

AN ANALYSIS OF THE ROLE OF GEOMETRY AT  
THE PRE-DEDUCTIVE LEVEL IN SCHOOL  
MATHEMATICS PROGRAMS

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## PREFACE

In the present period of change affecting pre-high school mathematics, those who are attempting to select or write programs must establish some guidelines regarding content, emphases, and presentation. These guidelines are of particular importance for pre-deductive geometry. Such content, at these grade levels, not only represents a radical departure from the traditional; its pre-deductive form raises fundamental questions regarding presentation. The following analysis of current programs, the comparisons made, and the questions raised are an attempt to provide a workable basis on which to make decisions regarding the role of pre-deductive geometry content.

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## CHAPTER I

### THE PROBLEM

#### Introduction

With a few exceptions, reform in elementary school mathematics programs that has occurred on this continent during the past decade has concerned itself primarily with the traditional content of arithmetic and, to some extent, the use of beginning techniques of algebra. Thus, a good "modern" elementary program is now expected to concern itself with the structuring of problems into arithmetic sentences. Considerable attention is also paid to the abstract concepts which underly the symbols for number, operation, and relation, all of which are used in these arithmetic sentences. The properties of number systems and numeration systems are studied both for their structure and as a means for presenting the 'why' as well as the 'how' of the various computational algorithms.

In recent years, more and more attention has been given to the role that geometry should play in the elementary grades. A review of the literature reveals that many individuals and groups, both here and in other countries, advocate a greatly increased amount of geometry

at the pre-deductive level; some have implemented their ideas in the development of new programs.

Bruner (8:33) is widely known for his statement of the hypothesis, "any subject can be taught effectively in some intellectually honest form to any child at any stage of development." However, for authors who are designing mathematics programs, decisions must still be made regarding the desirability of including topics at any given level. As Sand (59:21) points out: "The fact that very young children can learn relatively difficult aspects of science, mathematics, and other subjects provides at best an incomplete answer to the question of whether they should learn them."

Concerning pre-deductive geometry, authors are faced with a problem that is at least threefold:

1. There is necessity for some clear statement of the reasons for the inclusion of "pre-geometry." These objectives will surely be concerned with the mathematics program as a whole, with geometry in the world about us, and with preparation for deductive geometry in high school.

2. The decision regarding content and grade level will depend upon one or more of the following:

- (a) the place of the content in a pre-determined grade-by-grade, or vertical, geometry sequence

- (b) the use of the content to broaden understanding of other mathematical content at that level

- (c) the belief that the content is intrinsically



worthwhile for study or as an activity at that level

3. Authors cannot assume that the teacher has an adequate background of geometry. Background information must be provided; there is also a need for suggested methods of presentation of content that is very different from traditional content at this level.

#### Purpose of the Study

At the present time, school systems have reason to consider pre-deductive geometry content when decisions are made concerning adoption of new commercial mathematics programs. Some systems may even wish to design or revise their own programs to include pre-deductive geometry. In either case, the following results of a 1961 California survey by Lawrence (42) are pertinent:

1. The role of the textbook in nearly all districts in the state was reported to be that of providing the major organization and content of the course of study.

2. Over 73% of the districts in California reported that they had no stated guidelines for high school textbook selection.

3. Lack of thorough examination was most frequently named by administrators and teachers as the major deterrent to wise textbook selections.

With regard to textbook selection, there is little reason to assume that the situation is greatly different in other areas or at other grade levels. For geometry

content at the elementary level, there is a distinct possibility that "lack of thorough examination" may result in large part from an accompanying lack of knowing what to look for.

The purpose of this study is to analyze selected mathematics programs that contain pre-deductive geometry beginning in the primary grades. On the basis of this analysis the study provides some guidelines which, it is hoped, will be useful to those who are adopting or writing new pre-deductive geometry content. The selected programs, all in current use, are studied from three major standpoints. First, an analysis of pre-deductive geometry content (5) (9) indicates frequency of inclusion of topics and the extent of agreement regarding grade level for introduction. Second, consideration is given to the organization of the teachers' commentaries that accompany these programs, and to the kind of mathematical content and guidance that can be expected from this source. Finally, for the students' text or worksheet, the presentation of the geometric content is considered. The comparison will deal with the sequence, scope, and integration of the content into the whole program. Certain guidelines will be indicated for the avoidance of what appear to be weaknesses in current practice.

## CHAPTER II

### REVIEW OF THE LITERATURE

#### Introduction

At a meeting of the International Congress of Mathematics in Edinburgh, 1958, Fehr (23:37) presented a paper dealing with instruction in mathematics to youth from ages 6 to 15 which, he says, "gave a rather sorry picture of the type of study that was commonplace for most countries." This "sorry picture" is, of course, hardly news to anyone who has followed the development of new mathematics programs, particularly since 1960.

In the same article Fehr says further that "since instruction in geometry always presented the most troublesome problem, two special meetings were held to study the role of geometry." These meetings however, as indicated by reports (49) (50), gave first attention to the role of geometry at the secondary level. This concentration on deductive geometry first has been a general trend<sup>1</sup>, and

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<sup>1</sup>To a very large extent, reform in mathematics programs in the United States and in Western Europe began at the high school level. Two notable exceptions have been the Stanford Project, now in textbook (33) form, and the "Seeing Through Arithmetic" series, published by Scott, Foresman and Company, Chicago.

it is only recently that widespread attention has been given to the role of geometry at the pre-deductive level. From the first however, concern has been shown for improvement of geometric content at all levels.

By the beginning of this decade Fehr (24:427) believed:

That geometry, as a physical representation of the world in which we live, must play a larger role in elementary education, is now generally accepted. That an intuitive physical geometry is preparatory to the study of geometry as a deductive science is also generally accepted.

This "general acceptance" had certainly become more evident. In 1960, the Organization for European Economic Cooperation (OEEC)<sup>2</sup> "Group of Experts" (50:85-6), even though primarily concerned with secondary school mathematics, states that for "Geometry: Cycle I":

It is essential that the pupil learns to think creatively and intuitively. To this end, he must be given opportunities to find his own problems, to state his own solutions. He will naturally make many false starts and give invalid solutions. The teacher will be called upon many times to help reformulate the problem in a way which facilitates solution. The study, moreover, of various solutions to the same problem will lead to an understanding of what constitutes geometry and to an appreciation for elegance. With friendly encouragement, and careful arrangement of the context of the geometry under discussion, the teacher will go a long way to establish the confidence which plays an important role in fostering a liking for mathematics generally, and an

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<sup>2</sup>This organization later became known as the Organization for Economic Cooperation and Development (OECD)

attitude that mathematics is not only interesting, but also exciting.

Furthermore, the OEEC report continues, "for the young, however, a rich and varied concrete experience is a necessary preparation for abstraction."

In Canada, in the same year (1960), the Canadian Teachers Federation held a national seminar on "New Thinking in School Mathematics." Concerning geometry at the pre-deductive level, Linis (44:125) reported to the seminar as follows:

The comparison between geometry and algebra (which includes arithmetic) during the first period reveals a rather striking fact: the number concept, which is an abstract concept, is taught and presumably learned by the pupils at a very early stage, whereas the geometric concepts, in spite of their apparent "concreteness" are treated rather cautiously. Two explanations can be offered for this phenomenon: first, the number concept being rather limited and bare is enriched and extended gradually by repetition and widening its scope; the geometric concepts appear to be much richer in their possible interpretations even at a low level and may lead to a bewildering (for a child) complexity. Secondly, geometry as an independent subject was always associated with the "geometry à la Euclid" and was considered unpalatable for young gourmets, whereas algebra unshackled by a rigid scheme permitted any degree of plausibility and justification by manipulation.

Analyses of content such as those by Abeles (1) and Miller (46) show generally a greater geometry content for elementary grade levels in Western Europe as compared to the United States. However, in this country interest in

an expanded role for pre-deductive geometry has been evidenced from many quarters. In 1961 Brune (6:211), editor of The Mathematics Teacher, suggested that:

...geometry deserves a lifetime of interest. To study it in only the tenth grade hardly suffices. At that level pupils presumably study one or more kinds of geometry as deduction. There and in subsequent courses they also learn about applications. But the computing with geometric formulas that frequently represents the only planned experience that pupils have in geometry prior to Grade 10 seldom prepares them for Grade 10.

He further states (6:213):

Geometric preparedness begins at an early age. Tots in kindergarten enjoy plopping the cutout figures into their proper places. Children quickly discriminate between right triangles and equilateral triangles, between squares and oblongs, and between trapezoids and parallelograms. Already these youngsters are shape conscious.

Success in this sort of activity leads young children on. Their handling of squares, cubes, disks, triangles, spheres, and so on, prepares them for further work with forms.

All too frequently, however, such activities terminate abruptly. This occurs because courses of study encourage the pupils to put away "childish things" and settle down to the stern business of memorizing facts and practicing operations with numbers. Since perfection in these worthy matters eludes most learners, the study of facts and operations flourishes while the study of forms languishes. Of course, lessons in the upper grades deal with areas and volumes, but computing with numbers and distinguishing between area and perimeter and between volume and surface have been known to monopolize the act.

Fortunately, the trend today points to

geometry for the sake of geometry, rather than to geometry as further practice in calculating. In the elementary grades informal, or intuitive, studies get the emphasis. Drawing, counting, and measuring lead pupils to observing, inferring, and generalizing. Consciousness of forms continues to grow, and readiness for proofs in geometry also continues to grow.

The foregoing quotations indicate that, by the beginning of this decade, there was widespread agreement concerning the need for an increased role for geometry at the elementary school level. The quotations indicate some of the reasons why elementary school geometry is considered important; there is also considerable discussion concerning the informal intuitive presentation to be used. The pre-deductive geometry content of many current elementary mathematics programs reflects the kind of thinking evident in the above quotations.

#### Programs in Pre-Deductive Geometry

The first widely-known experimental program in pre-deductive geometry was the Stanford Project's "Geometry for the Primary Grades" (33). Content is developed on the basis of traditional straightedge and compass constructions. Comparatives as a basis for measurement of segments and angles are done with the compass. In 1961, the program for grades 4, 5, and 6 (62) of the School Mathematics Study Group (SMSG) was generally available; one major feature that distinguished it from traditional

programs at this level was the inclusion of several chapters of pre-deductive geometry, with an accompanying illustrated background section in the teachers' commentary. Rosenbloom (57:359), former director of MINNEMATH<sup>3</sup>, has approached elementary geometry through familiarity with the use of the coordinate system: "a vectorial approach to geometry, with thorough integration of algebra and geometry, and an intuitive approach to vectors beginning in grade 7." Other experimental programs followed (22) (74) (76), and by 1963, Deans (17:6), reporting for the United States Office of Education was able to generalize regarding pre-deductive geometry:

It is believed that the study of geometry can be expanded far beyond a meager knowledge of shapes, forms, and the computations required for finding areas and perimeters. It may include discovery of the principles underlying area and perimeter and the development of simple concepts with regard to points, lines, and planes in space. From their earliest school experiences with blocks and puzzles, children work informally with shapes and forms. While the lines, points, and planes are ideas, their representations in pictures or in the real environment of the child may be made concrete. In fact, it is possible that simple concepts of geometry are easier for the child to grasp than much of the abstract work with the operations of addition and subtraction which children have usually been expected to master during their first 2 or 3 years in school.

In Canada, the Ontario Mathematics Commission (2:4)

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<sup>3</sup>MINNEMATH: The Minnesota Mathematics and Science Teaching Project, Institute of Technology, University of Minnesota.



has received a report from its subcommittee concerned with mathematics for grades K - 6. Concerning geometry, the report says, in part:

The subcommittee was unanimous in its opinion that more geometry content is essential. The delay of any substantial consideration of geometry until Grade 6 or Grade 7 in our present curriculum is unquestionably bad. Some school systems are experimenting with geometric recognition in kindergarten and include some three-dimensional study and experiment. The subcommittee considered that two- and three-dimensional experience should be given at all grades, the three-dimensional aspects being delayed a little behind the corresponding two-dimensional work.

At each stage guided recognition through "pre-geometry" discovery by experiment, measurement with models, and verification of properties, (and later proofs of properties) should proceed in a spiral. In general, considerations of relations for plane figures will precede similar work for solids, but both two- and three-dimensional geometry should be studied in every grade.

In the past attention has concentrated mainly on single polygons or relations between two polygons. Attention must now be paid to the three-dimensional analogues and to the properties of sets of geometric figures and their spatial relations, e.g. patterns, symmetries, space filling properties.

Attention must also be given to the effects of translations and rotations and these ideas should be applied to the discovery of proofs of properties. It is not generally recognized in schools that proofs based on displacements and symmetries can be made as rigorous as the proofs of Euclidean deductive geometry.

Similarly coordinate geometry can be introduced earlier. A single number line is used from Grade I onwards, and two number lines can be used in the junior grades, certainly by Grades 5 and 6, possibly earlier.

After all in Grade 6 or 7 we have always taught latitude and longitude which is a more difficult coordinate system -- and on a sphere.

Such a study should occupy about 25% of the mathematics time in the early grades. Notice that this includes the topics of linear, area, and volume measurement at present taught. This geometrical study is one of the major changes from the present curriculum, but appears to be in line with the current trends in elementary mathematics.

In England, the School Mathematics Project (61:vi) has developed an extremely interesting intuitive introduction to certain geometric transformations and to other topics in geometry:

In this course, the objects of attention, and indeed the modes of geometrical thinking, are rather different. The pupils will learn to investigate what happens to a figure when it undergoes some kind of transformation -- reflection in some axis, rotation about some point, enlargement from some centre -- and they should do this with two distinct objectives in mind. When their attention is on the figure, they will find that the consideration of its behaviour under transformation will often throw light on its structure, exhibiting its properties clearly and sometimes even dramatically, and when their attention shifts to the transformations they will find that these are interesting in themselves and are related to one another in interesting ways. Later books will increasingly emphasize the group structure of various sets of transformations.

