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Alexander Polynomials of Tunnel Number One Knots

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Alexander Polynomials of Two-Bridge Knots and Links

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Abstract

Every two-bridge knot or link is characterized by a rational number p/q, and has a fundamental group which has a simple presentation with only two generators and one relator. The relator has a form that gives rise to a formula for the Alexander polynomial of the knot or link in terms of p and q [15]. Every two-bridge knot or link also has a corresponding "up-down" graph in terms of p and q. This graph is analyzed combinatorially to prove several properties of the Alexander polynomial. The number of two-bridge knots and links of a given crossing number are also counted.

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Chapter 1

Introduction

Two-Bridge knots are a special class of knots which have been completely classified [20] and studied in depth. A good, complete description of two-bridge knots and their many interesting properties is given in [3]. Two-bridge links are similar to two-bridge knots, but have two components instead of just one. They also have been studied extensively and are of interest to knot theorists.

Much is known about the form of the Alexander poynomials of two-bridge knot and links. Perhaps the most interesting property is the "trapezoidal" property, which was proved by Hartley [9] in 1979. Two-bridge knots are alternating knots [8], algebraic knots, and are tunnel number one knots. Fox conjectured that the trapezoidal property holds for all alternating knots, and Murasugi proved it for algebraic alternating knots [17]. It has also been proved [10] that tunnel number one knot groups have palindromic presentations similar to that of two-bridge knots. Several other papers, such as [11], [17], and [18], give other conditions on Alexander polynomials for two-bridge knots and links.

My research used two main tools to study the properties of two-bridge knots and links. The first tool is the up-down graph associated with every two-bridge knot or link. This thesis defines and combinatorially analyzes the up-down graph to prove several properties of the Alexander polynomials of two-bridge knots and links, some of which are original. Progress is also made toward a new proof of the trapezoidal property, including proofs of some special cases. Murasugi [15] has used the up-down graph to prove other things about two-bridge knots and links not related to their Alexander polynomials.

The second major tool is a set of three transformations defined on the set of "valid" (p, q) pairs for two-bridge knots and links. These transformations are defined for the first time in this thesis, and are used to count the number of two-bridge knots and links of a given crossing number. Two-bridge knots and links were first counted in [6], thereby providing the first proof that the number of knots and links grows exponentially with the crossing number.

Chapter 2

Two-Bridge Knots and Links

Mathematical knots and links are topological objects, and knot theory is a branch of topology. However, mathematicians have taken many approaches to knot theoretic questions, including combinatorial, algebraic, geometric, topological, and so on. Most of the work in this thesis takes an algebraic or combinatorial approach to a particular class of knots, the two-bridge knots.

2.1 Definitions

Knots and links can be defined in many different ways. The following are the simplest definitions, which were taken from [14].

Definition 1. A *knot* is a simple closed polygonal curve in $\mathbb{R}^3$. A *link* is a finite union of disjoint knots. The disjoint knots in a link are called the *components* of the link.

Knot theory is the study of equivalence classes of knots and links. Knots and links can be oriented or unoriented; in this paper we only consider unoriented knots and links unless otherwise specified. There are two common definitions for equivalence of unoriented knots and links:

Definition 2. Two unoriented knots or links K_1 and K_2 are *homeomorphism equivalent* if there is a homeomorphism ϕ of $\mathbb{R}^3$ to itself that takes K_1 to K_2. The knots or links K_1 and K_2 are *ambient isotopic*, or *equivalent*, if they are homeomorphism equivalent and the homeomorphism ϕ is orientation preserving.

Two knots or links are equivalent if they are related by Reidemeister moves (see Figure 2.1), which are operations defined on diagrams of knots. The Reidemeister moves are defined in [14].

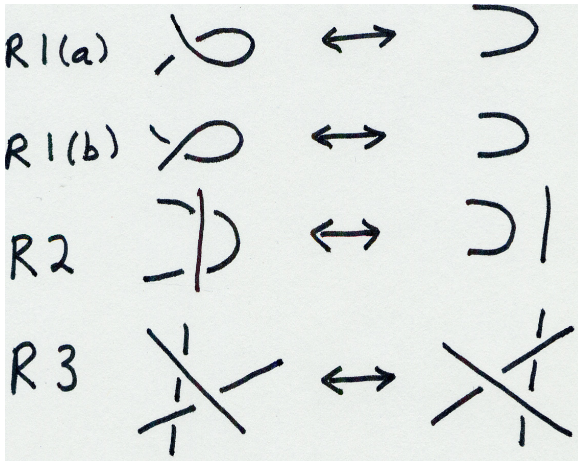

Figure 2.1: Reidemeister moves.

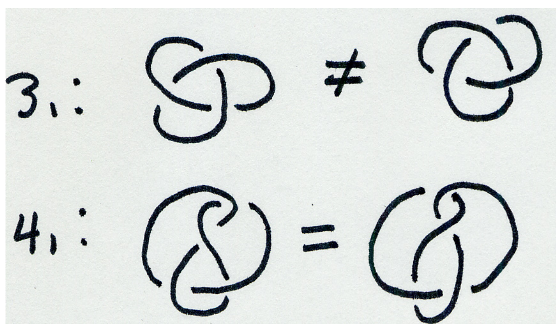

Figure 2.2: Examples of chiral and achiral knots.

Definition 3. A knot or link is *achiral* (or *amphicheiral*) if it is equivalent to its mirror image. Otherwise, it is *chiral*.

For example, the trefoil knot 3_1 is chiral, whereas the figure-eight knot 4_1 is achiral, as shown in Figure 2.2.

2.2 Two-Bridge Knots and Links

In this thesis, I am studying a particular class of knots and links called *two-bridge* knots and links. This name comes from the fact that they have two-bridge diagrams. A diagram is a regular two-dimensional projection of a knot in which crossings are drawn in a way that indicates which arc crosses over and which crosses under.

Definition 4. A *bridge* in a knot diagram is an arc between undercrossings that has at least one overcrossing. A *two-bridge* diagram for a knot or link is a diagram with exactly two bridges. A *two-bridge knot* or *link* is a knot or link that has a two-bridge diagram (see Figure 2.3).

In fact, the bridge number is defined for any knot or link. The only 1-bridge knot is the unknot, so the two-bridge knots and links are in a sense the simplest nontrivial knots and links.

A two-bridge knot or link is classified by a fraction p/q, and is denoted by $K_{p/q}$, where p and q are relatively prime and $q > 1$, and $p > 0$. Two bridge knots and links have been completely classified by Schubert [20], as given by the following theorem.

Theorem 2.1. *The two-bridge knots or links $K_{p/q}$ and $K_{p'/q'}$ are equivalent if and only if $q = q'$ and*

$$p^{\pm 1} \equiv p' \pmod{q}.$$

The two-bridge knots or links $K_{p/q}$ and $K_{p'/q'}$ are homeomorphism equivalent if and only if $q = q'$ and

$$p^{\pm 1} \equiv \pm p' \pmod{q}.$$

2.3 Two-Bridge Diagrams

The diagram of a two-bridge knot or link can always be drawn as follows (Figure 2.3): Draw two vertical lines for the bridges, bridge A from $(0,0)$ to $(0,1)$ and bridge B from $(1,0)$ to $(1,1)$. Let S be the unit square with vertices at $(0,0), (0,1), (1,0)$, and $(1,1)$. For $0 \le i \le q$, draw lines of slope p/q starting at $(0, i/q)$ and ending at the edge of the S. If a line intersects the edge of S at $(i/p, 1)$, draw an arc around to $(1, (q-p+i)/q)$ for $0 \le i \le p$. If a line intersects the edge of S at $(i/p, 0)$, draw an arc around to $(0, i/p)$, for $0 < i \le p$. If a line intersects on the left of S at $(0, i/q)$, draw an arc around to $(1, (p-i)/q)$, for $p < i \le q$. Draw each arc so that it crosses no previously drawn arc. The only crossings are at the bridges, which cross over the other arcs.

Another diagram for a two-bridge knot or link is a "pillow diagram" (Figure 2.3). The pillow diagram is constructed as follows: Draw all the line segments within S of slope p/q that pass through points $(0, i/q), 0 \le i < q$

Figure 2.3: Two-bridge and pillow diagrams for $K_{3/5}$.

or $(i/p, 0), 0 < i < p$. These are the overarcs. Next draw all the underarcs, which are the line segments within S of slope $-p/q$ that pass through points $(1, i/q), 0 \leq i < q$ or $(i/p, 0), 0 < i < p$. Lastly, draw arcs from $(0,0)$ to $(0,1)$ and from $(1,0)$ to $(1,1)$, not intersecting any other arc. These are the two bridges. If all the underarcs are pulled up and out to the side of the rest of the knot, the previously described diagram of the two-bridge knot results.

Define the extended 2D diagram (Figure 2.4) for $K_{p/q}$ as follows: For $i, j \geq 0$, associate the oriented line segment from (i, j) to $(i, j+1)$ to the bridge A if i is even and B if i is odd, oriented the same as the corresponding bridge if j is even and opposite if j is odd. Draw the line segment d from $(0,0)$ to (q, p). This diagram is an "unfolding" of half of $K_{p/q}$. Every point $(i, ip/q), 0 < i < q$ that d passes through corresponds to an arc crossing under a bridge. The point (q, p) corresponds to an arc connecting to the end of a bridge.

Lemma 2.1. *In the diagrams described above for a two-bridge knot or link $K_{p/q}$ every arc is in a link component in common with at least one of the two bridges.*

Proof. Count the number of line segments in S in the pillow diagram for $K_{p/q}$. There are $q + p - 1$ segments with positive slope and $q + p - 1$ with negative slope. Now count the number of squares the line segment d in the extended 2D diagram passes through. It must pass through $p - 1$ horizontal lines with integer y coordinate, and $q - 1$ vertical lines with integer x coordinate. Line segment d never crosses a vertical and horizontal line at

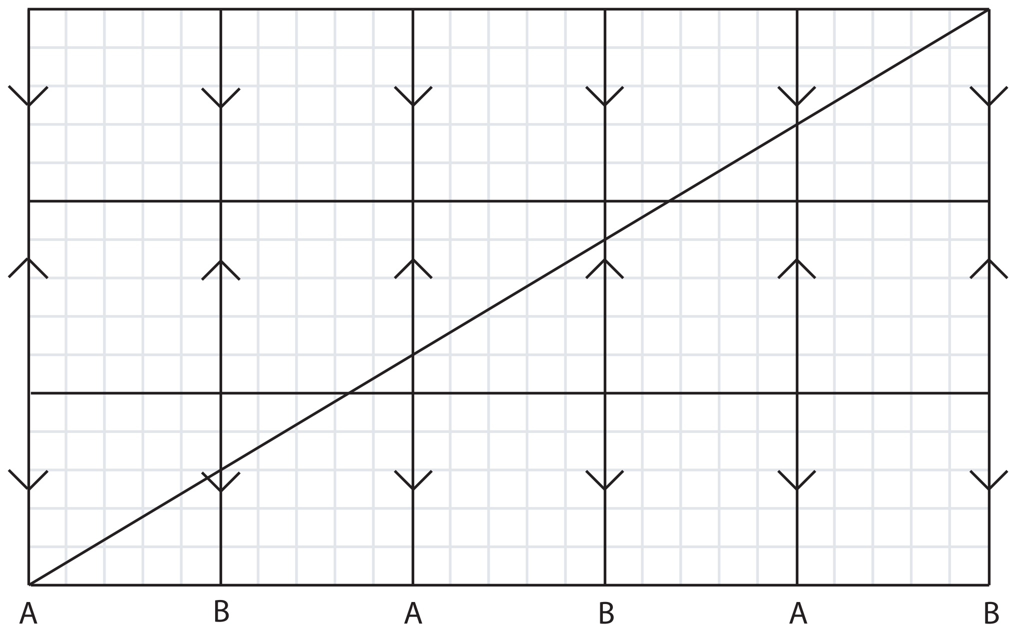

Figure 2.4: Extended 2D diagram for $K_{3/5}$.

the same point, since p and q are relatively prime. So the number of squares is $1 + (p-1) + (q-1) = q + p - 1$. Consider another line segment d' on the extended 2D diagram, from $(1, 0)$ to $(q + 1, p)$, which corresponds to tracing $K_{p/q}$ starting at bridge B rather than bridge A. Line segment d' also passes through $q + p - 1$ squares. Thus the line segments d and d' account for all arcs in the pillow diagram for $K_{p/q}$, and so all arcs are connected to a bridge. □

Proposition 1. *If q is odd then $K_{p/q}$ is a knot, and if q is even then $K_{p/q}$ is a link with two components.*

Proof. The pillow diagram for a 2-bridge knot or link has 180° rotational symmetry. Trace the knot or link starting at the two points $(0, 0)$ and $(1, 1)$ simultaneously, and moving away from the overcrossings on the bridges. The paths will eventually coincide at $(1/2, 1/2)$ iff there is a line from $(0, i)$ to $(1, q - i)$ for some i iff $q - i = i + p$ iff $2i = q - p$ iff q is odd (since p is assumed to be odd).

An alternative proof can be given as follows. In the extended 2D diagram for $K_{p/q}$, the lattice point that the line segment d passes through with the smallest positive coordinates is (q, p) since p and q are relatively prime. If q is odd, the point (q, p) corresponds to a point on bridge B, so $K_{p/q}$ is a knot. If q is even, the point (q, p) corresponds to a point on bridge A, so $K_{p/q}$ is a link. □

It is clear from the description of the diagram of a two-bridge knot or link that the mirror image of $K_{p,q}$ is $K_{-p/q}$, which is equivalent to $K_{(q-p)/q}$.

Proposition 2. *Every two-bridge knot or link $K_{p/q}$ is homeomorphism equivalent to a two-bridge knot or link $K_{p'/q}$ for which $0 < p' < q$ and p' is odd.*

Proof. There exists a unique p' such that $0 < p' < q$ and $p' \equiv p \pmod{q}$. At least one of p' and $q - p'$ must be odd, or else p and q are both even and hence not relatively prime. $K_{p/q}$ is equivalent to $K_{p'/q}$, and is the mirror image of $K_{(q-p')/q}$. □

Throughout this paper, unless otherwise stated, we will assume $0 < p < q$ and p is odd. That is, we will usually only consider one knot or link from each chiral pair.

An invariant of every knot or link is the fundamental group of its complement, henceforth called simply the *group of the knot or link*.

Theorem 2.2. *For $1 \leq i \leq q-1$, let*

$$\epsilon_i = (-1)^{\lfloor ip/q \rfloor}.$$

(i) The group of the two-bridge knot $K_{p/q}$ has the presentation $\langle a, b \mid aw = wb \rangle$ where

$$w = b^{\epsilon_1} a^{\epsilon_2} \cdots b^{\epsilon_{q-2}} a^{\epsilon_{q-1}}.$$

(ii) The group of the two-bridge link $K_{p/q}$ has the presentation $\langle a, b \mid aw = wa, bw^* = w^* b \rangle$, where

$$w = b^{\epsilon_1} a^{\epsilon_2} \cdots a^{\epsilon_{q-2}} b^{\epsilon_{q-1}},$$

and w^* is w with a and b interchanged.

