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GEOMETRIC EVOLUTION EQUATIONS AND p-HARMONIC THEORY WITH APPLICATIONS IN DIFFERENTIAL GEOMETRY

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GEOMETRIC EVOLUTION EQUATIONS AND p-HARMONIC THEORY WITH APPLICATIONS IN DIFFERENTIAL GEOMETRY

A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

BY

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DEDICATION

to

My parents

Li Lian-Min and Xiao Rong

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Contents

Abstract

In this dissertation, we consider parabolic (e.g. Ricci flow) and elliptic (e.g. p-harmonic equations) partial differential equations on Riemannian manifolds and use them to study geometric and topological problems. More specifically, to classify a special class of Ricci flow equations, we constructed a family of new entropy functionals in the sense of Perelman. We study the monotonicity of these functionals and use this property to prove that a compact steady gradient Ricci breather is necessarily Ricci-flat. We introduce a new approach to prove the monotonicity formula of Perelman's $\mathcal{W}$-entropy functional and we construct similar entropy functionals on expanders from this new viewpoint. We prove that a large family of complete non-compact Riemannian manifolds cannot be stably minimally immersed into Euclidean space as a hypersurface which serves as a non-existence theorem considering the Generalized Bernstein Conjecture. We give another yet simpler proof for a theorem of do Carmo and Peng, concerning stable minimal hypersurfaces in Euclidean space with certain integral curvature condition. In the study of p-harmonic geometry, we develop a classification theory of Riemannian manifolds by using p-superharmonic functions in the weak sense. We gave sharp estimates as sufficient conditions for a p-parabolic manifold. By developing a Generalized Uniformization Theorem, a Generalized Bochner's Method, and an iterative method, we approach various geometric and variational problems in complete noncompact manifolds of general dimensions.

Chapter 0 Introduction

In this chapter, we introduce the history, motivation and background of this thesis. We also briefly describe the main results we have obtained.

0.1 Background

In this thesis, we consider problems originated from geometry and topology. More specifically, we are interested in Riemannian manifolds (including submanifolds defined by variational principles) and functions defined on those manifolds which yield information about the topology and geometry of the underlying manifolds. The tools we use are from differential geometry and analysis of partial differential equations on Riemannian manifolds.

One of the fundamental problem in topology is the Poincaré Conjecture and its generalization : Thurston's program. Hamilton introduced the methods of Ricci flow by evolving the metric of a Riemannian manifold $(M^n, \mathbf{g}_0)$ around 1982 in his fundamental work [29]. In early 90's, he developed methods and theorems to understand the structure of singularities of the Ricci flow. And he crafted a very well-developed program to use these flows to resolve Thurston's Geometrization Conjecture for closed 3-manifolds, [28, 30]. Perelman's ground-breaking work [45, 46, 47] and Cao-Zhu's very recent work

[16] are aimed at completing that program. This spectacular development will bring tremendous influence on the study of geometry and topology. The theory of various geometric evolution equations and their applications in other branches of mathematics, in mathematical physics and computer science will be one of the most active topics in the area of geometric analysis.

One fundamental area in differential geometry is the study of minimal submanifolds. A geodesic is locally the shortest path connecting two points on a manifold. It also can be viewed as a critical point of the length functional via a variational approach. The study of geodesics has been very-well developed. It is hard to overestimate the importance of it. The natural analogue of geodesic in higher dimensional Riemannian manifolds is minimal submanifold which are critical points of volume functional. We quote the following comment made by S.T.Yau [60] in 2006 :

"Comment: The theory of higher dimensional minimal submanifolds is one of the deepest subjects in geometry. Unfortunately our knowledge of the subject is not mature enough to give applications to solve outstanding problems in geometry, such as the Hodge conjecture. But the future is bright."

One classical well-known result about minimal submanifolds in Euclidean space is the generalized Bernstein theorem, see [4, 17, 1, 50, 8] for complete proofs or see the introduction (3.1) in Chapter 3. A natural question was asked by S.T.Yau ([59], p.692, Problem 102):

Question 0.1.1. *Is a hyperplane the only stable minimal hypersurface in* $\mathbb{R}^{n+1}$ *(*$n \leq 7$*)?*

In the last thirty years, there are many interesting work related to this question. When $n = 2$, it is completely solved by Fischer-Colbrie and Schoen [22], Do Carmo and Peng [19], Pogorelov [48] around early 80's, see the introduction 3.1. For higher dimensions, it remains open. With certain volume growth condition on the hypersurface, it is solved by R.Schoen, L.Simon and S.T.Yau [53] for $n \leq 5$. With different integral curvature conditions, a stronger result was obtained by Do Carmo and Peng [20], P.Bérard [5], Y.B.Shen and X.H.Zhu [55] for all $n \geq 2$. Without any constraint, H-D.Cao, Y.Shen, and S. Zhu (also see the work of J. Mei, and S.Xu [41]) proved that the hypersurface must have only one end which yields topological information of the hypersurface. But in general, when $n > 2$, the problem still has not been completely solved.

The celebrated uniformization theorem of F. Klein, P. Keobe and H. Poincaré is a classification theorem that sharply divides complete noncompact surfaces into parabolic and hyperbolic ones and enables us to solve many geometric variational problems on surfaces. However, in dealing with higher dimensional geometric problems, the scope of the uniformization theorem and its related Laplace operator needs to be widened. For example, the manifolds R^3 and R^4 are both hyperbolic, and therefore can not be distinguished from each other in this way, while manifolds with Ricci curvature bounded below by a nonpositive constant behave like the Euclidean space $\mathbb{R}^n$ from the viewpoint of harmonic functions(cf. [61]). This motivates us to study the geometric significance and applications of the p-Laplace operator Δ_p $\Big($ defined by $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f)$, where ∇f denotes the gradient of f $\Big)$, as well as its perturbation, the $\mathcal{A}$-operator, which does not arise from the ordinary Laplace operator Δ.

0.2 Main results

In this section, we describe the main results in this thesis.

Our main results are distributed in five chapters and can be described in the following three different categories :

A. Ricci flow and Entropy functionals (Parabolic Methods).

A.1. Compact steady Ricci breathers and entropy functionals One of the key objects in the study of geometric flow (especially, Ricci flow) is soliton (Ricci soliton and its generalization Ricci breather) which is a special (self-similar) solution of the flow equations. It is crucial to classify Ricci solitons in Hamilton's program to solve Poincaré-Thurston conjecture. One of the applications of Perelman's amazing entropy functionals is that he used them to prove that there are no non-trivial breathers. In the spirit of Perelman's entropy functional, we introduce a family of new entropy functionals and use them to prove that compact steady Ricci breathers must be Ricci-flat. See Remark 1.1.2 for the history of this problem.

Theorem 1.1.1. *Suppose the Ricci flow of* $\mathbf{g}_{ij}(t)$ *exists for* $[0, T]$, *then the entropy functionals* $\mathcal{F}_k(\mathbf{g}_{ij}, f)$ *will be monotone along the time interval, where* $\int_M e^{-f} d\mu$ *is fixed under the Ricci flow. Furthermore, the monotonicity is strict unless the Ricci flow is a trivial Ricci soliton. Namely, it is Ricci-flat.*

Theorem 1.3.4. *There is no compact steady Ricci breather other than the one which is Ricci-flat.*

A.2. $\mathcal{W}$-Functional and new entropy functionals We introduce a new approach to prove the monotonicity formula of Perelman's $\mathcal{W}$-entropy functional. This method reveals the relation between $\mathcal{F}$ functional and $\mathcal{W}$ functional which greatly simplifies the computations. Inspired by the above idea, we introduce more new entropy functionals and study their monotonicity properties. This result was previously obtained by M. Feldman, T. Ilmanen, and L.Ni.[23]. (Informed by Professor B. Chow.)

B. Higher Dimensional Minimal Submanifolds (Elliptic L^2 Methods).

B.1. A p-Harmonic approach to the generalized Bernstein conjecture We made some progress toward the generalized Bernstein conjecture. A non-existence theorem is obtained as following.

Theorem 3.5.1. *Let $(M^n, \mathbf{g})$ be a complete Riemannian manifold, where $n > 2$.*

1. *If in the conformal class of the metric $\mathbf{g}$ on M^n, there exists a non-complete metric $\widetilde{\mathbf{g}}$ with non-positive scalar curvature $\widetilde{R} \leq 0$ and finite volume, then $Q(u) = 0$ with coefficients satisfying (3.44) and (3.46) does not have any essential positive supersolution.*

2. *Assume in the conformal class of $\mathbf{g}$, there exists a complete metric $\widetilde{\mathbf{g}}$ with non-positive total scalar curvature $\int_M \widetilde{R} d\mu_{\widetilde{\mathbf{g}}} \leq 0$ and quadratic volume growth. If $Q(u) = 0$ with coefficients satisfying (3.44) and (3.46) has an essential positive supersolution, then $(M^n, \mathbf{g})$ is isometric to $(M^n, const \cdot \widetilde{\mathbf{g}})$, and also has quadratic volume growth.*

B.2. Stable minimal hypersurface with finite integral curvature condition If the second fundamental form of a complete oriented stable minimal hypersurface in Euclidean space has finite L^2 norm, then it must be a hyperplane. This is a theorem of Do Carmo and Peng in the early 80's. We present another yet simpler proof in Chapter 4.

The main ingredients of our proof include: Bochner's method, stability inequality and minimal hypersurfaces has infinite volume. The major difference of our proof is that we do not use the main estimates by [53].

Theorem 4.0.1. *Let $x : M \to R^{n+1}$ be a complete oriented stable minimal immersion such that the integral curvature $\int_M |A|^2$ is finite, then $x(M) \subset R^{n+1}$ is a hyperplane.*

C. p-Harmonic theory with sharp estimates, generalized Uniformization theorem, and Bochner's methods (Elliptic L^p Methods [58]). We make sharp global integral estimates by a unified method, and find a dichotomy between constancy and "infinity" of weak sub- and supersolutions of a large class of degenerate and singular nonlinear partial differential equations on complete noncompact Riemannian manifolds. These lead naturally to a Generalized Uniformization Theorem, a Generalized Bochner's Method, and an iterative method, by which we approach various geometric and variational problems in complete noncompact manifolds of general dimensions.

Theorem 5.2.1. *Every locally bounded weak solution $f : M \to (-\infty, \infty)$ of the differential inequality $f\operatorname{div}(\mathcal{A}_x(\nabla f)) \geq 0$, with $1 < p < \infty$, is constant a.e. provided f has one of the following:* p-finite*,* p-mild*,* p-obtuse*,* p-moderate*, or* p-small *growth, for some $q > p - 1$.Or equivalently, for every $q > p - 1$, any nonconstant a.e. locally bounded weak solution $f : M \to (-\infty, \infty)$ of $f\operatorname{div}(\mathcal{A}_x(\nabla f)) \geq 0$ has f has* p-infinite *,* p-severe*,* p-acute*,* p-immoderate*, and* p-large *growth. See (1.1 -1.5) in [58].*

Theorem 5.2.2. *Let $u : M \to S^k_+ \subset R^{k+1}$ be a smooth (a) harmonic map (where $p = 2$) or (b) p-harmonic morphism with $p > 2$, where S^k_+ is an open hemisphere centered at pole y_0. If for some $q < p - 1$, the height function, defined by $f(x) = \langle u(x), y_0\rangle_{R^{k+1}}$ for $x \in M$, has one of the following:* p-finite*,* p-mild*,* p-obtuse*,* p-moderate*, or* p-small *growth, then u is constant.*

Theorem 5.2.3. *Every p-parabolic stable minimal hypersurface in R^{n+1} is a hyperplane for $n \leq 5$, where $p \in [4, 4 + \sqrt{8/n})$.*

Chapter 1 Gradient Ricci breathers and entropy functionals

In this chapter, we define a family of modified $\mathcal{F}$-entropy functionals of Perelman. We prove that these new functionals are also nondecreasing under the Ricci flow. As an application, we give a direct proof of the theorem that the only compact steady Ricci breather is the trivial one which is Ricci-flat.

1.1 Introduction

Let $(M, \mathbf{g})$ be a closed Riemannian manifold. In [45], Perelman introduced an entropy functional

$$\mathcal{F}(\mathbf{g}_{ij}, f) = \int_M (R + |\nabla f|^2) e^{-f} d\mu. \tag{1.1}$$

If $(M, \mathbf{g}(t))$ is a solution to the Ricci flow equation, Perelman proved that the $\mathcal{F}$-functional is nondecreasing under the Ricci flow and the Ricci flow can be viewed as the gradient flow of this functional. More precisely, he showed that under the following coupled system:

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{g}_{ij} = -2R_{ij} \\ \frac{\partial}{\partial t} f = -\Delta f - R + |\nabla f|^2, \end{cases} \tag{1.2}$$

the $\mathcal{F}$-functional is nondecreasing, since

$$\frac{\partial}{\partial t}\mathcal{F} = 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu \geq 0.$$

A metric $\mathbf{g}_{ij}(t)$ is called a steady breather, if for some time $t_1 < t_2$, the metric $\mathbf{g}_{ij}(t_1)$ and $\mathbf{g}_{ij}(t_2)$ differ only by a diffeomorphism. Trivial breathers, for which the metrics $\mathbf{g}_{ij}(t_1)$ and $\mathbf{g}_{ij}(t_2)$ differ only by diffeomorphism for each pair of t_1 and t_2 are called Ricci solitons.

If we define

$$\lambda(\mathbf{g}_{ij}) = \inf \mathcal{F}(\mathbf{g}_{ij}, f),$$

where the infimum is taken over all the smooth f which satisfies

$$\int_M e^{-f} d\mu = 1,$$

then the nondecreasing of the $\mathcal{F}$ functional implies the nondecreasing of $\lambda(\mathbf{g}_{ij})$. As an application, Perelman was able to show that there is no nontrivial steady breathers on compact manifolds.

In this chapter, we consider a family of new entropy functionals $\mathcal{F}_k$, for $k > 1$. Although Ricci flow or even modified Ricci flow are not gradient flow for these new functionals. They still carry the monotonicity formula. Similarly, for each of them, we could also define the related first eigenvalues $\lambda_k(\mathbf{g}_{ij})$. This leads us to prove the following theorem

Theorem 1.1.1. *There is no compact steady Ricci breather other than the one which is Ricci-flat.*

Remark 1.1.2. *A similar result appeared in a very recent preprint of [11]. Compare to* *their results, our approach is different. More importantly, our theorem drops the curvature constraint.*

This chapter is organized as following. In section two, we review briefly some basic facts about Ricci flow. In section three, we give the definition of the new functionals $\mathcal{F}_k$ and we prove that although the Ricci flow is not a gradient flow of $\mathcal{F}_k$ in the sense of Perelman, $\mathcal{F}_k$ are still nondecreasing under the flow. Furthermore, the monotonicity is strict unless we are on a Ricci-flat trivial gradient Ricci soliton. In section four, we give the detailed proof of Theorem 1.1.1. In section five, we give remarks on how to remove the the nonnegative curvature assumptions by using the approach in [11].

Throughout this chapter, we use Einstein convention: repeated index represents summations. Also, we frequently use the tensorial property of Riemannian tensors, i.e., Riemannian tensors are independent of the choice of local coordinates. Whenever an expression need an explanation about the upper or lower index, one can apply the tensorial property and choose normal coordinates at a point to fix it.

1.2 Preliminaries

In this section, we will briefly introduce the history of Ricci flow, two concepts: Ricci soliton and Ricci breather, and the very important role they play in Hamilton's program.

Given a 1-parameter family of metrics $\mathbf{g}(t)$ on a Riemannian manifold M^n, defined on a time interval $\mathcal{I} \subset \mathbb{R}$, Hamilton's **Ricci flow equation** is

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{g} &= -2Rc \\ \mathbf{g}(0) &= \mathbf{g}_0, \end{aligned} \tag{1.3}$$

where Rc denotes the Ricci curvature with respect to $\mathbf{g}$.

Hamilton introduced the methods of Ricci flow by evolving the metric of a Riemannian manifold $(M^n, \mathbf{g}_0)$ around 1982 in his first fundamental work [29]. And he crafted a very well-developed program to use these flows to resolve Thurston's Geometrization Conjecture for closed 3-manifolds. Perelman's ground-breaking work and Cao-Zhu's very recent work is aimed at completing this program. There are some very good reference for introductions to the history besides the original papers of Hamilton [28, 29, 30], Perelman [45, 46, 47], Cao-Zhu [16] , see e.g. [6],[7].

In early 90's, Hamilton developed methods and theorems to understand the structure of singularities of the Ricci flow [28, 30]. The fundamental theorem for understanding the singularity of Ricci flow is the following compactness theorem of Hamilton

Theorem 1.2.1. *(Hamilton's Compactness of Ricci flows). Let M_i be a sequence of manifolds of dimension n, and let $p_i \in M_i$ for each i. Suppose that $\mathbf{g}_i(t)$ is a sequnce of complete Ricci flows on M_i for $t \in (a, b)$, where $-\infty \leq a < 0 < b \leq \infty$. Suppose that*

1. $\sup_i \sup_{x \in M_i, t \in (a,b)} |Rm(\mathbf{g}_i(t))|(x) < \infty$; and
2. $\inf_i inj(M_i, \mathbf{g}_i(0), p_i) > 0$.

Then there exist a manifold M of dimension n, a complete Ricci flow $\mathbf{g}(t)$ on M for $t \in (a, b)$, and a point $p \in M$ such that, after passing to a subsequence in i,

$$(M_i, \mathbf{g}_i(t), p_i) \longrightarrow (M, \mathbf{g}(t), p), \tag{1.4}$$

as $i \to \infty$.

We next introduce the concepts of Ricci soliton (gradient Ricci soliton) and Ricci breather (gradient Ricci breather), see [6, 7].

Suppose that $(M^n, \mathbf{g}(t))$ is a solution Ricci flow on a time interval (α, ω) containing 0, and set $\mathbf{g}_0 = \mathbf{g}(0)$. One says $\mathbf{g}(t)$ is a **self-similar solution** of the Ricci flow if there exists scalars $\sigma(t)$ and diffeomorphisms $\psi(t)$ of M^n such that

$$\mathbf{g}(t) = \sigma(t)\psi(t)^*(\mathbf{g}(0)) \tag{1.5}$$

for all $t \in (\alpha, \omega)$.

Suppose that $(M^n, \mathbf{g}_0)$ is a fixed Riemannian manifold such that the identity

$$-2Rc(\mathbf{g}_0) = \mathcal{L}_X \mathbf{g}_0 + 2\lambda \mathbf{g}_0 \tag{1.6}$$

holds for some constant λ and some complete vector field X on M^n. In this case, we say $\mathbf{g}_0$ is a **Ricci soliton**. The three cases of $\lambda \in \{-1, 0, 1\}$ correspond to **Shrinking**, **steady** and **expanding** solitons, respectively. If the vector field X is the gradient field of a potential function, we say it is a **gradient Ricci soliton**.

There is a bijection between the families of self-similar solutions and Ricci solitons which allows us to regard the concepts as equivalent.

If the metric $\mathbf{g}(t)$ evolving by Ricci flow is self-similar only for a pair of moments t_1 and t_2, then we call it a **Ricci breather**. Similarly, we can define **gradient Ricci breather** and **Shrinking**, **steady** and **expanding** Ricci breathers, (see the definition in 1.4.1 or [45]).

In the study of the singularities, a fundamental notion is that of rescaling and applying monotonicity formulas to obtain self-similar solutions which model the solutions near the singularities. The singularity models which arise are usually ancient solutions, where the solutions exist all the way back to time minus infinity. Among such "long existing" solutions (i.e., solutions which exist on an infinite time interval) are the self-similar solutions, i.e. Ricci solitons. The classification of Ricci solitons (Ricci breathers) are very important subjects in the study of Ricci flow.

