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Intrinsic Linking and Knotting of Graphs

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Abstract

An analog to intrinsic linking, intrinsic even linking, is explored in the first half of this paper. Four graphs are established to be minor minimal intrinsically even linked, and it is conjectured that they form a complete minor minimal set. Some characterizations are given, using the simplest of the four graphs as an integral part of the arguments, that may be useful in proving the conjecture.

The second half of this paper investigates a new approach to intrinsic knotting. By adapting knot energy to graphs, it is hoped that intrinsic knotting can be detected through direct computation. However, graph energies are difficult to compute, and it is unclear whether they can be used to determine whether a graph is intrinsically knotted.

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Acknowledgments

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Chapter 1

Introduction

In 1983, John H. Conway and Cameron McA. Gordon proved that K_6 is intrinsically linked and K_7 is intrinsically knotted [2], bridging the areas of topology and graph theory. Sachs [21] independently arrived at the same result for linked graphs and gave a list of seven graphs that he conjectured classified all intrinsically linked graphs. These seven graphs were generated from K_6 by a sequence of $\triangle - Y$ and $Y - \triangle$ exchanges, which can be proved to preserve intrinsic linking.

While it is possible to prove certain connectedness properties of intrinsically linked graphs that are minor minimal - or "smallest," in some sense - for intrinsic linking, it took a series of papers by Roberston, Seymour, and Thomas [19, 20] to finally establish that Sachs' list was indeed complete. The major results are outlined in Chapter 2.

Chapter 3 investigates original research on an analog to the Conway-Gordon-Sachs-Robertson-Seymour-Thomas results on intrinsic linking. It can be shown that intrinsic linking is equivalent to intrinsic odd linking, and this can naturally be extended to a notion of intrinsic even linking. The list of four intrinsically even linked graphs was generated during the Clarkson / SUNY Potsdam REU in 2007. It was hypothesized that they constituted a complete set, similar to the set of Petersen-family graphs for intrinsic linking. The results in Chapter 3 attempt to establish characterizations for minor minimal intrinsically even linked graphs by adapting arguments for intrinsic linking through the use of a simple, disconnected intrinsically even linked graph. They suggest an elementary method for proving the completeness of the set by enumerated graphs in a specific manner.

In Chapter 4, intrinsic knotting results are reviewed, and an alternative method for determining whether a graph is intrinically knotted is explored. By generalizing knot energies to graphs, it is hoped that they can be used to detect intrinsically knotted graphs. Specifically, curvature energy, which extends naturally to graphs, is used as an example of a graph energy. Computation of graph energy is observed to be tedious, and it is unclear whether graph energy can be used to detect intrinsic properties of graphs.

Chapter 2

Intrinsic Linking

A graph, $G = (E, V)$, with edge set $E(G) = E$ and vertex set $V(G) = V$, abstractly, is simply a set of vertices and edges connecting vertices. In this paper, unless otherwise specified, all graphs will be considered to be *simple*, that is, they will not be allowed to have loops and repeated edges. Given a graph, G, we can represent G in space by placing the vertices in space and drawing arcs between the adjacent vertices in a way that no two distinct edges intersect (except possibly at common endpoints, which are vertices of the graph). Such a representation is called an *embedding* of G, and whenever the distinction is clear, *graph* and *graph embedding* will be used interchangeably.

We can represent embeddings of graphs in a two dimensional drawing by projecting to a two dimensional subspace V of $\mathbb{R}^3$. When two edges intersect in a projection, we use over- and under-crossings to illustrate the relative position of the edges along the axis normal to V. For example, in Figure 2.1, the edge A is drawn so that it crosses over the edge B.

When considering different embeddings of a particular graph G, we would like to know when two embeddings are equivalent. Figure 2.2(a) and Figure 2.2(b) are two embeddings of the same graph on four vertices.

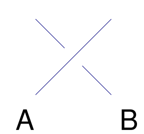


Figure 2.1: The edge A crosses over the edge B.

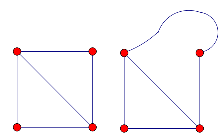


Figure 2.2: Two embeddings of the same graph that are equivalent under ambient isotopy.

The curved edge in Figure 2.2(b) can be continuously deformed to a straight edge, giving the same embedding as in Figure 2.2(a). The equivalence can be defined by *ambient isotopy*.

Definition 2.1. An *ambient isotopy* between two embeddings f and g of a graph G into $\mathbb{R}^3$ is a continuous function $F : \mathbb{R}^3 \times I \to \mathbb{R}^3$, where I is the unit interval $[0, 1]$ and $F_t(x) = F(x, t)$, such that

1. $F_0(x)$ is the identity,
2. $F_t(x)$ is a homeomorphism for all t,
3. $F_1(g(x)) = f(x)$.

Then f and g are said to be *ambient isotopic* embeddings of G.

In general, projections of graph embeddings will be representing a particular ambient isotopy class of embeddings. Consequently, it can be assumed that no two vertices are projected to the same point and that no three edges will mutually intersect in a projection. Therefore, the projection can be drawn so that there are no ambiguities, and the ambient isotopy class of graph embeddings can be uniquely defined from a two dimensional drawing.

When a graph contains two or more disjoint cycles, an embedding of the graph may have two cycles that cannot be pulled apart without breaking open one of the cycles. Two such cycles form a topological link. Formally, a two component link is an embedding of two disjoint copies, L_1 and L_2, of S^1 into (Euclidean) space. The link is trivial (also called the *unlink*) if each copy of S^1 is ambient isotopic to the unknot and if there exist topological open balls B_1 and B_2 that are disjoint, $L_1 \subseteq B_1$, and $L_2 \subseteq B_2$. Whenever it is clear, the word *link* will be used to mean a non-trivial two component link.

Definition 2.2. A graph G is *intrinsically linked* if for every embedding of G into $\mathbb{R}^3$, G contains two disjoint cycles whose image is a (non-trivial) two component link.

Any graph that contains another graph that is intrinsically linked as a subgraph is clearly intrinsically linked as well (we can look at the embedding's restriction to the intrinsically linked subgraph). In fact, an even stronger statement is true, which requires the definition of a graph *minor*.

Definition 2.3. Let G be a graph. If H is a graph such that H can be obtained from G by a sequence of the following three operations:

1. removal of an edge
2. removal of a vertex
3. contraction along an edge,

then H is called a *minor* of G, written $H \leq G$. If $H \leq G$ but $H \neq G$, then H is called a *proper minor* of G, written $H < G$.

Definition 2.4. A graph G is *minor minimal intrinsically linked* if it is intrinsically linked and every proper minor of G has a linkless embedding.

Conway and Gordon [2] proved in 1983 that K_6 is intrinsically linked in $\mathbb{R}^3$, beginning the study of intrinsically linked graphs.

Theorem 2.5 (Conway-Gordon 1983)**.** *The graph K_6 is minor minimal intrinsically linked.*

Proof. It is known that every knot can be turned into the unknot by a finite sequence of ambient isotopies and crossing changes. As a consequence, every embedding of a graph G can be obtained from a particular embedding of G via ambient isotopies and crossing changes.

It is clear that linking numbers of cycles in an embedding of G are invariant under ambient isotopies since ambient isotopies of topological links preserve linking number. Define

$$\lambda = \sum_{C_1 \cup C_2} lk(C_1, C_2)$$

where the sum is over all two component links (possibly trivial) $C_1 \cup C_2$ of an embedding of K_6, and $lk(C_1, C_2)$ is the linking number of the two component link $C_1 \cup C_2$. Take λ_2 to be the mod 2 equivalence class of λ.

We claim that λ_2 is invariant under crossing changes. Suppose the edge (v_1, v_2) crosses over the edge (v_3, v_4). Changing the crossing so that (v_1, v_2) now crosses underneath (v_3, v_4) affects the (oriented) linking number of any two component link with one cycle using the edge (v_1, v_2) and the

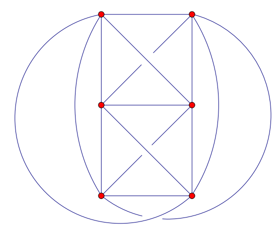


Figure 2.3: An embedding of K_6 with only one non-trivial link.

other cycle using the edge (v_3, v_4) by ± 1 since the oriented crossing number goes from $\pm\frac{1}{2}$ to $\mp\frac{1}{2}$.

Notice that (v_1, v_2) and (v_3, v_4) are contained in two such links: $v_1 - v_2 - v_5 \cup v_3 - v_4 - v_6$ and $v_1 - v_2 - v_6 \cup v_3 - v_4 - v_5$. Therefore, λ changes by $-2, 0$, or 2, so λ_2 is invariant. Combined with our previous observation that λ is invariant under ambient isotopy this shows that λ_2 is the same for all embeddings of K_6.

Consider the embedding given in Figure 2.3. All disjoint pairs of cycles in this embedding of K_6 have linking number 0, except for one, which has linking number ± 1 (depending on orientation). Therefore,

$$\lambda_2 \equiv 1 \pmod 2.$$

Moreover, since every cycle must contain at least three vertices, and K_6 only has six vertices, any choice of three vertices determines a link (the cycle using those three vertices and the disjoint cycle using the remaining three vertices). There are $\binom{6}{3} = 20$ cycles, and each link is double counted by the first cycle and its complementary cycle, so there are 10 disjoint pairs of cycles. Therefore, at least one pair of cycles in every embedding must have odd linking number, otherwise $\lambda \equiv 0 \pmod 2$.

Consequently, there exists a non-trivial link in every embedding of K_6.