Proof. (i) Let a and b be the elements of the knot group which pass beneath the two bridges at A and B, respectively, oriented down (the loops a and b pass under the left side and come up on the right side of the bridges). We say that a and b are "labels" for the arcs A and

B. Trace around the diagram, starting at $(0,0)$, and labeling each arc satisfying this conjugation relation at each crossing: if an arc labeled x crosses under an arc labeled y then the arc that emerges on the other side of y is labeled yxy^{-1} if the crossing is left-handed and $y^{-1}xy$ if the crossing is right-handed. As we can see from the extended 2D diagram, the crossings are alternately under bridge A and bridge B, until the end of a bridge is reached. The line segments in the extended 2D diagram are oriented down or up accordingly as $\lfloor ip/q \rfloor$ is even or odd. Thus the labels, which begins as a at bridge A, are conjugated by alternating a or a^{-1} and b or b^{-1}, beginning with b or b^{-1}. The labels of all arcs are determined by the labels of the two bridges, since every arc crosses under a bridge or is a bridge, so a and b generate the knot group. If q is odd and $K_{p/q}$ is thus a knot, then the conjugating elements are $\{a^{-\epsilon_i}, b^{-\epsilon_i}\}$, and the only relation is then $w^{-1}aw = b$, or equivalently $aw = wb$, where $w = b^{\epsilon_1}a^{\epsilon_2}\cdots b^{\epsilon_{q-2}}a^{\epsilon_{q-1}}$.

(ii) If q is even, then $K_{p/q}$ is a link. If a and b are oriented down, as in part (a) of the proof, the conjugating elements are $a^{-\epsilon_i}, b^{-\epsilon_i}$, and the relation obtained from tracing the knot is $w^{-1}aw = a$, or equivalently $aw = wa$, where $w = b^{\epsilon_1}a^{\epsilon_2}\cdots b^{\epsilon_{q-2}}a^{\epsilon_{q-1}}$. Since the link diagram has 180° rotational symmetry, the other relation is $bw^* = w^*b$. □

Chapter 3

Alexander Polynomials

The Alexander polynomial $\Delta(t)$ of a knot K is a polynomial associated with K that is invariant under the Reidemeister moves. The Alexander polynomial is defined only up to multiplication by a factor of $\pm t^k$ for integers k. In "standard form", $\Delta(t)$ has a nonzero constant term, a positive leading term, and no negative powers of t. For a link L, the Alexander polynomial is a polynomial in as many variables as there are components in L. Thus for a two-bridge link, the Alexander polynomial $\Delta(x, y)$ is in two variables x and y. Links also have reduced Alexander polynomials in one variable t, defined by $\Delta(t) = \Delta(t, t)(t - 1)$. Henceforth we will refer to the reduced Alexander polynomial of a link simply as the Alexander polynomial. The Alexander polynomials of a chiral pair of knots are equal, but not necessarily for a chiral pair of links. We will define Alexander polynomials more precisely a little later.

3.1 The Fox Calculus

The Alexander polynomial for a knot or link K can be computed in numerous ways. One way uses Fox's calculus applied to the relators of the fundamental group of a knot or link. Fox's calculus is developed in [5].

Let G be a group with presentation $G = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle$, where $x_1, \ldots, x_n$ are the generators of 4g4 and $r_1, \ldots, r_n$ are the relators. Let $F = \langle x_1, \ldots, x_n \rangle$ be the free group on the generators of G. The Fox calculus, which is a map from the group ring $\mathbb{Z}[G]$ to itself, follows these rules:

$$\begin{aligned}
\frac{\partial x_i}{\partial x_j} &= \delta_{ij} \\
\frac{\partial 1}{\partial x} &= 0 \\
\frac{\partial(\alpha\beta)}{\partial x_i} &= \frac{\partial \alpha}{\partial x_i} + \alpha\frac{\partial \beta}{\partial x_i} \\
\frac{\partial(\alpha+\beta)}{\partial x_i} &= \frac{\partial \alpha}{\partial x_i} + \frac{\partial \beta}{\partial x_i}
\end{aligned}$$

where α and β are elements of $\mathbb{Z}[G]$. It follows from these rules that

$$\frac{\partial x_i^{-1}}{\partial x_i} = -x_i^{-1}$$

because

$$0 = \frac{\partial 1}{\partial x_i} = \frac{\partial x_i x_i^{-1}}{\partial x_i} = \frac{\partial x_i}{\partial x_i} + x_i\frac{\partial x_i^{-1}}{\partial x_i} = 1 + x_i\frac{\partial x_i^{-1}}{\partial x_i}$$

3.2 The Alexander Matrix

The Fox calculus is used to construct the *Alexander matrix* of a group presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$. Let $F = \langle \mathbf{x} \rangle$ be the free group generated by $x_1, x_2, \ldots, x_n$. Let $\gamma : F \to \langle \mathbf{x} \mid \mathbf{r} \rangle$ be the quotient map. Let H be the abelianization of G and $\alpha : G \to H$ be the quotient map.

Definition 5. The *Alexander matrix* of $\langle \mathbf{x} : \mathbf{r} \rangle$ is the matrix (a_{ij}) where

$$a_{ij} = \alpha\gamma\left(\frac{\partial r_i}{\partial x_j}\right).$$

Note that the entries of the Alexander matrix are elements of $\mathbf{Z}[H]$.

The following definition of the first elementary ideal of a matrix is a special case of more general elementary ideals, which are discussed in [5].

Definition 6. Let R be a commutative ring with identity, and let A be an $m \times n$ matrix with entries in R, and assume $m \geq n-1$. The *first elementary ideal* $E_1(A)$ of A is the ideal generated by the determinants of all $(n-1) \times (n-1)$ submatrices of A.

Let $\langle \mathbf{x} \mid \mathbf{r} \rangle$ and $\langle \mathbf{y} \mid \mathbf{s} \rangle$ be finite presentations for a group G. A theorem in [5] states that the first elementary ideals of the Alexander matrices for these presentations are isomorphic. Now the Alexander polynomial can be defined in terms of the Alexander matrix.

Definition 7. The *Alexander polynomial* of a knot or link K is the greatest common divisor of the determinants of all $(n-1) \times (n-1)$ submatrices of the Alexander matrix of any finite presentation of the group of K.

The resulting polynomial has as many variables as there are components in K. In the case where K is a knot, it is proved in [5] that the Alexander polynomial $\Delta(t)$ is well-defined up to multiplication by units, in this case $\pm t^k$ for integers k. That is, the Alexander polynomial is the same for equivalent knots. When K is a 2-bridge knot, the group of K has a presentation with two generators and one relator, so the Alexander matrix is a 1×2 matrix. I will prove in the next section that either entry may be taken to be $\Delta(t)$.

In the case where K is a link, the Alexander polynomial in the variables $t_1, \ldots, t_k, k \leq n$ is well-defined up to multiplication by the units $\pm t_1^{k_1} \cdots t_n^{k_n}$. When K is a two-bridge link, the group of K has a presentation with two generators and two relators, so the Alexander matrix is a 2×2 matrix. The greatest common divisor of these four entries is the Alexander polynomial. Each link also has a *reduced Alexander polynomial*, defined as follows

Definition 8. Let K be a link, let A be the Alexander matrix of any finite presentation of the group of K. Define the *reduced Alexander matrix* A' to be the matrix that results when all the variables in A are replaced by the single variable t. The *reduced Alexander polynomial* of K is the greatest common divisor of the entries of A'.

For a two-bridge knot, with knot group $\langle a, b \mid r \rangle$, the Alexander matrix is

$$A = \begin{pmatrix} \dfrac{\partial r}{\partial a} & \dfrac{\partial r}{\partial b} \end{pmatrix}\Big|_{a=t, b=t}$$

and the Alexander polynomial is obtained by taking the greatest common divisor of the two entries. For a two-bridge link, with knot group $\langle x, y \mid r_1, r_2 \rangle$, the Alexander matrix is

$$A = \begin{pmatrix} \dfrac{\partial r_1}{\partial a} & \dfrac{\partial r_1}{\partial b} \\ \dfrac{\partial r_2}{\partial a} & \dfrac{\partial r_2}{\partial b} \end{pmatrix}\Big|_{a=x, b=y}$$

the Alexander polynomial is the greatest common divisor of these entries, and the reduced Alexander polynomial is obtained by letting x and y be t and then multiplying by $t-1$.

3.3 Equivalence of Entries

In this section we prove that for a two-bridge knot or link K, all entries in the Alexander matrix for a presentation for K are equivalent.

Let the symbol $|_t$ denote the result of letting all the generators of a presentation be set to t.

Lemma 3.1. *Let $\epsilon = \pm 1$. Then*

$$(t-1)\frac{\partial a^{\epsilon}}{\partial a}\Big|_t = t^{\epsilon} - 1$$

Proof. If $\epsilon = 1$, then

$$(t-1)\frac{\partial a^{\epsilon}}{\partial a}\Big|_t = t - 1 = t^{\epsilon} - 1.$$

If, on the other hand, $\epsilon = -1$, then

$$(t-1)\frac{\partial a^{\epsilon}}{\partial a}\Big|_t = (t-1)(-t^{-1}) = -1 + t^{-1} = t^{\epsilon} - 1.$$

□

The following theorem is very useful because it expresses the Alexander polynomial of a two-bridge knot or link $K_{p/q}$ entirely in terms of p and q. This formula seems to be commonly known [15], but I did not find a formula this explicit anywhere in the literature.

Theorem 3.1. *Let $K_{p/q}$ be a two-bridge knot or link. For $1 \le i \le q-1$, let $\epsilon_i = (-1)^{\lfloor ip/q \rfloor}$. Then the (reduced) Alexander polynomial of $K_{p/q}$ is*

$$\Delta(t) = \sum_{j=0}^{q-1}(-t)^{\sum_{i=1}^{j}\epsilon_i} = 1 - t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2} - t^{\epsilon_1+\epsilon_2+\epsilon_3} + \ldots + t^{\epsilon_1+\epsilon_2+\ldots+\epsilon_{q-1}}.$$

Proof. First consider the case where $K_{p/q}$ is a knot. The relator of the group of $K_{p/q}$ is $r = aw - wb$, where $w = b^{\epsilon_1}a^{\epsilon_2}\cdots b^{\epsilon_{q-2}}a^{\epsilon_{q-1}}$.

$$\begin{aligned}
\frac{\partial r}{\partial a} &= 1 + a\frac{\partial w}{\partial a} - \frac{\partial w}{\partial a} \\
&= 1 + (a-1)\frac{\partial w}{\partial a} \\
&= 1 + (a-1)b^{\epsilon_1}\left(\frac{\partial a^{\epsilon_2}}{\partial a} + a^{\epsilon_2}b^{\epsilon_3}\left(\frac{\partial a^{\epsilon_4}}{\partial a} + a^{\epsilon_4}\ldots\left(\frac{\partial a^{\epsilon_{q-1}}}{\partial a}\right)\right)\ldots\right)
\end{aligned}$$

It follows that

$$\begin{aligned}
\left.\frac{\partial r}{\partial a}\right|_t &= 1 + t^{\epsilon_1}\left(\left.(t-1)\frac{\partial a^{\epsilon_2}}{\partial a}\right|_t + a^{\epsilon_2}b^{\epsilon_3}\left(\left.(t-1)\frac{\partial a^{\epsilon_4}}{\partial a}\right|_t + a^{\epsilon_4}b^{\epsilon_5}\cdots\right.\right. \\
&\qquad \left.\left.\cdots\left(\left.(t-1)\frac{\partial a^{\epsilon_{q-1}}}{\partial a}\right|_t\right)\right)\cdots\right) \\
&= 1 + t^{\epsilon_1}(-1 + t^{\epsilon_2} + t^{\epsilon_2}t^{\epsilon_3}(-1 + t^{\epsilon_4} + t^{\epsilon_4}\ldots(-1 + t^{\epsilon_{q-1}}))\ldots) \\
&= 1 - t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2} - t^{\epsilon_1+\epsilon_2+\epsilon_3} + \ldots + t^{\epsilon_1+\epsilon_2+\ldots+\epsilon_{q-1}}
\end{aligned}$$

Now consider the other Fox derivative.

$$\begin{aligned}
\frac{\partial r}{\partial b} &= a\frac{\partial w}{\partial b} - \left(\frac{\partial w}{\partial b} + w\right) \\
&= (a-1)\frac{\partial w}{\partial b} - w \\
&= (a-1)\left(\frac{\partial b^{\epsilon_1}}{\partial b} + b^{\epsilon_1}a^{\epsilon_2}\left(\frac{\partial b^{\epsilon_3}}{\partial b} + b^{\epsilon_3}a^{\epsilon_4}\cdots\left(\frac{\partial b^{\epsilon_{q-2}}}{\partial b}\right)\right)\cdots\right) - w
\end{aligned}$$

It follows that

$$\begin{aligned}
\left.\frac{\partial r}{\partial b}\right|_t &= \left.(t-1)\frac{\partial b^{\epsilon_1}}{\partial b}\right|_t + t^{\epsilon_1}t^{\epsilon_2}\left(\left.(t-1)\frac{\partial b^{\epsilon_3}}{\partial b}\right|_t + t^{\epsilon_3}t^{\epsilon_4}\cdots\right. \\
&\qquad \left.\cdots\left(\left.(t-1)\frac{\partial b^{\epsilon_{q-2}}}{\partial b}\right|_t\right)\cdots\right) - w\Big|_t \\
&= -1 + t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2}\left(-1 + t^{\epsilon_3} + t^{\epsilon_3}t^{\epsilon_4}\left(\cdots\left(t^{\epsilon_{q-2}}\right)\right)\cdots\right) - t^{\epsilon_1+\epsilon_2+\ldots+\epsilon_{q-1}} \\
&= -(1 - t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2} - t^{\epsilon_1+\epsilon_2+\epsilon_3} + \ldots + t^{\epsilon_1+\epsilon_2+\ldots+\epsilon_{q-1}})
\end{aligned}$$

The greatest common divisor of the two Fox derivatives is

$$\Delta(t) = 1 - t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2} - t^{\epsilon_1+\epsilon_2+\epsilon_3} + \ldots + t^{\epsilon_1+\epsilon_2+\ldots+\epsilon_{q-1}}.$$

Notice that the signs of the coefficients alternate.

