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Extensions of Graph Pebbling

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Chapter 1

Introduction

1.1 Introduction

Within the past twenty years, mathematicians have analyzed combinatorial games on graphs, such as peg solitaire, checker jumping and graph pebbling. In most cases, the reason why these games have been studied is that we can apply analysis of the games as a way to solve more serious questions in number theory or other mathematical fields. I will now describe what graph pebbling is and how it arose.

In 1989, Lemke and Kleitman proved a conjecture of Erdős and Lemke from additive number theory in (21). It is known that for any set $N = \{n_1, n_2, \ldots, n_q\}$ of q natural numbers, there is a nonempty index set $I \subset \{1, \ldots, q\}$ such that $q | \sum_{i \in I} n_i$. The conjecture stated that the additional condition $\sum_{i \in I} n_i \leq \mathrm{lcm}(q, n_1, n_2, \ldots, n_q)$ could also be imposed (18). Although a correct proof was constructed, Lemke and Kleitman's argument was detailed and contained a considerable amount of case analysis. Glenn Hurlbert, a leading expert on graph pebbling, asserts in (18) that "it was the intention of Lagarias and Saks to introduce graph pebbling as a more intuitive vehicle for proving this theorem". Specifically, Erdős and Lemke's extension would follow if the conjectured pebbling number of the Cartesian product of paths was true. Before we continue with this exposition, we will provide some specific definitions in the field of pebbling.

Given a graph G, distribute k pebbles on its vertices in some *configuration*, call it C. Assume that in all cases G is connected. Specifically, a configuration on a graph G is a function from $V(G)$ to $\mathbb{N} \cup \{0\}$ representing an arrangement of pebbles on G. A *pebbling move* is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex.

A pebble can be moved to a *root vertex* v if given a sequence of pebbling moves, it is possible to place one pebble on v. Here is an example of a pebbling move: We define the *pebbling number*, $\pi(G)$ to be the minimum

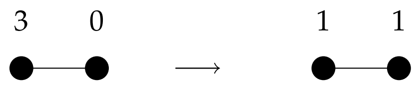

Figure 1.1: A pebbling move.

number of pebbles that are sufficient so that for any initial configuration of pebbles, it is possible to move to any root vertex v in G. After moving to a particular root vertex, if one desires to move to another root vertex, the pebbles reset to their original initial configuration. As a simple example, the pebbling number of K_n, the complete graph on n vertices, is n. Notice that if a pair of pebbles was placed on any vertex, then a pebbling move could place one pebble on any other vertex of the graph. By the pigeonhole principle, this means that $\pi(G) \leq |G| + 1$, where $|G|$ denotes the number of vertices of G. Since placing one pebble on every vertex means that any root vertex contains a pebble, we see that $\pi(G) \leq |G|$. To see that $\pi(G) > |G| - 1$ consider the configuration of pebbles that places one pebble on all but one vertex of G. The unpebbled vertex can never be reached via a pebbling move since there are no pebbling moves that can be made. Thus, $\pi(G) = |G|$.

As another example, it can be determined through case analysis that the pebbling number of the following graph is 4. From this example, we can see

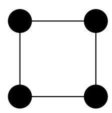

Figure 1.2: A graph where $\pi(G) = 4$.

how checking all the different initial configurations, even for a very basic graph, can get complicated quite easily. With this initial background information covered, we now continue the story of pebbling's "first moves."

In 1989, Fan Chung Graham published the first pebbling paper, which was a proof of the Cartesian product result for paths. One way to generalize Fan Chung Graham's result is the following famous conjecture of Ronald Graham:

Conjecture 1.1.1. *For all G_1, G_2, $\pi(G_1 \times G_2) \leq \pi(G_1)\pi(G_2)$.*

This conjecture is still open, though people have shown its validity on special cases of graphs. To explain one example of such a generalization of the path result, define the *support* of a distribution of pebbles D on a graph G as the number of vertices where $D(v) > 1$. A graph G has the *2-pebbling property* if, for any distribution D of size at least $2\pi(G) - q(D) + 1$, where $q(D)$ is the support of a distribution, it is possible to move two pebbles to any specified root vertex. Moews proves (22) that if G is a graph with the 2-pebbling property and T is a tree, then $\pi(G \times T) \leq \pi(G)\pi(T)$.

The remainder of this introduction will focus on other pebbling questions that I have considered in this thesis. This work was begun at the East Tennessee State University REU and continued over the course of the senior thesis year. Each of the following sections will introduce one branch of research I am conducting.

1.2 Cover Pebbling

Within the past few years the field of graph pebbling has rapidly grown. Mathematicians are now studying a multitude of related questions that involve pebbling moves on graphs. One such question relates to the cover pebbling number of a graph. The *cover pebbling number* $\gamma(G)$ was introduced in (7), and defined as the minimum number of pebbles needed to place a pebble on *every* vertex of the graph using a sequence of pebbling moves, regardless of the initial configuration. One application in (7) for $\gamma(G)$ is based on a military application where troops must be distributed simultaneously. If C is a configuration of pebbles onto the vertices of G and it is possible to cover pebble G, then we say C is *cover solvable*.

The main problem in cover pebbling is determining $\gamma(G)$ for all graphs. As of October 2004, this problem is essentially solved. After the first cover pebbling paper (7), two papers, written by Hurlbert and Munyan (19), and Watson and Yerger (12), provide cover pebbling results for specific classes of graphs. A month later, Vuong and Wyckoff proved a general theorem, that essentially says that for cover pebbling, the configuration that requires the most number of pebbles before a cover solution can be found occurs when all the pebbles are on a single vertex. Thus, they proved the following theorem, colloquially called the "stacking theorem," because all the pebbles can be placed on one vertex. Before we state the theorem, we must define a few terms. For $v \in V(G)$, define

$$s(v) = \sum_{u \in V(G)} 2^{d(u,v)},$$

where $d(u, v)$ denotes the distance from u to v, and let

$$s(G) = \max_{v \in V(G)} s(v).$$

Theorem 1.1. *(Vuong and Wyckoff) Given a graph G, the cover pebbling number on G is $s(G)$.*

In fact, Vuong and Wyckoff prove an analogous result for a more general statement of the theorem, where the number of pebbles required on each vertex is positive, but may be greater than one and may be different for every vertex. Finally, Godbole, Watson and Yerger have shown in Chapter 2 that given a configuration of pebbles C, and a graph G, the decision of whether the graph has a cover-solution or not is NP-complete.

1.3 Domination Cover Pebbling

After the cover pebbling problem was solved, the REU group considered other pebbling-type question and invariants. Alberto Teguia proposed the concept of *domination cover pebbling* as a way to connect the field of graph domination to pebbling. Recall that a set of vertices D in G is a *dominating set* if every vertex in G is either in D or adjacent to some element in D. The domination cover pebbling number of a graph G is the minimum number of pebbles required so that any initial configuration of pebbles can be transformed by a sequence of pebbling moves so that the set of vertices that contain pebbles form a dominating set S of G. The pebbles may be placed on any of the vertices of G. One reason why this problem is hard is that for different initial configurations of pebbles, different dominating sets may be required. Graphs can be easily constructed to show this property. Consider the following configurations of pebbles on P_4, the path on four vertices: For the graph on the left, we make pebbling moves so that the first and third vertices are selected (from left to right) are the vertices of the dominating set. However, for the graph on the right, we make pebbling moves so that the second and fourth vertices are selected (from left to

Figure 1.3: An example where two different initial configurations produce two different domination cover solutions.

right) are the vertices of the dominating set. We refer the reader to (16) for additional exposition on domination in graphs.

With this problem, two branches of study were proposed. The first task was to determine the domination cover pebbling number for various classes of graphs. In Chapter 3, we have determined the domination cover pebbling number for K_n, the complete graph on n vertices, P_n, the path graph on n vertices, C_n, the cycle graph on n vertices, $K_{c_1,c_2,\ldots,c_r}$, the complete r-partite graph with vertex classes $c_1, c_2, \ldots, c_r$ respectively, and B_n, the complete binary tree of height n. The second task was to prove more general structural results about domination cover pebbling. Specifically, given a graph with n vertices and diameter d, we want to determine that maximum value of $\psi(G)$. We have proven a sharp result for graphs of diameter 2 and 3, and an almost sharp result for graphs of diameter n. Recall that the diameter of a graph is the maximum number of edges that are traversed in the shortest path between two arbitrary vertices of a graph.

We also discovered that domination cover pebbling can be related to another graph invariant called the *vertex neighbor integrity* (VNI) of a graph. We describe this parameter using the definitions of Cozzens and Wu (6). Let $G = (V, E)$ be a graph and u be a vertex of G. The *open neighborhood* of u is $N(u) = \{v \in V(G) | \{u, v\} \in E(G)\}$; the *closed neighborhood* of u is $N[u] = \{u\} \cup N(u)$. Analogously, for any $S \subseteq V(G)$, define the *open neighborhood* $N(S) = \cup_{u \in S} N(u)$ and the *closed neighborhood* $N[S] = \cup_{u \in S} N[u]$. A vertex $u \in V(G)$ is *subverted* by removing the closed neighborhood $N[u]$ from G. Notice that this subversion is equivalent to the removal of a dominating set from G. For a set of vertices $S \subseteq V(G)$, the *vertex subversion strategy* S is applied by subverting each of the vertices of S from G. Define the *survival subgraph* to the the subgraph left after the subversion strategy is applied to G. The *order* of G is defined to be $|V(G)|$.

Definition 1.3.1. *The vertex neighbor integrity of a graph G is defined as*

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G \setminus S)\},$$

where $w(H)$ is the order of the largest connected component in the graph H.

We apply a variant of subversion in order to describe how VNI calculations relate to domination cover pebbling. Let $\Omega_\omega(G)$ be the minimum number of pebbles required such that it is always possible to construct an incomplete domination cover solution of G, where disjoint undominated components of G can have order at most ω. This corresponds to the $\omega(G)$ term in the VNI computation. Notice that domination cover pebbling corresponds to the case when $\omega = 0$. As a side note, the only people so far that have worked explicitly in the field of VNI have been women. There have only been a few results so far with respect to this VNI domination cover pebbling, but this may be a topic of further investigation.

1.4 Deep Graphs

Another variant of graph pebbling that has recently been constructed is the concept of deep graphs. In 2004, Hetzel and Isaksson (17) introduced the concept of *deep graphs*. A graph is *deep* if for each positive integer $n \leq \pi(G)$, there exists an induced subgraph H of G such that $\pi(H) = n$. At the REU, we extended this definition to prove stronger, more relevant results.

Before we describe these results we will define terminology used in this branch of the research. One classification of graphs that has been useful in obtaining results for pebbling problems is considering graphs of Class 0. A graph is *Class 0* if its pebbling number is equal to its number of vertices. In fact, in 1997, Clarke, Hurlbert and Hochberg proved in (4) that all 3-connected graphs with diameter 2 are Class 0. We say that a graph G is *Class 0 deep* if G is deep and Class 0. Recall that a graph is k-connected if it is possible to remove $k-1$ vertices from G and have the remaining graph be connected. A graph G is *profound* if it is deep and it is possible to construct G by a series of induced subgraphs $H_1 \subset H_2 \subset \cdots \subset H_{\pi(G)}$. By this construction each of these subgraphs are also profound. Extending this definition to Class 0 graphs, we say that a graph G is *Class 0 profound* if it is Class 0 deep and G can be constructed by a series of induced subsets, $H_1 \subset H_2 \subset \cdots \subset H_{|G|}$, each of which are Class 0 deep.

In the beginning of Chapter 5, we prove some interesting properties about deep graphs. For instance, deepness is not a *monotonic* property with respect to edges. That is, if a graph is deep, adding edges to it may mean the graph is now not deep. Also, not all graphs that are Class 0 deep are Class 0 profound. Notice the inclusion in the opposite direction holds by definition. The remainder of the chapter determines probabilistic results concerning $G(n, p)$, a random graph on n vertices, where each edge is placed independently with probability p. Specifically, we want to show that for some range of values of p, as n approaches infinity, the probability that $G(n, p)$ is Class 0 profound approaches 1 almost surely.

1.5 Cover Pebbling Thresholds

Instead of considering random graphs, now consider what happens when we randomly throw pebbles on a graph such that the each pebble has the same probability of landing on each vertex of the graph. We want to determine the probability that the graph is cover solvable. In order to tackle a doable problem, we consider placing the random pebbles on K_n, the complete graph on n vertices. Conjectures and results have been obtained in two ways. Watson and Yerger used somewhat simple techniques from analysis, and with the help of Godbole, also applied some more complex techniques from probability theory to solve the question.

One common occurrence when considering properties of random graphs is that there exists a threshold, a frequency such that the following occurs: If n, in this case the number of pebbles, but it could be other properties such as the number of edges in a graph or the probability of an edge in $G(n, p)$, is asymptotically less than the threshold, then the probability of the property occurring as n approaches infinity is 0 almost surely. Similarly, if the number of pebbles is asymptotically greater than the threshold, the probability that the property occurring approaches 1 almost surely as n approaches infinity. With this framework in mind, in Chapter 6, we prove the surprising result that to first order, when the pebbles are initially distributed in a random way, the threshold for cover pebbling the complete graph on n vertices is γn, where γ is the golden ratio, $\frac{1+\sqrt{5}}{2}$!

In addition, the random allocation of pebbles described above is one of generally two ways that the pebbles are distributed. The other way is for the probability of every distribution of pebbles to be uniform. Unfortunately, little emphasis has been put on this second distribution.

1.6 Optimal Pebbling

There is another branch of pebbling research that explores best-case pebbling scenarios. Instead of being concerned with the worst possible case, the optimal pebbling number tries to determine the fewest number of pebbles so that every vertex can be reached. Formally speaking, we have the following definition:

Definition 1.6.1. *The optimal pebbling number $\pi_{OPT}(G)$ is the minimum k such that some distribution of k pebbles so that every vertex can be reached.*

Studies of the optimal pebbling number were started by Pachter et al (23), who computed that the optimal pebbling number of the path graph was $\lceil 2n/3 \rceil$.

Recently, Douglas West et al (2) reexamined optimal pebbling numbers and made some simplifications to the literature. The most useful result of this paper is their "smoothing lemma." This lemma says that an optimal pebbling configuration will have at most two pebbles on any vertex of degree 2 and no pebbles on any leaf. Another active group in optimal pebbling is the research group of Narayan at Michigan Tech.

1.7 Structure of the Thesis

The thesis will be organized in the following manner. Each section will describe a particular problem, starting with results proved by other mathematicians, or drafted by me over the summer, and continues with new results or generalizations. Notice that most of my summer work was not completed in a final form by the end of the program, so a significant portion of my thesis time has been spent proofreading, editing and tightening the summer results. In addition, there are parts of the thesis that are part of the joint work I am continuing from the REU whose first drafts were not written by me. However, I have edited and revised all of the material present, and I will note others' contributions accordingly.

Chapter 2

Cover Pebbling

2.1 Cover Pebbling

This section will describe various non-probabilistic results related to cover pebbling. First, I will briefly discuss the invention of cover pebbling, which appeared in (7). Then I will describe the current status of the cover pebbling problem by proving various results from the papers of (27), (19), and (26). It is interesting that although cover pebbling forces every vertex to have a pebble on it instead of just one, which is the requirement of pebbling, it is a simple question to solve. If fact, one corollary of (26) is the proof of a modified from of Graham's conjecture that relates to cover pebbling.

As a first simple example, consider the following theorem shown in (7):

Theorem 2.1. $\gamma(K_n) = 2n - 1$.

Proof. Suppose that two pebbles are placed on each vertex of K_n except for one. Although the non-pebbled vertex, call it v, can be reached by any of the pairs, the vertex whose pair of pebbles is used to cover v is now empty. Thus, we can never have all the vertices covered, and so $\gamma(G) \geq 2n - 1$.

Now suppose that there are at least $2n - 1$ pebbles on the vertices. Suppose that some vertex, say w, has no pebbles on it, else we are finished. In this case, by the pigeonhole principle there exists some vertex that contains a pair of pebbles. Take a pair of pebbles on this vertex and cover w. Now, removing w we have $2n - 3$ pebbles left for the $n - 1$ vertices remaining. Thus, we can apply induction to see that every vertex will be able to be covered. □

2.2 Trees and the "Stacking Theorem"

For all graphs with n vertices and arbitrary diameter, the complete graph has the smallest cover pebbling number and the path graph has the largest cover pebbling number. Another relevant question that may be asked is, given a graph with k vertices, and diameter d, what is the graph with the largest cover pebbling number? This graph is the fuse graph and we will state the definition given in (7).

Definition 2.2.1. *Suppose that $l \geq 2$ and $n \geq 3$. Define the fuse graph, denoted by $F_l(n)$ as follows. The vertices of $F_l(n)$ are $v_1, \ldots, v_n$, such that the first l vertices form a path from v_1 to v_l, and the remaining vertices are independent and adjacent only to v_l.*

In the paper, they also show that $\gamma(F_l(n)) = (n - l + 1)2^l - 1$. In (27), Watson and Yerger use this bound and a greedy algorithm to show a sharp upper bound for graphs of fixed diameter and n vertices.

In order to accurately describe the results of (7), we must define a few more relevant terms.

Definition 2.2.2. *(7) Let T be a tree and let $V(T)$ be the vertex set of T. For $v \in V(T)$, define*

$$s(v) = \sum_{u \in V(T)} 2^{d(u,v)},$$

where $d(u, v)$ denotes the distance from u to v, and let

$$s(T) = \max_{v \in V(T)} s(v).$$

In (7) the main result is that for any tree T, $\gamma(T) = s(T)$. In fact, this result extends to include trees with positive weight functions. That is, given some weight function $w(G)$, which is defined as a function that maps some nonnegative integer onto each vertex of G, call it $w(v)$, we can also determine the cover pebbling number of G with respect to the given weighting. The authors prove that $\gamma(T)$ with respect to some weight function w is equal to

$$\max_{v \in V(T)} s_w(v),$$

where

$$s_w(v) = \sum_{u \in V(T)} w(u) 2^{d(u,v)}.$$

The proof of this assertion is based on classifying vertices in three types: demand vertices, that need more pebbles, (that is, $w(v) - C(v) > 0$) neutral vertices, where $w(v) = C(v)$, and supply vertices, where $C(v) - w(v) > 0$. Then the authors perform a series of case analysis steps to find a way to place enough pebbles on the tree's vertices. Notice that this result is the first evidence leading towards the stacking theorem, namely that the initial configuration that requires the most number of pebbles occurs when all the pebbles are initially placed on one vertex.

The next few sections will derive results obtained by Nathaniel Watson and the author in (27). Most of this work was completed over the summer and was submitted to *Bulletin of the Institute for Combinatorics and its Applications* in March.

2.3 Complete Multipartite Graphs

The first class of graphs after trees that we tackled was complete multipartite graphs.

Definition 2.3.1. *For $s_1 \geq s_2 \geq \cdots \geq s_r$, let $K_{s_1,s_2,\ldots,s_r}$ be the complete r-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $c_1, c_2, \ldots, c_r$ respectively.*

Definition 2.3.2. *For a complete r-partite graph $G = K_{s_1,s_2,\ldots,s_r}$, let $\phi(G) = 4s_1 + 2s_2 + \cdots + 2s_r - 3$.*

Theorem 2.2. $\gamma(K_{s_1,s_2,\ldots,s_r}) = \phi(G)$.

