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Ahmed Bilal

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Abstract

The Backward Induction strategy for the Centipede Game leads us to a counterfactual reasoning paradox, The Centipede Game paradox. The counterfactual reasoning proving the backward induction strategy for the game appears to rely on the players in the game not choosing that very same backward induction strategy. The paradox is a general paradox that applies to backward induction reasoning in sequential, perfect information games. The Centipede Game is a prime illustration of this paradox in counterfactual reasoning. As a result, this paper will use a material versus subjunctive/counterfactual conditional analysis to provide a theoretical and empirical resolution to the Centipede Game, with the hope that a similar solution can be applied to other areas where this paradox may appear. The solution involves delineating between the epistemic systems of the players and the game theorists.

1 Introduction

The Centipede Game was first introduced by Robert Rosenthal in 1981. It is a finite n-person extensive form game with perfect information. In its basic form, two players sequentially chose to either “Take” the larger share of an increasing pot which ends the game, or “Pass” on the pot to the opponent. Each time a player gets to chose either Take or Pass, they are at one of the terminal nodes of the game.

The payoffs of the game are configured in such a that, upon a player choosing Pass at her node, if her opponent chooses Take at the subsequent node, then the original player will receive less than the payoff she would have received, had she chosen Take. Since there is Perfect Information in the game, ”Common Knowledge of Rationality” is at play in the game.

1.1 Research Questions

This paper will focus on two main research questions:

1. **Theoretical solution to the Centipede Game Paradox**

   The Backward Induction theory for the Centipede Game leads us to a counterfactual reasoning paradox that is discussed at length later in the paper. The paradox can be summarized in the following way. Consider the following game:

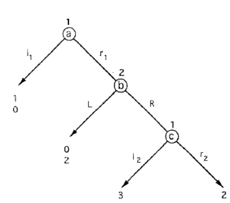

Fig. 1. A simple game.

Assume player 1 is at node c, where she knows that she gets a higher conditional payoff by choosing Take. Player 2 knows this; hence, player 2 will chose Take at node b. Inductively proceeding, player 1 will chose backward induction reasoning to chose Take at node a and end the game on the first move. The problem is that this reasoning assumes player 1 will reach the last node, c, which is only possible if both players deviate from the backward induction equilibrium path i.e player 1 and 2 choose Pass instead of Take on the initial nodes. Therefore, the counterfactual reasoning proving the backward induction strategy seems to rely on the players not choosing the backward induction strategy.

The paradox is a general paradox that applies to backward induction reasoning in sequential, perfect information games. Therefore, the paradox is not only problematic for the Centipede Game, but affects counterfactual reasoning solutions in games similar to the Centipede Game. The Centipede Game is a prime illustration of this paradox in counterfactual reasoning. As a result, this paper will attempt to provide a theoretical solution to the Centipede Game, with the hope that a similar solution can be applied to other areas where this paradox appears.

The solution involves delineating between the epistemic (relating to knowledge) systems of the players and the game theorists. The players in the game are allotted a distributed knowledge system, just enough knowledge at each node to compute the backward induction equilibrium solution for that node. This retains the players' ability to use backward induction reasoning to calculate the move with the highest conditional payoff for their respective node in all situations, in cases of deviations and in cases without. Therefore, in response to a deviation, the solution is not contradicted and in need of revision at the players' level. Instead, in response to a deviation, the revision to the solution happens at the meta-game level, at the level of the game theorists'. The game theorists can then use counterfactual conditional analysis to reveal the true game tree that is appropriate; hence, revise their theory of the game after every move that the players make.

Currently it is generally accepted that there is no clear consensus on resolving the paradox. Robert Aumann and Kenneth Binmore are the two main theorists on opposing camps in the debate. Both often contest that the other is misunderstanding their theory. There have been several individual attempts to solve the paradox, but it is hard to see how they all fit together towards explaining why the paradox exists and how to solve it.

In regards to this question, the purpose of this paper will be to:

(a) Explain Aumann's main proof supporting the backward induction outcome in the game, present an objection to it and then present a modified version of the proof in response

(b) Revisit the Aumann and Binmore debate and illustrate how the debate reduces to a problem of modelling material versus subjunctive/counterfactual conditionals. We will then employ a game tree analysis to clarify why the paradox exists

(c) Illustrate how epistemic system delineation solves the paradox

2. **How well do the Centipede Game backward induction theoretical results predict the empirical results?**

   The Centipede Game's theoretical solution is built upon backward induction reasoning, an instrumental tool in Game Theory. Hence, it is important to ask if such theoretical solutions predict human behavior or are they simply, theoretical tools, with no empirical bearing.

The general consensus tends to be that the empirical results of the Centipede Game deviate significantly from the theory. Therefore, it is generally believed that the theoretical project of the Centipede Game has little bearing on predicting human behavior, making it a weak economic theory. Recently, more empirical literature has come out supporting results that are different from the initial studies. However, it is unclear how all the empirical literature fits together, due to the different specifications of each of the experiments. Therefore, it is hard to draw any main conclusions from the empirical literature.
In regards to this question, the purpose of this paper will be to:

(a) Analyze the main Centipede Game empirical studies and illustrate how the particular specifications of each of the experiments affect their respective results

(b) Explain how Aumann's distinction between Rationality and Common Knowledge of Rationality shows up in the empirical literature and interpret experimental results in light of this distinction

(c) Present a statistical model that accounts for most of the empirical results of the game and illustrate how the model incorporates the main theoretical intuitions of both, Aumann and Binmore.

2 Common Knowledge of Rationality

In this section, we will start by revisiting Aumann's proof of how Common Knowledge of Rationality (CKR) leads to the Backward Induction (BI) equilibrium outcome in Perfect Information games. This will allow us to further infer how CKR leads to the BI equilibrium outcome in the Centipede Game.

Aumann sheds light on the concept of CKR in Perfect Information (PI) games in his seminal paper, *Backward Induction and Common Knowledge of Rationality*. Aumann (1995) aims to prove :

Theorem A: CKR $\subset$ **I** {I: Backward Induction outcome}

In order to understand Aumann's proof for Theorem A, it is important to outline some initial definitions, axioms and lemmas that will prove relevant later on.

2.1 Definitions

Knowledge (K) stands in contrast to Beliefs, which are often modelled as Bayesian probability functions. Knowledge, instead, can be understood as true belief. Knowledge is a belief with probability 1. This means that Knowledge is a belief that is necessarily true of the world inhabited by the believer. Therefore, a player i's Knowledge of an event (E) implies that (E is true $\cap$ i is certain that E is true in the world). As Aumann puts it, it requires "absolute certainty". Aumann also defines "time of the players' knowledge" (Aumann 1995, 8) to the start of the play.

E is a set of the states of the world. Common Knowledge of an E is defined as

$$\text{CKE} = \text{KE} \cap \text{KKE} \cap \text{KKKE} \cap ...$$

An important feature of the Knowledge operator K is that it can quantify over itself. This is because Knowledge of an Event counts as an event itself. Therefore, it is appropriate to use the K operator in the following way: KKE_i. Let b be "the strategy profile" that assigns the inductive choice (I) to each node v. Then for each node v of each player i, the following equation holds:

$$h_i^v(b) \geq h_i^v(b; a^v)$$

for all actions a^v at v.

This equation means that the conditional payoff at node v of playing the inductive choice action (I) strategy profile is greater or equal to the conditional payoff from playing any other strategy profile, b^v at v.

For example, if player i is at node v, then the move prescribed at v by the inductive choice strategy yields i a conditional payoff at v, greater or equal to the conditional payoff i would have received from playing any other move at v.

Rationality for i is defined as following:

$$R_i = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} (\sim K_i[h_i^v(s; t_i) > h_i^v(s)])$$

for all of i's nodes, it is not the case that i Knows that there exists an action at that node (from the set of all available actions on the node) that will yield i a conditional payoff greater than the one that i receives due to the action performed by i, as prescribed by the strategy played by i.

For example, if i is at node v and plays Take, then if i is rational, then it is the case that i does not Knows of any strategies at v and the nodes after v, that would have yielded i a higher conditional payoff than the one i receives from playing Take at v.