Now consider the case where $K_{p/q}$ is a link. Recall that the reduced Alexander polynomial is the greatest common divisor of the elements of the reduced Alexander matrix. The relators of the group of $K_{p/q}$ are $r_1 = aw - wa$ and $r_2 = bw^* - w^*b$, where $w = b^{\epsilon_1}a^{\epsilon_2}\cdots a^{\epsilon_{q-2}}b^{\epsilon_{q-1}}$, and w^* is w with a and b interchanged. Then clearly

$$\left.\frac{\partial r_1}{\partial a}\right|_t = \left.\frac{\partial r_2}{\partial b}\right|_t$$

and

$$\left.\frac{\partial r_1}{\partial b}\right|_t = \left.\frac{\partial r_2}{\partial a}\right|_t.$$

So we only need to compare $\left.\frac{\partial r_1}{\partial a}\right|_t$ and $\left.\frac{\partial r_1}{\partial b}\right|_t$.

$$\begin{aligned}
\frac{\partial r_1}{\partial a} &= 1 + a\frac{\partial w}{\partial a} - \frac{\partial w}{\partial a} - w \\
&= 1 + (a-1)\frac{\partial w}{\partial a} - w \\
&= 1 + (a-1)b^{\epsilon_1}\left(\frac{\partial a^{\epsilon_2}}{\partial a} + a^{\epsilon_2}b^{\epsilon_3}\left(\frac{\partial a^{\epsilon_4}}{\partial a} + a^{\epsilon_4}b^{\epsilon_5}\dots\right.\right. \\
&\qquad \left.\left.\left(\frac{\partial a^{\epsilon_{q-2}}}{\partial a}\right)\right)\dots\right) - w
\end{aligned}$$

It follows that

$$\begin{aligned}
\left.\frac{\partial r_1}{\partial a}\right|_t &= 1 + t^{\epsilon_1}\left(\left(t-1\right)\left.\frac{\partial a^{\epsilon_2}}{\partial a}\right|_t + a^{\epsilon_2}b^{\epsilon_3}\left(\left(t-1\right)\left.\frac{\partial a^{\epsilon_4}}{\partial a}\right|_t + a^{\epsilon_4}b^{\epsilon_5}\cdots\right.\right. \\
&\qquad \left.\left.\left(\left(t-1\right)\left.\frac{\partial a^{\epsilon_{q-1}}}{\partial a}\right|_t\right)\right)\cdots\right) \\
&\qquad -t^{\epsilon_1+\epsilon_2+\ldots+\epsilon_{q-1}} \\
&= 1 + t^{\epsilon_1}(-1 + t^{\epsilon_2} + t^{\epsilon_2}t^{\epsilon_3}(-1 + t^{\epsilon_4} + t^{\epsilon_4}\dots(-1 + t^{\epsilon_{q-2}}))\dots) \\
&\qquad -t^{\epsilon_1+\epsilon_2+\ldots+\epsilon_{q-1}} \\
&= 1 - t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2} - t^{\epsilon_1+\epsilon_2+\epsilon_3} + \ldots - t^{\epsilon_1+\epsilon_2+\ldots+\epsilon_{q-1}}
\end{aligned}$$

We also have

$$\begin{aligned}
\frac{\partial r_1}{\partial b} &= a\frac{\partial w}{\partial b} - \frac{\partial w}{\partial b} \\
&= (a-1)\frac{\partial w}{\partial b} \\
&= (a-1)\left(\frac{\partial b^{\epsilon_1}}{\partial b} + b^{\epsilon_1}a^{\epsilon_2}\left(\frac{\partial b^{\epsilon_3}}{\partial b} + b^{\epsilon_3}a^{\epsilon_4}\left(\cdots\left(\frac{\partial b^{\epsilon_{q-1}}}{\partial b}\right)\right)\right)\cdots\right)
\end{aligned}$$

It follows that

$$\begin{aligned}
\left.\frac{\partial r_1}{\partial b}\right|_t &= (t-1)\left.\frac{\partial b^{\epsilon_1}}{\partial b}\right|_t + t^{\epsilon_1}t^{\epsilon_2}\left((t-1)\left.\frac{\partial b^{\epsilon_3}}{\partial b}\right|_t + t^{\epsilon_3}t^{\epsilon_4}\cdots\right.\\
&\quad \left.\left((t-1)\left.\frac{\partial b^{\epsilon_{q-1}}}{\partial b}\right|_t\right)\cdots\right)\\
&= -1 + t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2}\left(-1 + t^{\epsilon_3} + t^{\epsilon_3}t^{\epsilon_4}\left(\cdots\left(t^{\epsilon_{q-1}}\right)\right)\cdots\right)\\
&= -(1 - t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2} - t^{\epsilon_1+\epsilon_2+\epsilon_3} + \ldots - t^{\epsilon_1+\epsilon_2+\ldots+\epsilon_{q-1}})
\end{aligned}$$

The greatest common divisor of these two Fox derivatives is

$$\Delta(t) = 1 - t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2} - t^{\epsilon_1+\epsilon_2+\epsilon_3} + \ldots - t^{\epsilon_1+\epsilon_2+\ldots+\epsilon_{q-1}}.$$

Also note that all entries in the reduced Alexander matrix are equal up to a factor of ± 1. $\square$

Chapter 4

The Up-Down Graph

The following definitions relate to the up-down graph of a knot or link $K_{p/q}$. The only place I have found the up-down graph described and used is in a preprint by Murasugi and Hirasawa [15]. They prove some of the basic properties of the graph, but they use it for different purposes than I do. In this section we describe the up-down graph and prove some of its properties. In the next section we use the up-down graph to prove properties of the Alexander polynomial.

4.1 Definitions

Definition 9.

Let p and q be integers such that $0 < p < q$ and $\gcd(p, q) = 1$.

(a) Define the *sequence of signs* $\{\epsilon_i\}$ by $\epsilon_i = (-1)^{\lfloor ip/q \rfloor}$.

(b) Define the *partial sums* $\{s_i\}$ by $s_0 = 0$ and for $i \geq 1, s_i = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_i$.

(c) The *up-down graph* $G(p, q)$ for the knot or link $K_{p/q}$ consists of the points $P_0, P_1, \ldots, P_{q-1}$ and the edges connecting P_i and $P_{i+1}, 0 \leq i \leq q-2$, where $P_i = (i, s_i)$.

(d) The *level* of P_i is the y coordinate s_i.

(e) Define the *break points* in the up-down graph as follows: Let $B_0 = P_0$, and for $i \geq 1$ let the ith break point B_i be the ith point where the graph changes direction (up or down), that is, where $\epsilon_i = -\epsilon_{i+1}$.

(f) For $i \geq 0$ define the ith *valley* of the graph to be B_{2i}.

(g) For $i \geq 1$ define the ith *peak* of the graph to be B_{2i-1}.

(h) Define a *summit* of the graph to be a peak with the highest level.

(i) Define the ith *segment* of the graph to be the edges between B_{i-1} and B_i.

(j) Define the ith *length* l_i to be the number of edges in the ith segment of the graph.

(k) Define the *degree* of $G(p,q)$ to be the number of distinct levels, minus 1.

4.2 Properties of the Up-Down Graph

By its very definition, the graph $G(p,q)$ contains q points and $q-1$ edges. Since $p < q$, $\lfloor ip/q \rfloor$ takes on each of the values $0, 1, \ldots, p-1$, for $1 \leq i \leq q-1$. Thus there are p segments in the up-down graph for $K_{p/q}$.

Proposition 1. $\epsilon_i = \epsilon_{q-i}$ *for* $1 \leq i \leq q-1$.

Proof.

$$\begin{aligned} \epsilon_{q-i} &= (-1)^{\lfloor (q-i)p/q \rfloor} = (-1)^{\lfloor p - ip/q \rfloor} = (-1)^{p + \lfloor -ip/q \rfloor} = (-1)^{p+1+\lfloor ip/q \rfloor} \\ &= (-1)^{\lfloor ip/q \rfloor} = \epsilon_i. \end{aligned}$$

The penultimate step is true since p is odd. Thus the sequence of exponents is symmetric.

□

Corollary 1. *The up-down graph* $G(p,q)$ *has* $180°$ *rotational symmetry.*

Proof. This follows immediately from Proposition 1.

□

The following proposition is one that I discovered, and it is very useful in proving several of the other propositions about the up-down graph. I suspect this proposition may also be a key to proving the trapezoidal property, Theorem 5.2.

Proposition 2. *For any integer k such that $1 \le k \le p$,*

$$\sum_{i=1}^{k} l_i \le \sum_{i=j+1}^{j+k} l_i \le 1 + \sum_{i=1}^{k} l_i$$

for $j \ge 0$.

Proof. The sum $\sum_{i=1}^{k} l_i$ is the number of multiples of p that are greater than 0 and less than or equal to kq, or $\lfloor kq/p \rfloor$. The sum $\sum_{i=j+1}^{j+k} l_i$ is the number of multiples of p that are greater than jq and less than or equal to $(j+k)q$, or $\lfloor (j+k)q/p \rfloor - \lfloor jq/p \rfloor$. But $\lfloor (j+k)q/p \rfloor = \lfloor jq/p + kq/p \rfloor$, which is equal to $\lfloor jq/p \rfloor + \lfloor kq/p \rfloor$ or $\lfloor jq/p \rfloor + \lfloor kq/p \rfloor + 1$. Therefore, either $\sum_{i=j+1}^{j+k} l_i$ equals $\sum_{i=1}^{k} l_i$ or $1 + \sum_{i=1}^{k} l_i$, for $j \ge 0$. $\square$

Another way to prove Proposition 2 relies on the following proposition and corollary, which compute the exact length of any segment or group of consecutive segments in an up-down graph.

Proposition 3. *For any integer k such that $1 \le k \le p$,*

$$\sum_{i=1}^{k} l_i = \left\lfloor \frac{kq-1}{p} \right\rfloor .$$

Proof. The sum on the left is the sum of the lengths of the first k segments, which equals the greatest i such that $\lfloor ip/q \rfloor < k$, which equals the greatest i such that $ip \le kq - 1$, which equals $\lfloor (kq-1)/p \rfloor$. Note that if $k < p$ then this formula is equivalent to $\lfloor kq/p \rfloor$. $\square$

This leads to the following formula:

$$\sum_{i=j+1}^{j+k} l_i = \sum_{i=1}^{j+k} l_i - \sum_{i=1}^{j} l_i = \left\lfloor \frac{(j+k)q-1}{p} \right\rfloor - \left\lfloor \frac{jq-1}{p} \right\rfloor .$$

It is well known that for any real numbers x and y,

$$\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x+y \rfloor < \lfloor x \rfloor + \lfloor y \rfloor + 1 \Rightarrow \lfloor x \rfloor \le \lfloor x+y \rfloor - \lfloor y \rfloor < \lfloor x \rfloor + 1.$$

Therefore, letting $x = (kq-1)/p$ and $y = jq/p$ proves Proposition 2.

Corollary 2. *(a) The length of the first segment in $G(p,q)$ is*

$$l_1 = \left\lfloor \frac{q-1}{p} \right\rfloor = \left\lfloor \frac{q}{p} \right\rfloor.$$

(b) For $i > 1$,

$$l_i = \left\lfloor \frac{iq-1}{p} \right\rfloor - \left\lfloor \frac{(i-1)q-1}{p} \right\rfloor.$$

Proof. This follows immediately from Proposition 3. □

Conjecture 1. *If G is an up-down graph that is rotationally symmetric and whose segments satisfy the condition in Proposition 2, then G is the up-down graph for the two-bridge knot or link $K_{p/q}$, where p is the number of segments of G and q is the number of vertices.*

Proposition 4. *Suppose $q = mp + r$, where $0 \leq r < p$.*

(a) $l_1 = l_p = m$.

(b) The lengths of the segments of $G(p,q)$ are all m or $m+1$.

Proof.

(a) If $p = 1$, the statement is true trivially. Otherwise, $m+1$ is the smallest number i such that $\epsilon_i = -1$. Thus $l_1 = m$. By Proposition 1, $l_p = m$ also, since there are p segments in $G(p,q)$.

(b) This follows from part (a) and Proposition 2 with $k = 1$. □

Definition 10. For the graph $G(p,q)$, let $q = mp + r$, where $0 \leq r < p$. Segments of length m are called *short* segments and segments of length $m+1$ are called *long* segments. By Proposition 4 (b), all segments in $G(p,q)$ are short or long.

Definition 11. Segments of a length l are *isolated* in $G(p,q)$ if no two consecutive segments of $G(p,q)$ are of length l.

Proposition 5. *Short segments are isolated or long segments are isolated in $G(p,q)$.*

Proof. Suppose that short segments were not isolated. Then there exist two consecutive short segments. Let m be the length of a short segment. Then by Proposition 2, the maximum possible sum of two consecutive lengths is $2m + 1$, which is less than $2(m + 1)$. Thus no two consecutive segments can be long, so they are isolated. Therefore short or long segments must be isolated.

□

Definition 12. A *cluster* of segments of length l in an up-down graph $G(p, q)$ is a nonempty set of consecutive segments of length l such that the previous segment (if it exists) and the following segment (if it exists) are not of length l. The *size* of a cluster is the number of segments of length l in the cluster.

Lemma 4.1. *For any up-down graph $G(p, q)$ that has segments of length l, all clusters of segments of length l have size s or $s + 1$ for some number $s > 0$.*

Proof. Suppose there were a cluster of size x and a cluster of size y of segments of length l, where $y \geq x + 2$. If there are segments in $G(p, q)$ before and after the cluster of size x, they must be of length k, where $k \neq l$. Then the sum of the lengths of these two segments plus the sum of the lengths of the segments in the cluster of size x is different by at least 2 from the sum of the lengths of $x + 2$ consecutive segments of length l within the cluster of size y, which contradicts Proposition 2. If instead there are not segments of another length before and after the cluster of size x, then that cluster must be the first or last cluster in the graph. Hence the segments in these clusters are short, by Proposition 4 (a). Then this violates Proposition 2 where $k = x$. Thus the two clusters cannot exist. □

Proposition 6. *Suppose $p > 1$, and $q = mp + r$, where $0 < r < p$.*

(a) *There are $p - r + 1$ segments of length m and $r - 1$ segments of length $m + 1$.*

(b) *Segments of length m always occur in clusters of $\lfloor p/r \rfloor$ or $\lfloor p/r \rfloor - 1$ consecutive segments. Segments of length $m + 1$ always occur in clusters of $\lfloor (p-1)/(p-r) \rfloor$ or $\lfloor (p-1)/(p-r) \rfloor - 1$ consecutive segments.*

Proof.

(a) Let x be the number of segments of length m and y the number of segments of length $m + 1$. Then $x + y = p$ as shown above, and

$xm + y(m+1) = q-1$ since the number of edges is $q-1$. The unique solution to these two equations is $y = q - mp - 1 = r - 1$ and $x = p - r + 1$.

(b) By Proposition 4 (b), there are two possible segment lengths. Thus clusters alternate between segments of length m and segments of length $m+1$. By Proposition 4 (a), the first and last clusters are of segments of length m, so there is one more cluster of length m than of length $m+1$.

