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An Incidence Approach to the Distinct Distances Problem

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Abstract

In 1946, Erdős posed the distinct distances problem, which asks for the minimum number of distinct distances that any set of n points in the real plane must realize. Erdős showed that any point set must realize at least $\Omega(n^{1/2})$ distances, but could only provide a construction which offered $\Omega(n/\sqrt{\log n})$ distances. He conjectured that the actual minimum number of distances was $\Omega(n^{1-\epsilon})$ fo any $\epsilon > 0$, but that sublinear constructions were possible. This lower bound has been improved over the years, but Erdős' conjecture seemed to hold until in 2010 Larry Guth and Nets Hawk Katz used an incidence theory approach to show any point set must realize at least $\Omega(n/\log n)$ distances. In this thesis we will explore how incidence theory played a roll in this process and expand upon recent work by Adam Sheffer and Cosmin Pohoata, using geometric incidences to achieve bounds on the bipartite variant of this problem. A consequence of our extensions on their work is that the theoretical upper bound on the original distinct distances problem of $\Omega(n/\sqrt{\log n})$ holds for any point set which is structured such that half of the n points lies on an algebraic curve of arbitrary degree.

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Chapter 1

Introduction

In 1946, Paul Erdős proposed the distinct distance problem by asking for the minimum number of different distances that any planar subset of n points may realize, ie

$$\min_{|\mathcal{P}|=n} \left|\left\{|pq| \text{ s.t. } p, q \in \mathcal{P} \text{ and } \mathcal{P} \subset \mathbb{R}^2\right\}\right|$$

where $|ab|$ is the Euclidean distance between a and b. Upon posing the problem Erdős showed that there existed a construction with $\Theta(n/\sqrt{\log n})$ (shown in Figure 1.1) distances by considering a cartesian grid with points located on integer combinations (x, y), which implies that $f(n) = O(n/\sqrt{\log n})$. Erdős conjectured that $f(n) = \Omega(n^{1-\epsilon})$ for any $\epsilon > 0$.

Although there has been no progress in creating a construction which contains less distances, the lower bound on $f(n)$ has been worked on and

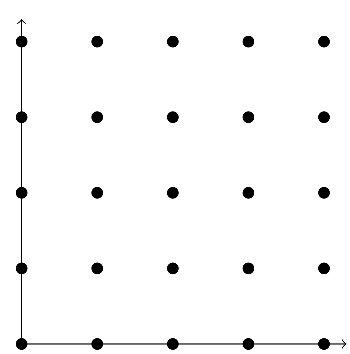

Figure 1.1 Erdős's construction of integer lattice points

Bound	Contributor
$\Omega(n^{1/2})$	Erdös (1946)
$\Omega(n^{2/3})$	Moser (1952)
$\Omega(n^{5/7})$	Chung (1984)
$\Omega(n^{4/5})$	Székely (1997)
$\Omega(n^{6/7})$	Solymosi and Tóth (2001)
$\Omega(n/\log n)$	Guth and Katz (2015)

Table 1.1 Selected Progression of the Lower Bound on $f(n)$

steadily increased over the years. When Erdős introduced the problem he also introduced a lower bound $f(n) = \Omega(n^{1/2})$. He showed this by considering a point p that acted as a vertex of the convex hull of $\mathcal{P}$ and considered the circle centered at p which contained the most points of $\mathcal{P}$ (say N points). He noted that by choosing one point on this circle he could show that there was $N - 1$ unique distances in the point set, while also there must be $n - 1/N$ different distances from points to p (as N is the most common distance from p). By maximizing the minimum of these two, Erdős found his lower bound.

The first major jump on this bound was made by Leo Moser six years later. Moser split his point set along the vector which represented the shortest distance in $\mathcal{P}$ and considered only the more populated side of this line. He then divided this space into semi-annuals of unit thickness centered at the midpoint of the shortest distance and considered the two cases of one of these divisions containing at least s points compared to none of the divisions containing s points. In each of these cases, Moser used geometric arguments to come up with a lower bound on $f(n)$ in terms of n and s. Knowing that one of these two cases must always hold, Moser was then able to create the bound $f(n) = \Omega(n^{2/3})$ by choosing a value for s which maximized the minimum of these two cases.

Both Erdős and Moser made explicit formulas for lower bounds of the number of points which I translated into asymptotic bounds. The next major bound increase was found by Chung 30 years later by forgoing an explicit bound and utilizing properties of asymptotics. After developing an extensive set of lemmas about distances using geometric arguments, Chung considered annular divisions of a $1°$ arc to develop his bound.

Over 10 years later Székeley increased the bound again using the first argument whose basis was not geometric. By creating a multigraph which maintained information about the pairwise distances in $\mathcal{P}$, Székeley was able to show that at least one point in $\mathcal{P}$ must realize $\Omega(n^{4/5})$ distinct distances by analyzing the crossing number of this multigraph. His starting point for this argument was Lemma 2.1.1, which we will use in the next chapter to explore methods used in proofs which rely on geometric incidences. A few years later Solymosi and Tóth expanded on Székeley's multigraph proof structure by adding in point-line incidence arguments to achieve their bound.

Erdős's conjecture was surpassed in 2015 when Larry Guth and Nets Hawk Katz were able to show that any set of n points in the plane must realize $\Omega(n/\log n)$ distances. Guth and Katz's result was made possible by clever use of incidence techniques which they manipulated in a way that they called the polynomial method. In the next chapter I will dive into what geometric incidences are, how they are applied within discrete geometry, and show a relatively simple example of the polynomial method.

Chapter 2

Incidence Theory

2.1 Using Geometric Incidences in Discrete Geometry

A **geometric incidence** between a point set $\mathcal{P}$ and a set of other geometric objects $\mathcal{O}$ occurs when a point $p \in \mathcal{P}$ lies on one of the objects $o \in \mathcal{O}$. We denote the number of incidences between the elements of $\mathcal{P}$ and $\mathcal{O}$ as $I(\mathcal{P}, \mathcal{O})$. For instance, if $\mathcal{P}$ is the set of four points and $\mathcal{L}$ is the set of four lines in Figure 2.1, then $I(\mathcal{P}, \mathcal{L}) = 9$.

Finding $I(\mathcal{P}, \mathcal{O})$ can be difficult, especially when the sets $\mathcal{P}$ and $\mathcal{O}$ become large, so we often calculate $I(\mathcal{P}, \mathcal{O})$ by summing over one of the involved sets, ie

$$I(\mathcal{P}, \mathcal{O}) = \sum_{p \in \mathcal{P}} I(p, \mathcal{O}) = \sum_{o \in \mathcal{O}} I(\mathcal{P}, o).$$

This technique becomes especially useful when iteratively stengthening

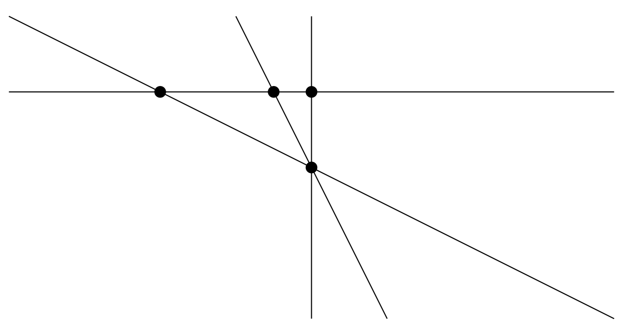

Figure 2.1 A demonstration of point-line incidences

bounds using the polynomial method.

In order to solve a problem in discrete geometry with incidences we encapsulate the problem in a general incidence problem. After this, we pick some quantity q and bound it from below using the number of incidences between our sets ($L(I(\mathcal{P}, \mathcal{O})) \leq q$). After this we bound q from above using the size of our point set $m = |\mathcal{P}|$ and object set $n = |\mathcal{O}|$, so that $q \leq U(m, n)$. As $L(I(\mathcal{P}, \mathcal{O})) \leq q \leq U(m, n)$, we can bound the number of incidences, $I(\mathcal{P}, \mathcal{O}) \leq L^{-1}(U(m, n))$. This second step is often referred to as "double counting".

Understanding how to effectively use this proof technique can be rather difficult, so we will illustrate it with an example. In 1946, Erdös posed the unit distance problem, which asks for the maximum number of distinct unit distances ($u(n)$) that a set of n points can realize in $\mathbb{R}^2$. For instance, $u(3) = 3$ as we can arrange the points to be the vertices of an equilateral triangle of unit side length. On the other hand, $u(4) = 5$ as we can form the vertices of a rhombus with side length 1 and one diagonal of length 1, but we cannot arrange the four points such that all 6 pair-wise distances are 1. When he posed the problem, Erdös (1946) proved an initial bound of $u(n) = O(n^{3/2})$. Since then, the bound has only been improved once, and was done so using the incidence technique described above.

Theorem 2.1.1 (Spencer et al. (1984))**.** *Let $u(n)$ be the maximum number of pair-wise unit distances a set of n points can realize in $\mathbb{R}^2$. Then $u(n) = O(n^{4/3})$*

Proof. Consider any configuration of n points and draw a unit circle around each one. For clarity, let's call our point set $\mathcal{P} = \{p_1, \ldots, p_n\}$ and our unit circle set $C = \{c_1, \ldots, c_n\}$ where c_i is the unit circle centered at p_i. Notice that a unit distance between points p_i and p_j realizes two incidences between $\mathcal{P}$ and C, (p_i, c_j) and (p_j, c_i). For instance in Figure 2.2 we have 3 points realizing 3 pair-wise unit distances and $I(\mathcal{P}, C) = 6$. Therefore, we can say that for this specific configuration of n points and n unit circles $u(n) \leq I(\mathcal{P}, C)$.

