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Richard Delaware

University of Missouri - Kansas City

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Richard Delaware

Department of Mathematics & Statistics, University of Missouri - Kansas City, USA
`delawarer@umkc.edu`

Synopsis

In writing a proof, a student surprises her sense of reality and logic.

She swept through my open office door around noon with a question.

The textbook statement she was asked to prove for homework [1, page 114, Exercise 4.10] was written as a theorem, which in mathematics means a true statement for which a proof is known. So, taking the authors at their word, barring any unintended error in this third edition of the text, she confidently engaged in the task of finding such a proof.

The theorem stated that a certain property was held by the integers, that familiar set of numbers 0, 1, 2, 3, $\cdots$, together with the negatives -1, -2, -3, and so on.

What provoked her visit to me was that while she proved two parts of the theorem true, for the third and final part she unexpectedly proved the opposite of what the theorem stated.

This is where our exchange began, with her writing on my office chalkboard as needed.

My words are indented.

  So, walk me through what you did.

The theorem as given was this. The "n" everywhere stands for an integer.

"If 2 divides the integer $n^4 - 3$, then 4 divides the integer $n^2 + 3$."

This theorem is written as a standard logical "If P, then Q" statement.

> Also known as "P implies Q", right?

Right. And, it is logically equivalent to the contrapositive statement "If (not Q), then (not P)."

> Meaning by saying that not having Q implies you can't have had P in the first place, we are just saying "P implies Q" using different words, right?

Yah. So, I decided to prove that equivalent contrapositive version of the theorem, meaning: "If 4 does not divide the integer $n^2 + 3$, then 2 does not divide the integer $n^4 - 3$."

> OK. By the way, why did you decide to recast the statement as contrapositive?

Well, in the contrapositive statement I can assume a hypothesis about n^2 and draw a conclusion about n^4, rather than the reverse. It just seemed that it might eventually be easier to move from n^2 to n^4 by squaring n^2.

> Good idea. Now, how did you proceed from there?

Well, I didn't like starting with the contrapositive hypothesis written as a negative:

"4 does not divide the integer $n^2 + 3$."

So, I decided to use something we discussed in class.

What happens when we divide integers by 4? The answer is that integers are either "multiples of 4" (having the form $4k$), "multiples of 4" plus 1 (having the form $4k+1$), "multiples of 4" plus 2 (having the form $4k+2$), or "multiples of 4" plus 3 (having the form $4k + 3$), where in each of those forms k is some integer. That covers all the integers because the next subset of integers following that pattern, "multiples of 4" plus 4 (having the form $4k + 4$), are really multiples of 4 because $4k + 4 = 4(k + 1)$, and so are contained in the previous "multiples of 4" subset.

This process partitions the set of integers into four disjoint sets.

> How does that help you avoid that negative contrapositive hypothesis?

Well, that hypothesis,

"4 does not divide the integer $n^2 + 3$",

just means $n^2 + 3$ is never of the $4k$ form. So, having partitioned the integers as above, I can rewrite that contrapositive hypothesis as a positive statement consisting of three cases, meaning

"4 does not divide the integer $n^2 + 3$"

is recast as the equivalent hypothesis:

"The integer $n^2 + 3$ has one of the forms $4k + 1$, or $4k + 2$, or $4k + 3$, where k is some integer."

As a first case, I assumed

"the integer $n^2 + 3$ has the form $4k + 1$ (where k is some integer)",

and as a second case, I assumed

"the integer $n^2 + 3$ has the form $4k + 3$ (where k is some integer)."

In each case I then used algebra to deduce the desired contrapositive conclusion:

"2 does not divide the integer $n^4 - 3$."

[*She then showed me these two proofs, which we verified to be correct.*]

The third and final case I had to prove was:

"If the integer $n^2 + 3$ has the form $4k + 2$ (where k is some integer), then 2 does not divide the integer $n^4 - 3$."

I used the same technique as before, but what I proved was that the opposite conclusion was true, meaning

"2 does divide the integer $n^4 - 3$."

Is it possible that the theorem is true for two of the cases, but not the third one? Or do I need to go back and take another look at what I'm doing?

So, you're asking whether in that third case the theorem is false, or you've made a mistake, right?

Yeah.

First, let's check your work. [*We verified that her surprising proof was indeed correct.*]

But if the theorem's correct, and I didn't make a mistake, then something's wrong. What am I missing?

Let's work through your reasoning with fresh eyes. Look again just at the hypothesis in that third case of the theorem:

"The integer $n^2 + 3$ has the form $4k + 2$ (where k is some integer)."

Notice that this particular written form of the hypothesis, because English is so flexible, disguises the fact that this is a logical "and" statement about integers, describing a subset of integers that have simultaneously the form $n^2 + 3$ and the form $4k + 2$, where n and k are integers.

You correctly proved that if there are any such integers possessing both properties, then the opposite conclusion

"2 does divide the integer $n^4 - 3$"

holds, and hence the theorem would be false. So, if you believe the theorem is true, meaning the conclusion

"2 does divide the integer $n^4 - 3$"

is impossible here, then what improbable fact about the hypothesis must be true, as Sherlock Holmes might say?

That there are no integers like that!

Right. Of course, integers do exist of the form $n^2 + 3$, and integers do exist of the form $4k + 2$, so ...

Yes, yes. I need to prove that no integers exist that possess both those properties at the same time. If I do that, then that third hypothesis, though logically possible, is never in fact satisfied by any subset of integers. So, the theorem is not proved false since that third case never occurs.

Wow! My hidden assumption is that something logically possible must be actually possible, which need not be true. My proof of that case is an argument in a vacuum!

> Correct! [*Afterward, proving there were no such integers was easy using the skills she'd already shown.*]

References

[1] Gary Chartrand, Albert D. Polimeni, Ping Zhang, *Mathematical Proofs, A Transition to Advanced Mathematics*, third edition, Pearson Education Inc., 2013