Application of matched asymptotic expansion techniques to the analysis of geothermal heat exchangers

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ABSTRACT

Most theoretical models for the thermal response of geothermal heat exchangers assume the mean azimuthal borehole wall temperature to be uniform along the boreholes. This simplifying assumption, closely related to the g-functions introduced by Eskilson in 1987, has dominated the research field for the past 30 years, allowing the analysis of large geothermal heat exchangers in reasonable amounts of time. The assumption, however, is not physically correct, which hinders the attainable accuracy. By using matched asymptotic expansion techniques, analytical models for the thermal response of geothermal heat exchangers are derived, which do not require the aforementioned simplification. The resulting expressions, applicable to geothermal heat exchangers with irregularly placed heterogeneous boreholes, show accuracy and flexibility levels comparable to SBM, but with a computational cost in line with the use of g-functions.

INTRODUCTION

Geothermal heat exchangers have an increasingly fundamental role in the design and construction of sustainable buildings. An HVAC system based on geothermal energy can only be considered truly renewable, however, if the geothermal heat exchanger and the heat injection/extraction strategy are properly designed. Otherwise, thermal exhaustion of the ground occurs, and the geothermal heat pump's efficiency decreases over time.

To ensure the long-term efficiency of geothermal HVAC systems, its thermal behavior after 25, 50, or even 100 years of operation must be known, reason why their theoretical and numerical modeling is extremely relevant to the industry. Extensive literature exists on this topic, being the contributions by the research group at Lund University the most influential ones (Bennet et al. 1987; Claesson and Hellström 2011; Eskilson 1986, Eskilson 1987; Eskilson and Claesson 1988; Hellström 1991). They proposed two models. The *Superposition Borehole Model* (SBM) solved numerically the governing equation in the ground and the one-dimensional energy conservation equations for the transfer of heat along the pipes. In order to connect both, a network of thermal resistances was used to describe the quasi-steady two-dimensional thermal response of the grout and of the ground close to the borehole (Eskilson 1986). The extension to borehole fields was done by superposing the exerted temperature perturbation of neighboring boreholes. This approach rendered the model very accurate but also very slow, especially for large borehole fields. The second one, the *g-function model*, assumed a uniform mean azimuthal borehole wall temperature along the boreholes (Eskilson 1987). This simplification allowed the uncoupling of the heat conduction problem in the ground from the heat transfer problem inside the boreholes. For the former problem, a large database of precomputed borehole field configurations, the so called *g-functions*, was obtained using a tailored version of SBM, while the latter problem was solved analytically thanks to the introduced simplifying assumption. The result was a less accurate and less flexible model than SBM, but an extremely fast one.

A lot of effort has been made to develop efficient but accurate methods to calculate the *g-functions* (Li and Lai 2015). In recent years, *g-functions* have been obtained using a commercial finite-element simulation tool (Monzó et al. 2016), but this approach is very time consuming. On the other hand, a family of models assimilates the borehole to a line of heat sources of uniform intensity (Eskilson 1987). The infinite line source (ILS) (Ingersoll et al. 1954), the infinite cylindrical-surface source (ICS) (Ingersoll et al. 1954) and the finite line source (FLS) (Zeng et al. 2002; Lamarche and Beauchamp 2007) models have been used respectively to model the borehole for increasing values of the characteristic injection time. All these models assume that the heat injection rate per unit borehole length q is uniform along the borehole, contrary to the original *g-function* definition proposed by Eskilson that assumes a uniform mean azimuthal borehole wall temperature T_b along the borehole. As the resulting value of T_b is not uniform along the borehole, its mean value is used instead to define the *g-functions* (Zeng et al. 2002; Lamarche and Beauchamp 2007; Claesson and Javed 2011; Cimmino and Bernier 2014; Cimmino et al. 2013) and its calculation can be reduced to a single integral (Lamarche and Beauchamp 2007; Claesson and Javed 2011) that is discretized using the midpoint rule with one (Cimmino et al. 2013) or more segments (Cimmino and Bernier 2014; Lazzarotto and Björk 2016) or the trapezoidal rule with multiple segments along the boreholes (Lamarche 2017). Furthermore, a primitive function for the integral can be obtained for the steady-state and time-harmonic cases (Hermanns and Ibáñez 2018) leading to simple analytical expressions. This alternative *g-function* definition results in very efficient methods. However, discrepancies up to 30% have been reported between Eskilson's original definition with uniform T_b and the alternative definition with uniform q (Cimmino and Bernier 2014; Malayappan and Spitler 2013) that grow as the number of thermally interacting boreholes increases (Cimmino and Bernier 2014; Cimmino et al. 2013).

