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Examples of Volume-Preserving Great Circle Flows of S3

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Abstract

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This summer Herman Gluck and Weiqing Gu proved the last step in a process that took conformal maps between two complex spaces and related them to Volume-Preserving Great Circle Fibrations of S^3. These fibrations, which are non-intersecting flows, break down under certain conditions. We obtained the fibrations by applying the process to different conformal maps then calculated the angles where they intersect. This paper centers around the developments in the method for converting the conformal maps and finding the critical angles. Finally, the examples are included in their various stages of completeness.

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Chapter 1

INTRODUCTION

This work looks at examples of a process Herman Gluck and Weiqing Gu [1] finished during the summer of 1999 that takes conformal maps and builds volume-preserving great circle fibrations on S^3. Fibrations are non-intersecting flows. The original purpose was to look at how the flows given by these conformal maps intersect and therefore fail to be fibrations. In addition to giving the background information on this process, this paper presents new methods used to simplify and make parts of the process possible for more complex maps. However, the methods fail at various points due to calculational intensity. All the worked examples are then presented as far as these methods could be solved starting with the only complete example and moving toward maps in which only the first step was possible.

1.1 The Process

In Gluck and Gu [1] the authors prove the following

Theorem 1.1.1 *Let V be a smooth unit vector field on the round 3-sphere, such that the orbits of V are great circles and the flow of V is volume-preserving. If V is defined on the entire 3-sphere, then V must be tangent to a Hopf fibration. But if V is only defined on a proper open subset of the 3-sphere, which for convenience we take to be connected, then there are many examples which are not tangent to any Hopf fibration. All these flows are analytic, and each can be extended to a volume-preserving great circle flow defined on a maximal connected open subset.*

Now the space of great circles on S^3 is isomorphic to the planes through the origin in R^4. The set of such planes forms the Grassmann manifold (G_2R^4). Gluck and Gu [1] use the natural diffeomorphism $G_2R^4 \cong S^2 \times S^2$ and the following characterization of great circle fibrations of S^3: *a submanifold of G_2R^4 is the base space of a fibration of S^3 by oriented great circles if and only if it is the graph of a distance-decreasing map from either S^2 factor to the other.* A distance-decreasing map in this case takes any curve on the first sphere and maps it to a shorter curve on the second sphere. Distance-decreasing maps are made by taking a map and restricting it to its distance-decreasing points on the first sphere. An oriented great circle fibration defined only on some connected open subset of the 3-sphere will correspond to a distance-decreasing map from a connected open subset of one S^2 factor to the other.

Gluck and Gu [1] continue with the following theorem.

Theorem 1.1.2 *Let F be a smooth fibration of a connected open subset of S^3 by oriented great circles. Then the unit speed flow along the fibers of F is volume-preserving if and only if the corresponding distance-decreasing map from a connected open subset of S^2 to S^2 is holomorphic.*

Conformal maps when converted to a map between S^2 and itself using the stereographic projection are holomorphic maps. Combined with the distance-decreasing condition above, the maps give volume-preserving great circle fibrations on S^3. In order to look at the breakdown of these fibrations, we look at where the maps are no longer distance-decreasing. Since the map can be distance-decreasing from either S^2 factor to the other to be valid and if at a point it is distance-increasing in one direction is distance-decreasing in the other, the only breakdown points are the points where it is distance-preserving.

To summarize, the whole process (shown in Fig. 1.1) starts with a conformal map and converts it to a map between S^2 and itself. Then, calculates where that map is distance-preserving. That section of the map is then converted into G_2R^4 where each plane is the wedge product of two vectors. Finally, we visualized the flows in S^3 and their intersections.

Figure 1.1: Pictorial View of the Full Process

The following example was given in [1] for the full process with some changes made in the names of the variables to fit with the rest of the examples.

1.2 *Original Example:* $f : W \to 1/2W$

We separate the map into its real and imaginary parts

$$\begin{aligned} W &= u + iv \\ 1/2W &= 1/2u + i1/2v \\ Re(f) &= 1/2u \\ Im(f) &= 1/2v \end{aligned}$$

The next portion converts the map to a map between two spheres ($(x, y, z) \to (\tilde{x}, \tilde{y}, \tilde{z})$) using the spherical projection formula (Figure 2.1)

$$\tilde{x} = (1 + \tilde{z})Re(f)$$

$$\begin{aligned}\tilde{y} &= (1+\tilde{z})Im(f)\\ x &= (1+z)u\\ y &= (1+z)v\end{aligned}$$

The individual coordinates on the target sphere are found in terms of the first.

$$\begin{aligned}1-\tilde{z}^2 &= \tilde{x}^2+\tilde{y}^2\\ &= (1+\tilde{z})^2 Re^2(f)+(1+\tilde{z})^2 Im^2(f)\\ &= 1/4(1+\tilde{z})^2(u^2+v^2)\\ &= 1/4(1+\tilde{z})^2\frac{x^2+y^2}{(1+z)^2}\\ 1-\tilde{z} &= 1/4(1+\tilde{z})\frac{1-z^2}{(1+z)^2}\\ \tilde{z} &= \frac{3+8z+5z^2}{5+8z+3z^2}\\ \tilde{x} &= (1+\tilde{z})Re(f)\\ &= (1+\frac{5z+3}{3z+5})(\frac{1}{2}\frac{x}{1+z})\\ &= \frac{4x}{3z+5}\\ \tilde{y} &= \frac{4y}{3z+5}\end{aligned}$$

The map is then changed to the spherical coordinates shown in Figure 2.2 the target sphere is the same with tilded coordinates.