1.3 Entropy functionals $\mathcal{F}_k$ and their monotonicity formula

Let $(M, \mathbf{g}(t))$ be a closed manifold with Riemnannian metric $\mathbf{g}(t)$. Consider the following evolution of the metric $\mathbf{g}(t)$:

$$\frac{\partial}{\partial t}\mathbf{g}_{ij} = h_{ij}. \tag{1.7}$$

Definition 1.3.1. *We define the following modified entropy functionals*

$$\mathcal{F}_k(\mathbf{g}_{ij}, f) = \int_M (kR + |\nabla f|^2)e^{-f} d\mu, \tag{1.8}$$

where $k > 1$.

We first derive the first variation of the functional $\int_M Re^{-f} d\mu$ under the evolution of (1.7).

Lemma 1.3.2. *If the metric evolves by*

$$\frac{\partial}{\partial t}\mathbf{g}_{ij} = h_{ij},$$

then

$$\frac{\partial}{\partial t}\int_M Re^{-f} d\mu = \int_M [-\Delta H + \nabla_i \nabla_j h_{ij} - R_{ij}h_{ij}]e^{-f} d\mu + \int_M R(-f_t + \frac{H}{2})e^{-f} d\mu.$$

Proof.

$$\tfrac{\partial}{\partial t}\int_M Re^{-f}d\mu \quad = \int_M \frac{\partial R}{\partial t}e^{-f}d\mu + \int_M R\frac{\partial}{\partial t}\Big(e^{-f}d\mu\Big)$$

Denote $H = g^{ij}h_{ij}$. Direct computations yield

$$\frac{\partial R}{\partial t} = -\Delta H + div(divh) - < Rc, h >,$$

(see the book of Ben Chow and Dan Knopf [6]) and

$$\frac{\partial}{\partial t}\Big(e^{-f}d\mu\Big) = (-f_t + \frac{H}{2})e^{-f}d\mu.$$

$$\tfrac{\partial}{\partial t}\int_M Re^{-f}d\mu \quad = \int_M[-\Delta H + \nabla_i\nabla_j h_{ij} - R_{ij}h_{ij}]e^{-f}d\mu + \int_M R(-f_t + \frac{H}{2})e^{-f}d\mu \tag{1.9}$$

□

Proposition 1.3.3. *As in Lemma 1.3.2, let $h_{ij} = -2R_{ij}$, assume $\mathbf{g}_{ij}(t)$ satisfies the Ricci flow equation over the time interval [0,T], and also function f satisfies the evolution equation (1.2), then*

$$\frac{\partial}{\partial t}\int_M Re^{-f}d\mu = 2\int_M |R_{ij}|^2 e^{-f}d\mu.$$

Proof. By using $h_{ij} = -2R_{ij}$, we get $H = g^{ij}h_{ij} = -2R$. Plug $h_{ij} = -2R_{ij}$ into (1.9), apply second Bianchi identity and integration by parts, we derive

$$\begin{aligned}
\tfrac{\partial}{\partial t}\int_M Re^{-f}d\mu \quad &= \int_M[-\Delta(-2R) + \nabla_i\nabla_j(-2R_{ij}) - R_{ij}(-2R_{ij})]e^{-f}d\mu \\
&\quad + \int_M R(-f_t - R)e^{-f}d\mu \\
&= \int_M[\Delta R + 2|R_{ij}|^2]e^{-f}d\mu + \int_M R(-f_t - R)e^{-f}d\mu \\
&= 2\int_M |R_{ij}|^2 e^{-f}d\mu + \int_M R\Delta e^{-f}d\mu + \int_M R(-f_t - R)e^{-f}d\mu \\
&= 2\int_M |R_{ij}|^2 e^{-f}d\mu + \int_M R[-\Delta f + |\nabla f|^2]e^{-f}d\mu + \int_M R(-f_t - R)e^{-f}d\mu \\
&= 2\int_M |R_{ij}|^2 e^{-f}d\mu + \int_M R[-\Delta f + |\nabla f|^2 - f_t - R]e^{-f}d\mu \\
&= 2\int_M |R_{ij}|^2 e^{-f}d\mu
\end{aligned}$$

The last equality comes from the second equation of the coupled system (1.2). $\square$

Next we derive the monotonicity formula for entropy functionals $\mathcal{F}_k(\mathbf{g}_{ij}, f)$.

Theorem 1.3.4. *Suppose the Ricci flow of* $\mathbf{g}_{ij}(t)$ *exists for* $[0, T]$*, then all the entropy functionals* $\mathcal{F}_k(\mathbf{g}_{ij}, f)$ *will be monotone along the time interval, where* $\int_M e^{-f} d\mu$ *is fixed under the Ricci flow. Furthermore, the monotonicity is strict unless the Ricci flow is a trivial Ricci soliton. Namely, it is Ricci-flat.*

Proof. Under the coupled system (1.2)

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{g}_{ij} = -2R_{ij} \\ \frac{\partial}{\partial t} f = -\Delta f - R + |\nabla f|^2, \end{cases}$$

we have shown in the above *Proposition 1.3.3* that,

$$\frac{\partial}{\partial t}\int_M Re^{-f} d\mu = 2\int_M |R_{ij}|^2 e^{-f} d\mu.$$

On the other hand, in [45], it was shown that under the same system (1.2),

$$\frac{\partial}{\partial t}\int_M (R + |\nabla f|^2) e^{-f} d\mu = 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu.$$

Put them together we get the following formula:

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{F}_k(\mathbf{g}_{ij}, f) &= \frac{\partial}{\partial t}\int_M (kR + |\nabla f|^2) e^{-f} d\mu \\ &= 2(k-1)\int_M |R_{ij}|^2 e^{-f} d\mu + 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu \qquad (1.10) \\ &\geq 0 \end{aligned}$$

where $k > 1$.

Using system (1.2) and divergence theorem, we have the following

$$\begin{aligned}
\tfrac{d}{dt}\int_M e^{-f}d\mu &= \int_M \frac{\partial}{\partial t}(e^{-f}d\mu) \\
&= \int_M (-f_t e^{-f} + e^{-f}\frac{H}{2})d\mu \\
&= \int_M (-f_t - R)e^{-f}d\mu \\
&= \int_M (\Delta f - |\nabla f|^2)e^{-f}d\mu \\
&= -\int_M \Delta e^{-f}d\mu \\
&= 0.
\end{aligned} \tag{1.11}$$

This finishes the proof of *Theorem 1.3.4*. □

Remark 1.3.5. *We notice that under the coupled system (1.2), the Ricci flow can be viewed as a L^2 gradient flow of Perelman's $\mathcal{F}$ functional up to a diffeomorphism where our functionals are not. But the monotonicity is still retained.*

Remark 1.3.6. *Our functionals yield information about the Ricci tensor itself directly.*

1.4 Compact steady breather

In this section, we discuss the applications of the monotonicity formula (1.10) we derived in Theorem 1.3.4.

First, we recall the definition of Ricci breathers, see also in [45].

Definition 1.4.1. *A metric $\mathbf{g}_{ij}(t)$ evolving by the Ricci flow is called a breather, if for some $t_1 < t_2$ and $\alpha > 0$ the metrics $\alpha\mathbf{g}_{ij}(t_1)$ and $\mathbf{g}_{ij}(t_2)$ differ only by a diffeomorphism; the cases $\alpha = 1$, $\alpha < 1$, $\alpha > 1$ correspond to steady, shrinking and expanding breathers, respectively.*

Trivial breathers are called Ricci solitons for which the above properties are true for each pair of t_1 and t_2.

Define $\lambda_k(\mathbf{g}_{ij}) = \inf \mathcal{F}_k(\mathbf{g}_{ij}, f)$, where infimum is taken over all smooth f, satisfying $\int_M e^{-f} d\mu = 1$. λ_k is the lowest eigenvalue of the corresponding operators $-4\Delta + kR$ for $k > 1$. By applying direct method and elliptic regularity theory (see [18],§8.12), the infimum is attained.

The property of steady breather yields the following proof of Theorem 1.1.1.

Proof. (**Theorem 1.1.1**) Suppose that the pair of time t_1 and t_2 are breather moments, namely, $\mathbf{g}_{ij}(t_1)$ and $\mathbf{g}_{ij}(t_2)$ differ only by a diffeomorphism. Hence, $\lambda_k(\mathbf{g}_{ij}(t_1)) = \lambda_k(\mathbf{g}_{ij}(t_2))$.

On the other hand, suppose that the first eigenvalue $\lambda_k(\mathbf{g}_{ij}(t_2))$ is attained by a function $f_k(x)$. Evolving under the backward Ricci flow, we get solutions $f_k(x, t)$ of the coupled system (1.2) which satisfies the initial condition $f_k(x, t_2) = f_k(x)$.

Using the monotonicity formula of (1.10),

$$\begin{aligned} \lambda_k(\mathbf{g}_{ij}(t_2)) &= \mathcal{F}_k(\mathbf{g}_{ij}(t_2), f_k(x, t_2)) \\ &\geq \mathcal{F}_k(\mathbf{g}_{ij}(t_1), f_k(x, t_1)) \\ &\geq \inf \mathcal{F}_k(\mathbf{g}_{ij}(t_1), f) \\ &= \lambda_k(\mathbf{g}_{ij}(t_1)) \end{aligned}$$

The equalities are obtained in each step. Consequently, in formula (1.10),

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{F}_k(\mathbf{g}_{ij}, f) &= 2(k-1)\int_M |R_{ij}|^2 e^{-f} d\mu + 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu \\ &= 0 \end{aligned}$$

This proves $Rc \equiv 0$ and finishes the proof of Theorem 1.1.1. □

Remark 1.4.2. *A similar result without the curvature assumption was obtained with a different approach by Perelman and Hamilton.*

1.5 A remark on the assumption of positive Ricci curvature operator

In [11], the author consider this problem from a different viewpoint. They study the eigenvalue λ and eigenfunction f of $-\Delta + \frac{R}{2}$ with $\int f^2 dv = 1$. They obtain the following monotonicity formula with non-negative curvature operator:

Theorem 1.5.1. *[11] On a Riemannian manifold with nonnegative curvature operator, the eigenvalues of the operator $-\Delta + \frac{R}{2}$ are nondecreasing under the Ricci flow, i.e.*

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = 2\int R_{ij}f_i f_j d\mu + \int |Rc|^2 f^2 d\mu \geq 0. \tag{1.12}$$

In this theorem, f_i denotes the covariant derivative $\nabla_i f$ of f with respect to $\frac{\partial}{\partial x_i}$, (also denoted as ∂_i).As a direct consequence of Theorem 1.5.1, they prove the following theorem

Theorem 1.5.2. *[11] There is no compact steady Ricci breather with nonnegative curvature operator, other than the one which is Ricci-flat.*

In this section, we go one step further based on (1.12) and remove the curvature constraint.

By a standard argument, we know that eigenfunction f is always positive. Let φ be a function such that $f^2(x) = e^{-\varphi(x)}$ and plug it into (1.12), we have

$$\begin{aligned} 2\tfrac{d}{dt}\lambda(f,t)|_{t=t_0} &= 4\int R_{ij}\nabla_i f \nabla_j f d\mu + 2\int R_{ij}^2 f^2 d\mu \\ &= \int R_{ij}\nabla_i\varphi\nabla_j\varphi e^{-\varphi} d\mu + 2\int R_{ij}^2 e^{-\varphi} d\mu \\ &= \int R_{ij}\nabla_i\varphi\nabla_j\varphi e^{-\varphi} d\mu + \int R_{ij}^2 e^{-\varphi} d\mu + \int R_{ij}^2 e^{-\varphi} d\mu. \end{aligned} \tag{1.13}$$

Using integration by parts and rearrangements, we derive the first term of the last identity

$$\begin{aligned}\int R_{ij}\nabla_i\varphi\nabla_j\varphi\, e^{-\varphi}d\mu &= -\int R_{ij}\nabla_i e^{-\varphi}\nabla_j\varphi d\mu \\ &= \int \nabla_i R_{ij}\nabla_j\varphi\, e^{-\varphi}d\mu + \int R_{ij}\nabla_i\nabla_j\varphi\, e^{-\varphi}d\mu.\end{aligned} \tag{1.14}$$

By the contracted second Bianchi identity "$\nabla^i R_{ij} = \frac{1}{2}\nabla_j R$" and integration by parts, we have

$$\begin{aligned}\int R_{ij}\nabla_i\varphi\nabla_j\varphi\, e^{-\varphi}d\mu &= \int \nabla_i R_{ij}\nabla_j\varphi\, e^{-\varphi}d\mu + \int R_{ij}\nabla_i\nabla_j\varphi\, e^{-\varphi}d\mu \\ &= \tfrac{1}{2}\int \nabla_j R\nabla_j\varphi\, e^{-\varphi}d\mu + \int R_{ij}\nabla_i\nabla_j\varphi\, e^{-\varphi}d\mu \\ &= -\tfrac{1}{2}\int \nabla_j R\nabla_j e^{-\varphi}d\mu + \int R_{ij}\nabla_i\nabla_j\varphi\, e^{-\varphi}d\mu \\ &= \tfrac{1}{2}\int R\nabla_j\nabla_j e^{-\varphi}d\mu + \int R_{ij}\nabla_i\nabla_j\varphi\, e^{-\varphi}d\mu \\ &= \tfrac{1}{2}\int R\Delta e^{-\varphi}d\mu + \int R_{ij}\nabla_i\nabla_j\varphi\, e^{-\varphi}d\mu.\end{aligned} \tag{1.15}$$

This implies the following

$$\int R_{ij}\nabla_i\nabla_j\varphi\, e^{-\varphi}d\mu = \int R_{ij}\nabla_i\varphi\nabla_j\varphi\, e^{-\varphi}d\mu - \frac{1}{2}\int R\Delta e^{-\varphi}d\mu. \tag{1.16}$$

On the other hand, using integration by parts and symmetry of the hessian of a function, we have

$$\begin{aligned}\int |\nabla\nabla\varphi|^2\, e^{-\varphi}d\mu &= \int \nabla_i\nabla_j\varphi\nabla_i\nabla_j\varphi\, e^{-\varphi}d\mu \\ &= -\int \nabla_j\varphi\nabla_i\nabla_i\nabla_j\varphi\, e^{-\varphi}d\mu - \int \nabla_j\varphi\nabla_i\nabla_j\varphi\nabla_i e^{-\varphi}d\mu \\ &= -\int \nabla_j\varphi\nabla_i\nabla_j\nabla_i\varphi\, e^{-\varphi}d\mu - \int \frac{1}{2}\nabla_i|\nabla\varphi|^2\nabla_i e^{-\varphi}d\mu \\ &= -\int \nabla_j\varphi\nabla_i\nabla_j\nabla_i\varphi\, e^{-\varphi}d\mu + \int \frac{1}{2}|\nabla\varphi|^2\nabla_i\nabla_i e^{-\varphi}d\mu \\ &= -\int \nabla_j\varphi\nabla_i\nabla_j\nabla_i\varphi\, e^{-\varphi}d\mu + \int \frac{1}{2}|\nabla\varphi|^2\Delta e^{-\varphi}d\mu.\end{aligned} \tag{1.17}$$

By the commutator formulas for covariant derivatives which are known as Ricci identities, see page 286 in [6], we have

$$\nabla_i\nabla_j\nabla_i\varphi = \nabla_j\nabla_i\nabla_i\varphi - R^k_{iji}\nabla_k\varphi, \tag{1.18}$$

where R^l_{ijk} represents the Riemann curvature $(3,1)$-tensor.

Combing (1.17), (1.18), and the contracted second Bianchi identity, we have

$$
\begin{aligned}
\int |\nabla\nabla\varphi|^2 \quad e^{-\varphi}d\mu &= -\int \nabla_j\varphi(\nabla_j\nabla_i\nabla_i\varphi - R^k_{iji}\nabla_k\varphi)\, e^{-\varphi}d\mu + \int \frac{1}{2}|\nabla\varphi|^2\Delta e^{-\varphi}d\mu \\
&= -\int \nabla_j\varphi\nabla_j\Delta\varphi\, e^{-\varphi}d\mu + \int R^k_{iji}\nabla_j\varphi\nabla_k\varphi\, e^{-\varphi}d\mu + \int \frac{1}{2}|\nabla\varphi|^2\Delta e^{-\varphi}d\mu \\
&= \int \nabla_j e^{-\varphi}\nabla_j\Delta\varphi d\mu - \int R^k_{iji}\nabla_j e^{-\varphi}\nabla_k\varphi d\mu + \int \frac{1}{2}|\nabla\varphi|^2\Delta e^{-\varphi}d\mu \\
&= -\int \Delta e^{-\varphi}\Delta\varphi d\mu + \int (\nabla_j R^k_{iji}\nabla_k\varphi + R^k_{iji}\nabla_j\nabla_k\varphi)\, e^{-\varphi}d\mu + \int \frac{1}{2}|\nabla\varphi|^2\Delta e^{-\varphi}d\mu \\
&= -\int \Delta e^{-\varphi}\Delta\varphi d\mu + \int (-\nabla_j R^k_{jii}\nabla_k\varphi - R^k_{jii}\nabla_j\nabla_k\varphi)\, e^{-\varphi}d\mu + \int \frac{1}{2}|\nabla\varphi|^2\Delta e^{-\varphi}d\mu \\
&= -\int \Delta e^{-\varphi}\Delta\varphi d\mu + \int (-\nabla_j R_{jk}\nabla_k\varphi - R_{jk}\nabla_j\nabla_k\varphi)\, e^{-\varphi}d\mu + \int \frac{1}{2}|\nabla\varphi|^2\Delta e^{-\varphi}d\mu \\
&= -\int \Delta e^{-\varphi}\Delta\varphi d\mu - \int \frac{1}{2}\nabla_k R\nabla_k\varphi\, e^{-\varphi}d\mu - \int R_{jk}\nabla_j\nabla_k\varphi\, e^{-\varphi}d\mu \\
&\quad + \int \frac{1}{2}|\nabla\varphi|^2\Delta e^{-\varphi}d\mu \\
&= -\int \Delta e^{-\varphi}\Delta\varphi d\mu - \frac{1}{2}\int R\Delta e^{-\varphi}d\mu - \int R_{jk}\nabla_j\nabla_k\varphi\, e^{-\varphi}d\mu \\
&\quad + \tfrac{1}{2}\int |\nabla\varphi|^2\Delta e^{-\varphi}d\mu \\
&= -\int \Delta e^{-\varphi}(\Delta\varphi + \frac{1}{2}R - \frac{1}{2}|\nabla\varphi|^2)d\mu - \int R_{jk}\nabla_j\nabla_k\varphi\, e^{-\varphi}d\mu.
\end{aligned}
\tag{1.19}
$$

We notice that we are free to change the dummy index from $\{i, j, k, l\}$ to other index or exchange among them whenever necessary.