2.2 Lemmas:

(a) CKE = K_iCKE

This means that if E is Common Knowledge, then i not only Knows E, but also, Knows that she Knows E is Common Knowledge.

(b) If E $\subset$ F, then K_iE $\subset$ K_iF

This means that if it is the case that E entails F, then if i Knows E, then i must Know F as well. For example, assume i is at node v. Since reaching v requires i to have traversed all {<v}: the set of nodes before v that are necessary to reach v. If Knows that i is at v, i must Know that i has traversed {<v} nodes.

(c) K_iE $\cap$ K_iF = K_i(E$\cap$F)

(i Knows E and i Knows F) is equivalent to i Knows (E and F), or that i Knows Event G : (G = E $\cap$ F). This means that events separately Known to i can always be jointly Known. This Lemma is instrumental for Common Knowledge. Jointly Knowing separately Known Events is what makes the CKE definition involving the intersections of Knowledge Events possible.

(d) CKE $\subset$ E

If it is the case that an event E is Common Knowledge, then it is the case that E is true. This follows directly from the definition of Common Knowledge. Knowledge of an event cannot be a false belief.

(e) $K_i{\sim}K_i$E = ${\sim}K_i$E

If it is not the case that i Knows E, then it is also true that i Knows that i does not Know E.

(f) K_iE $\subset$ E

This follows directly from the definition of Knowledge operator, K. K is a probability 1 belief.

(g) $I^v \subset K_i I^v$

For Lemma 7, the following definitions and axioms are necessary for its proof:

i. $I^v := [s^v = b^v]$

I^v is defined recursively. I^v, the inductive choice at v, is the Event that i chooses the "action that maximizes i's payoff" in the case that all players make the inductive choices at all nodes after v.

ii. Axiom A: $[s_i = s_i] \subset K_i\ [s_i = s_i]$ for all $s_i \in S_i$

for $\forall\ v_i$: if i actually did chose I^v at v, then i Knows that i chose I^v.

This means that for all strategies available to i (for all elements part of i's strategy profile, S_i), if it is the case that i chose s_i, then i Knows that the strategy chosen by i is s_i. There is no possibility of false Knowledge. In other words, i cannot be mistaken about the strategy that she chose. This again, follows from the definition of the Knowledge operator, K.

2.2.1 Lemma 7 Proof:

Let B_i^v denote the set of those strategies s_i of i for which $s_i^v = b^v$. Then by (2) and Lemma 5,

$\mathrm{I}^v = [s_i^v = b^v] = \bigcup_{s_i \in B_i^v} [\mathbf{s}_i = s_i] \subset \bigcup_{s_i \in B_i^v} K_i[\mathbf{s}_i = s_i] \subset \bigcup_{s_i \in B_i^v} K_i[\mathbf{s}_i^v = b^v] \subset \bigcup_{s_i \in B_i^v} K_i I^v = K_i I^v$

2.2.2 Lemma 7 Proof summary:

By definition, I^v is the event that the inductive choice, b^v is made at v. Therefore, I^v implies that i chooses a strategy s_i that prescribes b^v at v from the set of all such strategies, B^v. As a result, if i Knows that I^v, it entails that i Knows that i chose b^v at v, as prescribed by the s_i, from the set B^v. Following axiom A, we Know that since i Knows that i chose b^v at v, i actually did choose the inductive choice b^v at v. Hence, $I^v \subset K_i I^v$

These definitions, axioms, and Lemmas allow us to prove Theorem A: CKR $\subset I^v$:

2.3 Theorem A Proof:

CKR $\subset K_i[s^{>v} = b^{>v}] \cap \sim K_i[h_i^v(s; b_i) > h_i^v(s)] \subset \sim K_i[h_i^v(b) > h_i^v(s^v)] = \sim K_i[s^v \neq b^v] = \sim K_i \sim I^v = \sim K_i \sim K_i I^v = \sim\sim K_i I^v = K_i I^v \subset I^v$

2.4 Theorem A proof Explanation:

Assume player i is at v, CKR entails that, for all nodes after v : {>v}, i Knows that i chooses $b^{>v}$ at all {>v} nodes (Lemma 1) and i does not Know that i's conditional payoff would have been higher had she chosen some strategy other than the one that she actually chooses, as prescribed by the strategy that she actually chooses (Rationality principle). This entails that i does not Know that she chose an action other than the inductive choice, b^v at v. Hence, i does not Know that I^v did not occur (Lemma3 and Lemma 5). This implies that it is not the case that (i does not know I^v occurred). In other words, i does not know that any event other than I^v occurred. Following Lemma 5, this further means that i Knows that i does not Know that any event other than I^v occurred (Lemma 2). Therefore, I Knows that I^v must have occurred (Lemma 6). Following Lemma 6, since i Knows that I^v occurred, I^v must have occurred. Hence, CKR $\subset I^v$ (Lemma 7).

2.5 Objection:

Aumann's proof seems to depend on lemma 6. It can be argued that it is odd to claim that Knowledge of an E actually entails E, since it seems like it is E that entails Knowledge of E, not the other way around. This objection can be circumvented by replacing Lemma 6 by the much more intuitive, Axiom A:

2.6 Solution:

Let A_i^v denote the set of those strategies such that $s_x^v \neq b^v$. In words, s_x is a strategy that does not prescribe b^v at v. Moreover, as shown above, CKR $\subset\sim K_i \sim I^v$). $\sim I^v$ means that i picked a strategy that does not prescribe b^v at v. We can write this as the following proposition:

B_i : for all $s_x^v \in A_i^v$, CKR (i does not Know that i picked s_x^v)

Axiom A states that if i picked strategy s_i, then i Knows that he picked strategy s_i. The contrapositive of A is:

Axiom -A: if i does not Know that i picked strategy s^i then i did not choose s^i.

Using B_i and axiom -A: CKR $\subset$ ($forall s_x \in A_i^v$, i does not Know that i chose s_x^v). Therefore, i did not chose s_x^v, for all s_x^v $\in A_i^v$. Hence, I^v: i chose a strategy that prescribes b^v at v.

3 Conditionals

As Aumann notes, a crucial role in this paper is played by conditionals. Aumann briefly discusses the three different types of conditionals, material, subjunctive and counterfactuals:

"*Consider, for example, the statement "If White pushes his pawn, Black's queen is trapped." For this to hold in the material [conditional] sense, it is sufficient that White does not, in fact, push his pawn. For the substantive [conditional] sense, we ignore White's actual move, and imagine that he pushes his pawn. If Black's queen is then trapped, the substantive conditional is true; if not, then not. If White did not push his pawn, we may still say "If he had pushed his pawn, Black's Queen would have been trapped…This is a counterfactual [conditional].*" (Aumann 1995, 14)

Aumann claims that substantive/subjunctive conditionals and counterfactual conditionals "are important in interpreting four key concepts that were formally defined in Section 2: strategy, conditional payoff, rationality at a vertex, and rationality" Aumann 1995, 16). Substantive conditionals are especially needed to define rationality. For i to be rational at v, "means that he cannot knowingly increase his payoff if v is reached" (Aumann 1995, 16). Similarly, the substantive conditional is needed to define rationality at all of i' nodes. The importance of substantive conditionals is seen by the fact that material rationality (rationality at reached nodes only), does not imply Backward Induction, I^v. This means that Theorem A fails without substantive and counterfactual conditionals.

A good illustration of the need for substantive and counterfactual conditionals for Theorem A is players' judgments of conditional payoffs at each node. For Aumann's proof, a player is expected to occupy a very peculiar epistemic state in order to assess his conditional payoff at his nodes, "when evaluating his conditional payoff at v, the player must assume that v is reached, even when he knows that it is not" (Aumann 1995, 17). The reason such an epistemic state cannot be modelled with a regular material conditional is that a material conditional considers all "if" statements with a false antecedent to be true. This leads to the following problem: player i Knows the following proposition:

P : node v will not be reached.