$$\begin{aligned}\tilde{\sigma} &= \sigma\\ \cos\tilde{\theta} &= \frac{3+5\cos\theta}{5+3\cos\theta}\\ \sin\tilde{\theta} &= \frac{4\sin\theta}{5+3\sin\theta}\end{aligned}$$

Since the map just maps the two azimuthal coordinates to each other the map is distance-preserving when the map takes a circle at a particular polar angle to the same radius circle on the other sphere.

$$r = \tilde{r}$$

$$\begin{aligned} r &= \sin\theta \\ \tilde{r} &= \sin\tilde{\theta} \\ \sin\theta &= \sin\tilde{\theta} \\ &= \frac{4\sin\theta}{5+3\cos\theta} \end{aligned}$$

Canceling $\sin\theta$ gives the distance-preserving equation

$$1 = \frac{4}{5+3\cos\theta}$$

and the distance-preserving angles

$$\begin{aligned} \theta_0 &= arccos(-1/3) \\ \tilde{\theta}_0 &= \pi - arccos(-1/3) \end{aligned}$$

The following is the family of fibrations

$$W_p = (\cos\alpha\ e_1' + \sin\alpha\ e_3') \wedge (\cos\beta\ e_2' + \sin\beta\ e_4') \tag{1.1}$$

and in this example the simple map gives nice simplifications.

$$\begin{aligned} \mu &= (\tilde{\sigma}+\sigma)/2 \\ &= \sigma \\ \nu &= (\tilde{\sigma}-\sigma)/2 \\ &= 0 \\ e_1' &= \cos\sigma\ e_1 + \sin\sigma\ e_2 \\ e_2' &= -\sin\sigma\ e_1 + \cos\sigma\ e_2 \\ e_3' &= e_3 \\ e_4' &= e_4 \end{aligned}$$

Also, α and β work out nicely for this map at the distance-preserving angle

$$\alpha_0 = (\theta_0 - \tilde{\theta}_0)/2$$

$$
\begin{aligned}
&= (\theta_0 - (\pi - \theta_0))/2 \\
&= \theta_0 - \pi/2 \\
\beta_0 &= (\theta_0 - \tilde{\theta}_0)/2 \\
&= \pi/2
\end{aligned}
$$

giving the fibrations

$$
\begin{aligned}
W_p &= (\cos\alpha_0\ e_1' + \sin\alpha_0\ e_3) \wedge e_4 \\
&= (\sin\theta_0\ e_1' + \cos\theta_0\ e_3) \wedge e_4
\end{aligned}
$$

Using ρ as the parameterization of the circles gives a useful form of the fibration,

$$W_p = (\sin\theta_0 \cos\rho \cos\sigma, \sin\theta_0 \cos\rho \sin\sigma, \cos\theta_0 \cos\rho, sin\rho)$$

which all intersect at $(0,0,0,1)$ when $\rho = \pm\pi/2$. To get a better view we use the following simplifications in the (e_1, e_2) plane

$$x_1^2 + x_2^2 = \sin^2\theta_0 \cos^2\rho$$

this gives a circle parameterized by σ. Doing the same to the (e_3, e_4) plane gives an ellipse

$$(\frac{x_3}{\cos\theta_0})^2 + x_4^2 = 1$$

We map the ellipse to the x-y plane and rotating the circle around the ellipse to visualize the fibrations in three dimensions. Figure 1.2 shows part of the output of the Mathematica program that was written to visualize the intersections (Appendix A). The pinching of the ellipse at the critical angles is clearly visible showing the break down of the fibrations.

Figure 1.2: Visualization for $f : W \rightarrow 1/2W$

Chapter 2

METHOD

2.1 Converting a Conformal Map to a Map Between S^2 and Itself

We will call the initial conformal map f. This map can be split into its real and imaginary parts. Starting on the second of the unit spheres, which will be denoted from now on by tilde coordinates,

$$1 - \tilde{z}^2 = \tilde{x}^2 + \tilde{y}^2$$

Then using the north pole stereographic projection formula as shown in Figure 2.1

$$\begin{cases} \tilde{x} = (1 + \tilde{z})Re(f) \\ \tilde{y} = (1 + \tilde{z})Im(f) \end{cases}$$

combine as follows to give the coordinates of the 2nd unit sphere in terms of the original map.

$$\begin{aligned} 1 - \tilde{z}^2 &= (1 + \tilde{z})^2(Re(f)^2 + Im(f)^2) \\ 1 - \tilde{z} &= (1 + \tilde{z})|f|^2 \\ \tilde{z} &= \frac{1 - |f|^2}{1 + |f|^2} \\ \tilde{x} &= (1 + \tilde{z})Re(f) \\ &= \frac{2Re(f)}{1 + |f|^2} \\ \tilde{y} &= \frac{2Im(f)}{1 + |f|^2} \end{aligned}$$

Next, we find the new map in spherical coordinates (Figure 2.2) in terms of the map for the 2nd sphere.

$$
\begin{aligned}
\cos\tilde{\theta} &= \tilde{z} \\
&= \frac{1-|f|^2}{1+|f|^2} \\
\sin\tilde{\theta} &= (\tilde{x}^2+\tilde{y}^2)^{1/2} \\
&= \frac{2|f|}{1+|f|^2} \\
\sin\tilde{\theta}\cos\tilde{\sigma} &= \tilde{x}
\end{aligned}
$$

