IRREGULAR TRANSFORMATION GROUPS

ON COMPACT POLYHEDRA

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CHAPTER I

INTRODUCTION

Let (X, T, π) be a transformation group where the phase space X is a metric space with the metric d and the phase group T is a topological group. If T is effective, then we can regard T as a subgroup of the group of all homeomorphisms of X onto itself with an appropriate topology (5). In case that T is either Z, the additive group of integers with the discrete topology, or R, the additive group of real numbers with usual topology, (X, T, π) is called a <u>flow</u>. We call (X, R, π) a <u>continuous flow</u> and (X, Z, π) a <u>discrete flow</u>.

Regarding T as a family of homeomorphisms from X onto itself, from the point of view of equicontinuity, we have the following three cases.

1. T is regular (Reg (T) = X)
2. T is intermediate ($\emptyset \subsetneq$ Reg (T) $\subsetneq X$) or
3. T is irregular (Reg (T) = $\emptyset$)

The cases 1) and 2) have provided some of the most interesting theories and results in topological dynamics. For example, it has been shown that (X, T) is uniformly almost periodic if, and only if, Reg (T) = T (4). Also, one can show that if T is compact, Reg (T) = X. Perhaps one of the most interesting results obtained, in this line, is a theorem of Kérékjartó. In (10), he shows that if, in (S^2, Z), Z is regular, except possibly at a finite number of points, then Z is generated by a one point compactification of a homeomorphism which is topologically equivalent to a bilinear transformation of the complex plane.

Although the concept of expansiveness, as a special case of irregularity, has been studied extensively (2) (8) (9) (11) (13) (14), the irregularity in general has not been exploited. Gottschalk's (5) question about the existence of expansive homeomorphism on the unit disk motiviated the studies of expansiveness. These studies are, quite naturally, concerned with the existence of an expansive discrete flow (X, Z, π) where X is a compact manifold. The complexity of determining the existence of an expansive transformation group can well be illustrated by pointing out that the problem of determining the existence of such a transformation group on S^2 is still outstanding. Lam (12) asked for what spaces one can define an irregular homeomorphism. The purpose of this paper is to give a somewhat more complete solution to the question by relaxing the condition for expansiveness. This can be done by not insisting that all pairs of distinct points "move away" from each other (expansiveness) but only requiring that for each point x, there is a point y arbitrarily close to x such that x and y "move away" from each other (uniform irregularity).

As far as a discrete flow is concerned, we can completely ignore the topology on Z and just talk about the iterates of its generator. Since the topology of Z induced from the usual topology of R is the discrete topology, the existence of an irregular continuous flow can be established by constructing an irregular homeomorphism which can be embedded in a continuous flow.

In Chapter II, after establishing some notations and basic definitions, some of the basic properties, concerning irregularity, of homeomorphisms on one dimensional compact polyhedra will be discussed. Then a necessary and sufficient condition for the existence of an irregular flow on a compact polyhedron will be established in Chapter III. In Chapter IV, along with some examples, characterizations of expansive homeomorphisms and uniformly irregular homeomorphisms will be given. In addition lifts and projections of an irregular homeomorphism, via covering projections, will be discussed in Chapter IV. Finally, in Chapter V, some open questions about irregularity will be given.

For basic concepts and theorems used without specific references, the reader is referred to (3) for general topology, (15) for piecewise linear topology and (4) or (6) for topological dynamics.

CHAPTER II

IRREGULARITY IN ARCS AND SIMPLE CLOSED CURVES

All spaces considered in this paper are metric spaces and we let d denote the metric. All maps are continuous functions. By the n-dimensional Euclidean space R^n, we mean the set of all sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$ of real numbers such that $x_i = 0$ for $i > n$, with a topology induced by the norm given by $||x|| = (\sum_{i=1}^{\infty} x_i^2)^{\frac{1}{2}}$ If $x \varepsilon R^n$, we also write $x = (x_1, x_2, \ldots, x_n)$.

By the <u>n-ball</u> B^n, we mean the set of all points $x \varepsilon R^n$ such that $||x|| \leq 1$ and by an <u>n-cell</u>, we mean a space which is homeomorphic to B^n. In particular, a 1-cell is called an <u>arc</u>. We let I denote the unit closed interval in R^1. Other closed intervals in R^1 are denoted by [a,b] and corresponding open intervals are denoted by (a,b).

By the standard <u>n-sphere</u> S^n, we mean the set of all points $x \varepsilon R^{n+1}$ such that $||x|| = 1$. A space which is homeomorphic to the standard 1-sphere is called a <u>simple</u> <u>closed</u> <u>curve</u>.

