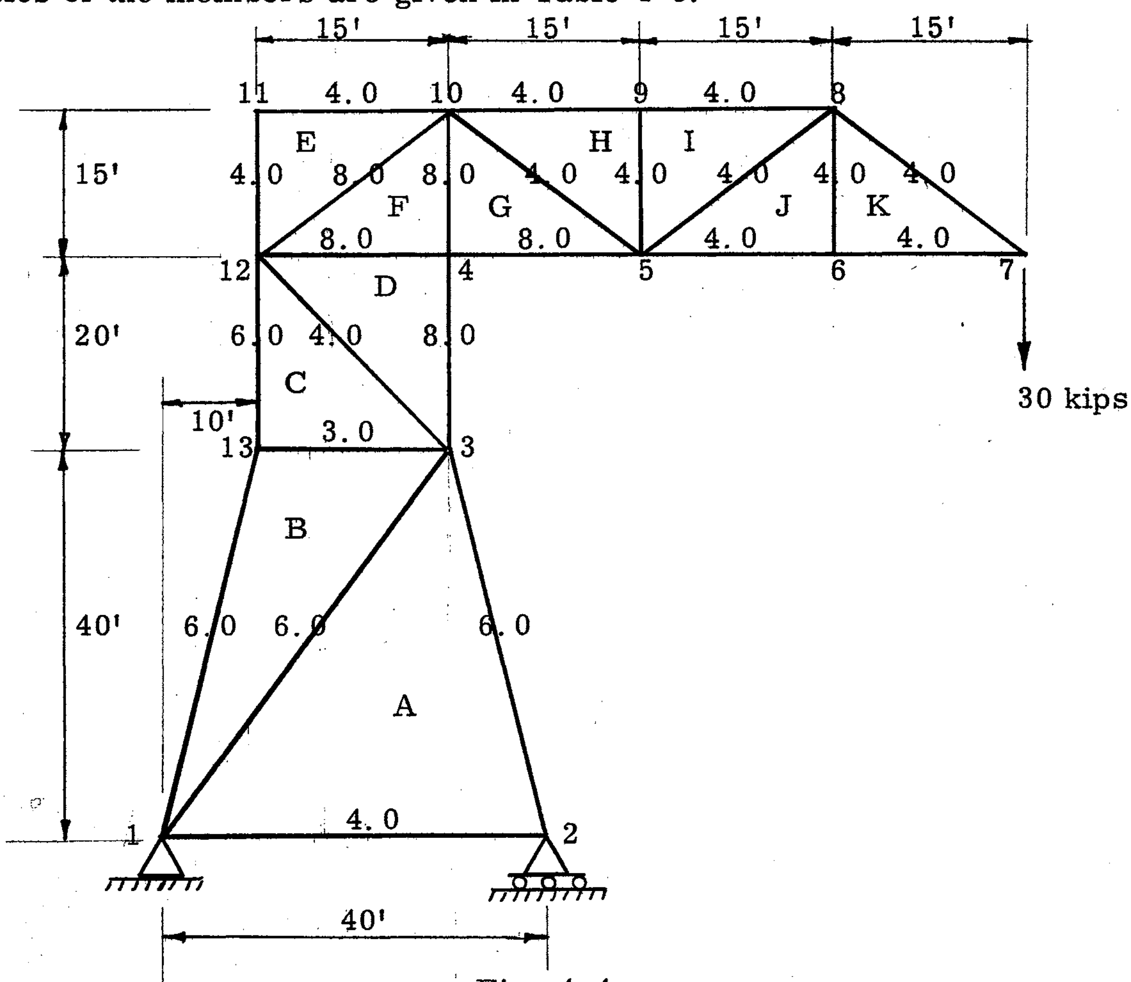


Fig. 4-4

Given Structure

TABLE 4-5
PROPERTIES OF MEMBERS

Member	Length (ft.)	Component (ft.)	
		X	Y
1 2	40	40	0
2 3	41.23	10	40
3 4	20	0	20
4 5	20	20	0
5 6	20	20	0
6 7	20	20	0
7 8	25	20	15
8 9	20	20	0
9 10	20	20	0
10 11	20	20	0
11 12	15	0	15
12 13	20	0	20
13 1	41.23	10	40
1 3	50	30	40
3 13	20	20	0
3 12	28.28	20	20
12 4	20	20	0
12 10	25	20	15
10 4	15	0	15
10 5	25	20	15
5 9	15	0	15
5 8	25	20	15
8 6	15	0	15

Part 2. Stereo-Statics

The stereo-static equilibrium matrix (Eq. 2-8) becomes:

+1.0	0	0	0	0	0	0	0	0	0	0	0	+.2425	0	0	0	0	0	0	0	+0.6	0	0	0	0	+1.0	$N_{1\,2}$		0	
-1.0	-.2425	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$N_{2\,3}$		0	
0	+.2425	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1.0	0	-0.6	-.707	0	0	0	0	$N_{3\,4}$		0	
0	0	0	+1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1.0	0	0	0	0	0	0	$N_{4\,5}$		0	
0	0	0	-1.0	+1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.8	+0.8	0	$N_{5\,6}$		0	
0	0	0	0	-1.0	+1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$N_{6\,7}$		0	
0	0	0	0	0	-1.0	-0.8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$N_{7\,8}$		0	
0	0	0	0	0	0	+0.8	-1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.8	0	$N_{8\,9}$		0	
0	0	0	0	0	0	0	+1.0	-1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$N_{9\,10}$		0	
0	0	0	0	0	0	0	0	+1.0	-1.0	0	0	0	0	0	0	0	0	0	0	0	0	-0.8	+0.8	0	0	$N_{10\,11}$		0	