Figure 2.1: North Pole Stereographic Projection

Figure 2.2: Spherical Coordinates on the First and Second Sphere

$$\begin{aligned}
&= \frac{Re(f)}{1+|f|^2} \\
\frac{2|f|}{1+|f|^2}\cos\tilde{\sigma} &= \frac{Re(f)}{1+|f|^2} \\
\cos\tilde{\sigma} &= \frac{Re(f)}{|f|} \\
\sin\tilde{\sigma} &= \frac{Im(f)}{|f|}
\end{aligned}$$

To summarize the relations between the points on the second sphere and the conformal map,

$$\cos\tilde{\theta} = \frac{1-|f|^2}{1+|f|^2} \tag{2.1}$$

$$\sin\tilde{\theta} = \frac{2|f|}{1+|f|^2} \tag{2.2}$$

$$\cos\tilde{\sigma} = \frac{Re(f)}{|f|} \tag{2.3}$$

$$\sin\tilde{\sigma} = \frac{Im(f)}{|f|} \tag{2.4}$$

The angles on the target sphere can now be found in terms of $|f|$, $Re(f)$ and $Im(f)$. The three quantities $|f|$, $Re(f)$ and $Im(f)$ can be put into the angles of the first sphere using the stereographic projection and then changed to spherical coordinates. We can do these two parts of the process after any step (as long as they are done in order) giving us the freedom to place them in the simplest spot. Often it is more convenient to do only the stereographic projection and then plug into formulas 2.1-2.4, saving the change into spherical coordinates for last.

Either way this process finds a formula for the angles of the target sphere in terms of the angles from the first sphere or in other words a map from S^2 into itself. However many conformal maps became much too complicated at the end of this step to complete the rest of the analysis.

2.2 Distance Preserving

Now that we have the map between the two spheres we need to find the distance preserving boundary. To do this generally we use the following method relating a differential length on each of the spheres. We used a differential method because we need to find if specific points are distance decreasing. We start with the method used by Gluck and Gu[1] comparing the distances of curves on the sphere using

$$s(t) = \int_0^t |\alpha\prime(t)|\, dt$$

which converts to the following by differentiating and using the first fundamental form,

$$\begin{aligned} \frac{ds}{dt} &= |\alpha\prime(t)| \\ \frac{ds}{dt} &= \sqrt{E\,\frac{du^2}{dt^2} + 2F\,\frac{du}{dt}\,\frac{dv}{dt} + G\,\frac{dv^2}{dt^2}} \\ ds^2 &= E\,du^2 + 2F\,du\,dv + G\,dv^2 \end{aligned}$$

For a unit sphere with the parameterization

$$\vec{x}_{(\theta,\sigma)} = (\sin\theta\cos\sigma, \sin\theta\cos\sigma, \cos\theta)$$

$E = 1$, $F = 0$ and $F = \sin^2\theta$. This gives the differential distance

$$\begin{aligned} ds^2 &= d\theta^2 + \sin^2\theta\; d\sigma^2 \\ d\tilde{s}^2 &= d\tilde{\theta}^2 + \sin^2\tilde{\theta}\; d\tilde{\sigma}^2 \end{aligned}$$

In order for the map to be distance decreasing the differential distances must be equal.

$$\begin{aligned} ds^2 &= d\tilde{s}^2 \\ d\theta^2 + \sin^2\theta\; d\sigma^2 &= d\tilde{\theta}^2 + \sin^2\tilde{\theta}\; d\tilde{\sigma}^2 \end{aligned}$$

We can get the forms for $\tilde{\theta}$ and $\tilde{\sigma}$ in terms of both θ and σ from formulas 2.1 - 2.4. Using those formulas and their forms after implicit differentiation we can plug in for the right side of the above equation giving

$$d\theta^2 + \sin^2\theta \; d\sigma^2 = g_{(\theta,\sigma)} \; d\theta^2 + h_{(\theta,\sigma)} \; d\sigma^2$$

where g and h depend on the map.

The coordinates θ and σ are independent so we can split the above into two equations

$$1 = g_{(\theta,\sigma)} \quad (2.5)$$

$$\sin^2\theta = h_{(\theta,\sigma)} \quad (2.6)$$

which can be simultaneously solved for the points at which the map is distance preserving. The differential length method agrees with answers given by an arc-length method, but also gives answers to maps which don't have azimuthal symmetry. This method gives ugly nonlinear equations which can often only be solved numerically. Numerical solutions make visualizing the flows even harder so most of the maps of average complexity stop at this point.

2.3 Visualization of the Flows

The final analysis of each map involves raising the distance-preserving points on the map from the spheres to their corresponding great circles on S^3 and then looking at how the great circle fibrations intersect. Unfortunately, visualizing in four dimensions requires the reduction to three dimensions to be understood while keeping the intersection information. I present the beginning of the method.

Finding the family of great circles (W_p) involves using the natural isomorphism relating the spaces of the map ($S^2 \times S^2$) and our target space G_2R^4. Following Gluck and Gu, we calculate the flows with the function

$$W_p = (\cos\alpha \; e_1' + \sin\alpha \; e_3') \wedge (\cos\beta \; e_2' + \sin\beta \; e_4') \quad (2.7)$$

where α, β and the primed axes are given by

$$\begin{aligned}
\alpha &= 1/2(\theta - \tilde{\theta}) \\
\beta &= 1/2(\theta + \tilde{\theta})
\end{aligned}$$

and the primed axes are rotated off the standard axes of R^4 by these angles.

$$\begin{aligned}
\mu &= 1/2(\tilde{\sigma} + \sigma) \\
\nu &= 1/2(\tilde{\sigma} - \sigma) \\
e_1' &= \cos\mu \; e_1 + \sin\mu \; e_2 \\
e_2' &= -\sin\mu \; e_1 + \cos\mu \; e_2 \\
e_3' &= \cos\nu \; e_3 + \sin\nu \; e_4 \\
e_4' &= -\sin\nu \; e_3 + \cos\nu \; e_4
\end{aligned}$$

Figure 2.3 is a picture of how this formula works in R^4.