Proof. First, we will show that not every configuration of size $\phi(K_{s_1,s_2,\cdots,s_r}) - 1$ on $K_{s_1,s_2,\cdots,s_r}$ is cover-solvable. Consider the case where all $\phi(K_{s_1,s_2,\cdots,s_r}) - 1$ pebbles are on one vertex of c_1, call it x. There are $k = s_2 + s_3 + \cdots + s_r$ vertices that are distance 1 from x and $l = s_1 - 1$ vertices that are distance 2 from x. For the k vertices a distance 1 from x, $2k$ pebbles are required to cover these vertices, and for the l vertices at distance 2 from x, there are $4l$ pebbles required to cover these vertices. We need one more pebble to remain on x, for a total of $2k + 4l + 1 = \phi(K_{s_1,s_2,\cdots,s_r})$ pebbles required, which is one more than we have. Thus, this configuration is not cover-solvable.

Now suppose that there exists some complete r-partite graph $K_{s_1,s_2,\ldots,s_r}$ which has a configuration of size $\gamma(K_{s_1,s_2,\ldots,s_r})$ that is not cover-solvable. Among such graphs, choose one (let it be $G' = K_{s'_1,s'_2,\ldots,s'_{r'}}$).

First, we will show that G' cannot be a star graph (that is, a $K_{s'_1,1}$.) In (7) it is shown that for any tree T, $\gamma(T) = s(T)$. Since G' is a tree, we can compute $\gamma(G')$ by evaluating $s(v)$ for all $v \in G$ to obtain $s(G)$. If $v \in c_1$ then

$s(v) = 4s_1' - 1$, and if $v \in c_2$ then $s(v) = 2s_1' + 1$. Thus, $s(G') = \gamma(G') = 4s_1' - 1 = 4s_1' + 2s_2' - 3 = \phi(G')$. Hence, for a star, every configuration of size $\gamma(G')$ is cover-solvable. Since G' is not a star, further suppose that for any G', each complete multipartite subgraph G of G' is cover-solvable with $\phi(G)$ pebbles.

Notice that for any complete p-partite graph with $p \geq 2$ other than a star graph, the removal of a vertex from the graph leaves a subgraph that is a complete q-partite graph with $q \geq 2$. Since G' cannot be a star, for any vertex $v \in G'$, $G' - v$ is a complete r^*-partite graph with $r^* \geq 2$. Furthermore, since by our assumption, for any complete r-partite graph G smaller than G', a configuration of size $\gamma(G)$ or greater must be cover-solvable, and since clearly $\gamma(G' - v) \leq \gamma(G') - 2$, any configuration of size $\gamma(G') - 2$ or greater on $G' - v$ is cover-solvable.

Let C be a configuration of size $\gamma(G')$ on G'. Suppose $C(v) = 1$ or 2 for some $v \in G'$. Then C restricted to $G' - v$ is a configuration of size at least $\gamma(G') - 2$ and thus is cover-solvable on $G' - v$. After we carry out the steps of the cover-solution of this subgraph, we will have cover-solved G', contradicting our hypothesis.

Otherwise, if $C(v) = 0$ or $C(v) \geq 3$ for all $v \in G'$, choose some v' for which $C(v') = 0$ (if no such v' exists, we are done). Then consider the vertices of G' which are in different vertex classes of G' from v'. If at least one of these is initially occupied, call it v''. Then since $C(v'') \geq 3$, we can cover v' with pebbles from v'', while leaving $\gamma(G') - 2$ pebbles on $G' - v'$. Thus, the configuration of pebbles on G' after this move, restricted to the subgraph $G' - v'$ is cover solvable, and after we carry out the steps of the cover-solution of this subgraph, we will have cover-solved G'. Otherwise, all the vertices in the vertex classes of G' that are different than the vertex class from v' are empty. Thus, all pebbles are on vertices in the vertex class of v', and in particular, some vertex w of this class has pebbles on it, so $C(w) \geq 2$. Thus, we can use pebbles on w to cover some vertex w' in another vertex class, as all these vertices are empty. Note that after this move, the configuration of pebbles on $G' - w'$ has size $\gamma(G') - 2$, and thus this configuration restricted to the subgraph $G' - w'$ is cover-solvable. Again, after we carry out the steps of the cover-solution of this subgraph, we will have cover-solved G'.

□

2.4 The Wheel Graph

In this section, we will compute $\gamma(W_n)$, where W_n is the wheel graph. The wheel graph is composed of a cycle consisting of n vertices, $v_1, \ldots, v_n$, which are all connected to a hub vertex, v_0, for a total of $v = n + 1$ vertices.

Theorem 2.3. *For $n \geq 3$, $\gamma(W_n) = 4n - 5 = 4v - 9$.*

Proof. Consider the configuration of pebbles where all the pebbles are on one vertex of W_n, say x, that is not the hub. In this case, 2 pebbles are required to cover each of the three vertices adjacent to x, and 4 pebbles are required to cover each of the $n - 3$ vertices that are a distance of 2 away from x. The total number of pebbles required to cover-solve these vertices is $4n - 6$. However, we require one more pebble to place on x. Hence, $\gamma(W_n) \geq 4n - 5$.

To complete the proof, we will show that if there is some configuration of pebbles on W_n with at least $4n - 5$ pebbles, then the configuration is cover-solvable. Suppose C is a configuration of pebbles on W_n and consists of at least $4n - 5$ pebbles. We now will describe a sequence of moves that will cover-solve any such configuration. First, if there are outer vertices on W_n that are empty but are adjacent to outer vertices w such that initially $C(w) \geq 3$, then if the adjacent vertices can be covered and w can also remained covered, then these adjacent vertices should be covered. Let k be the number of outer vertices that are covered after this process.

Case 1: Suppose that $k = 0$. In this case, all the pebbles are on the hub vertex. To cover-solve the remaining $v - 1$ vertices, we can cover $\lfloor \frac{4v-10}{2} \rfloor = 2v - 5$ vertices using the excess pebbles already on the hub vertex. Since $v \geq 4$ and $2v - 5 \geq v - 1$, we can cover-solve all of the outer vertices in this manner.

Case 2: Suppose that $k = 1$ or $k = 2$. Each outer vertex covered in the process above requires at most two pebbles to cover it. Since $v \geq 4$, there are at least $4v - 9 - 2k$ pebbles already on the hub vertex. After subtracting 1 pebble for the hub itself, there are $4v - 10 - 2k$ pebbles that can be used such that pebbles can be placed on the remaining $v - k - 1$ uncovered vertices. With these remaining pebbles on the hub, we can cover at least $\lfloor \frac{4v-10-2k}{2} \rfloor = 2v - 5 - k$ vertices. Since $2v - 5 - k \geq v - k - 1$ for $v \geq 4$, there are enough pebbles to cover-solve W_n in this situation.

Case 3: Suppose that $k \geq 3$. Again, each outer vertex in the process above requires at most two pebbles to cover it. If there are any pairs of pebbles remaining on outer vertices such that removing the pairs would not uncover that vertex, those pairs of pebbles should be moved to the

hub vertex. After this process, there are at least $\lceil \frac{4v-9-2k}{2} \rceil = 2v-4-k$ pebbles on the hub vertex. Notice that this bound is based on the worst case that occurs when no pebbles are initially on the hub vertex. From the hub vertex, it takes exactly 2 pebbles to cover each of the remaining outer vertices and one pebble to cover the hub vertex. So at most $\lfloor \frac{2v-5-k}{2} \rfloor = v-3-\lfloor \frac{k}{2} \rfloor$ outer vertices can be pebbled. Since there are at most $v-k-1$ outer vertices left to be pebbled, and for $k \geq 3$, $v-k-1 \geq v-3-\lfloor \frac{k}{2} \rfloor$, there are enough pebbles to cover-solve W_n in this case, and the proof is complete.
□

2.5 The Cover Pebbling Number of Graphs of Diameter d

In this section, we compute an upper bound for the cover pebbling number of graphs of a specified diameter. This was a step in the direction of the stacking theorem.

Definition 2.5.1. *A* binary weighting *on a graph G is a function from $V(G)$ to $\{0,1\}$. If B is a binary weighting on G, then let the order $|B|$ of B be $\sum_{v \in G} B(v)$.*

Definition 2.5.2. *For a graph G and binary weighting B, a configuration C on G will be called* permissible *(with respect to B) if for all $v \in G$, $B(v) = 0 \implies C(v) = 0$. A permissible configuration on a graph G with a binary weighting B will be called* cover-solvable *(with respect to B) if we can reach a configuration on which $B(v) = 1 \implies C(v) \geq 1$ for all $v \in G$ by a sequence of pebbling moves.*

Lemma 2.5.1. *Let G be a graph of diameter d, B a binary weighting on G, and C a configuration of size at least $(|B|-1)2^d + 1$ on G which is permissible with respect to B. Then C is cover-solvable with respect to B.*

Proof. Assume the opposite. Then for all pairs $\{G, B\}$ of a graph G together with a binary weighting on G such that there exists a non-cover-solvable configuration of size at least $(|B|-1)2^d + 1$ (where d is the diameter of G,) choose one for which $|B|$ is minimal, and call it $\{G', B'\}$. Let d' be the diameter of G', let $k = (|B'|-1)2^{d'} + 1$, and choose some configuration (call it C') on G' which is permissible with respect to B', has size at least k and is not cover-solvable.

Certainly we cannot have $|B'| = 1$, for then the only permissible configuration of size $|C'| \geq k = 1$ is the function which takes the value $|C'|$ on the lone vertex for which $B' = 1$, and is zero elsewhere. This configuration covers all vertices with non-zero weights, and so is trivially cover-solvable, creating a contradiction.

Now, suppose that $|B'| \geq 2$. If it is true that $C'(v) > 0$ whenever $B'(v) = 1$ we have a contradiction, for C' is then trivially cover-solvable. Otherwise, let v' be some vertex of G' for which $C'(v') = 0$ and $B'(v') = 1$. At most $|B'| - 1$ vertices are initially occupied, and there are at least $= (|B'| - 1)2^{d'} + 1$ total pebbles, so by the pigeonhole principle, there are at least $2^{d'} + 1$ pebbles on some vertex (call it v''). Since the diameter of G' is d', $d(v', v'') \leq d'$. Thus we can move $2^{d'}$ of the pebbles from v'' onto v', through a series of pebbling moves, losing half of these pebbles for each edge we must move across, but leaving at least one pebble on v' if we move all pebbles via one of the shortest paths.

Now, define a binary weighting B^* on G by

$$B^*(v) = \begin{cases} 0 & : \quad v = v' \\ B'(v) & : \quad v \neq v' \end{cases},$$

and define a configuration C^* on G by

$$C^*(v) = \begin{cases} 0 & : \quad v = v' \\ C'(v'') - 2^{d'} & : \quad v = v'' \\ C'(v) & : \quad \text{otherwise} \end{cases}.$$

This is the configuration after we have moved pebbles from v'' onto v', except that we ignore the pebbles on v' and designate it as a vertex which need not be covered by pebbles. Clearly $|B^*| = |B'| - 1$ and $|C^*| = |C'| - 2^{d'}$ so from $|C'| \geq ((|B'| - 1)2^{d'} + 1)$, we see $|C^*| \geq ((|B^*| - 1)2^{d'} + 1)$. C^* is permissible with respect to B^*, and so by our assumption of the minimality of B', C^* is cover-solvable with respect to B^*.

If we carry out the moves of the cover-solution of C^* on G starting with the configuration left on G' after our initial movement of pebbles from v'' to v', (certainly this is possible because this configuration is no smaller than C^* on any vertex,) we will have covered every vertex of G' for which $B^* = 1$. Also, we must still have $v' \geq 1$, because $C^*(v') = 0$, which does not permit any sequence of moves that decreases the number of pebbles on v'. Thus every vertex for which $B' = 1$ now has $C' \geq 1$, and we have cover-solved C' with respect to B', which contradicts the assumption that C' was not cover-solvable.

□

Theorem 2.4. *Let G be a graph of order n and diameter d, and let C be a configuration on G of size at least $2^d(n-d+1)-1$. Then G is cover-solvable (with respect to the weighting on G which is equal to 1 for each vertex.)*

Proof. First, notice that this bound is sharp because we can exhibit the following class of graphs where $\gamma(G) \geq 2^d(n-d+1)-1$. Suppose we have a graph consisting of n vertices and diameter d. Then we will construct a fuse graph (a path connected to a star) whose length is $d-1$ and has $n-d$ spokes at the end of the fuse. Here is an example for $n=7$ and $d=4$.

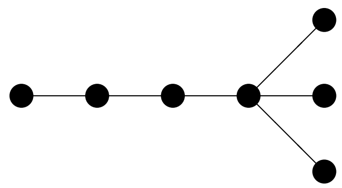

Figure 2.1: A graph where $n = 7$ and $d = 4$ such that $\gamma(G) = 2^d(n-d+1)-1$.

Suppose all the pebbles are on the last vertex of the path, which is at distance d from the spokes (in the figure, the leftmost vertex.) Then each of the $n-d$ spokes requires 2^d pebbles, and the path requires 2^d-1 vertices to cover-solve. (Note: In (7), the cover-pebbling number of all trees is found. Thus, we know for these particular trees that $\gamma(G) = 2^d(n-d+1)-1$ even before proving this theorem.)

We will prove this theorem by defining an algorithm by induction which will take us to a configuration, the solvability of which we can prove using the lemma. Let $R_0 = \{v \in G : C(v) > 0\}$, let $S_0 = \{v \in G : C(v) = 0\}$, and let $T_0 = \emptyset$. Let $C_0 = C$.

For illustrative purposes, we now describe the first step of the algorithm. If $S_0 = \emptyset$, we are clearly done, for C already covers G. Otherwise, note that since R_0 and S_0 are complementary, there exist vertices $r_0 \in R_0$ and $s_0 \in S_0$ such that $d(r_0, s_0) = 1$. If $C_0(r_0) = 1$ or $C_0(r_0) = 2$, then let $R_1 = R_0 \setminus \{r_0\}$, $S_1 = S_0$ and $T_1 = T_0 \cup \{r_0\} = \{r_0\}$. In this case, let $C_1 = C_0$.

If on the other hand $C_0(r_0) \geq 3$ then we move 2 pebbles from r_0 to s_0, and instead put s_0 in T_1 and define C_1 according to the following configuration. Explicitly, in this case let $R_1 = R_0$, $S_1 = S_0 \setminus \{s_0\}$, and $T_1 =$

$T_0 \cup \{s_0\} = \{s_0\}$. Define C_1 on G by

$$C_1(v) = \begin{cases} r_0 - 2 & : \quad v = r_0 \\ 1 & : \quad v = s_0 \\ C_0(v) & : \quad \text{otherwise} \end{cases} .$$

Define the sequences $R_0, R_1, \ldots, R_{d-1}$, $S_0, S_1, \ldots, S_{d-1}$, $T_0, T_1, \ldots, T_{d-1}$, and $C_0, C_1, \ldots, C_{d-1}$, recursively in an analogous manner. Suppose for some $m < d-1$ we have R_m, S_m, T_m, and C_m, such that the following hold:

1. $|T_m| = m$.
2. R_m, S_m and T_m are disjoint and $R_m \cup S_m \cup T_m = V(G)$.
3. For all $v \in R_m \cup T_n$, $C_m(v) > 0$ and for all $v \in S_m$, $C_m(v) = 0$.
4. C_m is a configuration which can be reached from C by a sequence of pebbling moves.
5. R_m and S_m are both non-empty.
6. $\sum_{v \in R_m} C_m(v) \geq [2^d(n-d+1) - 1] - [2^{m+1} - 2]$.

Note that all these conditions are trivially true for $m = 0$.

From condition 1, it is evident that the minimum distance between R_m and S_m is at most $m+1$. Take points $r_m \in R_m$ and $s_m \in S_m$ for which this minimum distance is achieved (and thus $d(r_m, s_m) \leq m+1$.) If $C_m(r_m) \leq 2^{m+1}$ then let $R_{m+1} = R_m \setminus \{r_m\}$, $S_{m+1} = S_m$ and $T_{m+1} = T_m \cup \{r_m\}$. In this case, let $C_{m+1} = C_m$.

Otherwise, if $C_m(r_m) > 2^{m+1}$ then we can move 2^{m+1} pebbles along a minimal path from r_m to s_m, which is of length at most $m+1$. We lose half of these pebbles for each edge we must move across, but we will be able to move $2^{(m+1)-d(r_m,s_m)} \geq 1$ onto s_m. Put s_m in T_{m+1} and define C_{m+1} according to the configuration after these moves. Explicitly, in this case let $R_{m+1} = R_m$, $S_{m+1} = S_m \setminus \{s_m\}$ and $T_{m+1} = T_m \cup \{s_m\}$. Define C_{m+1} on G by

$$C_{m+1}(v) = \begin{cases} r_m - 2^{m+1} & : \quad v = r_m \\ 2^{(m+1)-d(r_m,s_m)} & : \quad v = s_m \\ C_m(v) & : \quad \text{otherwise} \end{cases} .$$

For $m+1$, it is clear from our definitions that conditions 1, 2, 3, and 4 still hold. Condition 6 also holds, for in either of the two above cases, the total number of pebbles left on R_{m+1} is at most 2^{m+1} less than were on R_m. Thus,

$$\begin{aligned}\sum_{v\in R_{m+1}} C_{m+1}(v) &\geq \sum_{v\in R_m} C_m(v) - 2^{m+1} \\ &\geq [2^d(n-d+1)-1]-[2^{m+1}-2]-2^{m+1} \\ &= [2^d(n-d+1)-1]-[2^{m+2}-2].\end{aligned}$$

For condition 5, since always $m+1 < d$ and $n \geq d$, $[2^d(n-d+1)-1]-[2^{m+1}-2] > 0$. Thus, the fact that condition 6 is true for $m+1$ necessitates that $R_m \neq \emptyset$. Also, if $S_{m+1} = \emptyset$ then $C_{m+1}(v) > 0$ for all $v \in R_m \cup S_m \cup T_m = V(G)$, and since C_m is attainable from C by a sequence of pebbling moves, we have cover-solved C and we are done. So we may assume $S_{m+1} \neq \emptyset$ and condition 5 holds.

By this recursive definition, we now have R_{d-1}, S_{d-1}, T_{d-1}, and C_{d-1} for which conditions 1-6 hold. Now define a binary weighting B on G by

$$B(v) = \begin{cases} 1 & : \ v \in R_{d-1} \cup S_{d-1} \\ 0 & : \ v \in T_{d-1} \end{cases}.$$

Also, define C'_{d-1} on G by

$$C'_{d-1}(v) = \begin{cases} C_{d-1}(v) & : \ v \in R_{d-1} \cup S_{d-1} \\ 0 & : \ v \in T_{d-1} \end{cases}.$$

Clearly C'_{d-1} is permissible with respect to B. From condition 1, $|T_{d-1}| = d-1$ so $|B| = n-d+1$, and from condition 6 we have $|C'_{d-1}| \geq [2^d(n-d+1)-1]-[2^{(d-1)+1}-2] = 2^d(n-d)+1$. Thus, by the lemma, C'_{d-1} is cover-solvable with respect to B.

By condition 4, C_{d-1} is a configuration which can be reached from C by a sequence of pebbling moves. If after we carry out this sequence of moves, we carry out the moves of this cover-solution of C'_{d-1} on G (certainly this is possible because C'_{d-1} is no greater than C_{d-1} on any vertex,) we will have covered every vertex of G for which $B = 1$, that is every vertex in $R_{d-1} \cup S_{d-1}$. Also, every vertex $v \in T_{d-1}$ must remain covered, because for each of these vertices, $C'_{d-1}(v) = 0$, which does not permit any sequence of moves which decreases the number of pebbles on v. Applying, condition 2, we see for every vertex $v \in V(G) = R_{d-1} \cup S_{d-1} \cup T_{d-1}$, our final configuration after this sequence of moves is greater than zero, and so we have cover-solved C.