This means K_iP. Applying Lemma 6, $K_iP \subset$ P. In other words, since i Knows P and Knowing is a probability 1 belief, P is the case, Since P, any statement of the form "if node v is reached, strategy s_x is rational" will be true since the antecedent (node v is reached) is false. If all strategies, regardless of their conditional payoffs at v are equally rational, this violates the Rationality definition and Theorem A fails. As Aumann puts it, player i "cannot say "since I know that v is not reached, whatever I do there is rational."" (Aumann 1995, 16). As a result, there is a need for defining conditionals that can model the conditional payoff at v as needed. Analysis of substantive/counterfactual conditionals will become the central point of dispute in theoretical modelling of the Centipede Game.

4 Centipede Game

Aumann (1998) starts by referring to Theorem A, "in perfect information PI games, common knowledge of rationality implies that the backward induction outcome is reached (Aumann, 1995; henceforth [A])" . He is quick to mention that "conceptually, this result depends on the notion of counterfactual conditional", along with the notion of substantive rationality as opposed to material rationality. With these, he aims to show in this paper "that in Rosenthal's centipede game, if at the start of play there is common knowledge of rationality, then the backward induction outcome results:the first player ''goes out'' immediately" (Aumann 1998, 1).

4.1 New Vocabulary for Aumann (1998) Proof

(a) Ω^v: the event that "v is reached"

(b) K_i^v: represents i's information when he "learns whether v is reached; that is, the information he had at the start of play, updated by the information that v is or isn't reached."

(c) $K_i^v E$: i Knows E at v

(d) $K_i E$: i knows E at the start of play

In order to circumvent criticism of using subjunctive conditionals in his proof, Aumann modified his definition of Rationality from $[A]$ to make use of only material conditionals.

"In a given state ω, call i *ex post rational* at v if there is no strategy that i knows at v would have yielded him a conditional payoff at v larger than that which in fact he gets; call i *ex post materially rational* if he is ex post rational at each of his reached vertices. Denote by R_i^v the event ‘i is ex i post rational at v;’’ then the event ‘‘i is ex post materially rational,’’ denoted R_i^M, is given by:” (Aumann 1998, 99)

$$R_i^M = \bigcap_{v \in V_i} (\sim \Omega^v \cap R_i^v)$$

This means that assuming Common Knowledge of *ex post material rationality* (CKR^M) holds,I^v will occur in Rosenthal’s PI Centipede Game with N nodes or a game of the same type. The I^v in this game is Player 1 “‘‘goes out’’ (chooses Take) at the first move”. Reductio arguments can only be made with material conditionals. Since [A] relied on subjunctive/counterfactual conditionals, [A] was not a reductio argument. Since Aumann is making use of material conditionals here, he can make use of a reductio proof that relies of [A], but is different.

4.2 Centipede Game I^v proof:

Assume Ω^m: {m is the last node}, that is, the last node, m is reached in the game by player i (*w.l.o.g* since CKR^M holds at the start of the game, K_iR^M: i Knows that R^M holds. Hence, R^M (material rationality) holds. Therefore, R^M holds at the reached m node and it is “is commonly known that no vertex beyond m is reached.”

(a) If m is the first node, following R^M, i will play Take rather than Pass and the game will be finished.

(b) If m is not the first node, i will still play Take at m since R^M still holds. However, since m-1 node exists, player y at m-1 will have already anticipated i’s move at m because CKR^Mstill holds. It is clear than playing Take increases y’s conditional payoff at m-1. Therefore, y plays Take at m-1. However, i could have only reached m had y played Pass instead of Take at m-1. This leads to a contradiction.

As a result, following CKRM , *w.l.o.g* player i will play Take at the first node.

The reason for switching his proof from [A], was that Aumann anticipated the following subjunctive mode critique of [A] for the Centipede Game, “on the one hand, we are told that under common knowledge of rationality CKR., Ann [player x] must go out at her first move. On the other hand, the backward induction argument for this is based on what the players would do if Ann stayed in. But, if she did stay in, then CKR is violated, so the argument that she will go out no longer has a basis” (Aumann 1998, 103). Aumann claims that it is not necessary to make use of the subjunctive mood; the proof in this note refers only to rationality at nodes that are actually reached (material condition), not to whether players ‘‘would’’ play rationally if their nodes ‘‘were’’ reached (subjunctive condition).” The principle difference from the proof in [A] is that rather than reasoning from what happens after a given vertex is reached, it reasons from what happens before it is reached. This means that “if some vertex is the last that can possibly be reached, then already the one before it should have been the last” (Aumann 1989, 103).

5 Deviations

Binmore (1994) starts by attacking Aumann’s [*A*] proof by claiming that Aumann is wrong to argue that prior common knowledge of rationality implies backward induction. Binmore agrees with Aumann that it is the case that a rational player will begin the Centipede Game by playing Take under the conditions that Aumann specifies. However, Binmore believes that Aumann's setup does not model the game fully. The problem with Aumann's proofs is that it cannot explain the case of deviations from I^v path.

Binmore argues that “to assess the rationality of a 'rational player' who plays down [Take] at the opening move, we need to ask what payoff he would get if he were to play across [Pass].” Hence, we need to ask ourselves, within what possible world should we interpret a subjunctive conditional that begins with, “If a 'rational player' were to play across [Pass] ... ”?”

Binmore is asking how are we supposed to judge a subjunctive conditional in which a player who *should* play “down”[Take] (following Aumann's proof), due to some error, plays “across” [Pass]. Binmore criticizes Aumann's model for not incorporating a “theory of mistakes.” Binmore claims that Aumann fails to outline the kind of possible world we should conceive in order to analyze the consequent of a rational player erroneously playing an irrational move.

“In brief, I believe that the rationality of a 'rational player' must necessarily remain open as long as we have no idea what would be believed about him if he were to make an 'irrational' move. In consequence, we have no grounds for claiming that we know the 'solution' to games like the Centipede or the finitely repeated Prisoners' Dilemma” (Binmore 1994, 155). It is important to note that Binmore's critique that Aumann fails to account for a ”theory of mistakes” (erroneous deviations from I^v) is not only problematic for $[A]$, but also for Aumann's reductio proof. One might accept the reductio proof claiming that player 1 *should* play Take on the first move. However, in the case, that player 1 *actually* erroneously plays Pass instead, which is common in experimental findings, the reductio proof does not hold any longer, since it cannot accommodate deviations. In summary, consider the following game again,

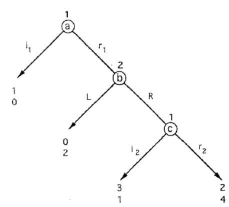

Fig. 1. A simple game.

We accept Aumann's proof that player 1 should chose Take at node a. Binmore asks us to consider the case of a deviation, player 1 makes the mistake of choosing Pass at node a instead. Aumann's theory is no longer helpful in determining the optimal move for player 2 at node b, since player 1 contradicted it. Hence, Aumann's theory fails.

5.1 Aumann's Reply

Aumann (1996) raises two issues with Binmore' reply ([B] from hereafter): Rationality versus Common Knowledge of Rationality There is a difference between Rationality and Common Knowledge of Rationality. This distinction is instrumental for Aumann's analysis. Aumann claims that it can be *perfectly rational* for a player to *not* play the backward induction solution. "Indeed, we go further: even if there has been no deviation from the backward induction path up to some point, a rational player may well deviate at that point" (Aumann 1996, 139). He criticizes [B] for conflating Rationality with CKR. Rationality is only a requirement that player i plays the move at each of his nodes that maximizes his conditional payoff at that node *relative* to his epistemic state or belief system. Player i's belief system can be composed of any belief about the opponent's strategy, in forming her beliefs at v, the player can take into account whatever she chooses to. If i believes her opponent will irrationally choose Pass for several nodes, then it would be "incumbent on rational players to ‘‘stay in’’ until quite late in the game" (Aumann 1996, 139).