Case 1: Suppose short segments are isolated. Then by Proposition 6 (a), the number of clusters of short segments is $p - r + 1$ and the number of clusters of long segments is at least $r - 1$, so

$$p - r + 1 \leq r \Rightarrow p \leq 2r - 1 \Rightarrow \lfloor p/r \rfloor = 1,$$

which is the number of segments in a cluster of short segments. Now, since the clusters of longs come in at most two sizes, these sizes are

$$\left\lfloor \frac{r-1}{p-r} \right\rfloor = \left\lfloor \frac{p-1}{p-r} - 1 \right\rfloor$$

and

$$\left\lceil \frac{r-1}{p-r} \right\rceil = \left\lceil \frac{p-1}{p-r} - 1 \right\rceil,$$

which is equal to

$$\left\lfloor \frac{p-1}{p-r} \right\rfloor$$

or

$$\left\lfloor \frac{p-1}{p-r} - 1 \right\rfloor.$$

Case 2: Suppose long segments are isolated. By part Proposition 6 (a), there are $r - 1$ segments of length $m + 1$, so there are r clusters of short lengths. Since the clusters come in at most two sizes, these sizes are $\lfloor (p - r + 1)/r \rfloor$ and $\lceil (p - r + 1)/r \rceil$. If r divides $p + 1$, then these numbers are both equal to $\lfloor p/r \rfloor$. Otherwise, they are equal to $\lfloor p/r \rfloor$ and $\lfloor p/r \rfloor - 1$. Since longs (if they exist) are isolated, $r - 1 \leq p - r$, which implies $p \geq 2r - 1$, which implies $\lfloor (r-1)/(p-r) \rfloor \leq 1$, and this is the number of long segments that appear in clusters.

$\square$

Proposition 7. *Consecutive peaks are at levels that are equal or differ by 1.*

Proof. The difference in the levels of two consecutive peaks is equal to the difference in the lengths of the two segments between them. By Proposition 4 (b), the lengths of these two segments are equal or differ by one, so the levels of the consecutive peaks are equal or differ by one.

□

Corollary 3. *If there are peaks at levels i and j in an up-down graph, then there are peaks at every level in between.*

Proof. Without loss of generality let $i < j$. Suppose to the contrary that there exists k such that $i < k < j$ and there are no peaks at level k. Then by 7, no peak above level k can be adjacent to any peak below level k. But all peaks are contiguous, which is a contradiction. Thus there are peaks at every level between i and j. □

Chapter 5

Applying the Up-Down Graph to the Alexander Polynomial

In this chapter we prove the fundamental relationship between the up-down graph $G(p,q)$ and the Alexander polynomial $\Delta(t)$ for the knot or link $K_{p/q}$. The up-down graph is easy to analyze combinatorially, and the connection allows us to prove many properties of $\Delta(t)$.

Throughout this chapter, the Alexander polynomial is assumed to be in standard form $\Delta(t) = \sum_{j=0}^{n} a_j(-1)^{n-j}t^j$.

5.1 Connection between the Up-down Graph and the Alexander Polynomial

To begin, the following theorem [15] provides the fundamental connection.

Theorem 5.1. *Let b_i be the number of vertices on the up-down graph for the two-bridge knot or link $K_{p/q}$ with y coordinate i. Let $\Delta(t) = \sum_{j=0}^{n} a_j(-1)^{n-j}t^j$ be the (possibly reduced) Alexander polynomial for $K_{p/q}$ in standard form, with $a_n \neq 0$ and $a_j \geq 0$ for all j. Then $a_i = b_{i+k}$ for $0 \leq i \leq n$ for some integer k.*

Proof. The number of i for which $0 \leq i \leq q-1$ and $s_i = j$ is b_j. By Theorem 3.1, the Alexander polynomial for $K_{p/q}$ in standard form is $\Delta(t) = t^k \sum_{i=0}^{q-1}(-t)^{s_i}$ for some integer k. The coefficient a_j is the number of terms in this sum such that $s_i = j$. Therefore $a_j = b_j$ for all j.

□

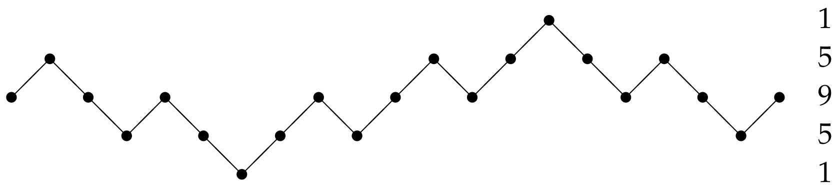

Figure 5.1: The up-down graph for $K_{13/21}$.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\cdots$	20
ϵ_i	1	-1	-1	1	-1	-1	1	1	-1	1	1	-1	1	1	$\cdots$	1

Table 5.1: The sequence of signs for $K_{13/21}$.

Figure 5.1 and Table 5.1 show the sequence of signs for $(p,q) = (13,21)$, the up-down graph $G(13,21)$, and the corresponding coefficients of the Alexander polynomial for $K_{13/21}$.

5.2 Properties of the Alexander Polynomial

The following theorem is the trapezoidal property of the Alexander polynomial and was proved by Hartley [9]. This theorem was a main motivation for my thesis, although I did not complete a proof of it.

Theorem 5.2. *The coefficients a_i of the (possibly reduced) Alexander polynomial for the two-bridge knot or link $K_{p/q}$ satisfy $a_i = a_{n-i}$ for $0 \le i \le n$, and*

$$a_0 < a_1 < \cdots < a_s = \cdots = a_{n-s} > a_{n-s+1} > \cdots > a_n.$$

for some integer s.

Theorem 5.2 gives just one interesting property of the Alexander polynomials of two-bridge knots and links. Using the up-down graph for $K_{p/q}$, many other interesting properties about their Alexander polynomials can also be proved. A few are given here, including some special cases in which the trapezoidal property is proven. Most of these are already well-known.

Proposition 8. *Let $\Delta(t)$ be the (possibly reduced) Alexander polynomial of the two-bridge knot or link $K = K_{p/q}$.*

(a) $\Delta(t)$ *has even degree if K is a knot and odd degree if K is a link.*

(b) $\Delta(t)$ *has degree at least 2 if K is a knot and at least 1 if K is a link.*

(c) The coefficients of $\Delta(t)$ *are symmetric and alternate in sign.*

(d) The coefficients of $\Delta(t)$ *are all nonzero, that is, there are no skipped terms.*

(e) The maximum coefficient of $\Delta(t)$ *(in absolute value) is at least* $\min\{p, 2\lfloor q/p \rfloor + 1\}$ *and at most* $\min\{p, \lfloor (q+1)/2 \rfloor\}$.

(f) $\Delta(-1) = q$.

(g) $\Delta(1) = \pm 1$ *if K is a knot and* $\Delta(1) = 0$ *if K is a link.*

(h) The degree of $\Delta(t)$ *is at least* $\lceil (q-1)/p \rceil$ *and at most* $\lfloor q/p \rfloor + (p-1)/2$.

(i) If K is a knot, the middle coefficient of $\Delta(t)$ *is odd.*

Proof.

(a) By the symmetry of the ϵ_i's, the up-down graph for a two-bridge knot has an odd number of rows and thus the degree of $\Delta(t)$ is even, and vice versa for a two-bridge link.

(b) Since $q \geq 1$ for any two-bridge knot or link $K_{p/q}$, by definition, the degree of $\Delta(t)$ is not 0. By part (a), the degree is at least 1 for a link and 2 for a knot.

(c) In the formula for the Alexander polynomial given in Theorem 5.1, the coefficients alternate in sign. Since the graph $G(p,q)$ is symmetric by Corollary 1, the Alexander polynomial is symmetric.

(d) Since the graph $G(p,q)$ is connected, the coefficients $a_0, a_1, \ldots, a_n$ of $\Delta(t)$ are all nonzero.

(e) If the highest valley is at a lower level than the lowest peak, then every segment has a vertex at the level of the lowest peak. Since the graph $G(p,q)$ has p segments, the max coefficient is p. Otherwise, let k be the level of the lowest peak. Since the highest valley is at a higher level, there must be peaks at levels k through $k+s$, where $s = \lfloor q/p \rfloor$. Both segments adjacent to any peak at level $k+1$ through

$k+s-1$, and both segments next two a given peak at level $k+s$, contain vertex at level k, with no vertex being double-counted. So the number of vertices at level k is at least $1+2(s-1)+2=2s+1$.

The graph $G(p,q)$ has p segments, and each segment can contribute at most 1 to the value of any coefficient of $\Delta(t)$. Thus the max coefficient is at most p. Also, since no two consecutive points in $G(p,q)$ have the same level, the most a coefficient can be is $\lfloor (q+1)/2 \rfloor$. Examples in which these upper bounds are reached are the knots $K_{3/7}$ and $K_{5/9}$, respectively.

(f) Since the Alexander polynomial has coefficients that alternate in sign, $\Delta(-1)$ is the number of points in $G(p,q)$, which is q.

(g) Since the parity of the level of the points of $G(p,q)$ alternate as the graph is traversed from left to right, $\Delta(1)$ is ± 1 if q is odd (when K is a knot) or 0 (when K is a link).

(h) Since the number of segments of $G(p,q)$ is p and the number of edges is $q-1$, there is a segment with at least $\lceil (q-1)/p \rceil$ edges, so the degree of $\Delta(t)$ is at least $\lceil (q-1)/p \rceil$. By Proposition 2, there are only two different lengths in $G(p,q)$, and by Proposition 7 consecutive peaks differ in level by at most 1. Traverse the graph from left to right. The first segment has length $\lfloor q/p \rfloor$, and after that, only every other length can extend the degree of the graph, and only by 1. The degree of $\Delta(t)$ is equal to the degree of $G(p,q)$. Thus an upper bound on the degree of $\Delta(t)$ is $\lfloor q/p \rfloor + (p-1)/2$. These bounds are reached in the cases of the knots $K_{3/7}$ and $K_{7/11}$, respectively.

(i) If K is a knot, then the number of points in $G(p,q)$ is q, which is odd. Since the degree is even, the middle coefficient of $\Delta(t)$ is the number of points on the middle row of $G(p,q)$. The middle row contains the middle point $P_{(q-1)/2}$, and by symmetry contains an even number of other points. Thus the middle coefficient of $\Delta(t)$ is odd. □

The following proposition is one part of Schubert's classification of two-bridge knots and links.

Proposition 9. *If $K_{p/q}$ and $K_{p'/q'}$ are equivalent, then $q=q'$.*

Proof. Suppose $K_{p/q}=K_{p'/q'}$. Since the Alexander polynomial is a knot invariant, it follows that $\Delta(-1)=q=q'$. □

5.3 Some Trapezoidal Special Cases

Using the up-down graph, the Alexander polynomials for some special cases of two-bridge knots and links are proven to be trapezoidal.

Lemma 5.1. *Let P_{q-1} in $G(p,q)$ be a summit, and $p \neq 1$. Then $G(p,q)$ has at least two summits.*

Proof. Proposition 4 (a) showed that l_p is a short segment. Then l_{p-1} must be short, or else P_{q-1} would not be a summit. But then the peak on the left end of the segment l_{p-1} must be a summit, so there are at least two summits. □

Lemma 5.2. *If two consecutive peaks in $G(p,q)$ are both summits, then the two segments between them are both short.*

Proof. If all peaks are summits, then all segments are short. Otherwise, there must be a peak that is not a summit. Suppose that P_i and P_j are two consectuive peaks that are summits, and they are connected by two long segments of length l. □

Proposition 10. *Let $K_{p/q}$ be a two-bridge knot or link, with Alexander polynomial $\Delta(t)$, and let n be the degree of $\Delta(t)$. Then the coefficients of $\Delta(t)$ satisfy one of the following: (1) $n = 1$, or (2) $a_0 = a_1 = \ldots = a_n = 1$, or (3) $a_1 > a_0, a_1 \geq 3$, and $a_1 \geq 2a_0 - 1$.*

Proof. $n = 1$ if and only if $q = p + 1$, in which case all segments are of length 1. Assume it is not the case that $n = 1$ or that $a_0 = a_1 = \ldots = a_n = 1$. Then $G(p,q)$ has at least three segments and at least two peaks. The highest level of $G(p,q)$ is level n. By the symmetry of $\Delta(t)$ (Proposition 8 (c)), a_0 is the number of summits in $G(p,q)$ and a_1 is the number of points at level $n-1$.

Suppose P_i and P_{i+2d} are two consecutive peaks which are at the same level. The two segments connecting them have length d. Thus any two consecutive segments of the same length in $G(p,q)$ must have length d, by Proposition 2. Therefore any two consecutive peaks which are at the same level are connected by two segments of length d.

Case 1: $a_0 = 1$. Since a_0 is the number of summits in $G(p,q)$, the last point P_{q-1} cannot be a summit, by Lemma 5.1. Also, P_0 is not a summit since it is a valley. Let P_i be the unique summit of $G(p,q)$. Then P_{i+1} and P_{i-1} are two points at level $n-1$, neither of which is a peak because they are next to P_i. Since there are at least two peaks, there must be a peak P_j at height $n-1$, by Proposition 7. Thus $a_1 \geq 3$. Since $a_0 = 1$, it follows that $a_1 \geq 2a_0 - 1$ and $a_1 > a_0$.

Case 2: $a_0 \geq 2$, and no two consecutive peaks are summits connected by two segments of length 1. Let $S = \{P_{k_1}, P_{k_2}, \ldots, P_{k_{a_0}}\}$ be the set of summit points in $G(p,q)$. Then the points $P_{k_1 \pm 1}, P_{k_2 \pm 1}, \ldots, P_{k_{a_0 - 1} \pm 1}, P_{k_{a_0} - 1}$ are $2a_0 - 1$ distinct points at level $n-1$. Therefore $a_1 \geq 2a_0 - 1$. Thus $a_1 \geq 3$ since $a_0 \geq 2$. Also, $a_1 \geq 2a_0 - 1 \geq a_0 + 1$, hence $a_1 > a_0$.

Case 3: $a_0 \geq 2$, and there exist two consecutive peaks that are summits connected by two segments of length 1. So short segments have length 1.

Define a set of summits to be *close* if it consists of a sequence of summits $P_i, P_{i+2}, \ldots, P_{i+2j}$, where P_{i-2} and P_{i+2j+2} (if they exist) are not summits. Partition all summits into close sets of summits. There are two cases to consider.