If we remove any vertex or edge, or contract along any edge of K_6, we can start with an embedding of K_6 (based on Figure 2.3) where elimination of that edge or vertex destroys the only link with odd linking number. Hence, any proper minor of K_6 has a linkless embedding, so K_6 is minor minimal intrinsically linked, as claimed. $\square$

Sachs [21] and Robertson, Seymour, and Thomas [20] showed that all graphs obtained by $\triangle - Y$ and $Y - \triangle$ exchanges (removing the edges between three pairwise adjacent vertices v_1, v_2, and v_3, and connecting them with a new vertex v, or vice-versa), starting with K_6 are minor minimal intrinsically linked. These graphs form the set of seven *Petersen-family graphs*, which characterize intrinsic linking. Furthermore, Robertson, Seymour, and Thomas [20] proved that a graph is intrinsically linked if and only if it contains a Petersen-family graph as a minor, making the characterization complete.

Chapter 3

Intrinsic Even Linking

3.1 Definitions

This paper explores an analog of the Conway-Gordon [2], Sachs [21], and Robertson-Seymour-Thomas [20] results. The fact that each of the Petersen-family graphs is minor minimal intrinsically linked can be proved directly by adapting the Conway-Gordon argument to each of the six other graphs. Consequently, we can see that not only is each graph minor minimal intrinsically linked - it must necessarily contain a link with odd linking number. This result is confirmed by Taniyama and Yasuhara [22].

Theorem 3.1 (Taniyama-Yasuhara)**.** *Let G be a graph in the set of Petersen-family graphs. Let C_1^i, C_2^i index the set of disjoint cycles of G, and let L_i be any two component link (possibly the unlink). Define*

$$\lambda_2 = \sum_i lk(C_1^i, C_2^i).$$

Then, there exists an embedding of G such that $C_1^i \cup C_2^i$ is ambient isotopic to L_i if and only if $\lambda_2 \equiv 1$ (mod 2).

A natural question to ask is whether there exist graphs such that in every embedding, there exists a two component unlink. The Taniyama-Yasuhara theorem shows that this is not true for any of the Petersen-family graphs. However, relaxing the condition to graphs that contain a link with even linking number in every embedding yields fruitful results.

Definition 3.2. A graph G is *intrinsically even linked* if every embedding of G contains a link with even linking number.

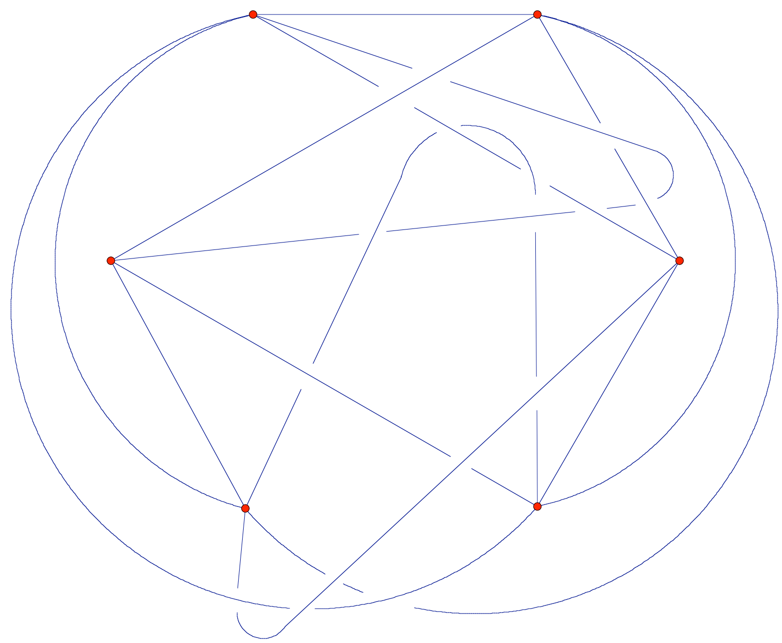


Figure 3.1: An embedding of K_6 with an edge removed such that all links are non-trivial.

Definition 3.3. A graph G is *minor minimal intrinsically even linked* if it is intrinsically even linked and every proper minor of G is not intrinsically even linked.

3.2 Intrinsically Even Linked Graphs

From the Conway-Gordon argument on K_6, we can see that the complete graph on six vertices must contain a pair of disjoint cycles whose linking number is even in every embedding. Otherwise $\lambda_2 \equiv 0 \pmod 2$ as there are an even number of disjoint pairs of cycles. Moreover, removing a vertex or contracting an edge results in a graph on five vertices, so there are no disjoint cycles. In addition, there exists an embedding of K_6 minus an edge where every disjoint pair of cycles has odd linking number. This is a direct consequence of the Taniyama-Yasuhara theorem, and one such embedding is demonstrated in Figure 3.1. As a result, we have the following theorem:

Theorem 3.4. *K_6 is minor minimal intrinsically even linked.*

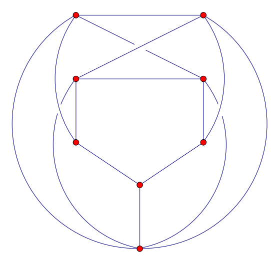


Figure 3.2: The minor minimal intrinsically even linked graph P_8.

A similar argument on P_8, shown in Figure 3.2, and the Petersen graph, shown in Figure 3.3, proves that those two graphs are minor minimal intrinsically even linked. However, the other four Petersen-family graphs, P_7, $K_{4,4} \setminus \{e\}$, $K_{3,3,1}$, and P_9 each contain an odd number of disjoint pairs of cycles. Thus, it is possible that all of them have odd linking number, and in fact, we can exhibit such embeddings (see Figures 3.4, 3.5, and 3.6). The embeddings can be obtained systematically by starting with any embedding of the graph and enumerating all of the disjoint pairs of cycles along with their linking numbers in that embedding. Then, for each pair of cycles $C_1 \cup C_2$ that has an odd linking number, find a pair of edges $e \in C_1, f \in C_2$ such that e and f are not in any disjoint pairs of cycles that have odd linking number. Using ambient isotopy, if necessary, to create a crossing involving e and f, the crossing can be inverted (i.e. if e crosses over f, then change the crossing so that f crosses over e, and vice-versa). This affects the linking number of any disjoint pairs of cycles that have e and f is different cycles by ± 1. Thus, any disjoint pairs of cycles containing e and f in disjoint cycles will now have odd linking number. Repeating this process, it is possible to obtain an embedding with no even linked cycles.

There is one other graph that is known to be minor minimal intrinsically even linked: the $\Delta - \theta$ graph. It is a two component graph, with one component isotopic to a theta, and the other isotopic to a loop (the Δ). Unlike in the case for intrinsically linked graphs, where there exists an elementary proof showing that there are no disconnected minor minimal intrinsically linked graphs, in the case of even linking, there are disconnected minor minimal graphs.

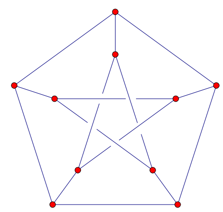


Figure 3.3: The minor minimal intrinsically even linked Petersen graph.

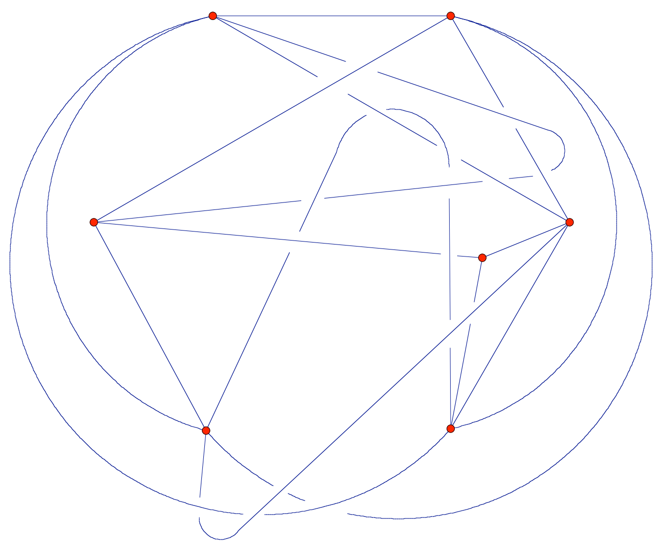


Figure 3.4: An embedding of P_7 with only odd linked cycles. A similar embedding of $K_{4,4}$ minus an edge can be obtained by a $\triangle - Y$ exchange on the triangle disjoint from the Y in P_7.

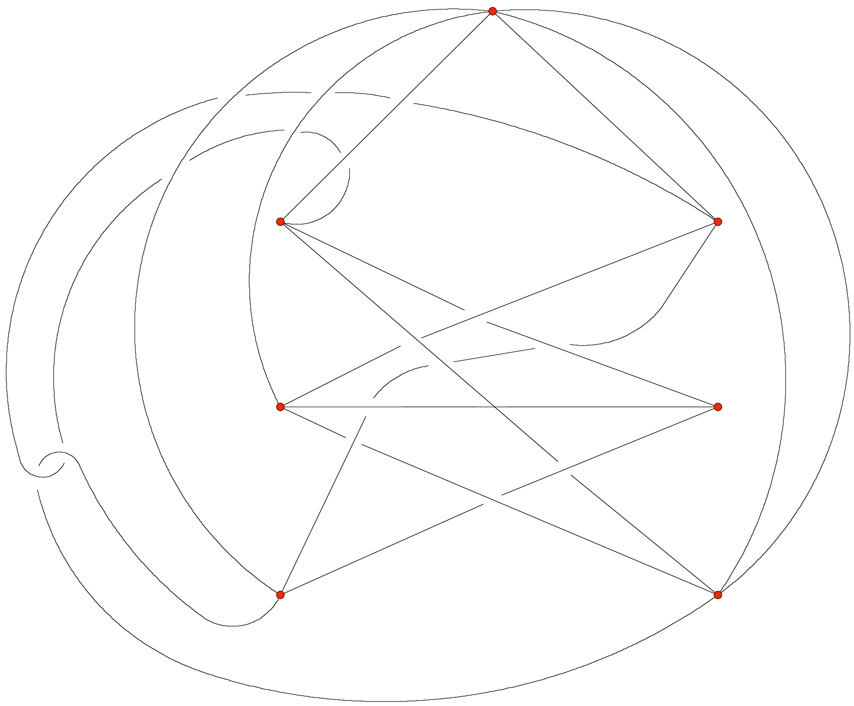


Figure 3.5: An embedding of $K_{3,3,1}$ with only odd linked cycles.