This configuration is just one of many for n points and n unit circles. Therefore, if we can bound $I(\mathcal{P}, C)$ for any set of n points $\mathcal{P}$ and any set of n unit circles C, that bound will also apply to $u(n)$.

Index $C = \{c_1, \ldots, c_n\}$ and let m_i be the number of points of $\mathcal{P}$ incident to c_i. Notice that $\sum_{i=1}^{n} m_i = I(\mathcal{P}, C)$ as we are counting the incidences circle by circle. Next, remove any circle from C if $m_i \leq 2$. We can ignore these circles as they will contribute at most $2n$ incidences and we are attempting to prove that $I(\mathcal{P}, C) = O(n^{4/3})$.

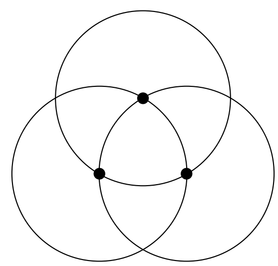

Figure 2.2 A set of 3 points a unit distance apart with a unit circle around each

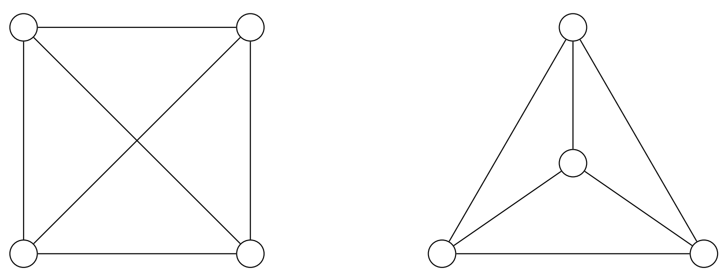

Figure 2.3 2 drawing of K_4 with each with a different number of crossings

Create a graph G such that every point in $\mathcal{P}$ has a corresponding vertex in our graph G. Let an edge exist between two vertices if their corresponding points are consecutive on at least one circle. We will double count the crossing number of this graph, cr(G).

In graph theory, a graph G can be drawn in an infinite number of ways, which may all look completely different. The crossing number of a graph G (cr(G)) is the minimum number of edge intersections required to draw G. For instance, cr(K_4) = 0 as there is a drawing of it with no crossings (as shown in Figure 2.3). On the other hand, cr(K_5) = 1 as every drawing of K_5 involves at least one intersection of edges and there exists a drawing with exactly 1 crossing.

Lemma 2.1.1 (The Crossing Lemma: Ajtai et al. (1982))**.** *Let G be a graph with e edges and v vertices. If $e \geq 4v$, then*

$$cr(G) = \Omega\left(\frac{e^3}{v^2}\right).$$

Note that $|V(G)| = n$. Recall that we placed an edge between consecutive points on a circle so each circle c_i corresponds to m_i edges of our graph.

At the same time, every edge is created by at most two unit circles, so we can say $|E(G)| \geq \sum_{i=1}^{n} m_i/2 = I(\mathcal{P}, \mathcal{C})/2$. In order to use the crossing lemma, it is sufficient to show $|E(G)| \geq 4|V(G)|$. Notice that if $|E(G)| < 4|V(G)|$ then we have that $I(\mathcal{P}, \mathcal{C})/2 \leq |E(G)| < 4n$, giving us the tighter bound of $I(\mathcal{P}, \mathcal{C}) = O(n)$. Now if $|E(G)| \geq 4|V(G)|$, the crossing lemma gives

$$\mathrm{cr}(G) = \Omega\left(\frac{I(\mathcal{P}, \mathcal{C})^3}{n^2}\right).$$

Draw G such that each vertex is on its corresponding point in $\mathcal{P}$. The circle arcs between points represent the edges. Though there may be some double edges between vertices, yet these could only contibute additional crossings, so any upper bound on the crossing number for this representation will still be valid for the graph without the double edges. In this drawing of our graph, two edges cross only if their corresponding circle arcs intersect. As we have n circles, and two circles have at most two intersections we know that the crossing number is at most $2\binom{n}{2}$, so

$$\mathrm{cr}(G) = O(n^2).$$

From these two bounds, we know that there exist constants $c_1 > 0$ and $c_2 > 0$ such that,

$$c_1 \frac{I(\mathcal{P}, \mathcal{C})^3}{n^2} \leq \mathrm{cr}(G) \leq c_2 n^2.$$

We can then rearrange this inequality to get the desired bound for $I(\mathcal{P}, \mathcal{C})$

$$I(\mathcal{P}, \mathcal{C}) = O(n^{4/3}).$$

■

2.2 Problem Reduction to Incidences

Geometric incidences can also bound problems outside of discrete geometry. Consider the following theorem:

Theorem 2.2.1 (Farber et al. (2014))**.** *Let M be an $n \times 2$ totally positive matrix. The number of 2×2 minors of M that are equal to 1 is $O(n^{4/3})$.*

A minor of M is the determinant of a matrix formed by taking a subset of x rows and x columns of M. All minors of a totally positive matrix are positive.

Proof. Consider the matrix

$$M = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}.$$

Each row represents a single point in $\mathbb{R}^2$. Let these points define the point set $\mathcal{P}$. In order for a minor to equal 1 we need the points corresponding to rows i and j to satisfy

$$x_i y_j - x_j y_i = 1.$$

Therefore for each row i we can create the following, implicitly defined line

$$x_i * y - y_i * x = 1.$$

The row corresponding to a fixed point in $\mathcal{P}$ would form a minor of value 1 with row i. If we create this line for every one of our rows we have a set $\mathcal{L}$ of n lines.

Notice that if rows i and j ($i < j$) form a minor equal to 1 then we create an incidence between $\mathcal{P}$ and $\mathcal{L}$ where the point associated to row j lies on the line associated to row i. Note that this matrix being totally positive prevents us from getting incidences which don't correspond to a minor which was not 1 (if row i and row j such that $i < j$ has a minor of -1, then the point associated to row i would be incident to the line associated to row j). Therefore we have a one to one correspondence between $I(\mathcal{P}, \mathcal{L})$ and 2×2 minors equal to 1.

Now we need to show that for a set of n points $\mathcal{P}$ and a set of n lines $\mathcal{L}$ in $\mathbb{R}^2$ that $I(\mathcal{P}, \mathcal{L}) = O(n^{4/3})$. This is provided by the following theorem.

Theorem 2.2.2 (Szemerédi and Trotter (1983))**.** *Let $\mathcal{P}$ be a set of m points and $\mathcal{L}$ a set of n lines in $\mathbb{R}^2$. Then $I(\mathcal{P}, \mathcal{L}) = O(m^{2/3}n^{2/3} + m + n)$.*

When we plug in $m = n$, we find that $I(\mathcal{P}, \mathcal{L}) = O(n^{4/3} + 2n) = O(n^{4/3})$. Therefore, as each incidence corresponds to a minor equal to 1, the number of 2×2 minors of M that are equal to 1 is $O(n^{4/3})$.

∎

2.3 Algebraic Geometry

We will take a quick moment to introduce a number of concepts from algebraic geometry required to understand the polynomial method.

A real polynomial ring $\mathbb{R}[x_1, \ldots, x_d]$ is the ring of polynomials on the variables $\{x_1, \ldots, x_d\}$ with real-valued coefficients. As the work we do will remain limited to the real plane, we will primarily focus on the polynomial ring $\mathbb{R}[x, y]$, which can have polynomials such as

$$y^3 + 2x^2 + \frac{3}{4}xy^2 - \pi xy + 2.$$

The affine variety of a polynomial f is the set of points P such that for $p \in P$, $f(p) = 0$. We denote the affine variety of f as $\mathbf{V}(f)$. For instance $\mathbf{V}(x^2 + y^2 - 1)$ is the unit circle centered at the origin. The only varieties we will use are affine varieties, so we will simple call them varieties from here on.

Often time varieties of a single polynomial consist of multiple connected components. For instance, let

$$f = x^3 + xy^2 + x^2y - y^3 - 5x^2 + 5y^2 - x + y + 5.$$

$\mathbf{V}(f)$ consists of the unit circle and the line $y = x - 5$. One way to see this is by factoring f:

$$f = (x^2 + y^2 - 1)(x - y - 5),$$

so f will equal 0 if and only if at least one of its factors equals 0. Therefore,

$$\mathbf{V}(f) = \mathbf{V}(x^2 + y^2 - 1) \cup \mathbf{V}(x - y - 5).$$

We say that these two varieties are subvarieties of $V(f)$. A variety V is reducible if it has two proper subvarieties S and T such that $V = S \cup T$. If no such proper subvarities exist, V is said to be irreducible. An irreducible variety is a curve in $\mathbb{R}^2$ if it is not a single point, the null set, or all of $\mathbb{R}^2$. Similarly, a reducible subvariety is a curve if all of its irreducible components are curves. The degree of a curve γ is the minimum integer k such that some polynomial f of degree k satisfies $\mathbf{V}(f) = \gamma$.