□

2.6 Hypercube Cover Pebbling Number

Recall that the first nontrivial family of graphs whose pebbling numbers were obtained was the Cartesian products of paths. In the same paper, Fan Chung Graham proves that the pebbling number of a hypercube in d dimensions is 2^d. In a recent paper, Hurlbert and Munyan (19)prove a corresponding result for cover pebbling the hypercube. They show that the number of pebbles required for this situation is 3^d. The proof utilizes induction on the size of the *support* of G, which is defined as the set of vertices that initially have pebbles on them.

Another property that was initially mentioned in (7) is the *cover pebbling ratio*, which is defined as $\gamma(G)/\pi(G)$. This ratio can be as small as 2 for complete graphs, or as large as $\frac{n}{\lg n}$ for fuses. For the d-cube, the cover pebbling ratio is $d^{\lg(3)-1}$.

2.7 The Stacking Theorem

In 2004, two independent efforts produced a proof of the stacking theorem, namely Vuong and Wyckoff (26) and Sjostrand (24). One of the main ideas in both papers is determining some process to distribute the pebbles in an optimal way. In this light, Sjostrand (24) defines a node to be *fat* if the number of pebbles on vertex v is greater than $w(v)$. Analogous definitions hold for *thin* and *perfect*. Another new definition is the *value* of a pebble, which is the number of pebbles that have gone into making it. To cover the entire graph Sjostrand uses the invariant that every pebble has a value no greater than the cost from its nearest fat node. Sjostrand then pebbles the graph in a way to preserve this invariant, specifically that among all pairs (f, t) of fat and thin vertices, take one that minimizes the distance $d(f, t)$.

2.8 Complexity of the Cover Pebbling Problem

(This section was joint work with Nathaniel Watson.) The normal way that mathematicians and computer scientists formalize the concept of the difficulty of a problem is to describe the problem in terms of *computational complexity*. Formally, we imagine a *decision problem* to be a set of infinite strings of characters (like data represented by bits in a computer.) A decision problem is said to *accept* a string if this set contains the string. Usually, we look for the best possible asymptotic upper bound (in terms of the length of the string) for the number of steps the fastest possible algorithm takes to determine whether a given string is in the set. Informally, we think of decision problems being yes-no questions about a property of some class of finite mathematical structures (graphs, integer matrices, etc.) and we ask how fast it is possible to correctly determine the yes or no answer in terms of the size of the input.

For instance, some problems can be solved by an algorithm which takes only a number of steps which is bounded by a polynomial in the size of the input, while others take at least an exponential amount of time to solve. The former class of decision problems is called P for "polynomial." The class *NP*, for "nondeterministic polynomial" is a bit more complicated; roughly speaking it is the set of decision problems for which a "yes" answer can be "checked" in polynomial time, given an appropriate piece of information. That is, if we call the class of inputs to the decision problem X, and the class of inputs which the decision problem accepts X', there exists a class Y of objects (called the *certificates*) and a function $A : X \times Y \to \{0,1\}$ which is computable in polynomial time, such that for any instance $x \in X$ of the decision problem, there exists a $y \in Y$ such that $A(x,y) = 1$ if and only if $x \in X'$. For instance, the decision problem which asks whether a given number is composite is easily seen to be in *NP*, because the composite numbers are exactly those with nontrivial divisor, and, given two numbers, it is easy to determine by division whether one is a divisor of the other. Also, any problem in P is also in *NP*, because any polynomial-time method of solving a problem is trivially also a polynomial-time method of verifying a yes answer. However, it is a celebrated open problem if the converse also holds and $P = NP$.

Within *NP*, there is a class of problems, called the *NP-complete* problems, which is thought of being a set of problems that are at least as hard as any other problem in *NP*. This is because any instance x of a problem D in *NP* can be translated by a polynomial-time algorithm to an instance x' of any *NP*-complete problem D' such that x' is accepted by D' if and only if x is accepted by D. Therefore, if we could solve any *NP*-complete problem in polynomial time, we could solve any problem in *NP* in polynomial time by translating it to an instance of this problem. Thus, the question of whether $P = NP$ reduces to the question of whether any particular *NP*-complete problem can be solved in polynomial time.

We now show that the problem which asks if a configuration of pebbles on a graph is cover solvable is *NP*-complete. It is worth noting that most complexity theorists speculate that $P \neq NP$, and therefore, when a problem is classified as *NP*-complete, it is usually thought of as evidence of its difficulty. See (13) for a comprehensive theory of *NP*-completeness.

Theorem 2.5. *Let G be a graph, and C a configuration on G. Let $|G| = m$ and label the vertices of G as $v_1, v_2, \ldots v_m$. Then C is cover solvable if and only if there exist integers $n_{ij} \geq 0$ with $1 \leq i, j \leq m$ and $n_{ij} = 0$ and $n_{ji} = 0$ whenever $\{v_i, v_j\} \notin E(G)$ such that for all $1 \leq k \leq m$,*

$$C(v_k) + \sum_{l=1}^{m} n_{lk} - 2\sum_{l=1}^{m} n_{kl} \geq 1.$$

Proof First, suppose C is cover solvable. Then we can find some sequence of pebbling moves which cover solves C. Let n_{ij} be the total number of pebbling moves from v_i to v_j in this sequence. Then after all the moves, there are exactly

$$C(v_k) + \sum_{l=1}^{m} n_{lk} - 2\sum_{l=1}^{m} n_{kl}$$

pebbles left on v_k, which is always at least 1 because of the fact that this sequence of moves cover solves C.

Conversely, suppose such numbers n_{ij} exist. This means that there *does* exist a sequence of moves that solves C, with n_{ij} moves being made from v_i to v_j, but possibly with some illegal "negative pebbling" along the way. That is, we could be removing pebbles from vertices that already have zero pebbles. We show, however, that for each i, j it is possible to *legally* make n_{ij} moves from v_i to v_j; since for each k,

$$C(v_k) + \sum_{l=1}^{m} n_{lk} - 2\sum_{l=1}^{m} n_{kl} \geq 1,$$

this leads to a cover solution of C. Thus, the main question is: In what order do we make these moves? We proceed in any arbitrary fashion, continuing to make pebbling moves as long as there exist vertices $v_{i'}$ and $v_{j'}$ such that less than $n_{i'j'}$ moves from $v_{i'}$ to $v_{j'}$ have already been made and there are at least two pebbles on $v_{i'}$. If no such pair $\{v_{i'}, v_{j'}\}$ exists, then for each (i, j), either n_{ij} moves have been made from i to j or else there is at most 1 pebble on vertex v_i. Let C' be the configuration left on G after these moves and S be the set of $v_i \in G$ for which the total number of moves from v_i is less than $\sum_{l=1}^{m} n_{il}$.

If $S = \emptyset$ then clearly for every $1 \leq i, j \leq m$ we have made n_{ij} moves from v_i to v_j and thus, for every k there are $C(v_k) + \sum_{l=1}^{m} n_{lk} - 2\sum_{l=1}^{m} n_{kl} \geq 1$ pebbles on v_k, so $C'(v) \geq 1$ for all $v \in G$ and we have solved C.

If $S \neq \emptyset$ then consider the total number of moves that remain to be made from a vertex of S (the total $\sum_{i \in S} \sum_{l=1}^{m} n_{il}$ minus the number of moves that have already been made from vertices in S.) By the definition of S, this total is at least $|S|$, since at least one move remains to be made from every vertex in S. The total is exactly $|S|$ only if exactly one move remains to be made from each vertex. Also, $C'(v) \leq 1$ for all $v \in S$, for a total of at most $|S|$ pebbles. Consider the remaining moves, each of which must originate from S. Each of these moves, if executed, would remove one pebble from S if they also end at a vertex in S, and two if they end at a vertex outside of S. Thus each move must both begin and end in S, for otherwise S would be left with a negative total number of pebbles at the *end* of the pebbling sequence, which is impossible. Even if all moves begin and end in S, however, we end up with at most 0 pebbles on S – which too is impossible since we have assumed that the moves cover-solve, and so there must be one pebble on each vertex of S. Thus we must have $S = \emptyset$ and we have solved C. □

Corollary 2.8.1. *The cover solvability decision problem which accepts pairs $\{G, C\}$ if and only if G is a graph and C is a configuration which is cover solvable on G is in NP.*

Proof The above theorem gives the appropriate certificate of cover solvability, any list of integers n_{ij} which satisfy the equation in Theorem 2.5. Indeed, Theorem 2.5 shows that cover-pebbling is equivalent to a special case of the *NP*-complete problem of integer programming, which asks, given an $n \times m$ integer matrix A and an n-dimensional integer vector b if there exists an m-dimensional integer vector x such that $Ax \geq b$, holds componentwise. Having reduced cover solvability to a special case of this *NP* problem, we know that cover solvability is also in *NP*.

We now pause to point out another corollary which will be needed later but is interesting on its own:

Corollary 2.8.2. *Let G be a graph, C a configuration on G. If the sequence of pebbling moves $Q = (q_1, q_2 \dots q_k)$ solves C, and it is possible to make the sequence $Q' = (q_{i_1}, q_{i_2}, \dots q_{i_l})$ of moves (with $1 \leq i_j \leq k$ for all j but with no particular requirement on the order of the i_j,) then the configuration C' obtained from C after the moves $(q_{i_1}, q_{i_2}, \dots q_{i_l})$ is solvable.*

Proof The point of this corollary is that the order of our pebbling moves can't matter. To show this, we simply note that if it were possible to somehow execute the remaining moves from Q which are not in Q', they would solve C'. By Theorem 2.5, it is thus possible to solve C'. □

Now we turn our attention to showing that the cover solvability decision problem is *NP*-hard, that is, that any instance of any problem in *NP* can be translated to an instance of cover solvability in polynomial time. The usual method of showing that a problem A is *NP*-hard is to find an *NP*-complete problem B for which any instance of B can be translated into an instance of A in polynomial time. Then for any instance of any problem in *NP* we can translate it in polynomial time to an instance of B, then translate this instance into an instance of A. For cover solvability, we will use a known *NP*-complete problem known as "exact cover by 4-sets." Indeed, the corresponding and seemingly simpler problem of exact cover by 3-sets is also *NP*-complete, but for our purposes, the 4-set problem is more useful. We state a theorem of Karp that tells us the 4-set problem is NP complete.

Theorem 2.6. *(20) Let the exact cover by 4-sets problem be the decision problem which takes as input a set S with $4n$ elements and a class A of at least n 4-element subsets of S, accepting such a pair if there exists an $A' \subseteq A$ such that A' is a class of disjoint subsets which make a partition of S, that is they are n subsets containing every element of S. This problem is NP-complete.*

Theorem 2.7. *The cover solvability decision problem is NP-complete*

Proof Having shown that this decision problem is in *NP*, it remains to be shown that it is *NP*-hard. Given a set $S = \{s_1, s_2, \ldots s_{4n}\}$ and a class $A = \{a_1, a_2, \ldots a_m\}$ of four-element subsets of S, that is, an instance of the exact cover by 4-sets problem, we construct a graph G' and a configuration C' on G' in the following manner: We create a set of vertices $T = \{t_1, t_2, \ldots t_{4n}\}$ which will be thought of as corresponding to the elements of S, and a set of vertices $B = \{b_1, b_2, \ldots b_m\}$ which will be thought of as corresponding to the members of A. Let $C'(t) = 0$ for all $t \in T$ and let $C'(b) = 9$ for all $b \in B$. We make edges between B and T in the intuitive way, including $\{b_i, t_j\}$ if $t_j \in b_i$. Additionally, create a vertex v and a path of length $m - n$ which has one terminal vertex v and the other called w. Let $C'(v) = 2^{m-n} - (m-n) + 1$, $C'(w) = 0$ and $C'(u) = 1$ for all u between v and w on the path. Finally, create vertex classes $B' = \{b'_1, b'_2 \ldots b'_m\}$ and $B'' = \{b''_1, b''_2 \ldots b''_m\}$, creating edges $\{b_i, b'_i\}$, $\{b'_i, b''_i\}$ and $\{b''_i, v\}$ for all i. Let $C'(u) = 1$ for all $u \in B' \cup B''$. (Figure 2.2.)

Clearly, this construction can be made in polynomial time in the size of the pair $\{S, A\}$. Indeed, an upper bound for creating the graph is $8|A|$ vertices and $8|A| - 1$ edges. In order to finish the proof, we claim that C' is solvable if and only if A contains a perfect cover of S.

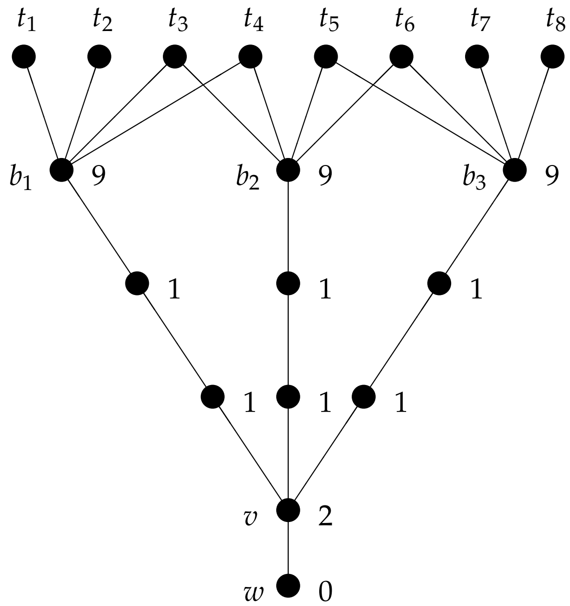

Figure 2.2: A graph that corresponds to the exact cover by four 4-sets problem, $a_1 = \{s_1, s_2, s_3, s_4\}$, $a_2 = \{s_3, s_4, s_5, s_6\}$, $a_3 = \{s_5, s_6, s_7, s_8\}$.

First suppose that A contains a perfect cover $A' = \{a_{i_1}, a_{i_2}, \ldots a_{i_n}\}$ of S. Then for each vertex in B which is a b_{i_j} for some $1 \leq j \leq n$, we use 8 of the pebbles on this b_{i_j}, two each to cover the four vertices of T to which it is adjacent. Because of the fact that A' is a perfect cover and the way we constructed G', we now have exactly one pebble on every vertex of T. Furthermore, we have $m - n$ vertices in B which still have 9 pebbles each on them. Because v is at distance v from each of these vertices, we can use 8 pebbles from each of these vertices to move one pebble each onto v from these $m - n$ vertices. This leaves $2^{m-n} + 1$ pebbles on v, which is enough to move one pebble onto w while leaving one pebble on v. After this is done, we have exactly one pebble on every vertex of G', and we thereby know that C' is solvable.

To show the converse, suppose that A does not contain a perfect cover of S. Suppose as well that C' is solvable on G'. Clearly, the sequence of pebbling moves which solves C' must contain (at least) one move to t for every $t \in T$. Clearly, each of these moves must originate from B, and no more than 4 can originate from any one vertex of B. Since A does not contain a perfect cover of S, it cannot be the case that these moves originate from exactly n vertices in B.

We make these $4n$ moves immediately from C', using Corollary 2.8.2 to see that the resulting configuration must be solvable (we use the fact that no more than 4 of these moves can originate from any one vertex in B to see that it is indeed possible to make these moves). In addition to the one pebble left on every vertex of B to ensure they remain covered, there are now $8(m-n)$ pebbles on B, but they are not in $m-n$ groups of 8 pebbles because the moves we made originated from more than n vertices of B. In order to reach w, we clearly need to move $m-n$ pebbles onto v while leaving the rest of the graph covered. Clearly, this is only possible if all $8(m-n)$ extra pebbles are moved by a path of length 3 onto v. Clearly, only one such path is available, but any group of less than 8 pebbles cannot increase the number of pebbles on v by moving along these paths while leaving the vertices of the path covered. By the fact that there are not indeed $m-n$ groups of 8 pebbles on B we see that it is impossible to gather the pebbles necessary to reach w and thus our configuration is not solvable, which is a contradiction.

Chapter 3

Domination Cover Pebbling

3.1 Domination Cover Pebbling

Professor Teresa Haynes, an East Tennessee State University professor, has written around fifty papers in the theory of graph domination. A significant percentage of graduate students at ETSU are collaborators with her and write masters theses about domination in graphs. Therefore, it was no surprise that Alberto Teguia, an ETSU graduate student who sat in on some of the REU discussions, proposed the concept of *domination cover pebbling.*

Recall that a set of vertices D in G is a *dominating set* if every vertex in G is either in D or adjacent to some element in D. The domination cover pebbling number of a graph G is the minimum number of pebbles required so that any initial configuration of pebbles can be transformed by a sequence of pebbling moves so that after the pebbling moves, the set of vertices that contain pebbles form a dominating set S of G. As stated in the introduction, the reason why domination cover pebbling is nontrivial is that the optimal choice of covered vertices may not always be the same.

Therefore Sjostrand's pebbling distribution idea for cover pebbling does not help us solve the domination cover pebbling problem because if the pebbles are initially distributed in different places, a different dominating set may be produced. For instance, take a path on four vertices arranged as a horizontal line and place pebbles only on the far left vertex. Now, from left to right, the domination cover would place a pebble on vertices 1 and 3. If the pebbles were initially on the far right vertex, then the domination cover would include vertices 2 and 4. This is one reason why the domination cover problem is difficult.

3.2 Preliminary results

We begin by determining the domination cover pebbling number for various types of graphs.

Theorem 3.1. *For the complete graph on n vertices, denoted by K_n, $\psi(K_n) = 1$.*

This result is obvious since placing a pebble on any vertex dominates K_n.

Theorem 3.2. *For $s_1 \geq s_2 \geq \cdots \geq s_r$, let $K_{c_1,c_2,\ldots,c_r}$ be the complete r-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $c_1, c_2, \ldots, c_r$ respectively. Then, for $s_1 \geq 3$, $\psi(K_{c_1,c_2,\ldots,c_r}) = s_1$.*

Proof. First, if there is one pebble each on every vertex of c_1 but one, the configuration does not produce a domination cover pebbling. So $\psi(K_{c_1,c_2,\ldots,c_r}) > s_1 - 1$. Notice that if there are vertices on two different $c_i's$, the graph contains a domination cover pebbling. Thus, any pair of pebbles on a vertex along with another pebbled vertex can force a domination cover pebbling. So if there are s_1 pebbles, the only configuration that has not been considered that does not force a domination cover pebbling is for there to be one pebble on every vertex in a vertex class that contains s_i vertices, but this also forces a domination cover pebbling. Hence, $\psi(K_{c_1,c_2,\ldots,c_r}) = s_1$. □

For the next theorem we define the wheel graph, denoted W_n, to be the graph with $V(W_n) = h, v_1, v_2, \ldots, v_n$, where h is called the hub of W_n, and $E(W_n) = C_n \cup \{hv_1, hv_2, \ldots, hv_n\}$. In this case C_n denotes the cycle graph on n vertices.

Theorem 3.3. *For $n \geq 3$, $\psi(W_n) = n - 2$.*

Proof. First, $\psi(W_n) > n - 3$ because placing one pebble on each of $n - 3$ consecutive outer vertices leaves a vertex of W_n undominated. If there is a pair of pebbles on any vertex, move it to the center, and the domination is complete. Likewise, if there is a pebble in the hub vertex, W_n is dominated. Thus, consider all configurations containing pebbled vertices that each contain only one pebble. If there are $n - 2$ vertices containing pebbles, the two non-pebbled outer vertices are forced to be dominated since there are only 3 vertices in all of W_n that contain no pebbles. Therefore, $\psi(W_n) = n - 2$. □

3.3 Domination cover pebbling for Paths

(This section was originally drafted by James Gardner and revised by the author.) We now show $\psi(P_n)$, where P_n denotes a path on n vertices.

Theorem 3.4.

$$\psi(P_n) = 2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor,$$

for $n \geq 3$ and where $n-2 \equiv \alpha_n \pmod 3$ and $n-2 = \alpha_n + 3k_n$.