A <u>neighborhood</u> of A in X is an open set in X which contains A. If $x \varepsilon X$, then the ε-neighborhood of x, $N_\varepsilon(x)$, is the set of all points y of X such that $d(x,y) < \varepsilon$. In case that $y \varepsilon N_\varepsilon(x)$, we say that y is ε-close to x. The symbol int(A) denotes the point-set interior of A.

A homeomorphism of a space onto itself is called a <u>homeomorphism on</u> X. If h is a homeomorphism on X and n is a positive integer, h^n

denotes the n-fold composition of h and we let $h^{-n} = (h^{-1})^n$, $h^o = 1_X$, the identity map on X.

A family F of maps on X is equicontinuous at $x \in X$ if for each positive number ε, there is a positive number δ such that $d(x,y) < \delta$ implies $d(f(x), f(y))<\varepsilon$ for all $f \in F$. A homeomorphism h on X is said to be regular at $x \in X$ if $\{h^n | n\varepsilon Z\}$ is an equicontinuous family at x and in this case x is called a regular point of h. We write Reg(h) for the set of all regular points of h. If x is not a regular point of h we call x an irregular point of h and write Irr(h) for the set of all irregular points of h. If $h(x) = x$, x is called a fixed point of h and we let Fix(h) denote the set of all fixed points of h.

Let h be a homeomorphism and $x \in X$. The set $\sigma(x) = \{h^n(x) | n\varepsilon Z\}$ is called the orbit of x under h. The orbit $\sigma(x)$ of x is positively asymptotic (negatively asympototic) to a set $A \subset X$ if for each positive number ε, there is an integer N such that $n > N$ $(n < N)$ implies $d(h^n(x),A)< \varepsilon$. If $\sigma(x)$ is both positively and negatively asymtotic to A, then we say that $\sigma(x)$ is asymptotic to A.

Two homeomorphisms h_1, h_2 on X are topologically equivalent if h_1 and h_2 belong to the same conjugacy class in the group of all homeomorphisms on X.

It is a known fact that if h is a homeomorphism on an arc or a simple closed curve then $\text{Reg}(h) \neq \emptyset$ (12). In this chapter we prove a slightly different version of this fact.

Lemma 2.1. Let X be a compact space and h a homeomorphism from X onto Y. Then for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $d(x,y) \geq \varepsilon$ then $d(h(x),h(y)) \geq \delta$.

Proof: By uniform continuity of h^{-1}, choose $\delta > 0$ so that if $d(h(x),h(y)) > \delta$ then $d(h^{-1}h(x),h^{-1}h(y)) = d(x,y) > \varepsilon$.

Definition 2.2. A homeomorphism h on X is said to be an irregular homeomorphism if $Irr(h) = X$.

Lemma 2.3. If h is a homeomorphism on a compact space X and ϕ is a homeomorphism from X onto Y, then $Irr(\phi h\phi^{-1}) = \phi(Irr(h))$.

Proof: Let $y \in Irr(\phi h\phi^{-1})$. Then there is an $\varepsilon > 0$ such that for each $\eta > 0$, we can find y' which is η-close to y but $(\phi h\phi^{-1})^n(y') = \phi h^n\phi^{-1}(y')$ is not ε-close to $\phi h^n\phi^{-1}(y)$ for some $n \in Z$. By lemma 2.1, there is a $\delta > 0$ such that if $d(y,y') \geq \varepsilon$ then $d(\phi^{-1}(y),\phi^{-1}(y')) \geq \delta$. Let $x = \phi^{-1}(y)$. For each $\lambda > 0$ there is a $\lambda' > 0$ such that if y' is λ'-close to y then $\phi^{-1}(y')$ is λ-close to $\phi^{-1}(y)$. Thus, if we choose y' which is λ'-close to y and $\phi h^n\phi^{-1}(y')$ is not ε-close to $\phi h^n\phi^{-1}(y)$, then $x' = \phi^{-1}(y')$ is a point which is λ-close to x but $h^n(x')$ is not δ-close to $h^n(x)$ so that $Irr(\phi h\phi^{-1}) \subset \phi(Irr(h))$.

To prove the other inclusion, note that $Irr(\phi^{-1}(\phi h\phi^{-1})\phi) \subset \phi^{-1}(Irr(\phi h\phi^{-1}))$. Thus, $Irr(h) \subset \phi^{-1}(Irr(\phi h\phi^{-1}))$ so that $\phi(Irr(h)) \subset Irr(\phi h\phi^{-1})$.

In view of lemma 2.3, we see that, for compact spaces, the property of supporting an irregular homeomorphism is a topological property. In particular, this property does not depend on the metric. By setting X = Y, we also see that if h_1 and h_2 are topologically equivalent, then h_1 is irregular if, and only if h_2 is irregular.

Lemma 2.4. Let h be a homeomorphism on a compact space X and let $x \in X$. Then for each $n \neq 0$, x is an irregular point of h if, and only if, x is an irregular point of h^n.