With this background we can introduce some theorems that will help us in our discussion of the polynomial method.

Theorem 2.3.1 (Bezout)**.** *If f and g are polynomials of degree k_f and k_g in $\mathbb{R}[x, y]$ which have no common factors, then $\mathbf{V}(f)$ and $\mathbf{V}(g)$ intersect in at most $k_f * k_g$ points.*

Theorem 2.3.2 (Harnack)**.** *If $f \in \mathbb{R}[x, y]$ is a degree k polynomial, then $\mathbf{V}(f)$ has $O(k^2)$ connected components in $\mathbb{R}^2$.*

In this case a connected component is a geometrically separate piece of the variety. For instance the polynomial $V(f)$ for the polynomial f discussed above has 2 connected components, the line and the circle.

Theorem 2.3.3 (Warren (1968))**.** *If f is a degree k polynomial on $\mathbb{R}[x_1, \ldots, x_d]$, then $\mathbf{V}(f)$ partitions $\mathbb{R}^d$ into $O(k^d)$ pieces.*

As we will show in the next section, the combination of these theorems is incredibly powerful.

2.4 The Polynomial Method

The polynomial method is a way of strengthening combinatorial incidence bounds by partitioning $\mathbb{R}^k$ and applying a known incidence bound in each section individually. It's power hinges on the following theorem,

Theorem 2.4.1 (Guth and Katz (2015))**.** *If $\mathcal{P}$ is a set of m points in $\mathbb{R}^d$, then for each r such that $1 < r \leq m$, there exists an r-partitioning polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ of degree $O(r)$.*

Here, an r-partitioning polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ on a set of m points partitions $\mathbb{R}^d$ with its variety $\mathbf{V}(f)$ in such a way that no connected component of $\mathbb{R}^d$ contains more than $\frac{m}{r^d}$ points. For instance a 2-partitioning polynomial on 12 points in $\mathbb{R}^2$ would break up the plane in such a way that no more than 3 points were in any component. Note that points on the variety itself are not counted in any partition of the space.

Guth and Katz's theorem essentially guarentees that there will be a polynomial to divide the plane as we desire. This is vital to the polynomial method, whose steps can losely be defined as

1. Combinatorially develop an incidence bound between a point set $\mathcal{P}$ and object set $\mathcal{O}$.
2. Evenly divide up $\mathcal{P}$ using an r-partitioning polynomial.
3. Apply the original combinatorial bound in these restricted spaces locally as well as on any points of $\mathcal{P}$ which lie on the r-partitioning polynomial to develop a stronger incidence bound globally.

At the moment this series of steps seems rather abstract, so we will solidfy them by proving a powerful theorem using the polynomial method. This theorem directly produces both Theorem 2.2.2 (Szemerédi-Trotter) and

Theorem 2.1.1 (unit distances bound) and makes use of the concept of an incidence graph. The **incidence graph** between our point set $\mathcal{P}$ and object set $\mathcal{O}$, $\mathcal{G}(\mathcal{P},\mathcal{O})$, is a graph in which each $p \in \mathcal{P}$ and $o \in \mathcal{O}$ corresponds to a vertex and an edge exists between two vertices if they are incident.

Theorem 2.4.2 (Pach and Sharir (1998))**.** *Let $\mathcal{P}$ be a set of m points and Γ a set of n distinct irreducible algebraic curves of degree at most k in $\mathbb{R}^2$. If the complete bipartite graph $K_{s,t}$ is not a subgraph of $\mathcal{G}(\mathcal{P},\Gamma)$, then*

$$I(\mathcal{P},\Gamma) = O\left(m^{\frac{s}{2s-1}} n^{\frac{2s-2}{2s-1}} + m + n\right)$$

Proof. First we prove the following bound

$$I(\mathcal{P},\Gamma) = O(mn^{1-\frac{1}{s}} + n). \tag{2.1}$$

Consider the following set

$$T = \{(a_1,\ldots,a_s,\gamma) | \gamma \in \Gamma \text{ and } a_1,\ldots,a_s \in \mathcal{P} \cap \gamma\}.$$

As there is no $K_{s,t}$ in the incidence graph, any subset of s points is completely contained within at most $t-1$ curves, so

$$|T| \le \binom{m}{s}(t-1) \;\text{ or }\; |T| = O(m^s).$$

Index the curves of Γ. For $\gamma_i \in \Gamma$, let $d_i = |\mathcal{P} \cap \gamma_i|$. Therefore, $I(\mathcal{P},\Gamma) = \sum_{i=1}^n d_i$. and

$$|T| = \sum_{i=1}^n \binom{d_i}{s} = \Omega\left(\sum_{i=1}^n (d_i - s)^s\right).$$

Note that, by Hölder's Inequality,

$$\sum_{i=1}^n (d_i - s) \le \left(\sum_{i=1}^n (d_i - s)^s\right)^{1/s} \left(\sum_{i=1}^n 1\right)^{(s-1)/s} = \left(\sum_{i=1}^n (d_i - s)^s\right)^{1/s} n^{(s-1)/s},$$

so we can say

$$\sum_{i=1}^n (d_i - s)^s \ge \frac{\left(\sum_{i=1}^n (d_i - s)\right)^s}{n^{s-1}} = \frac{(I(\mathcal{P},\Gamma) - sn)^s}{n^{s-1}},$$

meaning

$$|T| = \Omega\left(\frac{(I(\mathcal{P},\Gamma) - sn)^s}{n^{s-1}}\right).$$

Combining the upper and lower bounds for $|T|$ and reorganizing yields the desired bound for $I(\mathcal{P}, \Gamma)$.

Now we return to proving the bound in the theorem. By Theorem 2.4.1, there exists an r-partitioning polynomial, f, on $\mathcal{P}$ of degree $O(r)$. If f had multiple identical terms when factored, eliminate those aditional terms as the points defined by the variety will remain the same (note the degree of f is still $O(r)$).

Let $\mathcal{P}_0$ be the points which lie on $\mathbf{V}(f)$ and Γ_0 be the subset of Γ which is completely contained within $\mathbf{V}(f)$. Let c denote the number of cells of $\mathbb{R}^2$ that f's partitioning creates. Let $\mathcal{P}_i$ be the points of $\mathcal{P}$ in cell i and let Γ_i be the curves that intersect cell i. We can break up $I(\mathcal{P}, \Gamma)$ as a disjoint union of the incidences between points and curves contained in our variety $I(\mathcal{P}_0, \Gamma_0)$, incidences between points on the variety and all other curves $I(\mathcal{P}_0, \Gamma \setminus \Gamma_0)$, and incidences between points and curves within individual cells $I(\mathcal{P}_i, \Gamma_i)$.

$$I(\mathcal{P}, \Gamma) = I(\mathcal{P}_0, \Gamma_0) + I(\mathcal{P}_0, \Gamma \setminus \Gamma_0) + \sum_{i=1}^{c} I(\mathcal{P}_i, \Gamma_i).$$

so, we find each of these quantites individually.

According to Theorem 2.3.3, $c = O(r^2)$. If m_i is the number of points in cell i, then $m_i = |\mathcal{P}_i| \leq m/r^2$. Simiarly, if n_i is the number of curves which intersect cell i, $n_i = |\Gamma_i|$. If we apply the initial combinatorial bound in each cell, then we find

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \Gamma_i) = O\left(\sum_{i=1}^{c} (m_i n_i^{\frac{s-1}{s}} + n_i)\right) = O\left(\frac{m}{r^2} \sum_{i=1}^{c} n_i^{\frac{s-1}{s}} + \sum_{i=1}^{c} n_i\right).$$

Using Hölder's inequality on the first summation allows us to write this bound as

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \Gamma_i) = O\left(\frac{m}{r^2} \left[\sum_{i=1}^{c} n_i\right]^{\frac{s-1}{s}} \left[\sum_{i=1}^{c} 1\right]^{\frac{1}{s}} + \sum_{i=1}^{c} n_i\right) = O\left(\frac{m}{r^{2-\frac{2}{s}}} \left[\sum_{i=1}^{c} n_i\right]^{\frac{s-1}{s}} + \sum_{i=1}^{c} n_i\right)$$

so it remains to bound $\sum_{i=1}^{c} n_i$.