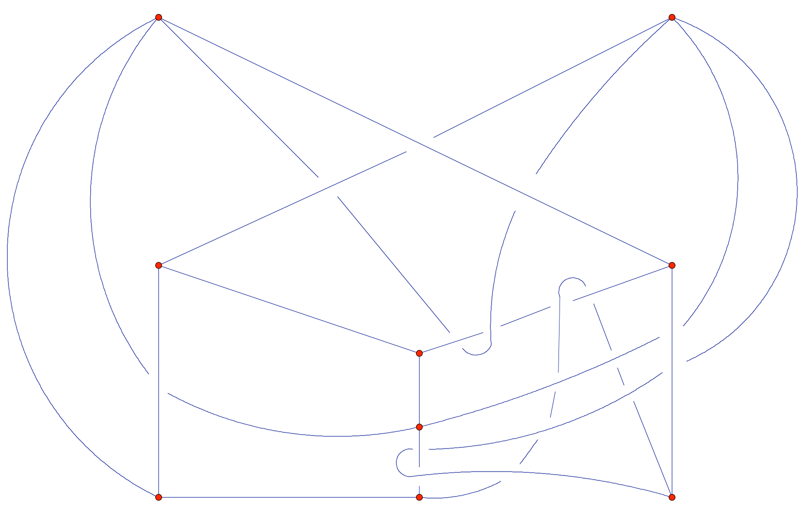


Figure 3.6: An embedding of P_9 with only odd linked cycles.

3.3 Connectedness of Minor Minimal Graphs

In $\mathbb{R}^3$, there are elementary arguments showing that any minor minimal intrinsically linked graph must be 3-connected. For intrinsic even linking, we have demonstrated that there exists a minor minimal intrinsically even linked graph that is disconnected, the $\Delta - \theta$ graph. We can, however, use this graph to show that any other minor minimal even linked graph must be 3-connected.

Before delving into the connectedness results, we define the induced subgraph and n-connectedness.

Definition 3.5. Let G be a graph with $A \subseteq V(G)$. The *subgraph of G induced by A* is the largest subgraph of G that contains none of the vertices in $V(G) \setminus A$.

Definition 3.6. A graph G is n-connected if at least n vertices must be removed from G to disconnect the graph.

Theorem 3.7. *The $\Delta - \theta$ graph is the only disconnected minor minimal intrinsically even linked graph.*

Proof. Let G be a disconnected intrinsically even linked graph consisting of components $P_1, \dots, P_n$. Suppose G is minor minimal and not the $\Delta - \theta$ graph. If either component contains edges or vertices that do not contribute to a cycle, the graph is not minor minimal. Thus, each component of G must consist of at least one cycle. If $P_1, \dots, P_n$ each contain only one cycle, they can be embedded with P_i laying on top of $P_1, \dots, P_{i-1}$, and then linked by any crossing change between an edge of P_i and an edge of $P_j, j < i$. As this embedding has an odd linking number for every pair of disjoint cycles, a component of G must contain at least two cycles.

Suppose P_1 contains more than one cycle. If an edge is shared between any two cycles in P_1, then the union of those two cycles contain the θ component of the $\Delta - \theta$ graph as a minor. Since P_2 has a cycle disjoint from P_1, then G contains $\Delta - \theta$ as a minor. Therefore, the cycles of P_1 must be wedged at vertices. Since the cycles have no edges in common, we are left with a tree of cycles (a cycle of cycles will form a θ, so representing each cycle as a vertex in a graph G', G' must not contain any cycles, so it is a tree) which can be embedded in a plane. Using ambient isotopy, we can stack the cycles on top of each other by rotating them about the shared vertex, lining them up along an edge. Braiding the cycles together along that edge by looping each cycle around the others as in Figure 3.7 will ensure that

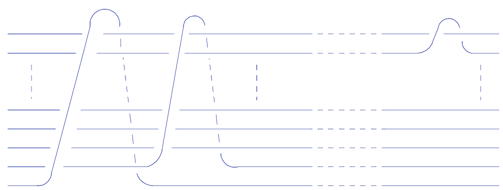


Figure 3.7: Linking cycles along an edge.

any two pair of cycles in P_1 have linking number ± 1. The same process can be applied to $P_i, 1 < i \leq n$.

Treating the braided cycles of each component as a single unit, this reduces to the previous case of having a component containing only one cycle. Thus, every intrinsically even linked graph that is disconnected must contain $\Delta - \theta$ as a minor. □

Definition 3.8. An edge in a graph G is called a bridge if removing that edge causes the graph to become disconnected.

Theorem 3.9. *A minor minimal intrinsically even linked graph cannot contain a bridge.*

Proof. Suppose that some minor minimal intrinsically even linked graph G contains a bridge, e_b. Then, removing e_b from G disconnects the graph into two disjoint components, A and B. The edge e_b cannot be contained in any cycles since after crossing from A to B via e_b, it is not possible to return to A as there is no other path between A and B. As a result, removing e_b will not affect any of the cycles contained in the graph. Thus, the graph will remain intrinsically even linked after we remove the bridge. Hence, G is not minor minimal. □

Definition 3.10. A vertex v in a graph G is called a *cut vertex* if removing v and all edges containing v from G increases the number of connected components of the graph.

Theorem 3.11. *A connected graph G that is minor minimal for intrinsic even linking cannot contain a cut vertex.*

Proof. G cannot contain any leaves (edges with an endpoint with degree one) since no cycles can use a leaf. Thus, we can remove any leaves from G to obtain a smaller graph with intrinsic even linking.

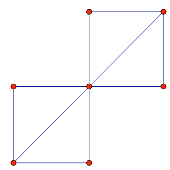


Figure 3.8: Two θ graphs wedged so that their diagonals meet.

Suppose that G contains a cycle C that does not share an edge with any other cycle in the graph. Then, for any cycle C' that is disjoint from C, it is possible to force $lk(C \cup C') \equiv 1 \pmod 2$ by threading C through the middle of C'. Because C does not share an edge with any other cycles and C' cannot share any edges with another cycle C'' disjoint from C without G containing $\Delta - \theta$ as a minor, this can be accomplished independently of all other links in the graph. Hence, if G is intrinsically even linked, then it must be that every embedding of G contains an evenly linked pair that does not involve C. So the edges in C can be removed from the graph.

Therefore, we can reduce this to the case that G contains at least two θ graphs, $K_4 \backslash \{e\}$ by taking minors (if there is only one, then G does not have a cut vertex). Because of Theorem 3.9, any two θ graphs cannot be bridged together. So any two 2-connected components must be wedged together along a cut vertex.

The only way to wedge two θs together so that the graph does not contain $\Delta - \theta$ as a minor is to wedge them so that the diagonals meet (Figure 3.8. However, then there are no disjoint cycles, so the graph cannot be intrinsically even linked. Hence, the graph cannot contain a cut vertex.

□

Theorem 3.12. *Every connected, minor minimal intrinsically even linked graph must be* 3*-connected.*

Proof. Let G be a minor minimal intrinsically even linked graph. We know by Theorem 3.11 that the connectivity of G must be at least 2. Suppose that the connectivity of G is 2. Then there exists two vertices $p, q \in G$ such that removing p and q from G results in a disconnected graph $A \cup B$. Let $\bar{A}$ and $\bar{B}$ be the subgraphs induced by the vertices of $A \cup \{p, q\}$ and the vertices of $B \cup \{p, q\}$ with respect to $\bar{G} = G \cup \{e_{pq}\}$, respectively. Note that $G \setminus A$ must contain a path from p to q, since otherwise, removing either p or q will disconnect G. By taking minors, we can reduce this path down to a single edge. Therefore, $\bar{A}$ is a proper minor of G. Similarly, $\bar{B}$ is a proper minor of G.

Let $\mathbb{R}^3_+$ denote the half-space $z \geq 0$ and $\mathbb{R}^3_-$ be the half-space $z \leq 0$. As $\bar{A}$ and $\bar{B}$ are proper minors of G, there exist embeddings $f_A : \bar{A} \to \mathbb{R}^3_+$ and $f_B : \bar{B} \to \mathbb{R}^3_-$ with no even links such that only p, q, and e_{pq} lie on the plane $z = 0$. Let $f : \bar{G} \to \mathbb{R}^3$ be an embedding such that $f_A = f|_{\bar{A}}$ and $f_B = f|_{\bar{B}}$, up to ambient isotopy.

Since G is intrinsically even linked, there exist cycles C_1 and C_1' that are even-linked in the embedding $f|_G$. C_1 and C_1' cannot both be contained in $\bar{A}$ because $f|_{\bar{A}} = f_A$ is an embedding of A with no even links. Similarly, they cannot both be contained in $\bar{B}$.

Suppose that one of the two cycles intersects both A and B. Without loss of generality, assume $C_1 \subset A$ and C_1' intersects both A and B. Then, the part of C_1' in B can be changed via edge-homotopy to the edge e_{pq} in the embedding f. Therefore, $C_1 \cup C_1'$ is isotopic to a link in the embedding f_A of $\bar{A}$. But f_A is an embedding with no even links.

Suppose that $C_1 \subset A$ and $C_1' \subset \bar{B}$. Let e_1 and e_1' be edges in C_1 and C_1', respectively. Allowing for ambient isotopy, e_1 and e_1' can be made to cross in a planar projection of $f(\bar{G})$. Inverting a crossing between e_1 and e_1' will form a Hopf link with C_1 and C_1'. If this is possible for all links $C_i \subset A, C_i' \subset \bar{B}$ (or $C_j \subset \bar{A}, C_j' \subset B$) such that the corresponding crossing change of e_j and e_j' only affects the crossing number of one link, then we can obtain an embedding of G that has no even link. Therefore, we may assume that some crossing change affects two different links.