Combing (1.16) and (1.19), we calculate $\int 2R_{ij}\nabla_i\nabla_j\varphi e^{-\varphi}d\mu + \int |\nabla\nabla\varphi|^2 e^{-\varphi}d\mu$ as following

$$
\begin{aligned}
\int 2R_{ij}\nabla_i\nabla_j\varphi e^{-\varphi}d\mu \quad &+ \int |\nabla\nabla\varphi|^2 e^{-\varphi}d\mu \\
&= \int R_{jk}\nabla_j\nabla_k\varphi\, e^{-\varphi}d\mu - \int \Delta e^{-\varphi}(\Delta\varphi + \frac{1}{2}R - \frac{1}{2}|\nabla\varphi|^2)d\mu \\
&= \int R_{ij}\nabla_i\varphi\nabla_j\varphi\, e^{-\varphi}d\mu - \frac{1}{2}\int R\Delta e^{-\varphi}d\mu - \int \Delta e^{-\varphi}(\Delta\varphi + \frac{1}{2}R - \frac{1}{2}|\nabla\varphi|^2)d\mu \\
&= \int R_{ij}\nabla_i\varphi\nabla_j\varphi\, e^{-\varphi}d\mu - \int \Delta e^{-\varphi}(\Delta\varphi + R - \frac{1}{2}|\nabla\varphi|^2)d\mu
\end{aligned}
\tag{1.20}
$$

We recall that f is the eigenfunction, λ is the eigenvalue of $-\Delta + \frac{R}{2}$, and $f^2 = e^{-\varphi}$. From the fact that $\lambda f = -\Delta f + \frac{R}{2} f$, a simple calculation yields

$$2\lambda = \Delta\varphi + R - \frac{1}{2}|\nabla\varphi|^2 \tag{1.21}$$

Plugging (1.21) into (1.20), by divergence theorem on closed manifold, we have

$$\begin{aligned}\int 2R_{ij}\nabla_i\nabla_j\varphi e^{-\varphi}d\mu \quad &+ \int |\nabla\nabla\varphi|^2 e^{-\varphi}d\mu \\ &= \int R_{ij}\nabla_i\varphi\nabla_j\varphi\, e^{-\varphi}d\mu - \int 2\lambda\Delta e^{-\varphi}d\mu \\ &= \int R_{ij}\nabla_i\varphi\nabla_j\varphi\, e^{-\varphi}d\mu - 2\lambda\int \Delta e^{-\varphi}d\mu \\ &= \int R_{ij}\nabla_i\varphi\nabla_j\varphi\, e^{-\varphi}d\mu.\end{aligned} \tag{1.22}$$

In the end, we plug (1.22) into (1.13) and have the following

$$\begin{aligned}2\tfrac{d}{dt}\lambda(f,t)|_{t=t_0} &= \int R_{ij}\nabla_i\varphi\nabla_j\varphi e^{-\varphi}d\mu + \int R_{ij}^2 e^{-\varphi}d\mu + \int R_{ij}^2 e^{-\varphi}d\mu \\ &= \int 2R_{ij}\nabla_i\nabla_j\varphi e^{-\varphi}d\mu + \int |\nabla\nabla\varphi|^2 e^{-\varphi}d\mu + \int R_{ij}^2 e^{-\varphi}d\mu + \int R_{ij}^2 e^{-\varphi}d\mu \\ &= \int |R_{ij} + \nabla_i\nabla_j\varphi|^2\, e^{-\varphi}d\mu + \int R_{ij}^2\, e^{-\varphi}d\mu \\ &\geq 0\end{aligned} \tag{1.23}$$

We then proved a new version of Theorem 1.5.1 in [11] without any curvature condition.

Theorem 1.5.3. *On a Riemannian manifold, the eigenvalues of the operator $-\Delta + \frac{R}{2}$ are nondecreasing under the Ricci flow.*

Similarly, as a direct consequence, we have the following theorem

Theorem 1.5.4. *There is no compact steady Ricci breather other than the one which is Ricci-flat.*

Chapter 2 Perelman's $\mathcal{W}$-entropy functional and new entropy functionals

In this chapter, we introduce a new approach to prove the monotonicity formula of Perelman's $\mathcal{W}$-entropy functional. This method reveals the relation between $\mathcal{F}$ functional and $\mathcal{W}$ functional which also greatly simplifies the computations . Inspired by this idea, we introduce a new entropy functionals on expanders and study their monotonicity properties. This result was previously obtained by M. Feldman, T. Ilmanen, and L.Ni.[23]. (Informed by Professor B. Chow.)

2.1 From $\mathcal{F}$-entropy functional to $\mathcal{W}$-entropy functional

In [45], Perelman also introduced another entropy functional $\mathcal{W}$-entropy besides $\mathcal{F}$-entropy functional. He then uses this $\mathcal{W}$-entropy to remove two stumbling blocks in Hamilton's program.

We defined $\mathcal{F}$-entropy functional in (1.1) Chapter 1 . $\mathcal{W}$-Entropy is defined as following

$$\mathcal{F}(\mathbf{g}_{ij}, f) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right](4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu. \tag{2.1}$$

In [45], Perelman proved the monotonicity formula for this $\mathcal{W}$-Entropy, which is the following theorem. This monotonicity is fundamental in understanding the local geometry of the solution $\mathbf{g}(t)$ to the Ricci flow.

Theorem 2.1.1. *If $(\mathbf{g}(t), f(t), \tau(t))$ is a solution of the following coupled system*

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{g}_{ij} = -2R_{ij} \\ \frac{\partial}{\partial t}f = -\Delta f - R + |\nabla f|^2 + \frac{n}{2\tau} \\ \frac{d\tau}{dt} = -1, \end{cases} \tag{2.2}$$

then

$$\frac{d}{dt}\mathcal{W}(\mathbf{g}(t), f(t), \tau(t)) = \int_M 2\tau\Big|R_{ij} + \nabla_i\nabla_j f - \frac{\mathbf{g}_{ij}}{2\tau}\Big|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu \geq 0. \tag{2.3}$$

Recall that a shrinking soliton solution of Ricci flow equation satisfies

$$R_{ij} + \nabla_i\nabla_j f - \frac{\mathbf{g}_{ij}}{2\tau} \equiv 0, \tag{2.4}$$

where $\tau = T - t$, T is a positive constant. It was proved in [45] that the $\mathcal{F}$-entropy has the monotonicity formula as below

Theorem 2.1.2. *If $(\mathbf{g}(t), f(t))$ is a solution of the following coupled system*

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{g}_{ij} = -2R_{ij} \\ \frac{\partial}{\partial t}f = -\Delta f - R + |\nabla f|^2, \end{cases} \tag{2.5}$$

then

$$\frac{d}{dt}\mathcal{F}(\mathbf{g}(t), f(t)) = \int_M 2\Big|R_{ij} + \nabla_i\nabla_j f\Big|^2 e^{-f} d\mu \geq 0. \tag{2.6}$$

Based on (2.5), suppose we want to guess an entropy functional which has a vanishing first variation on a shrinker $R_{ij} + \nabla_i\nabla_j f - \frac{\mathbf{g}_{ij}}{2\tau} \equiv 0$. Naturally, we would think that the first variation of it will be as the following in the same spirit of the first variational formula of $\mathcal{F}$-entropy

$$\int_M \Big|R_{ij} + \nabla_i\nabla_j f - \frac{\mathbf{g}_{ij}}{2\tau}\Big|^2 e^{-f} d\mu. \tag{2.7}$$

Remark 2.1.3. *$\mathcal{F}$-entropy has vanishing first variation on a steady breather (steady soliton).*

Here, we always normalize the volume as $\int_M e^{-f} d\mu = 1$, because under the coupled system (2.5), $\frac{d}{dt} e^{-f} d\mu = 0$. We next do some calculations under the coupled system (2.5)

$$\begin{aligned}
\int_M \left| R_{ij} \right. & \left. + \nabla_i \nabla_j f - \tfrac{\mathbf{g}_{ij}}{2\tau} \right|^2 e^{-f} d\mu \\
&= \int_M \left| R_{ij} + \nabla_i \nabla_j f \right|^2 e^{-f} d\mu - 2 \int_M (R_{ij} + \nabla_i \nabla_j f) \frac{\mathbf{g}_{ij}}{2\tau} e^{-f} d\mu + \int_M \left(\frac{\mathbf{g}_{ij}}{2\tau} \right)^2 e^{-f} d\mu \\
&= \int_M \left| R_{ij} + \nabla_i \nabla_j f \right|^2 e^{-f} d\mu - \frac{1}{\tau} \int_M (R + \Delta f) e^{-f} d\mu + \int_M \left(\frac{\mathbf{g}_{ij}}{2\tau} \right)^2 e^{-f} d\mu \\
&= \int_M \left| R_{ij} + \nabla_i \nabla_j f \right|^2 e^{-f} d\mu - \frac{1}{\tau} \int_M (R + |\nabla f|^2) e^{-f} d\mu + \frac{n}{4\tau^2} \int_M e^{-f} d\mu \\
&= \int_M \left| R_{ij} + \nabla_i \nabla_j f \right|^2 e^{-f} d\mu - \frac{1}{\tau} \int_M (R + |\nabla f|^2) e^{-f} d\mu + \frac{n}{4\tau^2}
\end{aligned} \tag{2.8}$$

Multiply (2.8) by 2τ and by Theorem 2.1.2, we obtain

$$\begin{aligned}
2\tau \int_M \left| R_{ij} \right. & \left. + \nabla_i \nabla_j f - \tfrac{\mathbf{g}_{ij}}{2\tau} \right|^2 e^{-f} d\mu \\
&= \tau \mathcal{F}' - \mathcal{F} + \tfrac{n}{2\tau} \\
&= \tau \mathcal{F}' - 2\mathcal{F} + \tfrac{n}{2\tau} \\
&= \tau \mathcal{F}' - \mathcal{F} - \mathcal{F} + \tfrac{n}{2\tau} \\
&= (\tau \mathcal{F})' - \mathcal{F} - \tfrac{n}{2} (\ln \tau)'.
\end{aligned} \tag{2.9}$$

The only question left for us is what is the anti-derivative of $\mathcal{F}$. It is given by the following lemma

Lemma 2.1.4. *Under the coupled system (2.5), $\frac{d}{dt} \int_M f e^{-f} d\mu = -\mathcal{F}$.*

Proof. (Lemma 2.1.4) We know under (2.5), $\frac{d}{dt}(d\mu) = -R d\mu$, e.g. see [6]. Using integration by parts, we obtain

$$
\begin{aligned}
-\tfrac{d}{dt}\int_M fe^{-f}d\mu &= \int_M \Big[-f' + ff' + fR\Big]e^{-f}d\mu \\
&= \int_M \Big[-(-\Delta f - R + |\nabla f|^2) + f(-\Delta f - R + |\nabla f|^2) + fR\Big]e^{-f}d\mu \\
&= \int_M \Big[\Delta f + R - |\nabla f|^2 - f\Delta f + f|\nabla f|^2\Big]e^{-f}d\mu \\
&= \int_M \Big[\Delta f + R - |\nabla f|^2 + f|\nabla f|^2\Big]e^{-f}d\mu - \int_M f\Delta f e^{-f}d\mu \\
&= \int_M \Big[\Delta f + R - |\nabla f|^2 + f|\nabla f|^2\Big]e^{-f}d\mu + \int_M (|\nabla f|^2 - f|\nabla f|^2)e^{-f}d\mu \\
&= \int_M (\Delta f + R)e^{-f}d\mu \\
&= \int_M (|\nabla f|^2 + R)e^{-f}d\mu \\
&= \mathcal{F}.
\end{aligned}
\tag{2.10}
$$

□

Remark 2.1.5. *We were informed that this lemma was first obtained by L.Ni.*

Combing (2.9) and (2.10), we have

$$
\begin{aligned}
2\tau\int_M \Big|R_{ij} &+ \nabla_i\nabla_j f - \tfrac{g_{ij}}{2\tau}\Big|^2 e^{-f}d\mu \\
&= (\tau\mathcal{F})' + (\int_M fe^{-f}d\mu)' - \frac{n}{2}(\ln\tau)' \\
&= \int_M \Big[\tau(R + |\nabla f|^2) + f - \frac{n}{2}\ln\tau\Big]e^{-f}d\mu
\end{aligned}
\tag{2.11}
$$

This suggests that we could define $\mathcal{W}$-entropy as following

$$
\mathcal{W} = \int_M \Big[\tau(R + |\nabla f|^2) + f - \frac{n}{2}\ln\tau\Big]e^{-f}d\mu. \tag{2.12}
$$

Then from (2.11), we know this entropy has monotonicity property and the first variation vanishes on shrinker.

We next want to make the following system scale invariant under simultaneous scaling of τ and $\mathbf{g}$

$$\begin{cases} \mathcal{W}(\mathbf{g}(t), f(t), \tau(t)) = \displaystyle\int_M \Big[\tau(R + |\nabla f|^2) + f - \frac{n}{2}\ln\tau\Big] e^{-f} d\mu \\ \displaystyle\int_M e^{-f} d\mu = 1. \end{cases} \tag{2.13}$$

We define a new function φ such that $\int_M e^{-f} d\mu = \int_M \tau^{-\frac{n}{2}} e^{-\varphi} d\mu = 1$ which makes the second identity scale invariant. Easy computation shows that $\varphi = f - \frac{n}{2}\ln\tau$ which also yields a new coupled

$$\begin{cases} \mathcal{W}(\mathbf{g}(t), f(t), \tau(t)) = \mathcal{W}(\mathbf{g}(t), \varphi(t), \tau(t)) = \displaystyle\int_M \Big[\tau(R + |\nabla\varphi|^2) + \varphi\Big]\tau^{-\frac{n}{2}} e^{-\varphi} d\mu \\ \displaystyle\int_M \tau^{-\frac{n}{2}} e^{-\varphi} d\mu = 1. \end{cases} \tag{2.14}$$

We notice that we have used the fact that $\tau(t)$ is a covariant-constant and $\nabla\varphi = \nabla f$. Recall that f satisfies the coupled system (2.5), i.e. $\frac{\partial}{\partial t} f = -\Delta f - R + |\nabla f|^2$. Easy computation yields

$$\frac{\partial}{\partial t}\varphi = -\Delta\varphi - R + |\nabla\varphi|^2 - \frac{n}{2\tau}. \tag{2.15}$$

The above process actually gives a complete proof for the following theorem

Theorem 2.1.6. *If* $(\mathbf{g}(t), \varphi(t), \tau(t))$ *is a solution of the following coupled system*

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{g}_{ij} = -2R_{ij} \\ \frac{\partial}{\partial t}\varphi = -\Delta\varphi - R + |\nabla\varphi|^2 - \frac{n}{2\tau} \\ \frac{d\tau}{dt} = -1, \end{cases} \tag{2.16}$$

then

$$\frac{d}{dt}\mathcal{W}(\mathbf{g}(t), \varphi(t), \tau(t)) = \int_M 2\tau\Big|R_{ij} + \nabla_i\nabla_j\varphi - \frac{\mathbf{g}_{ij}}{2\tau}\Big|^2 \tau^{-\frac{n}{2}} e^{-\varphi} d\mu \geq 0, \tag{2.17}$$

where $\mathcal{W}(\mathbf{g}(t), \varphi(t), \tau(t)) = \int_M \Big[\tau(R + |\nabla\varphi|^2) + \varphi\Big]\tau^{-\frac{n}{2}} e^{-\varphi} d\mu.$

The only difference between Theorem 2.1.1 and Theorem 2.1.6 are two constants $(4\pi)^{-\frac{n}{2}}$ and $\int_M n(4\pi)^{-\frac{n}{2}}e^{-f}d\mu = n(4\pi)^{-\frac{n}{2}}$. Hence, up to a constant, we also gave a proof for Theorem 2.1.1.

2.2 $\mathcal{W}$-Entropy over an expander

Following the same idea in the previous section, we want to ask is there a similar $\mathcal{W}$-entropy over an expander $R_{ij} + \nabla_i\nabla_j f + \frac{\mathbf{g}_{ij}}{2\tau} \equiv 0$ (see the definition in Pg150 of [7])?

Remark 2.2.1. *In the definition of an expander,* $\frac{d}{dt}\tau(t) = 1$ *instead of* $\frac{d}{dt}\tau(t) = -1$ *of a shrinker.*

In this section, we will follow the process of section 1 in this chapter and construct a similar $\mathcal{W}$-entropy which has monotonicity property and has vanishing first variation on an expander.

Similarly, we start with (2.5). We expect an entropy with the following formula of first variation

$$\int_M \left|R_{ij} + \nabla_i\nabla_j f + \frac{\mathbf{g}_{ij}}{2\tau}\right|^2 e^{-f}d\mu. \tag{2.18}$$

We next carry some calculations similar to (2.8) under the coupled system (2.5)

$$\begin{aligned}\int_M \Big|R_{ij} &+ \nabla_i\nabla_j f + \tfrac{\mathbf{g}_{ij}}{2\tau}\Big|^2 e^{-f}d\mu \\ &= \int_M \left|R_{ij} + \nabla_i\nabla_j f\right|^2 e^{-f}d\mu + 2\int_M (R_{ij} + \nabla_i\nabla_j f)\,\frac{\mathbf{g}_{ij}}{2\tau}e^{-f}d\mu + \int_M \left(\frac{\mathbf{g}_{ij}}{2\tau}\right)^2 e^{-f}d\mu \\ &= \int_M \left|R_{ij} + \nabla_i\nabla_j f\right|^2 e^{-f}d\mu + \frac{1}{\tau}\int_M (R + |\nabla f|^2)e^{-f}d\mu + \frac{n}{4\tau^2}\end{aligned} \tag{2.19}$$

Multiply (2.19) by 2τ and by Theorem 2.1.2, we obtain

$$
\begin{aligned}
2\tau \int_M \Big| R_{ij} & + \nabla_i \nabla_j f + \tfrac{\mathbf{g}_{ij}}{2\tau} \Big|^2 e^{-f} d\mu \\
& = \tau \mathcal{F}' + \mathcal{F} + \tfrac{n}{2\tau} \\
& = \tau \mathcal{F}' + 2\mathcal{F} + \tfrac{n}{2\tau} \\
& = \tau \mathcal{F}' + \mathcal{F} + \mathcal{F} + \tfrac{n}{2\tau} \\
& = (\tau \mathcal{F})' - (\int_M f e^{-f} d\mu)' + \frac{n}{2} (\ln \tau)'.
\end{aligned}
\tag{2.20}
$$

In the above, we have used the hypothesis $\frac{d}{dt}\tau(t) = 1$ and Lemma 2.1.4.

This suggests that we could define our new $\mathcal{W}$-entropy as following

$$
\mathcal{W} = \int_M \Big[\tau (R + |\nabla f|^2) - f + \frac{n}{2} \ln \tau \Big] e^{-f} d\mu. \tag{2.21}
$$

Then from (2.20), we know this entropy has monotonicity property and the first variation vanishes on an expander.