Figure 2.3: Creation of a Great Circle

For the simple map (like the $f : W \to 1/2W$ from the introduction) in which the azimuthal angle on the first sphere maps to the same azimuthal angle on the target sphere for all points as example one the dependence of the azimuthal angle just parameterizes ellipses in the (e_1, e_2) and (e_3, e_4) planes the fibrations can be visualized as a torus. The breakdown then shows as a singularity of the torus. For any map without that special symmetry this method gets rid of important azimuthal angle dependence and fails to show the intersections.

Attempts were made at analytically solving for intersections amongst the fibrations. Also, we tried many numerical visualization schemes, but too much information is lost viewing the 4-dimensional flows on a computer screen.

Chapter 3

EXAMPLES

3.1 Reading the Examples

Each of the examples begin with a explanation of how far the process progressed on that map. After that the process will be worked out in the order of the method from the beginning to whenever the example became too complicated to complete. The first example includes a full explanation so the reader can get an understanding of the processes. The rest of the examples have few words in the interests of brevity.

Table 3.1 shows how far each of the examples is completed.

Conformal Map	Conversion	Distance-preserving	Visualization
$f : W \to 1/2W^{-1}$	X	X	X
$f : W \to cW$	X	X	
$f : W \to 1/2\frac{W+1}{W-1}$	X	X	
$f : W \to W^c$	X	X	
$f : W \to W^2$	X	X	
$f : W \to e^W$	X	Numerically	
$f : W \to (1-W)^{-1}$	X		
$f : W \to \tan W$	X		
$f : W \to \sin W$	X		

Table 3.1: Steps Completed for Each Example

3.2 $f : W \to 1/2W^{-1}$

The intersection of the flows for this map are very similar to $f : W \to 1/2W$. First, we must put the map into component form from the first complex plane (u, v).

$$\begin{aligned} W &= u + iv \\ 1/2W^{-1} &= 1/2\frac{u - iv}{u^2 + v^2} \\ Re(f) &= 1/2\frac{u}{u^2 + v^2} \\ Im(f) &= 1/2\frac{-v}{u^2 + v^2} \end{aligned}$$

We set up the map in terms of x, y and z on the first sphere using the following stereographic projection equations

$$\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases}$$

remembering that the new coordinates are on a sphere for simplification.

$$
\begin{aligned}
Re(f) &= 1/2\frac{u}{u^2+v^2} \\
&= 1/2\frac{x}{1+z}\frac{(1+z)^2}{x^2+y^2} \\
&= 1/2\frac{x}{1+z}\frac{(1+z)^2}{1-z^2} \\
&= 1/2\frac{x}{1-z} \\
Im(f) &= 1/2\frac{-y}{1-z} \\
|f|^2 &= Re^2(f)+Im^2(f) \\
&= 1/4\frac{1-z^2}{(1-z)^2} \\
&= 1/4\frac{1+z}{1-z}
\end{aligned}
$$

These equations could be converted to spherical coordinates now, however that only makes the calculations more complicated so that will be done later.

Now we find the angles of the target sphere in terms of the coordinates of the first using equations 2.2 and 2.3.

$$
\begin{aligned}
\sin\tilde{\theta} &= \frac{2|f|}{1+|f|^2} \\
&= 2\frac{(1+z)^{1/2}}{2(1-z)^{1/2}}\frac{4(1-z)}{5-3z} \\
&= 4\frac{(1+z)^{1/2}(1-z)^{1/2}}{5-3z} \\
&= 4\frac{(1-z^2)^{1/2}}{5-3z} \\
\sin\tilde{\sigma} &= \frac{Im(f)}{|f|} \\
&= \frac{-y}{1-z}\frac{(1-z)^{1/2}}{(1+z)^{1/2}} \\
&= \frac{-y}{(1-z^2)^{1/2}}
\end{aligned}
$$

With these formula the coordinates of the first sphere can be changed to spherical coordinates giving the complete map between the two spheres.

$$
\begin{aligned}
\sin\tilde{\theta} &= 4\frac{(1-\cos^2\theta)^{1/2}}{5-3\cos\theta} \\
&= 4\frac{\sin\theta}{3\cos\theta-5} \\
\sin\tilde{\sigma} &= \frac{-\sin\theta\sin\sigma}{\sin\theta} \\
&= \sin(-\sigma)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\cos\tilde{\theta} &= \frac{5\cos\theta-3}{3\cos\theta-5} \\
\cos\tilde{\sigma} &= \cos\sigma
\end{aligned}
$$

The map can be simplified to the following three equations.

$$
\cos\tilde{\theta} = \frac{5\cos\theta-3}{3\cos\theta-5} \quad (3.1)
$$

$$
\sin\tilde{\theta} = \frac{4\sin\theta}{3\cos\theta-5} \quad (3.2)
$$

$$
\tilde{\sigma} = -\sigma \quad (3.3)
$$

We now find where the map is distance-preserving following section 2.2. First, we implicitly differentiate 3.1 and 3.2

$$
\begin{aligned}
\cos\tilde{\theta}\, d\tilde{\theta} &= 4\frac{\cos\theta(3\cos\theta-5)+3\sin^2\theta}{(3\cos\theta-5)^2}\, d\theta \\
\frac{5\cos\theta-3}{3\cos\theta-5}\, d\tilde{\theta} &= 4\frac{3-\cos\theta}{3\cos\theta-5}\, d\theta \\
d\tilde{\theta} &= \frac{-4}{3\cos\theta-5} d\theta \\
d\tilde{\sigma} &= -d\sigma
\end{aligned}
$$

so we can find the formula for the distance-preserving map.