Those two simplifications that assume uniform values along the borehole are inherently limited in accuracy as shown in Malayappan and Spitler (2013), Hermanns and Ibáñez (2017), and Ibáñez and Hermanns (2018) among others. Besides these accuracy problems, the mathematical problem that the original Eskilson's definition of the *g-function* gives rise to, becomes ill-posed under certain circumstances (Hermanns and Ibáñez 2018). Therefore, a well-posed theoretical model that can retain axial variations of the mean azimuthal borehole wall temperature and the heat injection rate per unit borehole length is needed.

Using matched asymptotic expansion techniques (Lagerstrom 1988), the leading author and his research group have derived analytical expressions for the thermal response of geothermal heat exchangers without requiring the aforementioned simplifications. The resulting model is comparable to SBM in terms of flexibility and accuracy, but with a computational cost in line with the *g-function model*. Despite being also an analytical method, the proposed method clearly differs from the *g-function model* in that the thermal responses of the pipes and the liquid inside it, the grout, and the ground, all are tackled at once without assuming neither T_b nor q to be uniform along the borehole, that is, the developed solutions directly give the thermal response of the geothermal heat exchanger.

The present conference paper is a summary of the work done so far by the leading author and his research group. First, a description of the characteristic times that appear in the problem is given. The resulting inner and outer regions and the asymptotic matching process that weaves them are then described. An overview of the process carried out to obtain an asymptotic solution is given afterwards. The extension of the proposed model from one to multiple boreholes is discussed here for the first time. The proposed model is compared against existing models for a test case of a time-harmonic sub-annual excitation applied to a borehole field comprised by two coaxial probes of different lengths. These preliminary results state clearly the differences between the *g-function model* and the asymptotic model developed by the authors. Finally, conclusions and indications on the expected achievements to accomplish in the coming years are given.

SLENDER GEOTHERMAL BOREHOLES

A vertical geothermal borehole consists of a set of pipes placed in a vertical drilling, filled with grout, and through which a liquid flows and exchanges heat with the surrounding ground. Typical boreholes present depths H of the order of hundreds of meters and radii r_b of the order of tens of centimeters, leading to aspect ratios $\Lambda \sim H / r_b$ of the order of thousands. This extreme slenderness is responsible for the appearance of two distinct regions, an inner one located at

radial distances to the borehole of order r_b, and an outer one located further away. To obtain the thermal response of the borehole, the solutions to these two regions need to be combined, something that is accomplished through asymptotic matching at an intermediate scale (Lagerstrom 1988).

Closely related to the previous two length scales are the characteristic transversal diffusion time, $t_b \sim r_b^2/\alpha$, and the characteristic longitudinal diffusion time, $t_H \sim H^2/\alpha$, where α is the thermal diffusivity of the ground. Considering the typical values attained by the different parameters, these two characteristic times are respectively of the order of hours and centuries (Hermanns and Pérez 2014). Two additional characteristic times can be identified in the problem. The first one is the residence time of the liquid in the pipes, $t_r \sim V/H$, with V being the mean flow velocity of the liquid. Due to the requirement of turbulent flow regime to enhance the heat exchange between the liquid and the pipe wall, this characteristic time is of the order of tens of minutes, so that $t_r \ll t_b \ll t_H$ (Hermanns and Pérez 2014). The second one is the characteristic heat injection time t_q. Since the heat injection/extraction from the ground depends on the cooling/heating needs of the building, this characteristic time varies from minutes to months. Also, characteristic values of years or even decades are common, as certain heat injections/extractions are sustained over time.

The thermal problems to solve in the inner and outer regions depend on the relationship between t_q and the other three characteristic times. The following discussion will focus on the realistic case in which $t_r \ll t_b \ll t_q \ll t_H$. For the sake of simplicity, the presentation will be limited to simple U-pipe or coaxial pipe configurations with two pipes connected at the bottom, one with the flow downwards (pipe 1) and one with the flow upwards (pipe 2). A straightforward extension to other borehole configurations, like double U-pipes and energy piles, is possible if certain thermal symmetries are preserved (Hermanns and Ibáñez 2017; Ibáñez and Hermanns 2018).

Inner region

The inner region encompasses the pipes and grout inside the borehole, and the ground located at radial distances to the borehole of order r_b. In the pipes, the convective transport of heat by the liquid prevails, while heat conduction takes place in the pipe walls, grout, and ground.