We next need to make the following system scale invariant under simultaneous scaling of τ and $\mathbf{g}$

$$
\begin{cases}
\mathcal{W}(\mathbf{g}(t), f(t), \tau(t)) = \displaystyle\int_M \Big[\tau (R + |\nabla f|^2) - f + \frac{n}{2} \ln \tau \Big] e^{-f} d\mu \\
\displaystyle\int_M e^{-f} d\mu = 1.
\end{cases}
\tag{2.22}
$$

As before, we define a new function $\varphi = f - \frac{n}{2} \ln \tau$ such that $\int_M e^{-f} d\mu = \int_M \tau^{-\frac{n}{2}} e^{-\varphi} d\mu = 1$, which makes the second identity scale invariant. Easy computation yields

$$
\begin{cases}
\mathcal{W}(\mathbf{g}(t), f(t), \tau(t)) = \mathcal{W}(\mathbf{g}(t), \varphi(t), \tau(t)) = \displaystyle\int_M \Big[\tau (R + |\nabla \varphi|^2) - \varphi \Big] \tau^{-\frac{n}{2}} e^{-\varphi} d\mu \\
\displaystyle\int_M \tau^{-\frac{n}{2}} e^{-\varphi} d\mu = 1.
\end{cases}
\tag{2.23}
$$

Using $\tau(t)$ is a covariant-constant and $\frac{\partial}{\partial t} f = -\Delta f - R + |\nabla f|^2$, we have the following

$$
\frac{\partial}{\partial t} \varphi = -\Delta \varphi - R + |\nabla \varphi|^2 - \frac{n}{2\tau}. \tag{2.24}
$$

We obtain a complete proof for the following theorem

Theorem 2.2.2. *If* $(\mathbf{g}(t), \varphi(t), \tau(t))$ *is a solution of the following coupled system*

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{g}_{ij} = -2R_{ij} \\ \frac{\partial}{\partial t}\varphi = -\Delta\varphi - R + |\nabla\varphi|^2 - \frac{n}{2\tau} \\ \frac{d\tau}{dt} = 1, \end{cases} \tag{2.25}$$

then

$$\frac{d}{dt}\mathcal{W}(\mathbf{g}(t), \varphi(t), \tau(t)) = \int_M 2\tau\Big|R_{ij} + \nabla_i\nabla_j\varphi + \frac{\mathbf{g}_{ij}}{2\tau}\Big|^2 \tau^{-\frac{n}{2}} e^{-\varphi} d\mu \geq 0, \tag{2.26}$$

where $\mathcal{W}(\mathbf{g}(t), \varphi(t), \tau(t)) = \int_M \Big[\tau(R + |\nabla\varphi|^2) - \varphi\Big]\tau^{-\frac{n}{2}} e^{-\varphi} d\mu.$

Remark 2.2.3. *We could modify our entropy by constants to obtain an exact form like Perelman's* $\mathcal{W}$*-entropy on shrinkers. One could also derive it from a differential inequality for a Harnack-like quantity for the conjugate heat equation and discuss the monotonicity of the forward reduced volume, see [23]. See applications also in [23].*

Chapter 3 A p-harmonic approach to the generalized Bernstein conjecture

In this chapter, we consider a problem of Yau. We give an affirmative answer to the question as a partial solution. We also study similar problems on general Riemannian manifolds in an intrinsic setting which is involved with nonlinear degenerate partial differential equations.

3.1 Introduction

Let M be an oriented n-dimensional manifold immersed in an oriented $(n + 1)$-dimensional Riemannian manifold N. M is called a minimal hypersurface in N if it is a critical point of the volume functional, and it is said to be stable, if the second variation of its volume is always nonnegative for any normal deformation with compact support. The classical Bernstein theorem [4] states that an entire minimal graph in $\mathbb{R}^3$ must be a plane. The generalized Bernstein Theorem which states that an entire minimal graph in $\mathbb{R}^{n+1}$ must be a hyperplane was proved by E.De Giorgi [17] when $n = 3$, by F.Almgren [1] when $n = 4$, and By J.Simons [50] when $5 \le n \le 7$. Simons actually gave a proof for all the cases of $2 \le n \le 7$. For $n > 7$, E.Bombieri, E.De Giorgi, and E.Giusti [8] constructed counter examples. Hence, the classical Generalized Bernstein problem is completely settled.

S.T.Yau [59] raised the following question about the generalized Bernstein Theorem

Question 3.1.1. *(A problem of Yau) Is a hyperplane the only stable minimal hypersurface in $\mathbb{R}^{n+1}$ ($n \leq 7$)?*

We notice that any minimal graph is area-minimizing, hence stable. This shows that the classical generalized Bernstein theorem is a special case of the Yau's problem. The first effort was made by R.Schoen, L.Simon and S.T.Yau [53]. They gave an affirmative answer to the question when dimension $n \leq 5$ under a p-th polynomial volume growth condition on the hypersurface. A special case of their results is the following theorem.

Theorem 3.1.2. *[53] Let M^n be a complete oriented stable minimal hypersurface in $\mathbb{R}^{n+1}$, where $n \leq 5$. If $\lim_{r\to\infty} \frac{Vol(B(r))}{r^p} = 0$, $p \in [4, 4+\sqrt{\frac{8}{n}})$, then M^n must be a hyperplane. Here $B(r)$ denote either geodesic balls on the hypersurface with radius r or the intersection with M of geodesic balls in $\mathbb{R}^{n+1}$ with radius r.*

In fact, Theorem 3.1.2 gives a simple differential geometric proof for the classical generalized Bernstein theorem ($n \leq 5$). The case of $n = 2$ of Yau's problem was completely solved by Fischer-Colbrie and Schoen [22], Do Carmo and Peng [19],Pogorelov [48]. Under relative strong conditions like finite L^2 or L^n norm of second fundamental form, several people confirmed Yau's question for all dimensions, see the work of Do Carmo and Peng [20], P.Bérard [5], Y.B.Shen and X.H.Zhu [55].

To generalize the inspiring theorem of Schoen, Simon and Yau [53], in [58], we studied a classification theory on higher dimensional Riemannian manifolds from a p-harmonic viewpoint. We divide general dimensional complete non-compact Riemannian manifolds into two classes: p-parabolic and p-hyperbolic, based on the existence of a positive supersolution of a p-Laplace equation on the manifold. We also characterized this classification with various sufficient conditions. As an application, We prove that Yau's problem has an affirmative answer if the hypersurface is p-parabolic, which is a generalization of the results in [53].

Theorem 3.1.3. *[58] Let M be a stable minimal hypersurface in a manifold N^{n+1} with constant sectional curvature $K \geq 0$. If M is p-parabolic then M is totally geodesic, where $p \in [4, 4 + \sqrt{8/n})$. In particular, every p-parabolic stable minimal hypersurface in R^{n+1} is a hyperplane for $n \leq 5$, where $p \in [4, 4 + \sqrt{8/n})$.*

In contrast to Schoen-Simon-Yau's theorem 3.1.2, the only difference of theorem 3.1.3 is that we replace the volume growth condition by the p-parabolicity condition. From the characterization of p-parabolic manifolds (see [58]), we understand that the p-th polynomial volume growth condition implies p-parabolicity. Furthermore, one can construct a p-parabolic manifold with arbitrary volume growth, even exponential volume growth. Hence, the condition in Theorem 3.1.3 is a strictly weaker condition.

One other motivation for us to consider this kind of classification is the beautiful uniformization theorem on Riemann surfaces. Recall how Yau's problem is solved when $n = 2$, the strategy is that one consider two cases of the open (complete non-compact) Riemann surfaces: parabolic and hyperbolic. One then proves that a parabolic stable minimal surface in $\mathbb{R}^3$ must be a plane and a hyperbolic minimal surface cannot be stably immersed into $\mathbb{R}^3$. This and Theorem 3.1.3 in [58] lead us to ask the following natural question:

Question 3.1.4. *If a minimal hypersurface in $\mathbb{R}^{n+1}$ is p-hyperbolic, can the minimal immersion be stable?*

A negative answer to this question combined with Theorem 3.1.3 will give a complete solution for the generalized Bernstein problem at least when $n \leq 5$. It seems that p-harmonic geometry will play an important role and may shed some lights on this problem.

The main Theorem of this chapter is the following

Theorem 3.1.5. *(Theorem 3.5.1) Let $(M^n, \mathbf{g})$ be a complete Riemannian manifold.*

1. *If in the conformal class of the metric $\mathbf{g}$ on M^n, there exists a non-complete metric $\widetilde{\mathbf{g}}$ with non-positive scalar curvature $\widetilde{R} \leq 0$ and finite volume, then $Q(u) = 0$ with coefficients satisfying (3.44) and (3.46) does not have any essential positive supersolution.*

2. *Assume in the conformal class of $\mathbf{g}$, there exists a complete metric $\widetilde{\mathbf{g}}$ with non-positive total scalar curvature $\int_M \widetilde{R} d\mu_{\widetilde{\mathbf{g}}} \leq 0$ and quadratic volume growth. If $Q(u) = 0$ with coefficients satisfying (3.44) and (3.46) has an essential positive supersolution, then $(M^n, \mathbf{g})$ is isometric to $(M^n, const \cdot \widetilde{\mathbf{g}})$, and also has quadratic volume growth.*

This chapter contains five sections. In section 2, we give a preliminary introduction to the terminology and notations. We give a short proof of a well-known lemma in differential geometry which interprets the relation of scalar curvatures and second fundamental form between immersed and ambient manifolds. In section 3, we study conformal class of the metrics on immersed hypersurfaces. By utilizing the conformal invariance of n-parabolicity of n-dimensional manifolds, we study several types of n-hyperbolic manifolds (n is the dimension of the immersed manifolds), and prove that they cannot be stably minimally immersed into $\mathbb{R}^{n+1}$. In section 4, we study the essential positive supersolutions of a class of general potentially non-linear degenerate partial differential equations on immersed hypersurfaces or Riemannian manifolds with non-positive scalar curvature. The existence of such solutions suggests a stability-like inequality. In the last section, we prove our main theorem by using a slightly different approach. We consider general manifolds without any curvature or extrinsic constraints. In this context, we don't restrict ourselves on **minimal hypersurfaces**.

3.2 Preliminaries

We follow the notations and terminology from [13] and [53].

Let M be a n-dimensional manifold immersed in a $n+1$-dimensional Riemannian manifold N. Given an orthonormal frame field $\{e_1, \cdots, e_{n+1}\}$ adapted to the immersion (i.e. e_{n+1} is normal to M). We denote h_{ij}, $i, j = 1, \cdots, n$, as the second fundamental form A of the immersion under this frame field and h_{ijk}, $k = 1, \cdots, n$ as the covariant derivative of A.

Let K_{ij} denote the sectional curvature of the 2-plane spanned by the orthonormal basis e_i and e_j and $\widetilde{K}_{ij}$. Let R be the scalar curvature of the immersed manifold M and $\widetilde{R}$ be the scalar curvature of the ambient manifold N. Under the above orthonormal frame field, we have the following identities

$$\begin{aligned} R &= 2\Sigma_{1\leq i<j\leq n} K_{ij} \\ \widetilde{R} &= 2\Sigma_{1\leq i<j\leq n+1} \widetilde{K}_{ij} \end{aligned} \tag{3.1}$$

The well-known Gauss equation yields the following

$$\begin{aligned} \widetilde{K}_{ij} &= K_{ij} - h_{ii}h_{jj} + h_{ij}^2 \\ K_{ij} &= \widetilde{K}_{ij} + h_{ii}h_{jj} - h_{ij}^2 \end{aligned} \tag{3.2}$$

Sum the second equation over i, j, we have

$$\sum_{1\leq i<j=n} K_{ij} = \sum_{1\leq i<j=n} \widetilde{K}_{ij} + \sum_{1\leq i<j=n} h_{ii}h_{jj} - \sum_{1\leq i<j=n} h_{ij}^2 \tag{3.3}$$

Let H be the mean curvature which is the trace of the second fundamental form. Hence we have

$$\begin{aligned} H^2 &= (h_{11} + \cdots + h_{nn})^2 = \sum_{1\leq i\leq n} h_{ii}^2 + 2 \sum_{1\leq i<j=n} h_{ii}h_{jj} \\ \sum_{1\leq i<j=n} h_{ii}h_{jj} &= \tfrac{1}{2}H^2 - \tfrac{1}{2} \sum_{1\leq i\leq n} h_{ii}^2 \end{aligned} \tag{3.4}$$

Plug (3.4) into (3.3), we have

$$\begin{aligned} \sum_{1\leq i<j=n} K_{ij} &= \sum_{1\leq i<j=n} \widetilde{K}_{ij} + \tfrac{1}{2}H^2 - \tfrac{1}{2} \sum_{1\leq i\leq n} h_{ii}^2 - \sum_{1\leq i<j=n} h_{ij}^2 \\ &= \sum_{1\leq i<j=n} \widetilde{K}_{ij} + \tfrac{1}{2}H^2 - \tfrac{1}{2}(2 \sum_{1\leq i<j=n} h_{ij}^2 + \sum_{1\leq i\leq n} h_{ii}^2) \\ &= \tfrac{1}{2}\widetilde{R} + \tfrac{1}{2}H^2 - \tfrac{1}{2} \sum_{i,j} h_{ij}^2 \end{aligned} \tag{3.5}$$

Multiply (3.5) by 2 yields

$$R = 2 \sum_{1\leq i<j=n} K_{ij} = \widetilde{R} + H^2 - \sum_{i,j} h_{ij}^2 \tag{3.6}$$

If we denote the norm of the second fundamental form A as $||A||$, under the orthonormal frame, $||A||^2 = \sum_{i,j} h_{ij}^2$. We have the following theorem

Theorem 3.2.1. *If M is a n-dimensional manifold immersed in a $(n+1)$-dimensional Riemannian manifold N. Let R, $\widetilde{R}$ be the scalar curvature of M and N respectively. A is the second fundamental form and H denotes the mean curvature of the immersion. Then we have the following identity,*

$$R = \widetilde{R} + H^2 - \|A\|^2. \tag{3.7}$$

Proof. (Theorem 3.2.1) See (3.6). □

The following well-known result is an immediate consequence of Theorem 3.2.1 (see [13]).

Corollary 3.2.2. *Let M be a minimal hypersurface immersed in Euclidean space $\mathbb{R}^{n+1}$. Then $R = -\|A\|^2$.*

Proof. (Corollary 3.2.2) Let $\widetilde{R}$ and H be 0 in (3.7). The proof is immediate. □

3.3 p-Hyperbolicity and stable minimal hypersurfaces

By the classification theorems in [58], we know that complete non-compact Riemannian manifolds can be classified into p-parabolic and p-hyperbolic manifolds. It can also be shown that when $p = n$, the p-parabolicity and p-hyperbolicity are conformally invariant, where n =dimension of the manifold. This is because of the fact that n-energy functional and n-superharmonicity (n-harmonicity) are preserved under conformal change of the Riemannian metric(see also in [38]). More precisely, given an n-dimensional Riemannian manifold $(M^n, \mathbf{g})$ where $\mathbf{g}$ is the Riemannian metric. If $(M^n, \mathbf{g})$ is n-parabolic (resp. n-hyperbolic), then for any $\varphi \in C^\infty(M^n)$, $(M^n, e^\varphi \mathbf{g})$ is also n-parabolic (resp. n-hyperbolic).

Proposition 3.3.1. *Let $n > 2$, then n-dimensional hyperbolic spaces $\mathbb{H}^n$ are n-hyperbolic.*

Proof. (Proposition 3.3.1) Assume that the hyperbolic space $\mathbb{H}^n$ is modeled on an Euclidean unit ball $(\mathbb{B}^n, \frac{4}{(1-|\mathbf{x}|^2)^2} d\mathbf{x}^n)$ where $d\mathbf{x}^n$ is Euclidean metric and $\mathbf{x} = (x_1, \cdots, x_n)$.

Denote $f(\mathbf{x}) = 1-x_1$. Direct computations show that $f(\mathbf{x})$ is a positive n-superharmonic function defined $\mathbb{B}^n$ and $f(\mathbf{x})$ is non-constant. The proposition follows from the definition of p-hyperbolic manifold immediately. □

As an immediate application of Proposition 3.3.1 and the conformal invariance of n-hyperbolicity, we have the following corollary:

Corollary 3.3.2. *Let $n > 2$ and $\varphi \in C^\infty(M^n)$. Then a non-compact manifold $\mathbb{B}^n$ with complete Riemannian metric $ds^2 = e^{\varphi(\mathbf{x})} d\mathbf{x}^n$ must be n-hyperbolic.*

Next, we prove that the n-hyperbolic manifold in Corollary 3.3.2 cannot be stably immersed into Euclidean space $\mathbb{R}^{n+1}$ as a minimal hypersurface.

Let M be a unit ball in $\mathbb{R}^n$ endowed with a complete Riemannian metric $\mathbf{g} = u^{-\frac{4}{n-2}} \mathbf{g}_0$, where $\mathbf{g}_0$ is the Euclidean metric and $u \in C^\infty(M)$ is a positive function. Clearly, $(M, \mathbf{g})$ lies in the conformal class of hyperbolic space $\mathbb{H}^n$.

From the definition, $\mathbf{g}_0 = u^{\frac{4}{n-2}} \mathbf{g}$. By the well-known transformation law of scalar curvature under conformal change of the metric, we have the following equation:

$$R_{\mathbf{g}_0} = u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \Delta_{\mathbf{g}} u + R_{\mathbf{g}} u \right), \tag{3.8}$$

where $R_{\mathbf{g}_0}$ and $R_{\mathbf{g}}$ denote the scalar curvature of the metrics $\mathbf{g}_0$ and $\mathbf{g}_0$ respectively, $\Delta_{\mathbf{g}}$ is the Laplace operator of a function with respect to the metric $\mathbf{g}$.

In the following a subscript $\square_0$ indicates the metric used is the Euclidean metric $\mathbf{g}_0$, e.g. $R_0 \equiv R_{\mathbf{g}_0}$, $\Delta_0 \equiv \Delta_{\mathbf{g}_0}$. A supscript $\widetilde{\mathbf{g}}$ indicates the metric used is $\widetilde{\mathbf{g}}$, e.g. $R_{\widetilde{\mathbf{g}}}$. We also denote $R_{\widetilde{\mathbf{g}}}$ as $\widetilde{R}$. By default we assume that all the geometric quantities are with respect to the metric $\mathbf{g}$, hence we omit the subscript $\mathbf{g}$.

Theorem 3.3.3. *Let $n > 2$ and $(M^n, \mathbf{g}) = (\mathbb{B}^n, u^{-\frac{4}{n-2}}\mathbf{g}_0)$ be a* complete *non-compact Riemannian manifold where $\mathbf{g}_0$ is the Euclidean metric. Assume M^n is a minimally immersed hypersurface in $\mathbb{R}^{n+1}$, then the minimal immersion cannot be stable.*

Proof. (Theorem 3.3.3) Since $R_0 \equiv 0$, (3.8) yields:

$$-\frac{4(n-1)}{n-2}\Delta u + Ru = 0, \tag{3.9}$$

We define a function $v(x)$ by $v = u^{\frac{n}{n-2}}$. It follows that

$$\begin{aligned} \Delta u &= div(\nabla v^{\frac{n-2}{n}}) \\ &= div(\tfrac{n-2}{n}v^{-\frac{2}{n}}\nabla v) \\ &= \tfrac{n-2}{n}v^{-\frac{2}{n}}\Delta v - \tfrac{n-2}{n}\cdot\tfrac{2}{n}v^{-\frac{n+2}{n}}|\nabla v|^2 \\ &= \tfrac{n-2}{n}v^{-\frac{n+2}{n}}\Big(v\Delta v - \tfrac{2}{n}|\nabla v|^2\Big) \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$-\frac{4(n-1)}{n}v^{-\frac{n+2}{n}}\Big(v\Delta v - \frac{2}{n}|\nabla v|^2\Big) + Rv^{\frac{n-2}{n}} = 0.$$

Equivalently,

$$\begin{aligned} -\tfrac{4(n-1)}{n}\Big(v\Delta v - \tfrac{2}{n}|\nabla v|^2\Big) + Rv^2 &= 0 \\ v\Delta v - \tfrac{2}{n}|\nabla v|^2 &= \tfrac{n}{4(n-1)}Rv^2 \end{aligned} \tag{3.11}$$

This yields

$$\begin{aligned} \tfrac{1}{2}\Delta v^2 &= \tfrac{1}{2}div(2v\nabla v) \\ &= v\Delta v + |\nabla v|^2 \\ &= \tfrac{n}{4(n-1)}Rv^2 + \tfrac{n+2}{n}|\nabla v|^2 \end{aligned} \tag{3.12}$$

We are now ready to prove the theorem. We prove it by contradiction. Assume M^n is a stable minimal immersion in $\mathbb{R}^{n+1}$. Let $D \subset M$ be any bounded domain and f be any smooth function supported in D.