In this book, however, we are still mainly interested in the figures and in their properties. Three length-preserving transformations, reflection, rotation and translation, are encountered ... We meet enlargement and study the geometry of similar figures.

## The Cambridge Report

A good deal of interest and discussion (pro and con) has centered around the so-called "Cambridge Report" (29). This interest has been evident not only on this continent, but in Europe and elsewhere (10:10). The report, described as a "discussion document not a prescription," is a result of a conference held during the summer of 1963 at Cambridge, Massachusetts. The participants were "a group of twenty-five professional mathematicians and mathematics-users." On page 7, for grades K - 6, the report states that the main objective of the mathematics program should be "familiarity with the real number system and geometry." Regarding presentation of the real number system the report says, on the same page:

The child usually learns quite early and easily how to count. As soon as he is able to count, he can begin to get experience with the number line. This line can be regarded from the first as a representation for all real numbers, even though the child will not be immediately able to give sophisticated names for most of these numbers. Nonetheless, he can speak of "a little more than three" and "a little less than five" and he can give a temporary name, like  $\star$  to any number.

With regard to geometry content for grades K - 2, the report is, by its own admission (page 33), "a far more tentative groping than was the case for the work in real numbers described earlier." On the same page, the following general statement is made:

Geometry is to be studied together with arithmetic and algebra from kindergarten on. Some of the aims of this study are to develop the planar and spatial intuition of the pupil, to afford a source of visualization for arithmetic and algebra, and to serve as a model for that branch of natural science which investigates physical space by mathematical methods.

For grades 3 - 6, the report says on pages 37 and 38:

In the later grades of elementary school, relatively little pure geometry would be introduced, but more experience with the topics from K - 2 would be built up. The pictorial representation of sets with Venn diagrams and the graphing of elementary functions using Cartesian coordinates would be continued. ... many geometrical questions are motivated by problems concerning solid bodies and the ways they fit together.

The actual listing of the proposed pre-geometry content (pages 33 - 38) will not be given here. To this writer, the lists show a good deal of evidence of the above-mentioned "groping."

#### The Intuitive Approach

In the presentation of pre-deductive geometry, much use is made of the word "intuitive," or as Deans (17:6) states it:

Proponents of these innovations believe that through a discovery approach to learning children often grasp an idea intuitively long before they are ready for the detailed step-by-step analysis of the process. By an intuitive approach is meant a method which yields possible hunches or rapidly formulated ideas which will later be subjected to more formal

analysis and proof. The method implies a freedom to make mistakes and to question. It makes use of what is known to arrive at a workable procedure as a starting place for solving a problem situation. Important aspects are the "critical question" and a low technical vocabulary. If the child can answer certain key questions, depth of understanding is assumed even though he cannot express his understanding in words.

Kaufman (39) has found evidence that "referential meaning is possible prior to attainment of a level of meaning by verbalization. Thus the exemplification of a mathematical term or symbol rightly precedes rigorous definition of it." The discovery method, or the drawing of probable conclusions based on several examples, is not the basic, distinguishing feature of mathematics. Teachers and students alike, when using the discovery method, should be aware that nothing is being "proved" in the mathematical sense (34). However, the use of the discovery method with accompanying pupil participation is surely a worthy attempt to escape from the "exposition -- examples -- exercises" approach that is virtually all that has been found in some texts (and some classrooms). Furthermore, the intuitive realization of what "should" be true is the real basis for the search for mathematical proof.

Brune (6:211) has written:

Teachers cherish in their pupils such traits as alertness, preparedness, and willingness. And possibly the greatest of these is willingness. Seldom though, do these traits develop overnight; rather,

they seem to stem from many things that pupils do. Through the situations that teachers encourage them to explore, pupils discover relations, achieve insight, and gain satisfactions for the moment as well as for later studies.

### The Need for Pre-Geometry

The present widespread interest and activity in the provision of pre-deductive geometry has been justified on many grounds. Some explicit justification is, of course, necessary; for many adults, including teachers, geometry has been associated exclusively with the upper grades, particularly in the senior high school.

Reasons for increased emphasis on geometry, particularly at the elementary level, seem to fall into three major categories. These categories are not mutually exclusive; they appear to represent different emphases taken by various authors and designers of both experimental and commercial programs.

First, authors point to the fundamental or basic position of geometry in mathematics and to the contribution geometry can make to the total school program. Thus for Hawley and Suppes (33:1), whose primary geometry programs are widely known:

Two important reasons for geometry in the primary grades are to deepen the mathematical experience of the young student and at the same time increase his reading accuracy and comprehension. The basis for the first reason is the fundamental position of geometry in the structure of mathematics. The basis for the second is the extensive

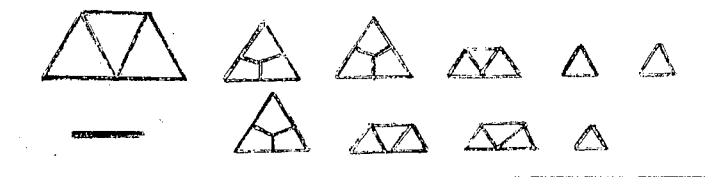
training provided by the lessons in reading, comprehending, and then executing a sequence of precise instructions.

Johnson (36:52) states:

Until recently we have assumed that mathematics for the primary grades should consist of counting, reading and writing numerals, and learning the basic number combinations. From several points of view, these may not be the most appropriate ideas for an introduction to mathematics. The psychologists would probably recommend that geometry would be a good beginning topic since the opportunities for concrete and visual experiences are abundant. The historians would point out that geometry was well developed over a thousand years before our Arabic numerals appeared on the scene ... Finally, some mathematicians are suggesting that some number concepts might be developed through geometric representations, such as number lines, blocks, etc.

So we have Dienes (20:56) illustrating the base three subtraction:

$$\begin{array}{r} 1212 \\ \underline{121} \end{array} \text{ as}$$



Wide use is made of segments and interiors of closed plane figures for illustrating fractional parts, and multiplication and division of whole numbers and fractions. Graphing in one- and two-dimensional spaces is also common (30) (41) (51) (60) (75). Finally, the geometric basis for measurement of length, area, volume, and of angles has received a good deal of attention. Felder (25:357) says:

By the time the child reaches Grade 3 he should have four abilities to draw on:

1. an intuitive awareness of difference in size
2. a choice of an arbitrary unit of measurement which is of the same nature as the thing to be measured, i.e. a unit of length to measure length, a unit of area to measure area
3. a selection of standard units for purposes of communication
4. a selection of suitable scales for convenience in measuring, e.g. he would not measure the length of a room in inches

This "choice of an arbitrary unit of measurement which is of the same nature as the thing to be measured" depends entirely on geometric definition, whether intuitive or rigorous. The choosing of the arbitrary unit is, of course, essentially using the function concept of "associating numbers with given lengths" (13) (65). This functional relationship is one that may give less difficulty to children than to adults accustomed to thinking of measurement as the use of an instrument. Here is what Syer (73:597), reviewing Hogben's "Mathematics in the Making" (35) states:

There is confusion in this book between the physical process or series of operations which are used to arrive at the length of an object and the mathematical, ideal concept that every line segment has a number associated with it that can be called its length.

The study of geometry at an intuitive, inductive level is also urged as the most effective preparation for



introduction to formal deduction. This is essentially the position taken by the Cambridge Report, dealt with earlier. It is also a basic premise for Bruner (8:43) in his "Process of Education." He quotes Inhelder<sup>4</sup> at length:

In view of all this it seems highly arbitrary and very likely incorrect to delay the teaching, for example, of Euclidean or metric geometry until the end of the primary grades, particularly when projective geometry has not been given earlier. So too with the teaching of physics, which has much in it that can be profitably taught at an inductive or intuitive level much earlier. Basic notions in these fields are perfectly accessible to children of seven to ten years of age, provided that they are divorced from their mathematical expression and studied through materials that the child can handle himself.

Many others (4) (7) (25) (32) (40) (58) (66) (67) (68) (75) have urged the study of geometry for this reason.

Finally, we are urged to allow the child geometry at the elementary level because it forms a vital and fascinating part of his natural and man-made environment:

Look all around you  
What do you see  
In the shape of a bird  
In the size of a bee?  
Ge--om--e--try! (25:356)

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<sup>4</sup>Mlle. Inhelder is a colleague of Professor Piaget at L'Institut Pedagogique in Geneva. The work of Piaget, Inhelder, and others of the "Geneva School" has had considerable effect upon present day thinking regarding the formation of number and spatial concepts by children.

Or, as Brune (6:211) says:

To enumerate and to describe man's uses of geometry would take a trip in time from prehistory to the present moment. The subject began in earth-measuring, it grew in planet-observing, it led the way in pure mathematics, and it pioneered in modern mathematics.

Man has always needed geometric principles, however dimly he may at first have perceived them. Similarly, children's lives cannot be devoid of geometry, however unaware they may be of its formal aspects. For, irrespective of its many applications and regardless of its value as a system of reasoning (and both of these phases merit attention), geometry embodies numerous ideas interesting in themselves.

We believe, therefore, that children of all ages should get ample opportunities to find out things about geometry. The goal is satisfaction, here and now, with things mathematical, and geometry abounds in such ideas.

There are many excellent texts and pamphlets (11) (16) (21) (27) (37) (45) (54) (55) (56) (58) (68) (72) that outline activities in which children can see, touch, and build representations of both two- and three-dimensional figures. Leonhardt (43) and Cohen (12) have found that such constructions can provide improved space perception as well as increased motivation. To many, this aspect of intuitive geometry is a more-than-sufficient justification for its inclusion in elementary school mathematics programs.

In actual practice, as stated earlier, no worthwhile program can take one approach in isolation from the others.

If the aim is primarily to develop concepts that will be of use in a later deductive course, good pedagogy will still require attention to geometrical content in other parts of the mathematics program and in the environment, particularly from a motivational standpoint. And if "the goal is satisfaction, here and now" (6:211), the course must still make every effort to avoid formation of concepts that will later lead to unnecessary confusion.

Irrespective of the aims an elementary mathematics program wishes to emphasize, there is one question which the architects of the program cannot avoid. Gibb (28) asks it in her article, "Do You Have a Mathematics Program?" Put another way, the authors of a well-known report (71) would ask, "What does the geometry strand look like? Is it, in fact, a strand? Is it sequential? If it is sequential, where is it going?"

#### The Problem of Teacher Education

In an Associated Press article (3) datelined Washington, John Goodlad of the University of California at Los Angeles is quoted as follows:

It is dangerous to assume that curriculum change has swept through all of our 85,000 public elementary schools and 24,000 public secondary schools during this past decade of reform.

Tens of thousands of schools have scarcely been touched, or not been touched at all, especially in areas of very sparse or very dense population.

Tens of thousands of teachers have had little opportunity to realize what advances in knowledge and changes in subject fields mean for them. Tens of thousands hold emergency certificates or teach subjects other than those in which they were prepared.

In elementary schools, teachers with backgrounds in science and mathematics constitute a species that is about as rare as the American buffalo.

While the rarity of the "species" may be somewhat exaggerated, pre-service or in-service training in mathematics for teachers has been of deep concern during the sixties. In 1960, the Committee on Undergraduate Programs (CUPM) of the Mathematics Association of America (14) made recommendations with regard to undergraduate course content for teachers of mathematics, "meant to be the minimum which should be required of teachers in any reasonable education program."

Five teaching levels are specified. On pages 983 and 984, the following are the particular recommendations for Levels I and II concerned with geometry at the intuitive level:

Level I (for teachers of elementary school mathematics):

#### Intuitive Foundations of Geometry

A study of space, plane, and line as sets of points, considering separation properties and simple closed curves; the triangle, rectangle, circle, sphere and the other figures on the plane and space considered as sets of points with their properties developed intuitively; the concept of deduction and the beginning of deductive theory based on properties

that have been identified in the intuitive development; concepts of measurement in the plane and space; angle measurement, circle measurement, volumes of familiar solids, treatment of coordinate geometry through graphs of simple equations.

Level II (for teachers of the elements of algebra and geometry):

Extension for level two to a 2-course sequence in geometry.

For the elementary teacher, a minimum of 4 mathematics courses is recommended; for level II, a minimum of 7 courses is recommended. However, as a survey by Hardgrove and Jacobson (31) shows, these recommendations are still far from being implemented by even a minority of teacher-training institutions.

For teachers already in the classroom, in-service programs are highly desirable; there may be some question as to whether they will be either available or effective, however. In light of the above, it would seem that some sort of teacher commentary to accompany a newly-adopted series of texts is a sine qua non.

#### Summary

There is no lack of evidence that pre-deductive geometry is considered to be a desirable part of mathematics education; this evidence comes both from mathematics educators and mathematicians in many parts of the western world.

There are now widely-tested experimental programs that contain large sections of pre-deductive geometry. From material available it would appear that the pertinent question at this stage is not "What can be taught?" but "What do we want to be taught? Why and how?" And in this discussion of why and how, the classroom teacher must emerge with some clear objectives in presenting the geometry content.

The inclusion of content unfamiliar at the elementary level has far-reaching implications. Within a school system the problem of teacher-training immediately arises. Teachers' commentaries can provide a basis for much in-service work. Very often, the only practicable way of making a start toward desirable changes in the mathematics program for any given system is to base studies on texts that are accompanied by useful teachers' commentaries. To wait until a majority of teachers have had sufficient pre-service or in-service training in mathematics is undoubtedly desirable in the long run; at the present time, however, such waiting can deprive pupils of the benefits of new approaches and new content in the teaching of elementary mathematics. Unless a school system has a large group of well-trained teachers able to provide in-service leadership, then the nature of the teachers' commentary should be carefully considered prior to adoption of a new series of texts.