Consider $\gamma \in \Gamma_i$. By Theorem 2.3.2, γ has $O(1)$ connected components. Entering a new partition on a connected component requires γ to intersect $\mathbf{V}(f)$, which by Theorem 2.3.1, occurs $O(r)$ times. Provided that γ does not have any points of multiple intersections with $\mathbf{V}(f)$, which would only count as 1 intersection by Bezout, this allows us to say γ lies in $O(r)$ components of the partition. At each of these multiple intersection points, consider a circle of sufficently small radius centered at the intersection point p (here sufficently small means that each intersection of γ with p creates a new intersection of γ with the circle). By Theorem 2.3.1, the number of intersections between the circle and the curve γ is $O(1)$. Therefore, even when we have these multiple intersection points, γ lies in $O(r)$ cells. Therefore,

$$\sum_{i=1}^{c} n_i = O(nr),$$

making our bound on the incidences inside of cells

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \Gamma_i) = O\left(\frac{mn^{\frac{s-1}{s}}}{r^{\frac{s-1}{s}}} + nr\right).$$

Note from above that we showed using Theorem 2.3.1 that each $\gamma \in \Gamma \setminus \Gamma_0$ intersects $\mathbf{V}(f)$ in $O(r)$ points. This suffices to show

$$I(\mathcal{P}_0, \Gamma \setminus \Gamma_0) = O(nr).$$

To bound $I(\mathcal{P}_0, \Gamma_0)$ we will break into two cases: regular and singular points of the partitioning polynomial f. A singular point of a polynomial is any point in which all partials of the polynomial are 0. At the beginning of this proof we reduced f to contain $O(r)$ irreducible components. As every curve in Γ is also irreducible, and all are distinct, each curve of Γ_0 is the variety of an irreducible factor of f. Suppose $f = f_1 \cdots f_k$ is the irreducible decomposition of f. If a point on $\mathbf{V}(f)$ lies on multiple varieties of irreducible factors of f (say f_i and f_j), then taking a partial of f (using the product rule) with respect to some variable z yields

$$\begin{aligned} f' &= (f_1 \cdots f_k)' \\ &= (f_1)'(f_2 \cdots f_k) + \cdots + (f_k)'(f_1 \cdots f_{k-1}) \\ &= f_i(\text{all terms but that with } f_i') + f_j(f_1 \cdots f_k)(f_i)'. \end{aligned}$$

As f_i or f_j is present in every term of this sum, and both evaluate to 0 at this point, every partial will be 0 so the point will be singular. As every curve in Γ_0 can be matched up to a component of f and every regular point exists inside at most one component, every point of $\mathcal{P}_0$ which is regular for f will lie on at most one curve of Γ_0. Therefore, the regular points of f create at most $O(m)$ incidences.

At every singular point of f, $f_x = 0$. As f has no repeated factors, f_x shares no common components with f. As $\gamma \in \Gamma_0$ is contained within f it also contains no common components with f_x, so by Theorem 2.3.1, γ intersects f_x in $O(r)$ points, therefore Γ_0 is incident to at most $O(nr)$ singular points of f. Therefore,

$$I(\mathcal{P}_0, \Gamma_0) = O(nr + m).$$

Overall, our bound on the number of incidences is

$$I(\mathcal{P}, \Gamma) = O\left(\frac{mn^{\frac{s-1}{s}}}{r^{\frac{s-1}{s}}} + nr + m\right).$$

Note that the first term decreases as r increases, while the second term increases with r. Setting these terms to be equal and solving for r therefore yields the minimum upper bound.

$$\frac{mn^{\frac{s-1}{s}}}{r^{\frac{s-1}{s}}} = nr \implies r = m^{\frac{s}{2s-1}} n^{\frac{-1}{2s-1}}$$

Substituting this value in for r then creates the bound

$$I(\mathcal{P}, \Gamma) = O\left(m^{\frac{s}{2s-1}} n^{\frac{2s-2}{2s-1}} + m + n\right).$$

■

Note that this bound ignores completely the value of t. If we choose to bound using t in our initial combinatorial bound and throughout the proof, the polynomial method does not improve the bound. While the polynomial method is a powerful tool, it is not always universally applicable. The general use of incidences in addressing problems in discrete geometry on the otherhand continues to consistently provide powerful results, as we will show in the next chapter.

Chapter 3

Bipartite Distances

3.1 Problem Introduction

Question 3.1.1 is often referred to as the bipartite variant of the distinct distances problem. In this chapter we will attack this problem and use the insights and results gained to draw conclusions about the original distinct distances problem.

Question 3.1.1. *If $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^2$, with $|\mathcal{P}_1| = m$, $|\mathcal{P}_2| = n$, what is the minimum number of distinct distances that must be realized between $\mathcal{P}_1$ and $\mathcal{P}_2$?*

We often refer to the number of distinct distances between $\mathcal{P}_1$ and $\mathcal{P}_2$ as $D(\mathcal{P}_1, \mathcal{P}_2)$.

3.2 Distance Energies

The concept of distance energies is useful in attacking discrete distance problems such as the distinct bipartite distances problem. The d^{th} distance energy of a point set $\mathcal{P}$ is

$$E_d(\mathcal{P}) = \left|\left\{(a_1, b_1, \ldots, a_d, b_d) \in \mathcal{P}^{2d} \text{ s.t. } |a_1b_1| = \ldots = |a_db_d| > 0\right\}\right|.$$

Note that the only restrictions on the points is that their Euclidean distance is greater than zero, so $a_i \neq b_i$, but it is valid for $a_i = a_j$ and $b_i = b_j$ for $i \neq j$. If Δ is the set of all positive distances that $\mathcal{P}$ realizes, then for every $\delta \in \Delta$ let

$$m_\delta = |\{(a, b) \text{ s.t. } |ab| = \delta \text{ and } a, b \in \mathcal{P}\}|.$$

m_δ represents the number of distances of length δ in our point set. Therefore,

$$E_d(\mathcal{P}) = \sum_{\delta \in \Delta} m_\delta^d.$$

A slight variation on this concept is necessary for our work with bipartite distances. Given $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^2$ let

$$E_d(\mathcal{P}_1, \mathcal{P}_2) = \left|\left\{(a_1, \ldots, a_d, b_1, \ldots, b_d) \in \mathcal{P}_1^d \times \mathcal{P}_2^d \text{ s.t. } |a_1 b_1| = \ldots = |a_d b_d| > 0\right\}\right|.$$

This variant of d^{th} distance energies only counts bipartite distances, so if Δ is the set of bipartite distances between $\mathcal{P}_1$ and $\mathcal{P}_2$, and for $\delta \in \Delta$,

$$p_\delta = |\{(a, b) \text{ s.t. } |ab| = \delta \text{ and } a \in \mathcal{P}_1, b \in \mathcal{P}_2\}|,$$

then

$$E_d(\mathcal{P}_1, \mathcal{P}_2) = \sum_{\delta \in \Delta} p_\delta^d.$$

Lemma 3.2.1 (Pohoata and Sheffer (2017))**.** *If $m = |\mathcal{P}_1|$ and $n = |\mathcal{P}_2|$, then*

$$E_d(\mathcal{P}_1, \mathcal{P}_2) = \Omega\left(\frac{m^d n^d}{D(\mathcal{P}_1, \mathcal{P}_2)^{d-1}}\right)$$

Proof. We know that $E_d(\mathcal{P}_1, \mathcal{P}_2) = \sum_{\delta \in \Delta} p_\delta^d$. Note that $\sum_{\delta \in \Delta} p_\delta$ is the number of bipartite distances, so $\sum_{\delta \in \Delta} p_\delta = mn$. If we apply Hölder's inequality, we find

$$\begin{aligned} E_d(\mathcal{P}_1, \mathcal{P}_2) &= \sum_{\delta \in \Delta} p_\delta^d \\ &= \left[\left(\sum_{\delta \in \Delta} p_\delta^d\right)^{1/d}\right]^d \\ &\geq \left[\frac{\sum_{\delta \in \Delta} p_\delta}{\left(\sum_{\delta \in \Delta} 1^{d/(d-1)}\right)^{(d-1)/d}}\right]^d \\ &= \left[\frac{mn}{D(\mathcal{P}_1, \mathcal{P}_2)^{(d-1)/d}}\right]^d. \end{aligned}$$

Therefore,

$$E_d(\mathcal{P}_1, \mathcal{P}_2) = \Omega\left(\frac{m^d n^d}{D(\mathcal{P}_1, \mathcal{P}_2)^{d-1}}\right).$$

■

3.3 Restricting the Problem

We begin by considering specific restrictions on the point sets $\mathcal{P}_1$ and $\mathcal{P}_2$. These proofs follow the double counting technique touched upon in Section 2.1 using Lemma 3.2.1 to find a lower bound for the number of distinct distances.

Theorem 3.3.1 (Pohoata and Sheffer (2017))**.** *Let $\mathcal{P}_1$ be a set of m points on a line l in $\mathbb{R}^2$ and let $\mathcal{P}_2$ be a set of n points in $\mathbb{R}^2$. Then*

$$D(\mathcal{P}_1, \mathcal{P}_2) = \begin{cases} \Omega\left(m^{1/2} n^{1/2} \log^{-1/2} n\right) & \text{when } m = \Omega\left(n^{1/2}/\log^{1/3} n\right) \\ \Omega\left(n^{1/2} m^{1/3}\right) & \text{when } m = O\left(n^{1/2}/\log^{1/3} n\right) \end{cases}$$

Proof. Consider a point $b \in \mathcal{P}_2$. For any distance $\delta > 0$, there are at most 2 points $a \in \mathcal{P}_1$ which could satisfy $|ab| = \delta$ as all the points of $\mathcal{P}_1$ lie on l. Therefore, as $D(\mathcal{P}_1, \mathcal{P}_2) \geq D(\mathcal{P}_1, \{b\}) \geq m/2$, we can say that as long as $m = \Omega\left(n/\log n\right)$, then

$$D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(m) = \Omega(m^{1/2} n^{1/2} \log^{-1/2} n)$$

which adheres to the limits stated. Therefore, it suffices to show our result when $m = O\left(n/\log n\right)$.