$$
\begin{aligned}
d\theta^2+\sin^2\theta\, d\sigma^2 &= d\tilde{\theta}^2+\sin^2\tilde{\theta} d\tilde{\sigma}^2 \\
&= (\frac{-4}{3\cos\theta-5})^2 d\theta^2 + (4\frac{\sin\theta}{3\cos\theta-5})^2 d\sigma^2
\end{aligned}
$$

Both independent parts reduce to the following equation

$$1 = (\frac{-4}{3\cos\theta - 5})^2$$

for which there is only one solution for $0 < \theta < 2\pi$.

$$\begin{aligned} \theta_0 &= \arccos 1/3 \\ &= 70.53^0 \end{aligned}$$

At this distance-preserving angle on the first sphere the corresponding angle on the target sphere comes from equations.

$$\begin{aligned} \sin\tilde{\theta}_0 &= \frac{4\sin\theta_0}{5-1} \\ &= \sin\theta_0 \\ \cos\tilde{\theta}_0 &= \frac{5/3 - 3}{1-5} \\ &= 1/3 \end{aligned}$$

So, the following nice relations hold that will help us find the flows (the second is just restated from above).

$$\begin{aligned} \theta_0 &= \tilde{\theta}_0 \\ \sigma &= -\sigma \end{aligned}$$

These formula for the map at the distance-preserving angles gives simple forms for equations 2.8-2.15

$$\begin{aligned} \alpha &= 1/2(\theta - \tilde{\theta}) \\ &= 0 \\ \beta &= 1/2(\theta + \tilde{\theta}) \\ &= \theta_0 \\ \mu &= 1/2(\tilde{\sigma} + \sigma) \end{aligned}$$

$$\begin{aligned}
&= 0 \\
\nu &= 1/2(\tilde{\sigma} - \sigma) \\
&= -\sigma \\
e_1' &= \cos\mu\, e_1 + \sin\mu\, e_2 \\
&= e_1 \\
e_2' &= -\sin\mu\, e_1 + \cos\mu\, e_2 \\
&= e_2 \\
e_3' &= \cos\nu\, e_3 + \sin\nu\, e_4 \\
&= \cos\sigma\, e_3 - \sin\sigma\, e_4 \\
e_4' &= -\sin\nu\, e_3 + \cos\nu\, e_4 \\
&= \sin\sigma\, e_3 + \cos\sigma\, e_4
\end{aligned}$$

The flows follow using equation (2.7).

$$\begin{aligned}
C &= (\cos\alpha\, e_1' + \sin\alpha\, e_3') \wedge (\cos\beta\, e_2' + \sin\beta\, e_4') \\
&= e_1 \wedge (\cos\theta_0 e_2 + \sin\theta_0(\sin\sigma\, e_3 + \cos\sigma\, e_4))
\end{aligned}$$

We parameterize the circles by ρ.

$$C_\rho = (\cos\rho, \sin\rho\cos\theta_0, \sin\rho\sin\theta_0\sin\sigma, \sin\rho\sin\theta_0\cos\sigma)$$

When $\rho = 0, \pi$ the flows all intersect at $(\pm 1, 0, 0, 0)$. The (e_1, e_2) plane forms an ellipse and the (e_3, e_4) plane forms a circle both parameterized by σ.

$$\begin{aligned}
x_1^2 + \frac{x_1^2}{\cos\theta_0} &= 1 \\
x_3^2 + x_4^2 &= \sin\rho\sin\theta_0
\end{aligned}$$

The Mathematica program in Appendix A gives the following visualizations for the flows.

Figure 3.1: Visualization for $f : W \rightarrow 1/2W^{-1}$

3.3 $f : W \rightarrow cW$

This map was a good test for the theory that reduces to the map $f : W \rightarrow 1/2W$ in the introduction.

$$\begin{aligned}
W &= u + iv \\
cW &= cu + icv \\
Re(f) &= cu \\
&= c\frac{x}{1+z} \\
Im(f) &= c\frac{y}{1+z} \\
|f|^2 &= c^2(u^2+v^2) \\
&= c^2\frac{x^2+y^2}{(1+z)^2} \\
&= c^2\frac{1-z}{1+z}
\end{aligned}$$

The map between the two spheres is

$$\begin{aligned}
\sin\tilde{\theta} &= \frac{2c\sin\theta}{1+c^2+(1-c^2)\cos\theta} \\
\cos\tilde{\theta} &= \frac{(1-c^2)+(1+c^2)\cos\theta}{(1+c^2)+(1+c^2)\cos\theta} \\
\tilde{\sigma} &= \sigma
\end{aligned}$$

which reduces to the example in the introduction when $c = 1/2$.