The decision to include this new geometry content

creates problems of organization and presentation in the text. Consideration must be given to the respective roles of geometry as represented in the child's environment and of pre-high school geometry. The intuitive approach, the use of discovery, and the "low technical vocabulary" referred to by Deans (17:6) all require great care in the design and presentation of course content.

Finally, the program as a whole should surely show continuity, integration with the whole mathematics program, and some indication that it will eventually "link up" with the first course in deductive geometry. It should be noted that there is no evidence to indicate that deductive geometry will be moved down in grade level to any significant extent.

## CHAPTER III

### PROCEDURES

#### Choice of Programs

As stated earlier, for a majority of school systems that wish to increase elementary geometry content in accordance with present day thinking, the first practicable step involves the choice of a suitable, available program. This study has chosen six programs for analysis and description of their pre-deductive geometry content. The programs are all in current use in various parts of the English-speaking world. Programs A, B, C, and D are American, E is Canadian, and F is British.

The reasons for choice are as follows:

1. With the exception of F, all programs include a development of pre-deductive geometry beginning at the primary level. This geometry includes content apart from that needed for teaching other mathematical topics, such as measurement.
2. In every case there is a teachers' manual or commentary that attempts to provide essential mathematical background information for the teacher and suggestions as to presentation of text content; most of these commentaries also provide other types of help.



3. With the exception of material for grades 3 to 6 in C, all programs have been written since 1960. While modernity in itself is no guarantee of excellence, from the standpoint of geometry content at the elementary level it becomes a very important factor. This importance derives not only from the amount of geometry as such that is introduced, but in the way geometry is integrated into the program generally.

4. (a) The American programs chosen are those of nationally known teams of authors whose mathematics texts are in wide use in this country.

(b) The Canadian program is an experimental one in the process of being developed. It is based upon proposals from the elementary curriculum committee for Ontario, Canada's most populous province.

(c) Program F, from the School Mathematics Project in England, was chosen primarily because of differences in emphasis on certain topics, as compared with the five other programs. The work with coordinates and transformations, both in the plane and in three dimensions, is of particular note. The texts are designed for pupils aged 11 to 13.

#### Tabular Analysis of Text Content

Three tables concerned with text content are presented. For each topic, the tables indicate those programs that include the topic and the grade level at which

it is first introduced. (It is necessary to specify first introduction since topics may be re-introduced at various grade levels in the same program.) Thus, read vertically, the table serves as a check on the sequential introduction of topics; read horizontally the table shows the extent of agreement regarding level of introduction of topics. Omission of a topic is indicated by a blank cell in the table.

The procedures used for construction of the tables and for obtaining data are as follows:

Table I is concerned with topics introduced to form an intuitive basis for formal treatment in deductive geometry. The major headings (in capital letters) for this table were obtained primarily from chapter titles of six high school texts (38) (47) (48) (63) (64) (69), written since 1960, and designed to provide a first course in deductive geometry. The subtopics under each major heading were obtained from the six elementary programs to be analyzed. Final choice of headings and subtopics was made using the following procedures:

1. Each different chapter heading from the six high school texts was first listed on a separate card.

2. The topics from the elementary texts were then entered on the card with the appropriate heading.

3. Major headings with three or more subtopics were retained, otherwise the major heading became a subtopic, and so is still included in the table. Thus, for example,

"perpendicular lines" which was in a chapter title (or part of one) in most high school texts, now appears under the major heading, "LINES."

4. Three major headings were added:

(a) CLOSED CURVE: This concept is basic to definitions of polygons (as distinguished from their interiors), and to the concept of region of a plane. Its three-dimensional analogue, "closed region of space," is a concept basic to understanding the common three-dimensional figures.

(b) GRAPHING: This topic, basic to analytical geometry, was not placed as a subtopic under any other heading.

(c) TRANSFORMATIONS IN THE PLANE AND IN SPACE: Since transformations can form a basis for proofs in deductive geometry (15), and since intuitive presentations are found in at least two programs, this topic has been included.

The grade level reported for introduction of a topic has been determined as follows:

1. When a defined or undefined term from formal geometry is used explicitly for the first time it is recorded as introduced, even though the text may not be concerned, at that point, with developing the geometric concepts associated with the term. Thus a text may use the terms "circle" and "square" in connection with its presentation

of number facts before the program ever deals with them as topics in geometry.

2. Where the topic is either a defined or undefined term in formal geometry, the topic is considered to be introduced where the concept is dealt with specifically, even though the geometric term may not be used. Thus, if a text asks the pupil to look for plane regions that are "the same size and shape," the topic of "congruence of plane regions" will be considered as introduced even though the word "congruence" may not be used; if, in a diagram two lines are said to "meet" or "cross" at a given point, "intersection of lines" will be recorded for that grade level.

3. Where the topic is either a postulate or theorem from deductive geometry, the topic is considered as introduced if there is content (or a formal statement) that attempts to generalize. This generalization may come from several examples, from questions that ask the student to generalize, or from a statement made in the text.

Table II is concerned with the use of geometric concepts to deepen understandings in other areas of the mathematics program; it demonstrates the fundamental position of geometry in mathematics as a whole. As noted above, comparisons between this table and table I will show that geometric concepts sometimes are used in other areas before they have been dealt with explicitly in the geometry section.

In those sections of each program not specifically devoted to teaching geometry concepts, a page-by-page inspection was made. The major headings of table II list the geometric concepts found in use. The subheadings represent the uses or purposes, as stated in the program, for each concept in these non-geometric sections.

This table is not concerned with presentation of geometric concepts as such. Thus a text may be recorded as using "regions of the plane" to illustrate fractional parts where the geometric concepts of "plane" or "region" have not been dealt with in any way.

Table III lists those topics from areas other than geometry that can be applied to the development and understanding of geometric concepts. Here the topics are those which are fundamental to many branches of mathematics. Topics from this table were selected on the same basis as those for table II.

Three topics that have geometric concepts as a basis are excluded from the above tables. The geometric concepts basic to these topics are included.

1. While the geometric bases for measurement of length, area, volume, and angle are included (table II), other topics concerned with the teaching of measurement are not. Thus topics such as standard units of measure, error of measurement, or precision and accuracy of measurement are not recorded.

2. Standard formulas from mensuration are not recorded. The basic properties of common plane and three-dimensional figures are included (table I); formulas for perimeters and areas of plane figures, and for surface areas and volumes of three-dimensional figures are not included.

3. Topics from trigonometry are not included.

#### The Teachers' Guide and Commentary

A page-by-page subjective analysis (5) (9) of the teachers' commentaries for each of the selected programs was made. Following this analysis, certain generalizations were made regarding each series. Comparisons among series were based upon these generalizations under each of the following headings:

1. provisions of mathematical background information:

(a) the use of illustrations in addition to those provided by the text

(b) provision of definitions to provide for precision in use of geometric terms

2. suggested procedures for presentation of course content:

(a) the nature and clarity of the stated objectives

(b) suggested motivational material and extra activities

3. (a) provision for individual differences

(b) provision of extra exercise material, particularly material that will lead teachers to stress fundamental concepts rather than a formalized presentation concerned merely with rules, definitions, and symbolization

### The Content of the Textbook

Using the procedures described in the previous section and inspection of the tabular content, generalizations and comparisons regarding the presentation of geometric content in pupils' texts were made under each of the following headings:

1. the use of illustrations of geometric concepts in the design and structure of the natural and man-made world

2. the inclusion of constructions as pupil activities:

These constructions may range from simple paper folding to multiple-step problems with straightedge and compass; they may be concerned with design and structure or with the traditional geometric constructions from Euclid.

3. nature of the sequential order of content:

This can be difficult to assess because of the revisions which some series are now making. In many cases, for example, elementary programs include as much or more content than junior high school texts written earlier; rather than sequence, repetition is observed. Where revisions

have affected an earlier sequence, the fact is noted. However, even among the most recently written series some odd practices (from a sequential standpoint) can be found. As an example, many series cannot resist the term "line" (sometimes incorrectly used) from the very first, even though they are shortly going to devote a good deal of time to the development of the accepted mathematical concepts associated with the word "line."

4. the use of geometric concepts throughout the program as a whole:

There is certainly some evidence of a tendency to include "undigested lumps" of geometry that are largely ignored in the rest of the course. For example, the concept of "segment" is basic to linear measurement; some courses which include intuitive geometry do not use this geometric basis in the presentation of linear measurement.

5. the scope of the content:

In dealing with scope, the discussion will not be concerned primarily with the number of topics introduced by a program. There is little virtue in quantity, as such. One major consideration will be the extent to which geometry is confined to the plane rather than dealing with the physical world of three dimensions. This would seem to be of particular concern in the elementary grades. Provisions for individual differences, work with designs, and the use of constructions are all further indications of the scope of presentation.



## Conclusions and Their Use

On the basis of conclusions derived from the foregoing analyses and comparisons of commentary and text content, a check list of questions was designed to assist in the selection of pre-deductive geometry content. Major consideration has been given to the selection of questions concerning topics which will be of continuing importance in geometry content at the pre-deductive level.

## CHAPTER IV

### TABULAR ANALYSIS OF TEXT CONTENT

Three tables concerned with text content are presented in this chapter. The accompanying comment concerns extent of agreement regarding level for introduction of a topic and the arrangement of subtopics within a given program.

Because of the length of table I, comment is given after each major heading and its subtopics. For tables II and III, the comment is at the end of each table.

#### The Intuitive Basis for Deductive Geometry

There are eighteen major headings listed in table I. Corresponding to these major headings are eighteen separate parts of table I; these parts are numbered I(a), I(b), and so forth. Thirteen of the major headings are concerned with the idea of point and with the properties of certain sets of points, called geometric figures. A further topic is concerned with constructions of these geometric figures.

Among the remaining topics, three are concerned with relations: congruence, similarity, and parallelism. The last topic, transformations, deals with mathematical

operations involving geometric figures, and is dealt with in only two of the programs.

The following distinctions are made regarding geometric figures and their introduction:

1. The "picture" of a figure as a concrete or pictorial representation is usually the first stage in presentation. Its objective is the ability to recognize and to associate word and picture. Table I refers to this presentation with the subtopics "idea of" or "recognition only." This is an intuitive presentation of the concept as opposed to formal definition. Of course, for a primitive term such as "point," an intuitive concept must suffice for geometry at any level.

2. The figure is named, usually by letters, occasionally by numerals. Thus the text may refer to "segment AB" with or without an accompanying pictorial representation.

3. The figure is symbolized as well as named, and the above example would now be written  $\overline{AB}$ .

In the discussions that follow each major topic, attention will be given mainly to wide variations in inclusion of topics and in level of introduction. Consideration is also given to differences in sequences of presentation. As noted earlier, program F is designed for students aged 11 to 13 years. It was included primarily because of its pre-deductive treatment of certain topics not generally included in the other programs. However,

there are several major topics not dealt with at all in program F. Since these omissions are quite evident from the table, the fact will not be noted specifically.

TABLE I(a)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
ANGLES

	A	B	C	D	E	F
acute	7	7	7		6	x
adjacent	7	8	7		8	
angle in semi-circle		8		3		x
complementary	8	7				
definition	2	3	7	5	2	
dihedral		8				
idea of				3		
interior-exterior	5	7	7	3	4	
linear pair			7			x
naming	2	5	7	5	3	x
obtuse	7	7	7		6	x
of polygon	2	4	7	4	2	x
reflex						x
right	2	3	4	3	3	x
straight		7			7	
supplementary	7	5	7		8	
symbolization	2	7	7	5	7	x
vertex	2	4	7		7	
vertical (opposite)	7	5	7		7	

The term "angle" can be formally defined once the concepts of line and ray have been established. A, B, C, and E do not refer to angles until a definition is possible. Only D presents the idea of an angle first, by

using the term in reference to an illustration.

At the grade two level, A introduces a considerable number of topics connected with angles. These are all used frequently in succeeding grades. The topics that are left until grades seven or eight deal with special classes of angles and special relations that are not introduced until that level. Program C, whose grades three to six series was published before 1960, leaves all but one topic until grade seven.

With regard to omissions (or inclusions) of topics, there are three worthy of note. "Angle in a semi-circle" is at the grade three level in D, largely because this program devotes a large part of its introductory geometry to geometry of the circle. (See topic following.) This same program never uses the term "vertex," as applied to angles, in the first seven grades. Programs A and C, which define "interior of an angle" in terms of the intersection of two half-planes, omit use of the term "straight angle." All programs except C and E interpret measurement of angles as "associating numbers with the interiors of angles;" for these programs, measurement of angles must be deferred until interiors and exteriors of angles have been introduced.

The most pronounced differences in level of introduction are for "naming" and "symbolization." While "symbolization" is not a necessity for any other topic, naming becomes important in listing correspondences between

polygons. No program attempts a statement of congruence or similarity before naming of angles by letters has been introduced.

TABLE I(b)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
THE CIRCLE

	A	B	C	D	E	F
arc: idea of	5	7	8		8	
naming	6	7	8		8	
center	3	4	7	3	4	
central angle	6	5	8	4		
chord	6	7	7	3	4	
circumference	7	5	8	5	8	
circumscribed about polygon	6	7	8	6	8	
concentric	7				8	
definition	7	4	7		8	
diameter	6	4	5	3	4	
idea of	1	2	3	1	1	x
inscribed angle(s)	6			4		
inscribed in polygon	3			4	8	
radius	3	4	7	3	4	
secant			8			
sector		5	7			
semi-circle	7	8	8		8	
tangent	7		7	7	8	

The idea of a circle and certain associated terms such as "center" and "radius" (or "diameter") are introduced by grade four in all programs except C. At the grade three to five level, D introduces a considerable amount of vocabulary and illustration associated with

deductive geometry of the circle (38) (47) (48) (63) (64) (69). At the same time this series has not introduced "arc" or a definition of "circle" by the end of grade seven. This definition is not introduced by any program, except B, before grade seven. The "angle in a semi-circle" for D is the only attempt to anticipate theorems from deductive geometry of the circle. For A and E the idea of "tangent" is used prior to "circle inscribed in polygon;" for D the reverse order holds. For both A and E the term "concentric" is introduced in connection with problems.