If at least half the points of $\mathcal{P}_2$ lie on l, then for $a \in \mathcal{P}_1$, $D(\mathcal{P}_1, \mathcal{P}_2) \geq D(\{a\}, \mathcal{P}_2 \cap l) = \Theta(n)$ as $|\mathcal{P}_2 \cap l| \geq n/2$ and at most two points share a distance δ to a. As $m = O(n/\log n)$ and $D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(n)$, $D(\mathcal{P}_1, \mathcal{P}_2) = \Omega\left(m^{1/2} n^{1/2} \log^{1/2} n\right)$, which adheres to all limits. Therefore, it remains to consider configurations of $\mathcal{P}_2$ where less than half the points lie on l.

Consider only the points of $\mathcal{P}_2$ which do not lie on l and call this $\mathcal{P}_2'$. Rotate and translate the plane such that l becomes the x-axis and at least half the points of $\mathcal{P}_2'$ have positive y-coordinates. Call this set of points $\mathcal{P}_2''$. Notice that $n/4 \leq |\mathcal{P}_2''| \leq |\mathcal{P}_2|$ as $\mathcal{P}_2'' \subseteq \mathcal{P}_2$ and at least half the points of $\mathcal{P}_2$ aren't on l and of those at least half now have positive y-coordinates. Therefore $|\mathcal{P}_2''| = \Theta(n)$, so $D(\mathcal{P}_1, \mathcal{P}_2)$ has the same asymptotic complexity as $D(\mathcal{P}_1, \mathcal{P}_2'')$. Therefore it suffices to bound $D(\mathcal{P}_1, \mathcal{P}_2'')$.

Consider the 3^{rd} distance energy between $\mathcal{P}_1$ and $\mathcal{P}_2''$. By Lemma 3.2.1,

$$E_3(\mathcal{P}_1, \mathcal{P}_2'') = \Omega\left(\frac{m^3 n^3}{D(\mathcal{P}_1, \mathcal{P}_2'')^2}\right).$$

If we let Δ be the set of distances between $\mathcal{P}_1$ and $\mathcal{P}_2''$, then $E_3(\mathcal{P}_1, \mathcal{P}_2'') = \sum_{\delta \in \Delta} p_\delta^3$. Recall that for $b \in \mathcal{P}_2$, at most two points in $\mathcal{P}_1$ realize the same distance. Therefore, for all $\delta \in \Delta$, $p_\delta \leq 2n$. Let Δ_j be all $\delta \in \Delta$ such that $p_\delta \geq j$ and let $k_j = |\Delta_j|$. Breaking our summation up using dyadic decomposition (into regions between powers of two), we find

$$\begin{aligned} E_3(\mathcal{P}_1, \mathcal{P}_2'') &= \sum_{\delta \in \Delta} p_\delta^3 \\ &\leq \sum_{j=0}^{\log_2 n} \sum_{\{\delta \in \Delta | 2^j \leq p_\delta \leq 2^{j+1}\}} p_\delta^3 \\ &< \sum_{j=0}^{\log_2 n} \sum_{\{\delta \in \Delta | 2^j \leq p_\delta \leq 2^{j+1}\}} (2^{j+1})^3 \\ &\leq 8 \sum_{j=0}^{\log_2(n)} 2^{3j} k_{2^j}. \end{aligned}$$

If $q = 2^j$, then the quantity inside the summand becomes $q^3 k_q$. We need to bound the size of this quantity to bound the summation above.

For a given q, let Γ_q be the set of circles centered at the points of $\mathcal{P}_1$ (which now all lie on the x-axis) of all distances δ such that $\delta \in \Delta_q$. Specifically,

$$\Gamma_q = \{(x, y) | (x - a_x)^2 + y^2 = \delta^2, \delta \in \Delta_j, (a_x, 0) \in \mathcal{P}_1\}.$$

This creates a set of mk_q circles, which, pairwise, have at most one possible intersection point which could coincide with a point of $\mathcal{P}_2''$ (as all of these have positive y-coordinates). Therefore $\mathcal{G}(\mathcal{P}_2'', \Gamma_q)$ contains no $K_{2,2}$, which, by Theorem 2.4.2, means

$$I(\mathcal{P}_2'', \Gamma_q) = O(m^{2/3} n^{2/3} k_q^{2/3} + n + mk_q).$$

If the term mk_q dominates this bound, then $m^{2/3} n^{2/3} k_q^{2/3} = O(mk_q)$, so $k_q = \Omega(n^2/m) = \Omega(n^{1/2} m^{1/2} \log^{3/2}(n))$ upon considering the bound $m = O(n/\log n)$. By definition, $D(\mathcal{P}_1, \mathcal{P}_2'') \geq k_q$, so $D(\mathcal{P}_1, \mathcal{P}_2'') = \Omega(n^{1/2} m^{1/2} \log^{3/2}(n))$. This meets the requirements of all bounds, so we need not consider this case moving forward.

If the term n dominates this bound, then $m^{2/3} n^{2/3} k_q^{2/3} = O(n)$, so $k_q = O(n^{1/2}/m)$. Therefore,

$$q^3 k_q = O(q^3 n^{1/2}/m).$$

If the term $m^{2/3}n^{2/3}k_q^{2/3}$ dominates this bound we once again consider $I(\mathcal{P}_2'', \Gamma_q)$. Every $\delta \in \Delta_q$ corresponds to at least q incidences between circles of radius δ and points in $\mathcal{P}_2''$ (as each distance in Δ_q must be realized q times), so $I(\mathcal{P}_2'', \Gamma_q) \geq qk_q$. In this case, that means $qk_q = O(m^{2/3}n^{2/3}k_q^{2/3})$, which leads to the bound

$$q^3 k_q = O(m^2 n^2).$$

Combining these last two cases reveals

$$\begin{aligned} E_3(\mathcal{P}_1, \mathcal{P}_2'') &< 8 \sum_{j=0}^{\log_2(n)} 2^{3j} k_{2j} \\ &= O\left(\sum_{j=0}^{\log_2(n)} \left(m^2 n^2 + \frac{2^{3j} n^{1/2}}{m} \right) \right) \\ &= O\left(m^2 n^2 \log n + \frac{n^{7/2}}{m} \right). \end{aligned}$$

Provided $m = \Omega(n^{1/2}/\log^{1/3} n)$, the term $m^2 n^2 \log n$ dominates this bound. Double counting the third energies with the bounds

$$E_3(\mathcal{P}_1, \mathcal{P}_2'') = \Omega\left(\frac{m^3 n^3}{D(\mathcal{P}_1, \mathcal{P}_2)^2} \right) \text{ and } E_3(\mathcal{P}_1, \mathcal{P}_2'') = O(m^2 n^2 \log n)$$

reveals that in this case

$$D(\mathcal{P}_1, \mathcal{P}_2'') = \Omega(n^{1/2} m^{1/2} \log^{-1/2} n).$$

As $D(\mathcal{P}_1, \mathcal{P}_2'')$ is asymptotically the same as $D(\mathcal{P}_1, \mathcal{P}_2)$ we have reached the desired bound.

Finally we need to consider the cases under which $m = O(n^{1/2}/\log^{1/3} n)$. Consider the existence of a δ such that $p_\delta \geq n^{1/2} m^{4/3}$. If we consider C to be the set of δ radius circles centered at the points of $\mathcal{P}_1$, then $I(\mathcal{P}_2'', C) \geq n^{1/2} m^{4/3}$, as each distance of length δ will correspond to one incidence. As there are m circles, the pigeon hole principle proves the existence of $\gamma \in C$ that is incident to $n^{1/2} m^{1/3}$ points of $\mathcal{P}_2''$. Let $a \in \mathcal{P}_1$ be a point not at the center of γ. At most two points in the part of γ which lies above the x-axis can be the same distance from a, so

$$D(\mathcal{P}_1, \mathcal{P}_2'') \geq D(\{a\}, \mathcal{P}_2'' \cap \gamma) \geq n^{1/2} m^{1/3}/2,$$

which matches our desired bound.

If no such δ exists, then it must be that case that $p_\delta < n^{1/2}m^{4/3}$ for all $\delta \in \Delta$. Every $\delta \in \Delta_j$ corresponds to at least j bipartite pairs, so we can say $k_j \leq mn/j$. Using a dyadic decomposition argument similar to above (but using 2^{nd} distances), we can show

$$\begin{aligned}
E_2(\mathcal{P}_1, \mathcal{P}_2'') &< 4 \sum_{j=0}^{\log_2 n^{1/2}m^{4/3}} 2^{2j} k_{2^j} \\
&= 4\left(\sum_{j=0}^{\log \sqrt{mn}} 2^{2j} k_{2^j} + \sum_{j=\log \sqrt{mn}}^{\log_2 n^{1/2}m^{4/3}} 2^{2j} k_{2^j} \right) \\
&= O\left(\sum_{j=0}^{\log \sqrt{mn}} mn2^j + \sum_{j=\log \sqrt{mn}}^{\log_2 n^{1/2}m^{4/3}} (2^{2j} n^{1/2} m^{-1} + m^2 n^2 2^{-j}) \right) \\
&= O\left(n^{3/2} m^{5/3} \right).
\end{aligned}$$

Note that we use the bound $k_j \leq mn/j$ for the first summation and bounds derived for k_j by reorganizing the 3^{rd} distance energies bounds for the second summation. By Lemma 3.2.1 we additionally have

$$E_2(\mathcal{P}_1, \mathcal{P}_2'') = \Omega\left(\frac{m^2 n^2}{D(\mathcal{P}_1, \mathcal{P}_2'')} \right).$$

Combining these bounds reveals $D(\mathcal{P}_1, \mathcal{P}_2'') = \Omega(n^{1/2} m^{1/3})$, so in both cases we have met our desired bound, thus ending the proof.