In the heat conduction equation to solve, the vertical heat conduction terms are of order Λ^{-2} times smaller than the radial ones, rendering them negligible. Additionally, when the characteristic heat injection time t_q is large compared to the characteristic transversal diffusion time t_b, the thermal inertia in the heat conduction equation is of order t_b/t_q times smaller than the radial heat conduction terms, rendering it also negligible. All this allows the heat conduction problem in the inner region to be interpreted as taking place in two-dimensional planes perpendicular to the borehole, only coupled to each other through the pipes and the outer region, and in which a quasi-steady two-dimensional heat conduction problem must be solved.

Far from the borehole, at large radial distances compared to r_b, the behavior of the inner and outer regions must coincide. The boundary condition to impose there, when solving the aforementioned quasi-steady two-dimensional heat conduction problem, is given by the first two terms of the asymptotic expansion of the inner solution for $r \gg r_b$ (Hermanns and Pérez 2014):

$$r \gg r_b: \quad T(r,z,t) = -\frac{q(z,t)}{2\pi k}\ln\left(\frac{r}{r_b}\right) + T_a(z,t) + o(1), \tag{1}$$

where q is the heat injection rate per unit borehole length and T_a is the apparent temperature at which the inner region perceives the ground. This temperature, whose value results from the asymptotic matching with the outer region, differs from the unperturbed ground temperature due to the presence of the borehole and its operation.

It was shown by Hermanns and Pérez (2014), that the heat injection rates per unit pipe length q_i, obtained as part of the solution to the quasi-steady heat conduction problem in the inner region, must obey the very specific mathematical structure

$$q_1 = \frac{1+S}{2}\frac{T_m - T_a}{R_b} + \frac{T_1 - T_2}{R_a}, \qquad\qquad q_2 = \frac{1-S}{2}\frac{T_m - T_a}{R_b} + \frac{T_2 - T_1}{R_a}, \tag{2}$$

where R_a and R_b are respectively the thermal resistance between the pipes and the thermal resistance between the borehole and the ground. The thermal skewness parameter S, introduced for the first time by Ibáñez and Hermanns (2018), represents how the borehole's thermal resistance R_b is distributed among the pipes. Its value lies in the range [-1,1], corresponding the extreme values to coaxial probes.

The weighted mean fluid temperature T_m, given in terms of the temperatures T_i of the liquid in the pipes by

$$T_m = \frac{T_1 + T_2}{2} - S\frac{T_1 - T_2}{2}, \tag{3}$$

was introduced for the first time by Hermanns and Pérez (2014) and represents the temperature at which the borehole exchanges heat with the ground. This interpretation can easily be inferred from adding q_1 to q_2:

$$q = q_1 + q_2 = \frac{T_m - T_a}{R_b}. \tag{4}$$

The use of T_m and S is very convenient, because several different borehole configurations can be described using the same system of equations. For instance, single U-pipes and coaxial pipes, which in the literature are treated using different sets of equations, are treated in a unified manner here.

By substituting the expressions for q_i into the energy conservation equations for the pipes, which are quasi-steady because of the characteristic residence time t_r being small compared to the characteristic heat injection time t_q, a set of ordinary differential equations for $T_1 - T_2$ and T_m can be obtained, which combined with (4) constitutes the mathematical problem to solve in the inner region. Using the nondimensional variables

$$\Delta = \frac{T_1 - T_2}{Q_c/(\dot{m}c)}, \qquad \Theta = \frac{T_m - T_{mean}}{Q_c/(kH)}, \qquad \overline{q} = \frac{q}{Q_c/H}, \qquad \Theta_a = \frac{T_a - T_{mean}}{Q_c/(kH)}, \tag{5}$$

in which Q_c is the characteristic heat injection rate, $\dot{m}$ the mass flow rate in the pipes, c the specific heat capacity of the liquid, k the thermal conductivity of the ground, and T_{mean} the mean annual temperature at the considered geographical location, the following system of differential and algebraic equations results (Hermanns and Ibáñez 2017; Ibáñez and Hermanns 2018):

$$\frac{d\Delta}{d\xi} = -\overline{q}, \qquad\qquad \frac{d\Theta}{d\xi} = \Omega S\overline{q} - \frac{\Omega^2}{A}\Delta, \qquad\qquad \Theta - \Theta_a = \overline{q}B, \tag{6}$$

where $A = kR_a$ and $B = kR_b$ are the nondimensional thermal resistances of the borehole, $\Omega = kH/(\dot{m}c)$ is a nondimensional parameter that compares the characteristic temperature difference $Q_c/(\dot{m}c)$ between the pipes with the characteristic temperature difference $Q_c/(kH)$ between the borehole and the ground located at radial distances of order $\sqrt{\alpha t_q}$, and $\xi = z/H$ is the nondimensional vertical coordinate.