This means that if player i believes that his opponent will erroneously chose Pass, then the rational move for i is to deviate from BI path, and chose Pass.
It is only when CKR is at play that I^v has to occur. Since when CKR is present, P1 Knows that "if P2's last vertex were reached, he would play down". This *necessarily* implies that if "P1's last vertex were reached, she would play down" (Aumann 1996, 141). $[B]$ is wrong to even question what would happen (what possible world we would be in) if the opponent played a particular move at any of the nodes. This is because unlike Rationality alone, CKR already assumes that each of the players Knows "without a shadow of a doubt" the move that the other player will play at each of the nodes. Therefore, whenever a deviation occurs, CKR must not be at play, and Aumann's theory has nothing to do with those situations.
In summary, if player 1 chooses Pass at node a, CKR is automatically not at play. CKR by definition, ensures that there are no mistakes. Hence, as soon as mistakes (deviations) are involved, CKR is absent. When CKR is absent, it may be perfectly rational for Player 1 and Player 2 to choose Pass at nodes a,b and c, depending on what they believe about their opponent's level of rationality. Hence, player 1's choice to play Pass at node a does not dispute Aumann's theory since Aumann's proof only involves CKR game play settings.

5.1.1 $[B]$'s methodological error

Aumann argues that $[B]$'s method of finding a flaw with $[A]$ is flawed. Aumann characterizes $[B]$ as first accepting CKR $\subset I^v$ (CKR implies P1 will will play Take on the first node), but then claiming "what if P1 does not play Take on the first node?" i.e a deviation occurs, $\sim I^v$. Now since CKR $\subset I^v$, and $\sim I^v$, CKR must be false. Aumann claims that this method of proving $\sim CKR$ is absurd. He characterizes this method as the following to showcase its absurdity.

"Suppose we have proved a theorem of the form ''p implies q.'' But our hypothetical reader is skeptical. ''The proof sounds right,'' he says, ''but let's look again. Assume p. Perhaps, after all, this could jibe with 'not q.' So suppose that it does"i.e., that q does not obtain. But then, since we have proved that p implies q, it cannot be that p obtains. So we conclude that after all, p doesn't obtain...Clearly, this argument is absurd" (Aumann 1996, 143).

5.2 Reply to Aumann

Binmore (1997) points out that the principle problem with Aumann's account is that it fails to account for any deviations from the I^V path. "Aumann (1995, sect. 5c), for example, is insistent that his conclusions say nothing whatever about what players would do if vertices of the game tree off the backward induction path were to be reached". The issue then persists, "if nothing can be said about what would happen off the backward-induction path, then it seems obvious that nothing can be said about the rationality of remaining on the backward induction path" (Binmore 1997, 24).

Binmore claims that it is not the case that he accepts CKR $\subset I^v$, and then asks "what if $\sim I^v$?". Instead, Binmore is making the claim that if Aumann's theory has no account for deviations (choosing Pass, instead of Take), which are the only way of reaching the last node in a > 1 node Centipede Game, then how can Aumann *assume* that the last node is reached, and then use backward induction to arrive at I^v. Similarly, Reny (1988) and Bicchieri (1989, 1992) argue that since CKR fails to account for any deviations, in the case of actual deviations in the Centipede Game, players will be unable to determine their optimal strategy at subsequent nodes.

In summary, Binmore is making the claim that it is not just the case that Aumann's theory fails *after* a deviation at node a. Instead, if Aumann's theory cannot say anything about the games when player 1 chooses Pass at node a, then how can Aumann inductively reason backwards from player 1 choosing at node c. Node c could have only been reached, had a deviation occurred at node a (Pass was chosen at a) and a deviation occurred at node b (Pass was chosen at b). Aumann claims that when Pass is chosen at a, CKR is not active; hence, his theory does not operate in such situations. Then Aumann cannot assume player 1 playing at c as the first premise of his proof. If Aumann's theory is fully inconsistent with gameplay involving any deviation, then how can his proof rely on deviations at node a and b, in order to reach c.

The Centipede Game Paradox can be summarized in the following way: it is initially true that under CKR conditions, I^v should result. However, if there were to actually be a deviation, we no longer have any tools to predict the optimal strategy. Moreover, now that we cannot explain the case of deviations, it casts doubt on our initial belief in CKR $\subset I^v$. This paradox is further reflected in the empirical findings of the Centipede Game.

6 Empirical

The following empirical studies assume homogeneity in player personality types. Hence, there is no heterogeneous player modelling based on their individual personality types.

6.1 M/P study

The first experimental study of the Centipede Game was conducted by Richard D. McKelvey and Thomas R. Palfrey (M/P from hereafter), and presented in *An Experimental Study of the Centipede Game.* M/P conducted 662 games, with two versions of the games, the four move and the six move centipede game. The payoffs in US $ amounts are shown below.

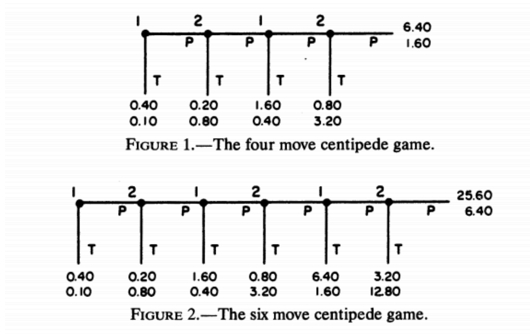

FIGURE 1.—The four move centipede game.

FIGURE 2.—The six move centipede game.

Game.png Game.bb

6.1.1 Experimental Design

A total of seven experimental sessions were conducted on students. In each session, twenty students, without any prior experience of the game, participated. At the beginning of each session, the subjects were divided into two groups, Red and Blue. The Red player moves first, while Blue makes the second move. Moreover, subjects did not communicate with other subjects except through the strategy choices they made. It was also Common Knowledge that no subject matched with another more than once. Further, a rotating machine was used to make sure that "no player i ever plays against a player who has previously played someone who has played someone that i has already played" (M/P, 807). The purpose of these matching procedures was to eliminate potential supergame or cooperative behavior, yet at the same time allow us to obtain multiple observations on each individual's behavior.

6.1.2 Results

The results indicated that an "Always Take" model; namely, P1 picking Take on the first node, could easily be rejected. This is because only 7 % of the four move games and 1 % of the six move game ended with I^v, the first player choosing Take on the first node. Moreover, the results illustrated that the probability of playing Take “increases as we get closer to the last move”. Even though, an “Always Take” model should be outright rejected, any theory of the data would have to account for the fact that the probability of Take per each node is changing.

Zauner (1999) finds similar evidence against the ”Always Take” model. His data illustrated that “the game theoretic prediction of taking is rarely the outcome of the experiments”. Nagel and Tang (1995) conclude that subjects deviate substantially from subgame-perfection when the number of stages is sufficiently large.

6.2 A.Rapoport et al.

6.2.1 Experimental Design

Rapoport et al. in contrast to M/P, use a 3-person centipede game, instead of a 2-person game. The underlying belief is that increasing the number of players in the game would make collusion harder (Dawes, 1980). The game then “includes three “innings” or rounds of play, each consisting of three moves (one per player) for a total of nine moves” (Rapoport et al., 242).

Empirical studies so far have shown that it is often the case that conditions for CKR are not met in experimental settings. Binmore at al. (2001) made the claim that if players do arrive at I^v, in most cases it will be for evolutive reasons, “If they find their way to an equilibrium, it is therefore by some process of trial-and-error adjustment” (Rapoport et al., 242). Since Common Knowledge of Rationality is posited as a requirement for BI outcome, it can be claimed that repeated iterations of the game would contribute to a stronger belief for each player in the rationality of the other player. However, there is also a reason to be skeptical about any learning effects towards I^v. Nagel and Tang (1998) give evidence that they found no convergence towards

I^v “within 100 periods as he [Zauner] hypothesized”. Therefore, it is important to account for the learning patterns of players across games. As a result, Rapoport et al. attempt to better catch learning by increasing number of trials to 60.

Further, (Rapoport and Chammah, 1965)) present evidence of “lock in” on the joint “cooperate” responses. In order to counteract this collusive effect, they opted to increase the number of players and rotate players in the game. The study included 60 trials ”thereby allowing for more learning.” The study was again conducted on college students. In order to further reduce chances of collusion, the study opted to continually change group composition in each iteration of the game by randomly assigning player 1, 2 and 3 roles randomly in each iteration.