Proof. Let $V = V(P_n) = \{v_1, v_2, \ldots, v_n\}$ with $E(P_n) = \{v_1v_2, \ldots v_{n-1}v_n\}$. Consider the configuration where all pebbles are placed on v_1. We need at least 2^{n-2} pebbles to dominate v_n. Likewise, we need at least $2^{n-2} + 2^{n-5} + 2^{n-8} + \cdots + 2^{\alpha_n}$ to dominate $\{v_n, v_{n-1}, \cdots, v_{\alpha_{n+1}}\}$. If $\alpha_n = 0$ or 1, then we have already dominated P_n. Otherwise, $\alpha_n = 2$ and we need one more pebble on either v_1 or v_2 to dominate P_n. Thus, under this configuration,

$$\begin{aligned}\psi(P_n) &\geq 2^{n-2}\sum_{i=0}^{k_n}\frac{1}{8^i} + \left\lfloor \frac{\alpha_n}{2} \right\rfloor \\ &= 2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor\end{aligned}$$

(since $\left\lfloor \frac{\alpha_n}{2} \right\rfloor = 0$ for $\alpha_n = 0$ or 1 and 1 for $\alpha_n = 2$).

We now use induction to show that

$$\psi(P_n) \leq 2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor.$$

The assertion is clear for $n = 3$. Therefore, we assume it is true for all P_m, where $3 \leq m \leq n$. Consider an arbitrary configuration of P_n having $2^{n+1}((1-8^{-(k_n+1)})/7) + \lfloor \alpha_n/2 \rfloor$ pebbles. Clearly, we can cover dominate $\{v_{n-2}, v_{n-1}, v_n\}$ in a finite number of moves with 2^{n-2} pebbles or less. Thus, we need to dominate P_{n-3} with the remaining

$$\begin{aligned}2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor - 2^{n-2} &= 2^{(n-3)+1}\left(\frac{1-8^{-(k_{n-3}+1)}}{7}\right) \\ &\quad + \left\lfloor \frac{\alpha_{n-3}}{2} \right\rfloor\end{aligned}$$

pebbles, since $\forall n, k_n = k_{n-3} + 1$ and $\alpha_n = \alpha_{n-3}$. This number of pebbles is enough to dominate P_{n-3} by hypothesis. Thus,

$$2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right)+\left\lfloor\frac{\alpha_n}{2}\right\rfloor \le \psi(P_n) \le 2^{n+1}\left(\frac{1-8^{-(k_n+1)}}{7}\right)+\left\lfloor\frac{\alpha_n}{2}\right\rfloor.$$

□

3.4 Domination cover pebbling for Cycles

(This section was originally drafted by James Gardner and Alberto Teguia but revised by the author.) To determine the value of $\psi(C_n)$ the cycle on n vertices, we begin by proving that placing the pebbles on one vertex gives the largest domination cover pebbling number.

Lemma 3.4.1. *The maximum value of $\psi(C_n)$ is obtained by placing all the pebbles on one vertex.*

Proof. Let $P = P_1, P_2, \ldots$ be the set of the sets of vertices with pebbles in the original configuration, with each $P_i = v_1, v_2, \ldots$ consisting of adjacent pebbled vertices and each P_i not adjacent to any other P_k. The minimum distance between any $P_1, P_i \in P$ is 2. Assume the maximum number of pebbles necessary to cover dominate the graph is ϕ pebbles. We can dominate the graph with ϕ pebbles in the disconnected configuration and we assume this is the worst case. If we would have placed all the pebbles on P_1, we would have required more pebbles to reach P_i, leaving a contradiction. Thus $|P| = 1$.

Now let P consist of the vertices that have pebbles on them. Since P is connected we now consider when $|P| > 2$. We let $P = \{p_1, p_2, \ldots, p_k\}$, where p_1 and p_k are the exterior vertices of P. We assume we can cover dominate C_n with ϕ pebbles and assume this is maximum necessary. We conclude that after the graph has been cover dominated there are either (1) adjacent vertices with pebbles, or (2) vertices with more than two pebbles. If there are not any vertices satisfying (1) or (2), then we could have increased ϕ by moving the pebbles on $p_1, p_k \in P$ to either p_2 or p_{k-1}, respectively. Thus either (1) or (2) occurs, but these pebbles are unnecessary. We could have cover-dominated with less than ϕ pebbles. Now consider when $|P| = 2$. Clearly the worst case is placing $\phi - 1$ pebbles on p_1 and 1 on p_2, where $p_1, p_2 \in P$, since it cost more pebbles to reach the p_2 side of the cycle. We can still dominate C_n with ϕ pebbles. We would have needed at least 2 more pebbles in order to dominate the vertex v_a on the v_2 side of the cycle, where $d(v_1, v_a) = 2$. Thus, ϕ is not the maximum number of pebbles necessary, raising a contradiction. The statement follows, since $|P| = 1$ is the worst case. □

Since placing all the pebbles on a single vertex is the worst case, we now determine the value of $\psi(C_n)$.

Theorem 3.5. *Let C_n be a cycle on n vertices. If $n = 2k+1$,*

$$\psi(C_n) = 2^{m+2}\left(\frac{1-8^{-(k_m+1)}}{7}\right) + \phi_1(m)$$

and if $n = 2k$,

$$\psi(C_n) = 2^{m+1}\left(\frac{1-8^{-(k_m+1)}}{7}\right) + 2^m\left(\frac{1-8^{-(k_{m-1}+1)}}{7}\right) + \phi_2(m),$$

where $\phi_1(m) = \lfloor \alpha_m/2 \rfloor - |\alpha_m - 1|$, $\phi_2(m) = \lfloor \alpha_m/2 \rfloor + \lfloor \alpha_{m-1}/2 \rfloor - |\alpha_m - 1||\alpha_{m-1} - 1|$, $m-2 \equiv \alpha_m \pmod 3$, and $m - 2 = \alpha_m + 3k_m$.

Proof. By Lemma 3.4.1 we assume all $\psi(C_n)$ pebbles are on $v_1 \in C_n$. If $n = 2k+1$, there are two identical paths to cover. We can cover these with $2\psi(P_m)$ pebbles. We notice that we may have dominated vertex v_1 twice. $|\alpha_m - 1| = 1$ if v_1 is dominated twice. If $n = 2k$, there are two paths $P_1, P_2 \in C_n$ with $|P_2| = |P_1| - 1$. Thus we can cover these two paths with $\psi(P_m) + \psi(P_{m-1})$. Likewise in this case, we may have dominated vertex v_1 twice. $|\alpha_m - 1||\alpha_{m-1} - 1| = 1$ if v_1 is dominated twice. Thus we compute the domination cover pebbling number as follows. When $n = 2k+1$

$$\begin{aligned}\psi(C_n) &= 2\psi(P_m) - |\alpha_m - 1| \\ &= 2^{m+1}((1-8^{-(k_m+1)})/7) + \lfloor \alpha_m/2 \rfloor - |\alpha_m - 1|\end{aligned}$$

and if $n = 2k$,

$$\begin{aligned}\psi(C_n) &= \psi(P_m) + \psi(P_{m-1}) - |\alpha_m - 1||\alpha_{m-1} - 1| \\ &= 2^{m+1}((1-8^{-(k_m+1)})/7) + 2^m((1-8^{-(k_{m-1}+1)})/7) \\ &\quad + \lfloor \alpha_m/2 \rfloor + \lfloor \alpha_{m-1}/2 \rfloor - |\alpha_m - 1||\alpha_{m-1} - 1|.\end{aligned}$$

The theorem follows. □

3.5 Binary Trees

In this section, we will compute the domination cover pebbling number for the family of complete binary trees. Since the first few trees can be dominated more efficiently than the general case, we will consider them separately.

Definition 3.5.1. *A complete binary tree, denoted by B_n is a binary tree that is of height n, with 2^i vertices a distance i from the root. Each vertex of B_n has two children, except for the set of 2^n vertices that are distance n away from the root, each of which have no children.*

Theorem 3.6. *For B_n described above, $\psi(B_0) = 1$, $\psi(B_1) = 2$, $\psi(B_2) = 11$, and $\psi(B_3) = 81$.*

Proof. Clearly, $\psi(B_0) = 1$. Since B_1 is just a path on 3 vertices, $\psi(B_1) = 2$. Now, the general formula for B_n, which we will state later in this section, predicts $\psi(B_2) = 12$, but actually, $\psi(B_2) = 11$. In the figure below, we describe a configuration of 10 pebbles on B_2 that does not force a domination cover solution.

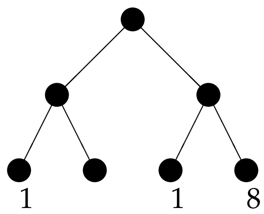

Figure 3.1: A configuration of 10 pebbles on B_2 that does not force a domination cover solution.

We will now show that $\psi(B_2) \leq 11$. Given a B_2, arbitrarily place 11 pebbles on it. Consider the following three subcases based on the number of pebbles on each of the two B_1's connected to the root of B_2.

Case 1: Suppose there are at least two pebbles on each of the two B_1's. It takes at most two pebbles for each of the B_1's to be dominated. Hence, after dominating each of the B_1's there are seven pebbles left. If there is a pebble on either of the two root vertices of the two disjoint copies of B_1, then we have dominated the root of B_2. Otherwise, move a pebble to the root of one of the B_1's, thus dominating the root vertex of B_2. This process induces a domination cover solution of B_2, completing this case.

Case 2: Suppose that neither B_1 contains two or more pebbles. Then there are at least 9 pebbles on the root of B_2. Pebble the root of each of the B_1's, and this case is complete.

Case 3: Suppose that one copy of B_1 contains two or more pebbles, call it B_1^*, and the other copy does not. Then all of the pebbles on B_1^* except for two can be used to move pebbles to the root of B_2. The 8 pebbles that can be moved from B_1^* are sufficient to get a pair of pebbles on the root. This pair can then dominate the other B_1, and the case is complete.

We next make a similar argument for B_3, because the distance between the top and the bottom of the tree is not large enough for the general method to apply. We now show that $\psi(B_3) = 81$. First, we have constructed below a configuration of 80 pebbles that does not produce a domination cover pebbling.

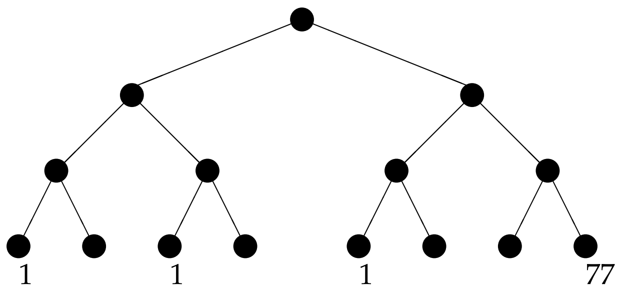

Figure 3.2: A configuration of 80 pebbles on B_3 that does not force a domination cover-solution.

Now suppose that we are given a configuration of 81 pebbles on B_3. We wish to force a domination cover pebbling on B_3. If there are fewer than 11 pebbles on each of the two disjoint B_2 subtrees in B_3, then we can use the 61 pebbles on the root vertex to produce a domination cover pebbling. If there are at least 11 pebbles on both of the disjoint B_2 subtrees, then with the 59 remaining pebbles, we can dominate the root vertex.

Next, consider the case when only one of the two disjoint B_2 subgraphs, call it B_2^*, contains at least 11 pebbles. There are at most 70 pebbles somewhere on the graph that can be used to dominate the other B_2, call it B_2', and the root vertex. Notice that two single pebbles on the bottom row of B_2', each with a different parent, does not decrease the number of pebbles required to dominate B_2' where all the pebbles are located at the root. Let c be the number of pebbles in B_2' that are not part of the two aforementioned vertices on the bottom row. In order to dominate B_2', we require at most $8 - b$ pebbles on the root vertex. If $b \geq 1$ we can move enough pebbles to the root vertex from B_2^*, so that we can dominate B_2' and leave a pebble on the root vertex. Similarly, if $b = 0$ and there is a pebble initially on the root vertex, we are finished.

Suppose that $b = 0$ and there is no pebble initially on the root vertex. There are then at least 79 pebbles in B_2^*. Suppose there at least two vertices, call them v_1 and v_2, that contain at least two pebbles, with the condition that these vertices are not adjacent to the same parent in the bottom row. If either v_1 or v_2 is not on the bottom row, then move a pair of pebbles from a v_i that is not on the bottom row to dominate the root vertex. This leaves 77 pebbles, 64 of which can be used to move 8 pebbles to the root. These 8 pebbles on the root vertex will dominate B_2', and the remaining 13 pebbles will dominate B_2^*.

If v_1 and v_2 are on bottom rows of different parents, it is possible to dominate B_2^* with 4 pebbles, by moving a pebble to the parent of each pair of children in the bottom row of B_2^*. The remaining 75 pebbles will be able to force 8 pebbles to be at the root, along with another pebble to dominate the root. If there is only one vertex with at least two pebbles on it, call it v_3, then it contains at least 75 pebbles, which happens when there is one pebble on every other vertex of B_2^*. Using at most 64 pebbles, place 8 pebbles on the root vertex. With the remaining 15 pebbles we must dominate B_2^* as well as the root vertex. To do this, place one pebble on the root of B_2^* unless it already has a pebble, costing us 4 pebbles. Also, place a pebble on the row second from the bottom on the vertex, call it v_4, that is at least a distance of 2 away from v_3 unless this vertex and its children are already dominated. This costs at most 8 pebbles. (If v_3 is the root vertex of B_2^*, then move a pair of pebbles to each of the two vertices in B_2^* that are on the second from the bottom row.) There are least three pebbles left, and at most one of them can be on the vertices that are children of v_4. This leaves 2 pebbles to dominate a path of 3 vertices, and since $\psi(B_1) = 2$, we are finished. Thus, $\psi(B_3) = 81$. $\square$

We now state and prove the general theorem for complete binary trees.

Theorem 3.7. *For $n \geq 4$,*

$$\psi(B_n) = 2^{n-1} + \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} [2^{3i+1} + \sum_{j=1}^{n-3i-2} 2^{j-1} 2^{3i+2j+1}] + \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{n-3k+1} 2^{2n-3k+2} + \gamma,$$

with $n \equiv 0 \mod 3$, $\gamma = 2^{n-1}$. Otherwise, $\gamma = 0$.

Proof. First we will prove that

$$\begin{aligned}\psi(B_n) \quad > \quad & 2^{n-1} + \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} [2^{3i+1} + \sum_{j=1}^{n-3i-2} 2^{j-1} 2^{3i+2j+1}] \\ & + \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{n-3k+1} 2^{2n-3k+2} + \gamma - 1.\end{aligned}$$

Consider the following initial configuration of pebbles. At each of $2^{n-1} - 1$ vertices on the bottom row each of which does not share a parent, place one pebble. Place all of the remaining pebbles on the remaining vertex in the bottom row who is unpebbled and so is its sibling. Call this vertex v. In order to find a domination cover solution for the subtree that is on the other side of the root vertex it takes $\sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{n-3k+1} 2^{2n-3k+2}$ pebbles. In this configuration, we will make pebbling moves so that one pebble is placed on every vertex in every third row, starting with the row that is next to the bottom row. If we consider rows as single vertices, this is analogous to the configuration of pebbles required to get an optimal domination cover pebbling bound for P_n.

We leave the details of proving this bound to the reader, but we will verify the term in the sum that counts the next to bottom row. There are 2^{n-2} vertices that must have a pebble placed on them. For each vertex, it takes 2^{2n-1} pebbles from vertex v, for a total of 2^{3n-3} pebbles. This is the number of pebbles counted in the $k = 1$ term of the sum. We obtain $2^{n-3+1} 2^{2n-3+2} = 2^{3n-3}$, as desired. A similar computation can be performed to verify the first sum. In order to obtain a domination cover solution, one more pebble is needed on vertex v to dominate the other sibling of v.

We now proceed to prove the bound by induction. Suppose that the bound for $\psi(B_{n-1})$ holds. Notice that if B_{n-1} were a case, such as B_2 or B_3, where the formula given in the theorem predicted a value for B_{n-1} larger than the actual value, we can assume the bound predicted in the theorem, and the induction still follows. We now show that $\psi(B_n)$ also holds. As before, we will consider three cases depending upon whether there are enough pebbles in each of the two disjoint copies of B_{n-1} connected to the root of B_n. First, suppose that neither copy contains $\psi(B_{n-1})$ pebbles. In this case, there are clearly enough pebbles on the root vertex in order to construct a domination cover solution.

Next, suppose that both copies contain at least $\psi(B_{n-1})$ pebbles. In this case we can use $\psi(B_{n-1})$ pebbles to construct a domination cover pebbling for this tree. Now, since most of the pebbles are left, (certainly 2^{2n}), the root vertex can be dominated.

Now, suppose that only one copy of B_{n-1}, call it B^*_{n-1}, contains at least $\psi(B_{n-1})$ pebbles. Then we move at most all but B_{n-1} pebbles, without wasting any pebbles (that is, we do not make pebbling moves that do not contribute to the number of pebbles that can reach the root vertex), in order to force a domination cover solution of the other B_{n-1}, call it B'_{n-1}. Let's compute $D(n) = \psi(B_n) - \psi(B_{n-1})$. We get:

$$
\begin{aligned}
D(n) &= 2^{n-1} + \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} [2^{3i+1} + \sum_{j=1}^{n-3i-2} 2^{j-1} 2^{3i+2j+1}] + \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{n-3k+1} 2^{2n-3k+2} + \\
&\quad \gamma - \left[2^{n-2} + \sum_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} [2^{3i+1} + \sum_{j=1}^{n-3i-3} 2^{j-1} 2^{3i+2j+1}] + \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k} 2^{2n-3k} \right] \\
&\geq 2^{n-2} + \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3i-3} 2^{2n-3i-3} + \frac{7}{8} \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{n-3k+1} 2^{2n-3k+2} + \gamma \\
&= 2^{n-2} + \sum_{i=1}^{\lfloor \frac{n}{3} \rfloor + 1} 2^{3n-6i} + \frac{7}{8} \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{3n-6k+3} + \gamma \\
&\geq 2^{n-2} + \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{3n-6k+3} + \gamma.
\end{aligned}
$$

For every 2^n extra pebbles on B^*_{n-1}, we can contribute at least one pebble on the root vertex. Notice that there can be 2^{n-2} pebbles on the bottom row of B_{n-1}, one in every pair, that does not assist the pebbling starting from the root. Also, any additional pebbles in B'_{n-1} can substitute for at least one pebble on the root vertex. Hence, there are at least $\sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{2n-6k+3} + \frac{\gamma}{2^n}$ pebbles either on the root vertex or in B^*_{n-1} such that each pebbles decreases the need for at least one pebble on the root of the tree. Now, starting from the root, it takes $\sum_{l=1}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3l+2} 2^{n-3l+1} \leq \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} 2^{2n-6k+3}$ pebbles to construct a domination cover solution of B'_{n-1}. Hence, we can construct a domination cover solution for the cases where $n \equiv 1, 2 \mod 3$ because the root vertex is already dominated. For the case when n is a multiple of 3, it is possible to move a pebble to a vertex adjacent to the root vertex since we have an gained an additional 2^{n-1} pebbles from the γ term. This completes the proof. □

3.6 Diameter 2 Graphs

In the next few sections, we will present structural domination cover pebbling results.

Theorem 3.8. *For all graphs G of order n with maximum diameter* 2, $\psi(G) \leq n - 1$.

Proof. First, we will exhibit a graph G such that $\psi(G) > n - 2$. Consider the star graph on n vertices, and place a pebble on all of the outer vertices except one. This configuration of pebbles does not dominate the last outer vertex. Hence, $\psi(G) > n - 2$.