Without loss of generality, we assume that changing the crossing of e_1 with e_1' affects both $C_1 \cup C_1'$ and $C_2 \cup C_2'$. Consider C_2'. If C_2' is contained within $\bar{B}$, then $C_1' \cup C_2'$ and C_1 form a $\Delta - \theta$ minor of G. Hence, G would not be minor-minimal. Thus, C_2' must cross into A and intersect C_1, going through both p and q. Now, consider C_2. C_2 cannot intersect C_2', so it must be that C_2 is contained entirely in A, as neither p nor q can be contained in C_2. Consequently, $C_1 \cup C_2$ and C_1' form a $\Delta - \theta$ minor of G.

Finally, suppose that $C_1 \subset \bar{A}$ and $C_1' \subset \bar{B}$, C_1 intersects p, and C_1' intersects q. Take e_1 to be an edge in C_1 that has p as an endpoint, and take e_2 to be an edge in C_1 that has q as an endpoint. If changing the crossing of e_1 and e_2 only affects the link $C_1 \cup C_1'$, then we are done. Otherwise, it must also affect the link $C_2 \cup C_2'$. C_2 cannot intersect q, as q is in C_2', so it is impossible for C_2 to intersect C_1'. Thus, $C_1 \cup C_2$ and C_1' form a $\Delta - \theta$ minor of G.

As a result, G must be 3-connected. $\square$

3.4 Completeness

The results in the previous section help characterize intrinsically even linked graphs by analyzing the connectedness of minor minimal graphs. It is hypothesized that the three Petersen-family graphs K_6, P_8, and the Petersen graph, along with the $\Delta - \theta$ graph, are the only minor minimal intrinsically even linked graphs. The goal of this section is to prove the completness of the minor minimal set. Although the proof of this conjecture remains unfinished, it appears that the general techniques applied in this section can be used to yield the desired result.

We begin by reducing to the case of two disjoint cycles connected by edges between vertices on the cycles. Then, we will make arguments based on the number of vertices in the cycles and the number of edges between the two cycles.

Lemma 3.13. *Let G be a minor minimal intrinsically linked graph that is not $\Delta - \theta$. Then, G is isomorphic to two disjoint cycles, C_1 and C_2 with any number of edges with one endpoint in C_1 and the other in C_2.*

Proof. First, we notice that if there are no disjoint cycles, then it is impossible to have a link. So at a minimum, G contains two disjoint cycles, C_1 and C_2. Requiring that the graph be minor minimal, we can reduce the situation even further.

On one hand, a minor minimal even linked graph G that is not $\Delta - \theta$ must be connected by Theorem 3.7, so there must be some path from C_1 to C_2 (in fact, there must be three). Moreover, suppose that there exists a vertex v not on C_1 nor C_2. Pick a vertex v_1 on C_1. Since G is three connected by Theorem 3.12, there must be three disjoint paths from v_1 to v by Menger's Theorem. For each of the three paths, start at v and follow the path until it first intersects either C_1 or C_2. Then two of these intersections occur on C_i for some $i = 1, 2$ by the pigeonhole principle. Without loss of generality, assume it is C_1. Since the paths are independent, the points of intersection must also be different. Then, the two paths from v to C_1, unioned with C_1, form a graph with θ as a minor. C_2 has Δ as a minor, so that G has $\Delta - \theta$ as a minor.

Consequently, a minor minimal even linked graph that does not contain $\Delta - \theta$ cannot contain any vertices outside of the pair of disjoint cycles C_1 and C_2. Hence, G is a pair of disjoint cycles joined by edges between the two cycles. If any edge had vertices in C_1 as both of its endpoints, then the subgraph induced by the vertices on C_1 contain θ as a minor, and C_2

contains Δ as a minor. Hence, it must be that the endpoints of any edges must be on different cycles. □

Based on this lemma, we make the following definitions.

Definition 3.14. Let G be a graph with two disjoint cycles C_1 and C_2, such that

1. the subgraph induced by the vertices in C_1 is C_1,
2. the subgraph induced by the vertices in C_2 is C_2,
3. the subgraph induced by the vertices in $C_1 \cup C_2$ is G.

Then, G is a *cross-cycle graph*, with *base cycles* C_1 and C_2.

Definition 3.15. Let G be cross-cycle graph with base cycles C_1 and C_2. If e_{pq} is an edge in G such that one endpoint p lies in C_1 and the other endpoint q lies in C_2, then e_{pq} is called a *crossedge*.

In the language defined above, Lemma 3.13 can be restated.

Lemma 3.13 *Every minor minimal intrinsically even linked graph that is not $\Delta - \theta$ is a cross-cycle graph.*

Lemma 3.13 provides a method for proving the completeness of the minor minimal set to classify intrinsically even linked graphs. By enumerated the types of graphs that are described in Lemma 3.13, it may be possible to show that they must all contain $\Delta - \theta$ as a minor, contain one of the three intrinsically even linked Petersen-family graphs as a minor, or has an embedding in which all links have odd linking number.

Suppose C_1 has n_1 vertices and C_2 has n_2 vertices. Then, for the graph to be minor minimal even linked, none of these vertices can be degree two - that is, each must be an endpoint of at least one crossedge. Otherwise, we can take a minor by contracting along one of its adjacent edges, resulting in a proper minor whose image under an embedding is ambient isotopic to an embedding of G. Hence, by breaking the problem into cases based on the number of vertices in each cycles as well as the number of crossedges, the crossedges will be forced into particular configurations. Then, checking that each intrinsically even linked configuration contains a known intrinsically even linked graph as a minor suffices.

First, consider the case where $n_1 = n_2 = 5$ and all crossedges are disjoint. By the above observation, there must be at least five disjoint crossedges

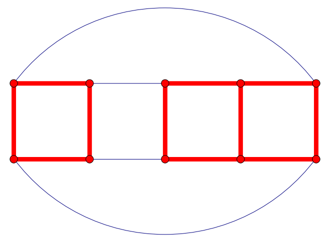


Figure 3.9: Five crossedges on two cycles of length five forming a $\Delta - \theta$ graph as a minor

connecting the vertices in C_1 to the vertices in C_2. If any three consecutive vertices in C_1 are connected to any three consecutive vertices in C_2 by three disjoint crossedges, then the subgraph induced by those six vertices forms a graph that is a subdivision of the θ graph. Furthermore, the subgraph induced by the other four vertices form a subdivision of the Δ graph, so G contains $\Delta - \theta$ as a minor (see Figure 3.9).

Otherwise, we can enumerate the possible edge configurations by associating to each valid edge configuration with a permutation. Numbering the vertices in C_1 and C_2, each edge configuration maps bijectively onto a permutation in S_5. Namely, a permutation in which n maps to n' corresponds with the edge configuration in which the nth vertex in C_1 maps to the n'th vertex in C_2. We are looking for permutations in which no two adjacent numbers are adjacent in the permutation, i.e. that

$$f(n+1) \neq f(n) \pm 1$$

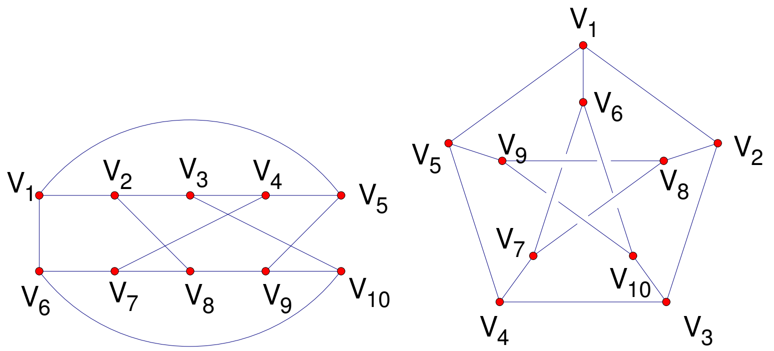
where $f \in S_n$. All such permutations are a rotation or reflection of the permutation (2354), written in cycle notation. We can see that this results in a graph isomorphic to the Petersen graph (see Figure 3.10).

In the case $n_1 \geq 5, n_2 \geq 5$, the same argument will yield a subgraph that is a subdivision of either $\Delta - \theta$ or the Petersen graph. Since $\Delta - \theta$ is a proper minor, then we get the following result.

Theorem 3.16. *Every minor minimal intrinsically even linked cross-cycle graph with at least five disjoint crossedges is isomorphic to the Petersen graph.*

Another simple case can be proved when $n_1 = n_2 = 3$.

Theorem 3.17. *Let G be a minor minimal intrinsically linked graph with two base cycles of length three. Then G is isomorphic to K_6.*



(a) (b)

Figure 3.10: Five crossedges on two cycles of length five isomorphic to the Petersen graph.

Proof. Notice that for simple graphs, there cannot be any cycles of length two or smaller, so this is the smallest case that needs to be considered. Since K_6 is minor minimal for intrinsic even linking and G has only six vertices in this case, it must be that G is isomorphic to K_6, otherwise, a proper minor (subgraph) of K_6 is intrinsically even linked. □

The previous theorems suggest a method for proving the completeness of the minor minimal set. We will first show that any cross-cycle graph consisting of four or less crossedges cannot be intrinsicaly even linked. Then, for every configuration of five crossedges, we will show that every intrinsically even linked graph contains a known minor minimal intrinsically even linked graph as a minor. Theorem 3.16 is one part of this approach.

Theorem 3.18. *Every cross-cycle graph with three of fewer crossedges is not intrinsically even linked.*

Proof. Let G be a cross-cycle graph with base cycles C_1 and C_2, and at most three crossedges. First, notice that the only cycle disjoint from C_1 is C_2, and vice-versa. It is easy to see that there exists an embedding of G such that C_1 and C_2 form the Hopf link so that they are not even linked.