Most strikingly, the amount of payoffs in this study was much higher than previous experiments. The maximum payoff in Rapoport et al. studies is 200 times higher than the one in M/P 4-move game and 50 times higher than the M/P 6-move game.

The goal of this study is measure the effect of the following four changes to the M/P study:

1. the number of players is increased from two to three
2. the game is played for unusually high stakes
3. the stage game is repeated for multiple periods
4. player roles are assigned randomly

Two experiments were conducted. Experiment 1 included changes 1-4, listed above. Experiment 2 mirrored Experiment 1 in all details “except the payoffs that were chosen to be of the same order of magnitude as in the 6-move 2-person centipede game of M/P.”

6.2.2 Limitations

This study has several limitations. Not only is the sample size limited, but also, confounding errors are present in their results. Confounding variable is when a variable influences the dependent variable and the independent variable. Hence, the presence of confounding variables means that confounding errors exist. The confounding errors make it harder to ascertain the exact affect of each of the dependent variables on the independent variable, with the exception of the ”the size of stakes” variable.

The most important variable,”the size of stakes” was specifically tested for, keeping other variables constant; hence, it is the only variable that does not suffer from confounding errors. The Rapoport et al. study conducted two experiments. Experiment 1 was the experiment that was conducted with higher number of players, rotating order of players, multiple stage game periods and higher monetary returns. Experiment 2 was the experiment with all of these variables, except high monetary returns. Hence, comparison of Experiment 1 and Experiment 2 illustrates the specific impact of higher monetary returns (”the size of stakes” variable).

The writers note the presence of these limitations were due to financial constraints. It is important to highlight that this study was not meant to provide a precise theory of the exact effects of all four variables in the centipede game outcome. It aimed at outlining the exact effect of higher monetary returns, and the aggregated impact of the four dependent variables outlined above.

The study presents new evidence showing that it is not that case that past empirical studies such as M/P contradicted Aumann’s theoretical results, but instead, they did not account for these four variables that are instrumental for CKR to operate. Aumann makes it clear that his theory only concerns CKR situations, and not rationality situations, no matter how high the degree of mutual rationality is. Only when CKR operates in the empirical settings, can we test Aumann's theory. Hence, past empirical literature failed to test Aumann's theoretical results for the game.

6.2.3 Results: Experiment 1 vs. Experiment 2

From the two experiments conducted, there is considerable difference in the results. In Experiment 2, with the lower monetary returns, across the 60 trials, the equilibrium outcome I^v was reached "on only 2.7%, 2.3%, 1.3%, and 4.0% of the times in sessions 1, 2, 3, and 4, respectively (with an average of 2.6%), compared to an average of 39.2% in Experiment 1." This difference is clearly statistically significant. Compared to Experiment 2, Experiment 1 shows considerable evidence of the games approaching equilibrium outcome. Moreover, while the "proportions of games that terminated at end-node j are seen to decrease monotonically in j throughout Experiment 1" (Rapoport et al., 262), meaning players are more likely to chose Take as they move forward in the game; in Experiment 2 they increase up to the sixth or seventh decision node and only then decrease quickly.

This gives us evidence that, keeping in mind the limitations of the study, higher monetary benefits result in a shift towards the equilibrium outcome. A possible explanation is that unless players value the returns enough, they are more willing to make erroneous moves, as we have seen in prior experimental literature such as in M/P. This gives some evidence to believe that CKR assumes that players value the gains from the game. CKR will only be a tool employed by each player in the game if they deem the returns of the game valuable enough to pursue.

6.2.4 Results: Learning Models

The experimental results showed that dynamic learning models, those that incorporated player learning across successive trials explained the data better than static models. An individual updating learning (IUL) model that updates player learning individually rather than on a population level (PUL), fared better. Therefore, increasing conditional probabilities of moving Down with successive iterations of the game, taking into account the ”type” of each player in the game explained the results better.

6.2.5 Results: Comparison with M/P

The 6-move 2-person M/P centipede game and the 9-move Rapoport at el. Experiment 2 (with low monetary returns) give very similar results, “the proportion of games that ended in the first decision node was 0.7% in the M/P study compared to 2.6% in Experiment 2”. This illustrates that the control experiment worked as expected, mirroring M/P results.

Therefore, results from Experiment 1, approaching equilibrium outcome, can largely be attributed to higher monetary returns. Hence, even with increasing number of players, assigning player roles randomly and high number of repeated stage games, but low monetary returns, the results were not significantly different from M/P results. Moreover, setting aside the confounding errors, there is evidence to suggest that, when changes 1-4, as stated above, are made to the M/P study, then players seem to approach equilibrium play. This result gives us limited empirical evidence to suggest that Aumann is right in claiming that given the right conditions approaching CKR in the Centipede Game, players start to approach I^v. Hence, there is some reason to believe that if the right factors for CKR are provided in empirical settings, players will tend to approach I^v. Aumann (1995) expressed that CKR is an ideal limit, that can generally be hoped to be approached in most empirical settings.

6.3 Palacios-Huerta and Volij

Palacios-Huerta et al. (2009) ((P/V) from hereafter) provide more empirical evidence to suggest that CKR $\subset I^v$. P/V start be espousing the claim in Aumann (1998) that in the Centipede Game, it is fully consistent for rationality to be mutually known to a high degree, but still for it to be the case that the players continue playing Pass for a few rounds before choosing Take. This is because extremely high mutual knowledge of rationality is still not equivalent to Common Knowledge of Rationality. (Elhanan BenPorath (1997) shows that several rounds of "continuation" are consistent with common certainty of rationality).

Keeping in mind Aumann (1995) claim that "even the smallest departure from common knowledge of rationality may induce rational players to depart significantly from equilibrium play" (P/V, 2). P/V attempt to mirror CKR conditions. As a result, instead of college student participants, they choose players who are characterized by a high degree of rationality, proficient chess players. Not only are these players known to exhibit really high degree of rationality, and to regularly employ backward induction reasoning in strategic form games, but also, these players are commonly known to do so as well.

The study was conducted on four types of chess players: Grandmasters [GM], International Masters [IM], Federation Masters [FM], proficient chess players with no titles. The sample consists of 422 chess players: 41 GMs, 45 IMs, 29 FMs, 307 players with no chess title.

The P/V study Experiment 1 takes place in field settings, at high-ranked chess tournaments. In order to fully minimize any collusive effects, each chess player participated in the experiment only once. Experiment 2 was a lab experiment conducted with chess players and college students. In Experiment 2, while each subject plays 10 rounds of the game, no subject plays against the same opponent more than once.

6.3.1 The Field Experiment

The reason for preferring the field setting, chess tournaments, is that it allows easy access to high ranked chess players. Moreover, P/V hypothesize that " a chess tournament may represent a more familiar and comfortable environment for chess players" (P/V, 6). However, the field setting greatly reduces the chances of a conducting a carefully designed experiment with repetitions. Hence, P/V focused on recording initial responses of the players. Even though, this eliminates any chances for observing learning patterns across games, chances of collusive behavior are severely limited.
Three rounds of the game were played with college student groups and chess player groups, where experimenters read the instructions to each player separately in separate rooms. Moreover, each player was placed in a separate room during game play and "the games were conducted through SMS messages" where each player "sent their decisions to the opponent, and received information on the decisions of the opponent." The goal of maintaining such a distance between the players was to completely minimize any collusive or altruistic behavior. The hypothesis was that if the players could not even see each other, it was likely that there would be no chance of any cooperative understanding being established between them.

6.3.2 The Laboratory Experiment

The goal of the lab experiment was to ascertain if players altered their game play based on their assessment of other players' level rationality. There were

two sessions of ten repetitions of the game. In each session, 20 subjects, none of whom had previously played this game, were split into two equal groups, black and white. Each player from each group (white/black), played each member of the other group, only once, without knowing their identity.
There were four treatments composed of different white and black groups, where a player of the white group played against a member of the black group and vice versa.