Suppose we are given a graph G on n vertices and $n - 1$ pebbles. To prove the theorem, we will show that after a sequence of pebbling moves G can always be dominated by the vertices that contain pebbles. Given an initial configuration c of $n - 1$ pebbles, let S_1 be the set of vertices $v \in G$ such that $c(v) > 1$. Suppose that S_2 is the set of dominated vertices $w \in G$ such that $c(w) = 0$ and w is adjacent to an undominated vertex, let S_3 be the set of dominated vertices $x \in G$ such that $c(x) = 0$, and x is not adjacent to an undominated vertex, and let S_4 be the set of vertices $y \in G$ such that y is not dominated.

Notice that if there exists some vertex $x \in S_3$, all of the vertices that are adjacent to it are either also in S_3 or contain pebbles. In either case, the set of unpebbled vertices connected to x in S_3 is completely surrounded by vertices with pebbles on them. Thus, these surrounding vertices form a dominating set of G, since the diameter of G is at most 2. Hence, if S_3 is nonempty, we are finished, and every vertex of G is an element of S_1, S_2 or S_4.

For the rest of this proof, let $|S_1| = a, |S_2| = b$, and $|S_4| = c$. Define the pairing number, $P(c)$, where c is a configuration of G, as $\sum_{v \in G} \max\{0, \frac{c(v)-1}{2}\} = \frac{b+c-1}{2}$. For instance, if exactly two pebbles are on a particular vertex, that vertex contributes $1/2$ to $P(c)$, and if three pebbles are on a particular vertex, that vertex contributes 1 to $P(c)$. For the purposes of this proof, we use the fact that if $P(c) = k$ then c contains at least $\lceil k \rceil$ pairs of pebbles, which means that we can definitely make $\lceil k \rceil$ pebbling moves, each of which requires a pair of pebbles.

Suppose that $b \leq c$. In this case, $P(c) \geq \frac{2b-1}{2}$. Hence, there are at least b pairs of pebbles that can be moved from elements in S_1 to S_2. For each element of S_2, if possible, move a pair of vertices from an element of S_1 to an element of S_2. This implies that every element of S_2 now contains a pebble, or there exists some $x \in L \in S_2$ where L is the set of vertices in S_2 that are a distance of 2 from all remaining pairs. If the latter is the case, S_4 is still dominated because if there were some vertex y that were only adjacent to elements in L with respect to S_2 then the minimum distance between y and a vertex with a pair of vertices is 3, a contradiction. However, it may be the case that the vertex in S_1 that y was adjacent to lost its pebbles, and if this is the case, move a pair of pebbles from S_1 so that y is dominated. Since there were $|L|$ vertices in S_2 where it was not possible to move a pair of pebbles, we have an extra $|L|$ pairs of pebbles. With these $|L$ pairs we can ensure that the $|L|$ vertices are also dominated if necessary, one for each $x \in L$. After this process, the graph will be completely dominated.

On the other hand, consider the case when $b > c$. In order to dominate G, we must place pebbles on at most c elements of S_2. We can apply the argument from the previous paragraph if there are at least c pairs of pebbles that we can remove from S_1, such that there are at least 3 pebbles on a particular vertex when a pair is removed from it. In this case, we are finished because the vertices in S_4 are dominated by the c pairs that were moved to elements of S_2. The sets S_1 and S_2 are dominated because every vertex that is in S_1 still contains pebbles. Thus, there are at most $c - 1$ pairs of pebbles that can be removed from S_1 with the condition that when a pair of pebbles is removed, there are at least 3 pebbles on that vertex.

After moving at most c pairs with the condition that we will first remove pairs from vertices that contain at least 3 pebbles, we know from the previous two paragraphs that set S_4 is dominated. Notice that after this process, every vertex in S_1 is dominated. By the previous paragraph, we know that there are at least $b - c$ vertices containing exactly two pebbles after the first c pebbling moves have been made. Once this is completed, we will dominate the remaining $b - c$ vertices in S_2 not pebbled of the domination of S_4 and that are not already dominated by using one of the $b - c$ or more remaining vertices containing exactly 2 pebbles to dominate it. We have enough vertices containing exactly 2 pebbles because initially, $P(c) = \frac{b+c-1}{2}$, and the first c pebbling moves remove at most $\frac{2c-1}{2}$ from $P(c)$, leaving a pairing number of $\frac{b-c}{2}$. At this point in the algorithm, we know that every vertex with more than one pebble contains exactly two pebbles. Thus there are exactly $b - c$ vertices in S_1 with two pebbles on it. Thus, G is dominated and therefore $\psi(G) \leq n - 1$. $\square$

We can apply this theorem to prove a result about the ratio between the cover pebbling number and the domination cover pebbling number of a graph. We conjecture that this ratio holds for all graphs, but this proof be much more difficult because we cannot rely on structural bounds that are simple. Instead, we need to find some invariant process that relates cover pebbling to domination cover pebbling.

Theorem 3.9. *For all graphs G of order n with diameter 2, $\lambda(G)/\psi(G) \geq 3$.*

Proof. First, suppose that the minimum degree of a vertex of G is less than or equal to $\lceil \frac{n-1}{2} \rceil$. By the previous theorem, we know that the maximum value of $\psi(G)$ is $n-1$. Place $3n-3$ pebbles on any vertex v that has a degree less than $\lceil \frac{n-1}{2} \rceil$. It takes 2 pebbles to cover-pebble each vertex adjacent to v, at most $\lceil \frac{n-1}{2} \rceil$, and all the remaining vertices require 4 pebbles. Since there are at least as many vertices a distance of 2 away from v as there are a distance of 1 away from G, $3n-3$ pebbles or more are required to cover pebble all of the vertices except for v. Thus for this class of graphs, $\lambda(G) > 3n-3 \geq 3\psi(G)$.

Now suppose that the minimum degree k of a vertex in G is greater than $\lceil \frac{n-1}{2} \rceil$. In this case, we will prove that $\psi(G) \leq \lceil \frac{n}{2} + 1 \rceil$. For each vertex that is pebbled in such a graph at least $\lceil \frac{n-1}{2} + 1 \rceil$ vertices are dominated. For every new pebbled vertex, the number of vertices dominated increases by at least one, and for every pair of pebbles added to a vertex, the number of vertices dominated increases by at least 2. Thus, the maximum number of pebbles needed to cover-dominate G is $\psi(G) \leq \lceil \frac{n}{2} + 1 \rceil$. The minimum number of pebbles needed to cover pebble a graph with minimum diameter k is $2k+1+4(n-k)$. So, for $n \geq 4$, since $k \leq n-2$, $2n+5 \geq 3\lceil \frac{n}{2} + 1 \rceil$. If $n = 3$, the only graph of diameter 2 is the path graph, where $\psi(P_3) = 2$ and $\lambda(P_3) = 7$. Thus, for all graphs G of diameter 2, $\lambda(G)/\psi(G) \geq 3$. □

We now prove a more general bound for graphs of diameter d.

3.7 Graphs of Diameter d

(This section is joint work with Nathaniel Watson.)

Theorem 3.10. *Let G be a graph of diameter $d \geq 3$ and order n. Then $\psi(G) \leq 2^{d-2}(n-2)+1$*

Proof. First, we define the *clumping number* χ of a configuration c' by

$$\chi(c') = \sum_{v \in G} 2^{d-2} \max \left(\left\lfloor \frac{c'(v)-1}{2^{d-2}} \right\rfloor, 0 \right).$$

Thus, the clumping number counts the number of pebbles in a configuration which are part of disjoint "clumps" of size 2^{d-2} on a single vertex, with one pebble on each occupied vertex ignored. Also, given a configuration c', we will say a vertex $v \in G$ is dominated by c' if v is adjacent to some vertex of G on which c' is nonzero.

Now let c be a configuration on G of size at least $2^{d-2}(n-2)+1$. We will show that c is domination-cover-solvable (henceforth, we shall simply say *solvable*) by giving a recursively defined algorithm for solving c through a sequence of pebbling moves. First, we make some definitions to begin the algorithm (We adopt the convention that G is a graph and V and W are subsets of $V(G)$ and $v \in V(G)$ then $d(v,W) = \min_{w \in W} d(v,w)$ and $d(V,W) = \min_{v \in V} d(v,W)$.) Also, for any set $S \subseteq V(G)$ we of course let $S^C = V(G) \setminus S$.

- $c_0 = c$.
- $A_0 = \{v \in G \,:\, c(v) > 0\}$.
- $B_0 = \{v \in G \,:\, c(v) \geq 2^{d-2}+1\}$.
- $C_0 = V(G) - A_0$.
- $D_0 = \emptyset$.

We will describe our algorithm by defining c_n, A_n, B_n, C_n, and D_n recursively. At each step, we will need to make sure a few conditions hold, to ensure that the next step of the algorithm may be performed. For each m, we will insist that:

1. For every $v \in C_m \cup D_m$, $c_m(v) = 0$ and for every $v \in A_m$ $c_m(v) > 0$.
2. $\chi(c_m) \geq 2^{d-2}(|C_m| - 1)$.
3. $|C_m| \leq |C_0| - m$.
4. $B_m = \{v \in G \,:\, c_m(v) \geq 2^{d-2}+1\}$.
5. If both $B_m \neq \emptyset$ and $D_m \neq \emptyset$, $d(B_m, D_m) = d$; If $D_m \neq \emptyset$, there always exists some $v \in G$ such that $d(v, D_m) = d$, even if $B_m = \emptyset$.
6. A_m, C_m, and D_m are pairwise disjoint and $A_m \cup C_m \cup D_m = V(G)$.
7. Every vertex of D_m is dominated by c_m.
8. There exists a sequence of pebbling moves transforming c to c_m.

Note by 1, 4, and 6, we will always have $B_m \subseteq A_m$. Also, by 1, 6, and 7, every vertex of G which is not dominated by c_m is in C_m. For $m = 0$, only condition 2 is not immediately clear. To see this, note that

$$\begin{aligned}
\chi(c) &= \sum_{v \in G} 2^{d-2} \max\left(\left\lfloor \frac{c(v)-1}{2^{d-2}} \right\rfloor, 0\right) \\
&= \sum_{v \in A_0} 2^{d-2} \left\lfloor \frac{c(v)-1}{2^{d-2}} \right\rfloor \\
&\geq \sum_{v \in A_0} 2^{d-2} \left(\frac{c(v)}{2^{d-2}} - 1\right).
\end{aligned}$$

and using the fact that the size of c is at least $2^{d-2}(n-2)+1$, and $|C_0| = n - |A_0|$

$$\chi(c) \geq (2^{d-2}(n-2)+1) - 2^{d-2}|A_0| = 2^{d-2}(|C_0|-2)+1.$$

From the definition of χ it is apparent that $2^{d-2} | \chi(c)$. Therefore, we indeed must have

$$\chi(c) = \chi(c_0) \geq 2^{d-2}(|C_0|-1).$$

Suppose for some $n-1 > 0$ we have defined $c_{n-1}, A_{n-1}, B_{n-1}, C_{n-1}$, and D_{n-1} and the above conditions hold when $m = n-1$.

We shall assume that there is some vertex in C_{n-1} which is not dominated by c_{n-1}, for otherwise, by conditions 6, 7 and 8, c is solvable and we are done. Thus $|C_{n-1}| \geq 1$. but suppose $|C_{n-1}| = 1$. Call this single vertex v. Since it is non-dominated, it is adjacent to only uncovered vertices. These vertices cannot be in C_{n-1} for $|C_{n-1}| = 1$, and they are not in A_{n-1}, because every vertex in A_{n-1} is covered by property 1. So every vertex adjacent to v is in D_{n-1}. Choose some element in B_{n-1} and call it w (and thus by 5, $d(w, D_{n-1}) = d$ or, if B_{n-1}, invoke property 5 to choose a w for which $d(w, D_m) = d$. Any path from w to v passes through one of the vertices in D_{n-1} which is adjacent to v, and is thus of length at least $d+1$, so $d(w, v) \geq d+1$, contradicting the assumption that G has radius d. We have now shown that, if C_{n-1} has a non-dominated vertex, then $|C_{n-1}| \geq 2$. In this case, we will have $\chi(c_m) \geq 2^{d-2}$, ensuring the existence of some clump of size 2^{d-2}, and thus, that B_{n-1} is non-empty. Thus, we will always implicitly assume that $B_{n-1} \neq \emptyset$

Case 1: $d(B_{n-1}, C_{n-1}) \leq d-2$

In this case, we choose $v' \in B_{n-1}$ and $w' \in C_{n-1}$ for which $d(v', w') \leq d-2$ and move $2^{d(v',w')}$ pebbles from v' to w', leaving one pebble on w' and at least one on v'. We let c_n be the configuration of pebbles resulting from this move. Let $C_n = C_{n-1} \setminus w'$. Thus $|C_n| = |C_{n-1}| - 1 \leq |C_0| - (m-1) - 1$ and we see that condition 3 holds when $m = n$. Furthermore, We have used at most one clump of 2^{d-2} pebbles so

$$\chi(c_n) \geq \chi(c_{n-1}) - 2^{d-2} \geq 2^{d-2}(|C_{n-1}| - 1) - 2^{d-2} = 2^{d-2}(|C_n| - 1)$$

and therefore condition 2 holds for n. Also, we let $A_n = A_{n-1} \cup \{w'\}$, let $C_n = C_{n-1}\ w'$, and $D_n = D_{n-1}$ (now, clearly condition 6 holds.) We again let $B_n = \{v \in G \ : \ c_n(v) \geq 2^{d-2} + 1\}$, which simply means that we have possible removed v' from B_{n-1} if v' now has less than $2^{d-2} + 1$ pebbles. Thus $B_n \subseteq B_{n-1}$, and now 1, 4, 5, 7, and, 8 are all easily seen to hold for $m = n$.

Case 2: $d(B_{n-1}, C_{n-1}) \geq d - 1$.

If every vertex in C_{n-1} is dominated by A_{n-1}, we are done. Otherwise, let w' be some non-dominated vertex in C_{n-1}. Of course, w' is at distance $d-1$ or d from B_{n-1}, but suppose $d(B_{n-1}, w') = d - 1$. Then w' is adjacent to some (non-covered) vertex w'' at distance $d-2$ from B_{n-1}. By 1, every vertex of G which is non-covered, (that is, for which $c_{n-1} = 0$) is in $C_{n-1} \cup D_{n-1}$. But $d(B_{n-1}, C_{n-1}) \geq d-1$ and by 5, $d(B_{n-1}, D_{n-1}) = d$ so $w'' \in C_{n-1} \cup D_{n-1}$ is impossible. This contradiction means that $d(w', B_{n-1}) \neq d-1$ and so $d(w', B_{n-1}) = d$. Choose some vertex in B_{n-1} and call it v'. We know $d(v', w') = d$ so consider some path of length d from v' to w'. Let v^* be the unique point on this path for which $d(v^*, v' = d-2)$. Thus $v^* \notin C_{n-1} \cup D_{n-1}$ and $v^* \in A_{n-1}$, and also $d(v^*, w') = 2$. Let w'' be some vertex which is adjacent to both v^* and w' so that $d(v', w'') = d-1$. Then because w'' is uncovered (else w' would be dominated,) it must be in C_{n-1}. This means that $v^* \notin B_{n-1}$ by condition 5.

We now move one clump of 2^{d-2} pebbles from v' to v^*, adding one pebble to v^*, which now, by condition 1, has at least two pebbles. We then move two pebbles from v^* and cover w'' with one pebble. We let c_n be the configuration resulting from these moves. We let $D_n = D_{n-1} \cup \{w'\}$ and we again let $B_n = \{v \in G \ : \ c_n(v) \geq 2^{d-2} + 1\}$, which just means we have possibly removed v' from B_{n-1}, so $B_n \subseteq B_{n-1}$. If now $c_n(v^*) = 0$, We let $A_n = A_{n-1} \cup \{w''\} \setminus v^*\}$ and $C_n = C_{n-1} \cup \{v^*\} \setminus \{w', w''\}$. Otherwise, if $c_n(v^*) > 0$ is still true, we let $A_n = A_{n-1} \cup \{w''\}$ and $C_n = C_{n-1} \setminus \{w', w''\}$. This ensures that 1 and 6 still hold for $m = n$. Also, $|C_n| \leq |C_{n-1}| - 1 \leq |C_0| - (m-1) - 1$ and so condition 3 holds for $m = n$. Also, we have used only one clump of 2^{d-2} pebbles, because $v^* \notin B_{n-1}$ so by using a pebble from v^*, we could not have destroyed a clump. So

$$\chi(c_n) = \chi(c_{n-1}) - 2^{d-2} \geq 2^{d-2}(|C_{n-1}| - 1) - 2^{d-2} \geq 2^{d-2}(|C_n| - 1)$$

and therefore condition 2 holds for n. Condition 5 also still holds for $m = n$ because we have added only the vertex w' to D_{n-1} and $d(B_{n-1}, w') = d$, so $d(B_{n-1}, D_n) = d$ and also by the fact that $B_n \subseteq B_{n-1}$. If $B_n = \emptyset$, then certainly $d(v', D_n) = d$ because $v' \in B_{n-1}$. To see condition 7 is still true, note that to get D_n we have only added w' to D_{n-1}, and certainly, w' is adjacent to w'', which is covered by c_n so w' is dominated by c_n. Also, the only previously covered vertex of G which is now uncovered is (possibly) v^* but $d(v^*, B_{n-1}) = d - 2$, and so v^* is not adjacent to any vertex in D_{n-1} for, by 5, $d(B_{n-1}, D_{n-1}) = d$. Thus, by possibly uncovering v^*, we did not cause any vertex in D_{n-1} to become non-dominated, so 7 still holds for $m = n$. That conditions 4 and 8 still hold for $m = n$ is easily seen.

The algorithm continues as long as there is some non-dominated vertex in C_n. By condition 3, it must terminate after at most $|C_0|$ steps, because when $n = |C_0|$, we would have $|C_n| = 0$, and certainly there could be no non-dominated vertex in C_n. Thus, the algorithm eventually stops, having created some c_m which dominates every vertex of G. By property 8, c_m is reachable from c by pebbling moves, so c must be solvable.

□

Chapter 4

Vertex Neighbor Integrity Domination Cover Pebbling

4.1 Vertex Neighbor Integrity DCP

Cozzens and Wu (6) created a graph parameter called the *vertex neighbor integrity*, or VNI, that has been the subject of numerous studies. We proceed to describe this parameter with the definitions of Cozzens and Wu (6). Let $G = (V, E)$ be a graph and u be a vertex of G. The *open neighborhood* of u is $N(u) = \{v \in V(G) | \{u, v\} \in E(G)\}$; the *closed neighborhood* of u is $N[u] = \{u\} \cup N(u)$. Analogously, for any $S \subseteq V(G)$, define the *open neighborhood* $N(S) = \cup_{u \in S} N(u)$ and the *closed neighborhood* $N[S] = \cup_{u \in S} N[u]$. A vertex $u \in V(G)$ is *subverted* by removing the closed neighborhood $N[u]$ from G. Notice that this subversion is equivalent to the removal of a dominating set from G. For a set of vertices $S \subseteq V(G)$, the *vertex subversion strategy* S is applied by subverting each of the vertices of S from G. Define the *survival subgraph* to the the subgraph left after the subversion strategy is applied to G. The *order* of G is defined to be $|V(G)|$.

Definition 4.1.1. *The vertex neighbor integrity of a graph G is defined as*

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G \setminus S)\},$$

where $w(H)$ is the order of the largest connected component in the graph H.

We apply a variant of subversion in order to describe how VNI calculations relate to domination cover pebbling. Let $\Omega_\omega(G)$ be the minimum number of pebbles required such that it is always possible to construct an incomplete domination cover pebbling of G, where disjoint undominated components of G can have order at most ω. This corresponds to the $\omega(G)$ term in the VNI computation. Notice that domination cover pebbling corresponds to the case when $\omega = 0$.