TABLE I(c)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
CLOSED CURVE

	A	B	C	D	E	F
as boundary	1	4	1	3	1	
idea of	1	3	1	3	2	
interior-exterior	1	4	1	3	2	
regions	1		7		2	
simple, not simple	2		7	7	7	

The concept of closed curve is basic in the definition (or description) of polygons and circles. By grade four, all courses have emphasized that "curve" refers only to the boundary. Either "interior of a closed curve" or "region of a plane" is the geometric concept which underlies the study of area. All courses except B introduce one or both of these topics before "measurement of

area." Texts for B, C, D, and E are able to defer or omit a discussion of simple and not simple closed curves by only showing curves that are simple.

TABLE I(d)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
CONGRUENCE

	A	B	C	D	E	F
and area	3				4	
angles	2	5	7	5	3	
as equivalence relation	6		7		5	
idea of (fitting)	1	5	3	5	3	
polygons (corresponding parts)	5	8	7	5	5	
segments	2	5	7	5	1	

Intuitive ideas of congruence of geometric figures, as figures that are "exactly the same size and shape," have wide use in elementary mathematics programs. Two of the most common uses have been in the teaching of area and in illustrations of fractional parts of regions. However, for B, D, and E, one or both of these uses of congruence is introduced before "idea of." All programs introduce congruence for segments and angles within two grade levels of introduction of the two concepts. Introduction of congruence of segments and angles in turn allows for a presentation of congruence of polygons as a correspondence. Only A and E show specifically that equality in area for two regions does not imply congruence. The symmetric and transitive properties of congruence for segments are dealt



with specifically by programs A, C, and E.

TABLE I(e)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
CONSTRUCTIONS

	A	B	C	D	E	F
angle bisection: paper fold.					4	
st.edge-comp.	6	5	8	5	8	
angle copy: octon paper fold.	5					
protractor	5	7	7	5	5	x
st.edge-comp.	5	5	8	5	8	
axis of symm. (paper fold.)					4	x
circle:	4	5	8	3	4	
circle through 2 pts.				3	8	
circle through 3 pts.				3	8	
parallel lines: paper fold.				4		
protractor			8			
st.edge-comp.		6			8	
perp.bisector (seg.):paper fold.					4	
st.edge-comp.	8	8	8	5	8	
perp.lines (rt.angle):paper fold.	3	6		3	3	
protractor					5	
st.edge-comp.	6	5	8	6	8	
segment copy: st.edge-comp.	4	4	8	5	4	
st.edge (paper fold.)	2					
surfaces (paper fold.): cone	4	8				
cylinder	4	5			8	x
polyhedra (reg.)	4	8		4	4	x
prisms	4	7		4	4	x
pyramid	4	5		4	4	x
triangle: inscribed circle				4		
ASA	8	8	8	6	6	
SAS	8	8	8	5	6	
SSS	5	6	7	5	6	

It is interesting to note that program C, where content for grades three to six has not been revised since 1958, contains the least construction activity, particularly in the lower elementary grades. Apart from C, program E shows a much later introduction of most straight-edge and compass constructions than programs A to D; program E uses paper folding to a slightly greater extent. Constructions of cones, polyhedra, and prisms for B, and cylinders for E are not done until the grade level at which mensuration formulas are introduced. Except for program D, with the early work with circles mentioned before, extensive use of the compass is not begun before grade five.

TABLE I(f)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
GRAPHING

	A	B	C	D	E	F
axes	6	8	8		5	x
on line	3	8	7	5	3	x
on plane	3	5	6	5	3	x
origin	6	8	8		7	x
quadrant	8		8		5	x

The use of graphing is closely associated with the use of coordinate planes and spaces in the study of geometric figures. Programs A, E, and F make use of the coordinate plane for topics from analytical geometry and for geometric transformations. For A and E, these topics

begin at the grade three level and so does graphing. D begins graphing of solution sets in grade five. None of the terms, "axes," "origin," or "quadrant" is essential.

TABLE I(g)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
LINES

	A	B	C	D	E	F
betweenness: idea of	7	2	7	7	2	
properties			7	7	7	
determined by 2 points	2	4	7	7	7	
idea of	2	3	2	5	2	x
intersecting (meeting):	2	3	2	4	3	
concurrent	8				7	
inf. number through point	2	4	7		7	
naming	2	7	7	5	2	
parallel	7	5	2	4	3	x
perpendicular	3	5	4	6	3	x
separation	5			7		
skew		7	7		5	x
symbolization	2		7	5		

The intuitive idea of line is introduced at the primary level by A, B, C, and E; naming and symbolization show much the same pattern as was shown for angles. Perhaps the most surprising divergence in level of introduction is shown for "parallel lines." Program A introduces intersecting lines in grade two. Representations of coplanar lines that do not meet are common in the physical world; yet the term describing this relationship is left until grade seven in program A. Both A and B use "between" in

defining a segment but do not list the properties associated with this term. The other notable omission is that of "skew lines" for A and D. Both of these programs use representations of three-dimensional space where skew lines are common.

TABLE I(h)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
PARALLEL LINES

	A	B	C	D	E	F
alternate angles		6			8	
corresponding angles	7	6	7		8	
transversal	7	6	7		8	

The concept of congruent corresponding angles (or alternate angles) provides a method for constructing a line parallel to a given line. Program D introduces a paper folding method that uses corresponding angles but never names them as such. On the other hand, A introduces "corresponding angles" and "transversal" but never carries out a construction of parallel lines. "Alternate angles" (omitted by A, C, and D) are not needed for construction where "corresponding angles" are available. There is no marked difference in level of introduction.

All courses except C and D introduce the idea of plane at the primary level. Handling and constructing three-dimensional surfaces with plane faces (which all courses include) is used by E to afford an extra perception basic

to the concept of plane; only program E speaks explicitly of a face as a subset (or part of) a plane. E also makes an early introduction of parallel planes as an extension of parallel plane faces of solids.

TABLE I(j)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
PLANES

	A	B	C	D	E	F
determined by 3 non-collinear pts.	7	7	7	5	7	x
figure:	3	3		7	7	
symmetry: axis of	6				4	x
idea of	3	3	7	5	3	x
inf. number containing given line	4	7	7	5	7	
intersection of	4	7	7	5	4	x
intersection with line	4	7		7		x
parallel	8	7	7		4	x
separation (half-plane)	4	8	7	7		
subset of (face)					3	

The term "plane figure" is a useful one in definition; only programs A and B introduce it at the primary level. For intersections involving planes, level of introduction is lower for those programs, A and E, where work has been done with three-dimensional surfaces since grade one. Axial symmetry (or lack of it), as an important property of plane figures, is dealt with by programs A, E, and F. The concept of half-plane is used in grade five by program A for a definition of "interior of angle."

TABLE I(k)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
POINTS

	A	B	C	D	E	F
coordinate (line)	3	3	7	3	2	x
coordinates (plane)	3	8	6	5	3	x
idea of	2	3	2	3	1	x
naming	1	4	2	3	2	x
representation	1	4	2	3	1	
symmetry (rotational)						x

With the exception of plane coordinates the levels of introduction for topics are relatively uniform. As stated earlier, use of coordinates in the plane is closely associated with graphing.

TABLE I(m)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
RAYS

	A	B	C	D	E	F
idea of	2	3	7	5	2	
naming	2	7	7	5	2	
opposite			7			
subset of line	2	3	7	5	2	
symbolization	2		7	5		

For every program, grade level of introduction is the same as that for definition of "angle." Naming and symbolization follow previous patterns regarding level of introduction.

TABLE I(n)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
POLYGONS

	A	B	C	D	E	F
angle of	2	4	7	4	2	x
diagonal(s):			6	4	3	x
number of				4	7	x
general (definition)	2	7	2	4	1	x
naming by letters	2	2	2	5	2	x
naming, generally	6	6	8	4	7	x
parallelogram:	7	4	6	4	3	x
altitude of	7	6	7	5	7	
properties of	7	4	6		7	x
quadrilateral	3	4	7	3	3	x
rectangle:	1	2	3	3	1	x
defined	3	7	8	3	3	x
properties of	3	4	6			x
regular			8			x
rhombus	8	4	8	7	7	x
square	1	2	3	1	1	
trapezoid	8	5	8	7	7	x
triangle	1	2	3	1	1	x
vertex	2	2	2	4	1	x

All courses use and name squares, rectangles, and triangles in the primary grades. All except course B use the general term "polygon" at this level. Again, for purposes of definition, the general term would appear to be a useful one. With the exception of C, level of introduction is relatively uniform. For "parallelogram" introduction follows introduction of parallel lines and segments.

TABLE I(o)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
SEGMENTS

	A	B	C	D	E	F
definition (betweenness)	7	6	7	7	7	
endpoints	1	4	2	5	1	
idea of	1	3	2	3	1	x
midpoint of			8	5	4	
naming	1	3	2	4	2	x
side of polygon	1	3	2	5	1	
subset of line	2	4	2	6	2	
symbolization	1	7	7	5		

Two programs, A and E, introduce the idea of "segment" in grade one, before the idea of "line." This is done by feeling or picturing a segment as a "straight side" of common polygons such as rectangles and triangles. The idea of line is then developed through indefinite extensions of segments. If the idea of line is dealt with first, as in B, C, D, and F, then a segment is part of a line.

Plane and three-dimensional figures are considered as sets of points when texts speak of points on (or in) the figure and points not on the figure. D does not make this distinction for plane figures; only A and B make it for three-dimensional figures. Closed figures are considered as sets of points when the text distinguishes points inside (or in the interior of),



points in (or on), and points outside (or in the exterior of) the figure.

TABLE I(p)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
SET OF POINTS

	A	B	C	D	E	F
curve: betweenness			1			
closed and open	2	4	1		1	
idea of (continuous)	2					
interior-exterior: angles	5				4	
closed curves	2	4	2	4	1	
line	2	4	7	5	2	x
plane	3	7	7	5	3	
ray	2	4	7	5	2	
segment	2	4	7	5	2	
space:	4	4	7		7	
half space	7					
three-dimensional surfaces	4	4				

The concept of space as "the set of all points" is omitted by D. Only A refers to the continuity of curves generally; all programs infer continuity for a closed curve.

Program C uses "betweenness" to name a property of open curves that is not possessed by closed curves. This usage of the term is not the customary one, which program C itself uses in grade seven.

In the elementary grades, only A and E use the idea of enlargement which forms an intuitive basis for similarity. In A this concept is developed in connection

with the study of graphing on the coordinate plane; for polygonal figures constructed on the plane, coordinates of vertices are doubled, tripled, and so on, to obtain an enlargement. For C and E enlargement is based on the idea of ratio for corresponding sides.

TABLE I(q)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
SIMILARITY

	A	B	C	D	E	F
idea of (enlargement)	3		8		5	x
polygons: (corresponding parts)	8	8	8	6	5	x
ratio: corresponding sides	8		8	6	6	x

Programs A, C, D, and E introduce similarity as a correspondence near the grade level at which congruence is introduced as a correspondence. B merely states the "AAA property" for two triangles and calls them similar.

Some programs, mainly A and E, introduce three-dimensional figures, the "geometry of the physical world," in the first or second grade. In all cases, where properties of these figures are studied formally, mensuration formulas are being developed. D omits this development for cones, cylinders, prisms, and spheres. A and E introduce many "solid" figures for recognition well before the level for formal study of properties. Only A and E introduce "great circle of a sphere;" E "obtains" a great circle from a plane of symmetry for the sphere.

TABLE I(r)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
THREE-DIMENSIONAL SURFACES

	A	B	C	D	E	F
closed (as boundary)	4	3	8		4	
cone: properties	8	8	8			
cube: properties	4	4	6	3	4	x
cylinder: properties	7	5	8		8	
edges, faces, vertices:	1	2	8	4	1	x
Euler's formula	8					x
interior-external	4	5			4	
intersections with planes	8		8	6	4	x
parallel edges				6	4	
parallel faces			8		4	
prisms: altitude of	7	6	6		8	
oblique	8	7		7		
right	4	5	5	5	4	
recognition only: cone	4	5	6	5		
cone: truncated				5		
conic sections		7		6		
cylinder	1	4		5	1	
pyramid (square base)	4	5	6	4	1	
rectangular prism	1	2		3	1	
sphere	1	4	8	5	1	
tetrahedron	4	5	6	4	1	x
sphere: definition	8	4	8			
great circle of sphere	8				4	
hemisphere		8				
symmetry: plane of	8				4	x

Transformations in the plane, developed most fully in Program F, are concerned primarily with investigation of properties of geometric figures on a coordinate plane.

Program A uses "enlarging" as an intuitive basis for similarity of plane figures. Only the operation "shearing" is dealt with in three dimensions.

TABLE I(s)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
TRANSFORMATIONS IN PLANE AND SPACE

	A	B	C	D	E	F
enlarging: 1-dimensional	3					x
2-dimensional (area)	3					x
rigid motions: idea of						x
reflections:	6				6	x
and symmetry	6				6	x
rotations					6	x
translations	6					x
shearing: plane						x
3-dimensional						x

While the idea of a triangle is introduced early (see Polygons), little of the geometry of the triangle is dealt with before grade five. Programs A and D show exceptions to this, particularly with their topics (and technical terms) introduced at the primary level. Programs B, C, and D never present the fact that an isosceles triangle has two congruent angles. Program D presents special triangles (classified according to sides), but never the term "scalene" for the general case. "Triangle property" refers to the fact that the sum of the measures of any two sides of a triangle is greater than the measure of the third.