We claim that G contains no other links. Any other cycle in G must use at least one crossedge. In order to complete the cycle, it must use another crossedge. Hence, every cycle C that is not C_1 nor C_2 must use two crossedges. In order to have a cycle C' that is disjoint from C, it must also use two crossedges that are disjoint from the two crossedges in C. But this is clearly impossible since there are only three crossedges.

Hence, G has an embedding in which its only link is not even linked. □

Theorem 3.19. *Every cross-cycle graph with exactly four crossedges is not intrinsically even linked.*

Proof. Every cross-cycle graph with exactly four crossedges is a minor of a cross-cycle graph with exactly four disjoint crossedges. Hence, if every cross-cycle graph with exactly four disjoint crossedges has an embedding with no evenly linked cycles, then every cross-cycle graph with exactly four (not necessarily disjoint) crossedges also has an embedding with no evenly linked cycles. So we will assume that the four crossedges are disjoint.

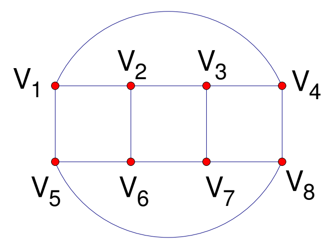
As in the five disjoint crossedges case of Theorem 3.16, each configuration can be assigned to a permutation on $\{1,2,3,4\}$. Up to cycling (multiplying by a power of (1234)) and mirror inversion (multiplying by (13)), there are three possible permuations: the trivial permutation, the permutation (12), and the permutation $(12)(34)$. These permutations correspond to the graphs in Figure 3.11. In each graph, we can list all of the links and compute the linking numbers.

In the graph in Figure 3.11(a), changing the crossing between edges (v_1, v_2) and (v_5, v_6) as well as the crossing between edges (v_2, v_6) and (v_4, v_8) (using ambient isotopy to create those crossings in the projection) results in an embedding where all links have odd linking number. Changing the crossing between edges (v_1, v_2) and (v_7, v_8) will do the same for the graphs in Figure 3.11(b) and Figure 3.11(c). □

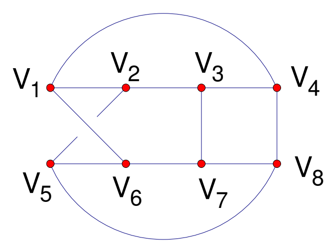
The previous two theorems show that any cross-cycle graph which is intrinsically even linked must have at least five crossedges.

In the following theorem, we consider when the base cycles have length four and five, and five crossedges that are as spread out as possible in the sense that they are placed so that each vertex on the base cycles is an endpoint to at least one of the five crossedges.

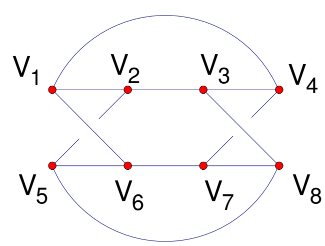
Theorem 3.20. *There exists no minor minimally intrinsically even linked graph consisting of base cycles of length four and five, respectively, and five crossedges such that only one vertex, which is on the 4-cycle, is shared by multiple crossedges, and that this vertex is shared by exactly two crossedges.*



(a) Graph corresponding to trivial permutation



(b) Graph corresponding to permutation (12)



(c) Graph corresponding to permutation (12)(34)

Figure 3.11: Graphs, up to isomorphism, with 4 disjoint crossedges.

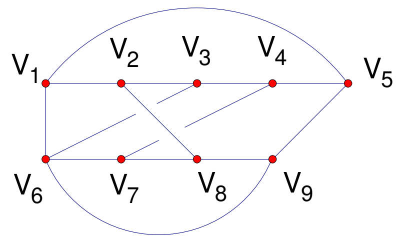


Figure 3.12: Embedding with five edges, only two sharing a vertex as an endpoint

Proof. Suppose such a graph G exists. Let C_1 be the 5-cycles, C_2 be the 4-cycle, and v be the vertex in the 4-cycle that is the endpoint of two crossedges. If the two crossedges have adjacent vertices in C_1 as endpoints, then the two crossedges and the edge between the two vertices form a Δ. In addition, three adjacent vertices in C_1 have crossedges to some permutation of three adjacent vertices in C_2, forming a subdivision of a θ graph. So G contains $\Delta - \theta$ as a proper minor, so it is not minor minimal. Otherwise, the two edges with v as an endpoint are separated by one vertex, u, in C_1. If the edge with u as an endpoint goes to a vertex adjacent to v, then there is a θ in the graph. The other two vertices in C_1 and the other two vertices in C_2 are adjacent and there are two crossedges between them, so this forms a subdivision of a Δ. Hence, G contains $\Delta - \theta$ as a proper minor.

The only remaining case is when the crossedges are configured as in Figure 3.12. While this graph does not contain any of the four known minor minimal intrinsically even linked graphs, it is possible to find an embedding such that all disjoint pairs of cycles have odd linking number by passing the edges (v_1, v_2) through (v_4, v_7) and (v_2, v_3) through (v_6, v_9). The only edges we can add to this graph without creating a $\Delta - \theta$ minor are (v_1, v_7) and (v_3, v_9). Adding both edges creates a $\Delta - \theta$ minor. By symmetry of the graph in Figure 3.12, adding either one gives isomorphic graphs. Without loss of generality, add the edge (v_1, v_7) (see Figure 3.13). Then, there exists an embedding of G such that all links have odd linking number. This can be obtained by passing the edges (v_1, v_2) through (v_4, v_7), (v_2, v_3) through (v_6, v_9), and (v_1, v_7) through (v_2, v_3).

□

We can extend the ideas in the proof, and couple it with our previous result from Theorem 3.16 to classify another subset of intrinsically even

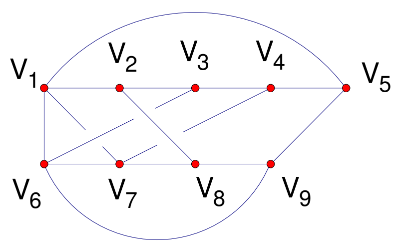


Figure 3.13: The graph in Figure 3.12, with the only edge added that does not create a $\Delta - \theta$ minor.

linked graphs. First, we will make two definitions.

Definition 3.21. Let v be a vertex in a cross-cycle graph G with any number of crossedges. The *crossdegree* of v is the number of crossedges with v as an endpoint.

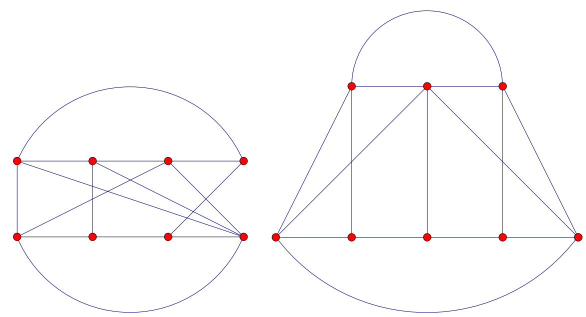
Definition 3.22. Let v be a vertex in a cross-cycle graph G with crossdegree at least n. Then, a set of n edges with v as an endpoint form a n-fan, with v as its fanning point.

Notice that the crossdegree of a vertex v is two less than the degree of v, so it is invariant of the choice of base cycles. We now state our next result.

Theorem 3.23. *There exists no minor minimal intrinsically even linked cross-cycle graph G such that G has a subgraph H with five crossedges where only one vertex has crossdegree at least two.*

Proof. Suppose such a G exists. As in the previous theorem, the only subgraph H that does not contain $\Delta - \theta$ as a minor is the subgraph in Figure 3.12. This subgraph has an embedding that has no even links. Thus, in order for G to be intrinsically even linked G must contain at least one more edge. If we add an edge that is completely disjoint from the crossedges in H, then there is a proper subgraph with five disjoint crossedges, so by Theorem 3.16, it must have a proper subgraph that contains the Petersen graph as a minor. Hence, G cannot be minor minimal intrinsically even linked.

Therefore, it must be that G contains an additional crossedge that shares an endpoint with one of the five crossedges in H. By Theorem 3.20, if the other endpoint also coincides with an endpoint of one of the five crossedges in H, G cannot be minor minimal intrinsically even linked. Thus, it must be the that other endpoint does not share an endpoint with one of the five



(a) (b)

Figure 3.14: Cross-cycle graphs for P_8.

crossedges in H. The only such edges that do not result in $\Delta - \theta$ as a proper minor are the edge with endpoints at v_1 and a vertex between v_7 and v_8 and the edge with endpoints at v_3 and a vertex between v_8 and v_9. If both edges are present, $\Delta - \theta$ is a proper minor.

By symmetry, we may assume that G contains a crossedge from v_1 to a vertex p between v_7 and v_8. Then, the crossedges (v_1, p) , (v_2, v_8), (v_3, v_6), (v_4, v_7), and (v_5, v_9) are disjoint. So by Theorem 3.16, G contains the Petersen graph as a proper minor. □

So far, each cross-cycle graph has contained $\Delta - \theta$, the Petersen graph, or K_6 as a minor. As we progress to the other cases, we should expect that P_8 will show up as a minor. Therefore, it will be useful to know the cross-cycle graph structure of P_8. In order to do so, pick any two disjoint cycles as the base cycles. We can notice that the P_8 graph, shown in Figure 3.2, is symmetric along the vertical axis. Moreover, switching the top two vertices on either side results in an isomorphic graph. Hence, by symmetry, any two disjoint 4-cycles will result in the same cross-cycle graph. This cross-cycle graph structure is as shown in Figure 3.14(a).

Similarly, every cross-cycle graph structure arising from base cycles of length three and five are the same. In this case, the cross-cycle graph is as in Figure 3.14(b).