1. Treatment 1: both groups consisted of college students
2. Treatment 2: white group: college students, black group: chess players
3. Treatment 3: white group: chess players, black group : college students
4. Treatment 4: both groups consisted of chess players

The goal of Treatment 1 was to act as a control. It was supposed to mirror M/P results as closely as possible. Treatment 4 was meant to provide empirical support for the results from the field experiment. Treatment 2 and 3 were of primary concern in this experiment. The four treatments were meant to ensure that the only significant feature that differentiates the four experiments is the type of the subjects in the pool. Moreover, just like the field experiment, players sent in their choices through SMS messages.

6.3.3 Results

6.3.3.1 Field Experiment Results

The results from the students lab experiments were very similar to the results from the student experiment in M/P experiment. Only 3 of the 40 players who played the role of Player 1 chose to stop in the first node, while close to two-thirds of the games ended in nodes 3 and 4. This shows that the experimental features of this study did not differ by much compared to MP. If a statistically significant difference is found in the experimental results with the chess player, it has to be attributed to the players themselves.

The chess players approached the equilibrium outcome in significantly more games than college students did. The proportion of total games that resulted in the equilibrium outcome is 69 percent.

Further, for the participants holding no chess titles, the proportion is 61 percent. For Federation Masters and International Masters the proportions are 73 percent and 76 percent, respectively. If we restrict our attention to Grandmasters, the proportion is a remarkable 100 percent. It is interesting to note that these proportions increase with the Elo rating of the players. The Elo rating is the official method according to which chess players are ranked. The graphs below illustrate the start difference between the results from college students versus chess players, according to rank.

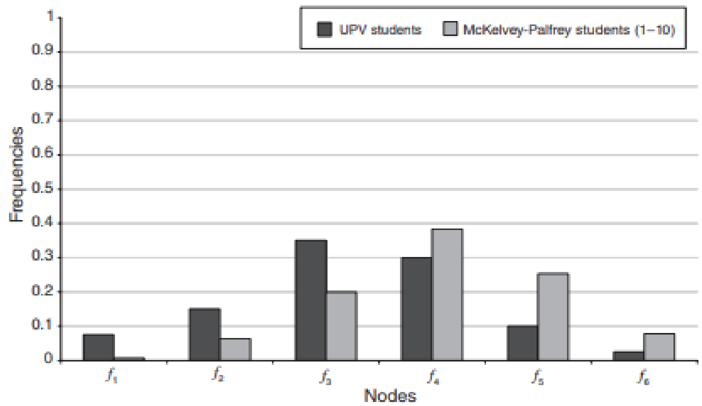

Figure 2. College Students: Proportion of Observations at Each Terminal Node

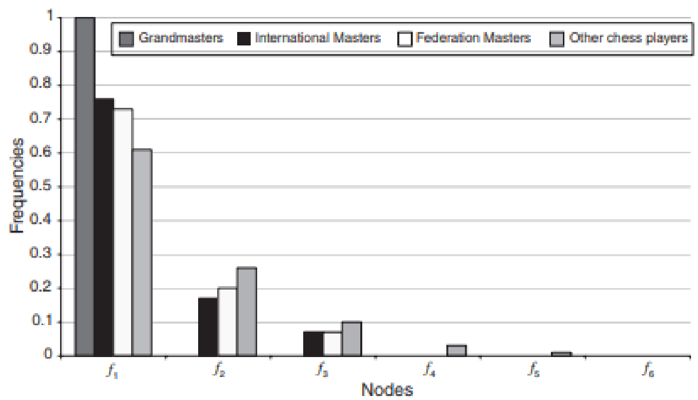

Figure 3. Chess Players: Proportion of Observations at Each Terminal Node by Type of Player 1 in the Pair

6.3.3.2 Laboratory Experiment Results

In Treatment 1: students against students, the results were similar to those from M/P. However, the students exhibit drastic changes in their play, when made aware that they are playing against chess players (Treatment 2 and 3). Especially when college students in Treatment 1, play as player 1, they chose the equilibrium outcome around 10 times more frequently than when they play against students. As a result, “the main observation one can infer from

these results is that students' behavior depends on whether they face a highly rational opponent or a fellow student" (P/V, 12). This gives evidence to suggest that knowledge of the opponent's level of rationality plays a major role in determining the moves a player plays.
In Treatment 4, the results corroborated the field experiment findings. In the lab settings, when chess players played ten repetitions of the centipede game against each other, more than 70 percent of the games ended at the first node. Most significantly, at the fifth repetition, every chess player converges to equilibrium. This provides significant empirical evidence so far that supporting the claim that there are settings when CKR is approached to such an extent, that the equilibrium outcome does result. Hence, there is now significant evidence to believe Aumann's Theorem A: CKR $\subset I^v$.

6.3.4 Main Results

The two experiments so far have shown that chess players on average play very differently from the way students do. Moreover, "a significant majority of chess players chose the only action that is consistent with equilibrium". This supports the "theory that gives a central role to the principles of self-interested rational economic agents" (P/V, 10). However, it is still unclear if Common Knowledge of the opponent's rationality plays a part in the results from the experiment with chess players. It is unclear whether the results can be attributed to just the higher level of individual rationality of the players or common knowledge of the opponent's rationality.

7 Models

We have seen that given the right empirical settings, there is evidence supporting Aumann's theorem. Now we will move on to providing a statistical model that can best represent the results commonly obtained from the game experiments. The goal is to have a general model that can not only model results from games with CKR active such as *The Field Experiment*, but also, games with CKR absent, such as M/P. Moreover, we will illustrate how the model is built in such a way, as to incorporate intuitions from both, Aumann's Theorem A and Binmore's case of deviations.

Fey et al. (1996) conduct a series of constant-sum centipede games to test statistical models that best explain the results from the game. Constant-sum games are games in which the total amount of money that the two players can receive at any time during the game remains constant. Since there are no Pareto improvements during the game due to the constant-sum nature of the games, the results cannot be explained by altruistic behavior, as was the case in M/P. As expected, the results from the experiments are very similar to M/P, with subjects frequently not playing the equilibrium outcome. Now Fey et al. attempt to best model these results.

The "Always Take" model and the "Random" model were readily dismissed after they failed to model the results. The "Learning" model is a modified Always Take Model. The model assumes that players do not start off with the Always Take model. As we have seen, this is because CKR is generally not active in most experimental settings at this stage. The players start off with the Random model and with experience playing the game, they start to approach the Always Take model. This learning is caused by the players slowly building trust in the opponent's level of rationality with experience; hence, they start to approach CKR.

For the Learning model, Fey et al. define P_t as the probability that a player will choose Take at any node of the match she plays in match number t. Therefore, P_t changes over time, with experience of playing the game. However, the

P_t does not change *during* a game.

$$p_t = 1 - (1 - p_0)e^{-\alpha t}$$

The equation above illustrates how inter-game learning affects the probability of P_t at game t, with P_0 and α being parameters to be estimated. The exponential function allows us to code non-linear learning rate across games. The "Always Take" model is obtained when $P_0 = 1$, while the Random model is obtained with $\alpha = 0$. The following log likelihood function is used to estimate P_0 and α of the data.

$$LogL = \sum_{i=1}^{M} log(1 - (1 - p_0)e^{-\alpha t_i}) + \sum_{i=1}^{M} n_i log((1 - p_0)e^{-\alpha t_i}$$