By the assumption, we have the following stability inequality,

$$\int_M \|A\|^2 f^2 d\mu_{\mathbf{g}} \leq \int_M |\nabla f|^2 d\mu_{\mathbf{g}}, \tag{3.13}$$

where $d\mu_{\mathbf{g}}$ denotes the volume element of M^n with respect to the metric $\mathbf{g}$.

Using Corollary 3.2.2, we have the equivalent inequality

$$0 \leq \int_M |\nabla f|^2 + Rf^2 d\mu, \tag{3.14}$$

for any smooth function f with compact support.

Let ξ be a smooth function compactly supported in D and let $f = v\xi$. Then $f = v\xi$ satisfies

$$0 \leq \int_M |\nabla v\xi|^2 + R(v\xi)^2 d\mu. \tag{3.15}$$

We calculate the RHS of (3.15)

$$\begin{aligned} \int_M |\nabla v\xi|^2 + R(v\xi)^2 d\mu &= \int_M |v\nabla\xi + \xi\nabla v|^2 + R(v\xi)^2 d\mu \\ &= \int_M v^2|\nabla\xi|^2 d\mu + 2\int_M v\xi < \nabla v, \nabla\xi > d\mu \\ &\quad + \int_M \xi^2|\nabla v|^2 d\mu + \int_M R(v\xi)^2 d\mu \\ &= \int_M v^2|\nabla\xi|^2 d\mu + \frac{1}{2}\int_M < \nabla v^2, \nabla\xi^2 > d\mu \\ &\quad + \int_M \xi^2|\nabla v|^2 d\mu + \int_M R(v\xi)^2 d\mu \end{aligned} \tag{3.16}$$

Using integration by parts, we get

$$\begin{aligned} \int_M |\nabla v\xi|^2 + R(v\xi)^2 d\mu &= \int_M v^2|\nabla\xi|^2 d\mu - \frac{1}{2}\int_M \xi^2 \Delta v^2 d\mu \\ &\quad + \int_M \xi^2|\nabla v|^2 d\mu + \int_M R(v\xi)^2 d\mu \end{aligned} \tag{3.17}$$

Combing (3.12) and (3.17), we have

$$\begin{aligned} 0 &\leq \int_M v^2|\nabla\xi|^2 d\mu + \frac{3n-4}{4(n-1)}\int_M R\xi^2 v^2 d\mu - \frac{2}{n}\int_M \xi^2|\nabla v|^2 d\mu \\ &\leq \int_M v^2|\nabla\xi|^2 d\mu - \frac{2}{n}\int_M \xi^2|\nabla v|^2 d\mu. \end{aligned} \tag{3.18}$$

The last step follows from the fact "$R = -\|A\|^2 \leq 0$" (see Corollary 3.2.2) and $n \geq 3$.

If we abuse the notations of $\xi(\rho) = \xi(d(x, x_0))$ with $\xi(x)$ where $d(x, x_0) = d_{\mathbf{g}}(x, x_0)$ denotes the distance from x to a fixed point x_0 on $(M^n, \mathbf{g})$. Choose ξ to be the standard cut-off function. Namely, $0 \leq \xi \leq 1$ and satisfies

$$\xi(\rho) = \begin{cases} 1, & \text{for} \quad \rho \leq \frac{\mathrm{r}}{2} \\ 0, & \text{for} \quad \rho \geq \mathrm{r}, \end{cases} \tag{3.19}$$

and $|\nabla\xi(x)|_g \leq \frac{3}{r}$.

Apply this test function to (3.18), we have

$$\begin{aligned} \tfrac{2}{n}\int_M \xi^2|\nabla v|^2 d\mu &\leq \tfrac{9}{r^2}\int_M v^2 d\mu \\ &= \tfrac{9}{r^2}\int_M u^{\frac{2n}{n-2}} d\mu \\ &= \tfrac{9}{r^2}\int_{\mathbb{B}^n} u^{\frac{2n}{n-2}}\sqrt{det\ \mathbf{g}}\, d\mathbf{x}^n \\ &= \tfrac{9}{r^2}\int_{\mathbb{B}^n} u^{\frac{2n}{n-2}}(u^{-\frac{4}{n-2}})^{\frac{n}{2}} d\mathbf{x}^n \end{aligned} \tag{3.20}$$

The last step follows from the definition of g. After simplification, we have

$$\begin{aligned} \tfrac{2}{n}\int_M \xi^2|\nabla v|^2 d\mu &\leq \tfrac{9}{r^2}\int_{\mathbb{B}^n} d\mathbf{x}^n = \frac{9}{r^2}Vol_0(\mathbb{B}^n) \\ &= \tfrac{9\pi}{r^2}. \end{aligned} \tag{3.21}$$

Let $r \to \infty$, we get $v(x) \equiv constant$. Hence $u(x) \equiv constant$ too, which implies that $(M^n, \mathbf{g}) = (\mathbb{B}^n, u^{-\frac{4}{n-2}} d\mathbf{x}^n) = (\mathbb{B}^n, constant \cdot d\mathbf{x}^n)$ is complete. The contradiction finishes the proof. □

The following Proposition is a generalization of Theorem 3.3.3.

Proposition 3.3.4. *Let $n > 2$ and $(M^n, \mathbf{g}) = (\Omega, u^{-\frac{4}{n-2}}\mathbf{g}_0)$ be a* complete *non-compact Riemannian manifold where Ω is any* **bounded** *domain in $\mathbb{R}^n$ and $\mathbf{g}_0$ is the Euclidean metric. Assume M^n is a minimally immersed hypersurface in $\mathbb{R}^{n+1}$, then the minimal immersion cannot be stable.*

Proof. (Proposition 3.3.4) By the assumption that Ω is bounded in $\mathbb{R}^n$, we have

$$\int_{M^n} d\mu_0 = \int_{\Omega} d\mu_0 < \infty. \tag{3.22}$$

Following the proof in Theorem 3.3.3, applying the fact (3.22) in (3.20) and (3.21), we proved that the stability of minimal immersion will contradict to the completeness of the immersed manifold. $\square$

Remark 3.3.5. *M^n has the same topological structure of Ω. Hence it may not necessarily be simply connected.*

Remark 3.3.6. *It is also easy to see that M^n is n-hyperbolic.*

Next, we consider a manifold modeled on a general domain in $\mathbb{R}^n$ which generalizes Proposition 3.3.4.

Theorem 3.3.7. *Let $n > 2$ and $(M^n, \mathbf{g}) = (\Omega', u\mathbf{g}_0)$ be a* complete *non-compact Riemannian manifold where $u(x) \in C^\infty(M)$ is a positive function and Ω' is any domain in $\mathbb{R}^n$ which satisfies that $(\mathbb{R}^n \backslash \Omega')^\circ$ is non-empty. Assume M^n is a minimally immersed hypersurface in $\mathbb{R}^{n+1}$, then the minimal immersion cannot be stable.*

Proof. (Theorem 3.3.7) Since $(\mathbb{R}^n \backslash \Omega')^\circ \neq \emptyset$, there exist a point $p \in (\mathbb{R}^n \backslash \Omega')^\circ$ and a positive $r \in \mathbb{R}$, such that $p \in B(r) \subset (\mathbb{R}^n \backslash \Omega')^\circ$.

We define a conformal diffeomorphism

$$\Phi : \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\},$$

where Φ is a reflection with respect to the $(n-1)$-sphere $\partial B(r)$ as an element of the Möbius transformation group $\mathfrak{M}(n)$.

From the definition, we have

$$\begin{aligned} \Phi(\partial B(r)) &= \partial B(r) \\ \Phi(\Omega') &\subset B(r). \end{aligned} \tag{3.23}$$

We next consider the pullback metric on Ω' by Φ, which can be denoted as $(\Omega', \Phi^*\mathbf{g}_0)$. $(\Omega', \Phi^*\mathbf{g}_0)$ is isometrically equivalent to $(\Phi(\Omega'), \mathbf{g}_0)$, which implies that

$$\begin{aligned} \int_{\Omega'} d\mu_{\Phi^*\mathbf{g}_0} &= \int_{\Phi(\Omega')} d\mu_{\mathbf{g}_0} \\ &\leq \int_{B(r)} d\mu_{\mathbf{g}_0} \\ &< \infty. \end{aligned} \tag{3.24}$$

On the other hand, since $\Phi : \Omega' \longrightarrow \Phi(\Omega')$ is a conformal diffeomorphism, there exists a positive function $\varphi(x) \in C^\infty(\Omega')$, such that $\Phi^*\mathbf{g}_0 = \varphi\mathbf{g}_0$.

This follows that

$$(\Omega', \mathbf{g}) = (\Omega', u\mathbf{g}_0) = (\Omega', (u\varphi^{-1}) \cdot \varphi\mathbf{g}_0) = (\Omega', (u\varphi^{-1})\Phi^*\mathbf{g}_0).$$

Up to an isometry, applying Proposition 3.3.4, we finish the proof. $\square$

Remark 3.3.8. *M^n in Theorem 3.3.7 is also n-hyperbolic.*

Following the same spirit of the above results, one has the following more general theorem.

Theorem 3.3.9. *Let $(M^n, \mathbf{g})$ be a complete minimal hypersurface immersed in Euclidean space $\mathbb{R}^{n+1}$ with the induced metric $\mathbf{g}$. Assume that in the conformal class of the metric $\mathbf{g}$, there exists a non-complete metric $\widetilde{\mathbf{g}}$ with non-positive scalar curvature $\widetilde{R}$ and the manifold $(M^n, \widetilde{\mathbf{g}})$ has finite volume. Then the minimal immersion cannot be stable.*

Proof. (Theorem 3.3.9) Since $\widetilde{\mathbf{g}}$ lies in the conformal class of $\mathbf{g}$, there exists a positive function $u \in C^\infty(M)$ and $\widetilde{\mathbf{g}} = u^{\frac{4}{n-2}}\mathbf{g}$. Recall (3.8), we have the following identity

$$\begin{aligned} \widetilde{R} &= u^{-\frac{n+2}{n-2}}\big(- \tfrac{4(n-1)}{n-2}\Delta_{\mathbf{g}} u + R_{\mathbf{g}} u\big) \\ \tfrac{4(n-1)}{n-2}\Delta u &= uR - u^{\frac{n+2}{n-2}}\widetilde{R}. \end{aligned} \tag{3.25}$$

We define a function $v(x)$ by $v = u^{\frac{n}{n-2}}$. By the same calculation as before, it follows that

$$\Delta u = \frac{n-2}{n} v^{-\frac{n+2}{n}}\left(v\Delta v - \frac{2}{n}|\nabla v|^2\right) \tag{3.26}$$

Combining (3.25) and (3.26), we have

$$\tfrac{4(n-1)}{n-2} \cdot \tfrac{n-2}{n} v^{-\frac{n+2}{n}}\left(v\Delta v - \tfrac{2}{n}|\nabla v|^2\right) = v^{\frac{n-2}{n}} R - v^{\frac{n+2}{n}}\widetilde{R}. \tag{3.27}$$

Equivalently,

$$\begin{aligned} \tfrac{4(n-1)}{n} \cdot \left(v\Delta v - \tfrac{2}{n}|\nabla v|^2\right) &= v^2 R - \widetilde{R} \\ v\Delta v - \tfrac{2}{n}|\nabla v|^2 &= \tfrac{n}{4(n-1)} v^2 R - \tfrac{n}{4(n-1)}\widetilde{R}. \end{aligned} \tag{3.28}$$

This yields

$$\begin{aligned} \tfrac{1}{2}\Delta v^2 &= \tfrac{1}{2} div(2v\nabla v) \\ &= v\Delta v + |\nabla v|^2 \\ &= \tfrac{n}{4(n-1)} v^2 R - \tfrac{n}{4(n-1)}\widetilde{R} + \tfrac{n+2}{n}|\nabla v|^2 \end{aligned} \tag{3.29}$$

We carry out the rest of the proof as in Theorem 3.3.3 and prove by contradiction.

Assume M^n is a stable minimal immersion in $\mathbb{R}^{n+1}$. Let $D \subset M$ be any bounded open set and ξ be a smooth function compactly supported in D. Then the function $v\xi$ satisfies the following stability inequality,

$$0 \leq \int_M |\nabla v\xi|^2 + R(v\xi)^2 d\mu. \tag{3.30}$$

The calculations in (3.16) and (3.17) yields the following calculation about the RHS of (3.30),

$$\begin{aligned}\int_M |\nabla v\xi|^2 + R(v\xi)^2 d\mu &= \int_M v^2|\nabla\xi|^2 d\mu - \frac{1}{2}\int_M \xi^2 \Delta v^2 d\mu \\ &\quad + \int_M \xi^2|\nabla v|^2 d\mu + \int_M R(v\xi)^2 d\mu.\end{aligned} \tag{3.31}$$

Combining with (3.29) and (3.30), we get

$$0 \leq \int_M v^2|\nabla\xi|^2 d\mu + \frac{3n-4}{4(n-1)}\int_M R\xi^2 v^2 d\mu + \frac{n}{4(n-1)}\int_M \widetilde{R}\xi^2 d\mu - \frac{2}{n}\int_M \xi^2|\nabla v|^2 d\mu \tag{3.32}$$

Since $R \leq 0$ and $\widetilde{R} \leq 0$ point-wise, we have

$$\begin{aligned} 0 &\leq \int_M v^2|\nabla\xi|^2 d\mu - \frac{2}{n}\int_M \xi^2|\nabla v|^2 d\mu \\ \tfrac{2}{n}\int_M \xi^2|\nabla v|^2 d\mu &\leq \int_M v^2|\nabla\xi|^2 d\mu.\end{aligned} \tag{3.33}$$

We let ξ be the same cut-off function as in the proof of Theorem 3.3.3. We have

$$\begin{aligned}\tfrac{2}{n}\int_M \xi^2|\nabla v|^2 d\mu &\leq \tfrac{9}{r^2}\int_M v^2 d\mu \\ &= \tfrac{9}{r^2}\int_M u^{\frac{2n}{n-2}} d\mu_{\mathbf{g}} \\ &= \tfrac{9}{r^2}\int_M u^{\frac{2n}{n-2}}(u^{-\frac{4}{n-2}})^{\frac{n}{2}} d\mu_{\widetilde{\mathbf{g}}} \\ &= 9\tfrac{vol_{\widetilde{\mathbf{g}}}(M)}{r^2}\end{aligned} \tag{3.34}$$

Let $r \to \infty$, $\lim\limits_{r\to\infty} \dfrac{vol_{\widetilde{\mathbf{g}}}(M)}{r^2} = 0$ by the hypothesis. This implies that the LHS must also converge to zero as $r \to \infty$. Hence, v must be a constant and u is also a constant, which implies that $(M^n, \mathbf{g})$ is isometric to $(M^n, const\cdot\widetilde{\mathbf{g}})$ is non-complete. The contradiction completes the proof. $\square$

As an application of Theorem 3.3.9, we give the following non-existence theorem for the Generalized Bernstein Conjecture ($n \leq 5$), under a relatively weak condition.

Theorem 3.3.10. *Let $(M^n, \mathbf{g})$ be a complete minimal hypersurface immersed in Euclidean space $\mathbb{R}^{n+1}$ with the induced metric $\mathbf{g}$, where $n > 2$. Assume that in the conformal class of the metric $\mathbf{g}$, there exists a metric $\widetilde{\mathbf{g}}$ with non-positive scalar curvature $\widetilde{R}$ and finite volume. Then $(M^n, \mathbf{g})$ cannot be stably immersed into $\mathbb{R}^{n+1}$.*

Remark 3.3.11. *In the above theorem, we don't have any constraint on whether the metric $\tilde{\mathbf{g}}$ is complete or non-complete.*

Proof. (Theorem 3.3.10) We consider the following two cases:

1. $\widetilde{\mathbf{g}}$ is non-complete.
2. $\widetilde{\mathbf{g}}$ is complete.

The first case is ruled out, because if $\widetilde{\mathbf{g}}$ is non-complete, by Theorem 3.3.9, $\mathbf{g}$ cannot be stable minimal.

If $\widetilde{\mathbf{g}}$ is complete and has quadratic volume growth, we choose the function u as before, i.e., $\widetilde{\mathbf{g}} = u^{\frac{4}{n-2}}\mathbf{g}$. This time we let the function $v = u$. v then satisfies identity (3.25)

$$\begin{aligned}
\widetilde{R} &= v^{-\frac{n+2}{n-2}}\Big(-\tfrac{4(n-1)}{n-2}\Delta_{\mathbf{g}} v + R_{\mathbf{g}} v\Big) \\
\tfrac{4(n-1)}{n-2}\Delta v &= vR - v^{\frac{n+2}{n-2}}\widetilde{R}.
\end{aligned}$$

On the other hand, from the stability inequality, we have the inequality (3.16)

$$\begin{aligned}
0 &\leq \int_M |\nabla v\xi|^2 + R(v\xi)^2 d\mu \\
&= \int_M v^2|\nabla\xi|^2 d\mu + 2\int_M v\xi < \nabla v, \nabla\xi > d\mu + \int_M \xi^2|\nabla v|^2 d\mu + \int_M R(v\xi)^2 d\mu.
\end{aligned}$$

Plugging (3.25) into (3.16), after integration by parts and rearrangement, we have

$$
\begin{aligned}
0 \ \leq & \int_M v^2|\nabla\xi|^2 d\mu + 2\int_M v\xi < \nabla v, \nabla\xi > d\mu + \int_M \xi^2|\nabla v|^2 d\mu \\
& -\tfrac{4(n-1)}{n-2}\int_M 2v\xi < \nabla v, \nabla\xi > d\mu - \frac{4(n-1)}{n-2}\int_M \xi^2|\nabla v|^2 d\mu + \int_M \widetilde{R}\xi^2 v^{\frac{2n}{n-2}} d\mu \\
\leq & \int_M v^2|\nabla\xi|^2 d\mu - \frac{3n-2}{n-2}\int_M 2v\xi < \nabla v, \nabla\xi > d\mu \\
& -\tfrac{3n-2}{n-2}\int_M \xi^2|\nabla v|^2 d\mu + \int_M \widetilde{R}\xi^2 v^{\frac{2n}{n-2}} d\mu \\
\leq & \int_M v^2|\nabla\xi|^2 d\mu + \frac{3n-2}{n-2}\int_M 2v\xi|\nabla v||\nabla\xi| d\mu - \frac{3n-2}{n-2}\int_M \xi^2|\nabla v|^2 d\mu + \int_M \widetilde{R}\xi^2 v^{\frac{2n}{n-2}} d\mu \\
\leq & \int_M v^2|\nabla\xi|^2 d\mu + \frac{3n-2}{n-2}\int_M 2v\xi|\nabla v||\nabla\xi| d\mu - \frac{3n-2}{n-2}\int_M \xi^2|\nabla v|^2 d\mu.
\end{aligned}
\tag{3.35}
$$

The last step follows from $\widetilde{R} \leq 0$.