Case 3a: P_{q-1} is a summit. Then P_0 is at level 0. Since short segments have length 1, and $n > 1$, P_1 is a peak that is not a summit. Consider any close set of summits $S = \{P_i, P_{i+k}, \ldots, P_{i+2k-2}\}$, where $k = |S|$. By Proposition 4.1, segments of length 1 are in clusters of $2k-3, 2k-2$, or $2k-1$ segments in $G(p,q)$, and segments of length 2 are isolated. By Proposition 7, the last peak before P_i is at level $n-1$, so it is P_{i-3}. Thus the $2k$ points $P_{i-2k+1}, P_{i-2k+3}, \ldots, P_{i+2k-3}, P_{i+2k-1}$ are all at level $n-1$, unless $i + 2k - 2 = q - 1$, in which case only the first $2k-1$ of these are points in $G(p,q)$. Let T_S be this set of points at level $n-1$. For different close sets of summits S_1 and S_2, the sets T_{S_1} and T_{S_2} are disjoint. Thus a_1, the total number of points at level $n-1$, is at least $2a_0 - 1$. Since $a_0 \geq 2$, this also implies that $a_1 \geq 3$ and $a_1 > a_0$.

Case 3b: P_{q-1} is a not summit. Then repeat the procedure and analysis as in Case 3a, except that each close set of k summit points is associated with the level $n-1$ points that come after, not before. Thus there at least $2a_0 - 1$ points at level $n-1$. Hence $a_1 \geq 2a_0 - 1$, $a_1 \geq 3$ and $a_1 > a_0$. □

Corollary 4. *The Alexander polynomial $\Delta(t)$ is trapezoidal if it is quadratic or cubic.*

Proof. If $\Delta(t)$ is quadratic or cubic, then $a_1 > a_0$ by Proposition 10. If $\Delta(t)$ is quadratic, then $a_0 = a_2$, so $\Delta(t)$ is trapezoidal. Similarly, if $\Delta(t)$ is cubic then $a_0 = a_3$ and $a_1 = a_2$, so $\Delta(t)$ is trapezoidal. □

The natural question, then, is for which p and q is the Alexander polynomial for $K_{p/q}$ quadratic. The following proposition answers that question.

Proposition 11. *The Alexander polynomial $\Delta(t)$ for the two-bridge knot $K_{p/q}$ is quadratic if and only if $2(q-p)$ divides $q-1$ or $q+1$.*

This proposition implies that $\Delta(t)$ is quadratic if and only if p/q is "close" to one of the fractions $1/2, 3/4, 5/6, 7/8, \ldots (2m-1)/(2m) \ldots$ in the sense that either $(p+1)/(q+1)$ or $(p-1)/(q-1)$ is one of these fractions. This is because

$$\frac{p \pm 1}{q \pm 1} = \frac{2m-1}{2m} \iff \frac{q-p}{q \pm 1} = \frac{1}{2m} \iff 2m(q-p) = q \pm 1,$$

which is true if and only if $2(q-p)$ divides $q \pm 1$.

Proof. Suppose $\Delta(t)$ is quadratic. Then the up-down graph $G(p,q)$ contains segments of lengths 1 and 2 only. The center segment must be of length 2, by symmetry, so the number of length 2 segments is an odd number l. The length 2 segments must alternate ascending and descending, or else they would span more than three rows. There are $l+1$ sections in $G(p,q)$ consisting of segments of length 1, separated by the segments of length 2 (these sections could be empty). Each section has $2c$ segments of length 1, where $c \geq 0$, except the two end sections may contain $2c+1$ segments of length 1 (there are no other possibilities, by Propositions 2 and 4 (b)). The number of points in $G(p,q)$ is q, and it also equals $2c(l+1) + 2l + 2 \pm 1 = 2(c+1)(l+1) \pm 1$. The number of segments in $G(p,q)$ is p, and it also equals $2c(l+1) + l + 1 \pm 1$. Thus $q - p = l + 1$, so that $2(q-p) = 2(l+1)$ divides either $q+1$ or $q-1$.

Conversely, suppose that $2(q-p)$ divides $q \pm 1$. Then $2(q-p)(c+1) = q \pm 1$ for some $c \geq 0$. After some algebra, this turns into $2c(q-p)+1$ equals $2c(q-p)$ or $2c(q-p)+2$. Now, the number of long segments is $q-p-1$, so there are $q-p$ (possibly empty) sections of short segments. The number of short segments is $q-p$. Thus the up-down graph $G(p,q)$ consists $q-p-1$ long segments separating $q-p$ sections containing $2c$ short segments apiece, except for the end sections which may contain $2c+1$ short segments. Since $2c$ is even, the segments of length 2 are alternately ascending and descending. Thus $\Delta(t)$ is quadratic. $\square$

Lemma 5.3. *Let a_k denote the number of vertices at level k. Then $a_{k+1} - a_k$ equals the number of peaks at both levels minus the number of valleys at both levels, not counting a final peak at level $k+1$ or an initial valley at level k.*

Proof. Consider a horizontal line at level $k+1/2$, and let $a_{k+1/2}$ denote the number of segments the line intersects. Every non-final peak at level $k+1$ is divided into two segments at level $k+1/2$. Every valley at level $k+1$ has no segment extending downward from it to level $k+1/2$. Every segment passing through level $k+1$ that does not terminate in a peak or valley at level $k+1$, or a segment that terminates in a final peak at level $k+1$, also passes through level $k+1/2$. Therefore $a_{k+1}-a_{k+1/2}$ equals the number of valleys minus the number of non-final peaks at level k.

Exactly the same argument can be used in reverse to show that $a_k - a_{k+1/2}$ equals the number of peaks minus the number of non-initial valleys at level k. Combining these two equations proves the lemma. □

Lemma 5.4. *If every peak is at or above the level of every valley in $G(p,q)$, then the Alexander polynomial for $K_{p/q}$ is trapezoidal.*

Proof. By 1, $G(p,q)$ is symmetric. Let j be the level of the highest valley, let i be the level of the lowest peak, and let n be the level of the highest peak. It is given that $i \geq j$. For $i < k < n$, there is a peak at level k, by Corollary 3, but no valley at level k. Thus $a_k > a_{k+1}$. We consider two cases for i and j.

Case 1: $i = j$. Then level i is the middle level. Since $G(p,q)$ is symmetric, there are equally many peaks and valleys at level i. There is a peak at level $i+1$, as mentioned above. The number of peaks minus the number of non-initial valleys at level i is 0 or 1. Thus $a_i > a_{i+1}$. Thus $a_l > a_{l+1}$ for $i \leq l < n$. By the symmetry of $G(p,q)$, $\Delta(t)$ is trapezoidal.

Case 2: $i > j$. Since there is a peak at level i but no valley, $a_i > a_{a+1}$. Since there is no peak or valley at levels $j+1$ through $i-1$, $a_{j+1} = a_{j+2} = \ldots = a_{i-1} \geq a_i$. By the symmetry of $G(p,q)$, $\Delta(t)$ is trapezoidal. □

Proposition 3. *If $q > p^2/2$, then $\Delta(t)$ for $K_{p/q}$ is trapezoidal.*

Proof. Suppose that $G(p,q)$ is an up-down graph for which the lowest peak P_1 is at level i, the highest valley P_2 is at level j, and $i < j$. Let $s = \lfloor q/p \rfloor$ be the length of short segments and $l = s+1$. The number of edges between two adjacent peaks at different levels must be $s+l$, since there are only two segment lengths. P_1 is adjacent to a valley at level $i-s$ or lower. Thus there are at least $s+1$ different levels with peaks between P_1 and P_2 (inclusive). Thus the number of edges between P_0 and P_1 is at least $s+(s+1)(l+s) = 2l^2-1$. Since $q-1$ is the number of edges in $G(p,q)$, $q > 2l^2-1$.

Now let $G(p,q)$ be an up-down graph such that $q > p^2/2 \Rightarrow 2(q/p)^2 > q \Rightarrow 2(\lfloor q/p \rfloor + 1)^2 > q \Rightarrow q \leq 2l^2 - 1$. Therefore by the above argument, the lowest peak of $G(p,q)$ is at least as high as the highest valley. By Lemma 5.4, $\Delta(t)$ is trapezoidal. □

Proposition 4.

(a) If $q \equiv 0 \pmod{p}$, then $\Delta(t)$ for $K_{p/q}$ is trapezoidal.

(b) If $q \equiv 1 \pmod{p}$, then $\Delta(t)$ for $K_{p/q}$ is trapezoidal.

(c) If $q \equiv -1 \pmod{p}$, then $\Delta(t)$ for $K_{p/q}$ is trapezoidal.

Proof.

(a) If $q \equiv 0 \pmod{p}$, then $p = 1$ since $\gcd(p, q) = 1$. Then $a_i = 1$ for all i, so $\Delta(t)$ is trapezoidal.

(b) Suppose $q \equiv 1 \pmod{p}$. Then $q = mp + 1$ for some m. By Corollary 2,

$$l_1 = \left\lfloor \frac{q-1}{p} \right\rfloor = \left\lfloor \frac{mp+1-1}{p} \right\rfloor = m,$$

and for $k > 1$,

$$\begin{aligned} l_k &= \left\lfloor \frac{kq-1}{p} \right\rfloor - \left\lfloor \frac{(k-1)q-1}{p} \right\rfloor \\ &= \left\lfloor \frac{kmp+k-1}{p} \right\rfloor - \left\lfloor \frac{kmp+k-mp-1-1}{p} \right\rfloor \\ &= m. \end{aligned}$$

Since $l_k = m$ for all k, $a_0 = a_n = (p+1)/2$, and for $0 < i < n$, $a_i = p$. Thus $\Delta(t)$ is trapezoidal.

(c) Suppose $q \equiv -1 \pmod{p}$. Then $q = mp - 1$ for some m. By Corollary 2,

$$l_1 = \left\lfloor \frac{mp-1-1}{p} \right\rfloor = m - 1,$$

since $p > 1$. For $1 < k < p$,

$$l_k = \left\lfloor \frac{kmp-k-1}{p} \right\rfloor - \left\lfloor \frac{kmp-k-mp+1-1}{p} \right\rfloor = m.$$

Therefore $a_0 = a_n = (p-1)/2$, and for $0 < i < n$, $a_i = p$. Thus $\Delta(t)$ is trapezoidal.

$\square$

Chapter 6

Three Transformations

In this section we introduce three transformations on up-down graphs. Similar transformations have been used in [9] and [15]. Hartley uses three similar transformations in [9] to prove the trapezoidal theorem inductively. However, the three transformations here, and their applications, are original to this thesis.

6.1 The Three Transformations

Definition 13. Define a *valid* pair (p, q) to be a pair such that $\gcd(p, q) = 1$, $0 < p < q$, and p is odd. Define the pair $(1, 1)$ to be *pseudo-valid*.

For every two-bridge knot or link $K_{p/q}$, (p, q) is a valid pair. Conversely, for any valid pair (p, q), $K_{p,q}$ is a two-bridge knot or link.

Definition 14. Define the following three transformations on the space of all valid and pseudo-valid (p, q) pairs:

$$\begin{aligned} T_1(p, q) &= (p, p + q) \\ T_2(p, q) &= (2q - p, 3q - p) \\ T_3(p, q) &= (2q - p, 3q - 2p) \end{aligned}$$

The inverses of these transformations are:

$$\begin{aligned} T_1^{-1}(p, q) &= (p, q - p) \\ T_2^{-1}(p, q) &= (2q - 3p, q - p) \\ T_3^{-1}(p, q) &= (3p - 2q, 2p - q) \end{aligned}$$

Note that $T_1(1,1) = T_2(1,1) = (1,2)$, and $T_3(1,1) = T_3^{-1}(1,1) = (1,1)$.

Three other transformations S_1, S_2, and S_3 are used in [9], defined by $S_1(p,q) = (p+q,q)$, $S_(p,q) = (p,2p+q)$, and $S_3(p,q) = (p,2p-q)$. So $T_1 = S_1$ and $T_2 = S_1 \circ S_2$. Two transformations τ_1 and τ_2 are used in [15], defined by $\tau_1(q,p) = (p,q-2p)$ and $\tau_2(q,p) = (2p-q,3p-2q)$. So $T_3 = \tau_2^{-1}$.

Proposition 5. *Let (p,q) be a valid pair.*

(a) $T_1(p,q), T_2(p,q)$, and $T_3(p,q)$ are valid pairs.

(b) $T_1^{-1}(p,q)$ is valid if and only if $0 < p < (1/2)q$. $T_2^{-1}(p,q)$ is valid if and only if $(1/2)q < p < (2/3)q$. $T_3^{-1}(p,q)$ is valid if and only if $(2/3)q < p < q$.

(c) There exists a unique transformation T that is a finite composition of T_1's, T_2's, and T_3's, such that $T(1,2) = (p,q)$.

Proof. (a) We check the three properties of a valid pair for the result of each of the three transformations.

(1) T_1:

(i) p is odd.

(ii) $\gcd(p, p+q) = \gcd(p,q) = 1$.

(iii) $0 < p < p+q$.

(2) T_2:

(i) p is odd, so $2q-p$ is odd.

(ii) $\gcd(2q-p, 3q-p) = \gcd(2q-p, q) = \gcd(-p,q) = 1$.

(iii) Since $0 < p < q$, $0 < q-p < 2q-p < 3q-p$.

(3) T_3:

(i) p is odd, so $2q-p$ is odd.

(ii) $\gcd(2q-p, 3q-2p) = \gcd(2q-p, q-p) = \gcd(q, q-p) = \gcd(q,-p) = 1$.

(iii) Since $0 < p < q$, $0 < q-p < 2q-p < 3q-p$.

(b) We check the three properties for each of the three transformations.

(1) T_1^{-1}:

(i) p is odd.

(ii) $\gcd(p, q-p) = \gcd(p,q) = 1$.

(iii) $0 < p < q - p \iff 0 < p$ and $2p < q \iff 0 < p < (1/2)q$.

(2) T_2^{-1}:

(i) p is odd, so $2q - 3p$ is odd.

(ii) $\gcd(2q - 3p, q - p) = \gcd(p, q - p) = \gcd(p, q) = 1$.

(iii) $0 < 2q - 3p < q - p \iff 3p < 2q$ and $q < 2p \iff (1/2)q < p < (2/3)q$.

(3) T_3^{-1}:

(i) p is odd, so $3p - 2q$ is odd.

(ii) $\gcd(3p - 2q, 2p - q) = \gcd(-p, 2p - q) = \gcd(-p, -q) = 1$.

(iii) $0 < 3p - 2q < 2p - q \iff 3p > 2q$ and $p < q \iff (2/3)q < p < q$.

(c) Let $(p', q') = T(p, q)$, where (p, q) is a valid pair and T is T_1, T_2, or T_3. If $T = T_1$, then $q' = p + q > q$. If $T = T_2$, then $q' = 3q - p = 2q + (q - p) > 2q > q$. If $T = T_3$, then $q' = 3q - 2p = q + 2(q - p) > q$. In each case, $q' > q$.

We prove the Proposition 5 (c) by strong induction on q.

Base case: $q = 2$. $(1, 2)$ is the only valid pair for $q = 2$. Thus the identity $T(p, q) = (p, q)$ is the unique (empty) composition of T_1, T_2, and T_3, such that $T(1, 2) = (1, 2)$.