TABLE I(t)

LEVELS OF INTRODUCTION OF PRE-DEDUCTIVE GEOMETRY CONTENT:  
TRIANGLES

	A	B	C	D	E	F
altitude (height)	5	6	5	6	7	
angles of (isosceles)	6				7	
angle-sum	3	6	7	3	8	x
base	5	6	5	6	7	
definition: equilateral	3	5	6		6	x
isosceles	3	6	6	4	6	x
right	3	6	7	3	7	
scalene	6	7	7		6	
lengths of sides: "triang.prop."	5				8	
median			8			
Pythagorean property:	8	8	8	6	8	x
hypotenuse	8	8	8	4	8	

From the data in table I, certain generalizations are indicated:

1. Level of introduction of topics within a series is affected by the order in which various parts of the series have been written.

(a) For A particularly, grade levels shown might seem to indicate that very little geometry is included for grades between the primary level and grade seven. In actual fact, the primary program is the most recently published; a great many geometry topics formerly introduced in grades four through six are now introduced in the primary grades.

(b) As stated earlier, the grades three through

six materials for C have not been revised since 1958. Their content illustrates very clearly the widespread change in thinking, since 1960, regarding geometry content at the elementary level.

2. The level of introduction of content is also affected by the order of presentation, particularly at the beginning.

(a) While the primary levels for A, B, and E include the primitive or undefined terms, point, line, plane, from deductive geometry, program D begins with emphasis on circles and their construction.

(b) The inclusion of three-dimensional geometrical shapes, from the beginning, is characteristic of programs A and E.

3. Decisions regarding inclusion of certain terms, such as "dihedral angle" or "secant of a circle," particularly in the upper grades, seem to be largely arbitrary.

#### Geometry Concepts Used to Deepen Understanding of Other Areas

In other areas of the mathematics programs, common geometric figures and various other configurations of dots are used for illustration. Geometry also supplies the concepts basic to measurement. Many of these uses of geometric figures have been common to elementary programs for a long time. For such topics there tends to be

a good deal of agreement among the various courses with regard to grade level of introduction.

TABLE II  
LEVELS OF INTRODUCTION OF GEOMETRY CONCEPTS USED IN OTHER AREAS

		A	B	C	D	E	F
ARRAY	fractional parts	1		3	4	3	x
	magic squares	3					
	number operations	1	1	3	2	2	x
	number properties	2	5		3	3	
	numeration	3	1	4	2	3	
	square numbers (arith. prog.)					7	x
CIRCLE	circular disks					1	
	graph	6	5	6	5	4	x
	modular arithmetic	7	8				x
	Venn diagrams	6	7	7	4	7	x
CONGRUENCE	fractional parts of regions	1	1	3	1	2	
COORDINATES	fund. operations	1	1	1	1	1	
	graphing (generally)	6	5	6	5	4	x
	latitude, longitude	8					x
	ordinals		2	2		2	x
GEOMETRIC BASIS OF MEASUREMENT	general	5	7		3		
INTERIOR OF ANGLE	comparatives	4	7		5	3	
	measurement: additivity	5	7		5	4	
	subtractive			7			
	use of protractor	5	7		5	4	

TABLE II (continued)

		A	B	C	D	E	F
LINE (NUMBER)	comparatives (inequalities)	1	2	2	4	2	
	decimal numeration	5	6	5	5	5	x
	dense set (betweenness)	7		7			
	equiv. fractions	2	4	5	4	4	
	fund. operations	1	2	5	1	2	
	horizontal	6		8	5	5	x
	integers	2	2	6	5	8	x
	number properties	2	4	8	3	7	
	ordering numbers	1	2	7	4	2	
	real numbers (irrationals)	8	8	8			
	ruler (linear scale)	1	2		1	2	x
	vertical	6		8	5	5	x
RAY	as non-negative "number line"		8		1		
REGIONS OF PLANE	comparatives (inequalities)	5	2	5	3	1	x
	fractional parts	1	1	3	1	2	
	measure. of area: additivity	3	2	5	3	4	x
	number properties	4	5	6		4	
	numeration		4	6	6	3	
	operations with numbers	6	6	5	4	5	
REGIONS OF SPACE	comparison	6				5	
	measurement of surface area	7	4	6	4	7	
	measurement of volume	6	6	6	3	5	
	number properties	3				5	
	numeration		4	3	3	7	
SEGMENT	comparatives	1	4	7	6	1	
	directed segment (vector)		6	8			x



TABLE II (continued)

		A	B	C	D	E	F
SEGMENT (continued)	linear measurement:	1	2	1	1	1	x
	length of curve	3	4		6	7	
	perim.of polygon	3	3	5	4	4	x
	ratio of lengths				4	5	
SIMILARITY	scale drawing	3	5	5	4	5	x

## ARRAY

"Magic squares" and "square numbers," in A and E, are topics introduced partly for their intrinsic interest. For the other subtopics, illustrations with arrays are widely used in these and other programs. B does not use them for fractional parts of collections of discrete objects. The most common number property illustrated with arrays is commutativity for multiplication of whole numbers; C omits this use.

## CIRCLE

All programs use the circle or "pie" graph. Venn diagrams are the common visual illustration for set notation. Modular or "clock" arithmetic is used by A, B, and F as an introduction to finite arithmetic.

## CONGRUENCE

A common method of illustrating fractional parts is to divide a pictured region of the plane into congruent parts. This is done at the primary level in all courses.

## COORDINATES

The so-called number line is used at the primary level in all courses to illustrate fundamental operations, particularly addition and subtraction. "Graphing (generally)," found in all programs, refers to graphing on the plane; it requires at least an intuitive introduction to plane coordinates. The use of coordinates for ordinal number involves location of points (or objects) which are, for example, "fourth from the left, second from the bottom." This approach not only uses ordinal number, but emphasizes intuitively the idea that the coordinates of a point depend on an arbitrary choice of position for the origin. No use of this approach is made by A or D.

## GEOMETRIC BASIS OF MEASUREMENT

In Chapter II, reference was made to a report by Felder (25:357) stating that measuring depends upon "choice of an arbitrary unit of measurement which is of the same nature as the thing to be measured." C, E, and F make no specific reference to these ideas. Only D introduces them prior to the measurement of angle, area, and volume.

## INTERIOR OF AN ANGLE

For angle measurement, the approach used by A, B, D, and E is essentially "associating numbers with the interiors of angles." Program C associates a number with each ray of the angle; the angle measure is the absolute difference.

## LINE

In addition to its use in presentation of number operations, the number line is used by all programs for ordering numbers and introducing ideas of inequalities. Number lines are used by all programs for a representation and intuitive presentation of each number system introduced. Commutativity and associativity are presented in a manner comparable to that used for fundamental operations. The terms "horizontal" and "vertical," usually applied to the axes in graphing, are not introduced by B.

## RAY

While usually referred to as a number "line" the representation used may be that of a horizontal ray.

## REGIONS OF PLANE

The topics listed are well known. Additivity, an assumption for areas as it was for interior of angles, is used by all programs. All programs use areas of rectangular regions to illustrate commutativity and to illustrate multiplication and division of fractions. All programs, except A, use illustrations of regions of one hundred square units and ten square units for decimal numeration.

## REGIONS OF SPACE

For all programs, measurement of volume begins with a standard or non-standard unit that is a cube. Two

programs, A and E, compare volumes of "irregular solids" by the liquid displacement method. Cubes with a measure of ten (units) to a side are useful in illustrating decimal numeration, either for whole numbers or fractions; such illustration is used for fundamental operations in programs A and E. Program D, which introduces the general concept of measurement in grade three, features early introduction of measurement of both surface area and volume.

#### SEGMENT

While programs B, C, and F use directed segments, there is no study of vectors as such. A and B introduce the topic "length of curve" using the "string method." C, D, and E introduce it in connection with circles, in order to obtain an empirical approximation of the constant ratio, circumference to diameter. In D and E, ratios of lengths are studied, not only as an application of ratio, but in connection with properties of similar polygons.

#### SIMILARITY

Although "scale drawing" is an application of ratio to similar figures, all programs introduce the topic before "similarity." In these programs "scale drawing" is primarily an exercise in computation.

Of the fifty subtopics listed in table II, thirty-eight are used by four or more programs. Clearly, there is an extensive amount of agreement as to the uses of

geometric illustration in all areas of the programs.

While all programs make use of the geometric concepts basic to measurement, only program D presents the topic prior to actually using it.

### Concepts from Other Areas Used in Geometry

All of the major headings in this table represent concepts of a fundamental nature. As such, they are used throughout mathematics.

TABLE III  
LEVELS OF INTRODUCTION OF CONCEPTS FROM OTHER AREAS  
USED IN GEOMETRY SECTIONS

		A	B	C	D	E	F
EQUIVALENCE	relation	2	4	4	4	4	
FUNCTION	measurement	5	3	7	3	2	
INEQUALITIES	number line	2	3	2	4	2	x
NUMBER	number line	1	2	2	1	2	x
RATIO	similarity	3		8	6	6	x
SETS	renaming segments	2	3	2	5	2	
	sets of points	2	4	2	4	1	
SET INTERSECTION	interior of angle			7			
	lines	2	3	2	4	3	x
	line and plane	4	7	7	5	7	x
	planes	4	7	7	5	4	
	3-dim. figure and plane	8		8	6		x
SET UNION	angle	2	3	7	5	2	
	polygon	2	7	2	4	2	

## EQUIVALENCE

None of the programs treat this topic formally. Its most common use is in discussing equivalent fractions. However, A, C, and E draw specific attention to the symmetric and transitive properties of congruence. Program E does this for similarity.

## FUNCTION

This use of the function concept is inherent in the geometric basis for measurement which was discussed earlier.

## INEQUALITIES

For this table the general statement is: "If  $p$  is less than  $q$ , then  $p$  is to the left of  $q$  on the number line." The level of introduction depends upon a previous introduction to inequalities, usually through correspondences between groups of objects.

## NUMBER

As a teaching aid and in its use for graphing, the number line has near to universal use in elementary mathematics programs.

## RATIO

The concept of ratio is necessary for both parts of the "if and only if relation" for corresponding sides of polygons. In grade three, A introduces the ratio relation for similar polygons by enlargement on the

coordinate plane. As noted earlier, B does not state this ratio relationship.

## SETS

This concept, fundamental to many parts of mathematics, is now introduced at primary levels in many programs. Its use in geometry was discussed under table I.

## SET INTERSECTION

The lack of uniformity for levels of introduction is due primarily to differences in level at which planes are studied. One exception is program B which introduces the idea of plane in grade three and then does little more with it until grade seven.

## SET UNION

Program B does not introduce polygons as a general class until the seventh grade. For this reason there is no definition, before grade seven, of a polygon that indicates a union of segments. Similarly, thinking of an angle as a union depends upon introduction of the idea of ray.

Table III shows wide agreement among courses regarding concepts from other areas to be used in geometry. Level of introduction, as reported here, is entirely dependent upon the level at which the geometry concepts are introduced.

## CHAPTER V

### THE TEACHERS' GUIDE AND COMMENTARY

As has been stated earlier, the content of the teachers' commentary can play an extremely important role in the presentation of pre-deductive geometry. All of the commentaries under consideration here have content that can be used for several purposes.

The following analysis of commentaries is concerned with the nature of the mathematical background information provided, the suggested procedures for presentation of course content, and the inclusion of content designed to provide for individual differences and for extra exercise material. Under each subheading a comparison of programs will be made.

#### Mathematical Background Information

Mathematical background to unfamiliar geometry content found in elementary texts is available from many sources. However, if this information is provided in commentaries for texts that are in use by teachers, it can be of considerable use. The background material provided in a commentary is organized on a basis of text content and objectives. As such it can have the advantage



of immediacy for teachers; it is there when needed, and its application to the actual classwork is provided by the text material. Such commentary material, besides being of help to the individual teacher, can serve as a basis for useful in-service classwork for groups of teachers.

The general plan for program A is to provide a separate section on mathematical background for each chapter. In these sections there is extensive use of illustration to supplement illustrations from pupils' texts. In addition, the sections on "Suggested Procedures" include a detailed discussion of text material. Particular note is taken of questions with more than one possible answer; common misinterpretations are discussed; concepts to be emphasized are also noted. These sections also serve to deepen the teacher's understanding of the mathematical background involved.

New vocabulary for each section is introduced in two places. It is italicized (and in many cases illustrated) in the "Background." In addition, for the discussion of each lesson, the new vocabulary to be introduced is listed. The terms are then italicized in the "Suggested Procedures," to show their actual use in presentation to pupils.

In regard to extra illustrations, precision of definitions, and discussion of concepts, the background

sections of the commentary go well beyond the content of the pupils' texts. This is particularly true for grades one through six. The background sections for grades one through three form a well-integrated sequential presentation. Due to the order in which the materials for the grades were written, there is repetition of content for grades four through seven.

For program B, mathematical background information is provided from two main sources:

1. There is an illustrated glossary of mathematical terms. It is identical for all commentaries from grades one through eight. This is the chief source of background information.

2. In the discussions concerning presentation of specific lessons there is some background information given under two headings, "Emphasis" and "Procedures." The section, "Emphasis" sometimes gives a mathematical statement or definition of the concepts to be developed in the lesson. "Procedures" discusses mathematical concepts underlying the questions presented.

For exercise answers, which are included in the commentary, there is further illustration where necessary. Vocabulary is listed for each lesson; definitions of terms are in the above-mentioned glossary.

The glossary only partially fulfills its function as the source for definitions. There are less than 75

geometry terms in it; table I of Chapter IV shows that there are more than 140 such terms for this program. Thus the pupils' texts are relied on to provide nearly half the definitions.