Denote $c = \frac{3n-2}{n-2}$. Clearly, $c > 0$ when $n \geq 2$. By Young's inequality, we have the following

$$
\begin{aligned}
0 \ \leq & \int_M v^2|\nabla\xi|^2 d\mu + c\int_M 2v\xi|\nabla v||\nabla\xi| d\mu - c\int_M \xi^2|\nabla v|^2 d\mu \\
\leq & (1+\tfrac{c}{\epsilon})\int_M v^2|\nabla\xi|^2 d\mu - c(1-\epsilon)\int_M \xi^2|\nabla v|^2 d\mu
\end{aligned}
\tag{3.36}
$$

Choose ϵ small enough, we obtain

$$
\int_M \xi^2|\nabla v|^2 d\mu \leq C\int_M v^2|\nabla\xi|^2 d\mu, \tag{3.37}
$$

where C is a constant only depends on n. Now, if we conformal change the metric $\mathbf{g}$ to $\widetilde{\mathbf{g}} = u^{\frac{4}{n-2}}\mathbf{g}$ and recall $u = v$, we have

$$
\begin{aligned}
\int_M \xi^2|\nabla u|^2 d\mu \ & \leq C\int_M u^2|\nabla\xi|^2_{\mathbf{g}} d\mu_{\mathbf{g}} \\
& = C\int_M u^2 u^{\frac{4}{n-2}}|\nabla\xi|^2_{\widetilde{\mathbf{g}}}(u^{-\frac{4}{n-2}})^{\frac{n}{2}} d\mu_{\widetilde{\mathbf{g}}} \\
& = C\int_M |\nabla\xi|^2_{\widetilde{\mathbf{g}}} d\mu_{\widetilde{\mathbf{g}}}.
\end{aligned}
\tag{3.38}
$$

Choose ξ to be the standard cut-off function on $(M^n, \widetilde{\mathbf{g}})$. Comparing with the one in (3.19), the cut-off function we choose here is with respect to metric $\widetilde{\mathbf{g}}$.

we have $0 \leq \xi \leq 1$ and it satisfies

$$\xi(\rho) = \begin{cases} 1, & \text{for} \quad \rho \leq \frac{\mathrm{r}}{2} \\ 0, & \text{for} \quad \rho \geq \mathrm{r}, \end{cases} \tag{3.39}$$

where $|\nabla \xi(x)|_{\widetilde{\mathbf{g}}} \leq \frac{3}{r}$, and ρ is the distance function under metric $\widetilde{\mathbf{g}}$.

It then follows from (3.38)

$$\begin{aligned} \int_M \xi^2 |\nabla u|^2 d\mu &\leq \tfrac{9C}{r^2} \int_{B(r)} d\mu_{\widetilde{\mathbf{g}}} \\ &\leq 9C \tfrac{Vol_{\widetilde{\mathbf{g}}}(B(r))}{r^2} \\ &\leq 9C \tfrac{Vol_{\widetilde{\mathbf{g}}}(M)}{r^2}. \end{aligned} \tag{3.40}$$

Since $Vol_{\widetilde{\mathbf{g}}}(M) < \infty$, $\lim\limits_{r \to \infty} \dfrac{Vol_{\widetilde{\mathbf{g}}}(M)}{r^2} = 0$ by the hypothesis. This implies that the LHS must also converge to zero as $r \to \infty$. Hence, u must be a constant, which implies $(M^n, \mathbf{g})$ is isometric to $(M^n, const \cdot \widetilde{\mathbf{g}})$. Hence, the immersed minimal hypersurface $(M^n, \mathbf{g})$ has quadratic volume growth too. By Theorem 3.1.2 , $(M^n, \mathbf{g})$ is a hyperplane which has n-th polynomial volume growth. When $n > 2$, this yields a contradiction and finishes the proof. □

3.4 Nonlinear partial differential equations with geometric applications

In this section, we consider some nonlinear partial differential equations defined on complete non-compact Riemannian manifolds and their applications in geometry.

In [22], Fischer-Colbrie and Schoen studied the positive solution f of a second order linear partial differential equation $\Delta f - qf = 0$, where q is a smooth function on the complete non-compact manifold M. They have shown that the existence of the positive function f satisfying the above equation is equivalent to the condition that the first eigenvalue of $\Delta - q$ be positive on each bounded domain in M.

In [57], following the same spirit, S.W.Wei studied the existence of essential positive supersolutions of potentially nonlinear degenerate partial differential equations on complete non-compact manifolds. In this section, we will study the potentially nonlinear degenerate partial differential equations in [57] and extend our results in the previous section to their setting.

First, let's recall the notations and terminology Wei used in [57].

We define a second order partial deferential operator

$$Q(v) \equiv div(A(x, v, \nabla v)\nabla v) + b(x, v, \nabla v)v, \tag{3.41}$$

on a Riemannian n-manifold M, where A denotes a smooth section in the bundle whose fiber at each point x in M is a nonnegative linear transformation on the tangent space $T_x(M)$ into $T_x(M)$, b is a smooth real-valued function, and ∇v denotes the gradient of v. (Thus in terms of normal coordinates $\{x_1, \cdots, x_n\}$ at x_0 in M, $\nabla v = \sum_{i=1}^n \frac{\partial v}{\partial x_i}$ and $Q(v) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial v}{\partial x_j}) + bv$ at x_0, where $(a_{i,j})$ is the nonnegative matrix of $A(x, v, \nabla v)$ with respect to the orthonormal frame field $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$ on M.)

By an essential positive supersolution of $Q(u) = 0$, we mean a C^2 function v on M which is positive almost everywhere, and satisfies $Q(v) \leq 0$ on M. In contrast to many geometric linear elliptic partial differential equations (e.g., $\Delta u + pu = 0$), the equation $Q(u) = 0$ can be nonlinear and degenerate. However, the essential positivity of a supersolution can transform the nonlinear differential inequality

$$Q(v) \leq 0 \tag{3.42}$$

to the following fundamental integral inequality:

$$\int_M b\varphi^2 dv \leq \int_M < A\nabla\varphi, \nabla\varphi > dv, \quad \text{for every} \quad \varphi \in C_0^\infty(M), \tag{3.43}$$

where dv is the volume element on M and $<, >$ denotes the Riemannian metric on M.

Proposition 3.4.1. *[57] Suppose there exists an essential positive supersoulution of $Q(v) \leq 0$ with coefficient satisfying*

$$< A\nabla\varphi, \nabla\varphi > \leq c_1 |\nabla\varphi|^2 \tag{3.44}$$

for every $\varphi \in C_0^\infty(M)$, where $c_1 > 0$ is a constant, then for each φ

$$\int_M b\varphi^2 dv \leq c_1 \int_M |\nabla\varphi|^2 dv. \tag{3.45}$$

Proof. (Proposition 3.4.1) See Proposition 3.1 and Proposition 2.1 in [57]. □

Next, we study the geometric applications of the existence of essential positive supersolution of $Q(u) = 0$. Let M^n be a complete n-dimensional Riemannian manifold with non-positive scalar curvature R. We obtain the following result in which the case of $n = 2$ is due to S.W.Wei [57].

Theorem 3.4.2. *If a Riemannian manifold with non-positive scalar curvature R admits an essential supersolution of $Q(u) = 0$ with coefficients satisfying (3.44) and*

$$b(x, u, \nabla u) \geq -c_2 R(x) + c_3 p(x), \tag{3.46}$$

where $c_2 \geq c_1$, $p(x) \geq 0$ and $c_3 > 0$ is a constant, then M is not conformal to a disk $\mathbb{B}^n$.

Proof. (Theorem 3.4.2) By Proposition 3.4.1, (3.46), and $c_3 p(x) \geq 0$, (3.45) implies that for every $\varphi \in C_0^\infty(M)$,

$$-c_2 \int_M R\varphi^2 dv \leq \int_M (-c_2 R(x) + c_3 p(x))\varphi^2 dv \leq c_1 \int_M |\nabla\varphi|^2 dv. \tag{3.47}$$

Equivalently, we obtain the following stability-like inequality

$$0 \leq \int_M |\nabla\varphi|^2 + c_4 R\varphi^2 dv, \tag{3.48}$$

where $c_4 = \frac{c_2}{c_1}$.

Recall the stability inequality of minimal hypersurface in Euclidean space (3.14)

$$0 \leq \int_M |\nabla f|^2 + Rf^2 d\mu.$$

By the hypothesis that $c_2 \geq c_1$, we have $c_4 \geq 1$. Hence, (3.48) implies (3.14). The rest of the proof will be carried out as in the proof of Theorem 3.3.3. Using the conformal property, we choose functions u and v as the conformal factor. Combing the inequality (3.48) and that $\mathbb{B}^n$ has finite volume, we prove that u and v must be constants which implies that the manifold is non-complete. The contradiction finishes the proof. □

Suppose M is a complete minimal hypersurface in $\mathbb{R}^{n+1}$ with a unit normal vector $\nu = (\nu_1, \cdots, \nu_{n+1})$ and the coefficient b of $Q(u)$ satisfies

$$b(x, u, \nabla u) \geq c_2|\nabla\nu|^2 (\text{ or } c_2\|A\|^2) \quad (c_2 \geq c_1). \tag{3.49}$$

Then Theorem 3.4.2 still holds, namely,

Proposition 3.4.3. *If a complete minimal hypersurface in $\mathbb{R}^{n+1}$ admits an essential supersolution of $Q(u) = 0$ with coefficients satisfying (3.44) and (3.49), then M is not conformal to a disk $\mathbb{B}^n$.*

Indeed, if the a Riemannian manifold admits the stability-like inequality of (3.14), then we can always achieve the non-existence properties of certain kind. Let's consider a more general class of hypersurfaces in general ambient manifolds.

Theorem 3.4.4. *Let M^n be a complete Riemannian manifold with non-positive scalar curvature $R \leq 0$ and N^{n+1} be a complete Riemannian manifold with non-negative scalar curvature $\widetilde{R} \geq 0$. If M^n can be immersed into N^{n+1} as a hypersurface, and admits an essential supersolution of $Q(u) = 0$ with coefficients satisfying (3.44) and (3.49), then M is not conformal to a disk $\mathbb{B}^n$.*

Proof. (Theorem 3.4.4) Recall in Theorem 3.2.1, we have

$$||A||^2 = \widetilde{R} + H^2 - R. \tag{3.50}$$

Combining Proposition 3.4.1 and condition (3.49)we obtain

$$\int_M c_4 ||A||^2 \varphi^2 dv \leq \int_M |\nabla\varphi|^2 dv, \tag{3.51}$$

where $c_4 = \frac{c_2}{c_1} \geq 1$ as before.

Plugging (3.50) into (3.52) we have

$$\int_M (\widetilde{R} + H^2 - R)\varphi^2 dv \leq \int_M c_4(\widetilde{R} + H^2 - R)\varphi^2 dv \leq \int_M |\nabla\varphi|^2 dv. \tag{3.52}$$

By the hypothesis on R and $\widetilde{R}$, and the non-negativity of H^2, we obtain

$$0 \leq \int_M |\nabla\varphi|^2 + R\varphi^2 dv. \tag{3.53}$$

The rest of the proof will be carried out as in Theorem 3.3.3. □

As an analogy to Theorem 3.3.9, we have the following theorem concerning general immersed hypersurfaces.

Theorem 3.4.5. *Let $(M^n, \mathbf{g})$ be a complete Riemannian manifold with non-positive scalar curvature $R \leq 0$ and N^{n+1} be a complete Riemannian manifold with non-negative scalar* *curvature $\widetilde{R} \geq 0$. Assume M^n can be immersed into N^{n+1} as a hypersurface, and in the conformal class of the metric* $\mathbf{g}$*, there exists a non-complete metric* $\widetilde{\mathbf{g}}$ *with non-positive scalar curvature $\widetilde{R} \leq 0$ and the manifold has finite volume, then $Q(u) = 0$ with coefficients satisfying (3.44) and (3.49) does not have any essential positive supersolution.*

Proof. (Theorem 3.4.5) Combining the proof of Theorem 3.3.9 and Theorem 3.4.4, we first obtain a stability-like inequality as in (3.53) if the immersed manifold admits an essential positive supersolution. Then prove by contradiction by utilizing the condition of quadratic volume growth condition. □

As a special case of Theorem 3.4.5, if we consider the case that the hypersurface is a complete manifold modeled on a bounded domain in Euclidean space or more generally any domain $\Omega \subset \mathbb{R}^n$ such that $(\mathbb{R}^n \backslash \Omega)^\circ \neq \emptyset$, we have the following corollary.

Corollary 3.4.6. *Let M^n and N^{n+1} be as in Theorem 3.4.4 and Theorem 3.4.5 . If $Q(u) = 0$ with coefficients satisfying (3.44) and (3.49) admits an essential positive supersolution, then it cannot be conformal to a domain $\Omega \subset \mathbb{R}^n$ such that $(\mathbb{R}^n \backslash \Omega)^\circ \neq \emptyset$.*

Proof. (Corollary 3.4.6) See the proof of Theorem 3.4.5 and Theorem 3.3.7. □

Remark 3.4.7. *Results in this section consider a broad spectrum of partial differential equations which includes nonlinear and degenerate cases.*

3.5 The second method : an intrinsic approach

In this section, we consider a different approach to the problems. We want to develop a new method without appealing to Corollary 3.2.2 in the section of Preliminaries.

The aim of doing this has two-folds. On the one hand, we can study general manifolds instead of immersed hypersurfaces, i.e., there will be no immersion and we don't need to assume any conditions on the scalar curvature of the manifold. On the other hand, if the conformal metric is non-complete, we relax the condition from point-wise non-positive scalar curvature $\widetilde{R} \leq 0$ to non-positive total scalar curvature $\int_M \widetilde{R} d\mu_{\widetilde{R}} \leq 0$. We then give a proof to our main Theorem 3.5.1.

Theorem 3.5.1. *Let* $(M^n, \mathbf{g})$ *be a complete Riemannian manifold, where* $n > 2$*.*

1. *If in the conformal class of the metric* $\mathbf{g}$ *on* M^n*, there exists a non-complete metric* $\widetilde{\mathbf{g}}$ *with non-positive scalar curvature* $\widetilde{R} \leq 0$ *and finite volume, then* $Q(u) = 0$ *with coefficients satisfying (3.44) and (3.46) does not have any essential positive supersolution.*

2. *Assume in the conformal class of* $\mathbf{g}$*, there exists a complete metric* $\widetilde{\mathbf{g}}$ *with non-positive total scalar curvature* $\int_M \widetilde{R} d\mu_{\widetilde{\mathbf{g}}} \leq 0$ *and quadratic volume growth. If* $Q(u) = 0$ *with coefficients satisfying (3.44) and (3.46) has an essential positive supersolution, then* $(M^n, \mathbf{g})$ *is isometric to* $(M^n, const \cdot \widetilde{\mathbf{g}})$*, and also has quadratic volume growth.*

Proof. (Theorem 3.5.1) Recall in the proof of Theorem 3.4.2, we obtain an inequality (3.48)

$$0 \leq \int_M |\nabla \varphi|^2 + c_4 R \varphi^2 dv,$$

where $c_4 = \frac{c_2}{c_1} \geq 1$.

Equivalently, we have

$$0 \leq \int_M \frac{1}{c_4} |\nabla \varphi|^2 + R\varphi^2 dv \leq \int_M |\nabla \varphi|^2 + R\varphi^2 dv, \tag{3.54}$$

which is exactly the inequality (3.14).

For case 1, we follow the proof of Theorem 3.3.9, but without appealing to Corollary 3.2.2.

Choosing the same functions u, v , and test function ξ as in the proof of Theorem 3.3.9, after the same calculation, we get identity (3.28)

$$v\Delta v - \frac{2}{n}|\nabla v|^2 = \frac{n}{4(n-1)}v^2R - \frac{n}{4(n-1)}\widetilde{R}.$$

and inequality (3.16)

$$\begin{aligned} 0 &\leq \int_M |\nabla v\xi|^2 + R(v\xi)^2 d\mu \\ &= \int_M v^2|\nabla\xi|^2 d\mu + 2\int_M v\xi < \nabla v, \nabla\xi > d\mu + \int_M \xi^2|\nabla v|^2 d\mu + \int_M R(v\xi)^2 d\mu \end{aligned}$$

Plugging (3.28) into (3.16), after integration by parts and rearrangement, we have

$$\begin{aligned} 0 &\leq \int_M v^2|\nabla\xi|^2 d\mu - \frac{3n-4}{n}\int_M 2v\xi\nabla v\cdot\nabla\xi d\mu - \frac{3n^2+4n-8}{n^2}\int_M \xi^2|\nabla v|^2 d\mu \\ &+ \int_M \widetilde{R}(v\xi)^2 d\mu \\ &\leq \int_M v^2|\nabla\xi|^2 d\mu - \frac{3n-4}{n}\int_M 2v\xi\nabla v\cdot\nabla\xi d\mu - \frac{3n^2+4n-8}{n^2}\int_M \xi^2|\nabla v|^2 d\mu \end{aligned} \tag{3.55}$$

The last step follows from $\widetilde{R} \leq 0$.

As before, we assume $n > 2$. Let $c_5 = \frac{3n-4}{n} > 0$ and $c_6 = \frac{3n^2+4n-8}{n^2} > 0$, by using Cauchy-Schwartz inequality and Young's inequality, we have the following

$$\begin{aligned} 0 &\leq \int_M v^2|\nabla\xi|^2 d\mu + c_5\int_M 2v\xi|\nabla v||\nabla\xi| d\mu - c_6\int_M \xi^2|\nabla v|^2 d\mu \\ &\leq (1+\tfrac{c_5}{\epsilon})\int_M v^2|\nabla\xi|^2 d\mu - (c_6 - c_5\epsilon)\int_M \xi^2|\nabla v|^2 d\mu. \end{aligned} \tag{3.56}$$

Choose ϵ small enough, we obtain

$$\int_M \xi^2|\nabla v|^2 d\mu \quad \leq c\int_M v^2|\nabla\xi|^2 d\mu \tag{3.57}$$

where c is a positive constant depends only on n.

We let ξ be the same cut-off function as in the proof of Theorem 3.3.3. We have

$$\begin{aligned}\int_M \xi^2|\nabla v|^2 d\mu &\leq \tfrac{9c}{r^2}\int_M v^2 d\mu\\ &= \tfrac{9c}{r^2}\int_M u^{\frac{2n}{n-2}} d\mu_{\mathbf{g}}\\ &= \tfrac{9}{r^2}\int_M u^{\frac{2n}{n-2}}(u^{-\frac{4}{n-2}})^{\frac{n}{2}} d\mu_{\widetilde{\mathbf{g}}}\\ &= \tfrac{9cVol_{\widetilde{\mathbf{g}}}(M)}{r^2}\end{aligned} \tag{3.58}$$

Let $r\to\infty$, $\lim\limits_{r\to\infty}\dfrac{Vol_{\widetilde{\mathbf{g}}}(M)}{r^2}=0$ by the hypothesis. Hence, v must be a constant and u is also a constant, which implies that $(M^n,\mathbf{g})$ is isometric to $(M^n, const\cdot\widetilde{\mathbf{g}})$ is non-complete. The contradiction completes the proof of case 1.

For case 2, we follow the second case in the proof of Theorem 3.3.10, but consider total scalar curvature $\int_M \widetilde{R} d\mu_{\widetilde{\mathbf{g}}}\leq 0$ and quadratic volume growth of $(M^n,\widetilde{\mathbf{g}})$.