Inductive hypothesis: Assume that for any valid (p, q) pair with $2 \leq q \leq q_0$, there exists a unique transformation T that is a finite composition of T_1, T_2, and T_3, such that $T(1, 2) = (p, q)$.

Let $(p, q_0 + 1)$ be a valid pair. $p \neq (1/2)(q_0 + 1)$ because $\gcd(p, q_0 + 1) = 1$ and $q_0 + 1 > 2$. p is odd, so $p \neq (2/3)(q_0 + 1)$. Thus exactly one of $T_1^{-1}(p, q_0 + 1), T_2^{-1}(p, q_0 + 1)$ and $T_3^{-1}(p, q_0 + 1)$ is a valid pair, since $0 < p < q$. Let S be this transformation and let (p_1, q_1) be this valid pair. Since $q_1 \leq q_0$, by the inductive hypothesis there is a unique transformation T that is a finite composition of T_1, T_2, and T_3, such that $T(1, 2) = (p_1, q_1)$. Thus $S \circ T$ is the unique transformation such that $S \circ T(1, 2) = (p, q_0 + 1)$.

By strong induction on q, Proposition 5 (c) is true.

□

Given Proposition 5, it is possible to construct a ternary tree that lists every valid (p, q) pair and the unique transformation that produces it from

(1,2). Figure 6.1 displays this tree, which we will call the Transformation Tree.

6.2 Continued Fractions

Define a *continued fraction* to be either a positive integer a or the sum $a + 1/b$ where $a \geq 0$ is an integer and $b \geq 1$ is a continued fraction. A continued fraction terminates if and only if it is a rational number. Let $[a_0, a_1, \ldots, a_k]$ denote the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}$$

We call the numbers $a_0, a_1, \ldots, a_k$ the *elements* of the continued fraction, and the sum $a_0 + a_1 + \ldots + a_k$ the *diagonal sum* of the continued fraction.

Proposition 6. *Every positive rational number has exactly two representations as a continued fraction, which are of the form* $[a_0, a_1, \ldots, a_k]$ *and* $[a_0, a_1, \ldots, a_k - 1, 1]$.

Proof. This is proved in [7]. □

Note that the diagonal sum is the same for both continued fractions for a rational number.

Proposition 7. *Let* a *and* b *be integers such that* $0 < a < b$. *Then the diagonal sums of the continued fractions for* $\frac{a}{b}$, $\frac{b}{a}$, *and* $1 - \frac{a}{b}$ *are equal.*

Proof. Let $a/b = [0, a_1, \ldots, a_k]$ with diagonal sum s. Then

$$\frac{b}{a} = [a_1, a_2, \ldots, a_k],$$

which also has diagonal sum s.

Suppose that $a/b \geq 1/2$. Then

$$\frac{a}{b} = [0, 1, a_2, \ldots, a_k] = \frac{1}{1 + \frac{1}{x}}$$

where $x = [a_2, \ldots, a_k]$, so

$$1 - \frac{a}{b} = 1 - \frac{1}{1 + \frac{1}{x}} = \frac{1}{1 + x}$$

$$T_1(p,q) = (p,\ p+q)$$
$$T_2(p,q) = (2q-p,\ 3q-p)$$
$$T_3(p,q) = (2q-p,\ 3q-2p)$$

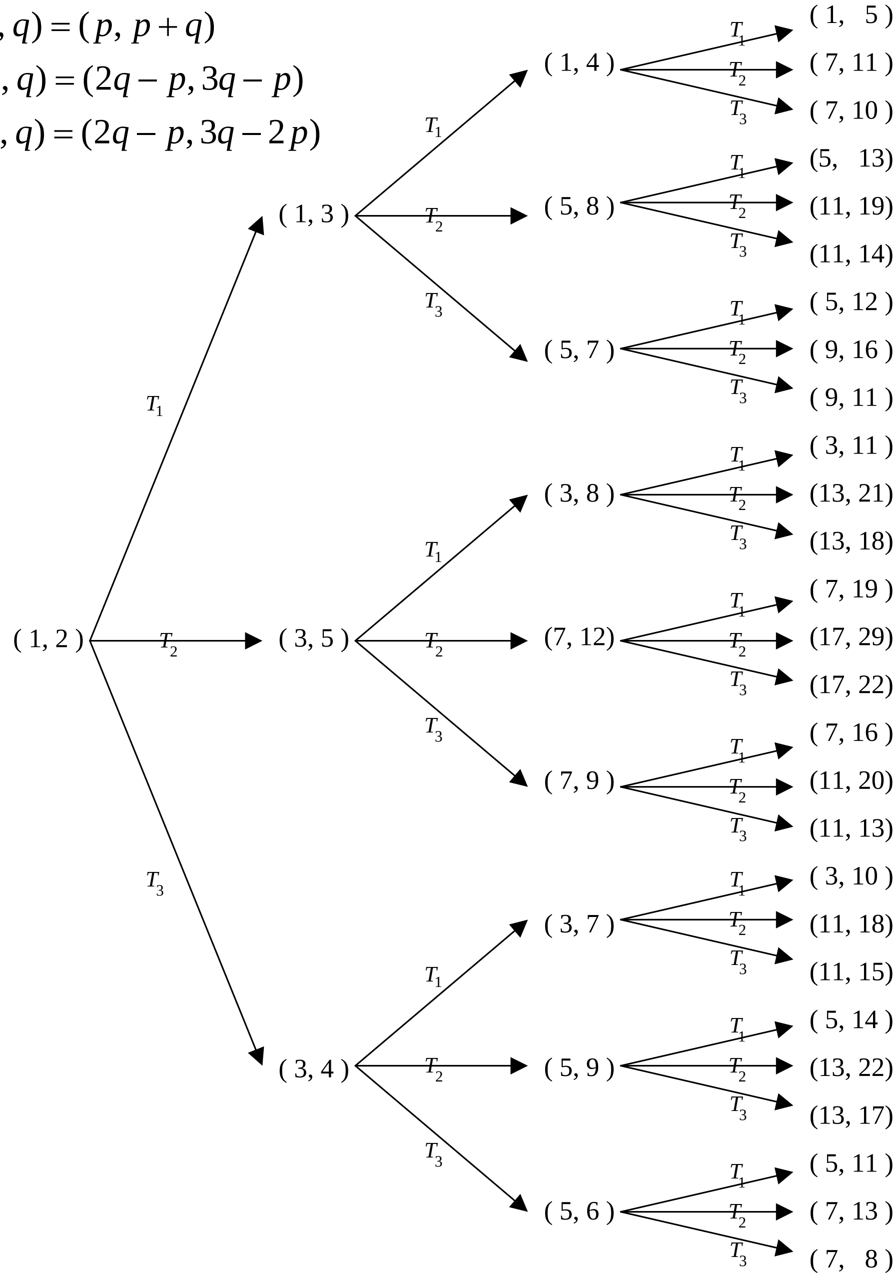

Figure 6.1: Transformation tree.

If $k > 2$, then $a/b = [0, a_2 + 1, a_3, \ldots, a_k]$, otherwise $a/b = [0, a_2 + 1]$. In either case, $1 - a/b$ has diagonal sum s.

Suppose instead that $a/b > 1/2$. Then reversing the roles of a/b and $1 - a/b$ in the above argument gives the same result. □

Definition 15. Define the *crossing number* of a fraction p/q or a pair (p, q) as the diagonal sum of the continued fraction for p/q.

Theorem 6.1. *The crossing number of a two-bridge knot or link $K_{p/q}$ is the sum $a_1 + a_2 + \ldots + a_k$, where $p/q = [0, a_1, a_2, \ldots, a_k]$.*

Proof. This is proved in [4]. □

Thus a fraction p/q has the same crossing number as the two-bridge knot or link $K_{p/q}$, so there is no confusion of the term "crossing number."

6.3 Effects of the Transformations

Conjecture 2. *Let (p, q) be a valid pair. These are the effects of the three transformations on the up-down graph $G(p, q)$:*

(a) *T_1 lengthens every segment by 1 edge.*

(b) *T_2 does the following sequence of moves:*

- *Flip the graph upside down.*
- *Disconnect all edges.*
- *Lengthen each edge to a segment of length 2 whose bottom point is at the same level as before.*
- *Connect any two consecutive segments of the same slope with an edge of opposite slope.*
- *Add an edge to the start and end of the graph.*

(c) *T_3 inserts two edges (one ascending and one descending) at each point in $G(p, q)$ that is not a break point or the end point.*

One main reason for studying T_1, T_2, and T_3 is that they are the most likely key for proving Theorem 5.2. To prove Theorem 5.2, it is necessary and sufficient that each of these transformations, when applied to a valid pair (p, q) such that $\Delta(t)$ for $K_{p/q}$ is trapezoidal, yields a (p', q') pair such that $\Delta(t)$ for $K_{p'/q'}$ is also trapezoidal. This proof would work by induction on q.

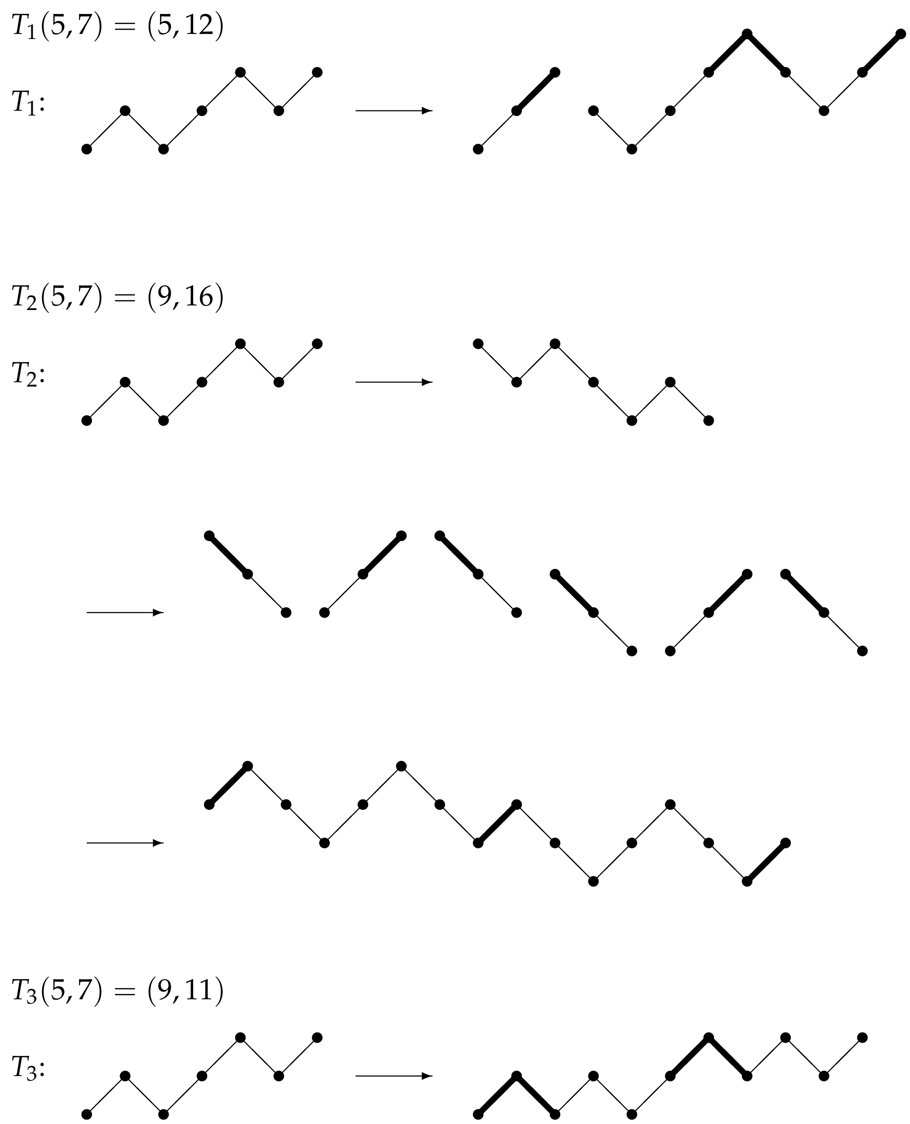

Figure 6.2: Effects of the transformations on $K_{5/7}$.

Proposition 8. *Let (p,q) be a valid pair.*

(a) Let n be the crossing number of $K_{p/q}$.

(1) Let $(p',q') = T_1(p,q)$. Then $K_{p'/q'}$ has crossing number $n+1$.
(2) Let $(p',q') = T_2(p,q)$. Then $K_{p'/q'}$ has crossing number $n+2$.
(3) Let $(p',q') = T_3(p,q)$. Then $K_{p'/q'}$ has crossing number $n+2$.

(b) Let d be the degree of $\Delta(t)$ for $K_{p/q}$.

(1) Let $(p',q') = T_1(p,q)$. Then $\Delta(t)$ for $K_{p'/q'}$ has degree $d+1$.
(2) Let $(p',q') = T_2(p,q)$. Then $\Delta(t)$ for $K_{p'/q'}$ has degree $d+1$.
(3) Let $(p',q') = T_3(p,q)$. Then $\Delta(t)$ for $K_{p'/q'}$ has degree d.

Proof. (a) By Proposition 7, q/p and $1-p/q$ have continued fractions whose digaonal sums are n.

If $(p',q') = T_1(p,q)$, then

$$\frac{q'}{p'} = \frac{p+q}{q} = 1 + \frac{q}{p}$$

which has a continued fraction with diagonal sum $1+n$.

If $(p',q') = T_2(p,q)$, then

$$\frac{q'}{p'} = \frac{3q-p}{2q-p} = 1 + \frac{1}{1+\left(1-\frac{p}{q}\right)}$$

which has a continued fraction with diagonal sum $2+n$.

If $(p',q') = T_3(p,q)$, then

$$\frac{q'}{p'} = \frac{3q-2p}{2q-p} = 1 + \frac{1}{1+\frac{1}{\left(1-\frac{p}{q}\right)}}$$

which has a continued fraction with diagonal sum $2+n$.

(b) This follow immediately from what effect each transformation has on $G(p,q)$, according to Conjecture 2. $\square$

Chapter 7

Counting Two-Bridge Knots and Links

Because of the simple effect that each transformation has on the crossing number, it is possible to count the two-bridge knots of a given crossing number. It is more difficult to count the two-bridge links of a given crossing number. In this chapter we first count the chiral and achiral two-bridge knots of a given crossing number, using the three transformations from Chapter 6. Then we use Conway's continued fraction notation for two-bridge knots and links [4] to count the total chiral knots and links and total achiral knots and links of a given crossing number. Combining this information with the number of two-bridge knots of a given crossing number gives us the number of links of that crossing number. Two-bridge knots and links were first counted in [6] in 1987.