In program C, the mathematical background information is under "Comments" for each lesson in the commentaries for grades one through six. For grades seven and eight the comparable section is "Content Overview." In these sections the background material is presented in a short expository form. While the text is profusely illustrated in the geometry sections, extra illustrations are provided in both the "Comments" and "Content Overview" sections of the commentary. These sections form a well-integrated sequential presentation of mathematical background for the geometry sections. In general, this content goes beyond that of the textbooks.

In a section on "Reference Materials" a bibliography of selected texts, articles, monographs, yearbooks, and pamphlets is included. The bibliography is not annotated. A complete overview of the organization of mathematical content for the series is supplied.

New vocabulary is listed for grades one through six; for grades seven and eight the vocabulary is italicized throughout the content overview, but is not listed separately.

In program D, each of the teachers' commentaries

for the elementary grades contains a section entitled "Mathematics Text for Teachers." The contents of this section are common for the first three grades; extra sections are added for grades four through six. Although grades three through six all have sections or chapters devoted specifically to the introduction of geometry concepts, no geometry is dealt with in the "mathematics text" section.

For the geometry sections or chapters, very brief sections called "Mathematics" have been written. To a large extent, these sections, which are not illustrated, are concerned with a discussion of the presentation in the textbook, rather than with the presentation of mathematical content to the teacher. In connection with the "Directions," written for each lesson, there is some discussion of the mathematical concepts underlying the material presented in the text. The vocabulary for the geometry chapters of grades five and six is listed at the beginning of the chapter. No attempt is made to give the teacher any definitions. Some of the terms are defined in the pupils' texts.

For grades one through six of program E, a section identified as "Background" is included for each unit devoted to geometry. These background sections are well illustrated. Further discussion of mathematical concepts is given under a section entitled "Comments," which is

largely concerned with purposes for introduction of material. New vocabulary is listed for each unit.

The above organization is followed for grades seven and eight with one exception. Since the geometry content is in two chapters for each text, the commentary for each chapter begins with an overview of the mathematical concepts to be covered. "Background" and "Comments" accompany each lesson. Illustration is confined to answers to exercises however. Here again, the content extends beyond that of the textbook.

For program F, the teachers' guide presents a series of commentaries, one for each of the geometry chapters. Furthermore, these geometry chapters are grouped together, (chapters 4, 7, 8, and 16 for one text) and an introduction is written for the group. This background material is well illustrated. Vocabulary and symbolization are listed under "Terminology." On the topic of "Motion Geometry," or the geometry of transformations, the material presented constitutes a short introductory course.

The six programs show a wide variation in the extent of background material provided. The following points of comparison seem to be indicated:

1. A, C, E, and F assume complete or near complete responsibility for providing the teacher with background information that is of considerably greater depth and breadth than the corresponding text material.

2. Among these four there is considerable difference in practice:

(a) F presents background for all the geometry chapters from the text under one section of the guidebook. As indicated earlier, the introduction, comment, and illustration provide a short course.

(b) In general, A discusses the geometry of a chapter under one introductory background section. This section attempts to show the relation of all parts studied. With this organization and the wealth of illustration, these commentaries can also provide a short course in the content covered.

(c) C and E also attempt to provide all necessary background information. Much of it is distributed on a lesson by lesson basis. C provides more illustrations.

3. The glossary in B, common to all texts, is well illustrated. As noted earlier it covers about half the new terms introduced. For the rest, the teacher presumably relies on the text, and where this is not sufficient, looks elsewhere. The new concepts are discussed, lesson by lesson.

4. Program D makes no attempt at a complete presentation of background material required for its geometric content. Introductory comments to chapters are general; there are no illustrations and vocabulary is not defined. The necessity for looking up definitions elsewhere is referred to explicitly.

## Presentation of Course Content

The elementary school mathematics text or worksheet can serve different purposes. Texts can be used for independent study, for information and illustration during a lesson involving both pupils and teacher, or for review following a class lesson. Exercise material may be thought of as an evaluation of the learning activities, as an extension of these activities, or both. For a variety of reasons, an expository development may be rarely used by texts; on some pages, particularly at the primary level, the printed word may be entirely missing. For certain topics, some programs are written so that the activities associated with the learning of new concepts are carried out before the text is ever used. In all such cases the teachers' commentary can play an important role as a guide for suggested procedures in the presentation of course content. Irrespective of text organization, for content that is very different from what has been usual, there may be a great necessity for guidance in preparation.

In program A, while the lesson objectives as stated are clear with regard to mathematical concepts to be learned, they do not always indicate the pupil achievement that should result. In the commentaries for the most recently written part of the program, grades one through three, the objectives are generally outlined in

behavioral terms such as, "to distinguish...", "to recognize...", and "to observe..." For this level, worksheets are generally designed to follow content presentation as an evaluation of the success of the objectives. For grades four and above, objectives are generally stated as "understandings and skills to be developed." The "Explorations" and comparable exercises for grades seven and eight, although included in the text, are essentially pre-textbook activities and discussion in that they precede the regular exercise work. By their nature, these exercises emphasize outcomes in terms of pupil performance.

For the primary grades, pre-textbook activities are included in the teachers' commentaries. Beyond the primary level there is little attempt to use geometry in its various applied forms for motivational purposes; activities designed to follow textbook presentations are common. For all grade levels, various models are suggested as an aid to presentation of content.

The program B teachers' edition speaks of both "Objectives" and "Aims." For each geometry chapter, the objectives for the chapter are listed and discussed in general terms. The "Discussion of Objectives" presents reasons for selection and presentation of content and its relation to the geometry content of the program as a whole. "Aims" are specified for each section of the chapter. They are stated in overprint form for each



section on textbook pages included in the teachers' edition. Statements are behavior-oriented and begin with phrases such as, "to analyze..." and "to investigate..."

The notes regarding presentation are largely concerned with the development as presented in the text. There appear to be no pre-textbook activities suggested; one text presentation suggests use of certain pictures as motivation. The overprint notes, both as comment and suggested procedure, are numerous; they are used for exposition, developmental questions, and exercises.

The objectives of program C, as stated for each lesson, are in a very concise and useful form. Statements begin with forms such as, "the child learns to...", or "the child discovers..." For the primary grades, practice tablets provide an evaluation of the extent to which objectives are attained.

The teachers' commentary contains page facsimiles to which instructional notes are keyed. The suggested activities are seldom pre-textbook.

For the upper grades, detailed instructional notes are provided, even though the text presentation is generally in the form of developmental questions, allowing for a discovery approach. This combination of text and commentary permits a wide variation in presentation.

The reference section mentioned earlier includes a bibliography on pedagogical background. Grade placement

and a suggested time schedule for topics in the series is indicated.

In program D the geometry objectives are listed as "Purposes." Statements begin with phrases such as, "to introduce..." and "to show..." A check with table I, Chapter IV, however, will indicate that the content from the beginning is construction oriented. Thus the above-mentioned "purposes" are often realized through pupil performance of these activities.

Suggestions regarding pre-textbook presentation of content are concerned largely with the listing of topics that should be reviewed in preparation for each lesson. Other notes regarding presentation are keyed directly to textbook pages.

Objectives for program E are not stated as such. For each lesson "Concepts" that are to be developed are listed. There is considerable explicit reference to pupil performance to be developed, however.

For the primary level the student worksheets are designed as a behavioral check or evaluation after class activities are completed. Presentation of new concepts is discussed with the teacher under "Comments," and detailed suggestions are given under "Procedures." By its very nature this section makes wide use of pre-textbook activities, largely concrete in nature.

For the upper grades in this series, there is a

shift to the use of developmental questions in the text for presentation of new concepts. Presentation, including suggested activities, is discussed under "Comments." A separate heading, "Activity," is concerned with motivation and, in some cases, follow-up to presentation. As with program C, a considerable variety of presentation is made possible by the content and organization of the commentary and text.

In program F, for each section of the geometry chapters a "Commentary" is given in the teachers' guide. There are no separate headings and objectives are not stated separately. A large part of each commentary is devoted to discussion of text questions. It is here that purposes and objectives are commonly mentioned. Suggested pre-textbook activities are common and the opening text exercises of each section are construction oriented.

As is to be expected, suggested procedures found in teachers' commentaries are affected to a large extent by the organization of the accompanying text. The following comparisons among programs are made with reference to this text organization.

1. (a) Programs B and C state objectives in concise behavioral terms; they use text pages both as "teaching" pages and for evaluation.

- (b) At the primary level, A and E use text pages almost entirely for evaluation. Expected pupil performance

is included in the pre-textbook development outlined in the commentary.

At the grade seven and eight level the behavioral objectives for A seem most evident in the textbook "Explorations." At this level, E makes use of developmental questions and discovery for presentation of content in text; this is where objectives become most evident.

(c) Objectives for D and F are clearest where the textbook content is concerned with constructions or where there are comments regarding specific questions.

2. Pre-textbook motivation suggestions are almost entirely confined to A and E.

3. (a) For reasons stated above, A and E use pre-textbook activity for presentation of material at the primary level.

(b) Above the primary level, A suggests many follow-up activities.

(c) At the grade seven and eight level, C and E make extensive use of developmental questions and discovery in presentation of text content; these approaches are found in some sections of A. In all cases this is accompanied by a detailed discussion in the commentary. A wide flexibility of presentation thus becomes possible; approaches range from independent study of developmental questions by the student to a teacher presentation with pupils' texts closed.

(d) Program F suggests some pre-textbook activities

both for teacher and pupils.

(e) Procedures for B and D are almost entirely concerned with text pages.

4. Programs A, C, and E make specific references to the possible alternate initial approaches referred to in 3(a) above. B and D give one set of suggested procedures for introducing each lesson.

#### Special Activities and Extra Exercise Material

The special activities of concern here are those designed to provide for individual differences in ability. All the programs under consideration state explicitly that they are designed for all ability levels in the schools.

Extra exercise material may be designed for individual differences, for extra practice, or as test material.

All of the above kinds of materials can be provided in texts; one common procedure is to use "special challenge" questions. The present discussion is concerned with content of the commentary.

Comparison of programs indicates the following points:

1. For the first six grades, C devotes special sections in each lesson to providing for "the able pupil" and "the slow learner." Suggestions for the able pupil are generally enrichment activities; different approaches, particularly those involving concrete activities, are

suggested for the slow learner. The commentaries also list various "Activities" that can be used for a variety of purposes such as re-teaching, review, and enrichment. Commentaries for grades seven and eight feature a schedule of text content that indicates sections that may be omitted for slow learners. These suggestions appear to be designed primarily for ability grouping. Tests are provided from grade three on.

The reference section of the commentary provides a bibliography concerned with learning theory, and one entitled "Books for Children."

2. For grades four through eight, A provides extra practice exercises, special challenge questions, and a great deal of extra exercise material useful for testing and review.

3. For grades one through six, E provides extra activities for both able and slow learners and for extra practice. These are at the end of the topic "Procedures" for worksheets. They include suggestions for re-teaching, and many concrete activities of a "game" or "puzzle" nature. Many can be used for extra practice or review. Grades seven and eight provide chapter tests.

4. The commentaries for B, D, and F do not provide extra material designed for individual differences; B and D make reference to special challenge work in text-book exercises. Only D provides extra exercise material and this is almost completely computational.

## CHAPTER VI

### THE CONTENT OF THE TEXTBOOK

Geometric content, by its nature, lends itself to visual interpretations. The use and quality of the illustrations, models, and designs can have an important effect on the interest and appeal of textbook materials. Furthermore, if the texts or worksheets are used as the main source of exercises, the organization and scope of the content will have a major effect in determining the nature of the course and the extent to which goals are achieved.

In this chapter the analysis is concerned with the use of geometric design and construction, the scope and sequence of geometric content, and the integration of this content into the whole program. Comparisons are made among programs.

#### Geometry in Design and in the Physical World

Aside from its use on the cover, design for its own sake is by no means common in mathematics texts. For the various series under consideration, such design has either extensive use throughout the series or very little use.

Illustrations of physical applications of geometric concepts are a different matter however. Poincaré (53:1) has written: "If there were no solid bodies in Nature, there would be no geometry." All the texts considered use pictures from the physical world to suggest various geometric concepts.

In program A, designs, geometrical or otherwise, are not featured. In grade one representations of common three-dimensional figures are introduced. By the end of the primary, the "common" pictures of physical objects such as pencil tips and table tops are used to suggest points and planes. In grade four space is interplanetary and embellished with space craft. Intersections of planes and lines are shown physically. The coordinate plane is used to produce pictures and letters after plotting; latitude, longitude, and great circles are pictured on the globe. Symmetry in the front of a bus and in the body of an insect are pictured.

The above examples represent most of the uses of pictures of physical objects. Representations of uses of plane and three-dimensional figures are not used to any great extent.

For program B, geometry is basic to the design of all textbook covers in the series. Colored representations of lines, rays, angles, plane regions, and solid figures are all prominent. Many pages that open chapters



use these ideas in an opaque, geometrical "landscape;" the geometry in art, particularly architecture, and in nature, is effectively portrayed.

Physical representations of figures are more numerous than for A; the use of color is effective. The purpose served by these illustrations is the same as in A, however.

In program C, geometric designs are not used. In addition to the "standard" physical representations referred to above, the primary texts in particular make effective use of pictures of paths, streets, and fences as representations of closed and open curves, betweenness, parallel and intersecting lines.

For program D, geometric design is used for some covers of texts. It is not used in the texts themselves.

There is wide use of effective illustration from the physical world, however, and much of this is geometrically oriented. By grade six the geometric figures and relations that students have studied are evident in all parts of the texts. Effective use can be made of such illustrations as the bathysphere on the ocean floor, the lunar landing craft that is the frustum of a cone, or the leaning tower of Pisa that illustrates the relation, "is not perpendicular to." Many other examples could be cited.