0	0	0	0	0	0	0	0	0	+1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$N_{11\,12}$		0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+1.0	0	+.707	+0.8	0	0	0	$N_{12\,13}$		0	
0	0	0	0	0	0	0	0	0	0	0	0	-.2425	0	0	0	0	0	-1.0	0	0	0	0	0	0	0	$N_{13\,1}$	+	0	= 0
0	0	0	0	0	0	0	0	0	0	0	0	+.9702	-1.0	0	0	0	0	0	0	+0.8	0	0	0	0	0	$R_{1\,y}$		0	
0	+.9702	0	0	0	0	0	0	0	0	0	0	0	0	-1.0	0	0	0	0	0	0	0	0	0	0	0	$R_{2\,y}$		0	
0	-.9702	+1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.8	+.707	0	0	0	0	$N_{4\,10}$		0	
0	0	-1.0	0	0	0	0	0	0	0	0	0	0	0	0	+1.0	0	0	0	0	0	0	0	0	0	0	$N_{10\,6}$		0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+1.0	0	0	0	0	0	0	+0.6	+0.6	0	$N_{6\,8}$		0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+1.0	0	0	0	0	0	0	0	0	$N_{13\,3}$		0	
0	0	0	0	0	0	+0.6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$N_{12\,4}$		-30	
0	0	0	0	0	0	-0.6	0	0	0	0	0	0	0	0	0	0	-1.0	0	0	0	0	0	0	-0.6	0	$N_{1\,3}$		0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1.0	0	0	0	0	0	0	0	0	0	$N_{3\,12}$		0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1.0	0	0	0	0	0	0	-0.6	-0.6	0	0	$N_{10\,12}$		0	
0	0	0	0	0	0	0	0	0	0	-1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$N_{10\,5}$		0	
0	0	0	0	0	0	0	0	0	0	+1.0	-1.0	0	0	0	0	0	0	0	0	0	-.707	+0.6	0	0	0	$N_{5\,8}$		0	
0	0	0	0	0	0	0	0	0	0	0	+1.0	-.9702	0	0	0	0	0	0	0	0	0	0	0	0	0	$R_{1\,x}$		0	