Fey et al. then compare the Learning model to the**Quantal Response Equilibrium Model** (QREM). QREM assumes that sophisticated players Know that ideally they would adopt the "Always Take" strategy. However, they play "mutually consistent strategies with the knowledge that other players may make mistakes." Further, there is additional Knowledge that the opponent is less likely to make "costlier" (in terms of expected payoff) mistakes. Hence, there is a changing probability of choosing Take at different nodes in the game. Therefore, instead of a constant P_t for all nodes in game t, P_t^j represent the probability of choosing Take at node j. For example, a player at node j in match t, with payoff u_T for choosing Take and u_P for choosing Pass, will have the following logistic response function:

$$P_t^j = \frac{e^{\lambda u_T}}{e^{\lambda u_T} + e^{\lambda u_P}} =$$

where $\lambda \geq 0$. When $\lambda = 0$, the player is randomizing between Take and Pass, while $\lambda = \infty$ means the player's choice "is perfectly rational and exhibits no error; the highest expected payoff choice will be played with certainty." Such a specification allows the probability to chose Take to increase with expected payoff. Hence, the more costly a mistake would be (in expected payoff), the less likely a player is to make that mistake. Starting from the last node N, we can go backwards to get a vector probability for the game: $P_t = \{P_t^1 ... P_t^N\}$. The likelihood function for QREM is given by:

$$LogL = \sum_{i=1}^{M} log[\hat{f}_t^{n_i}(\lambda_0, \beta)$$

where $\hat{f}_t^{n_i}$ is defined recursively. For example, $\hat{f}_t^1 = p_t^1$, $\hat{f}_t^2 = (1\text{-}p_t^1)p_t^2$, $\hat{f}_t^3 = (1\text{-}p_t^2)(1\text{-}p_t^2)p_t^2$, and so on. Moreover, not only does QREM account for varying probability of choosing Take intra-game, but it can also incorporate inter-game varying P_t. We can do this by "by supposing that the (common) λ value changes over time", through the following equation:

$$\lambda_t = \lambda_0 + \beta t$$

QREM performs better than the Learning model, especially since it models the changing probability of playing Take at every node, which the Learning Model is unable to do. As illustrated above, the inclusion of varying levels of probability at each node in QREM allows us to mirror Theorem A results by supposing that the probability of playing Take increases at successive nodes. Moreover, our ability to model the erroneous moves of each player and their opponent allows us to fulfill the Binmore criteria of including erroneous moves (deviations) in the model. As a result, this model incorporates both Binmore and Aumann intuitions. Hence, it does justice to both sides of the argument in the paradox and presents a solution that includes the main findings of each view. Doing so, allows it to best explain the results from the game. Hence, QREM is the statistical solution to the Centipede Game Paradox. Now that we have a statistical model that explains empirical results from the game, all that is needed is a theoretical resolution of the initial Centipede Game paradox.

8 Solving the paradox

Contrary to prevailing views, it is not the case that experimental results fail to show evidence for the equilibrium outcome. Instead, experimental results have given us considerable evidence to suggest that when the conditions for CKR are fulfilled, equilibrium outcome is often reached. The problem is that even in the face of evidential results supporting equilibrium outcome, there is no clear theory that supports the equilibrium outcome in the Centipede Game while not getting undone with a deviation from the BI path in the game. Since we have seen that Theorem A has significant empirical backing, we have reason to insist on retaining it. Therefore, any successful theory of the Centipede Game must reconcile Theorem A with the problem of deviations, just like our statistical model does. In order to provide a successful theoretical solution to the paradox, we will first use a conditional analysis to explain why the theoretical impasse was reached between Aumann and Binmore. After realizing why the paradox exists, we will present a solution to it.

8.0.1 Explaining the problem of conditionals

Brian Skyrms (1998) analyzes the case of subjunctive/counterfactual and material conditionals to explain the paradox that is encountered when reconciling Theorem A with case of deviations. Only if the case of these two conditionals is better analyzed, will the debate between Aumann and Binmore be better understood.

Skyrms begins by demarking cases where subjunctive/counterfactuals are actually involved from cases where only material conditionals are. Skyrms claims that the issue is that often times material conditionals are *expressed* as subjunctive/counterfactual conditionals - ”If Column were to play Up”, ”If Column had played Up”. This gives the illusion that even a reductio argument makes use of subjunctive/counterfactual conditionals. However, this is not true. Reductio arguments only make use of material conditionals.

What distinguishes Centipede Game reductio arguments (Theorem A) from those that involve subjunctive/counterfactual conditionals involving deviations is that the reductio arguments assume the following two premises.

(1) Complete Information: Payoffs (+ structure) individuate the game.

(2) Revealed Preference: Payoffs are in terms of utilities (interpreted as dispositions to choose).

Skyrms illustrates that when (1) and (2) are operational for any game tree with depth greater than 1, “there is associated a unique backward induction *reduced tree* gotten from the original tree.” This reduced tree is reached by Skyrms’ *marking function*. In a single choice node, “the edge leading to the leaf which maximizes payoff to the chooser is marked and the other edges are unmarked”. While for a tree with depth n+1, “all the edges which are marked in the reduced tree are marked and for each terminal choice node, the edge leading to the leaf associated with the greatest payoff for the chooser is marked” (Skyrm, 556). Skyrm then presents a short reductio proof that states that after assuming (1) and (2) in games of PI, one can inductively show for n node games that BI path is the only possible path. In other words, there is therefore no possible world at which player II reaches her first decision node. The proof is very similar to Aumann’s original reductio argument for BI solution to the Centipede Game. Since it is now apparent, that reductio shows that BI is the only path when (1) and (2) are assumed, the marking function picks out the BI path. This marked path picks out the reduced tree. The following diagram illustrates how the marking function reduces the original game tree to the reduced tree allowed for by (1) and (2) .

article

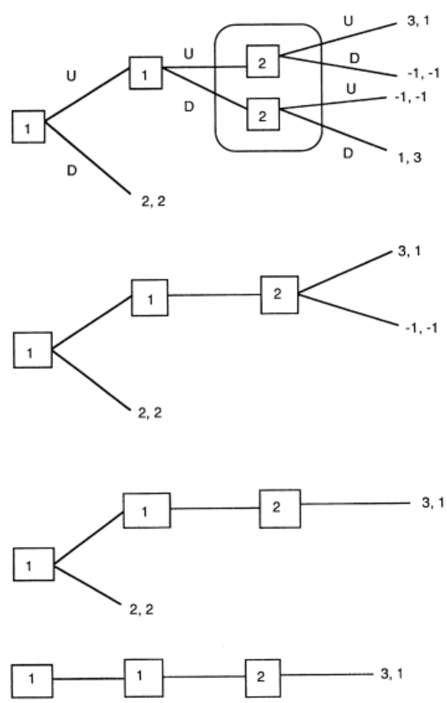

Figure 3.

tree.PNG tree.bb

The reduced tree is the only tree allowed under (1) and (2). It is only the *illusion* of the original tree that makes it seem like subjunctive/counterfactual conditional is needed. The reductio argument has successfully shown that the unmarked edges in the tree are not possible. Therefore, only the reduced tree is allowed. The marking function illustrates that under (1) and (2), the real game tree is the reduced tree and not the original tree. Hence, when Aumann makes his case for Theorem A, he is actually making the case for the reduced tree and not the original tree.

In the real world, (1) and (2) are often times relaxed, resulting in deviations. After a deviation, the original theory is no longer valid and in need of revision. In the presence of deviations, the reductio argument is no longer valid. Hence, the marking function cannot reduce the original game tree then. As a result, only having material conditionals does not work. The case of deviations is the only time when it seems a subjunctive/counterfactual conditional analysis is needed.

It is now clear that the theoretical impasse is reached because Aumann and Binmore are actually talking about different Centipede Game trees to begin with. Aumann's tree is a shrunken version of the game tree that Binmore is talking about, the original tree. Both of them are theorizing about different game trees, while believing that they are talking about the same tree. As a result, both are making valid claims about their respective trees, Aumann about the reduced tree, and Binmore about the original tree. Hence, there is no contradiction between the two views since they are referring to different trees. Therefore, the paradox unravels when we clarify which game tree each theory applies to. Since, there is no contradiction between the two views, and both are making valid statements about their respective trees, we need a theory of Centipede Game that incorporates both the intuition from Aumann's and Binmore' theories about their respective trees.

8.0.2 Solving the problem of conditionals

Bicchieri et al. (1995) ((B/G) from hereafter) distinguish between game theorist's and the players' own "theory of the game". The players' own theory of the game refers to a minimal set of axioms available to a player at a node, that allows the player to compute the BI equilibrium move for that node.