4.2 Basic Results

Theorem 4.1. *For $\omega \geq 0$, $\Omega_\omega K_n = 1$.*

When a pebble is placed on K_n, the entire graph is dominated. The result follows.

Theorem 4.2. *For $s_1 \geq s_2 \geq \cdots \geq s_r$, let $K_{s_1,s_2,\ldots,s_r}$ be the complete r-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $c_1, c_2, \ldots, c_r$ respectively. Then for $\omega \geq 1, \Omega_\omega(K_{s_1,s_2,\ldots,s_r}) = 1$.*

Proof. Place a pebble on any vertex in c_i. All the vertices in the other c_i's are dominated. The other vertices in c_1 that are undominated are disjoint from each other. Thus, the result follows.
□

Theorem 4.3. *For $\omega \geq 1$, $n \geq \omega + 3$, $\Omega_\omega(W_n) = n - 2 - \omega$, where W_n denotes the wheel graph on n vertices.*

Proof. First, we will show that $\Omega_\omega(W_n) > n - 3 - \omega$. Place a single pebble on each of $n - 3 - \omega$ consecutive outer vertices so that all of the pebbled vertices form a path. This leaves a connected undominated set of size $\omega + 1$. Hence, $\Omega_\omega(W_n) > n - 3 - \omega$. Now, suppose that we place $n - 2 - \omega$ pebbles on W_n. If any vertices have a pair of pebbles on them, the entire graph can be dominated by moving a single pebble to the hub vertex. Hence, each vertex can contain only one pebble. Since every outer vertex is of degree 3, at least 3 vertices must be dominated but unpebbled before any other vertices can be undominated. Hence, in order to obtain an undominated set of size $\omega + 1$, there must be $4 + \omega$ vertices that are unpebbled. By the pigeonhole principle, we obtain a contradiction because there are not enough vertices for this constraint to hold. Thus, for $\omega \geq 1$, $n \geq \omega + 3$, $\Omega_\omega(W_n) = n - 2 - \omega$.
□

4.3 Graphs of Diameter 2 and 3

Theorem 4.4. *Let G be a graph of diameter 2 with n vertices. For $\omega \geq 1$, $\Omega_\omega(G) \leq n-1-\omega$.*

Proof. To show that $\Omega_\omega(G) \geq n-2-\omega$ consider the following family of graphs which is a modification of the star graph.

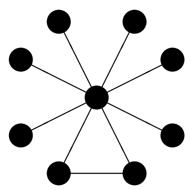

Figure 4.1: An example of the construction for $n = 9$, $\omega = 1$.

By placing a single pebble on each of the tendrils of the star that are not connected to any other tendrils, the domination set that results still has a set of undominated vertices that is at least size ω. Hence, $\Omega(G) > n-2-\omega$.

Let G be a graph with n vertices. Suppose there is an arbitrary configuration of pebbles $c(G)$ that contains exactly $n-1-\omega$ pebbles. We now show that a domination cover pebbling of G can be constructed such that the maximum order of an undominated component of G is ω.

Let A be the set of all vertices $a \in G$ such that vertex a contains a single pebble. Let B be the set of vertices $b \in G$ such that vertex b contains two or more pebbles. Let C be the set of vertices of all $c \in G$ such that c is dominated but contains no pebbles. Let D be the set of all vertices in G such that if $d \in G$, then vertex d is undominated. Thus, all vertices in D are a distance of 2 from every element of $A \cup B$.

We now describe a process that forces $n-\omega$ vertices to be dominated. Let F be the set of vertices that are forced to be dominated and will remain dominated throughout the process. Since we never move pebbles from vertices with a single pebble on them, we have forced all of the vertices in A to be dominated. Thus for all $a \in A$, $a \in F$. If D is empty, then we have dominated the entire graph and the proof is complete. So suppose there exists some vertex v that is in D. Since G has a diameter of 2, then v can be dominated by moving a pair of pebbles from any vertex in B that still has at least 2 vertices on them.

For every vertex v dominated in such a manner, two vertices become elements of F, namely v, and the empty vertex that the pair of pebbles moved to in order to dominate v. Perform this process repeatedly until the entire graph is dominated or there is only one vertex v^*, that has exactly 2 or 3 pebbles left on it and no other vertices have contain more than one pebble. Notice that the only vertices in F that are unpebbled are those that are a distance of two from every pair of vertices. Except for v^*, the vertices in B now either have zero or one pebble on it. If a vertex in B has one pebble on it, then that vertex also gets put into F. So far, for every pebble of the initial configuration of G except for the ones remaining on v^*, one pebble has forced at least one vertex to be in F.

First, consider the case where v^* has two pebbles. If there are $n - \omega$ vertices already in L, we are then finished because the maximum number of undominated vertices left is ω. Also notice that the only unpebbled vertices in F are those that are a distance of two away from the set of all pairs. Since the graph is undominated, there exists some vertex, d' in D that is undominated. In this case, moving the last pair of pebbles to dominate a vertex means that we have forced 3 verties not in F to be dominated, namely d', v^* and a vertex not already in F connecting them. Thus, since we have dominated at least $n - \omega$ total vertices, one vertex for each pebble plus an additional vertex, the largest undominated set possible is of size ω, and this case is complete.

If v^* has 3 pebbles and there is only one undominated vertex left, then moving a pair of pebbles to dominate v^* dominates the entire graph. Otherwise, there are at least two vertices that are undominated. If there is some common unpebbled vertex, x, that would dominate at least two undominated vertices, then using the last pair of pebbles to move a pebble to x will force at least 4 vertices to be dominated that are not members of F. These vertices are v^*, x, and two undominated vertices a distance of two away from v^*. Thus, after this operation, at least $n - \omega$ vertices are dominated. If there is no common unpebbled vertex, then there are at least two unpebbled vertices of distance 1 from v^* that have not been placed in F. Notice that the only unpebbled vertices that have been forced are those that are a distance of 2 away from the set of all pairs. So take the pair of pebbles and place a pebble on a vertex that forces two more vertices to be placed in F. The remaining pebble on v^* will force v^* and at least one more vertex adjacent to v^* that is empty and has not been forced to be dominated. Again, at least $n - \omega$ vertices are dominated, whence there cannot exist an undominated component of G that contains $\omega + 1$ or more vertices, and the proof is complete.

□

We conclude this section by conjecturing an analogous result for graphs of diameter 3, along with a valid lower-bound construction for this conjecture.

Conjecture 4.3.1. *Let G be a graph of diameter 3 with n vertices. For $i \geq 1$, $\Omega(G) \leq \lfloor \frac{3}{2}(n-2-\omega)+1 \rfloor$.*

To see that this result is reasonable, we will show that $\Omega_i(G) > \lfloor \frac{3}{2}(n-2-\omega) \rfloor$. Consider the following family of graphs that are of size $\lfloor \frac{3}{2}(n-3-\omega)+1 \rfloor$ but contain an undominated component of order $\omega+1$. Take a $K_{\omega+1}$ and attach each of the ω vertices to some other vertex v. Connect v to each vertex of a $K_{\lceil \frac{n-\omega-2}{2} \rceil}$, call it H. Connect each of the remaining $\lfloor \frac{n-\omega-2}{2} \rfloor$ vertices to a vertex of H, so that each vertex in H has only one tendril off it. Now, place three pebbles on each of the tendril vertices, and if there is one vertex in H without a tendril, place one pebble on it. This is a total of $3\lfloor \frac{n-\omega-2}{2} \rfloor$ (+1 if $n-\omega-2$ is odd) pebbles in this configuration, which is equivalent to $\lfloor \frac{3}{2}(n-2-\omega) \rfloor$. Since it is not possible to dominate the vertices in $K_{\omega+1}$, the graph still has an undominated component of order $\omega+1$. Thus, $\Omega_i(G) > \lfloor \frac{3}{2}(n-2-\omega) \rfloor$.

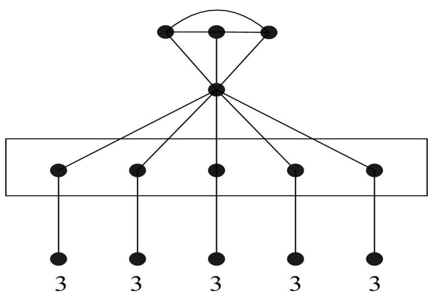

Figure 4.2: An example of the construction for $n = 14$, $\omega = 3$.

Chapter 5

Deep Graphs

5.1 Introduction

In this chapter, we will consider another pebbling property that has interesting characteristics. In 2004, Hetzel (17) introduced the concept of *deep graphs*. A graph is *deep* if for each positive integer $n \leq \pi(G)$, there exists an induced subgraph H of G such that $\pi(H) = n$. A simple class of deep graphs of diameter 1 is the family of complete graphs. For example, in K_n, the complete graph on n vertices, there clearly exist an induced $K_{n-1}, K_{n-2}, \ldots,$ and K_1. In this chapter, we will extend this definition in order to answer natural questions about deep graphs.

One classification of graphs that has been useful in obtaining results for various pebbling problems is considering graphs of *Class* 0. A graph G is *Class 0* if its pebbling number is equal to $|V(G)|$. We can extend the definitions of Class 0 and deep to say that a graph is *Class 0 deep* if a graph G is deep and $\pi(G) = |G|$. In fact, we can extend this definition even more by saying that a graph G is *profound* if it is deep and it is possible to construct G by a series of induced subgraphs $H_1 \subset H_2 \subset \cdots \subset H_{\pi(G)}$, each of which are deep. Finally, applying this definition to Class 0 graphs, we say that a graph G is *Class 0 profound* if it is Class 0 deep and G can be constructed by a series of induced subsets, $H_1 \subset H_2 \subset \cdots \subset H_{|G|}$, each of which are Class 0 deep. Notice that if G is profound, then it must be Class 0 profound because we remove a vertex from G between every subgraph, and there are only $|G|$ vertices in the graph.

5.2 Basic Results

In this section, we will describe some basic results that Hetzel proved about deep graphs. (17) We now state a necessary condition for deep graphs G where $\pi(G) \geq 3$.

Theorem 5.1. *There exists a K_3 in every deep graph G where $\pi(G) \geq 3$.*

Proof. Since K_3 is the only graph with a pebbling number of 3, it is necessary to be included in any deep graph with at least 3 vertices. □

In addition, Hetzel (17) creates the following construction:

Theorem 5.2. *It is possible to construct a deep graph with arbitrary diameter, n.*

Proof. Consider the following construction. Take a path of length $n > 2$. Take $2^{n-1} - 1$ additional vertices, and, using these vertices and the last 2 vertices of the path, build an almost complete graph, making every possible connection among these $2^{n-1} - 1 + 3$ vertices except do not connect the third to last vertex of the path with the last vertex of the path. There are other, less complex examples of such graphs, which contain $O(n)$ edges instead of $O(n^2)$ edges. We outline one such example.

We simplify this construction by considering a path of length $n - 1$ and attaching to it a star with $2^{n-1} + 2$ outer vertices, where two of the outer vertices of the star are connected to each other, forming a K_3. This is certainly deep because for every vertex of the star that we remove, the pebbling number of the graph decreases by one. Once we remove all the vertices of the star, we can remove one vertex of the path and then remove vertices of the star as needed to decrease the pebbling number even further. Notice that to create a graph with pebbling number 3, we only use the center of the star and the two outer vertices of the star connected to each other, thus creating a K_3.

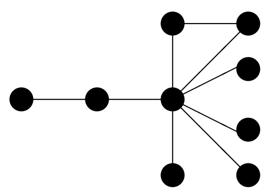

Figure 5.1: An example of the arbitrary-diameter graph construction for n=3.

In the next section, we present constructions describing various properties of deep graphs, such as an arbitrary diameter Class 0 deep graph. We conclude the paper in the following section by showing that $G(n,p)$ is Class 0 profound as $n \to \infty$.

5.3 Graph Constructions

Given a graph G, a property of a G is *monotonically increasing* when adding edges to G preserves the property. If deep graphs were monotonically increasing, then we could apply a variant of the FKG inequality that applies to graphs. The FKG inequality is a probabilistic theorem used to understand correlations between probabilities. For our purposes, we only require the following theorem dealing with properties of graphs. (1)

Theorem 5.3. *Let Q_1, Q_2, Q_3 and Q_4 be graph properties, where Q_1 and Q_2 are monotonically increasing, and Q_3 and Q_4 are monotonically decreasing. Let $G = (V, E)$ be a random graph on V obtained by picking every edge, independently, with probability p. Then the following statements hold:*

$$\Pr(G \in Q_1 \cap Q_2) \geq \Pr(G \in Q_1) \times \Pr(G \in Q_2),$$
$$\Pr(G \in Q_3 \cap Q_4) \geq \Pr(G \in Q_3) \times \Pr(G \in Q_4),$$
$$\Pr(G \in Q_1 \cap Q_3) \leq \Pr(G \in Q_1) \times \Pr(G \in Q_3).$$

This would take care of dependency problems when considering results concerning random graphs that involve both deep graphs and other graph properties. However, we can prove the following statement:

Theorem 5.4. *Deepness is not a monotonic property.*

Proof. Consider the graphs G_1 and G_2 as shown.

First we will show that G_1 is deep. We can see that $\pi(G_1) = 11$. We now will show that the G_1 is deep but G_2 is not. The following table describes which vertices must be removed in order to obtain a subgraph whose pebbling number is n.

$\pi(H)$	Vertices to Remove	$\pi(H)$	Vertices to Remove
10	4	5	1, 2, 3
9	4, 5	4	1, 2, 3, 4
8	3, 4, 5	3	1, 4, 5, 7
7	1	2	1, 2, 3, 4, 5
6	1, 2	1	1, 2, 3, 4, 5, 6

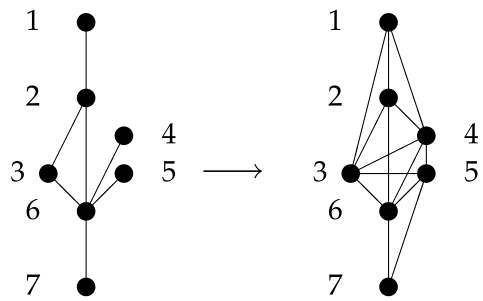

Figure 5.2: This is G_1, which is deep and G_2, which is not.

Now, we must show that G_2 is not deep. Notice that $\pi(G_2) = 8$. Further, unless we remove either vertex 1 or 7 we will always have a graph of diameter 3, which forces a pebbling number of at least 8. Thus, we will consider the cases when we remove either vertex 1 or 7. First, consider the case where vertex 1 is removed. Just removing vertex 1 gives a pebbling number of 6. If we remove any vertex of the K_5 subgraph, then the pebbling number of the remaining graph is at most 6. If we also remove vertex 1, then the remaining graph is a Class 0 profound graph whose pebbling number is at most 5. Therefore, we cannot obtain a pebbling number of 7 by removing vertex 1.

Now, suppose that vertex 7 is removed. Just removing this vertex gives a pebbling number of 6. Similarly, if any vertex of the K_5 subgraph is removed, the pebbling number will still be at most 6. Also, if we remove both vertex 1 and 7 the pebbling number of the remaining graph is at most 5. Thus, we cannot obtain a pebbling number of 7 by removing vertex 7, and we conclude that G_2 is not deep.

□

Theorem 5.5. *Not all Class 0 deep graphs are Class 0 profound.*

Proof. Consider the following Class 0 graph, call it G_3. We first show that this graph is Class 0 deep. Since this graph is Class 0, we know that $\pi(G_3) = |G_3| = 6$. For other values of n we have:

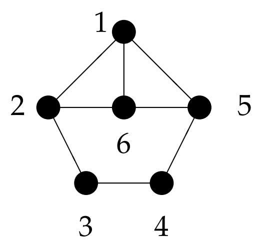

Figure 5.3: This is G_3, a Class 0 deep graph that is not Class 0 profound.

$\pi(H)$	Vertices to Remove
5	6
4	1, 2, 6
3	3, 4, 5
2	3, 4, 5, 6
1	2, 3, 4, 5, 6

Next, we prove that G_3 is not Class 0 profound. If we remove any of the five outer vertices (1,2,3,4 or 5), the graph has a diameter of 3, implying that the pebbling number is at least 8 for such subgraphs. Now, removing vertex 6 leaves a 5-cycle, which has a pebbling number of 5, but when another vertex is removed from the cycle, the remaining graph is a path of 4 vertices, whose pebbling number is 8. Hence, it is not possible to construct a chain of subgraphs $H_1 \subset H_2 \subset \cdots \subset H_6$ such that H_i is a graph with i vertices that is Class 0. Hence, not all Class 0 deep graphs are Class 0 profound.
□

Recall in (17), a deep graph with arbitrary diameter was constructed. We now present the following theorem.

Theorem 5.6. *There exists a graph with arbitrary diameter that is Class 0 deep.*

Proof. Suppose we want to construct a Class 0 profound graph of diameter k. Now, take $k+1$ sets, labeled $p_1, \ldots, p_{k+1}$ each containing 2^{2k+3} vertices. Connect every vertex of p_1 to every vertex of p_2 and every vertex of p_k to every vertex of p_{k+1}. For all other sets p_i, connect every vertex of p_i to every vertex of p_{i-1} and p_{i+1}. Also, connect two vertices of $p_{\lceil \frac{k}{2} \rceil}$. This can be thought of as the cartesian product of a path of length k and a set of 2^{2k+3} disconnected vertices, with the addition of an edge in order to produce a K_3. The diameter of this graph is certainly k. The connectivity of this graph is at least 2^{2k+3}. Thus, this graph is Class 0 (8). This graph is also Class 0 profound because when we remove vertices one at a time, starting at the two end sets of 2^{2k+3} vertices and moving inward, we always have a Class 0 graph. □

Another relevant lower bound to determine is the minimum number of edges of a Class 0 profound graph. Consider the following construction of M_n, a Class 0 profound graph with a minimal number of edges. Take a K_3, with vertices m_1, m_2, and m_3. Connect m_4 to m_1 and m_3, connect m_5 to m_1 and m_4, and in general connect m_i to m_1 and m_{i-1}, for $i \geq 4$. This constructs a Class 0 profound graph with $2n - 3$ edges.

This idea leads to the following result:

Theorem 5.7. *Any Class 0 profound graph must contain at least* $2n - 3$ *edges.*

Proof. First, given a Class 0 profound graph G with order at least 3, there cannot be any vertices with degree 1, because otherwise we could place three pebbles on such a vertex, call it v, and one pebble on every other vertex of G except for two vertices and still not be able to pebble the unpebbled vertex a distance of 2 away from v. Now, in the removal of every vertex of G for $n \geq 3$, since every vertex has degree at least 2, at least two edges are removed with the removal of every vertex except for the last two vertices, which must share at least one edge. This implies that at least $2n - 3$ vertices are required for this construction. We will use this fact in the next section. □

5.4 Random Graphs

Another reason for the study of Class 0 profound graphs instead of deep graphs in probabilistic investigations are that they are monotonic. That is, given a Class 0 profound graph G, by adding edges to G to make G', we observe that G' is Class 0 profound. This is apparent because the pebbling number of a graph is a monotonically decreasing property and at every point in the induced sequence of subgraphs that makes G Class 0 profound, the subgraph already has the minimum pebbling number. Thus, if we can find a subgraph of a random graph that is Class 0 profound, then G is also Class 0 profound. This is the basis for this section.

Let $G(n, p)$ be a random graph on n vertices where each edge is placed independently with probability p.