Normal Forces and Reactions

$N_{1\ 2} = +\ 16.9$ kips

$N_{2\ 3} = -\ 69.6$

$N_{3\ 4} = -\ 120.0$

$N_{4\ 5} = -\ 120.0$

$N_{5\ 6} = -\ 40.0$

$N_{6\ 7} = -\ 40.0$

$N_{7\ 8} = +\ 50.0$

$N_{8\ 9} = +\ 80.0$

$N_{9\ 10} = +\ 80.0$

$N_{10\ 11} = 0$

$N_{11\ 12} = 0$

$N_{12\ 13} = +\ 90.0$

$N_{13\ 1} = +\ 92.9$

$R_{1\ y} = -\ 37.5$

$R_{2\ y} = +\ 67.5$

$N_{4\ 10} = -\ 120.0$

$N_{5\ 9} = 0$

$N_{6\ 8} = 0$

$N_{3\ 13} = +\ 22.5$

$N_{4\ 12} = -\ 120.0$

$N_{1\ 3} = -\ 65.6$

$N_{3\ 12} = 0$

$N_{10\ 12} = +\ 150.0$

$N_{5\ 10} = +\ 50.0$

$N_{5\ 8} = -\ 50.0$

$R_{1\ x} = 0$

Part 3. Elasto-Statics

Axial Deformations $(E = 30 \times 10^6)$

$\Delta_{1\ 2}$	=	$+\ 5.6 \times 10^{-3}$	(feet)
$\Delta_{2\ 3}$	=	$-\ 15.9$	
$\Delta_{3\ 4}$	=	$-\ 10.0$	
$\Delta_{4\ 5}$	=	$-\ 10.0$	
$\Delta_{5\ 6}$	=	$-\ 6.7$	
$\Delta_{6\ 7}$	=	$-\ 6.7$	
$\Delta_{7\ 8}$	=	$+\ 10.4$	
$\Delta_{8\ 9}$	=	$+\ 13.3$	
$\Delta_{9\ 10}$	=	$+\ 13.3$	
$\Delta_{10\ 11}$	=	0	
$\Delta_{11\ 12}$	=	0	
$\Delta_{12\ 13}$	=	$+\ 10.0$	
$\Delta_{13\ 1}$	=	$+\ 21.3$	
$\Delta_{4\ 10}$	=	$-\ 7.5$	
$\Delta_{5\ 9}$	=	0	
$\Delta_{6\ 8}$	=	0	
$\Delta_{3\ 13}$	=	$+\ 5.0$	
$\Delta_{4\ 12}$	=	$-\ 10.0$	
$\Delta_{1\ 3}$	=	$-\ 18.2$	
$\Delta_{3\ 12}$	=	0	
$\Delta_{10\ 12}$	=	$+\ 15.7$	
$\Delta_{5\ 10}$	=	$+\ 10.4$	
$\Delta_{5\ 8}$	=	$-\ 10.4$	

<u>Determinant D</u>

D_A = -1600 ft.^{2}

D_B = -800

D_C = -400

D_D = -400

D_E = -300

D_F = -300

D_G = -300

D_H = -300

D_I = -300

D_J = -300

D_K = -300

The sum of component of deformation and the numerical constants D' are given in Table 4-6. The joint elastic weights are given in Table 4-7.