We notice that, different from case 1, we choose $v\equiv u$ instead of $u^{\frac{n}{n-2}}$. From inequality (3.35), we have

$$\begin{aligned}0 \leq &\int_M v^2|\nabla\xi|^2 d\mu + \frac{3n-2}{n-2}\int_M 2v\xi|\nabla v||\nabla\xi| d\mu - \frac{3n-2}{n-2}\int_M \xi^2|\nabla v|^2 d\mu\\ &+\int_M \widetilde{R}\xi^2 v^{\frac{2n}{n-2}} d\mu\end{aligned}$$

Applying Young's inequality and after rearrangement , we obtain the following inequality with the choice of $v=u$,

$$\begin{aligned}\int_M \xi^2|\nabla u|^2 d\mu &\leq C\int_M u^2|\nabla\xi|^2_{\mathbf{g}} d\mu_{\mathbf{g}} + \int_M \widetilde{R}\xi^2 u^{\frac{2n}{n-2}} d\mu_{\mathbf{g}}\\ &= C\int_M u^2 u^{\frac{4}{n-2}}|\nabla\xi|^2_{\widetilde{\mathbf{g}}}(u^{-\frac{4}{n-2}})^{\frac{n}{2}} d\mu_{\widetilde{\mathbf{g}}} + \int_M \widetilde{R}\xi^2 u^{\frac{2n}{n-2}}(u^{-\frac{4}{n-2}})^{\frac{n}{2}} d\mu_{\widetilde{\mathbf{g}}}\\ &= C\int_M |\nabla\xi|^2_{\widetilde{\mathbf{g}}} d\mu_{\widetilde{\mathbf{g}}} + \int_M \widetilde{R}\xi^2 d\mu_{\widetilde{\mathbf{g}}}.\end{aligned} \tag{3.59}$$

Choose ξ to be the standard cut-off function with respect to $\widetilde{\mathbf{g}}$, see (3.39). It then follows from (3.59)

$$\begin{aligned}\int_M \xi^2|\nabla u|^2 d\mu &\leq \tfrac{9C}{r^2}\int_{B(r)} d\mu_{\widetilde{\mathbf{g}}} + \int_M \widetilde{R}\xi^2 d\mu_{\widetilde{\mathbf{g}}}\\ &\leq 9C\tfrac{Vol_{\widetilde{\mathbf{g}}}(B(r))}{r^2} + \int_M \widetilde{R}\xi^2 d\mu_{\widetilde{\mathbf{g}}}.\end{aligned} \tag{3.60}$$

Let $r \to \infty$, $\lim_{r\to\infty} \frac{Vol_{\widetilde{\mathbf{g}}}(B(r))}{r^2} = 0$ by the hypothesis. On the other hand, $\lim_{r\to\infty} \int_M \widetilde{R}\xi^2 d\mu_{\widetilde{\mathbf{g}}} = \int_M \widetilde{R} d\mu \leq 0$. This implies that the LHS must also converge to zero as $r \to \infty$. Hence, u must be a constant, which implies $(M^n, \mathbf{g})$ is isometric to $(M^n, const \cdot \widetilde{\mathbf{g}})$. Hence, the immersed minimal hypersurface $(M^n, \mathbf{g})$ has quadratic volume growth too. □

Remark 3.5.2. *It would be interesting if one can improve the volume growth condition in our theorem to the maximal volume growth condition, i.e., n-th polynomial volume growth, where n is the dimension of the manifold.*

Chapter 4 Stable minimal hypersurface with finite integral curvature condition

If the second fundamental form of a complete oriented stable minimal hypersurface in Euclidean space has finite L_2 norm, then it must be a hyperplane. This is a theorem of Do Carmo And Peng in the early 80's. We present a simpler proof in this chapter. The main ingredients of our proof include: Bochner's method, stability inequality and infinite volume of minimal hypersurfaces. The major difference of our proof is we do not use the main estimates by [53].

Theorem 4.0.1. Let $x : M \to R^{n+1}$ be a complete oriented stable minimal immersion such that the total curvature $\int_M |A|^2$ is finite, then $x(M) \subset R^{n+1}$ is a hyperplane.

Proof. (**Proof of Theorem 4.0.1**)We know the following three well-known facts of minimal hyper-surface in Euclidean space.

1.the stability inequality

$$\int_M |A|^2\varphi^2 \leq \int_M |\nabla\varphi|^2$$

2.the Simons' inequality

$$|A|\triangle|A| + |A|^4 \geq \frac{2}{n}|\nabla|A||^2$$

3.M has infinite volume.

Let $\varphi = |A|f$,

$$
\begin{aligned}
\int_M |A|^2\varphi^2 &\le \int_M |\nabla\varphi|^2 \\
&= \int_M \left|\nabla|A|f\right|^2 \\
&= \int_M \left|f\nabla|A| + |A|\nabla f\right|^2 \\
&= \int_M f^2\left|\nabla|A|\right|^2 + \int_M |A|^2|\nabla f|^2 + \int_M < \nabla f^2, |A|\nabla|A| > \\
&= \int_M f^2\left|\nabla|A|\right|^2 + \int_M |A|^2|\nabla f|^2 - \int_M f^2\Delta(|A|\nabla|A|) \\
&= \int_M f^2\left|\nabla|A|\right|^2 + \int_M |A|^2|\nabla f|^2 - \int_M f^2\left|\nabla|A|\right|^2 - \int_M f^2|A|\triangle|A| \\
&= \int_M |A|^2|\nabla f|^2 - \int_M f^2|A|\triangle|A|
\end{aligned}
\tag{4.1}
$$

Then from the stability inequality:

$$
\begin{aligned}
\int_M |A|^4 f^2 = \;& \int_M |A|^2\varphi^2 \\
&\le \int_M |\nabla\varphi|^2 \\
&= \int_M |A|^2|\nabla f|^2 - \int_M f^2|A|\triangle|A|
\end{aligned}
\tag{4.2}
$$

equivalently

$$\int_M (|A|\triangle|A| + |A|^4)f^2 \le \int_M |A|^2|\nabla f|^2$$

Using Simons' inequality we get

$$\frac{2}{n}\int_M (|\nabla|A||^2)f^2 \le \int_M |A|^2|\nabla f|^2$$

Let ξ be the standard cutoff function,

$$
\begin{aligned}
\xi(r) &= 1 \; for\; r \le \tfrac{1}{2}R, \\
\xi(r) &= 0 \; for\; r \ge R, \\
\xi &\ge 0 \; for\; all\; r, \\
|\xi'| &\le \tfrac{3}{R} \; for\; all\; r.
\end{aligned}
\tag{4.3}
$$

If r measures the metric distance to 0, and R is any positive number, then $\xi(r)$ defines a Lipschitz function on M with support in $B_R(0)$.

Under the assumption the total curvature $\int_M |A|^2$ is finite, let R go to infinity, we get $\nabla|A| \equiv 0$,then $|A| = C$, which is a constant.

Also we know, $\int_M |A|^2 = C\ Vol(M) < \infty$, from the fact 3, we know this can only happen when $|A| \equiv 0$.

$\square$

Chapter 5 p-Harmonic theory with sharp estimates, generalized uniformization theorem, and Bochner's method [58]

In this chapter, we make sharp global integral estimates by a unified method, and find a dichotomy between constancy and "infinity" of weak sub- and supersolutions of a large class of degenerate and singular nonlinear partial differential equations on complete noncompact Riemannian manifolds, by introducing the concepts of their corresponding "*p-finite*, *p-mild*, *p-obtuse*, *p-moderate*, and *p-small*" growth, and their counter-parts "*p-infinite*, *p-severe*, *p-acute*, *p-immoderate*, and *p-large*" growth. These lead naturally to a Generalized Uniformization Theorem, a Generalized Bochner's Method, and an iterative method, by which we approach various geometric and variational problems in complete noncompact manifolds of general dimensions.

5.1 Introduction and preliminaries

It is well-known that on a complete Riemannian manifold, any $\mathbf{L}^2$ harmonic function is **constant**. A proof of this result can be based on essentially the arguments of Andreotti-Vesentini [3], or Stampacchia inequality ([3] Proposition 3 on page 90). R. Greene and

H. Wu prove the following:

Theorem (Greene-Wu [26]) *If for some $p \geq 1$, f is a continuous nonnegative* $\mathbf{L^p}$ *subharmonic function on a complete Riemannian manifold M of nonnegative sectional curvature outside a compact set , then* $\mathbf{f \equiv 0}$.

A similar result was obtained by Yau ([62] Theorem 3, and Appendix) for $p > 1$, and $f \in C^\infty$ without any curvature condition. More precisely, he showed that if f is a smooth nonnegative subharmonic function on M with $\liminf_{\mathbf{r}\to\infty} \frac{\mathbf{1}}{\mathbf{r}} \int_{\mathbf{B(x_0;r)}} \mathbf{f^p dv} = \mathbf{0}$ for some $p > 1$ and some x_0 in M, then f is **constant.** In particular, it follows that if f is in L^p then f is constant. Here and throughout this chapter, unless stated otherwise, we assume M is a complete manifold with volume element dv, and denote the geodesic ball of radius r centered at x_0 in M by $B(x_0; r)$, its volume by $\mathrm{vol}(B(x_0; r))$, and its boundary by $\partial B(x_0; r)$.

Ahlfors [2] and Nevanlinna [42] using the conformal mapping argument prove that every positive superharmonic function on a Riemann surface M is **constant**, or equivalently M is **parabolic**, if for some $x_0 \in M$, and $a > 0$,

$$\int_a^\infty \frac{1}{\mathrm{vol}(\partial B(x_0; r))} dr = \infty . \tag{5.1}$$

J. Milnor [40], R. Osserman [43][44], Blanc-Fiala-A. Huber [35], Greene-Wu [27], P. Yang [63], J. Jenkins [36], H. Huber [34], and R. Rüedy [49] have made further studies on deciding whether or not a surface is parabolic or hyperbolic under a curvature or analytic assumption. In general dimensions, Cheng-Yau [15] prove that M is **parabolic** if $\limsup_{r\to\infty} \frac{1}{r^2} \mathrm{vol}(B(x_0; r)) < \infty$, based on the following Cheng-Yau estimate: Let f be a positive C^2 function on $B(x_0; b)$ and suppose $\int_{B(x_0;r)} \frac{\Delta f}{f} dv$ is decreasing in $r \geq a$. Then for any sequence $a = r_0 < r_1 < \cdots < r_n = b$,

$$\int_{B(x_0;a)} |\nabla \log f|^2 dv \le \int_{B(x_0;a)} \frac{\Delta f}{f} dv + \Big(\sum_{j=0}^{n-1}\Big(\frac{(r_{j+1}-r_j)^2}{\mathrm{vol}(B(x_0;r_{j+1})) - \mathrm{vol}(B(x_0;r_j))}\Big)\Big)^{-1}. \quad (5.2)$$

Karp then proves the following:

Theorem (Karp[37]) *Every continuous nonnegative subharmonic function f on M is either* **constant or** *both* $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B(x_0;r)} f^p dv = \infty$ *and*

$$\limsup_{r\to\infty} \frac{1}{r^2 F(r)} \int_{B(x_0;r)} f^p dv = \infty \quad (5.3)$$

hold for every $p > 1$, every $x_0 \in M$, and every positive **nondecreasing** *function F satisfying $\int_a^\infty \frac{dr}{rF(r)} = \infty$ for some $a > 0$.*

Karp also refines the results of Greene-Wu, Blanc-Fiala-A. Huber and others for surfaces, and of Cheng-Yau for Riemannian manifolds, by showing that M is parabolic if M has **moderate volume growth**, i.e. there exist $x_0 \in M$, and a function $F(r)$ as above, such that

$$\limsup_{r\to\infty} \frac{1}{r^2 F(r)} \mathrm{vol}(B(x_0;r)) < \infty.$$

This result is further refined to state that M is parabolic, if for some $x_0 \in M$, and $a > 0$,

$$\int_a^\infty \frac{r}{\mathrm{vol}(B(x_0;r))} dr = \infty,$$

by Grigor'yan [24] [25] and Varopoulos [56], or if (5.1) holds, for some $x_0 \in M$, and $a > 0$, by Lyons-Sullivan [39], Varopoulos [56], Grigor'yan [24] [25], Sturm [52], and Carron [12] for Riemannian manifolds. Sturm [51] generalizes and further refines the works of Greene-Wu, Yau and Karp by showing that if either $p > 1$ and f is a nonnegative subharmonic function , or $p < 1$ and f is a nonnegative superharmonic function, then f

is **constant or**

$$\int_a^{\infty} \frac{r}{\int_{B(x_0;r)} f^p dv} dr < \infty \tag{5.4}$$

holds for every $x_0 \in M$, and $a > 0$.

The celebrated uniformization theorem of F. Klein, P. Keobe and H. Poincaré is a classification theorem that sharply divides complete noncompact surfaces into parabolic and hyperbolic ones and enables us to solve many geometric variational problems on surfaces (e.g. [22],[19],[45],[33],[57]). However, in dealing with higher dimensional geometric problems, the scope of the uniformization theorem and its related Laplace operator needs to be widened. For example, the manifolds R^3 and R^4 are both hyperbolic, and therefore can not be distinguished from each other in this way, while manifolds with Ricci curvature bounded below by a nonpositive constant behave like the Euclidean space $\mathbb{R}^n$ from the viewpoint of harmonic functions(cf. [61]). This motivates us to study the geometric significance and applications of the p-Laplace operator Δ_p $\Big($ defined by $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f)$, where ∇f denotes the gradient of f $\Big)$, as well as its perturbation, the $\mathcal{A}$-operator, which does not arise from the ordinary Laplace operator Δ. This also motivates us, for our subsequent study, to state

A Generalized Uniformization Theorem: *A complete noncompact Riemannian manifold is either p-**parabolic** or p-**hyperbolic**, for* $1 < p < \infty$.

Recall a C^2 function $f : M \to R$ is said to be p-**harmonic** (resp. p-**superharmonic,** p-**subharmonic**) if its p-Laplacian $\Delta_p f =$ (resp. $\leq$, $\geq$) 0, and a complete noncompact Riemannian manifold M^n is said to be p-**parabolic** if it admits no nonconstant positive p-superharmonic function, and p-**hyperbolic** otherwise. Accordingly, R^3 is 3-parabolic, but R^4 is not; and the hyperbolic space H^n is n-hyperbolic, whereas n-

manifolds with $Ric^M \geq 0$ are n-parabolic.

5.2 Main results

We initiate this chapter by observing that while Karp optimally refines Caccioppoli inequalities and manipulates them beautifully in an algebraic and analytic manner, his assumption in (5.3) that the auxiliary function F is nondecreasing can be removed, and the new result becomes optimal and is equivalent to the Sturm's result (5.4). Distilling further the basic ideas of Karp inequalities, we develop a unified method to refine and generalize the Cheng-Yau estimate and all other results displayed above, as well as their L^p versions due to I.Holopainen [32] T. Coulhon, I.Holopainen, and L. Saloff-Coste [14]. In fact, we find a dichotomy between constancy and "infinity" of weak sub- and supersolutions of a large class of degenerate and singular nonlinear partial differential equations on complete noncompact Riemannian manifolds, by introducing the concepts of their corresponding *"p-finite, p*-mild*, p*-obtuse*, p*-moderate*, and p*-small*" growth*, and their counter-parts ***"p-infinite*, *p-severe*, *p-acute*, *p-immoderate*, and *p-large*" growth**:

Definition 5.2.1. A function f has *p-finite growth* (or, simply, *is p-finite*) if there exists $x_0 \in M$ such that

$$\liminf_{r\to\infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q dv < \infty\,, \tag{5.5}$$

and has *p-infinite growth* (or, simply, *is p-infinite*) otherwise.

Definition 5.2.2. A function f has *p-mild growth* (or, simply, *is p-mild*) if there exists $x_0 \in M$, and a strictly increasing sequence of $\{r_j\}_0^\infty$ going to infinity, such that for every

$l_0 > 0$, we have

$$\sum_{j=\ell_0}^{\infty} \left(\frac{(r_{j+1}-r_j)^p}{\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)} |f|^q dv} \right)^{\frac{1}{p-1}} = \infty\,, \tag{5.6}$$

and has *p-severe growth* (or, simply, *is p-severe*) otherwise.

Definition 5.2.3. A function f has *p-obtuse growth* (or, simply, *is p-obtuse*) if there exists $x_0 \in M$ such that for every $a > 0$, we have

$$\int_a^{\infty} \left(\frac{1}{\int_{\partial B(x_0;r)} |f|^q dv} \right)^{\frac{1}{p-1}} dr = \infty\,, \tag{5.7}$$

and has *p-acute growth* (or, simply, *is p-acute*) otherwise.

Definition 5.2.4. A function f has *p-moderate growth* (or, simply, *is p-moderate*) if there exist $x_0 \in M$, and $F(r) \in \mathcal{F}$,such that

$$\limsup_{r\to\infty} \frac{1}{r^p F^{p-1}(r)} \int_{B(x_0;r)} |f|^q dv < \infty\,, \tag{5.8}$$

and has *p-immoderate growth* (or, simply, *is p-immoderate*) otherwise (cf. (**??**)), where

$$\mathcal{F} = \{F : [a, \infty) \longrightarrow (0, \infty) | \int_a^{\infty} \frac{dr}{rF(r)} = \infty \;\; for \;\; some \;\; a \geq 0\}\,. \tag{5.9}$$

(Notice that the function in $\mathcal{F}$ are not necessarily monotone.)

Definition 5.2.5. A function f has *p-small growth* (or, simply, *is p-small*) if there exists $x_0 \in M$, such that for every $a > 0$,we have

$$\int_a^{\infty} \left(\frac{r}{\int\limits_{B(x_0,r)} |f|^q dv} \right)^{\frac{1}{p-1}} dr = \infty\,, \tag{5.10}$$

and has *p-large growth* (or, simply, *is p-large*) otherwise (cf. (**??**)).

The above definition of "*p-finite*, *p-mild*, *p-obtuse*, *p-moderate*, *p-small*" and "*p-infinite*, *p-severe*, *p-acute*, *p-immoderate*, *p-large*" growth depends on q, and q will be specified in the context in which the definition is used. This has extended the scope of previous L^2 or L^p function growth due to Andreotti-Vesentini [3], Greene-Wu [26], Yau [62] and others, via moderate growth due to Karp [37], to 2-*moderate* growth and more generally "*p-finite*, *p-mild*, *p-obtuse*, *p-moderate*, and *p-small*" growth. Such a dichotomy between constancy and "infinity" also occurs for $\mathcal{A}-$sub- and $\mathcal{A}-$ superharmonic functions, which are central to the theory of quasiconformal and quasiregular maps, and for a global version of a Reverse Cheng-Yau inequality (cf. Theorem 1.8 in [58]) on complete noncompact Riemannian manifolds. Here and throughout this chapter, unless stated otherwise, we denote by $\mathcal{A}$ a measurable cross section in the bundle whose fiber at a.e. x in M is a continuous map $\mathcal{A}_x$ on the tangent space $T_x(M)$ into $T_x(M)$. We assume further that there are constants $1 < p < \infty$ and $0 < \alpha \leq \beta < \infty$ such that for a.e. x in M and all $h \in T_x(M)$,

$$\langle \mathcal{A}_x(h), h \rangle_M \geq \alpha |h|^p \tag{5.11}$$

$$|\mathcal{A}_x(h)| \leq \beta |h|^{p-1} \tag{5.12}$$

$$\langle \mathcal{A}_x(h_1) - \mathcal{A}_x(h_2), h_1 - h_2 \rangle_M > 0, \quad h_1 \neq h_2, \tag{5.13}$$

and

$$\mathcal{A}_x(\lambda h) \equiv |\lambda|^{p-2} \lambda \mathcal{A}_x(h), \quad \lambda \in R \backslash \{0\}. \tag{5.14}$$

A function $f \in L^{1,p}_{loc}(M)$ is a weak solution (resp. supersolution, subsolution) of the equation

$$\mathrm{div}\mathcal{A}_x(\nabla f) = 0(resp. \quad \le 0, \quad \ge 0) \tag{5.15}$$

if for all nonnegative $\varphi \in C_0^\infty(M)$

$$\int_M \langle \mathcal{A}_x(\nabla f), \nabla\varphi\rangle dv = 0(resp. \quad \ge 0, \quad \le 0) \tag{5.16}$$

Here $L^{1,p}_{loc}(M)$ is the Sobolev space whose functions are locally p-integrable and have locally p-integrable partial distributional first derivatives. Continuous solutions of (5.9) are called $\mathcal{A}-$harmonic. In the case $\mathcal{A}_x(h) \equiv |h|^{p-2}h$, $\mathcal{A}-$harmonic functions are p-harmonic. A lower (resp. upper) semicontinuous function $f : M \to R \cup \{\infty\}$ (resp. $\{-\infty\} \cup R$) is $\mathcal{A}$-superharmonic (resp. $\mathcal{A}$-subharmonic) if it is not identically infinite, and satisfies the $\mathcal{A}$-comparison principle: i.e., for each domain $D \subset\subset M$ and for each function $g \in C(\overline{D})$ which is $\mathcal{A}$-harmonic in D, $g \le f$ (resp. $g \ge f$) in ∂D implies $g \le f$ (resp. $g \ge f$) in D. An $\mathcal{A}$-superharmonic (resp. $\mathcal{A}$-subharmonic) function f is called p-superharmonic (resp. p-subharmonic) if $\mathcal{A}_x(h) \equiv |h|^{p-2}h$. $\mathcal{A}$-superharmonic and $\mathcal{A}$-subharmonic functions are closely related to subsolutions and supersolutions of (5.9). For a discussion of the $\mathcal{A}$-harmonic equation, we refer the reader to I. Holopainen, T. Kipeläinen and O. Martio's book ([31]).