■

Mimicking this proof allows us to augment the restrictions for $\mathcal{P}_1$ and come up with similar results.

Theorem 3.3.2. *Let $\mathcal{P}_1$ be a set of m points on a circle c in $\mathbb{R}^2$ and let $\mathcal{P}_2$ be a set of n points in $\mathbb{R}^2$. Then*

$$D(\mathcal{P}_1, \mathcal{P}_2) = \begin{cases} \Omega\left(m^{1/2} n^{1/2} \log^{-1/2} n \right) & \text{when } m = \Omega\left(n^{1/2} / \log^{1/3} n \right) \\ \Omega\left(n^{1/2} m^{1/3} \right) & \text{when } m = O\left(n^{1/2} / \log^{1/3} n \right) \end{cases}$$

Proof. Consider a point $b \in \mathcal{P}_2$ not at the center of c. For any distance $\delta > 0$, there are at most two points such that $|ab| = \delta$, where $a \in \mathcal{P}_1$. To see this, consider drawing a circle of radius δ around b. As this circle and c do not share the same center, they can only intersect in two places. Therefore, as $D(\mathcal{P}_1, \mathcal{P}_2) \geq D(\mathcal{P}_1, \{b\}) \geq m/2$, we can say that as long as $m = \Omega\left(n/\log n\right)$, then

$$D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(m) = \Omega(m^{1/2} n^{1/2} \log^{-1/2} n)$$

which adheres to the limits stated. Therefore, it suffices to show our result when $m = O\left(n/\log n\right)$.

If at least half the points of $\mathcal{P}_2$ lie on c, then for $a \in \mathcal{P}_1$, $D(\mathcal{P}_1, \mathcal{P}_2) \geq D(\{a\}, \mathcal{P}_2 \cap c) = \Theta(n)$ as $|\mathcal{P}_2 \cap c| \geq n/2$ and at most two points share a distance δ to a (this is fairly easy to see with a law of cosines argument). As $m = O(n/\log n)$ and $D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(n)$, $D(\mathcal{P}_1, \mathcal{P}_2) = \Omega\left(m^{1/2} n^{1/2} \log^{1/2} n\right)$, which adheres to all limits. Therefore, it suffices to show our results when less than half the points of $\mathcal{P}_2$ lie on c.

Consider only the points of $\mathcal{P}_2$ which do not lie on c or at the center of c and call this $\mathcal{P}_2'$. Provided that $n \geq 4$ (which will be true as we consider the asympotic limits), this means that $|\mathcal{P}_2'| \geq \frac{n}{4}$. $|\mathcal{P}_2'| = \theta(n)$, so any asymptotic use of $|\mathcal{P}_2|$ can be considered to be n. Therefore, it suffices to bound $D(\mathcal{P}_1, \mathcal{P}_2')$ from below as $D(\mathcal{P}_1, \mathcal{P}_2) \geq D(\mathcal{P}_1, \mathcal{P}_2')$.

Consider the 3^{rd} distance energy between $\mathcal{P}_1$ and $\mathcal{P}_2'$. By Lemma 3.2.1,

$$E_3(\mathcal{P}_1, \mathcal{P}_2') = \Omega\left(\frac{m^3 n^3}{D(\mathcal{P}_1, \mathcal{P}_2')^2}\right).$$

If we consider Δ to be the set of distances between $\mathcal{P}_1$ and $\mathcal{P}_2'$, then $E_3(\mathcal{P}_1, \mathcal{P}_2') = \sum_{\delta \in \Delta} p_\delta^3$. Recall that for $b \in \mathcal{P}_2$ such that b is not the center of c, at most two points in $\mathcal{P}_1$ realize the same distance. Therefore, for all $\delta \in \Delta$, $p_\delta \leq 2n$. Let Δ_j be all $\delta \in \Delta$ such that $p_\delta \geq j$ and let $k_j = |\Delta_j|$. Breaking our summation up using dyadic decomposition, we find

$$\begin{aligned}
E_3(\mathcal{P}_1, \mathcal{P}_2') &= \sum_{\delta \in \Delta} p_\delta^3 \\
&\leq \sum_{j=0}^{\log_2 n} \sum_{\{\delta \in \Delta | 2^j \leq p_\delta \leq 2^{j+1}\}} p_\delta^3 \\
&< \sum_{j=0}^{\log_2 n} \sum_{\{\delta \in \Delta | 2^j \leq p_\delta \leq 2^{j+1}\}} (2^{j+1})^3 \\
&\leq 8 \sum_{j=0}^{\log_2(n)} 2^{3j} k_{2^j}.
\end{aligned}$$

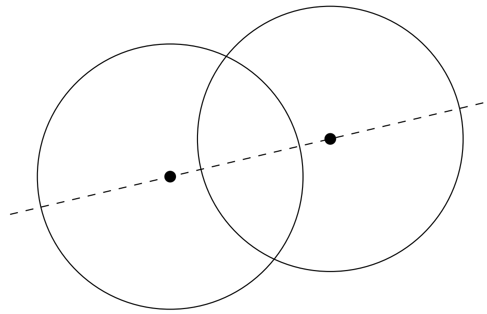

Figure 3.1 For a third circle to intersect in the same two places as these two circles intersect, it would need to lie on the line connecting the centers of the circles (as the radial vector perpendicular to the chord between these points would need to bisect the chord). As no three points on a circle all lie on the same line (under Thm 2.3.1 since circles are degree 2 and lines are degree 1), 3 circles with centers on c cannot have two common intersection points.

If $q = 2^j$, then the quantity inside the summand becomes $q^3 k_q$. We need to bound the size of this quantity to bound the summation above.

For a given q, let Γ_q be the set of circles centered at the points of $\mathcal{P}_1$ of all distances δ such that $\delta \in \Delta_q$. This creates a set of mk_q circles. These circles can intersect pairwise in two valid points of $\mathcal{P}_2'$, but 3 circles cannot have two common points of $\mathcal{P}_2'$ as valid intersections (as explained by Fig 3.1). Therefore, $\mathcal{G}(\mathcal{P}_2'', \Gamma_q)$ contains no $K_{2,3}$. Using Thm 2.4.2, we can bound the number of incidences,

$$I(\mathcal{P}_2', \Gamma_q) = O(m^{2/3} n^{2/3} k_q^{2/3} + n + mk_q).$$

The rest of this proof is nearly identical to that of Thm 3.3.1, but we will restate it here for convinience.

If the term mk_q dominates this bound, then $m^{2/3} n^{2/3} k_q^{2/3} = O(mk_q)$, so $k_q = \Omega(n^2/m) = \Omega(n^{1/2} m^{1/2} \log^{3/2}(n))$ upon considering the bound $m = O(n/\log n)$. By definition, $D(\mathcal{P}_1, \mathcal{P}_2') \geq k_q$, so $D(\mathcal{P}_1, \mathcal{P}_2') = \Omega(n^{1/2} m^{1/2} \log^{3/2}(n))$. This meets the requirements of all bounds, so we need not consider this case moving forward.

If the term n dominates this bound, then $m^{2/3} n^{2/3} k_q^{2/3} = O(n)$, so $k_q = O(n^{1/2}/m)$. Therefore,

$$q^3 k_q = O(q^3 n^{1/2}/m).$$

If the term $m^{2/3}n^{2/3}k_q^{2/3}$ dominates this bound we once again consider $I(\mathcal{P}_2', \Gamma_q)$. Every $\delta \in \Delta_q$ corresponds to at least q incidences between circles of radius δ and points in $\mathcal{P}_2'$ (as each distance in Δ_q must be realized q times), so $I(\mathcal{P}_2', \Gamma_q) \geq qk_q$. In this case, that means $qk_q = O(m^{2/3}n^{2/3}k_q^{2/3})$, which leads to the bound

$$q^3 k_q = O(m^2 n^2).$$

Combining these last two cases reveals

$$\begin{aligned} E_3(\mathcal{P}_1, \mathcal{P}_2') &< 8 \sum_{j=0}^{\log_2(n)} 2^{3j} k_{2j} \\ &= O\left(\sum_{j=0}^{\log_2(n)} \left(m^2 n^2 + \frac{2^{3j} n^{1/2}}{m} \right) \right) \\ &= O\left(m^2 n^2 \log n + \frac{n^{7/2}}{m} \right). \end{aligned}$$

Provided $m = \Omega(n^{1/2}/\log^{1/3} n)$, the term $m^2n^2 \log n$ dominates this bound. Double counting the third energies with the bounds

$$E_3(\mathcal{P}_1, \mathcal{P}_2') = \Omega\left(\frac{m^3 n^3}{D(\mathcal{P}_1, \mathcal{P}_2)^2} \right) \text{ and } E_3(\mathcal{P}_1, \mathcal{P}_2') = O(m^2 n^2 \log n)$$

reveals that in this case

$$D(\mathcal{P}_1, \mathcal{P}_2') = \Omega(n^{1/2} m^{1/2} \log^{-1/2} n).$$

As $D(\mathcal{P}_1, \mathcal{P}_2')$ is asymptotically the same as $D(\mathcal{P}_1, \mathcal{P}_2)$ we have reached the desired bound.