7.1 Reverse Transformations

Each of the next lemmas, propositions, and corollaries use the following terms. Let $T_{a_1}, T_{a_2}, \ldots, T_{a_k}$ be a sequence of k transformations, where $a_i \in \{1,2,3\}$ for all i. Let

$$T = T_{a_1} \circ T_{a_2} \circ \cdots \circ T_{a_k}$$

and let

$$\overline{T} = T_{a_k} \circ T_{a_{k-1}} \circ \cdots \circ T_{a_1}.$$

We call $\overline{T}$ the *reverse* of T, and say T is *symmetric* if $T = \overline{T}$. Let $(p,q) = T(1,2)$ and let $(\overline{p},\overline{q}) = \overline{T}(1,2)$. Let n be the crossing number of $K_{p/q}$.

Lemma 7.1. *The determinant of T is* $\det T = (-1)^{q+n}$.

Proof. The determinants of the three transformations are:

$$\begin{aligned}\det T_1 &= (1)(1)-(1)(0)=1\\ \det T_2 &= (-1)(3)-(-1)(2)=-1\\ \det T_3 &= (-1)(3)-(-2)(2)=1\end{aligned}$$

Thus

$$\det T = \prod_{i=1}^{k} \det T_{a_i} = (-1)^r$$

where r is the number of values of i for which $a_i = 2$.

Now we show how the determinant of T is related to the parities of q and n. Let (a, b) be a valid pair, and let c be the crossing number of $K_{a/b}$.

Let $(a', b') = T_1(a, b) = (a, a+b)$. Then b' has the opposite parity from b since a is odd. Let c' be the crossing number of $K_{a'/b'}$. From Proposition 8, $c' = c+1$, so $b'+c'$ and $b+c$ have the same parity.

Let $(a', b') = T_2(a, b) = (2b-a, 3b-a)$ and let c' be the crossing number of $K_{a'/b'}$. Then b' has the opposite parity from b, and $c' = c+2$, so $b'+c'$ and $b+c$ have opposite parity.

Let $(a', b') = T_3(a, b) = (2b-a, 3b-2a)$ and let c' be the crossing number of $K_{a'/b'}$. Then b' has the same parity as b, and $c' = c+2$, so $b'+c'$ and $b+c$ have the same parity.

Therefore the parity of $q+n$ is even if there are an even number of values of i for which $a_i = 2$, and odd otherwise. Hence $\det T = (-1)^{q+n}$. □

Lemma 7.2. *Let the matrix representation of T be*

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\overline{T} = \begin{bmatrix} d-2b & b \\ c+2d-2a-4b & a+2b \end{bmatrix}$$

Proof. We prove this by induction on l, the number of transformations that T is composed of.

Base Case: Let T be the identity transformation. Then

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1)-2(0) & 0 \\ (0)+2(1)-2(1)-4(0) & (1)+2(0) \end{bmatrix} = \overline{T}.$$

For the inductive step assume that the formula for $\overline{T}$ in Lemma 7.2 is true for any T that is an l-fold composition of the transformations T_1, T_2, or T_3.

$$\begin{aligned}
T_1 \circ T &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix} \\
\overline{T_1 \circ T} &= \overline{T} \circ T_1 = \begin{bmatrix} d-2b & b \\ c+2d-2a-4b & a+2b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} d-b & b \\ c+2d-a-2b & a+2b \end{bmatrix} \\
&= \begin{bmatrix} (b+d)-2(b) & b \\ (a+c)+2(b+d)-2(a)-4(b) & a+2b \end{bmatrix} \\
T_2 \circ T &= \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2c-a & 2d-b \\ 3c-a & 3d-b \end{bmatrix} \\
\overline{T_2 \circ T} &= \overline{T} \circ T_2 = \begin{bmatrix} d-2b & b \\ c+2d-2a-4b & a+2b \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} b-d & 2d-b \\ a+2b-c-2d & 2c+4d-a-2b \end{bmatrix} \\
&= \begin{bmatrix} (3d-b)-2(2d-b) & (2d-b) \\ (3c-a)+2(3d-b)-2(2c-a)-4(2d-b) & (2c-a)+2(2d-b) \end{bmatrix} \\
T_3 \circ T &= \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2c-a & 2d-b \\ 3c-2a & 3d-2b \end{bmatrix} \\
\overline{T_3 \circ T} &= \overline{T} \circ T_3 = \begin{bmatrix} d-2b & b \\ c+2d-2a-4b & a+2b \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -d & 2d-b \\ -c-2d & 2c+4d-a-2b \end{bmatrix} \\
&= \begin{bmatrix} (3d-2b)-2(2d-b) & (2d-b) \\ (3c-2a)+2(3d-2b)-2(2c-a)-4(2d-b) & (2c-a)+2(2d-b) \end{bmatrix}
\end{aligned}$$

In each of these three cases, the formula in Lemma 7.2 holds. Since every $(l+1)$-fold composition of the transformations T_1, T_2, and T_3 can be expressed uniquely as a composition of an l-fold composition and one of these transformations, Lemma 7.2 is true by induction on l. □

Proposition 12. *Let $(p,q) = T(1,2)$. Then $q = \overline{q}$ and $p\overline{p} \equiv (-1)^{q+n} \pmod{q}$.*

Proof. Given Lemma 7.2,

$$(p,q) = T(1,2) = (a+2b, c+2d)$$

and

$$(\overline{p},\overline{q}) = \overline{T}(1,2) = (d-2b+2b, c+2d-2a-4b+2a+4b) = (d, c+2d).$$

Thus $q = \overline{q}$, and

$$\begin{aligned} p\overline{p} &= (a+2b)d \\ &= ad+2bd \\ &= ad-bc+b(c+2d) \\ &\equiv ad-bc \pmod q \\ &\equiv \det T \pmod q \\ &\equiv (-1)^{q+n} \pmod q. \quad \square \end{aligned}$$

Corollary 5. *If q is odd, then $T = \overline{T}$ if and only if $p^2 \equiv \pm 1 \pmod q$. If q is even, then $T = \overline{T}$ implies $p^2 \equiv \pm 1 \pmod q$.*

Proof. First, suppose that $T = \overline{T}$. Then $p = \overline{p}$, so $p^2 = p\overline{p} \equiv \pm 1 \pmod q$.

Suppose conversely that $p^2 \equiv \pm 1 \pmod q$, and q is odd. Since $p\overline{p} \equiv \pm 1 \pmod q$, $p \equiv \pm \overline{p} \pmod q$. Thus $\overline{p} = q - p$ or $\overline{p} = p$. But if $\overline{p} = q - p$, then q is even, a contradiction. Therefore $\overline{p} = p$. By Proposition 12, $\overline{q} = q$, and hence $\overline{T} = T$ by Proposition 5. $\square$

Proposition 13. *The two-bridge knot or link $K_{p/q}$ is achiral if and only if $p^2 \equiv -1 \pmod q$.*

Proof. The mirror image of $K_{p/q}$ is $K_{-p/p}$, so $K_{p/q}$ is achiral iff $K_{p/q}$ is equivalent to $K_{-p/q}$ iff $-p \equiv p^{\pm 1} \pmod q$. If $-p \equiv p \pmod q$, then $\gcd(p,q) = p$. Then (p,q) is only valid if $(p,q) = (1,2)$, in which case $p^2 = 1 \equiv -1 \pmod q$. Otherwise, $-p \equiv p^{-1}$, and therefore $p^2 \equiv -1 \pmod q$. $\square$

Proposition 14. *Given $q > 1$, there exists a p such that $K_{p/q}$ is achiral if and only if q is not a multiple of 4 and all odd prime divisors of q are congruent to* 1 $\pmod 4$. *The number of of distinct achiral knots $K_{p/q}$ with q fixed is 2^{k-1} where k is the number of distinct odd prime divisors of q.*

Proof. Suppose that $K_{p/q}$ is achiral. Then $p^2 \equiv -1 \pmod q$ by Proposition 13. Let q_1 be any divisor of q. Then $p^2 \equiv -1 \pmod{q_1}$. Therefore $p^2 \equiv 0$ or $1 \pmod 4$, so $q_1 \equiv 1$ or $2 \pmod 4$. Thus 4 does not divide q, and all prime divisors of q are congruent to $1 \pmod 4$.

Suppose conversely that q is not a multiple of 4 and all odd prime divisors of q are congruent to $1 \pmod 4$. Let q_i be an odd prime divisor of q. In [7], it is proved that the Legendre symbol

$$\left(\frac{-1}{q_i}\right) = (-1)^{(q_i-1)/2} = 1.$$

Thus there exists p such that $p^2 \equiv -1 \pmod{q_i}$. Since q_i is prime, $a^2 \equiv b^2 \pmod{q_i}$ if and only if $a \equiv \pm b \pmod{q_i}$.

Let $q_i^{r_i}$ be the greatest power of q_i that divides q. Let $f(x) = x^2 + 1$. By Hensel's Lemma [7], there is a unique t such that $t^2 \equiv -1 \pmod{q_i^{r_i}}$ and $t \equiv p \pmod{q_i}$. Since there were exactly two numbers p such that $p^2 \equiv -1 \pmod{q_i}$, there are two number t_{i1} and t_{i2} such that $t_{i1}^2 \equiv t_{i2}^2 \equiv -1 \pmod{q_1^{r_i}}$. If q is even then the only such number is 1, that is, $1^2 \equiv -1 \pmod 2$.

Let $q_1, q_2, \ldots, q_l$ be the distinct prime divisors of q. By the Chinese Remainder Theorem, for each set of numbers t_i such that $t_i^2 \equiv -1 \pmod{q_i}$, there is a unique number $t \pmod q$ such that $t^2 \equiv -1 \pmod q$. Thus there are 2^k values of t for which $t^2 \equiv -1 \pmod q$. These correspond to 2^{k-1} pairs of equivalent (achiral) knots or links. $\square$

7.2 The Number of Two-Bridge Knots and Links

Definition 16. Let $AK(n)$ be the number of achiral two-bridge knots with crossing number n, and let $CK(n)$ be the number of pairs of chiral two-bridge knots with crossing number n. Similarly define $AL(n)$ and $CL(n)$ for two-bridge links of crossing number n.

Formulas for $AK(n) + CK(n)$, $AK(n) + 2CK(n)$, $AL(n) + CL(n)$, and $AL(n) + 2CL(n)$ are given in [6]. Exactly the same information is contained in these four functions as in the four functions defined in Definition 16. That is, the formulas for either set of four functions is sufficient to find the formulas for the other set of four functions.

Thus the following theorem is equivalent to the main theorems in [6].

Theorem 7.1. *Let $AK(n)$, $CK(n)$, $AL(n)$, and $CL(n)$ be as defined in Definition*

16. Then the following formulas are true for $n \geq 3$:

$$AK(n) = \begin{cases} \frac{1}{3}(2^{(n-2)/2}+1) & \textit{if } n \equiv 0 \pmod 4 \\ 0 & \textit{if } n \equiv 1 \pmod 4 \\ \frac{1}{3}(2^{(n-2)/2}-1) & \textit{if } n \equiv 2 \pmod 4 \\ 0 & \textit{if } n \equiv 3 \pmod 4 \end{cases}$$

$$CK(n) = \begin{cases} \frac{1}{3}(2^{n-3}-2^{(n-4)/2}-1) & \textit{if } n \equiv 0 \pmod 4 \\ \frac{1}{3}(2^{n-3}+2^{(n-3)/2}) & \textit{if } n \equiv 1 \pmod 4 \\ \frac{1}{3}(2^{n-3}-2^{(n-4)/2}) & \textit{if } n \equiv 2 \pmod 4 \\ \frac{1}{3}(2^{n-3}+2^{(n-3)/2}+1) & \textit{if } n \equiv 3 \pmod 4 \end{cases}$$

$$AL(n) = \begin{cases} \frac{1}{3}(2^{(n-4)/2}-1) & \textit{if } n \equiv 0 \pmod 4 \\ 0 & \textit{if } n \equiv 1 \pmod 4 \\ \frac{1}{3}(2^{(n-4)/2}+1) & \textit{if } n \equiv 2 \pmod 4 \\ 0 & \textit{if } n \equiv 3 \pmod 4 \end{cases}$$

$$CL(n) = \begin{cases} \frac{1}{3}(2^{n-4}+2^{(n-4)/2}+1) & \textit{if } n \equiv 0 \pmod 4 \\ \frac{1}{3}(2^{n-4}+2^{(n-5)/2}) & \textit{if } n \equiv 1 \pmod 4 \\ \frac{1}{3}(2^{n-4}+2^{(n-4)/2}) & \textit{if } n \equiv 2 \pmod 4 \\ \frac{1}{3}(2^{n-4}+2^{(n-5)/2}-1) & \textit{if } n \equiv 3 \pmod 4 \end{cases}$$

This theorem will be proved in two parts. In the first part, we count the number of chiral and achiral knots $CK(n)$ and $AK(n)$ using properties of the Transformation Tree. In the second part, we use continued fractions to compute the sums $AK(n) + AL(n)$ and $CK(n) + CL(n)$, which in turn give the formulas for $CK(n)$ and $CL(n)$.

7.3 Counting Using Transformations

Define an *n-board* to be a checkerboard of size $1 \times n$, with *n cells*. We can *tile* an n-board with 1×1 *squares* and 1×2 *dominoes* (both shapes are called *tiles*). It is well known [1] that the number of ways to tile an n-board with squares and dominoes, with no other restrictions, is the Fibonacci number F_{n+1}. We will consider tilings with a certain set of restrictions.

Suppose (p, q) has crossing number n. By Proposition 8, $T_1(p, q)$ has crossing number $n+1$, $T_2(p, q)$ has crossing number $n+2$, and $T_3(p, q)$ has crossing number $n+2$. The base pair is $(1, 2)$. So we can consider a T_1 to be a square, and T_2 and T_3 to be dominoes of different colors. Thus the number of pairs (p, q) of crossing number n in the Transformation Tree is the number of ways to tile an n-board with squares and dominoes of two colors, white and black. We let a white domino represent a T_2 and black domino represent a T_3. Figure 7.1 shows the tiling representation of

$$T = T_3 \circ T_1 \circ T_2 \circ T_2 \circ T_3 \circ T_1.$$

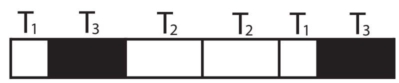

Figure 7.1: Tiling representation of a transformation.

Define $E(n)$ to be the number of ways to tile an n-board with squares and black and white dominoes, in which the number of squares plus white dominoes is even. Define $O(n)$ to be the number of tilings in which this sum is odd.

Proposition 15.

$$E(n) = \begin{cases} \frac{1}{3}(2^n + 2) & \textit{if } n \textit{ is even} \\ \frac{1}{3}(2^n - 2) & \textit{if } n \textit{ is odd} \end{cases}$$

$$O(n) = \begin{cases} \frac{1}{3}(2^n - 1) & \textit{if } n \textit{ is even} \\ \frac{1}{3}(2^n + 1) & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. We prove Proposition 15 by induction on n. Clearly $E(0) = 1 = (1/3)(2^0 + 2)$ and $O(0) = 0 = (1/3)(2^0 - 1)$.