Returning to the proof of completness, the next case to consider is when the cross-cycle graph contains two disjoint 2-fans. There are two subcases, when the fanning points are on disjoint base cycles, and when they are on the same base cycle. Notice that from Figure 3.14, P_8 has no disjoint fans. Hence, we should expect to find P_8 as a minor only when there are fans

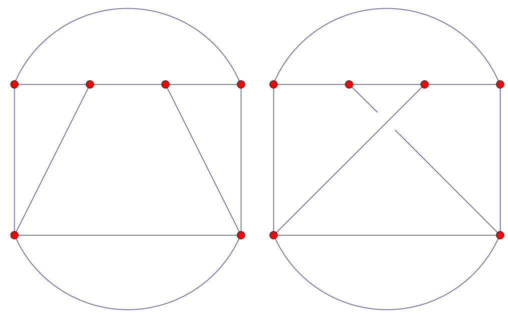


Figure 3.15: Possible configurations for 2-fans with fanning points on the same base cycle.

whos fanning points have an edge between them that is part of the fans. The cases when they are disjoint are addressed below.

Theorem 3.24. *Let G be a minor minimal intrinsically even linked cross-cycle graph with two disjoint 2-fans such that the fanning points are on disjoint base cycles. Then G is isomorphic to K_6.*

Proof. We know that every intrinsically even linked cross-cycle graph must have at least five crossedges. If the fifth crossedge is disjoint from both 2-fans, there will be a $\Delta - \theta$ minor. Similarly, if the fifth crossedge does not intersect both fans at its endpoints, then $\Delta - \theta$ appears as a minor. If all other crossedges have endpoints that lie on either of the fans, then the base cycles only have three non-trivial vertices (a vertex is trivial if it has crossdegree 0, so that one of its adjacent edges can be contracted without changing the topology of the graph embedding). By Theorem 3.17, then the graph has K_6 as a minor. So if G is minor minimal intrinsically even linked, it must be isomorphic to K_6. □

If the two 2-fans have fanning points that lie on the same base cycle, they can either be interwoven, as in Figure 3.15(b), or not, as in Figure 3.15(a).

Theorem 3.25. *There are no minor minimal intrinsically even linked cross-cycle graphs with two disjoint 2-fans whose fanning points lie on the same base cycle as in Figure 3.15(a).*

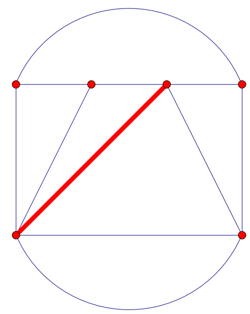


Figure 3.16: The red edge in bold does not create a $\Delta - \theta$ minor.

Proof. The only crossedges that can be added to Figure 3.15(a) without creating a $\Delta - \theta$ minor are those which go from a fanning point to an endpoint of an edge in the fan that is disjoint from it (see Figure 3.16).

Adding one such edge does not create any more links, so the graph is not intrinsically even linked because all of the links can be generated by a subgraph with only four crossedges and all graphs with only four crossedges are not intrinsically even linked. Adding all four possible crossedges of the type highlighted in Figure 3.16 still does not result in an intrinsically even linked graph. Each additional crossedge is used only in one link, so it can be embedded in a way that the links have odd linking number. □

The case in Figure 3.15(b) is more complicated, as there are more places that the fifth crossedge can be placed without creating a $\Delta - \theta$ minor. On the lower cycle, the possibilities are that the crossedge's endpoint is on a fanning point or it is not. On the top cycle, the crossedge may also intersect one of the fans at a vertex or it may not. Any of the possibilities has a very small number of links, so it is likely that there is an embedding without even links. It will probably require the addition of a couple more crossedges before forcing a minor other than the $\Delta - \theta$ graph. This case has not yet been fully explored.

To complete the proof of the conjecture that the four minor minimal graphs in Section 3.2 fully characterize intrinsic even linking, the case presented above must be finished. In addition, it is necessary to investigate the possible crossedge configurations for 2-fans that intersect at a common edge or vertex. The final cases that must be explored are 3-, 4-, and 5-fans. These n-fan cases for large n are likely to produce many $\Delta - \theta$ minors since they already contain Δ as a minor of the n-fans.

Chapter 4

Intrinsic Knotting

4.1 Definitions and Known Results

As with linking in graphs, it may be possible that a graph contain cycles that are knotted in particular embeddings. Now, we define intrinsic knotting of graphs.

Definition 4.1. Let G be a graph. If for every embedding $f : G \to \mathbb{R}^3$ of G, there exists a cycle C in G such that $f(C)$ is not the unknot (i.e. $f(C)$ is knotted), then G is *intrinsically knotted*.

Clearly, if H is a subgraph of G and H is intrinsically knotted, then G is intrinsically knotted as well. However, a more powerful concept than subgraph (and subdivision) will be helpful in classifying graphs that are intrinsically knotted.

Motwani, Raghunathan, and Saran [15] showed that if $H \leq G$ and H is intrinsically knotted, then G is intrinsically knotted. The proof follows easily by noting that if C is a knotted cycle in H, then there is a cycle C' in G that is ambient isotopic to C so that C' and C have the same knot type. Hence, in order to classify the set of all intrinsically knotted graphs, we need only to look for intrinsically knotted graphs that are "smallest" in the minor sense. Such graphs are called *minor minimal intrinsically knotted graphs*.

Definition 4.2. A graph G is a *minor minimal intrinsically knotted graph* if the following hold:

1. G is intrinsically knotted.

2. whenever $H < G$ is a proper minor of G, then H is not intrinsically knotted.

Robertson and Seymour [18] recently proved the following result, which guarantees that a characterization of intrinsically knotted graphs by a finite number of minor minimal intrinsically knotted graphs exists.

Theorem 4.3 (Roberston and Seymour)**.** *Let $\mathcal{G}$ be an infinite set of graphs. Then, there exist $H, G \in \mathcal{G}$ such that $H \leq G$.*

Known as the Minor Theorem (or Wagner's Conjecture), a corollary of this result shows that intrinsic knotting can be characterized by a finite number of graphs because the property is minor monotonic (i.e. if a graph contains another graph with the property, then the original graph has the property as well).

At this time, there are 16 graphs that are known to be minor minimal intrinsically knotted, and there are 25 other graphs that are intrinsically knotted, but it is not known whether they are minor minimal intrinsically knotted.

The known intrinsically knotted graphs, with the exception of the graph in [8], are all obtained from K_7 and $K_{3,3,1,1}$, which Conway and Gordon [2] and Foisy [7], respectively, showed to be minor minimal intrinsically knotted. Thirteen of the minor minimal intrinsically knotted graphs are obtained from K_7 by a series of $\triangle Y$ moves [15, 13]. The 25 intrinsically knotted graphs (not necessarily minor minimal) are obtained by $\triangle Y$ moves on $K_{3,3,1,1}$. This list of intrinsically knotted graphs relies heavily on the following theorem from Motwani et al [15].

Theorem 4.4. *Let G be a graph and suppose G' is obtained from G by removing the edges between three mutually adjacent vertices v_1, v_2, v_3 and adding a new vertex v to the graph along with the edges $e_{vv_1}, e_{vv_2}, e_{vv_3}$. The operation $G \to G'$ is called a $\triangle Y$ move, and G' is intrinsically knotted if G is intrinsically knotted.*

The argument relies on the observation that an embedding of G' has all of its cycles ambient isotopic to cyles in an embedding of G. We can see this by contracting the edges $e_{vv_i}, i = 1, 2, 3$ so that they lie on a disc that does not intersect the rest of the graph. Then, we can replace the Y with a triangle to obtain an embedding of G. Since G is intrinsically knotted, it contains a knotted cycle in this embedding. But if a cycle passes through any of the edges in the triangle v_1, v_2, v_3, we can isotope it to a cycle passing through the Y section of G'. Hence, G' is intrinsically knotted.

The proof that K_7 is intrinsically knotted uses a combinatorial argument, showing that any crossing change on a projection of K_7 affects the arf invariant of an even number of Hamiltonian cycles in K_7. Also, any embedding of K_7 can be obtained by applying crossing changes (and ambient isotopy) to a single, "standard" embedding of K_7. Then, by looking at

$$\sigma = \sum \alpha(C)$$

where the sum is over all Hamiltonian cycles, C, and $\alpha(C)$ is the arf invariant of C, we can see that $\sigma \equiv 1 \mod 2$ in all embeddings of K_7. Because K_7 contains an even number of Hamiltonian cycles, it must contain a cycle with odd arf invariant. This implies that some cycle (the cycle with odd arf invariant) is knotted.

The proof that $K_{3,3,1,1}$ is intrinsically knotted is similar [8].

In addition, the following reuslt will be helpful in identifying minor minimal intrinsically knotted graphs. Results on connectivity will rule out a large class of graphs.

Proposition 4.5. *Let G be a minor minimal intrinsically knotted graph. Then, G must be 2-connected.*

Proof. Let v be a vertex, and suppose that $G \setminus v = G_1 \cup G_2$, where G_1 and G_2 are disjoint and non-empty. Let H_1 be the subgraph induced by $V(G_1) \cup \{v\}$ and H_2 be the subgraph induced by $V(G_2) \cup \{v\}$.

Since H_1 is a proper minor of G, then there exists an embedding of H_1 into an open ball $B_1 \cong \mathbb{R}^3$ that is knotless. Using ambient isotopy, we may assume that v lies on the boundary of B. Similarly, H_2 has a knotless embedding into an open ball B_2, and we can isotope v to the boundary.

Identify the two balls in space at v. This gives an embedding of G, which we claim is knotless. Clearly, the embedding contains no knot that lies entirely in H_1 or entirely in H_2. However, no knot can lie in both G_1 and G_2 since the only path from G_1 to G_2 passes through v. If there were such a cycle, without loss of generality, we may assume it starts in G_1. If it lies partially in G_2, it must pass through v to get to G_2. To close the cycle, it must pass through v again to get back to the starting vertex in G_1. This is not a proper cycle since it has a self-intersection at v.