Theorem 5.8. *The probability that $G(n, p)$, where $p = \frac{1}{\log\log n}$, is a Class 0 profound graph approaches 1 almost surely as $n \to \infty$.*

Proof. Recall that a graph is always Class 0 if the diameter is at most 5 and $\kappa(G) \geq 2^{13}$, where $\kappa(G)$ denotes the vertex-connectivity of G (8). We will construct a diameter 5 graph that is Class 0 profound with probability 1 as $n \to \infty$.

Let $p = \frac{1}{\log\log n}$. From (14) we know that the largest clique size of $G(n, p)$ is $k = 2\log_b n - 2\log_b \log_b n + 2\log_b\left(\frac{e}{2}\right) + 1$ as $n \to \infty$, where $b = 1/p$. Further, given some β there exists an α such that as $n \geq \beta$, the probability of a clique of size k is at least $1 - \alpha$. Let this largest clique be $K_{\max}$. So we can assume there exists a complete graph of size k in $G(n, p)$. For simplicity in this instance, suppose that $k = \log_b n$.

Next, we will determine the number of vertices of G that connect to at least 2^{13} vertices of $K_{\max}$. We know that the number of vertices of $K_{\max}$ that each vertex of G connects to can be described by a binomial distribution with expectation $p \log n$. Let X be the random variable that counts the number of vertices that a particular vertex of G is connected to in $K_{\max}$. We will use the Chernoff tail approximation to determine the probability that a given vertex connects to fewer than 2^{13} vertices of $K_{\max}$. Here is the theorem in a general form:

Theorem 5.9. *Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials with $\Pr(X_i = 1) = p_i$. Then if X is the sum of the X_i's and if μ is $E[X]$, for any $\delta \in (0, 1]$:*

$$(1)\ \Pr[X < (1-\delta)\mu] < \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

Thus, applying the Chernoff tail approximation, we obtain the following (Notice that as $n \to \infty, \delta$ approaches 1):

$$\begin{aligned} P(X < 2^{13}) &< e^{-\frac{\log_b n}{\log\log n}(\delta)^2/2} \\ &= e^{\frac{-\log n}{\log\log\log n}\frac{1}{\log\log n}(\delta)^2/2}. \end{aligned}$$

Now, as $n \to \infty$, by applying L'Hopital's rule twice, we obtain that $\lim_{n\to\infty} \frac{\log_b n}{\log\log n} = \infty$. Also note that as $n \to \infty$, that $\frac{\delta^2}{2}$ is at least $1/3$.

So, for $p = \frac{1}{\log\log n}$ the expected number of vertices that have fewer than 2^{13} neighbors in $K_{\max}$ is $E' = n(e^{\frac{-\log_b n}{\log\log\log n}\frac{1}{3\log\log n}})$. Hence, the expected number of vertices that have at least 2^{13} neighbors in $K_{\max}$ is $E = n(1 - e^{\frac{-\log_b n}{\log\log\log n}\frac{1}{3\log\log n}}) - k$. Notice that as $n \to \infty$, the proportion of vertices that do not have at least 2^{13} neighbors in $K_{\max}$ approaches 0.

The variance of E is at most $[n(1 - e^{\frac{\log n}{\log\log\log n}\frac{1}{3\log\log n}}) - k](\frac{1}{\log\log n})(1)$. Thus, the standard deviation of E is at most $\sqrt{n}$. We obtain this variance by the formula $np(1-p)$. Thus, given some $\gamma > 0$, there exists an A such that Pr(The number of vertices with at least 2^{13} connections to $K_{\max}$ is at least $n - E' - A\sqrt{n}) > 1 - \gamma$.

Let F be the set of vertices that either are in $K_{\max}$ or connect to at least 2^{13} vertices of $K_{\max}$. Now, suppose there are $E' + A\sqrt{n}$ vertices not in F, and call this set H. We compute the probability that each vertex not in F connects to fewer than 2^{13} vertices of F via Chernoff's approximation. Let X be the random variable that counts the number of vertices that a particular vertex of H is connected to in F. Notice that again we have a binomial distribution with expectation $\frac{1}{\log\log n}(F - A\sqrt{n})$. Again, δ is a constant from 0 to 1, and as $n \to \infty$, $\delta \to 1$. Applying the approximation, we obtain:

$$P(X < 2^{13}) \quad < \quad e^{-\frac{\gamma^2}{2}(\frac{1}{\log\log n})(F - A\sqrt{n})}.$$

Notice that since $F \gg A\sqrt{n}$ or $\log\log n$, this probability clearly approaches zero as $n \to \infty$. Let r be the expected number of vertices in R, the set of vertices of E', where for each $e' \in R$, $X < 2^{13}$. Then r, as $n \to \infty$ is equal to $(E')e^{-\frac{1}{3}(\frac{1}{\log\log n})(F - A\sqrt{n})}$. As $n \to \infty$ this expectation approaches 0. So given an $\eta > 0$ there exists an $n = \lambda$ such that the probability that $r < 1$ (that is, there is no vertex that does not connect to 2^{13} vertices) is at least $1 - \eta$.

Before we continue this analysis, we must notice that the structures we have created are not independent, though in our analysis, we treat each property as independent. In actuality, there is a positive correlation between the events. First, notice the vertices of the clique are more likely to have a higher number of edges than average. Thus the number of vertices in the graph that connect to 2^{13} of them will be higher than average. Finally, the vertices connecting to E will have more edges connected to E than to vertices not in E or K_{max} by this same reasoning.

Notice that the graph we have constructed out of $G(n, p)$, where $p = \frac{1}{\log\log n}$ has connectivity 2^{13} and diameter at most 5. Then by the theorem of Czygrinow et al (8), the graph is Class 0. This graph is Class 0 profound because we can successively remove vertices, starting with vertices in E, then the vertices in F not in $K_{\max}$, and finally the vertices in $K_{\max}$, and still have a graph that is Class 0.

We now show that the probability that G exists approaches 1 as $n \to \infty$. Notice that G is constructed through three separate constructions, namely, first the clique, then the other vertices in F and finally the vertices in H. Let L, M and N be the events that these three events occur. By the arguments presented, $P(L \cap M \cap N) \leq P(L) + P(M) + P(N) - 2 = (1-\alpha) + (1-\gamma) + (1-\eta) = 3 - \alpha - \gamma - \eta$. Let $\mu = \alpha + \gamma + \eta$. Now, as $n \to \infty$, since α, γ and η are arbitrary and can approach 0, $P(L \cap M \cap N) \to 1$ as $n \to \infty$. Therefore with $p = \frac{1}{\log \log n}$, almost all graphs are Class 0 profound. Notice this also implies that if p is some fixed positive real number, then it also holds that almost all graphs are Class 0 profound. Indeed, any value for p that is asymptotically smaller than $\log n$ would also work.
□

Theorem 5.10. *The probability that $G(n,p)$, where $p \leq \frac{1}{n^{.5+\epsilon}}$, is a Class 0 profound graph approaches* 0 *almost surely as $n \to \infty$.*

Proof. Let $G = G(n,p)$, where each edge is placed independently with probability p. Let M_n be the Class 0 profound graph with exactly $2n-3$ edges and n vertices. Given a graph with $2r-3$ edges, there are $r!$ ways to arrange the edges in such a way that an M_r is created. There are r choices for the hub vertex (the vertex of degree $r-1$), and there are $\frac{(r-1)!}{2}$ ways to arrange the other vertices up to isomorphism. Then the expected number of graphs M_r in G can be calculated as follows:

$$\begin{aligned} E(M_r) &= \binom{n}{r} p^{2r-3} \left(\frac{r!}{2}\right) \\ &< n^r p^{2r-3} = n^r \left(\frac{1}{n^{.5+\epsilon}}\right)^{2r-3}. \end{aligned}$$

Since Class 0 profoundness is a monotonic property, we can drop the $(1-p)$ term because even if some more edges in M_r are present, the graph is still Class 0 profound. Notice, as $n \to \infty$ and $r \to \infty$, that $E(M_r) \to 0$. For any other graph with at least $2r-2$ vertices, this expectation will also go to zero. Thus, since there does not exist a subgraph of size r that is Class 0 profound, the entire graph must almost surely can not be Class 0 profound.
□

Chapter 6

Threshold Results for Cover Pebbling

6.1 Introduction

At the end of the last chapter, we considered probabilistic questions relating to deep graphs. Professors Glenn Hurlbert and Zsuzsanna Szaniszlo, in personal communications with probabilist Anant Godbole, wanted to know whether any probabilistic results could be obtained related to cover pebbling. As part of (15) Godbole, Watson and Yerger obtain a probabilistic threshold result for cover pebbling the complete graph, though the proofs ultimately used for the threshold part of the paper came from work by Godbole. Watson and Yerger used more elementary techniques to obtain the same results, but there were some flaws in the analysis that led to Godbole's more elegant presentation.

This section is an attempt to understand these techniques more completely by recreating Godbole's proofs with additional exposition. Before discussing the specifics of cover pebbling questions I have considered, a brief exposition explaining probability thresholds will be presented. Following this background information, the probability threshold results will be presented.

6.2 Probability Thresholds

To understand what a probability threshold is in the context of graph theory, we will recount an example of Erdős and Renyi presented in (10) and later retold in (1). Before we begin, we will provide some relevant definitions. Given a random graph, $G(n, p)$, let n be the number of vertices of the graph and let each edge of the graph have probability p of occurring. Also, the event of each particular edge occurring is independent of the existence of other edges. Consider a property of $G(n, p)$, for instance that G is connected, call it A. The probability that $G(n, p)$ satisfies property A is denoted by $\Pr[G(n, p)| = A]$. If the function is monotonic, then the probability that this property occurs is also monotonic.

Suppose that A is the property that G is triangle-free. That is, G is triangle-free if there is no K_3 in $G(n, p)$. Suppose that X is the number of triangles that are present in $G(n, p)$. Since the existence of an edge is an independent event, we know the probability of any edge occurring is p^3. There are $\binom{n}{3}$ possible triangles, so by applying linearity of expectation, we see that $E[X] = \binom{n}{3}p^3$.

Now suppose that $p = \frac{c}{n}$. Then we see that as $n \to \infty$,

$$\lim_{n\to\infty} E[X] = \lim_{n\to\infty} \binom{n}{3} p^3 = \frac{c^3}{6}.$$

According to (1), the distribution of this expectation is asymptotically Poisson, so we can say that

$$\lim_{n\to\infty} \Pr[G(n, p)| = A] = \lim_{n\to\infty} \Pr[X = 0] = e^{-\frac{c^3}{6}}.$$

So, $G(n, p)$ almost always has a triangle when c is large and very unlikely to have a triangle when c is small. In fact, as c approaches 0 the probability of a triangle is 1 and if c approaches infinity, the probability of a triangle is 0. Notice that if c were anything but a constant, if $\lim_{n\to\infty} c = \infty$, then the event would almost never happen, and if $\lim_{n\to\infty} c = \infty$, the event would almost always happen. There is something special going on at $p = c/n$, namely that it acts as a threshold. Once the threshold of a graph property is reached, the property is very likely to occur, but the property is very unlikely before the threshold is reached. Rigorously speaking, consider the following definition, taken from (1).

Definition 6.2.1. *$r(n)$ is called a threshold function for a graph theoretic property A if*

$$(i)\text{When } p(n) \ll r(n), \lim_{n\to\infty} \Pr[G(n, p)| = A] = 0.$$

$$(ii)\text{When } p(n) \gg r(n), \lim_{n\to\infty} \Pr[G(n, p)| = A] = 1.$$

or vice versa.

Now we are ready to consider threshold problems related to cover pebbling.

6.3 Preliminaries

Throughout this chapter we will be concerned with placing pebbles on a complete graph and determining the probability of the existence of a cover solution. The pebbles are placed on the graph via a function called a *configuration*, denoted by C. The total number of pebbles placed on a graph is called the *weight* of the graph, call it t. Historically, the distribution of random objects has been described in two ways, and the notation used is derived from usage in the quantum mechanics community.

First, suppose that all distributions of pebbles are equally likely. This is the *Bose Einstein* configuration of pebbles. This means that if there are n vertices on graph G with weight t, there are $\binom{n+t-1}{t}$ equally likely configurations of pebbles. So any distribution of pebbles is likely as any other distribution of pebbles. As an example, consider the following two distributions of pebbles:

Figure 6.1: An example of two equivalent Bose Einstein distributions.

On the other hand, if we assume that the pebbles are distinguishable, then we have a different probability distribution. In this case, there are n^l configurations, each with different probability. In our example above, the configuration on the right occurs $90 = \frac{6!}{2!2!2!}$ times more often than the example on the left. This distribution is called the *Maxwell Boltzmann* probability distribution. Little if anything has been done with Maxwell Boltzmann pebbling in other papers. For the purposes of the thesis, we will compute probability thresholds for both distributions.

6.4 Proof of the Maxwell Boltzmann Threshold

In this section, we will compute the probability threshold for cover pebbling the complete graph on n vertices placing pebbles on the graph via the Maxwell Boltzmann distribution described in the previous section. Before constructing a proof for the threshold result itself, we will prove some useful lemmas that give necessary and sufficient conditions for the cover-solvability of the complete graph.

Suppose that $X = X_{n,t}$ is the number of vertices that an odd number of pebbles are placed. Colloquially, we say that X is the number of *odd stacks*. The heuristic reason why these odd stacks are important is because we "save" a pebble by having the vertex have the right parity, for if we have exactly two pebbles on a vertex, it is optimal to leave both pebbles on the vertex even though only one is necessary for a cover-solution.

Theorem 6.1. *(Watson and Yerger) A configuration of t pebbles on the n vertices of K_n is cover solvable if and only if*

$$X + t \geq 2n.$$

Proof. Suppose that C is a configuration that is cover solvable. Thus, after a sequence of pebbling moves, each previously unpebbled vertex has two pebbles associated with it - one on it, and one that was removed from the game. In the same light, any vertex that previously had a non-zero and even number of pebbles on it must have at least two pebbles left on it. However, any vertex that previously had an odd number of pebbles on it now must have at least one pebble on it (not two!). Thus, if E denotes the number of pebbles that initially had an even number of pebbles on them, it must be the case that $t \geq 2E + X$. This is equivalent to our condition.

On the other hand, if C is not cover solvable, then after a series of pebbling moves, we must reach a point where there are unpebbled vertices. Each vertex initially covered in E or unpebbled is associated with two pebbles, and each vertex in X is associated with one pebble. Thus, $t < 2E + X$. This completes the proof. □

We now offer an alternate, more concise, proof of this theorem.

Proof. Again, suppose that C is a configuration of pebbles. Let Y_i be the number of vertices with i pebbles, where $0 \leq i \leq t$. Notice that a vertex can cover exactly i other vertices if it contains exactly $2i + 1$ or $2i + 2$ pebbles.

Now, the configuration C is a cover solution if and only if

$$\sum_{i\geq 0} i(Y_{2i+1} + Y_{2i+2}) \geq Y_0.$$

This can be expanded to look closer to our desired form. So we get:

$$\sum_{i\geq 0}(2i+1)Y_{2i+1} + \sum_{i\geq 0}(2i+2)Y_{2i+2} \geq 2Y_0 + \sum_{i\geq 0} Y_{2i+1} + 2\sum_{i\geq 0} Y_{2i+2}.$$

Even more suggestive is:

$$\sum_{i\geq 0}(2i+1)Y_{2i+1} + \sum_{i\geq 0}(2i+2)Y_{2i+2} \geq 2(Y_0 + \sum_{i\geq 0} Y_{2i+2}) + \sum_{i\geq 0} Y_{2i+1}.$$

This is equivalent to $t \leq 2E + X$, completing the proof. □

Before we prove our threshold result, we will provide some evidence for its discovery in a heuristic argument. In doing so we make the assumption that X is sharply concentrated around $\mathbb{E}(X)$ if $X \sim \mathbb{E}(X)$. So suppose that a cover solution to K_n exists when $\mathbb{E}(X) \geq 2n - t$. Recall that X simply counts the number of odd stacks, so $X = \sum_{j=1}^{n} I_j$, where $I_j = 1$ if vertex j contains an odd stack of pebbles, and $I_j = 0$ if vertex j does not. We can now compute $\mathbb{E}(X)$. By linearity of expectation we have:

$$\begin{aligned}\mathbb{E}(X) &= n\mathbb{P}(I_1 = 1) \\ &= n \sum_{j\text{odd}} \binom{t}{j} (\frac{1}{n})^j (1 - \frac{1}{n})^{t-j}.\end{aligned}$$

The terms of the sum can be explained by noting that there are $\binom{t}{j}$ ways to place j vertices on a vertex out of t pebbles. At each of these vertices, there must be j pebbles on it, and $t - j$ pebbles not on it. This gives the terms of the sums. Rearranging and simplifying using the binomial expansion, we obtain:

$$\mathbb{E}(X) = \frac{n}{2}(1 - (1 - \frac{2}{n}))^t.$$

In order for $\mathbb{E}(X)$ to satisfy the necessary and sufficient condition for a cover solution, $\mathbb{E}(X) \geq 2n - t$. Thus, a cover solution exists if

$$t - \frac{n}{2}(1 - \frac{2}{n})^t \geq \frac{3n}{2}.$$

If we look for a solution for t to first order, then we will parameterize this equation by letting $t = An$. Now, a cover solution exists if

$$A - \frac{1}{2}(1 - \frac{2}{n})^{An} \geq \frac{3}{2}.$$

But notice that $(1 - 2/n)^n \sim e^{-2}$, so we can guess that the threshold, call it A_0, occurs at the point where

$$A - \frac{1}{2}e^{-2A} = \frac{3}{2}.$$

It turns out that $A_0 = 1.5238\ldots$.

6.5 Threshold via Tchebychev's Inequality

One of the ways the threshold can be obtained is via the second-moment method, which is based on Tchebychev's inequality. We now recount the statement of this theorem:

Theorem 6.2. *Let X be a random variable and let $\lambda > 0$. Then,*

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \lambda) \leq \frac{\mathbb{V}(X)}{\lambda^2}.$$

Recall that we computed $\mathbb{E}(X)$ in the last section, so we will now compute $\mathbb{V}(X)$, which is the variance of X. We now compute $\mathbb{V}(X)$.

$$\mathbb{V}(X) = \mathbb{V}(\sum_{j=1}^{n} I_j).$$

Recall that the variance of a sum of random variables is not just the sum of the variances. In this case, a covariance term must also be computed. Specifically this entails computing $\mathbb{E}(I_i I_j)$. Therefore,

$$\mathbb{V}(\sum_{j=1}^{n} I_j) = \sum_{j=1}^{n} \mathbb{V}(I_j) + \sum_{i \neq j} (\mathbb{E}(I_i I_j) - \mathbb{E}(I_i)\mathbb{E}(I_j).$$

We already can compute the first term from the results from the last section, so

$$\begin{aligned} \sum_{j=1}^{n} \mathbb{V}(I_j) &= n\mathbb{P}(I_1 = 1)(1 - \mathbb{P}(I_1 = 1)) \\ &= \frac{n}{4}(1 - (1 - \frac{2}{n})^{2t}). \end{aligned}$$

Now, we compute the covariance term of the sum. Specifically, we must compute $\mathbb{E}(I_i I_j)$, which is just $n(n-1)\mathbb{P}(I_i I_j = 1)$. We only want to count configurations when both stack i and j are odd, but all we have is an expression that is more general. This expression for $\mathbb{P}(I_i I_j = 1)$ is:

$$A = \sum_{r,s \text{ odd}} \binom{t}{r, s, t-r-s} \left(\frac{1}{n}\right)^r \left(\frac{1}{n}\right)^s \left(1 - \frac{2}{n}\right)^{t-r-s}.$$

We see this is just an application of the multinomial distribution. Godbole splits this sum up in a creative way so that only the cases when r and s are odd are counted.

$$\begin{aligned}
A &= \frac{1}{4}(A_1 - A_2 - A_3 + A_4) \text{ where,} \\
A_1 &= {}_{r,s}\binom{t}{r, s, t-r-s} \left(\frac{1}{n}\right)^r \left(\frac{1}{n}\right)^s \left(1 - \frac{2}{n}\right)^{t-r-s} \\
A_2 &= {}_{r,s}\binom{t}{r, s, t-r-s} \left(\frac{1}{n}\right)^r \left(-\frac{1}{n}\right)^s \left(1 - \frac{2}{n}\right)^{t-r-s} \\
A_3 &= {}_{r,s}\binom{t}{r, s, t-r-s} \left(-\frac{1}{n}\right)^r \left(\frac{1}{n}\right)^s \left(1 - \frac{2}{n}\right)^{t-r-s} \\
A_4 &= {}_{r,s}\binom{t}{r, s, t-r-s} \left(-\frac{1}{n}\right)^r \left(-\frac{1}{n}\right)^s \left(1 - \frac{2}{n}\right)^{t-r-s}.
\end{aligned}$$

We can see that this works by looking at the four cases of parity for r and s. Notice, that if r and s are odd, A_1 and A_4 are positive and A_2 and A_3 are negative. However, if say, r is even and s is odd, then A_1 and $-A_3$ are positive, and $-A_2$ and A_4 are negative. Notice that the absolute value of each of the A_i's are exactly the same, so in the case where r is even and s is odd, we obtain a cancelation. An analogous argument holds for the other two parity cases.