TABLE 4-6

CELL DEFORMATION CONSTANTS

H	$\Sigma\Delta_{Hx}$ (in.)	$\Sigma\Delta_{Hy}$ (in.)	D'_{Hx} (in.$^{-1}$)	D'_{Hy} (in.$^{-1}$)
A	$- 20.46\times10^{-3}$	$+ 0.86\times10^{-3}$	$- 12.28\times10^{-6}$	$- 0.54\times10^{-6}$
B	+ 21.05	+ 35.20	+ 26.40	+ 44.00
C	- 5.00	+ 10.00	- 12.50	+ 25.00
D	- 10.00	+ 10.00	- 25.00	+ 25.00
E	- 12.55	- 9.42	- 41.80	- 31.40
F	+ 22.56	+ 16.92	+ 75.20	+ 56.50
G	+ 18.33	- 13.74	+ 61.10	- 45.80
H	+ 5.00	+ 6.24	+ 16.70	+ 20.80
I	+ 21.62	+ 6.24	+ 72.10	+ 20.80
J	- 1.62	- 6.24	- 5.40	- 20.80
K	+ 15.02	- 6.24	+ 50.10	- 20.80

TABLE 4-7

JOINT ELASTIC WEIGHTS

i	A	B	C	D	E	F	G	H	I	J	K
1	- 150	+ 528									
2	- 362										
3	+ 512	+ 2024	+ 500	- 500							
4				+ 1000		+ 2352	+ 1900				
5							- 688	+ 333	+ 1442	+ 312	
6										- 420	+ 1314
7											- 312
8									+ 312	+ 108	- 1002
9								- 21	- 1754		
10					- 471	- 1504	- 1213	- 312			
11					+ 1307						
12			+ 250	- 500	- 836	- 848					
13		- 2552	- 750						(All values $\times 10^6$)		