Finally we need to consider the cases under which $m = O(n^{1/2}/\log^{1/3} n)$. Consider the existence of a δ such that $p_\delta \geq n^{1/2}m^{4/3}$. If we consider C to be the set of δ radius circles centered at the points of $\mathcal{P}_1$, then $I(\mathcal{P}_2', C) \geq n^{1/2}m^{4/3}$, as each distance of length δ will correspond to one incidence. As there are m circles, the pigeon hole principle proves the existence of $\gamma \in C$ that is incidence to $n^{1/2}m^{1/3}$ points of $\mathcal{P}_2'$. Let $a \in \mathcal{P}_1$ be a point not at the center of γ. At most two points on γ can be the same distance from a, so

$$D(\mathcal{P}_1, \mathcal{P}_2') \geq D(\{a\}, \mathcal{P}_2'' \cap \gamma) \geq n^{1/2} m^{1/3}/2,$$

which matches our desired bound.

If no such δ exists, then it must be that case that $p_\delta < n^{1/2}m^{4/3}$ for all $\delta \in \Delta$. Every $\delta \in \Delta_j$ corresponds to at least j bipartite pairs, so we can say $k_j \leq mn/j$. Using a dyadic decomposition argument similar to above (but using 2^{nd} distances), we can show

$$\begin{aligned} E_2(\mathcal{P}_1, \mathcal{P}_2') &< 4 \sum_{j=0}^{\log_2 n^{1/2}m^{4/3}} 2^{2j} k_{2^j} \\ &= 4\left(\sum_{j=0}^{\log\sqrt{mn}} 2^{2j} k_{2^j} + \sum_{j=\log\sqrt{mn}}^{\log_2 n^{1/2}m^{4/3}} 2^{2j} k_{2^j} \right) \\ &= O\left(\sum_{j=0}^{\log\sqrt{mn}} mn2^j + \sum_{j=\log\sqrt{mn}}^{\log_2 n^{1/2}m^{4/3}} (2^{2j} n^{1/2} m^{-1} + m^2 n^2 2^{-j}) \right) \\ &= O\left(n^{3/2}m^{5/3}\right). \end{aligned}$$

Note that we use the bound $k_j \leq mn/j$ for the first summation and the bound for k_j found while looking at 3^{rd} distance energies for the second summation. By Lemma 3.2.1 we additionally have

$$E_2(\mathcal{P}_1, \mathcal{P}_2') = \Omega\left(\frac{m^2 n^2}{D(\mathcal{P}_1, \mathcal{P}_2')} \right).$$

Combining these bounds reveals $D(\mathcal{P}_1, \mathcal{P}_2') = \Omega(n^{1/2}m^{1/3})$, so in both cases we have met our desired bound, thus ending the proof.

∎

By applying the relatively simple tools of algebraic geometry discussed in Section 2.3, we are able to extend the result to a quite general restriction.

Theorem 3.3.3. *Let $\mathcal{P}_1$ be a set of m points on an curve of degree r, called γ, in $\mathbb{R}^2$ and let $\mathcal{P}_2$ be a set of n points in $\mathbb{R}^2$. Then*

$$D(\mathcal{P}_1, \mathcal{P}_2) = \begin{cases} \Omega\left(m^{1/2} n^{1/2} \log^{-1/2} n\right) & \textit{when } m = \Omega\left(n^{1/2}/\log^{1/3} n\right) \\ \Omega\left(n^{1/2} m^{1/3}\right) & \textit{when } m = O\left(n^{1/2}/\log^{1/3} n\right) \end{cases}$$

Proof. Consider a point $b \in \mathcal{P}_2$ which is not at the center of any circular component of γ. We know such a b exists as γ has at most $O(r^2)$ connected components by Theorem 2.3.2, so as n grows, $\mathcal{P}_2$ will have more points than γ has connected components.

For any distance $\delta > 0$, we can draw a circle c_b of radius δ around b to represent all point of distance δ from b. Due to our choice of b, c_b and γ share no common factors in their polynomial representations, so by Theorem 2.3.1, c_b and γ intersect in at most $2r$ points. Therefore, for any $\delta > 0$, there are at most $2r$ points in $\mathcal{P}_1$ which realize a distance δ to b. Thus, as $D(\mathcal{P}_1, \mathcal{P}_2) \geq D(\mathcal{P}_1, \{b\}) \geq m/(2r)$, if $m = \Omega(n/\log n)$, then

$$D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(m) = \Omega(m^{1/2} n^{1/2} \log^{-1/2} n).$$

Therefore, we only need to consider when $m = O(n/\log n)$.

If at least half the points of $\mathcal{P}_2$ lie on γ, consider $a \in \mathcal{P}_1$ such that a is not the center of any circular component of γ (exists by same argument as before for b). As at most $2r$ points on γ share a distance δ to a, $D(\mathcal{P}_1, \mathcal{P}_2) \geq D(\{a\}, \mathcal{P}_2 \cap \gamma) \geq \frac{n/2}{2r} = \Omega(n)$. As $m = O(n/\log n)$ and $D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(n)$, $D(\mathcal{P}_1, \mathcal{P}_2) = \Omega\left(m^{1/2} n^{1/2} \log^{1/2} n\right)$, which adheres to all limits. Therefore, we only need to continue considering configurations of $\mathcal{P}_2$ where less than half the points lie on γ.

Consider $\mathcal{P}_2'$ to be the points of $\mathcal{P}_2$ which do not lie on γ. As $|\mathcal{P}_2'| \geq \frac{n}{2}$ we can say $|\mathcal{P}_2'| = \Theta(n)$, so in all asymptotics bounds we can consider $|\mathcal{P}_2'|$ to be n. Now let $\mathcal{P}_2''$ to be the subset of points of $\mathcal{P}_2'$ which do not lie at the center of some circular component of γ. As this is at most $O(r^2)$ points by Theorem 2.3.2, $|\mathcal{P}_2''| = \Theta(n)$, so the same considerations can be made during asymptotic bounds. Additionally as $D(\mathcal{P}_1, \mathcal{P}_2) \geq D(\mathcal{P}_1, \mathcal{P}_2'')$, we can focus on bounding $D(\mathcal{P}_1, \mathcal{P}_2'')$.

If at least half the points of $\mathcal{P}_1$ lie on linear components of γ, then, by the Pigeon Hole Principle, there must be a single linear components l which has at least $\frac{m/2}{r} = \Theta(m)$ (linear components have degree at least one so there are at most r linear components in γ). Therefore, $D(\mathcal{P}_1, \mathcal{P}_2'') \geq D(\mathcal{P}_1 \cap l, \mathcal{P}_2'')$. As l has $\Theta(m)$ points lying on it, our bounds are given by the linear restriction of this theorem, Theorem 3.3.1. Thus, we only need to continue considering configurations of $\mathcal{P}_1$ where less than half the points lie on linear components of γ.

Let $\mathcal{P}_1'$ be the points of $\mathcal{P}_1$ which do not lie on linear components of γ. As $|\mathcal{P}_1'| \geq \frac{m}{2} = \theta(m)$, we may consider $|\mathcal{P}_1'|$ to be m in all asymptotic bounds. From here on we will focus on bounding $D(\mathcal{P}_1', \mathcal{P}_2'')$ as any lower bounds will also apply to $D(\mathcal{P}_1, \mathcal{P}_2)$.

Consider the 3^{rd} distance energies between $\mathcal{P}_1'$ and $\mathcal{P}_2''$. By Lemma 3.2.1,

$$E_3(\mathcal{P}_1', \mathcal{P}_2'') = \Omega\left(\frac{m^3 n^3}{D(\mathcal{P}_1', \mathcal{P}_2'')^2}\right).$$

If we consider Δ to be the set of distances between $\mathcal{P}_1'$ and $\mathcal{P}_2''$ and p_δ to be the number of bipartite pairs the realize a distance δ, then $E_3(\mathcal{P}_1', \mathcal{P}_2'') = \sum_{\delta \in \Delta} p_\delta^3$. As no point of $\mathcal{P}_2''$ lies at the center of a circular component of γ, Theorem 2.3.1 guarentees that no more than $2r$ points of $\mathcal{P}_1'$ realize the same distance to a single point of $\mathcal{P}_2''$. Therefore, for all $\delta \in \Delta$, $p_\delta \leq 2rn$. Let Δ_j be all $\delta \in \Delta$ such that $p_\delta \geq j$ and let $k_j = |\Delta_j|$. Using a dyadic decomposition we find

$$\begin{aligned} E_3(\mathcal{P}_1', \mathcal{P}_2'') &= \sum_{\delta \in \Delta} p_\delta^3 \\ &\leq \sum_{j=0}^{\log_2 rn} \sum_{\{\delta \in \Delta | 2^j \leq p_\delta \leq 2^{j+1}\}} p_\delta^3 \text{ then take the max of } p_\delta \text{ over the region} \\ &< \sum_{j=0}^{\log_2 rn} \sum_{\{\delta \in \Delta | 2^j \leq p_\delta \leq 2^{j+1}\}} (2^{j+1})^3 \\ &\leq 8 \sum_{j=0}^{\log_2 rn} 2^{3j} k_{2^j}. \end{aligned}$$

If $q = 2^j$, then the quantity inside the summand becomes $q^3 k_q$. We need to bound the size of this quantity to bound this summation.