Again, applying the binomial theorem, we obtain that

$$\mathbb{P}(I_i I_j = 1) = \frac{1}{4}\left(1 + \left(1 - \frac{4}{n}\right)^t - 2\left(1 - \frac{2}{n}\right)^t\right).$$

Now, we can compute the covariance term. This means we must subtract $\mathbb{E}(I_i))\mathbb{E}(I_j)$ from $\mathbb{E}(I_i I_j)$, where $i \neq j$. With some manipulations, we see that

$$Cov(I_i, I_j) = \frac{1}{4}\left(\left(1 - \frac{4}{n}\right)^t - \left(1 - \frac{2}{n}\right)^{2t}\right).$$

Therefore, we can now compute the variance. Thus,

$$\mathbb{V}(X) = \frac{n}{4}\left(1-\left(1-\frac{2}{n}\right)^{2t}\right) + \frac{n(n-1)}{4}\left(\left(1-\frac{4}{n}\right)^{t} - \left(1-\frac{2}{n}\right)^{2t}\right).$$

Given that we now know the mean and the variance term, we can use the second moment approximation to obtain a threshold for the Maxwell Boltzmann distribution. Now we will prove the following theorem:

Theorem 6.3. *Consider t distinct pebbles that are thrown onto the vertices of the complete graph K_n on n vertices according to the Maxwell Boltzmann distribution. Set $A_0 = 1.5238\ldots$. Then*

$$t = A_0 n + \phi(n)\sqrt{n} \sim \mathbb{P}(K_n \text{ is cover solvable }) \to 1 \text{ as } n \to \infty.$$

and

$$t = A_0 n - \phi(n)\sqrt{n} \sim \mathbb{P}(K_n \text{ is cover solvable }) \to 0 \text{ as } n \to \infty.$$

where $\phi(n) \to \infty$ is arbitrary.

Proof. Our goal is to show that $\mathbb{P}(X \geq 2n - t) \to 1$ as $n \to \infty$. Let t be of the form, $t = A_0 n + \phi(n)\sqrt{n}$. We want to find the value of $\mathbb{P}(X \geq 2n - t)$. Subtracting $\mathbb{E}(X)$ from both sides, we see that:

$$\mathbb{P}(X \geq 2n - t) = \mathbb{P}\left(X - \mathbb{E}(X) \geq 2n - t - \frac{n}{2}\left(1-\left(1-\frac{2}{n}\right)^{t}\right)\right).$$

Substituting for $t = A_0 n + \phi(n)\sqrt{n}$, we obtain:

$$= \mathbb{P}\left(X - \mathbb{E}(X) \geq \frac{3}{2}n - A_0 n - \phi(n)\sqrt{n} - \frac{n}{2}\left(1-\frac{2}{n}\right)^{A_0 n + \phi(n)\sqrt{n}}\right).$$

Now, recall that A_0 satisfied the equation $A_0 + \frac{1}{2}e^{-2A} = \frac{3}{2}$. Therefore, in our equation, we can substitute $\frac{3n}{2}$ by $A_0 n + \frac{n}{2}e^{-2A}$. This gives us:

$$= \mathbb{P}\left(X - \mathbb{E}(X) \geq -\frac{n}{2}e^{-2A_0} - \phi(n)\sqrt{n} + \frac{n}{2}\left(1-\frac{2}{n}\right)^{A_0 n + \phi(n)\sqrt{n}}\right).$$

Continuing to be clever, we now use the well-known inequality $1 - x \leq e^{-x}$. If $x = \frac{2}{n}$, then we can substitute $e^{\frac{-2}{n}}$ for $1 - \frac{2}{n}$ to obtain:

$$\geq \mathbb{P}\left(X - \mathbb{E}(X) \geq -\frac{n}{2}e^{-2A_0} - \phi(n)\sqrt{n} + \frac{n}{2}\left(\exp\{-2A_0 - \frac{2\phi(n)}{\sqrt{n}}\}\right)\right).$$

This simplifies to:

$$= \mathbb{P}\left(X - \mathbb{E}(X) \geq \frac{n}{2}e^{-2A_0}\left(\exp\{-\frac{2\phi(n)}{\sqrt{n}}\} - 1\right) - \phi(n)\sqrt{n}\right).$$

We now use a lesser known inequality, $e^{-x} - 1 \leq -x/(1+x)$ to get the next line of the proof. Specifically, the term $\exp\{-\frac{2\phi(n)}{\sqrt{n}}\} - 1$ is in the form $e^{-x} - 1$. After the use of this inequality, we have:

$$\geq \mathbb{P}\left(X - \mathbb{E}(X) \geq \frac{n}{2}e^{-2A_0} - \frac{2\phi(n)}{\sqrt{n}} - 1(1+o(1)) - \phi(n)\sqrt{n}\right)$$

$$= \mathbb{P}\left(X - \mathbb{E}(X) \geq -\phi(n)\sqrt{n}(1 + e^{-2A_0}(1+o(1))\right).$$

Taking the absolute value of $X - \mathbb{E}(X)$, we get to a usable answer, namely

$$\geq \mathbb{P}\left(|X - \mathbb{E}(X)| \leq \phi(n)\sqrt{n}(1 + e^{-2A_0}(1+o(1))\right).$$

Before completing the manipulations required to obtain the threshold, we must look at the variance to see how concentrated the data is. It turns out that we can show $\mathbb{V}(X) = \Theta(n)$, and so X is concentrated with high probability in some interval of length $\Omega(\sqrt{n})$ around $\mathbb{E}(X)$. Recall the meanings of these various asymptotic notations. $\Theta(n)$ is an asymptotically tight upper and lower bound, $\Omega(n)$ is an asymptotic lower bound, $O(n)$ is an asymptotic upper bound. Also $o(n)$ is a non-asymptotically tight upper bound.

The variance term calculated before was:

$$\mathbb{V}(X) = \frac{n}{4}\left(1 - \left(1 - \frac{2}{n}\right)^{2t}\right) + \frac{n(n-1)}{4}\left(\left(1 - \frac{4}{n}\right)^{t} - \left(1 - \frac{2}{n}\right)^{2t}\right).$$

For some constant, call it K, the first term of this variance can be simplified to $\frac{n}{4}(1 + K(1+o(1)))$. Let us now focus on the remaining part of the variance term. Expanding out the last exponent, we get:

$$\frac{n(n-1)}{4}\left(\left(1 - \frac{4}{n}\right)^{t} - \left(1 - \frac{2}{n}\right)^{2t}\right) =$$

$$\frac{n(n-1)}{4}\left(\left(1 - \frac{4}{n}\right)^{t} - \left(1 - \frac{4}{n} + \frac{4}{n^2}\right)^{t}\right).$$

This second expression can be bounded using the inequality $t(b-a)a^{t-1} \leq b^t - a^t \leq t(b-a)b^{t-1}$ to obtain an asymptotic bound for the second part of the sum. So

$$\frac{n(n-1)}{4}\left(\left(1-\frac{4}{n}\right)^t - \left(1-\frac{4}{n}+\frac{4}{n^2}\right)^t\right)$$

$$\begin{array}{lc} \leq & \frac{n(n-1)}{4}\left(\left(1-\frac{4}{n}+\frac{4}{n^2}\right)^t - \left(1-\frac{4}{n}\right)^t\right) \\ = & \frac{n(n-1)}{4}\left(1-\frac{4}{n}+\frac{4}{n^2}-1+\frac{4}{n}\right)^t\left(1-\frac{4}{n}+\Theta\left(\frac{1}{n^2}\right)\right)^{t-1} \\ = & \frac{n(n-1)}{4}\frac{4t}{n^2}\left(1-\frac{4}{n}+\Theta\left(\frac{1}{n^2}\right)\right)^{t-1} \\ = & \Theta(t) = \Theta(n). \end{array}$$

Thus, $\mathbb{V}(X) = \Theta(n)$. Now, if K is a constant, we can apply Tchebychev's inequality, to get that

$$\mathbb{P}(X \geq 2n-t) \geq \mathbb{P}(|X - \mathbb{E}(X)| \leq K \times \sqrt{n}\phi(n) \geq 1 - \frac{1}{K^2\phi^2(n)}.$$

Notice as $n \to \infty$, this probability approaches 1. This proves one of the two parts of the threshold. The proof for the other direction of the threshold is analogous to this one.
□

6.6 Bose Einstein Cover Pebbling Threshold

In this section, we will begin to derive the threshold for Bose Einstein cover pebbling. At the present time there are issues with the proof, but we can describe some important preliminary ideas. For Maxwell Boltzmann distributions, it was difficult to obtain an explicit formula for the probability of the number of odd stacks. However, we can get a nice closed form Bose Einstein cover pebbling.

Theorem 6.4. *Let t be the number of pebbles placed on K_n and let x be the number of odd stacks. Suppose that t and x have the same parity and that $x \geq \min t, n$. Then,*

$$\mathbb{P}(X = x) = \frac{\binom{n}{x}\binom{\frac{t-x}{2}+n-1}{n-1}}{n+t-1} n - 1.$$

Proof. If t and x have the same parity, we can place pebbles on K_n in a way that combinatorially explains the preceding identity. First, place one pebble on each of the x vertices that are odd stacks. Then, with the remaining $t - x$ pebbles, we can place $\frac{t-x}{2}$ indistinguishable pairs of pebbles on the n vertices. This accounts for the numerator of the preceding expression and counts the total number of configurations of pebbles with x odd stacks and t pebbles. The denominator of the theorem counts the total number of configurations possible given any number of odd stacks. This is derived from the classic combinatorial "stars and bars" interpretation, where there are $n - 1$ lines dividing a set of t pebbles for a total of $n + t - 1$ spaces. $\square$

Recall that in Bose Einstein pebbling each of the $\binom{n+t-1}{n-1}$ configurations are just as likely. In this case, there is no obvious sequential process that describes how the pebbles are placed. However, if we recast the problem in another probabilistic viewpoint, it turns out that we can describe Bose Einstein pebbling via a sequential process. Instead of "throwing balls into boxes" (15) we "draw balls *from* boxes". This process can be modeled in a procedure known as Polya sampling.

In Polya sampling, we begin with a box that contains one ball each of n colors. When a ball is chosen from the box, it is recorded and then that ball as well as another ball of that same color is placed into the box. In this case order matters, that is, the sequence of balls selected influences subsequent ball selections. It would be wonderful if the number of times a ball of color i was selected corresponded to the number of pebbles placed on vertex i. This seems unlikely because the order of the ball/pebble selection matters. Surprisingly, however, it turns out this procedure does allow the probability of each configuration of pebbles to be equal.

Theorem 6.5. *Let X_j be the number of times the color j is drawn among the t draws. Then for any $x_1, x_2 \dots, x_n$ where $\sum x_j = t$,*

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \frac{1}{\binom{n+t-1}{t}}.$$

Proof. First suppose the balls are drawn from the box so that the first x_1 balls are color X_1, the next x_2 balls are colored X_2 and so forth. Call this event B and notice that the order the balls were chosen is part of the event.

Then

$$\begin{aligned}\mathbb{P}(B) &= \frac{(1\cdot 2\cdots x_1)(1\cdot 2\cdots x_2)\cdots(1\cdot 2\cdots x_n)}{(n)(n+1)\cdots(n+t-1)} \\ &= \frac{x_1!x_2!\ldots x_n!}{(n)(n+1)\cdots(n+t-1)}.\end{aligned}$$

Notice that this probability is the same regardless of the order the balls are drawn in. The probabilities of the denominator will remain the same because the first ball is drawn out of n balls, the next ball is drawn out of $n+1$ and so forth. To see that the numerator is the same regardless of order, notice that when the k^{th} ball of a particular color is drawn, there are k balls in the box. Thus, regardless of order, the component of the numerator associated with X_1 is $x_1!$.

To finish the proof we must compute $\mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$. We can order the selection of pebbles in

$$\frac{t!}{x_1!x_2!\cdots x_n!}$$

ways. Multiplying this by the probability that a particular configuration occurs will equal the probability we desire. Let C be the event that $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$. Then

$$\begin{aligned}\mathbb{P}(C) &= \frac{t!}{x_1!x_2!\cdots x_n!}\cdot\frac{x_1!x_2!\ldots x_n!}{(n)(n+1)\cdots(n+t-1)} \\ &= \frac{1}{\binom{n+t-1}{t}}.\end{aligned}$$

This completes the proof of the theorem. □

Before stating the theorem that computes the Bose Einstein cover pebbling threshold, we will describe a recent probabilistic technique, called the method of bounded differences. This is also called the Azuma-Hoeffding inequality. This inequality is based on a probabilistic concept called a *martingale*.

Definition 6.6.1. *A martingale is a sequence $X_0, \ldots X_m$ of random variables so that for $0 \le i < m$,*

$$E[X_{i+1}|X_i] = X_i.$$

Martingales can be applied to describe random graph theoretic processes, understand deviations from means, such as the Chernoff bounds discussed in Chapter 5, and even in a recent paper of Bollobas that describes a bound for the chromatic number of a graph.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where $\{Y_n\}$ is a sequence of random variables on the probability space, which is not necessarily independent. For our application of the Azuma-Hoeffding inequality, the draws made via Polya sampling is the sequence of random variables $\{Y_n\}$. More generally, for a collection of random variables $\{\tau_i\}_i = 1^n$, a sequence of random variables $X_0, X_1, \ldots, X_m$ is a *martingale sequence* with respect to $\{\tau_i\}_i = 1^n$

$$E[X_{i+1}|\tau_0, \tau_1, \ldots, \tau_i] = X_i.$$

Now, let $X = X_t = X(Y_1, \ldots, Y_t)$ be the number of even stacks of pebbles. Also, consider the filtration, essentially a sequence of σ-algebras, which is just a set of subsets of $\mathcal{F}$. Recall the following definition of a σ-algebra.

Definition 6.6.2. *A σ-algebra is a set of subsets of Ω that satisfy the following three properties:*
1. The empty set is in $\mathcal{F}$.
2. If A is in $\mathcal{F}$, then so is the complement of A.
3. If A_n is a sequence of elements of $\mathcal{F}$, then the union of the A_n's is in $\mathcal{F}$.

So, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and in general $\mathcal{F}_i = \sigma(Y_1, \ldots, Y_i)$. Also, denote $\mathbb{E}_i X$ as the conditional expectation of X with respect to $\mathcal{F}_i$, and let $d_i = \mathbb{E}_i X - \mathbb{E}_{i-1} X$. This sequence $(d_i, \mathcal{F}_i)$ is called a Martingale difference sequence. On one side of the Azuma-Hoeffding inequality, we have $X - \mathbb{E}(X)$, and in the language of the Martingale difference sequence, it is just $\sum_{i=1}^{t} d_i$. We can finally state the inequality. (1)

Theorem 6.6. *For all $\lambda > 0$,*

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \lambda) \geq 2 \exp\left\{\frac{-\lambda^2}{2\sum ||d_i||_\infty^2}\right\}.$$

We're still not finished with analysis notation and definitions. Define $s = ||Z||_\infty$ as the essential supremum of $Z(\omega)$. For the purposes of this thesis, we will simply say that the essential supremum is just the smallest number for which s only exceeds $Z(\omega)$ on a set of measure zero.

We can directly use the inequality by manipulating d_i terms. This allows us to determine a bound for the interval that describes the location of the concentration for the number of odd stacks in Bose-Einstein pebbling. Suppose that Y_i^* is an independent copy of Y_i. Then,

$$\mathbb{E}_{i-1}X(Y_1,\ldots,Y_t) = \mathbb{E}_i X(Y_1,\ldots,Y_{i-1},Y_i^*,Y_{i+1},\ldots,Y_t).$$

We see this since the probability of each vertex being an even stack is equivalent. Now, we know that d_i is just the following difference:

$$d_i = \mathbb{E}_i(X(Y_1,\ldots,Y_t) - X(Y_1,\ldots,Y_{i-1},Y_i^*,Y_{i+1},\ldots,Y_t)).$$

This implies that

$$\begin{aligned} ||d_i||_\infty &= ||\mathbb{E}_i(X(Y_1,\ldots,Y_t) - X(Y_1,\ldots,Y_{i-1},Y_i^*,Y_{i+1},\ldots,Y_t))||_\infty \\ &\leq ||X(Y_1,\ldots,Y_t) - X(Y_1,\ldots,Y_{i-1},Y_i^*,Y_{i+1},\ldots,Y_t)||_\infty. \end{aligned}$$

Notice that this last term is at most two, since by adding or removing one pebble, we can change the number of even stacks by at most two. So, using this and the Azuma-Hoeffding inequality we see that

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{8t}}.$$

With this at hand we can do some more computations and manipulations to obtain the threshold result. This result will be formally completed by Godbole in May. Hence, to conclude this section, we state the result.

Theorem 6.7. *Consider t distinct pebbles that are placed on the vertices of the complete graph K_n according to the Bose Einstein distribution. Then, with γ representing the golden ratio $(1+\sqrt{5})/2$,*

$$t = \gamma n + \varphi(n)\sqrt{n} \sim \mathbb{P}(K_n \text{ is cover solvable}) \to 1 \quad (n \to \infty)$$

and

$$t = \gamma n - \varphi(n)\sqrt{n} \sim \mathbb{P}(K_n \text{ is cover solvable}) \to 0 \quad (n \to \infty),$$

where $\varphi(n) \to \infty$ is arbitrary.

Chapter 7

Conclusion

We examined a wide variety of extensions to graph pebbling, which included structural, probabilistic and computational results in cover pebbling, domination cover pebbling and deep graphs. These results can serve as starting points to many more complex investigations in pebbling because most of the results are the first of their kind. The cover pebbling threshold result for the complete graph is one such example. Also, Nathaniel Watson is currently working on other complexity issues with cover pebbling and extending the results proved in the thesis. Questions about domination cover pebbling and its relationship to the vertex neighbor integrity of a graph remain quite open and would be a suitable topic for further research.

Another interesting characteristic of pebbling problems is that they form new links between areas of mathematics, such as additive number theory and graph theory. Domination cover pebbling serves as a connection between graph domination, a substantial subfield of graph theory and pebbling, a new but rapidly growing field on its own. Deep graphs help us to understand more about the structure of Class 0 graphs, which may be useful in trying to finally prove Graham's conjecture. Complexity results link graph pebbling to the theory of computation. Finally, the well-known relationships between probability theory, graph theory and combinatorics are highlighted in the cover pebbling and deep graph threshold results.

7.1 Conclusion

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