Thus, it must be that G has a knotless embedding.

The case for disconnected graphs is similar. □

As a corollary of the above, we find that a minor minimal intrinsically knotted graph cannot contain a cut vertex, bridge, leaves, or multiple connected components.

4.2 Knot Energies

Although the Conway and Gordon argument outlined in the previous section gives a standard method for proving that a graph is intrinsically knotted, it must be adapted to each graph. For a graph that does not have much symmetry, it becomes increasing complicated because each type of edge crossing must be considered individually. Thus, we would like to find an alternative way of detecting intrinsically knotted graphs.

One candidate for such a method is to adapt knot energies to graphs. Since a particular knot may have different knot diagrams for which it is not immediately obvious that they represent equivalent knots, energy methods help to identify knots by providing a standard embedding (or diagram). First, an energy is defined on a knot, and then the energy is minimized for all knots of that particular knot type.

In this section, we will define a few different knot energies taken from a thorough survey of the subject by van Rensburg [11]. In all cases, $\alpha(t)$ will be taken to be a simple, closed curve.

1. **Mobius energy** Let $\alpha(t)$ be a parametrization of a simple, closed, C^2 curve. Then, the Mobius energy of α is

   $$E_M(\alpha) = \int_\alpha \int_\alpha \left(\frac{1}{||\alpha(s) - \alpha(t)||^2} - \frac{1}{D^2(\alpha(s), \alpha(t)} \right) ||\alpha'(s)||||\alpha'(t)||dsdt$$

   where $D(\alpha(s), \alpha(t))$ is the shortest distance from $\alpha(s)$ to $\alpha(t)$ along the curve α.

2. **O'Hara energy** Let $\alpha(t)$ be an arc-length parametrization of a curve with total length 1. Then, the O'Hara energy of α is

   $$E_j^p = \frac{1}{j} \left[\int_0^1 \int_0^1 \left(\frac{1}{||\alpha(s) - \alpha(t)||^j} - \frac{1}{D^j(\alpha(s), \alpha(t))} \right)^p dsdt \right]^{\frac{1}{p}}.$$

3. **Curvature energy** Let $\alpha(t)$ be C^2 with arc-length parametrization and unit length. Then, the curvature of α at t_0 is $\kappa(t_0) = |\alpha''(t_0)|$. The p-th curvature energy of α is

   $$\kappa_p(\alpha) = \int_0^1 \kappa^p(t)dt.$$

4. **Polygonal curvature energy** Let $\alpha(t)$ be piecewise linear segments $s_i, i = 1, \ldots, n$. Define θ_i to be the excluded angle between s_i and s_{i+1}. An analog to curvature energy can be defined by

$$E_p = \sum_{i=1}^{n} \theta^p.$$

For a given knot type $\mathcal{K}$, the energy of $\mathcal{K}$ is defined to be the minimum (or infimum) of its energy over all embeddings with that particular knot type. For example, the Mobius energy of the unknot is 4 [10].

It has been shown that the circle has the lowest energy of all knots for most types of knot energies [10, 1]. Consequently, we hope that if we put some of these energies on graphs, then a graph that is not intrinsically knotted will have no non-trivial knots in a minimum energy embedding since knots (that are not the unknot) have high energies.

4.3 Graph Curvature Energy

The curvature energy of a knot, κ, measures how much it turns as we travel along it once. Thus, if $\alpha(t)$ parametrizes an embedding of the knot such that α is piecewise C^2 and is parametrized by arc-length, we can define a variant of the curvature energy by

$$\kappa'(\alpha) = \int_{\alpha} \kappa(t)dt + \sum_{i=1}^{n} \theta_i$$

where θ_i is the turning angle of α at the points x_i where α'' is discontinuous. That is, θ_i is the angle between $\lim_{t \to x_i^-} \alpha'(t)$ and $\lim_{t \to x_i^+} \alpha'(t)$.

Then, if f is a graph embedding of G whose edges e are parametrized by arc-length by C^2 curves $\alpha_e(t)$, we can define the *total curvature energy* of f by

$$\kappa(f; G) = \sum_{e \in E(G)} \int_e \kappa_e(t)dt + \sum_{v \in V(g)} \sum_{\substack{e,f \in E(G) \\ e \cap f = v}} \theta_{ef}$$

where $\kappa_e(t) = |\alpha''(t)|$ and θ_{ef} is the exterior angle between e and f. It is important to note that θ_{ef} is the exterior angle, not the interior angle, because the exterior angle measures the turning of a curve traveling through v from e to f. For the remainder of this chapter, we will assume that all embeddings of graphs have edges which are C^2.

It is easy to see that $\kappa(f;G)$ is scale invariant since scaling the embedding by c also scales the curvature everywhere by $\frac{1}{c}$. This follows from the fact that the curvature, κ, at a point x satisfies $\kappa = \frac{1}{r}$ where r is the radius of the osculating circle at x.

Proposition 4.6. *Let f be an embedding of a graph G, and let $g = cf$ be a scaling of the embedding, with $c > 0$. Then, we have that $\kappa(f;G) = \kappa(g;G)$. That is to say, total curvature energy is scale invariant.*

Proof. Let $\alpha(s), s \in [0,a]$ be parametrized by arclength. Scale α by $c > 0$. Then, the scaled curve, parametrized by arclength, is $\gamma(s) = c\alpha\left(\frac{s}{c}\right), s \in [0, ca]$. Then,

$$\int_0^{ca} |\gamma''(s)| ds = \int_0^{ca} \left| c\alpha''\left(\frac{s}{c}\right)\left(\frac{1}{c^2}\right)\right| ds = \int_0^{a} |\alpha''(s)| ds.$$

So the energy contribution from each edge is invariant. The angles between adjacent edges also does not change by scaling, so

$$\kappa(f;G) = \kappa(g;G).$$

□

Definition 4.7. Let G be a graph. Then the *total curvature energy of G*, denoted $\kappa(G)$, is $\min_f\{\kappa(f;G)\}$.

So the total curvature energy of G is the minimum curvature energy of embeddings of G over all possible embeddings.

If the graph G is a cycle, then its embedding is equivalent to a closed, piecewise C^2 curve. For the cycle graph C_n on n vertices, we can find the minimum energy of the graph. A special case of the result follows from the Gauss-Bonnet Theorem.

Theorem 4.8 (Gauss-Bonnet Theorem)**.** *Let $\mathcal{R} \subseteq S$ be a regular region of an oriented surface and let $C_1, \ldots, C_n$ be the closed, simple, piecewise regular curves which form the boundary $\partial\mathcal{R}$ of $\mathcal{R}$. Suppose that each C_i is positively oriented and let $\theta_1, \ldots, \theta_p$ be the set of all external angles of the curves $C_1, \ldots, C_n$. Then,*

$$\sum_{i=1}^{n} \int_{C_i} k_g(s) ds + \iint_{\mathcal{R}} K d\sigma + \sum_{i=1}^{p} \theta_i = 2\pi\chi(\mathcal{R})$$

where s denotes the arc length of C_i, and the integral over C_i means the sum of integrals in every regular arc of C_i. [3]

Here k_g denotes the *geodesic curvature*, K is the *Gaussian curvature* of the surface, and $\chi(\mathcal{R})$ is the *Euler characteristic* of $\mathcal{R}$. As a corollary, we obtain a result about the total curvature of a simple, closed, planar, piecewise C^2 curve.

Corollary 4.9. *Let C be a planar, simple, closed, piecewise C^2 curve with external angles $\theta_1, \ldots, \theta_n$. Then,*

$$\int_C \kappa(s)ds + \sum_{i=1}^{n} \theta_i \geq 2\pi$$

Proof. Without loss of generality, let C lie on the xy plane. Let S be the surface on the xy plane that lies in the interior of C. Then, we have that

$$\begin{aligned} |k_g(s)| &= \kappa(s) \\ K(S) &= 0. \end{aligned}$$

Moreoever, S is homeomorphic to the closed disk, so it has Euler characteristic $\chi(S) = 1$.

Therefore, the Gauss-Bonnet Theorem tells us that

$$\begin{aligned} \int_C \kappa(s)ds + \sum_{i=1}^{n} \theta_i &= \int_C |k_g(s)|ds + \sum_{i=1}^{n} \theta_i \\ &\geq \int_C k_g(s)ds + \sum_{i=1}^{n} \theta_i \\ &= 2\pi\chi(S) \\ &= 2\pi. \end{aligned}$$

□

The general result for closed curves, stated below, was proved by Fenchel [4]. Fenchel's theorem immediately implies Proposition 4.11, which characterizes curvature energy for cycle graphs.

Theorem 4.10. *Let C be a simple, closed, piecewise C^2 curve with external angles $\theta_1, \ldots, \theta_n$. Then,*

$$\int_C \kappa(s)ds + \sum_{i=1}^{n} \theta_i \geq 2\pi$$

with equality holding if and only if C is planar convex.

Proposition 4.11. *Let C_n be the cycle graph on n vertices. Then $\kappa(C_n) = 2\pi$. Moreover, $\kappa(f, C_n) = 2\pi$ if and only if f is a planar convex embedding of C_n.*

Now, we wish to find $\kappa(G)$ for some non-cyclic graphs. We begin by considering the graph K_4. We expect that in a minimum energy embedding, the embedded graph will be symmetric under permutation of the vertices, so we will assume that they are equidistributed on the unit sphere, $S^2 \subset \mathbb{R}^3$. This assumption will need to be verified at a later time. We begin by calculating the position of the points.

Without loss of generality, we locate the first point at $p_0 = (0, 0, 1)$ and the second point on the xz-plane. Then, we have that the three points not on the z-axis are located at

$$\begin{aligned}
p_1 &= (\sin\phi, 0, \cos\phi), \\
p_2 &= (\cos\frac{2\pi}{3}\sin\phi, \sin\frac{2\pi}{3}\sin\phi, \cos\phi), \\
p_3 &= (\cos\frac{2\pi}{3}\sin\phi, -\sin\frac{2\pi}{3}\sin\phi, \cos\phi),
\end{aligned}$$

where ϕ is the angle from the positive z-axis.