Therefore, the players' own theory of the game makes use of two notions, "rationality at a node" and "knowledge at a node". The game theorist's theory of the game is supposed to be the justification the sequence of moves. B/G argue that the reason that a paradox arises is because we have been attributing the Theorem A to the players themselves, instead of attributing it to the game theorists.

The B/G solution involves modelling players as "automatic theorem provers". This means that they can be thought of as computational bots that, when given a set of axioms at a node, the players will be able to compute the solution (Take or Pass) at that node. Theories that attempt to justify a series of moves by players in the game such as Backward Induction, fall into the game theorists' theory and not the players' theory of the game.

Instead of complete game CKR at the begining of the game, the authors employ the notion of "CKR at a node". The Knowledge possessed by the players is a ""distributed" knowledge of the theory of the game." Distributed knowledge means that "whereas the first player to move has information about all subsequent nodes - the second player has slightly less information. She will have full information about all subsequent nodes, but not about the first node" (B/G, 147). Then, "at every node, the player who chooses at that node has a minimal theory that is just sufficient to infer an optimal move at that node, but does not imply anything about the preceding nodes."

B/G argue that the paradox only occurs when we rely on a complete game CKR that involves "group-knowledge (i.e., each player knows the theory of the whole game). Then when a deviation occurs, it makes the original Theorem A theory inconsistent and forces a revision. Instead, B/G propose a modular theory of Knowledge. In the modular theory of Knowledge, for each subgame, G' of game, G, a player must contain *just* enough information to infer an equilibrium for G'. As a result, the level of Knowledge relative to G' must be a *proper* subset of the level of Knowledge relative to G. Hence, B/G claim that the players have a *distributed* Knowledge of the game.

A central point made by B/G is that "to compute a solution is a different task than justifying it" (B/G, 150). A justification makes use of contrary-to-fact situations as to why a player should play the game according to a set of axioms. Then at the player level, the players simply need to compute the move at their axiom according to those axioms. In conditional terms, a justification makes use of counterfactual conditionals, while computing a solution only requires material conditionals when given a set of axioms required to computer the solution at each node.

B/G explain deviation from the point of view of the players as "contrary-to-fact event". Hence, a deviation must be explained with counterfactual conditionals. A *contrary-to-fact* analysis can only occur at the meta-theoretic level. This is because at the players level, only the actual (*factual*) is occurring. Hence, any counterfactual analysis cannot occur at the players' level. Moreover, this is why no counterfactuals are needed at the players' level to compute the optimum solution at their node.

The following example will illustrate the difference between a modular and a non-modular theory of Knowledge in gameplay. Consider the following game again,

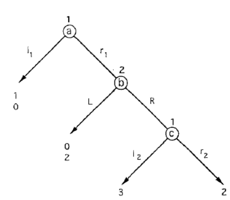

Fig. 1. A simple game.

Assume player 1 and 2 are at node a. Following our traditional views, player 2 had complete Knowledge of the game at node a. As a result, player 2 follows Theorem A reasoning to reduce the original game tree to a single reduced game tree for the entire game. However, player 1, due to some error chooses Pass at a. The reduced tree simply cannot accommodate this deviation. Following the reduced tree, it is a contradiction to play Pass at a; hence player 2's epistemic system consisting of the reduced game tree cannot accommodate the deviation.

B/G's solution involves a modular theory of Knowledge for each player instead of complete game Knowledge at the beginning of the game, which led to the above problem. Now player 2 has partial Knowledge of the game that has just enough information to compute the solution at a particular node. Therefore, following B/G, each player's epistemic system is updated with only node-specific partial information. This means that at no point does any player have a single reduced game tree for the entire game. Instead, each player uses their partial Knowledge to have only node-specific *sub-game* reduced game trees. As a result, when the deviation happens at a, player 2's epistemic system is automatically updated to have just enough information to infer the sub-game optimal move at b, and so on. The benefit of this is that at no point in the game does any player's epistemic system contain a single reduced tree for the entire game, which is contradicted in the case of a deviation. Instead, each player's epistemic system is constantly updated to contain just enough information for node a, b and c.

The benefit of modular Knowledge is that in the case of deviations, there is no need of a theory revision at the level of the players. This is because the local theory allowed to players infers an optimal move at their node alone. This is not the case when the general from CKR is assumed, "when the theory of the game is common knowledge, a deviation at any node forces an extensive revision of the whole theory" (B/G, 150). Moreover, even in the cases of a much lesser degree of CKR, a deviation still forces an revision (albeit less extensive). This is because a player still has to consider whether she believes the deviating player to be rational or not. Only in the case of modular knowledge is a theory revision not required at the level of the players.

The following is a more specific version of the B/G solution to the above game. In the case of a non-modular theory of Knowledge, similar to the ones being discussed up until now, if player 1 finds herself at node c, a major theory revision is required. Following CKR, each node is attributed the non-modular unified game (G) group theory, Φ_G: $\Phi_a \cap \Phi_b \cap \Phi_c$. However, since player 1 is actually present at c, it is understood that Φ_a which calculates π*(a) at a and prescribes the equilibrium move $_1$, is false, since player 1 must have made r_1 move instead i_1 to reach c. Since Φ_a is false, Φ_G must be false as well. Hence, Φ_G must be revised; otherwise, the move to make at c remains undetermined. Now theory revision naturally brings the problem of counterfactual conditionals, that is hard to resolve.

B/G's modular theory of Knowledge for the game that avoids any theory revision in the face of deviations, such as player 1 finding herself at node c. Each node: n $\in$ G is assigned a theory Φ_n. Φ_n is a minimal set of axioms that is "sufficient to infer an equilibrium for the corresponding subgame." For example, in the above game, if player 1 finds herself at node c, she will use Φ_c to infer her optimal BI solution π*(c) at c. Although, Φ_a is inconsistent, since a deviation has taken place resulting in player 1 being at c, Φ_c is still sufficient for player 1 computing r_2 at c. When the theory of the game is modular, knowledge is "distributed" across Φ_n, instead of unified group knowledge Φ_G at each node. This avoids the need for any theory revision. Hence, this further avoids the need for the controversial counterfactual conditional analysis.

Such a minimal modular theory for the players, maintains their ability to infer the BI solution situation at every node, even in the face of any deviations. The theoretical revision after a deviation can now occur at the level of the game theorists who can reformulate the general theory of the game in the case of a deviation by using theoretical tools such as the QREM equation, which allows us to tweak several variables in order to better model the game. Therefore, upon a deviation at a, player 2 will be able to calculate her next optimal solution and the game theorist will take the deviation into consideration to reformulate the QREM model for the game by increasing the error margin in the model.

9 Conclusion

The Centipede Game Paradox collapses to a distinction of material and counterfactual conditionals. These conditionals then help us determine the respective game trees that theories of Aumann and Binmore are operating on. Once we understand the distinct game trees that each theory is operating on, there is no contradiction that gives rise to the initial paradox. If we introduce a modular theory of Knowledge, instead of a complete game CKR at the start of the game, we can not only, employ Aumann's Theorem A reasoning to infer the optimal move at each node, but also, update each player's modular epistemic system in the case of deviations to accommodate Binmore's case of deviations. The theory revision can now occur at the game theorist's level where QREM can be tweaked with each deviation to better model the game at hand. A topic of further research is formalizing a more sophisticated theory of split epistemic systems of the players' and the game theorists'. This is important so we can be more precise in delineating the exact specifications of the information sets the players and the game theorists are supposed to operate with.

The empirical findings illustrated that some of the most famous Centipede Game experiments such as M/P did not respect the distinction between CKR and Rationality. As such, they did not control well for four dependent variables such as monetary gains, number of players, number of iterations of the stage game and player role assignments. Moreover, we saw that when CKR settings are actually emulated, like in the case of the chess player experiments, the I^voutcome does follow. After presenting the empirical evidence supporting Aumann's Theorem A, we presented the QREM that best models the data. A topic of further research would be to conduct the Palacios-Huerta et al. chess player experiments while altering the four dependent variables highlighted above. Each of the four variables should be tested separately to gauge its exact impact, which will further allow us to better the specifications of QREM.

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