The boundary conditions to impose at the top, $\xi = 0$, and bottom, $\xi = 1$, of the borehole are

$$\xi = 0: \ \Delta = f(\tau), \qquad\qquad \xi = 1: \ \Delta = 0, \tag{7}$$

being $f = Q(t)/Q_c$ the nondimensional heat injection rate and $\tau = t/t_q$ the nondimensional time.

The formulated mathematical problem requires an additional relationship between the four unknowns, Δ, Θ, $\overline{q}$, and Θ_a, to be solvable. This relationship is supplied by the outer region and its asymptotic matching with the inner region.

Outer region

Far from the borehole, at radial distances of order $\sqrt{\alpha t_q}$, the thermal inertia of the ground becomes important, and a slender borehole is perceived in first approximation as a finite line source of heat, as inferred from the behavior of the inner region far from the borehole, given in (1) (Hermanns and Pérez 2014; Ibáñez and Hermanns 2018). Thus, the unsteady outer region is in first approximation axisymmetric, and an analytical solution for the nondimensional ground temperature, $\Theta_g = (T_g - T_{mean})/(Q_c/(kH))$, can be obtained by superposition of point sources of heat (Carslaw and Jaeger 1959):

$$\Theta_g(\xi, \eta, \tau) = \Theta_\infty(\xi, \tau) + \frac{1}{8\epsilon\pi^{3/2}} \int_0^\tau \int_0^1 \frac{\overline{q}(\xi_0, \tau_0)}{(\tau - \tau_0)^{3/2}} \left[e^{-\frac{\eta^2 + \frac{(\xi - \xi_0)^2}{\epsilon^2}}{4(\tau - \tau_0)}} - e^{-\frac{\eta^2 + \frac{(\xi + \xi_0)^2}{\epsilon^2}}{4(\tau - \tau_0)}} \right] d\xi_0 d\tau_0, \tag{8}$$

where $\eta = r/\sqrt{\alpha t_q}$, $\epsilon = \sqrt{\alpha t_q}/H$, and Θ_∞ is the nondimensional unperturbed ground temperature:

$$\Theta_\infty(\xi, \tau) = G\xi + \sum_{n=-\infty}^{\infty} \hat{T}_{sn} e^{i\omega_n t_q \tau} e^{-\sqrt{i\omega_n t_q}\frac{\xi}{\epsilon}}, \tag{9}$$

being $G = q_{geo}H^2/Q_c$ the nondimensional geothermal heat flux at the considered geographical location and $\hat{T}_{sn}$ the nondimensional harmonic modes, with nondimensional angular frequency $\omega_n t_q$, of the ground surface temperature oscillations with respect to the mean annual temperature T_{mean}.

Asymptotic matching

The asymptotic matching procedure states that both the inner and outer solutions are valid in an intermediate region defined by $r_b \ll r \ll \sqrt{\alpha t_q}$, or expressed in non-dimensional variables $(\epsilon\Lambda)^{-1} \ll \eta \ll 1$, in which their asymptotic expansions must coincide. The asymptotic behavior of the outer solution close to the borehole can be obtained by taking the limit $\eta \ll 1$ in (8). The resulting expression depends on the position along the borehole. Near the top and the bottom of the borehole, asymptotic boundary layers appear that must be analyzed separately. Also, the logarithmic singular behavior near the borehole must be extracted carefully before comparing with expression (1). Thus, the remaining non-logarithmic terms that appear in the outer solution near the borehole correspond to the nondimensional apparent temperature Θ_a that expressed in compact form in terms of the nondimensional heat injection rate per unit borehole length $\overline{q}$ is

$$\Theta_a = F(\xi, \tau, \overline{q}). \tag{10}$$

All the details for the cases of steady-state excitation and time-harmonic sub-annual excitation have already been completed and submitted for publication (Hermanns and Ibáñez 2017; Ibánez and Hermanns 2018) while the case of arbitrarily time-varying heat injection rate is still under development.