Then,

$$\begin{aligned}
|p_1 - p_0| &= 2\sin\frac{\phi}{2} \\
&= \sqrt{2 - 2\cos\phi}.
\end{aligned}$$

We also have

$$|p_2 - p_1| = \sqrt{3}\sin\phi.$$

Hence, setting the two quantities together, we have

$$3\sin^2\phi = 3(1 - \cos^2\phi) = 2 - 2\cos\phi.$$

This simplifies and factors to

$$(3\cos\phi + 1)(\cos\phi - 1) = 0.$$

Thus, $\phi = 0, \cos^{-1} -\frac{1}{3}$. If $\phi = 0$, then the $p_i, i = 1, 2, 3, 4$ are not distinct, so it must be that $\phi = \cos^{-1} -\frac{1}{3}$.

Now, we can calculate the total curvature energy of G for the straight edge embedding, f_s, and for the embedding, f_g, where the edges are geodesics of S^2.

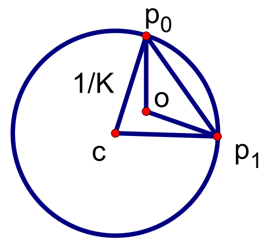


Figure 4.1: Circle of curvature centered at c passing through the points p_0 and p_1 of the regular tetrahedron inscribed in S^2.

For the straight edge embedding, $\kappa_e(t) = 0$ for all edges e, so that $\kappa(f_s; K_4) = \sum_{i=1}^{n} \theta_i$. Since each face of the tetrahedron is equilateral, the exterior angle between two adjacent edges is $\frac{2\pi}{3}$. Thus,

$$\begin{aligned}\kappa(f_s, K_4) &= 4(3)\left(\frac{2\pi}{3}\right)\\ &= 8\pi = 25.13274\ldots.\end{aligned}$$

In an embedding onto S^2, the curvature of each edge is 1. The arc-length of each edge is $\cos^{-1} -\frac{1}{3}$. At any vertex v, each edge incident to v has its tangent vector on the tangent plane at v. Moreover, the three edges cut the tangent plane into three equiangular pieces, so the exterior angle of any two edges that meet is $\frac{\pi}{3}$. Then,

$$\begin{aligned}\kappa(f_g, K_4) &= 6\cos^{-1} -\frac{1}{3} + 4(3)\left(\frac{\pi}{3}\right)\\ &= 6\cos^{-1} -\frac{1}{3} + 4\pi = 24.03017\ldots.\end{aligned}$$

Now, let f_κ be the embedding where all edges have contant curvature, κ, and the angles between all pairs of adjacent edges is the same. The central angle $\sigma = \angle p_0 c p_1$ of the circle of curvature subtended by an edge can be computed by setting the length of the chord associated with σ to be the distance between two vertices of the regular tetrahedron inscribed in S^2.

$$\begin{aligned}&\sin\frac{\sigma}{2} = \kappa \sin\left(\frac{1}{2}\cos^{-1} -\frac{1}{3}\right)\\ &\Rightarrow \sigma = 2\sin^{-1}\left(\kappa \sin\left[\frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)\right]\right).\end{aligned}$$

The angle between the circle's perimeter and the chord p_0p_1 is $\frac{\sigma}{2}$.

Letting $\psi = \angle op_0p_1$ be the angle between the radius of S^2 and the straight line between two vertices of K_4, and letting k the distance from the origin, o, to the straight line, p_0p_1 between two vertices, we have that

$$k = \cos\left[\frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)\right],$$

so

$$\psi = \sin^{-1}\cos\left[\frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)\right].$$

Then, the tangent vectors of two adjacent edges at a shared vertex are (up to rigid transformation),

$$\begin{aligned} u &= (\sin(\frac{\sigma}{2}+\psi), 0, \cos(\frac{\sigma}{2}+\psi)) \\ v &= (-\frac{1}{2}\sin(\frac{\sigma}{2}+\psi), \frac{\sqrt{3}}{2}\sin(\frac{\sigma}{2}+\psi), \cos(\frac{\sigma}{2}+\psi)). \end{aligned}$$

So the angle between two edge is given by

$$\omega = \cos^{-1}(u \cdot v).$$

We can combine these results to obtain the total curvature energy of the constant curvature embedding of K_4.

$$\begin{aligned} \kappa(f_\kappa; K_4) = & \sum_{e\in E(K_4)} \int_e \kappa_e(t)dt + \sum_{v\in V(K_4)} \sum_{\substack{e,f\in E(K_4) \\ e\cap f = v}} \theta_{ef} \\ =& 6\left(2\sin^{-1}\left(\kappa\sin\left[\frac{1}{2}\xi\right]\right)\right) + \\ & 12\cos^{-1}\left(\frac{1}{2} - \frac{3}{2}\cos^2\left(\sin^{-1}\cos\left[\frac{1}{2}\xi\right] + \sin^{-1}\left(\kappa\sin\left[\frac{1}{2}\xi\right]\right)\right)\right) \end{aligned} \tag{4.1}$$

where $\xi = \cos^{-1}\left(\frac{1}{3}\right)$.

Equation (4.1) has a minimum at $\kappa = \frac{1}{\sqrt{2}}$[1]. So for constant curvature embeddings, the minimum energy configuration of K_4 has curvature $\frac{1}{\sqrt{2}}$.

The energy associated with the embedding is

$$\kappa(f_{\frac{1}{\sqrt{2}}}; K_4) = 22.15726951\ldots$$

[1] Solved numerically using Mathematica to 24 digits of precision.

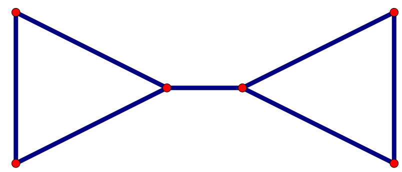


Figure 4.2: Graph obtained by bridging two triangles.

4.4 Remarks on Graph Curvature Energy

We have seen that for the cycle graph C_n and for K_4 (with some assumptions), a minimum energy configuration exists, although the configuration may not be unique. In particular, any convex planar embedding of C_n has minimum energy. We would like to know if a minimum energy configuration always exists. The following example shows that this will not always be the case, unless some restrictions are made.

Let G_0 be the graph obtained by bridging two triangles by an edge (see Figure 4.2). Then, the energy of an embedding f of G_0 is

$$\kappa(f; G_0) = \kappa(f|T_1; T_1) + \kappa(f|T_2; T_2) + \int_b \kappa_b(s)ds + \sum_{n=1}^{4} \theta_n$$

where T_1 and T_2 are the two triangle subgraphs of G_0, b is the bridge, and $\theta_1, \theta_2, \theta_3, \theta_4$ are the four exterior angles determined by b and the edges which are adjacent to it.

Note that the energy of an embedding of G is at least equal to the sum $\kappa(f|T_1; T_1) + \kappa(f|T_2; T_2)$. By Proposition 4.11, any planar, convex embeddings of the triangular subgraphs have minimum energy. Let T_1 and T_2 be convex triangles with the two vertices that are endpoints of b lying on the x axis. Then, the energy of the two subgraphs are at a minimum. Let b be the straight line between the two triangles. Then,

$$\int_b \kappa_b(s)ds = 0.$$

Consider the four edges that are adjacent to b and the angles $\theta_i, i = 1, 2, 3, 4$ associated to them. For $\epsilon > 0$, we can let $\theta_i < \epsilon$ for each i in an embedding of G_0 while keeping the embeddings of each of the two triangles planar convex. In the limit, we have that

$$\lim_{\epsilon \to 0} \kappa(f; G) = 4\pi.$$

However, the limit does not result in a proper embedding since two edges will overlap. Moreover, it is not possible for any proper embedding to have $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$ so that $\kappa(f; G_0) > 4\pi$ for all embeddings f of G_0.

Thus, $\kappa(G_0)$ is not well-defined for this graph.

We can rule out this particular case by Proposition 4.5, since minor minimal intrinsically knotted graphs are 2-connected. Since we are looking for minor minimal intrinsically knotted graphs, G_0 is not a candidate graph.

Unfortunately, the same problem arises for the graph obtained by joining two copies of C_3 along a common edge.

Chapter 5

Final Thoughts

While the work to classify intrinsically even linked graphs was not taken to completion, it is the author's opinion that the approach outlined and pursued in Section 3.4 will lead to a proof that the four graphs—K_6, P_8, the Petersen graph, and the $\Delta - \theta$ graph—form a complete minor minimal set for intrinsic even linking. At first, the proof attempted to adapt the methods in Theorems 3.16 and 3.17 to cycles of specific length. It was hoped that all cross-cycle graphs with base cycle lengths of at least five would contain the Petersen graph as a minor. However, it was realized that it is possible to have a cross-cycle graph with no trivial vertices without having five disjoint edges. This seemed to crush the idea for proving completeness, making the previous results useless.

However, the idea was salvaged by the insights of David Bachman, suggesting that the author look at specific subsets of graph where the size and number of fans are limited. In this way, it is possible to systematically generated graph one edge at a time by avoiding $\Delta - \theta$ as a minor. This realization and subsequent alteration occurred near the end of the term, so it was not possible to complete the work, but it does appear that it can be finished without looking at many more cases. The cases that remain are the last case for disjoint 2-fans, intersecting 2-fans, and n-fans for larger n.

The intrinsic knotting and graph energy chapter outlines work done in the first semester. Although the idea had potential, calculation of graph energies, even for the simplest graphs, is extremely difficult. There were many heuristic arguments made for calculating the energy of K_4 which do not seem very easy to prove. Moreover, connecting graph energy to intrinsic knotting does not seem to be very easy, as edges may be shared by multiple cycles. Thus, it appears that further insights are necessary before such an approach becomes useful.

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