In the geometry sections themselves the usual physical

representations of fundamental concepts are given. In addition there are exercises that require matching of physical objects, suggested geometrical terms, abstract representations, and symbolization.

In program E, geometric figures are used in the design of textbook covers and of pages for chapter openings. Physical illustrations are largely confined to representations of point, line, plane, space, and to relations and intersections applied to these concepts.

In program F, geometric design is both used and studied as such. There are effective color designs using a kaleidoscope pattern, various polyhedra, and lattices. The study of patterns, particularly as tessellations and on lattices, is used as a basis for area of plane figures.

The chapter introductions represent, to the writer, a remarkable attempt to relate graphic art, literature and geometry. Thus, for example, the chapter on symmetry is introduced by Blake's poem that begins, "Tiger! Tiger! burning bright...", and there is a suitable accompanying illustration.

Both photographs and drawings of the physical world are widely used throughout the geometry sections. All of the plane transformations are "pictured" through the use of many drawings and photographs of natural or man-made objects. Coordinate geometry is associated pictorially with street plans and maps.

The following comparisons are worthy of note:

1. In the study and presentation of design, and in the use of illustration for interest and exposition, F far exceeds any of the other programs.

2. Programs B and D make extensive use of both geometric design and of illustrated physical objects that suggest geometric concepts. As was noted in Chapter V, for both of these programs introduction of new concepts begins with the text page.

3. Programs A and E make least use of pictures from the physical world. For the primary grades in particular, both of these programs use most of the text pages for practice and evaluation following the learning of the concept. In such cases, physical illustrations are assumed to have been given in concrete form during developmental activities.

### Constructions

Constructions have three major uses in the pre-deductive programs under consideration.

1. They are used to focus attention on the properties of various plane and three-dimensional figures. When figures such as cylinders, polyhedra, and prisms are constructed, their properties can be felt as well as seen. Plane figures can become more meaningful as well; the special property of a rectangle, as compared to other parallelograms, can be made clearer by constructing any

parallelogram and any rectangle.

Most such constructions can be done by paper folding or by ruler and protractor. Compasses are only necessary where circular regions are required.

2. The traditional Euclidean straightedge and compass constructions are done without proof. A major purpose here is to provide an intuitive basis for the first course in deductive geometry.

The constructions themselves do not have to be done with straightedge and compass, however. For example, many pupils "make a right angle" by paper folding, or with a protractor, long before they do the formal construction of a perpendicular.

3. Designs in two and three dimensions may be constructed. The purpose here may be motivational, or such designs may be constructed for their intrinsic interest. Again, paper folding, ruler and protractor, or straightedge and compass may be used.

In program A, most paper folding construction is concerned with common three-dimensional surfaces. Angle measurement is introduced by using a non-standard unit angle, the octon, which is obtained by paper folding. The unit angle is placed in the interior of the angle to be measured to form a series of adjacent angles. "Additivity" refers to the assumption that the sum of the measures of these adjacent angles is equal to the measure of the whole angle. While the topic of "symmetry" for

plane figures is introduced in grade six, there is no use of paper folding to obtain axes of symmetry for these figures.

Use of the compass is introduced briefly in grade four. The straightedge and compass constructions are begun in grade five. There is no construction of designs.

In program B, construction is used for all three purposes discussed above. Paper folding is confined to construction of three-dimensional surfaces, and is not used as a basis for work with straightedge and compass. The protractor gets brief use in grade seven. A familiarization with the compass is included for grade four; all other work is for grades five and above. Some designs in circles are included in exercises for grades six and seven.

Program C omits paper folding completely. All the familiar straightedge and compass constructions except "parallel lines" are included. Parallel lines are constructed using the protractor.

All construction is confined to grades seven and eight.

In program D, lessons devoted entirely to geometry begin in grade three, and they begin with straightedge and compass construction. The early concentration on geometry of the circle for this series has been noted earlier. Some use is made of paper folding, particularly

for three-dimensional surfaces; little use is made of the protractor other than to introduce it. The extent to which the series is construction oriented may be gauged from the fact that, except for E, it introduces as many construction topics by the end of grade six as any of the other series do by the end of grade eight.

In program E, methods of construction tend to be placed in a hierarchy determined by grade levels for introduction. Grades three and four make wide use of paper folding; the protractor is introduced and used in grade five; straightedge and compass construction is largely confined to grades six through eight, although the course suggests that the compass be "available" from grade four on.

This course provides many exercises in construction of designs. These are done both with the compass and with paper folding (and cutting).

In program F, while the protractor and paper folding are used as indicated in table I(e), there is no reference to the use of a compass or to the traditional constructions. There are many exercises concerned with construction of patterns and designs.

The following comparisons seem to be indicated:

1. There is remarkable agreement regarding the content of constructions. All courses except F are concerned

with the traditional straightedge and compass constructions from deductive geometry. All courses except C use paper folding to construct three-dimensional surfaces. Certain constructions with circles are unique to D.

2. Methods of construction differ mostly with regard to level of introduction. First use of the compass varies from grade three to grade five. Regarding level of introduction, D uses the lowest grade level in nearly all cases.

3. Constructions of designs or patterns are used in F as part of the study of these topics. In B and E such constructions are included largely because of their intrinsic interest and for motivation.

#### The Nature of the Sequential Organization of Content

As discussed in this section, "sequential organization" does not refer only to order of topics. Attention is given to the reasons for introduction of a topic, as indicated by its uses in following content. Sometimes topics are not used or referred to for one or more grade levels after introduction. As stated earlier, this may be due to program revisions that are not yet complete.

Weaknesses in order, such as the use of a term before it has been introduced, are not common in the geometry sections themselves. Such practices are sometimes evident when the geometry content is considered in relation to the rest of the course content.

For program A, the materials for the primary grades postdate the rest of the course. As was noted in connection with table I of Chapter IV, the result is that many topics from the primary are re-introduced in grades four through six.

In the primary grades both the terms "line" and "point" are used in connection with "number line" a full year before they are introduced in a geometry section. Care is taken to make all diagrams of number lines conform to the standard representation that is used after the term is introduced. The indefinite extension to the left is shown, even though the numbers do not extend indefinitely to the left. No attempt is made to distinguish "dot" and "point." While many terms in pre-deductive geometry are introduced before definition, it should be noted that "line" and "point" present a special case; they remain undefined throughout geometry.

All texts in program B give 1963 as the year of publication. As with A, "number line" is used in grade one, "line" is introduced in grade two.

Reasons for inclusion of terms are sometimes obscure. In grade three "plane figures" are introduced and illustrated, yet in grade four triangles and quadrilaterals are described as "figures on a flat surface." The term "curve" is used in grade four but never defined in any text or in the glossary; "figures made only of line



segments" are distinguished from "figures made of curves or line segments." "Sector" and "central angle" are introduced in grade five without further apparent use.

Reasons for organization of certain topics is not always clear. "Measurement" is begun in grade two and continued grade by grade; there is no discussion of its geometric basis until grade seven, even though geometric figures are frequently measured. "East" and "North" coordinates for points on a plane are introduced in grade two; no further work in graphing is carried out until grade five. "Parallelogram" is introduced in grade four, "parallel lines" come in grade five.

For program C the great majority of the topics in table I, Chapter IV, occur in grades one, two, seven, and eight. For these grades texts have been published since 1961. The sequential development of geometric content at both primary and junior high school levels seems clear and straightforward. The most evident fact is the "gap" between grades two and seven.

As noted earlier, program D begins geometry in grade three with constructions, most of which involve the circle. Introduction of the concept of "line" is left until grade five. This has raised some difficulties. In grade three, "segment" is introduced and illustrated in connection with measurement; later, in the geometry section of grade three, an illustration of a segment is

called a "line." In this same text "parallel lines" are discussed and illustrated; there is also mention of "distance between" parallel lines with no discussion of what this means.

The term "angle" is used from grade three on, but is not defined until grade five. After definition, no distinction is made at any time among the terms "angle," "angle in a circle," and "angle in a polygon." In connection with constructions of circles, the terms "central angle," "inscribed angle," "inscribed circle" and "circumscribed circle" are all introduced; no further use is made of these terms until grade seven.

Program E generally introduces terms when they are to be used. The reasons for some "gaps" are not clear, however. Cylindrical solids are in use from grade one, yet no construction is outlined until grade eight. "Parallel lines" are introduced in grade three and no construction is outlined until grade seven. "Planes" are in grade three, but "plane figure" is not used until grade seven.

It is significant that for program F, "set of points" is applied only to the subtopic, "line." Although the topic of "sets" is covered thoroughly, it is not widely used in the geometry sections. Thus "ray," "angle," and "segment" are not defined. Distinctions between "closed" and "open" figures are not made.

The foregoing discussion indicates the following comparisons:

1. Programs E, C, and F show least tendency to use geometric terms before they have been introduced. This practice is found in A with regard to "line" and "point;" it is somewhat more frequent in B and D.

2. Programs A, C, E and F seem to avoid the introduction of topics several grade levels before they are used.

3. Repetition of topics due to course revision is most evident in A; it also occurs in C.

#### Geometric Concepts in Each Program as a Whole

As table II of Chapter IV indicates, geometric concepts can be used to deepen understandings in many areas of mathematics. The effectiveness with which geometry is used for this purpose will depend upon the extent to which it is part of the total design. If the geometry content tends to be a separate "course within a course," its total effectiveness is weakened.

In most parts of program A, geometry is used widely. Furthermore, this use is integrated into the general sequence of topics.

No use is made of the concepts of plane coordinates for practice in the use of ordinal number. As indicated earlier, the discussion of the "geometric basis of

measurement" follows most of the work in the measurement of length, area, and volume.

In program B, the "geometric basis of measurement" is delayed even later than in A. There is no reference to "congruence" by fitting until grade five, although its use for fractional parts begins in grade one. "Graphing (generally)" begins in grade five; the introduction of plane coordinates comes in grade eight.

Program C makes no explicit statement of the "geometric basis for measurement." The number property most commonly illustrated by "arrays" is commutativity for multiplication; the property illustrated by "regions of space" is associativity for multiplication. C omits both of these. "Modular arithmetic" is also omitted. This topic can provide an intuitive foundation for ideas from algebra, such as congruence classes and non-unique divisors of zero.

At the grade three level program D makes a full statement of the "geometric basis of measurement." This principle states that the arbitrary unit chosen must be of the same nature as the thing to be measured. D illustrates the choice of a segment, a square region, and a cubic region, all of which are non-standard units, for measuring length, area, and volume respectively.

This work in measurement bears no relation to the

geometry that is, or has been, introduced by the end of grade three, however. At this level none of the concepts of segment, square region, and region of space has been dealt with in geometry.

This program also omits modular arithmetic, ordinals in a plane, and number properties as illustrated by plane regions.

In program E, the "geometric basis of measurement" is not dealt with explicitly. Fractional parts of regions is used prior to introduction of "congruence."

With one exception, the content of program F is not concerned with the teaching of measurement or mensuration. Angle measure is presented in terms of rotation. The other uses of geometry are largely concerned with graphing, the coordinate plane, and number lines.

The above discussions indicate that all courses, except D, have succeeded in providing a large measure of consistency in usage of terms in the geometry sections and in the rest of their programs. As noted earlier, there is wide agreement regarding choice of geometric concepts used in other areas of the program.

#### The Scope of the Content

Some aspects of scope of content have already been discussed. For the programs under consideration, there

will be no further discussion, in this section, concerning the geometric content omitted or included, or the use of construction and design. Of primary concern here will be the nature and extent of the use of three-dimensional geometry and the provision of exercise material or suggested activities designed to provide for individual differences in achievement.

In program A, emphasis upon geometry as part of the child's physical world is begun in grade one with recognition of cylinders, spheres, and rectangular prisms. The stated purpose is recognition and discussion of properties. Except for constructions by paper folding, there is no attempt to establish a link with geometry of the plane until grade eight, where intersections of three-dimensional figures and planes are introduced.

As indicated in the discussion regarding the teachers' commentaries for this series, the amount of extra exercise material provided in the texts increases as the grade level increases. There are special challenge exercises at all levels above grade three, and some texts contain "optional" sections. The sections entitled "Working Together," and "Class Exercises and Discussion" emphasize development through questioning and discovery. It should be noted that such sections can be used to help in providing for individual differences. They can be used for independent study or as a guide to group study for under-

achievers. Further activities and further exercise materials designed to help the under-achiever are found mainly in the commentary.

In program B, the only three-dimensional figures included at the primary level are rectangular prisms. In the text these are represented by drawings that are abstractions, not pictures from the physical world. For grades four through six attention is drawn to objects in the real world and their geometric equivalents. Measurement of volume is also included. Intersections of planes is left until grade seven.

The main provision for individual differences is stated to be through exercises graded in difficulty. There are "starred" questions for special challenge. For grades five and six there are special pages entitled "Enrichment" and in grades seven and eight pages of special activities or problems. These are not necessarily for advanced students. Some of the activities, such as paper folding, could be done by all students.

For program C, solids are presented in connection with measurement of volume at the grade five and six level. Other than this, three-dimensional geometry is confined to grades seven and eight, where various mensuration formulas are developed.

The commentary is the major source for exercises and activities concerned with provision for individual

differences. In the texts for grades three to six there are "Side Trips;" these are not necessarily for the above-average achievers. In grades seven and eight the use of developmental questions allows for variations in presentation, as noted in A above.

Program D first introduces three-dimensional figures in grade three. Except for rectangular prisms, introduction of these figures is for recognition only. As has been noted, this course does a good deal of work with geometry of the circle; none is done with the circle's three-dimensional analogue, the sphere. In grade six, "plane faces" of solids are illustrated by considering intersections of three-dimensional surfaces with planes.

The provision of material for individual differences is limited to specially "starred" questions in the exercises. These are not questions designed for below-average achievers.