Solution procedure

The solution to the integro-differential problem formed by (6), (10), and the boundary conditions given in (7) is obtained through asymptotic expansions of the unknown variables in terms of the small parameter δ present in the problem (Hermanns and Ibáñez 2017; Ibáñez and Hermanns 2018):

$$\Delta = \Delta_0 + \delta\Delta_1 + o(\delta), \qquad \delta\Theta = \Theta_0 + \delta\Theta_1 + o(\delta), \qquad \bar{q} = \bar{q}_0 + \delta\bar{q}_1 + o(\delta). \qquad (11)$$

The small parameter δ is proportional to $log(\epsilon\Lambda)^{-1} \ll 1$ and its exact value depends on the case studied. In this expansion, the nondimensional weighted mean fluid temperature Θ is preceded by a factor δ that is foreseen by analyzing the order of magnitude of that variable in (6) (Hermanns and Ibáñez 2017; Ibáñez and Hermanns 2018). The expansions in (11) are introduced in (6) and similar order terms are grouped to obtain simpler systems of equations to be solved for each order of the expansions. The resulting analytical expressions for Δ, Θ, and $\bar{q}$ describe the thermal response of vertical geothermal boreholes and are computationally inexpensive to evaluate.

Worth to mention that the obtained apparent temperature is not uniform along the borehole, presenting deviations of around ±20% with respect to a mean value. This result is important, as the mean azimuthal borehole wall temperature coincides in value with the apparent temperature (Hermanns and Pérez 2014), showing that the simplifying assumption introduced by the *g-function model*, that the mean azimuthal borehole wall temperature is uniform along the borehole, is not accurate.

EXTENSION TO BOREHOLE FIELDS

Although most of the work done so far has been on single boreholes, the extension to geothermal heat exchangers comprised of tens or hundreds of boreholes has already been initiated, and preliminary results for the case of a time-harmonic sub-annual excitation will be presented for the first time in this conference.

When a time-harmonic excitation is imposed at the top of the borehole, the characteristic time of variation of the heat injection rate becomes $t_q = 1/\omega$, where ω is the angular frequency of the oscillation. The total heat injection rate can then be expressed as $Q(\tau) = \text{Re}[\hat{Q}e^{i\tau}]$. Consequently, all variables can be expressed as harmonic functions,

$$\Delta(\xi,\tau) = \text{Re}[\hat{\Delta}(\xi)\,e^{i\tau}], \qquad \Theta(\xi,\tau) = \text{Re}[\hat{\Theta}(\xi)\,e^{i\tau}], \qquad \bar{q}(\xi,\tau) = \text{Re}[\hat{q}(\xi)\,e^{i\tau}], \qquad (12)$$

and only the complex-valued harmonic modes $\hat{\Delta}$, $\hat{\Theta}$, and $\hat{q}$ need to be obtained. These complex values, that are functions of the nondimensional vertical coordinate ξ, include information of the amplitude of the oscillation as well as of the phase difference with respect to the total heat injection rate at any section of the borehole. A system of equations equivalent to (6) must be solved now for each borehole, and additional conditions fix the total heat injection rate Q transferred to the ground by the N_b boreholes and the common inlet temperature of the boreholes. The apparent temperature of each borehole contains now $N_b - 1$ new terms which correspond to the temperature perturbations exerted by the remaining boreholes. These contributions are obtained by setting $\eta = B_{ij}/\sqrt{\alpha t_q}$ in (8), where B_{ij} is the distance between the borehole and the remaining boreholes. Finally, the solution of the problem is obtained by expanding $\hat{\Delta}$, $\hat{\Theta}$, and $\hat{q}$ in terms of a set of N_b small parameters as in (11). Details of the solution procedure and the resulting expansions can be found in Hermanns and Ibáñez (2018b).