We define function $f \in L^{1,p+1}_{loc}(M)$ to be a weak solution of the differential inequality

$$f\mathrm{div}\mathcal{A}_x(\nabla f) \ge 0 \tag{5.17}$$

if for all nonnegative $\varphi \in C_0^\infty(M)$

$$\int_M \langle \mathcal{A}_x(\nabla f), \nabla(\varphi f)\rangle dv \quad \leq 0 \tag{5.18}$$

As an example, we prove the following:

Theorem 5.2.1. *Every locally bounded weak solution $f : M \to (-\infty, \infty)$ of the differential inequality $f\mathrm{div}(\mathcal{A}_x(\nabla f)) \geq 0$, with $1 < p < \infty$, is constant a.e. provided f has one of the following:* p-finite*,* p-mild*,* p-obtuse*,* p-moderate*, or* p-small *growth, for some $q > p-1$.Or equivalently, for every $q > p-1$, any nonconstant a.e. locally bounded weak solution $f : M \to (-\infty, \infty)$ of $f\mathrm{div}(\mathcal{A}_x(\nabla f)) \geq 0$ has f has* p-infinite *,* p-severe*,* p-acute*,* p-immoderate*, and* p-large *growth. See (1.1 -1.5) in [58].*

The estimates (1.1 -1.5) are sharp with respect to (1) the range of values $q > p-1$, (2) the rate of divergence: (i) the exponent p of the base r, and (ii) the exponent $p-1$ of the base F, (3) the family $\mathcal{F}$ of auxiliary functions F as in (0.13), and (4) the exponent $\frac{1}{p-1}$, (cf. Remarks (1.1-1.4.)) These estimates lead to The Generalized Bochner's Method, based on a maximum principle from the p-harmonic viewpoint, and allow us to extend the scope from compact manifolds, analytic (or holomorphic) functions between complex planes, subharmonic functions, and linear potential theory, to complete noncompact manifolds, quasiregular maps and p-harmonic morphisms between higher dimensions, p-subharmonic functions, and nonlinear potential theory:

The Generalized Bochner's Method : *On a complete manifold, every nonnegative p-subharmonic (resp. p-superharmonic) function f is constant provided f has one of the following:* p-finite*,* p-mild*,* p-obtuse*,* p-moderate*, or* p-small *growth. for $p > q-1$ (resp. $p < q-1$).*

We derive composition formulas (cf. Lemma 4.1 and Proposition 4.1 in [58]) and find the **first** set of nontrivial **geometric quantities** that are p-subharmonic (resp. p-superharmonic) functions, i.e. the composition of p-harmonic morphisms with convex (resp. concave) functions are p-subharmonic (resp. p-superharmonic) for $p \geq 2$(cf. Theorem 3.2 in [58]). As an application of The Generalized Bochner's Method, we prove

Theorem 5.2.2. *Let $u : M \to S^k_+ \subset R^{k+1}$ be a smooth (a) harmonic map (where $p = 2$) or (b) p-harmonic morphism with $p > 2$, where S^k_+ is an open hemisphere centered at pole y_0. If for some $q < p - 1$, the height function, defined by $f(x) = \langle u(x), y_0\rangle_{R^{k+1}}$ for $x \in M$, has one of the following:* p-finite*,* p-mild*,* p-obtuse*,* p-moderate*, or* p-small *growth, then u is constant.*

In an attempt to extend Sacks-Uhlenbeck's work [54] beyond the conformal dimension, FangHua Lin recently has made a breakthrough. In particular, he proves that if there is no smooth nonconstant harmonic map from S^2 into a compact manifold N, then the the Hausdorff dimension of the singular set of any stationary harmonic map from an n-manifold into N does not exceed $n-4$. The composition formulas lead to the **existence of smooth nonconstant harmonic maps from** S^3 into such manifolds N or into $K(\pi, 1)$ manifolds of dimension no less than 3. In particular, this solves **a problem** of FangHua Lin and ChangYou Wang in dimension 3 (cf. Remark 4.7 in [58]).

We develop an L^p version of the method employed in [57] (cf. [57] p. 152-153), so that one can use p-(super)harmonic functions to study minimal surfaces. In [53], Schoen-Simon-Yau prove that if M is a stable minimal hypersurface in a manifold N^{n+1}

with constant sectional curvature $K_2 \geq 0$, and if M satisfies the volume growth condition

$$\lim_{R\to\infty} R^{-p}\mathrm{vol}(B_R) = 0 \tag{5.19}$$

for some $p \in (0, 4 + \sqrt{8/n})$, then M is totally geodesic. This solves a generalized Bernstein Problem in Euclidean space under the volume growth condition (5.19) on M for $n \leq 5$. In fact, Yau raised **the question** ([59], p.692, Problem 102) whether one can prove that a complete minimal hypersurface M in R^{n+1} is a hyperplane for $n \leq 7$. For the case $n = 2$, it is proved by Colbrie-Fisher - Schoen [22], do Carmo - Peng [19], and Pogorelov [45] that M is a plane . The condition (5.19), which is stronger than a *p-moderate* volume growth condition, implies that M is p-parabolic (cf. Corollary 2.2 in [58]). On the other hand, a p-parabolic manifold can have exponential volume growth by the example we constructed (cf. Example 2.1 in [58]). Just as we use parabolicity and the uniformization theorem to study variational problems on surfaces in R^3 and their related differential equations (cf. [57] p.153), so do we use p-parabolicity and the generalized uniformization theorem to study stable minimal hypersurfaces in R^{n+1}. Thus, extending the work of Schoen -Simon-Yau, we solve the **generalized Bernstein Problem** under the p-parabolicity condition on M for $n \leq 5$:

Theorem 5.2.3. *Every p-parabolic stable minimal hypersurface in R^{n+1} is a hyperplane for $n \leq 5$, where $p \in [4, 4 + \sqrt{8/n})$.*

The p-harmonic theory with sharp estimates, the generalized uniformation theorem, the generalized Bochner's method, the iterative method, and the L^p- method, together with the geometric applications seem to provide a new perspective in approaching geometric and variational problems occurring in complete noncompact manifolds of all dimensions.

Theorem 5.2.4. *Let u be a smooth biharmonic map from a complete manifold M to a complete manifold N of nonpositve sectional curvature. If for some $q > 2$, its* **tension field** *$\tau(u)$ has one of the following:* 2-finite*,* 2-mild*,* 2-obtuse*,* 2-moderate*, or* 2-small *growth, then $\tau(u)$ is parallel; and if $q = 2$, its* **differential** *du has one of the following:* 2-finite*,* 2-mild*,* 2-obtuse*,* 2-moderate*, and* 2-small *growth, then u is an energy minimizing harmonic map in its homotopy class; and if for some $q > 2$, and y_0 in a simply-connected N, its* **distance function***, defined by $f(x) = dist(u(x), y_0)$ for $x \in M$, has one of the following:* 2-finite*,* 2−mild*,* 2-obtuse*,* 2-moderate*, or* 2-small *growth, then u is constant.*

Theorem 5.2.5. *Every p-harmonic map u from a p-parabolic manifold M into an open upper half-ellipsoid $E^k_{a,+}$ is constant.* **2.** *Every nonconstant p-stable map u into a closed upper half-ellipsoid $\overline{E^k_{a,+}}$ is supported by a p-hyperbolic manifold M, with p*-infinite*, p*-severe*, p*-acute*, p*-immoderate*, and p*-large *volume growth. Furthermore, if M has nonnegative Ricci curvature, then M is q-hyperbolic, with q*-infinite*, q*-severe*, q*-acute*, q*-immoderate*, and p*-large *volume growth. where q is as in (6.5) in [58] .*

Bibliography

[1] F. J. ALMGREN, JR.. *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. (2) 84 (1966), 277-292.

[2] L.V. Ahlfors. *Sur le type d'une surface de Riemann*, C.R. Acad. Sci. Paris, 201 (1935) 30-32.

[3] A. Andreotti and E. Vesentini. *Carleman estimate for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. 25 (1965), 81-130.

[4] S. BERNSTEIN. *Sur un theor'eme de geometrie et ses application aux equations aux derivees partielles du type elliptique*, Comm. Soc. Math. Kharkov (2) 15 (1915-C1917), 38-C45.

[5] P. Bérard. *Remarques sur l'équation de J. Simons, DifferentialGeommetry (A symposium in honour of M. do Carmo on his 60th birthday)*, Pitman Monographs Surveys Pure Appl. Math., vol. 52, Longman Sci. Tech., Harlow, 1991, pp. 47-57.

[6] Ben Chow and Dan Knopf. *The Ricci flow : An introduction*. Mathematical Surveys and Monographs, AMS, Providence, RI, 2004

[7] Ben Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci flow*. Preprint of the book.

[8] E. BOMBIERI, E. DE GIORGI, and E. GIUSTI, *Minimal cones and the Bernstein problem*, Invent. Math. 7 (1969), 243-268.

[9] K. Burns. *Convex supporting domains on surfaces*, Bull. London Math. Soc. 17 (1985), 271-274.

[10] H-D.Cao, Y.Shen, and S. Zhu. *The structure of stable minimal hypersurfaces in* R^{n+1}. Math. Res. Lett. 4 (1997), no. 5, 637–644.

[11] Xiaodong Cao. *Eigenvalues of* $(\Delta + \frac{R}{2})$ *on manifolds with nonnegative curvature operator*. Submitted preprint, 2005.

[12] G. Carron. *Inalit isopimriques de Faber-Krahn et consuences*. (French) [Faber-Krahn isoperimetric inequalities and consequences] Actes de la Table Ronde de Gmrie Diffentielle (Luminy, 1992), 205–232, Sin. Congr., 1, Soc. Math. France, Paris, 1996

[13] S. S. Chern. *Minimal submanifolds in a Riemannian manifold*, University of Kansas, Department of Mathematics Technical Report 19 (New Series), 1968

[14] T. Coulhon, I. Holopainen, and L. Saloff-Coste. *Harnack inequality and hyperbolicity for subelliptic p-Laplacians with applications to Picard type Theorems*, Geom. Funct. Anal. 11 (2001), no 6, 1139-1191; *Erratum to "Harnack inequality and hyperbolicity for the p-Laplacians with applications to Picard type Theorems"*, Geom. Funct. Anal. 12 (2002), no 1, 217.

[15] S. Y. Cheng and S.-T. Yau. *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math., 28 (1975) 333-354.

[16] Huai-Dong Cao and Xi-Ping Zhu. *A Complete Proof of the Poincaré and Geometrization Conjectures - application of the Hamilton-Perelman theory of the Ricci flow*. Volume 10, Number 2 (June 2006) Asian Journal of Mathematics.

[17] E. DE GIORGI. *Una estensione del teorema di Bernstein*, Ann. Scuola Norm. Sup. Pisa (3) 19 (1965), 79-85.

[18] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. (Second edition.) Springer-Verlag, 1983.

[19] M. do Carmo and C.K. Peng. *Stable complete minimal surfaces in R^3 are planes*, Bull. Amer. Math. Soc. **1**, (1979), 903–906.

[20] M. do Carmo and C.K. Peng. *Stable complete minimal hypersurfaces*. Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980), 1349–1358, Science Press, Beijing, 1982.

[21] W. FLEMING. *On the oriented plateau problem*, Rend. Circ. Mat. Palermo (2) 11 (1962), 69-90.

[22] D. Fischer-Colbrie and R. Schoen. *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33**, (1980), 199–211.

[23] Feldman, Michael; Ilmanen, Tom; and Ni, Lei. *Entropy and reduced distance for Ricci expanders*. (English. English summary) J. Geom. Anal. 15 (2005), no. 1, 49–62.

[24] A. Grigor'yan. *On the existence of a Green function on a manifold*, (in Russian) Uspekhi Matem. Nauk, 38 (1983) no.1, 161-162. Engl. transl. Russian Math. Surveys, 38 (1983) no.1, 190-191.

[25] A. Grigor'yan. *On the existence of positive fundamental solution of the Laplace equation on Riemannian manifolds*, (in Russian) Matem. Sbornik, 128 (1985) no.3, 354-363. Engl. transl. Math. USSR Sb., 56 (1987) 349-358.

[26] R.Greene and H. Wu. *Integral of subharmonic functions on manifolds of nonnegative curvature*, Inventiones Mathematicae vol **27** (1974), 265-298.

[27] R.Greene and H. Wu. *Function theory of manifolds which possess a pole*, Lecture Notes Math 699, Springer, 1979.

[28] Richard S. Hamilton. *Four-manifolds with positive curvature operator*. J. Differential Geom., 24(2): 153-178, 1986.

[29] Richard S. Hamilton. *Three-manifolds with positive Ricci curvature*. J. Differential Geom., 17(2): 255-306, 1982.

[30] Richard S. Hamilton. *The formation of singularities in the Ricci flow*. Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7-136, Internat. Press, Cambridge, MA, 1995.

[31] I. Holopainen, T. Kipeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, Clarendon Press, Oxford-New York-Tokyo, (1993).

[32] I. Holopainen. *A sharp L^q Liouville theorem for p-harmonic functions*, Israel J. Math. 115 (2000), 363-379.

[33] D.A. Hoffman, R. Ossorman and R. Schoen. *On the Gauss map of complete surfaces of constant mean curvature in R^3 and R^4*, Comment. Math. Helv. **57**, (1982) 519–531.

[34] H. Huber. *Riemannsche Flhen von hyperbolischem Typus im euklidischen Raum*. (German) Math. Ann. 139 1959 140–146 (1959).

[35] A. Huber. *On subharmonic functions and differential geometry in the large*. Comment. Math. Helv. 32 1957 13–72.

[36] James A. Jenkins. *On hyperbolic surfaces in three-dimensional Euclidean space*, Michigan Math. J. 8 (1961) 1–5.

[37] L. Karp. *Subharmonic functions on real and complex manifolds*, Math.Z. 179 (1982), no 4, 535-554.

[38] V.M. Kesel'man and V.A. Zorich. *On the conformal type of a Riemannian manifold* (in Russian), Funktsional. Anal. i Prilozhen. 30(1996), 40-55, 96; English transl. in Functional Anal. Appl. 30(1996), 106-117.

preprint, 2006.

preprint, 2006.

[39] T. Lyons and D. Sullivan. *Function theory, random paths and covering spaces*, J. Diff. Geom., 19 (1984) 299-323.

[40] J. Milnor. *On deciding whether a surface is parabolic or hyperbolic*. Amer. Math. Monthly 84 (1977), no. 1, 43–46.

[41] Mei, Jiaqiang and Xu, Senlin. *On minimal hypersurfaces with finite harmonic indices*. (English. English summary) Duke Math. J. 110 (2001), no. 2, 195–215.

[42] R. Nevanlinna. *Ein Satz ber offenen Riemannsche Flhen*, Ann. Acad. Sci. Fenn. Part A., 54 (1940) 1-18.

[43] R. Osserman. *Riemann surfaces of class* A. Trans. Amer. Math. Soc. 82 1956 217–245.

[44] R. Osserman. *Hyperbolic surfaces of the form* , Math. Ann., 144 (1961) 77-79.

[45] Grisha Perelman. *The entropy formula for the Ricci flow and its geometric applications*. arXiv:math.DG/0211159.

[46] Grisha Perelman. *Ricci flow with surgery on three-manifolds*. arXiv:math.DG/0303109.

[47] Grisha Perelman. *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*. arXiv:math.DG/0307245.

[48] A. V. Pogorelov. *On the stability of minimal surfaces*, Dokl. Akad. Nauk SSSR **260**, 293–295 (1981, Zbl. 495.53005), English transl.: Soviet Math. Dokl. 24, (1981), 274-276.

[49] R. Redy. *Embeddings of open Riemann surfaces*. Comment. Math. Helv. 46 (1971), 214–225.

[50] J. SIMONS. *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) 88 (1968), 62-105.

[51] K-Th. Sturm. *Analysis on local Dirichlet spaces I. Recurrence, conservativeness and -Liouville properties*, J. Reine. Angew. Math., 456 (1994) 173-196.

[52] K-Th. Sturm. *Sharp estimates for capacities and applications to symmetrical diffusions*, Probability theory and related fields, 103 (1995) no.1, 73-89.

[53] R. Schoen, L. Simon and S.T. Yau. *Curvature estimates for minimal hypersurfaces*, Acta Math 134, (1975), 275–288.

[54] J. Sacks and K. Uhlenbeck. *The existence of minimal immersions of* 2*-spheres*, Ann. of Math. 113, (1981), 1–24 .

[55] Shen, Yi-Bing and Zhu, Xiao-Hua. *On stable complete minimal hypersurfaces in* R^{n+1}. (English. English summary) Amer. J. Math. 120 (1998), no. 1, 103–116.

[56] N.Th. Varopoulos. *"Potential theory and diffusion on Riemannian manifolds" in Conference on Harmonic Analysis in honor of Antoni Zygmund*, Vol I, II, Wadsworth Math. Ser., Wadsworth, Belmont, Calif., (1983), 821–837.

[57] S. W. Wei. *Nonlinear partial differential systems on Riemannian manifolds with their geometric applications*, Journal of Geometric Analysis Vol 12, No 1, (2002) 147–182

[58] W.W. Wei, J-.F. Li, and L. Wu. *p-Harmonic Generalizations of the Uniformization Theorem and Bochner's Method, and Geometric Applications*,preprint, 2006.

[59] S.T. Yau. *Problem section, Seminar on Differential Geometry*, Ann. of Math. Stud., No. 102, Princeton Univ. Press, 1982, pp. 669-706.

[60] S.T. Yau. *Perspectives on Geometric Analysis*, preprint,2006.

[61] S.T. Yau. *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28, (1975), 201-228.

[62] S.T. Yau. *Some function theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. 25, (1976), 659-670.

[63] P. Yang. *Curvatures of complex submanifolds of C^n*. J. Differential Geom. 12 (1977), no. 4, 499–511 (1978).