By conditioning on the last tile of each tiling, we get the following recurrence relations for $E(n)$ and $O(n)$, for $n \geq 1$:

$$E(n) = O(n-1) + O(n-2) + E(n-2)$$

and

$$O(n) = E(n-1) + E(n-2) + O(n-2)$$

Assume the formulas are true for some $n \geq 0$. Then

$$\begin{aligned}
E(n+1) &= O(n) + O(n-1) + E(n-1) \\
&= \begin{cases} \frac{1}{3}(2^n - 1 + 2^{n-1} + 1 + 2^{n-1} - 2) & \text{if } n \text{ is even} \\ \frac{1}{3}(2^n + 1 + 2^{n-1} - 1 + 2^{n-1} + 2) & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} \frac{1}{3}(2^{n+1} - 2) & \text{if } n+1 \text{ is odd} \\ \frac{1}{3}(2^{n+1} + 2) & \text{if } n+1 \text{ is even} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
O(n+1) &= E(n) + E(n-1) + O(n-1) \\
&= \begin{cases} \frac{1}{3}(2^n + 2 + 2^{n-1} - 2 + 2^{n-1} + 1) & \text{if } n \text{ is even} \\ \frac{1}{3}(2^n - 2 + 2^{n-1} + 2 + 2^{n-1} - 1) & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} \frac{1}{3}(2^{n+1} + 1) & \text{if } n+1 \text{ is odd} \\ \frac{1}{3}(2^{n+1} - 1) & \text{if } n+1 \text{ is even} \end{cases}
\end{aligned}$$

Thus the formulas are true for $n+1$. By induction on n, Proposition 15 is true for all $n \geq 0$. □

Let $T(n) = E(n) + O(n)$. Note that

$$\begin{aligned}
T(n) &= \begin{cases} \frac{1}{3}(2^{n+1} + 1) & \text{if } n \text{ is even} \\ \frac{1}{3}(2^{n+1} - 1) & \text{if } n \text{ is odd} \end{cases} \\
&= O(n+1).
\end{aligned}$$

As we will see, these tilings can be used to count two-bridge knots and links. But first, we will define some terms.

Definition 17. Let $KP(n)$ be the number of entries in the Transformation Tree for which q is odd and $p^2 \equiv 1 \pmod{q}$, let $KN(n)$ be the number of entries for which q is odd and $p^2 \equiv -1 \pmod{q}$, and let $KT(n)$ be the total number of entries for which q is odd. Similarly define $LP(n)$, $LN(n)$, and $LT(n)$ when q is even.

We say a tiling is *symmetric* if the ith tile from the left end is identical to the ith tile from the right end for each i. Clearly a symmetric tiling corresponds to a symmetric transformation T. A symmetric tiling with an odd number of tiles and a central square corresponds to a (p,q) pair (where

$(p,q) = T(1,2))$ for which q is odd and $p^2 \equiv 1 \pmod q$. A symmetric tiling with an odd number of tiles and a central white domino corresponds to a (p,q) pair for which q is odd and $p^2 \equiv -1 \pmod q$. Any other symmetric tiling (with a central black domino or an even number of tiles) corresponds to a (p,q) pair for which q is even and $p^2 \equiv 1 \pmod q$.

Thus we have the following formulas:

$$\begin{aligned} KP(n) &= \begin{cases} 0 & \text{if } n \text{ is even} \\ T((n-1)/2) & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ \frac{1}{3}(2^{(n+1)/2}+1) & \text{if } n \equiv 1 \pmod 4 \\ 0 & \text{if } n \equiv 2 \pmod 4 \\ \frac{1}{3}(2^{(n+1)/2}-1) & \text{if } n \equiv 3 \pmod 4 \end{cases} \end{aligned}$$

and

$$\begin{aligned} KN(n) &= \begin{cases} T((n-2)/2) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{1}{3}(2^{n/2}+1) & \text{if } n \equiv 0 \pmod 4 \\ 0 & \text{if } n \equiv 1 \pmod 4 \\ \frac{1}{3}(2^{n/2}-1) & \text{if } n \equiv 2 \pmod 4 \\ 0 & \text{if } n \equiv 3 \pmod 4 \end{cases} \end{aligned}$$

By Proposition 8, $KT(n) = O(n-2)$, and $LT(n) = E(n-2)$, since the base pair $(1,2)$ has crossing number 2. If $K = K_{p/q}$ is achiral, then the Transformation Tree contains one entry for K if K is a knot and two entries if K is a link, because of Proposition 5. By 13, a two-bridge knot or link $K_{p/q}$ is achiral if and only if $p^2 \equiv -1 \pmod q$. Hence

$$AK(n) = KN(n)$$

and

$$2AL(n) = LN(n).$$

If $K_{p/q}$ is a chiral knot, then the tree contains one entry for K if $p^2 \equiv 1 \pmod q$ and two entries otherwise. Hence

$$2CK(n) = KT(n) - KN(n) + KP(n).$$

If $K_{p/q}$ is a chiral link, then the tree contains two entries for K if $p^2 \equiv 1 \pmod{q}$ and four entries otherwise. Hence

$$4CL(n) = LT(n) - LN(n) + 2LP(n).$$

Thus $AK(n)$ and $AL(n)$ are given in terms of the six functions in Definition 17. This proves, with some messy algebra, the formulas for $AK(n)$ and $AL(n)$ given in Theorem 7.1.

7.4 Counting Using Continued Fractions

We will now use continued fractions to prove the formulas for $CK(n)$ and $CL(n)$ given in Theorem 7.1. As was stated above in Theorem 6.1, the crossing number of a two-bridge knot or link $K_{p/q}$ is the diagonal sum of the continued fraction for p/q.

Lemma 7.3. *Let (p,q) be a valid pair, and let $p/q = [0, a_1, a_2, \ldots, a_k]$. Let $p'/q' = [0, a_k, a_{k-1}, \ldots, a_1]$. Then $q = q'$ and $pp' \equiv (-1)^{k+1} \pmod{q}$.*

Proof. This follows from Theorem 2 in [2]. □

Definition 18. The continued fraction $[a_0, a_1, \ldots, a_k]$ is *Type 1* if $a_k > 1$ or if $k = 0$, and is *Type 2* otherwise.

Definition 19. A continued fraction $[0, a_1, \ldots, a_k]$ is *symmetric* if $a_i = a_{k+1-i}$ for $1 \leq i \leq k$.

Note that although a fraction p/q has two representations as a continued fraction, only at most one of them can be symmetric. This is because the last element of a Type 1 continued fraction is greater than 1 and the last element of a Type 2 continued fraction is 1, while the element a_1 is the same for both continued fractions.

Lemma 7.4. *Let (p,q) be a valid pair.*

(a) The fraction p/q has a symmetric continued fraction $[0, a_1, \ldots, a_k]$, where k is even, if and only if $p^2 \equiv -1 \pmod{q}$.

(b) The fraction p/q has a symmetric continued fraction $[0, a_1, \ldots, a_k]$, where k is odd, if and only if $p^2 \equiv 1 \pmod{q}$.

Proof. Of the two continued fractions for p/q, one has an even number of elements and the other has an odd number of elements.

(a) Suppose that p/q has a symmetric continued fraction $[0, a_1, \ldots, a_k]$, where k is even. Define p'/q' as in Lemma 7.3. Then $p = p'$, so $p^2 \equiv (-1)^{k+1} \equiv -1 \pmod{q}$, by Lemma 7.3.

Suppose conversely that $p^2 \equiv -1 \pmod{q}$. Then $p \equiv -p^{-1} \pmod{q}$. Let the Type 1 continued fraction for p/q be $[0, a_1, \ldots, a_k]$. Let

$$p_1/q = [0, a_k, a_{k-1}, \ldots, a_1]$$

and let

$$p_2/q = [0, 1, a_k - 1, a_{k-1}, \ldots, a_1].$$

By Lemma 7.3,

$$pp_1 \equiv (-1)^{k+1} \pmod{q} \Rightarrow p_1 \equiv (-1)^{k+1} p^{-1} \equiv (-1)^k p \pmod{q}$$

and

$$pp_2 \equiv (-1)^{k+2} \pmod{q} \Rightarrow p_2 \equiv (-1)^k p^{-1} \equiv (-1)^{k+1} p \pmod{q}.$$

If k is even, then $p = p_1$, so $[0, a_1, \ldots, a_k]$ is a symmetric continued fraction for p/q. If k is odd, then $p = p_2$, so $[0, a_1, \ldots, a_k - 1, 1]$ is a symmetric continued fraction for p/q. In either case, the number of elements after the leading 0 is even.

(b) The proof to (b) is similar, with the parity of k switched. □

Lemma 7.5.

(a) The number of continued fractions with diagonal sum n is 2^{n-1}.

(b) If k is even, then the number of symmetric continued fractions $[0, a_1, \ldots, a_k]$ with diagonal sum n is $2^{(n-2)/2}$ if n is even and 0 if n is odd.

(c) If k is odd, then the number of symmetric continued fractions $[0, a_1, \ldots, a_k]$ with diagonal sum n is $2^{(n-2)/2}$ if n is even and $2^{(n-1)/2}$ if n is odd.

Proof.

(a) Let N be the number of continued fractions $[0, a_1, \ldots, a_k]$ with diagonal sum n. Each continued fraction is determined by its elements. So N is the number of sequences of positive integers that sum to n. Equivalently, N is the number of ways to tile an n-board with tiles of any length. There are $n - 1$ divisions between cells, each of which may or may not be a division between tiles. Therefore $N = 2^{n-1}$.

(b) Let N be the number of symmetric continued fractions $[0, a_1, \ldots, a_k]$ with diagonal sum n. Let k be even. If n is odd, then $N = 0$ since k is even. If n is even, then N is the number of ways to tile an $(n/2)$-board with tiles of any length. Thus $N = 2^{(n-1)/2-1} = 2^{(n-2)/2}$.

(c) Let N be the number of symmetric continued fractions $[0, a_1, \ldots, a_k]$ with diagonal sum n. Let k be odd. Then N is the number of ways to tile a board of length $(n-2)/2$ if n is even and length $(n-1)/2$ if n is odd, with tiles of any length. Thus $N = 2^{(n-2)/2}$ if n is even and $N = 2^{(n-1)/2}$ if n is odd. □

Lemma 7.6. *In each part of this lemma, assume (p, q) is a valid pair.*

(a) The number of fractions p/q with crossing number n is 2^{n-2}.

(b) The number of fractions p/q with crossing number n such that $p^2 \equiv -1 \pmod q$ is $2^{(n-2)/2}$ if n is even and 0 if n is odd.

(c) The number of fractions p/q with crossing number n such that $p^2 \equiv 1 \pmod q$ is $2^{(n-2)/2}$ if n is even and $2^{(n-1)/2}$ if n is odd.

Proof. This follows from Lemma 7.5 and the fact that each fraction has exactly two continued fraction representations, only at most one of which can be symmetric. □

Lemma 7.7. *Let $K_{p/q}$ be a two-bridge knot or link with crossing number n. If $p^2 \equiv \pm 1 \pmod q$, then there are two valid pairs (p', q') for which $K_{p'/q'}$ is homeomorphism equivalent to $K_{p/q}$. Otherwise, there are four such pairs (p'/q').*

Proof. By Theorem 2.1, $K_{p'/q'}$ is homeomorphism equivalent to $K_{p/q}$ if and only if $q = q'$ and p' is one of p, $q - p$, p^{-1}, and $q - p^{-1}$. If $p^2 \equiv \pm 1 \pmod q$, then either $p = p^{-1}$ or $p = q - p^{-1}$, so there are only two distinct possibilities for p'. Otherwise there are four distinct possibilities for p'. □

Lemma 7.8.

(a) The total number of achiral two-bridge knots and links of crossing number n is

$$AK(n) + AL(n) = 2^{(n-4)/2}.$$

(b) The total number of chiral two-bridge knots and links of crossing number n, counting chiral pairs together, is

$$CK(n) + CL(n) = \begin{cases} 2^{n-4} & \text{if } n \text{ is even} \\ 2^{n-4} + 2^{(n-5)/2} & \text{if } n \text{ is odd} \end{cases}$$

Proof.

(a) By Proposition 13, $K_{p/q}$ is achiral if and only if $p^2 \equiv -1 \pmod{q}$. By Lemma 7.7, there are twice as many fractions p/q with crossing number n such that $p^2 \equiv -1 \pmod{q}$ as there are achiral knots and links with crossing number n. By Lemma 7.6 (b), the number of achiral knots and links of crossing number n is $2^{(n-4)/2}$.

(b) By Lemma 7.6 (a), there are 2^{n-2} fractions p/q with crossing number n. By Lemma 7.7, there are two fractions for each knot or link $K_{p/q}$, except one if $p^2 \equiv 1 \pmod{q}$, where chiral pairs are counted separately. By Lemma 7.6 (a) and (c),

$$\begin{aligned} AK(n) &+ 2CK(n) + AL(n) + 2CL(n) \\ &= \frac{1}{2}\left(2^{n-2} + \begin{cases} 2^{(n-2)/2} & \text{if } n \text{ is even} \\ 2^{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}\right) \\ &= \begin{cases} 2^{n-3} + 2^{(n-4)/2} & \text{if } n \text{ is even} \\ 2^{n-3} + 2^{(n-3)/2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Taking half of this formula minus the formula in part (a) gives the desired result. □

Proof of Theorem 7.1. In Section 7.3, $AK(n)$ and $AL(n)$ were expressed in terms of other functions whose formulas were computed. In Lemma 7.8, formulas were computed for $AK(n) + AL(n)$ and $CK(n) + CL(n)$. Subtracting the first pair of formulas from the second pair gives the formulas for $CK(n)$ and $CL(n)$ given in Theorem 7.1. □

7.5 Future Work

The three transformations are a very promising tool, and I believe it is possible to prove the Trapezoidal Theorem using them. This will require a more exact description, with proof, of the effects of each transformation on the Alexander polynomial, in terms of the Alexander polynomials of the two-bridge knots and links which were along the transformation "path" from $(1,2)$ to (p,q). The final step would be to prove that each of these effects preserves the trapezoidal property of the Alexander polynomial.

It also may be possible to prove the Trapezoidal Theorem without using the three transformations, just using the uniform properties of the up-down graph. If so, then the proof might be extendable to tunnel number one knots, of which two-bridge knots are a subset. Tunnel number one knot groups also have two-generator, one-relator presentations [10]. Thus a new kind of up-down graph corresponds to each tunnel number one knot. In this graph, each edge could would go up or down one of two possible amounts, rather than just one level as for two-bridge knots. This is one approach to proving Fox's conjecture that the trapezoidal property holds for all tunnel number one knots.

I believe that it is possible to count all the achiral and chiral two-bridge knots and links using only continued fractions, without the need for the three transformations. It also might be possible to count them only using the three transformations, without continued fractions.

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