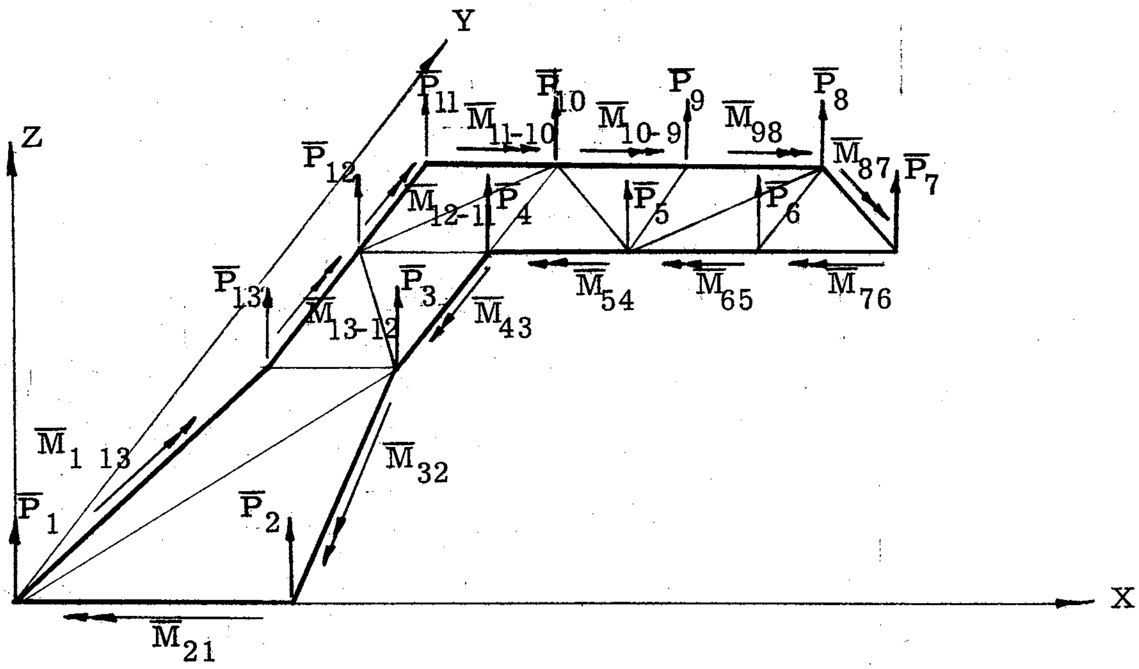


Fig. 4-5

Conjugate Tower with Elastic Loads

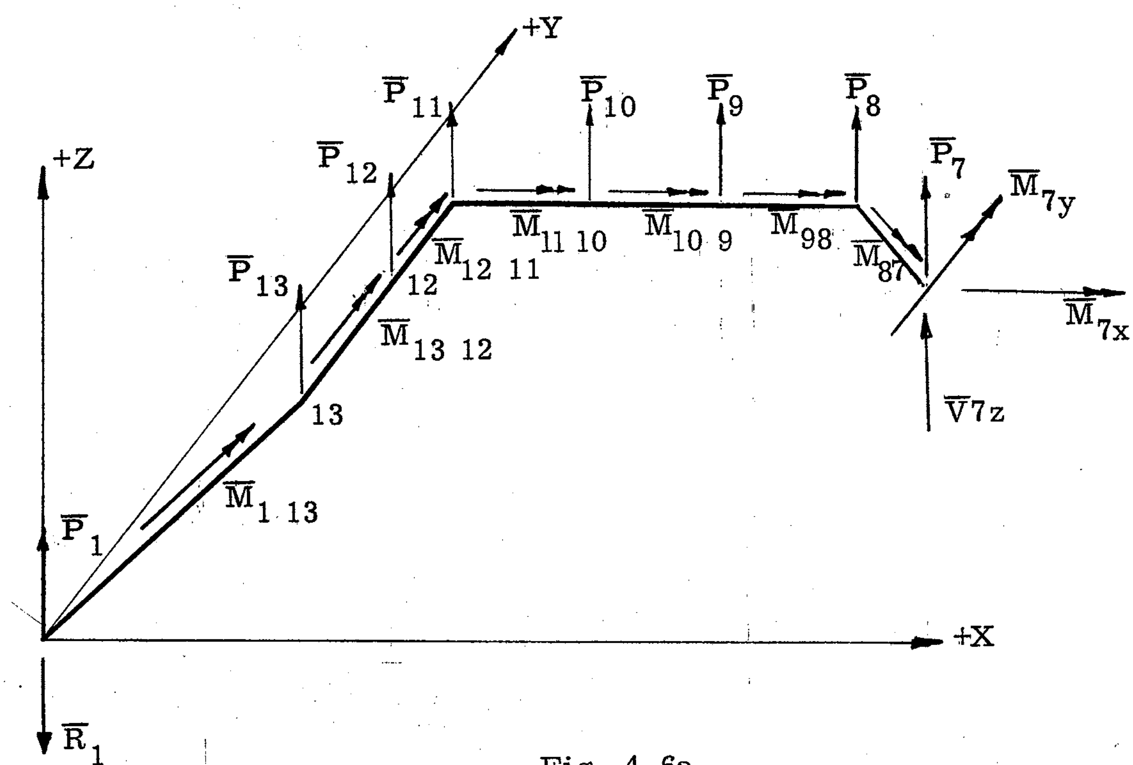


Fig. 4-6a

Conjugate Structure with Elastic Loads

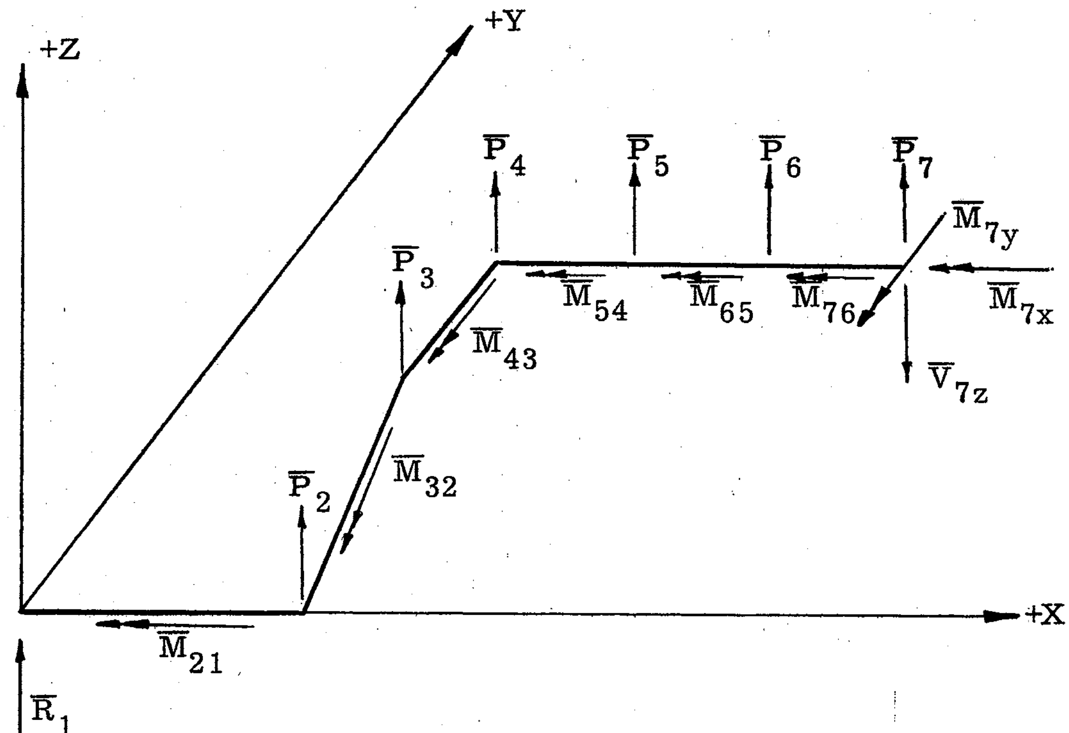


Fig. 4-6b

Conjugate Structure with Elastic Loads

The closed conjugate structure with elastic loads is shown in Fig. 4-5. This conjugate structure is cut at points 1 and 7 and shown as a string polygon (Fig. 4-6a, b). The required deformations are obtained using either Fig. 4-6a or b. The total joint elastic weights are shown in Table 4-8, and the conjugate moments are taken from the axial deformation listing.

Conjugate Reactions and Cross-Sectional Elements

$$\overline{R}_{1z} = 0$$

$$\overline{V}_{7z} = +.00972 \text{ Rod.} = \theta_{7z}$$

$$\overline{M}_{7x} = +.0150 \text{ ft.} = \delta_{7x}$$

$$\overline{M}_{7y} = -.5485 \text{ ft.} = \delta_{7y}$$

These deformations are identical with those obtained by Scordelis (4).

TABLE 4-8

TOTAL JOINT ELASTIC WEIGHTS

i	$\overline{P}_i$
1	$+\ 378 \times 10^{-6}$
2	$-\ 362$
3	$+\ 2530$
4	$+\ 5250$
5	$+\ 1405$
6	$+\ 900$
7	$-\ 312$
8	$-\ 582$
9	$-\ 1775$
10	$-\ 3500$
11	$+\ 1307$
12	$-\ 1934$
13	$-\ 3302$

4-4. Conclusions.

It has been shown in this thesis that the extension of the string polygon method to the calculation of deformations of coplanar statically determinant truss systems is possible. Three new ideas have been developed in this connection:

(a) The algebraic matrix form of the joint elastic weight for a joint of a triangular truss cell.

(b) The algebraic matrix formulation for the total joint elastic weight for a statically determinant coplanar truss.

(c) The representation of the geometry of deformation of a coplanar statically determinate truss by means of a conjugate string polygon.

The advantages of the string polygon method over some other well-known methods are:

(a) Displacements of all joints in any direction may be calculated.

(b) Relative displacements of one joint with respect to any other joint may be calculated.

(c) All deformations can be expressed algebraically and serve as influence values.

The application of the string polygon to this type of a problem gives results which are in good agreement with results based on other methods.

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