A first implementation of the developed model in a computer program shows that the solution to a geothermal heat exchanger comprised of 100 boreholes can be obtained in 30 milliseconds using a single CPU core on an Intel Core i7-7700K PC. Thus, the advantage of having analytical expressions for the thermal response of geothermal heat exchangers is that they can be used for the analysis of heterogeneous borehole fields with irregular placement of the boreholes without requiring the excessive computational cost of SBM or the loss in accuracy of the *g-function model*.
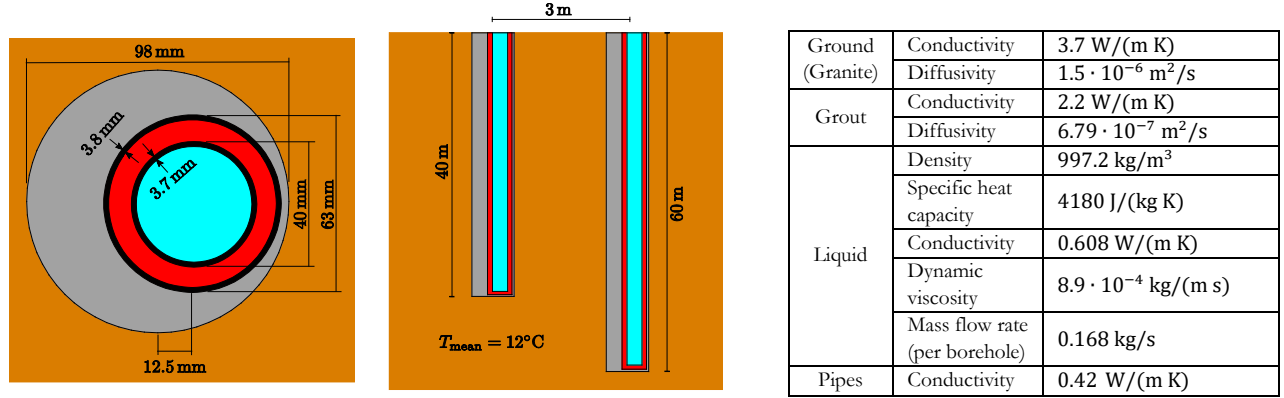
Figure 1 Diagram of the configuration used for the comparison of the derived asymptotic model with three existing models. (Left) Cross section of the coaxial borehole. The liquid flows downwards through the pipe shown in red and upwards through the pipe shown in cyan. The grout is shown in grey and the ground in brown. (Middle) Longitudinal section of the borehole field. Borehole 1 is at left and Borehole 2 is at right. (Right) Table of properties of the different materials used in the simulation.

COMPARISON WITH EXISTING MODELS

To assess the accuracy of the derived asymptotic expressions, comparisons with three existing models with different levels of accuracy are carried out. The first and most accurate model is a detailed numerical simulation (DNS) of the governing equations using a commercial finite-element simulation tool. The second model is the *Superposition Borehole Model* (SBM) developed by Eskilson and Claesson (Eskilson 1986; Eskilson and Claesson 1988). Since the original implementation is not meant for steady-state or time-harmonic excitations, a reimplementation of the method has been performed (Hermanns and Ibáñez 2017; Ibáñez and Hermanns 2018). The third model is the *g-function model* proposed by Eskilson (Eskilson 1987). For the steady-state case, the *g-functions* have been taken from the literature (Eskilson 1987; Zeng, et al. 2003; Conti 2016), while for the other regimes they have been computed using the reimplemented SBM and following the specifications given in the literature (Eskilson 1987; Eskilson and Claesson 1988; Hellström 1991; Blomberg, et al. 2017). The *g-function* solution has been implemented using an equally-spaced discretization with 20 segments along the boreholes, as in Cimmino and Bernier (2014).

The results from all the comparisons performed so far can be found in the work submitted for publication (Hermanns and Ibáñez 2017; Ibáñez and Hermanns 2018). Here, some preliminary results are shown for the time-harmonic sub-annual excitation of a geothermal heat exchanger comprised by two parallel coaxial boreholes of different lengths. Figure 1 contains a diagram of the borehole field studied as well as a table including the properties of all materials involved in the simulation. A total heat injection rate of modulus $|\hat{Q}| = 2100$ W and period of one year (365.25 days) is distributed among the boreholes through which a mass flow rate of $\dot{m} = 0.168$ kg/s per borehole is supplied.