As in A, program E begins geometry in grade one with physical representations of three-dimensional solids. Emphasis is placed on plane faces of solids; it is from these plane faces that the common plane figures, square, rectangle, circle, and triangle are obtained by tracing. By grade four, emphasis is on such properties as parallel edges and parallel faces as intuitive bases for the concepts of parallel lines and parallel planes. At this level intersections of planes and three-dimensional figures



are illustrated; planes of symmetry for various three-dimensional surfaces are included. Mensuration formulas are developed in grades six through eight.

At the grades one and two levels, activities designed for individual differences are confined to the commentary. The same is true for grades three through six except that the texts contain sections entitled "Explorations." These are not specifically designed for students whose achievement is above average. The grades seven and eight texts contain special challenge questions; presentation of new material is through the use of developmental questions.

In program F, work with three-dimensional surfaces is largely confined to properties of various regular polyhedra, and intersections of planes. Planes of symmetry are also studied in connection with properties of certain three-dimensional surfaces.

Examples and exercises considered most suitable for classroom discussion are denoted by a "D." There are special challenge questions and sections marked with an asterisk.

The following comparisons appear to follow from the above discussions:

1. Three-dimensional geometry serves a variety of purposes in the programs studied.

- (a) For programs A and E these figures serve to

introduce the child to geometry; they are also studied for their specific properties and for purposes of mensuration.

(b) In B, C, D, and F, properties are studied and mensuration formulas are developed.

2. The extent to which texts provide for individual differences also varies.

(a) Texts for A, C, and E contain enrichment sections, special challenge questions and sections, and developmental questions.

(b) B and F use enrichment and exercises of graduated difficulty.

(c) D provides special challenge questions.

## CHAPTER VII

### CONCLUSIONS AND THEIR USE

The results of the analysis recorded in the preceding chapters point to certain conclusions regarding content and presentation in current mathematics programs. In this chapter these conclusions are presented in connection with certain questions. The questions are designed to form guidelines for an approach to evaluation of content, teachers' commentary, and the textbook presentation of pre-deductive geometry.

The purpose of such evaluation is to make decisions regarding the inclusion of content or selection of a program. Various bases for such decisions are discussed in the first section.

#### Bases for Decisions Regarding Pre-Geometry

In Chapter II, following an extensive review of the literature, the statement is made that for pre-deductive geometry the most pertinent questions to be answered are "What is to be taught? Why and how?" Stated in other words, and in slightly different order, the concern must be with objectives, content, and presentation.

Few, if any, school systems can afford to make

decisions regarding these questions without a careful consideration of the opinions of others active in mathematics education. Examination of present day mathematics programs and review of the literature in general both indicate that, for a mathematics program, decisions regarding objectives, content, and organization are rarely made by individuals. Opinions are shared and decisions tend to be those of a "committee" or "team."

Where the writing is not part of actual mathematics programs, the literature, as reviewed in Chapter II, is largely concerned with general statements or with findings regarding details whose relation is not necessarily evident. There are broad objectives such as, "The goal is satisfaction, here and now, with things mathematical, and geometry abounds in such ideas" (6:211); general statements regarding content are made: "two- or three-dimensional experience in pre-geometry... at all grades" (2:4); and generalizations are applied to presentation: "... constructions can provide increased space perception as well as increased motivation" (12) (43). All of these suggestions merit attention in that they represent thinking that is fundamental to decision making. By themselves however, such suggestions are not a sufficient guide to the writing or selection of pre-deductive geometry.

With regard to content in particular, there is sometimes a tendency for reports or articles to simply list topics for introduction. For example, the Cambridge

Report states, on page 31, that from kindergarten the child should "get experience with the number line" and further, on page 32, that "this line can be regarded from the first as a representation for all real numbers." However, as Stone (70) points out in his evaluation of the report, the intention of this recommendation is not at all clear. Does it mean the inclusion of irrationals? If so, on what basis? For Stone, it is difficult to imagine a meaningful intuitive approach to this topic.

As a second basis for decisions, close attention should be given to what has been done in programs already published. These sources represent thinking of the above-mentioned "committees" that has been translated into actual organization of content and into a presentation of such content to both teachers and pupils. Worthy of particular attention are those series based on experimental programs or the work of well-known authors, or both. All of the six mathematics programs analyzed in the previous chapters meet one or both of the above specifications. The following sections are designed to provide for the "careful examination" called for in Lawrence's survey (42), referred to in Chapter I.

#### Consideration of "Pre-Geometry" Content

Table I provides a convenient check list for content of geometric sections and for level of introduction. In the consideration of the content, the following questions

will serve as a useful guide.

I. What dates for first publication are shown for each grade level?

As has been pointed out, for Program A material for early grades postdates later material; this leads to repetition of introduction. For C, grades three to six material predates that for grades one, two, seven and eight; here there are some "gaps" in level of presentation. Such programs may be considered very worthwhile, however. If so, the next questions become pertinent.

II. Are any parts of the program under revision? If they are, what are the projected publication dates? Publishers' representatives have such information readily available.

III. For program content being considered, which included topics are found in fewer than three programs in table I?

Frequency of inclusion (as determined by table I) does not in itself necessarily determine the desirability of the included topic. Table I itself includes the major topic "Transformations in the Plane and in Space" which is only dealt with to any extent by two programs. However, for a topic that is not frequently included, the following further questions are indicated:

(a) Does the commentary deal with it satisfactorily? (See section following.)

(b) If the topic is introduced at the primary level where is it used again? How much use is made of it?

In grade one Program C uses the term "betweenness" as a property of any open curve; it is next used in grade seven as a property of lines that is not possessed by other open curves. Program D introduces "central angle" in grade five; no further reference is made to the term in the texts analyzed.

IV. Which omitted topics are found in more than two programs in table I?

The importance of an omission will involve judgment in answering two other questions:

(a) If the omission is a subtopic, how essential is it in the development of the major topic?

If other programs include the topic before grade five, the importance of inclusion can be gauged by the use these programs make of it in later grades. As mentioned earlier, the term "axis" is not essential to the topic "Graphing." On the other hand, C omits "Constructions of Three-Dimensional Surfaces," a topic which the rest of the programs consider important.

(b) If the omission is a major topic, how useful is it? Do the other programs use it in geometry sections and elsewhere? Do they indicate the prominence of the topic in deductive geometry?

V. For what topics does the level of introduction

differ markedly from that of table I?

It is, of course, not always possible to answer this question; some topics show no consistent level of introduction. However, if this introduction is two or more grades above or below the level for half the programs in table I, investigation is warranted. Answers to these further questions should prove useful:

(a) Where introduction is at a lower grade level than usual, does the commentary give any reasons?

At the beginning level, programs A and E include much wider use of three-dimensional figures than is found in the other programs. In both cases, reasons are given for this.

(b) Where introduction is later than usual does the text use the term or concept earlier?

Program D does not introduce "line" until grade five. As a result, the pupil is exposed to inconsistent usage of the term throughout a good deal of his mathematics in the lower grades.

#### The Adequacy of the Teachers' Commentary

The importance of the teachers' commentary at the various grade levels has been discussed in Chapter V. As with the textbook, the usefulness of a commentary can be judged only after careful examination. Such an examination should be made in conjunction with the worksheets or text for pupil use.



On the basis of the discussion in Chapter V, the following questions can serve as a useful guide in evaluation:

I. Does the commentary assume responsibility for provision of complete or nearly complete background material for geometry content?

The analysis in Chapter V shows that such responsibility is not assumed by program D, and is only partially assumed by B.

II. If in-service work became necessary, would the organization of background material lend itself to use as a short course?

Chapter V shows that A and F present in one section the background for a chapter or group of chapters. On the other hand, for B this material may be in at least three different parts of the commentary: the keyed text pages, the section "Procedures," and the glossary.

III. Is the exposition written and illustrated to provide a greater depth of background than that provided by text pages?

Teachers need to have an idea of the mathematical significance of text materials. Program A, for example, introduces considerable work (including transformations) on the coordinate plane. If teachers are expected to approach these sections with intelligence and enthusiasm, then these teachers should be familiar with some basic

ideas from analytical geometry. Extensive illustration is also necessary.

IV. Does the section on background information, where one exists, include discussion, definition, and illustration of new vocabulary to be introduced?

New vocabulary is listed in all commentaries, A through F. Sometimes there is no further reference to such terms, outside the pupils' texts. If teachers are to use precise mathematical language in the classroom they need access to clear and precise exposition and definition of mathematical terms.

For approximately half of the new vocabulary introduced in B, the teacher relies on the pupils' texts for definitions. In the grade four text two pages are devoted to parallelograms and their properties. However, "parallelogram" is not defined in terms of the word "parallel" either in text or commentary.

V. Are general objectives or purposes stated for the teacher?

Some commentaries do little to give the teacher a clear sense of purpose regarding material introduced. Program D, for example, gives no indication as to why its geometry content for grades three and four is largely concerned with straightedge and compass constructions, involving the geometry of the circle. The authors must be aware that some people have doubts about wide use of such

constructions at this level. In addition, there is a good deal of advanced technical vocabulary in this content. These points are never mentioned or discussed with the teacher; and, as indicated earlier, little or no mathematical background is provided.

VI. Are the specific objectives for lessons stated in useful behavioral terms? Is the teacher given a clear idea of what pupils should be able to do as a result of the projected learning activities?

In most primary materials, the worksheet is designed as an evaluation of the lesson activities. These materials provide the behavioral outcomes to be checked. Where texts use developmental questions in presenting new materials, behavioral outcomes are a direct result of the presentation.

VII. (a) What suggestions are made regarding different approaches to content for re-teaching or remedial purposes?

(b) Are there suggested exploratory and enrichment exercises?

Program C, for example, includes extra "activities" that are numbered. These are specifically keyed to the content and feature both new approaches to topics or extension of topics. Some activities can be used first for extension, and then for review at a later time.

VIII. Does the commentary contain extra exercise material designed for review or testing?

#### The Effectiveness of Textbook Presentation

As indicated in Chapter VI, the appearance and organization of the text or workbook and the nature and scope of the content can play a large role in determining the geometry course that a student receives. The following questions are concerned with these facets of the textbook presentation:

I. Is there use of geometric design that represents an attempt to make the text artistically pleasing?

Programs B and F make effective use of such designs, in color.

II. Are there pupil exercises in construction of geometric designs?

Programs B, E, and F all use such exercises. Here again color can be used effectively.

III. Does the text make wide use of geometric representations from the physical world?

Programs B, D, and F contain many such illustrations throughout their texts. In contrast, program A appears drab with its few illustrations entirely confined to the geometry sections. On the basis of this aspect, it is difficult to imagine an equal interest or enthusiasm on the part of students for the geometric content in A as

for the content in B or D.

IV. To what extent are constructions confined to the traditional straightedge and compass exercises?

As indicated in table I(e), constructions using paper folding or protractor can serve many purposes. E uses paper folding and tracing for "congruence," "plane symmetry," "right angle," and "angle bisection."

On the basis of data in table I(e), extensive use of compasses during the first four grades is not common. As stated previously, D is the major exception and no reasons for the early introduction of compass constructions are given. The value of such constructions must be weighed against the time and attention that must be focused on mastering the necessary motor skills.

V. To what extent are constructions confined to the plane?

Where properties of three-dimensional figures are of concern, constructions of such figures are used extensively. Program C, which does not include construction of three-dimensional surfaces, is concerned mainly with mensuration.

VI. To what extent are the geometric concepts used in the non-geometric sections of the program?

This "extent" can be gauged by checking against

table II. In many cases the presence or absence of one of these "uses" of geometry depends on whether the text includes the non-geometric topic.

VII. Is the "geometric basis of measurement" made clear?

This use of geometry has been discussed at length on pages 17, 60, 94 and 95 of the preceding chapters; it is dealt with specifically by A, B, and D. As noted earlier, D is the only program that introduces these concepts, basic to measurement in general, at the beginning of the teaching of measurement in the elementary grades.

VIII. To what extent, if any, are three-dimensional "solids" used to introduce geometry? For how many grade levels is three-dimensional geometry included?

Table I(r) shows that programs A and E make extensive use of representations of three-dimensional figures at the grade one level. For A there is little more three-dimensional work done until grade four.

The use of three-dimensional "solids" is sometimes urged (6) because of the fact that geometry itself has its origin in the physical world. In addition, however, the results of learning research carried out by both Piaget (26) (52) and Dienes (18) (19) attach great importance to provision of perceptions that are not just visual. The use of "solid" figures in E emphasizes tactile as well as visual sensory perceptions. The use of

such perceptions is not confined to three-dimensional figures; plane faces of "solids" are investigated to provide a basis for introduction of conventional illustrations of plane figures.

IX. What provisions are made for individual differences?

A "complete picture" of the provision for individual differences must include material provided by both textbook and commentary. As noted in Chapter VI, texts use exercises of graduated difficulty, "special challenge questions," and special "exploratory" sections. The use of developmental questions for presentation of new material can help to provide for individual differences. Such questions may be used for independent study for the able pupil; they may also serve as review for the slow learner after the teacher has taken up the topic. This type of presentation is widely used in programs C and E; program A makes some use of it.

X. In the geometry sections, what is the extent of the use of concepts fundamental to all mathematics?

A check with table III indicates the nature of such topics and the wide extent to which they are used.

#### Use of the Questions as a Guide

The questions listed in the previous sections have been based on an analysis of content of programs in

present day use. For some time in the future, there will doubtless be continuing revision of programs, changes in emphases, and additions or deletions to pre-deductive geometry content.

However, even where such changes in programs occur, certain considerations concerning these programs seem likely to remain. To this writer, the organization and content of the teachers' commentary, its relation to the text content, and the presentation and content of the text material, will remain topics of paramount concern regarding mathematics programs. The questions presented in this chapter have been designed to check these topics that are likely to be of continuing importance.



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