We differentiate to find where the map is distance-preserving.

$$\begin{aligned}
d\tilde{\theta} &= \frac{2c}{(1+c^2)+(1-c^2)\cos\theta}d\theta \\
d\tilde{\sigma} &= d\sigma \\
d\theta^2 + \sin^2\theta\, d\sigma^2 &= d\tilde{\theta}^2 + \sin^2\tilde{\theta}\, d\tilde{\sigma}^2 \\
&= [\frac{2c}{(1+c^2)+(1-c^2)\cos\theta}]^2 d\theta^2 \\
&\quad +[\frac{2c\sin\theta}{(1+c^2)+(1-c^2)\cos\theta}]^2 d\sigma^2
\end{aligned}$$

Again, the distance-preserving formula is independent of σ and gives the following circles on the first sphere.

$$\cos\theta = \begin{cases} \frac{c+1}{c-1} & \text{if } c < 0 \\ \frac{c-1}{c+1} & \text{if } c > 0 \end{cases}$$

3.4 $f : W \to W^c$

This map is an interesting generalization that gives a distance-preserving equation.

$$\begin{aligned} Re(f) &= (\frac{1-z}{1+z})^c \cos(c \arctan(y/z)) \\ Im(f) &= (\frac{1-z}{1+z})^c \sin(c \arctan(y/z)) \\ |f| &= (\frac{1-z}{1+z})^c \end{aligned}$$

Which gives the map between the two spheres

$$\sin\tilde{\theta} = \frac{2\sin^c\theta}{(1+\cos\theta)^c + (1-\cos\theta)^{-c}} \tag{3.4}$$

$$\tilde{\sigma} = c\sigma \tag{3.5}$$

and the distance-preserving equation

$$2c\sin^{c-1}\theta = (1+\cos\theta)^c + (1-\cos\theta)^{-c} \tag{3.6}$$

3.5 $f : W \to W^2$

This specific case of the above map has fairly simple flow equations, but our method for visualization does not work. Starting with 3.4 and 3.5 we get the map between the spheres

$$\begin{aligned} \sin\tilde{\theta} &= \frac{2\sin^2\theta}{(1+\cos\theta)^2 + (1-\cos\theta)^{-2}} \\ &= \frac{\sin^2\theta}{1+\cos^2\theta} \\ \cos\tilde{\theta} &= \frac{2\cos\theta}{1+\cos^2\theta} \\ \tilde{\sigma} &= 2\sigma \end{aligned}$$

From equation 3.6 the distance-preserving equation is

$$4\sin\theta \;=\; 1+\cos^2\theta$$

which gives the distance-preserving angles.

$$\begin{aligned}\sin\theta_0 &= \sqrt{3}-1\\ \sin\tilde{\theta}_0 &= \frac{\sqrt{3}-1}{2}\end{aligned}$$

For the visualization only the μ and ν work well for this map.

$$\begin{aligned}\mu &= 1/2(\tilde{\sigma}+\sigma)\\ &= 3/2\sigma\\ \nu &= 1/2(\tilde{\sigma}-\sigma)\\ &= 1/2\sigma\\ e_1' &= \cos 3/2\sigma\ e_1+\sin 3/2\sigma\ e_2\\ e_2' &= -\sin 3/2\sigma\ e_1+\cos 3/2\sigma\ e_2\\ e_3' &= \cos 1/2\sigma\ e_3+\sin 1/2\sigma\ e_4\\ e_4' &= -\sin 1/2\sigma\ e_3+\cos 1/2\sigma\ e_4\end{aligned}$$

We parameterize the great circles by ρ and get these flows.

$$\begin{aligned}C_\rho = &(\cos\alpha\cos\rho\cos 3/2\sigma-\cos\beta\sin\rho\sin 3/2\sigma,\\ &\cos\alpha\cos\rho\sin 3/2\sigma+\cos\beta\sin\rho\cos 3/2\sigma,\\ &\sin\alpha\cos\rho\cos 1/2\sigma-\sin\beta\sin\rho\sin 1/2\sigma,\\ &\sin\alpha\cos\rho\sin 1/2\sigma+\sin\beta\sin\rho\cos 1/2\sigma)\end{aligned}$$

We were not able to solve for the intersection of these flows.

3.6 $f : W \rightarrow 1/2\frac{W+1}{W-1}$

This example is still under construction because I just got an idea.

$$\begin{aligned} Re(f) &= 1/2\frac{z}{x-1} \\ Im(f) &= 1/2\frac{y}{x-1} \\ |f|^2 &= Re^2(f) + Im^2(f) \\ &= \frac{z^2+y^2}{4(x-1)^2} \\ &= 1/4\frac{1+x}{1-x} \end{aligned}$$

Completing the preparation gives the following map between the two spheres.

$$\cos\tilde{\sigma} = -\frac{\cos\theta}{(1-\sin^2\theta\cos^2\sigma)^{1/2}} \tag{3.7}$$

$$\sin\tilde{\sigma} = -\frac{\sin\theta\sin\sigma}{(1-\sin^2\theta\cos^2\sigma)^{1/2}} \tag{3.8}$$

$$\sin\tilde{\theta} = 4\frac{(1-\sin^2\theta\cos^2\sigma)^{1/2}}{5-3\sin\theta\cos\sigma} \tag{3.9}$$

$$\cos\tilde{\theta} = \frac{3-5\sin\theta\cos\sigma}{5-3\sin\theta\cos\sigma} \tag{3.10}$$

The following two quantities are introduced to make the calculations easier to follow

$$\begin{aligned} \zeta &= 1-\sin^2\theta\cos^2\sigma \\ \eta &= 5-3\sin\theta\cos\sigma \end{aligned}$$

We differentiate to find where the map is distance preserving.

$$\begin{aligned} d\tilde{\theta} &= 4\frac{\cos\theta\cos\sigma}{\eta\sqrt{\zeta}}d\theta - 4\frac{\sin\theta\sin\sigma}{\eta\sqrt{\zeta}}d\sigma \\ d\tilde{\sigma} &= -\frac{\sin\sigma}{\zeta}d\theta - \frac{\sin\theta\cos\theta\cos\sigma}{\zeta}d\sigma \end{aligned}$$