Figure 2 shows the modulus and argument of the complex-valued harmonic modes $\hat{\Delta}$, $\hat{\Theta}$, and $\hat{q}$ for each of the two boreholes as functions of $\xi = z/\bar{H}$, where $\bar{H} = 50$ m is the mean depth of the boreholes. The modulus of each harmonic mode is a measure of the maximum value attained as it oscillates. On the other hand, the argument of each harmonic mode is a measure of the difference in phase that appears between that variable and the total heat injection rate.
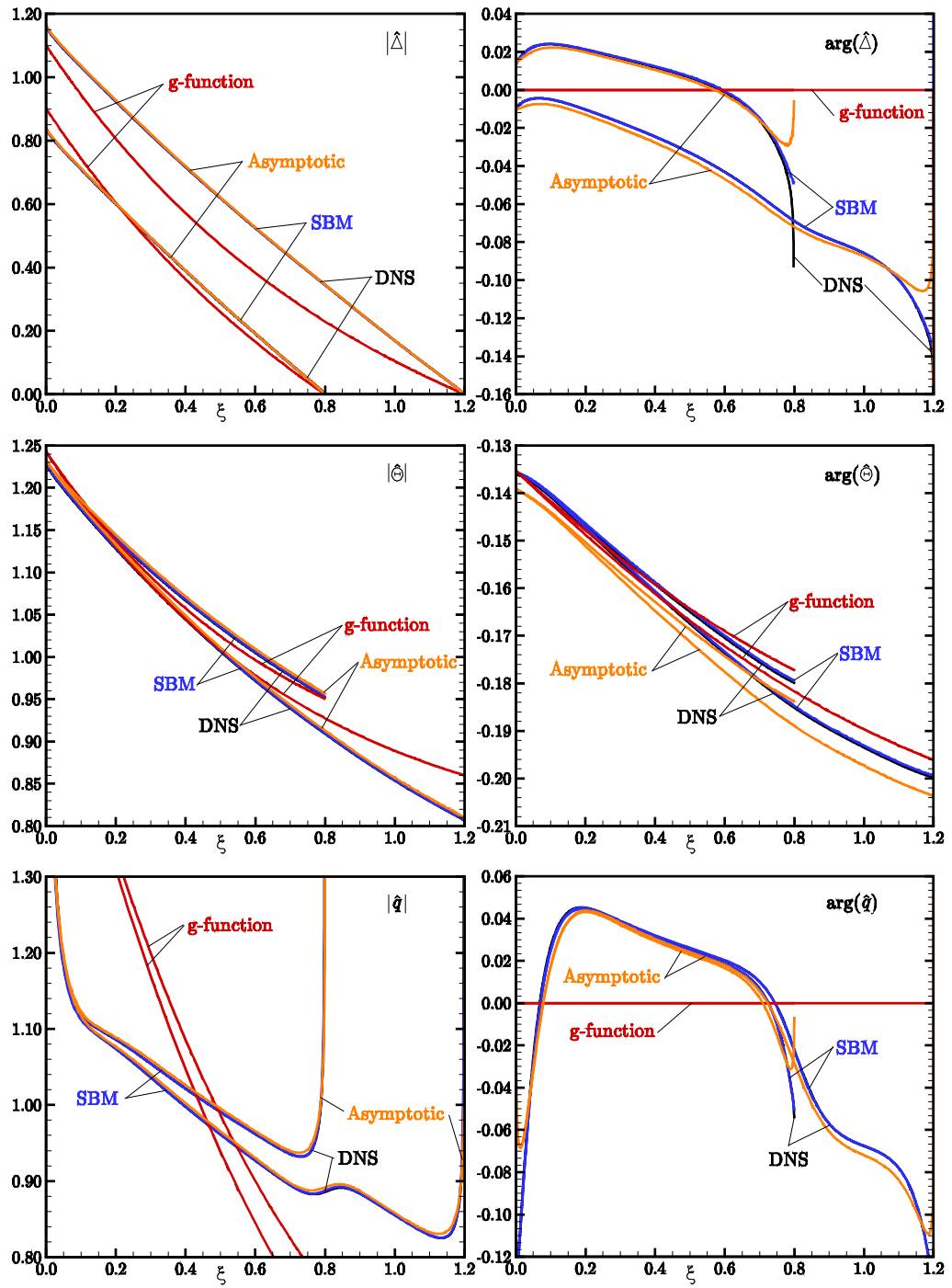
Figure 2 Time-harmonic thermal response of a geothermal heat exchanger, comprised of two heterogeneous coaxial boreholes with nondimensional depths h equal to 0.8 and 1.2, to a time-harmonic heat injection rate with a one-year period. (Left) modulus and (right) argument of the (top) nondimensional temperature difference $\widehat{\Delta}$, (middle) nondimensional weighted mean temperature $\widehat{\Theta}$, and (bottom) nondimensional heat injection rate per unit borehole length $\widehat{q}$ of the two boreholes as functions of the nondimensional axial coordinate ξ and for the considered models.
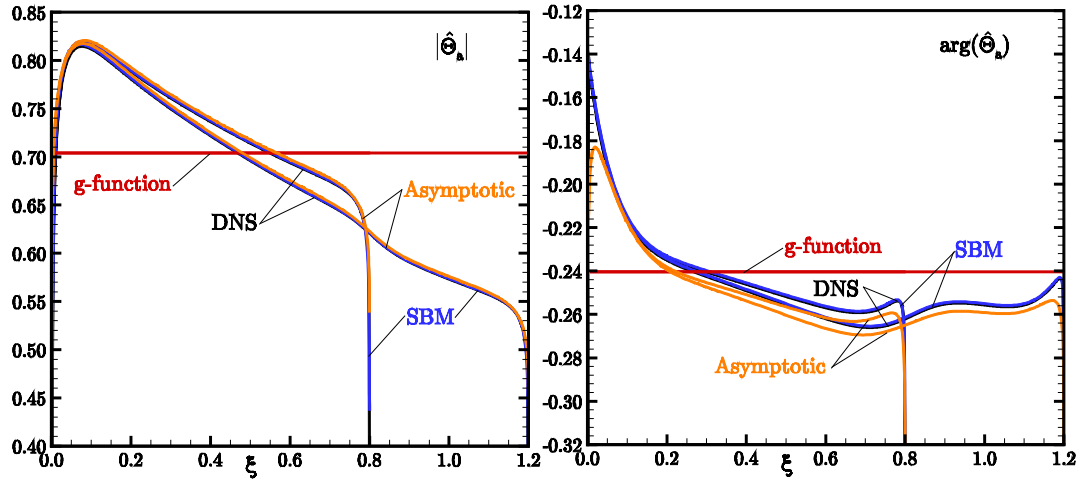
Figure 3 Time-harmonic thermal response of a geothermal heat exchanger, comprised of two heterogeneous coaxial boreholes with nondimensional depths h equal to 0.8 and 1.2, to a time-harmonic heat injection rate with a one-year period. (Left) modulus and (right) argument of the nondimensional apparent temperature $\widehat{\Theta}_a$ of the two boreholes as functions of the nondimensional axial coordinate ξ and for the considered models.