For a given q, let Γ_q be the set of circles centered at the points of $\mathcal{P}_1'$ whose radii correspond to distances δ such that $\delta \in \Delta_q$. This creates a set of mk_q circles. Notice that if a set of circles all share two common points of intersection (which is the maximum number of intersections for two distinct circles), then their centers must lie on the same line (as explain by Figure 3.1). As none of the points of $\mathcal{P}_1'$ lie on the linear components of γ we can construct γ', being γ without its linear components, and all the points of $\mathcal{P}_1'$ will lie on γ'. As γ' has no linear components, any line can intersect γ' in only r locations by Theorem 2.3.1. As only r points of $\mathcal{P}_1'$ could be colinear, $\mathcal{G}(\mathcal{P}_2'', \Gamma_q)$ contains no $K_{2,r+1}$ as a subgraph, so by Theorem 2.4.2

$$I(\mathcal{P}_2'', \Gamma_q) = O(m^{2/3} n^{2/3} k_q^{2/3} + n + mk_q).$$

If the term mk_q dominates this bound, then $m^{2/3} n^{2/3} k_q^{2/3} = O(mk_q)$, so $k_q = \Omega(n^2/m) = \Omega(n^{1/2} m^{1/2} \log^{3/2}(n))$ upon considering the bound $m = O(n/\log n)$. By definition, $D(\mathcal{P}_1', \mathcal{P}_2'') \geq k_q$, so $D(\mathcal{P}_1', \mathcal{P}_2'') = \Omega(n^{1/2} m^{1/2} \log^{3/2}(n))$. This meets the requirements of all bounds, so we need not consider this case moving forward.

If the term n dominates this bound, then $m^{2/3}n^{2/3}k_q^{2/3} = O(n)$, so $k_q = O(n^{1/2}/m)$. Therefore,

$$q^3 k_q = O(q^3 n^{1/2}/m).$$

If the term $m^{2/3}n^{2/3}k_q^{2/3}$ dominates this bound we once again consider $I(\mathcal{P}_2'', \Gamma_q)$. Every $\delta \in \Delta_q$ corresponds to at least q incidences between circles of radius δ and points in $\mathcal{P}_2''$ (as each distance in Δ_q must be realized q times), so $I(\mathcal{P}_2'', \Gamma_q) \geq q k_q$. In this case, that means $q k_q = O(m^{2/3}n^{2/3}k_q^{2/3})$, which leads to the bound

$$q^3 k_q = O(m^2 n^2).$$

Combining these last two cases reveals

$$\begin{aligned}
E_3(\mathcal{P}_1', \mathcal{P}_2'') &< 8 \sum_{j=0}^{\log_2(rn)} 2^{3j} k_{2^j} \\
&= O\left(\sum_{j=0}^{\log_2(rn)} \left(m^2 n^2 + \frac{2^{3j} n^{1/2}}{m} \right) \right) \\
&= O\left(m^2 n^2 \log_2(rn) + \frac{(rn)^3 n^{1/2}}{m} \right) \\
&= O\left(m^2 n^2 \log n + \frac{n^{7/2}}{m} \right).
\end{aligned}$$

Provided $m = \Omega(n^{1/2}/\log^{1/3} n)$, the term $m^2 n^2 \log n$ dominates this bound. Double counting the third energies with the bounds

$$E_3(\mathcal{P}_1', \mathcal{P}_2'') = \Omega\left(\frac{m^3 n^3}{D(\mathcal{P}_1', \mathcal{P}_2'')^2} \right) \text{ and } E_3(\mathcal{P}_1', \mathcal{P}_2'') = O(m^2 n^2 \log n)$$

reveals that in this case

$$D(\mathcal{P}_1', \mathcal{P}_2'') = \Omega(n^{1/2} m^{1/2} \log^{-1/2} n).$$

As $D(\mathcal{P}_1', \mathcal{P}_2'')$ is asymptotically the same as $D(\mathcal{P}_1, \mathcal{P}_2)$ we have reached the desired bound.

Finally we need to consider the cases under which $m = O(n^{1/2}/\log^{1/3} n)$. Consider the existence of a δ such that $p_\delta \geq n^{1/2} m^{4/3}$. If we consider C to be the set of δ radius circles centered at the points of $\mathcal{P}_1'$, then $I(\mathcal{P}_2'', C) \geq n^{1/2} m^{4/3}$, as each distance of length δ will correspond to one incidence. As there are at most m circles, the pigeon hole principle proves the existence of $\gamma \in C$ that is incidence to $n^{1/2}m^{1/3}$ points of $\mathcal{P}_2''$. Let $a \in \mathcal{P}_1'$ be a point not at the center of γ. At most two points on γ can be the same distance from a, so

$$D(\mathcal{P}_1', \mathcal{P}_2'') \geq D(\{a\}, \mathcal{P}_2'' \cap \gamma) \geq n^{1/2}m^{1/3}/2,$$

which matches our desired bound.

If no such δ exists, then it must be that case that $p_\delta < n^{1/2}m^{4/3}$ for all $\delta \in \Delta$. Every $\delta \in \Delta_j$ corresponds to at least j bipartite pairs, so we can say $k_j \leq mn/j$. Using a dyadic decomposition argument similar to above (but using 2^{nd} distances), we can show

$$\begin{aligned}
E_2(\mathcal{P}_1', \mathcal{P}_2'') &< 4 \sum_{j=0}^{\log_2 n^{1/2}m^{4/3}} 2^{2j}k_{2^j} \\
&= 4\left(\sum_{j=0}^{\log\sqrt{mn}} 2^{2j}k_{2^j} + \sum_{j=\log\sqrt{mn}}^{\log_2 n^{1/2}m^{4/3}} 2^{2j}k_{2^j}\right) \\
&= O\left(\sum_{j=0}^{\log\sqrt{mn}} mn2^j + \sum_{j=\log\sqrt{mn}}^{\log_2 n^{1/2}m^{4/3}} (2^{2j}n^{1/2}m^{-1} + m^2n^2 2^{-j})\right) \\
&= O\left(n^{3/2}m^{5/3}\right).
\end{aligned}$$

Note that we use the bound $k_j \leq mn/j$ for the first summation and the bound for k_j found while looking at 3^{rd} distance energies for the second summation. By Lemma 3.2.1 we additionally have

$$E_2(\mathcal{P}_1', \mathcal{P}_2'') = \Omega\left(\frac{m^2n^2}{D(\mathcal{P}_1', \mathcal{P}_2'')}\right).$$

Combining these bounds reveals $D(\mathcal{P}_1', \mathcal{P}_2'') = \Omega(n^{1/2}m^{1/3})$, so in both cases we have met our desired bound, thus ending the proof.

■

This result has been heavily generalized from Sheffer and Pohoata's original result, to the point at which it has significant consequences on the distinct distances problem as a whole, as will discuss in the next chapter.

Chapter 4

Conclusion

Consider the distinct bipartite distances problem where $\mathcal{P}_1$ lies on an algebraic curve of arbitrary degree, $\mathcal{P}_2$ is unrestricted, and $|\mathcal{P}_1| = |\mathcal{P}_2| = \frac{a}{2}$. By our theorem, the number of bipartite distances is

$$\begin{aligned}\Omega(m^{1/2}n^{1/2}\log^{-1/2} n) &= \Omega((a/2)^{1/2}(a/2)^{1/2}\log^{-1/2}(a/2)) \\ &= \Omega\left(\frac{a/2}{\sqrt{\log a - \log 2}}\right) \\ &= \Omega\left(\frac{a}{\sqrt{\log a}}\right)\end{aligned}$$

and there are a total points in consideration. Notice that the bipartite distances we are considering are a subset of the total set of distances that exist between these a points. This leads us to the following corollary.

Corollary 4.0.1. *Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$ such that at least half the points lie on a algebraic curve of arbitrary degree. There exist $\Omega(n/\log n)$ distinct pairwise distances between the points of $\mathcal{P}$.*

Note that this means any construction that meets the criteria laid out in this corollary will have at least asymptotically as many distinct distances as Erdős's construction. Therefore, we can draw conclusions about what types of constructions could possibly lower the theoretical maximum of $f(n)$.

As we are working in $\mathbb{R}^2$, any point set can have at most two dimensions of organization. Corollary 4.0.1 essentially implies that any point set with 1-dimension of organization (ie the points lying along some singular geometric object) will not contain fewer distinct distances than Erdős's construction. Having 0 dimensions of organization means that the points lie in complete chaos. It is unlikely that one of these organizations will provide an asymptotically smaller number of distances as complete randomness of point placement in the real numbers will likely lead to very few repeated distances. This means that we can focus our efforts on looking at constructions with 2 dimensions of organization. These constructions grow in grids, such as Erdős's, where two sets of geometric objects are overlapped so that they intersect, and points are placed at these intersections (in Erdős's case these are two families of lines). These objects could be any type of algebraic curve such as lines, parabolas, circles, or more complex multi-componented varieties. By adding this restriction to the point set, it is my hope that we will be able to close the gap and discover the true asymptotic behavior of $f(n)$.

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