The cross terms ($d\theta\ d\sigma$) cancel nicely in the following equation giving the distance-preserving formula

$$\begin{aligned} d\theta^2 + \sin^2\theta\ d\sigma^2 &= d\tilde{\theta}^2 + \sin^2\tilde{\theta}\ d\tilde{\sigma}^2 \\ &= 16\frac{\cos^2\theta\cos^2\sigma+\sin^2\sigma}{\zeta\eta^2}d\theta^2 \\ &\quad +16\frac{\sin^2\theta\sin^2\sigma+\sin^2\theta\cos^2\theta\cos^2\sigma}{\zeta\eta^2}d\sigma^2 \end{aligned}$$

Both of the independent parts of that equation give the following equation to solve

$$\begin{aligned} 16(\cos^2\theta\cos^2\sigma + \sin^2\sigma) &= \zeta\eta^2 \\ 16\zeta &= \zeta\eta^2 \end{aligned}$$

Converting to Cartesian coordinates $\zeta = 1 - x^2$ which only equals zero at the two points on the sphere where

$$x = \pm 1$$

These values when plugged into the equations for the map between the spheres give

$$\begin{aligned} \sin\tilde{\theta} &= 0 \\ \cos\tilde{\theta} &= \mp 1 \end{aligned}$$

which are the top and bottom points on the target sphere. The map does not include these points so we continue. When the map is not at those points the distance-preserving equation reduces to

$$\begin{aligned} 16 &= \eta^2 \\ 16 &= 9\sin^2\theta\cos^2\sigma - 30\sin\theta\cos\sigma + 25 \\ 0 &= 3x^2 - 10x + 3 \end{aligned}$$

Which gives only one solution in the valid range for x $(-1 \leq x \leq 1)$

$$(\sin\theta\cos\sigma =)\ x = 1/3$$

which is a circle at a constant x on the first sphere.

We simplify the process of finding the flows at these points by finding the points on the target sphere where the map is distance-preserving using 3.4-3.7

$$\begin{aligned} \sin\tilde{\theta} &= 2\sqrt{2}/3 \\ \cos\tilde{\theta} &= 1/3 \\ \sin\tilde{\sigma} &= \frac{-\sqrt{2}}{4}(9\sin^2\theta - 1)^{1/2} \\ \cos\tilde{\sigma} &= \frac{-3\sqrt{2}}{4}\cos\theta \end{aligned}$$

The flow equations from here on out get very complicated.

3.7 $f : W \to e^W$

The formula for where this map is distance-decreasing is too complicated for an analytical solution. Therefore the flows cannot be found.

$$\begin{aligned} W &= u + iv \\ e^W &= e^u(\cos v + i \sin v) \\ Re(f) &= e^u \cos v \\ Im(f) &= e^u \sin v \\ |f| &= e^u \end{aligned}$$

The map becomes

$$\begin{aligned} \sin \tilde{\theta} &= 2\frac{e^\zeta}{e^{2\zeta} + 1} \\ \tilde{\sigma} &= \frac{\sin\theta \sin\sigma}{1 + \cos\theta} \end{aligned}$$

where ζ has the value

$$\begin{aligned} \zeta &= \frac{x}{1+z} \\ &= \frac{\sin\theta\cos\sigma}{1+\cos\theta} \end{aligned}$$

The complicated distance-preserving formula

$$\begin{aligned} d\tilde{\theta} &= \frac{2e^\zeta}{e^{2\zeta}+1}\left(\frac{\cos\sigma}{1+\cos\theta}d\theta - \frac{\sin\theta\sin\sigma}{1+\cos\theta}d\sigma\right) \\ d\tilde{\sigma} &= \frac{\sin\theta\cos\sigma}{1+\cos\theta}d\theta + \frac{\sin\sigma}{1+\cos\theta}d\sigma \end{aligned}$$

reduce to this formula to find the distance-preserving angle.

$$(e^{2\zeta} + 1)^2(1 + \cos\theta)^2 = 4e^{2\zeta} \tag{3.11}$$

This equation has no analytic solutions, but using the Maple program in Appendix C we can get a look at where in the θ-σ space (represented by t and s respectively) the map is distance-preserving.

Figure 3.2: Distance-Preserving Points for $f : W \to e^W$

This elliptical object has no analytical form that can be plugged into the fibration equations to do the visualization.

3.8 $f : W \to \sin W$

This map is very similar to $f : W \to e^W$. It has only been completed to the map between the two spheres.

$$\begin{aligned} W &= u + iv \\ \sin W &= \sin u \cosh v + i \cos u \sinh v \end{aligned}$$

$$\begin{aligned} Re(f) &= \sin u \cosh v \\ Im(f) &= \cos u \sinh v \\ |f|^2 &= \sin^2 u \cosh^2 v + \cos^2 u \sinh^2 v \\ &= \sin^2 u + \sinh^2 v \end{aligned}$$

The map between the two spheres is

$$\begin{aligned} \sin\tilde{\theta} &= 2\frac{(\cos^2\zeta + \cosh^2\eta)^{1/2}}{\sin^2 u + \cosh^2\eta} \\ \cos\tilde{\sigma} &= \frac{\cos\zeta \sinh\eta}{(\cos^2\zeta + \cosh^2\eta)^{1/2}} \end{aligned}$$

where ζ and η are

$$\begin{aligned} \zeta &= \frac{x}{1+z} \\ &= \frac{\sin\theta\cos\sigma}{1+\cos\theta} \\ \eta &= \frac{y}{1+z} \\ &= \frac{\sin\theta\sin\sigma}{1+\cos\theta} \end{aligned}$$

This formula looks like the $f : W \to e^W$ formula and poses many of the same problems.