While the developed asymptotic model, SBM, and DNS present small differences, the *g-function model* is unable to correctly represent the temperature and heat injection rate distributions along the boreholes. The *g-function model* predicts a uniform zero value of the argument of both $\widehat{\Delta}$ and $\widehat{\Theta}$ while the remaining models display notable axial variations and non-zero mean values of those variables. The most noticeable discrepancies between the *g-function model* and the remaining models arise in the modulus of the nondimensional heat injection rate per unit borehole $\hat{q}$. In a small region near the top and the bottom of each borehole, the nondimensional heat injection rate per unit borehole $\hat{q}$ increases abruptly (Hermanns and Ibáñez 2017). These regions, which coincide with the asymptotic boundary layers mentioned before, are correctly represented by all models except the *g-function* one.

Figure 3 shows the nondimensional apparent temperature $\widehat{\Theta}_a$ for the same configuration. The developed asymptotic model, SBM, and DNS deliver very similar results, with deviations of up to $\pm23\%$ with respect to a mean value. These results confirm that neither the nondimensional apparent temperature $\widehat{\Theta}_a$ on Figure 3, nor the nondimensional heat injection rate per unit borehole length $\hat{q}$ on Figure 2, are uniform along the borehole. Hence, the simplifying assumptions made by Eskilson's definition of the *g-function* (uniform mean azimuthal borehole wall temperature) and the alternative definition found in the literature (uniform heat injection rate per unit borehole length) are not justified by the physics of the problem.

CONCLUSIONS AND FUTURE WORK

By means of scale analysis and matched asymptotic expansion techniques, a new approach for the modeling of the thermal response of geothermal heat exchangers was initiated in 2011. A first publication, whose main focus was the asymptotic matching process and the structure of the solution to the quasi-steady heat conduction problem in the inner region, was published in 2014 (Hermanns and Pérez 2014), followed by a recent extension of that analysis to the steady-state thermal response of complete boreholes (Ibáñez and Hermanns 2018). Further work related to the time-harmonic thermal response of single boreholes (Hermanns and Ibáñez 2017) and its extension to whole borehole fields (Hermanns and Ibáñez 2018b) is in reviewing process at the moment. Also, a critical analysis of the *g-function model* and its different versions was recently sent for publication (Hermanns and Ibáñez 2018). In it, a mathematical explanation is given to the ill-posedness that appears in the original definition of the *g-function model* when no initial insulated length is used.

Still in the present year, the analysis of the thermal response of single boreholes to arbitrarily time-varying heat injection rates will be finished and sent for publication. Then, in the next year, the case in which the characteristic heat injection time t_q is comparable to the characteristic transversal diffusion time t_b will be addressed. This regime of operation is relevant for peak loads, in which the inner region is unsteady. The same applies to the flow in the pipes in case the characteristic heat injection time t_q is comparable to the characteristic residence time t_r. Also, the extension to borehole fields of the thermal response to arbitrarily time-varying heat injection rates will be performed.

The outcome of all this work will ultimately be a coherent theoretical framework for the thermal response of geothermal heat exchangers with irregularly placed heterogeneous boreholes, in which extensions such as stratified grounds, time-varying mass flow rates, and groundwater flow will be possible, and whose reduced computational cost will allow the implementation of automatic optimization strategies at borehole field level.

ACKNOWLEDGMENTS

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