3.9 $f : W \to (1 - W)^{-1}$

Again, we completed this map only to the map between the spheres.

$$W = u + iv$$

$$\begin{aligned}(1-W)^{-1} &= \frac{1-u+iv}{u^2+v^2-2u+1}\\ Re(f) &= \frac{1+z-x}{2(1+x)}\\ Im(f) &= \frac{y}{2(1+x)}\\ |f|^2 &= \frac{1+z}{2(1+x)}\end{aligned}$$

The corresponding map between the two spheres is the following.

$$\begin{aligned}\sin\tilde{\theta} &= 2\frac{\sqrt{2}(\cos\theta-\sin\theta\cos\sigma+1)^{1/2}}{\cos\theta-2\sin\theta\cos\sigma+3}\\ \cos\tilde{\sigma} &= (\frac{\cos\theta-\sin\theta\cos\sigma+1}{2})^{1/2}\end{aligned}$$

3.10 $f : W \to \tan W$

This map has been converted to a map between the two spheres. The map starts with

$$\begin{aligned}W &= u+iv\\ \tan W &= \frac{\sin\zeta\cos\zeta-i\cosh\eta\sinh\eta}{\sin^2\zeta+\sinh^2\eta}\\ Re(f) &= \frac{\sin\zeta\cos\zeta}{\sin^2\zeta+\sinh^2\eta}\\ Im(f) &= \frac{\cosh\eta\sinh\eta}{\sin^2\zeta+\sinh^2\eta}\\ |f|^2 &= \frac{\sin^2\zeta\cos^2\zeta+\cosh^2\eta\sinh^2\eta}{(\sin^2\zeta+\sinh^2\eta)^2}\end{aligned}$$

where

$$\begin{aligned}\zeta &= \frac{x}{1+z}\\ &= \frac{\sin\theta\cos\sigma}{1+\cos\theta}\\ \eta &= \frac{y}{1+z}\\ &= \frac{\sin\theta\sin\sigma}{1+\cos\theta}\end{aligned}$$

The corresponding map between the two spheres is the following.

$$
\begin{aligned}
\sin\tilde{\theta} &= \frac{-2}{2\cosh^2\eta}\sqrt{(\cos^2\zeta - \cosh^2\eta)(\sin^2\zeta - \cosh^2\eta)} \\
\cos\tilde{\sigma} &= \frac{\sin\zeta\cos\zeta\sqrt{\sin^2\zeta - \cosh^2\eta}}{\sqrt{\cos^2\zeta - \cosh^2\eta}}
\end{aligned}
$$

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.1 Appendix A: Mathematica Visualization Programs

.1.1 Visualization Program for $f : W \to 1/2W$

This Mathematica code plots the flows of $f : W \to 1/2W$ onto a torus.

```
Do[
t1 = ArcCos[(3+5Cos[t])/(5+3Cos[t])];
a1 = (t-t1)/2;
b1 = (t+t1)/2;
a = Sin[a1];
b = Sin[b1];
Rt[a1_,b1_][p_] := Cos[a1]^2 Cos[p]^2 + Cos[b1]^2 Sin[p]^2;
r[x_][a1_,b1_][p_] := x + Rt[a1,b1][p];
surf[p,p1] := {Cos[p] (r[a][a1,b1][p] + Rt[a1,b1][p] Cos[p1]),
 (r[b][a1,b1][p] + Rt[a1,b1][p] Cos[p1]) Sin[p], Rt[a1,b1][p] Sin[p1]};

ParametricPlot3D[surf[p,p1] // Evaluate, {p, 0, Pi}, {p1, 0, 2Pi},
 PlotPoints -> {20,40}, PlotRange -> {{-2.5,2.5}, {-2.5,2.5}, {-1,1}},
 PlotLabel -> ""];,
{t, 0, ArcCos[-1/3],ArcCos[-1/3]/6 }];
```

.1.2 Visualization Program for $f : W \to 1/2W^{-1}$

This Mathematica code plots the flows of $f : W \to 1/2W^{-1}$ onto a torus.

```
Do[
t1 = ArcSin[(4Sin[t])/(3Cos[t]-5)];
a1 = (t-t1)/2;
b1 = (t+t1)/2;
a = Cos[a1];
```

```
b = Cos[b1];

Rt[a1_,b1_][p_] := (Sin[a1]^2 Cos[p]^2 + Sin[b1]^2 Sin[p]^2)^(1/2);
r[x_][a1_,b1_][p_] := x + Rt[a1,b1][p];
surf[p,p1] := {Cos[p] (r[a][a1,b1][p] + Rt[a1,b1][p] Cos[p1]),
 (r[b][a1,b1][p] + Rt[a1,b1][p] Cos[p1]) Sin[p], Rt[a1,b1][p] Sin[p1]};

ParametricPlot3D[surf[p,p1] // Evaluate, {p, 0, Pi}, {p1, 0, 2Pi},
 PlotPoints -> {20,40}, PlotRange -> {{-2.5,2.5}, {-2.5,2.5}, {-1,1}},
 PlotLabel -> ""];,
{t, 0, ArcCos[1/3],ArcCos[1/3]/6 }];
```

.2 *Appendix B: Finding the Distance-Preserving Points of* $f : W \to e^W$

This Maple code finds the Distance-Preserving Points of $f : W \to e^W$.

```
y := sin(t)*sin(s)/(1+cos(t));
z := sin(t)*cos(s)/(1+cos(t));

plot3d(2*exp(z)/(1+exp(2*z))*(z+y)/sin(t),t=0..Pi,s=0..2*Pi,view=.99..1.01);
```