Proof: Sufficiency is trivial by definition.

To prove the necessity, assume that $x \in Irr(h)$. There is then an $\eta > 0$ such that for each $\delta > 0$, we can find $y \in X$ with $d(x,y) > \delta$ and $d(h^m(x), h^m(y)) \geq \eta$ for some $m \in Z$. Since n is not zero, we can find integers k and r such that $0 \geq r < n$ and $m + r = kn$. For each $i = 0, 1, \ldots, n$, there is a $\delta_i > 0$ such that if $d(x',y') \geq \eta$ then $d(h^i(x'), h^i(y')) \geq \delta_i$. Let $\varepsilon = \min \{\delta_i | i = 0, 1, \ldots, |n|-1.\}$ Then $d((h^n)^k(x), (h^n)^k(y)) = d(h^{nk}(x), h^{nk}(y)) = d(h^{r+m}(x), h^{r+m}(y)) = d(h^r(h^m(x)), h^r(h^m(y)) \geq \varepsilon$. Therefore $x \in Irr(h^n)$.

By the above lemma and the definition, h is an irregular homeomorphism on X if, and only if, h^n is an irregular homeomorphism for each $n \neq 0$.

If h is a map defined from X into X, it is trivial to see that Fix(h) is closed in X.

Lemma 2.5. If h is a homeomorphism on I and if $h(0) = 0$, $h(1) = 1$, then $Irr(h) \subset Fix(h)$.

Proof: If $x \notin Fix(h)$, then there is a neighborhood of x which contains no fixed points of h. Let $x_o = \sup \{y \in I | y \leq x, h(y) = y\}$ and $x_1 = \inf \{y \in I | y \geq x, f(y) = y\}$. Note that both x_o and x_1 exist, $x_o \neq x_1$, $x_o, x_1 \in Fix(h)$, $x \in (x_o, x_1)$ and (x_o, x_1) contains no fixed points of h. Thus, $h(y) > y$ or $h(y) < y$ for all $y \in (x_o, x_1)$. We may assume that $h(y) > y$ for all $y \varepsilon (x_o,x_1)$. Then $\sigma(y)$ is positively asymptotic to x_1 and negatively asymptotic to x_o. Let $\varepsilon > 0$ be given and choose any $y_o < x$, $y_1 > x$. There are integers N_o and N_1 such that if $n < N_o$ then $h^n(x)$ and $h^n(y_1)$ are both ε-close to x_o and if $n > N_1$, then $h^n(x)$ and $h^n(y_o)$ are both ε-close to x_1. By uniform continuity of $h^{N_o+1}, \ldots, h^{N_1}$, we can find δ_o such that if y is δ_o-close to x then $h^i(y)$ is ε-close to $h^i(x)$ for all $i = N_o = 1, \ldots, n_1$. Let $\delta = \min\{\delta_o, y_1 - x, x-y_o\}$ then, for each y which is δ-close to x, $h^n(y)$ is ε-close to $h^n(x)$ for all $n \varepsilon Z$.

Lemma 2.5 does not hold true in general as we will see in Chapter III, that, on a 2-cell, we can define a homeomorphism with irregular points which are not fixed points.

<u>Proposition 2.6</u>. For each homeomorphism h on I, Irr(h) is nowhere dense subset of I.

<u>Proof</u>: In view of lemma 2.4, we may assume that $h(0) = 0$ and $h(1) = 1$ so that $Irr(h) \subset Fix(h)$. If Fix(h) is nowhere dense then there is nothing to show. On the other hand, if $\overline{Fix(h)} = Fix(h)$ contains an open set, then $int\,(Fix(h)) \subset Reg\,(h)$. Thus, Irr(h) is nowhere dense in I.

Since Irr(h) is a topological invariant for compact spaces, lemma 2.5 and proposition 2.6 remain valid for any arc.

By a <u>cantor set</u>, we mean a totally disconnected compact perfect metric space (3). If E is a cantor subset of an arc or a simple closed curve, we can of course think of the complement of E as a sequence of disjoint open intervals whose diameters converge to zero.

Definition 2.7. By a transformation group (X, T, π) we mean a space X, a topological group T, and a map $\pi : X \times T \to x$ satisfying the following conditions:

1) $\pi(x,0) = x$ for all $x \in x$ and the identity 0 of T

2) $\pi(\pi(x,t),t') = \pi(x,t+t')$ for all $x \in x$, and all $t,t' \in T$.

A set $A \subset X$ is said to be invariant under T if $\pi(A \times T) = A$. A closed subset A of X is called a minimal set if A is invariant under T and A contains no proper closed subset which is invariant under T.

In case that X is compact, the existence of a minimal set can be easily established (5).

We need following results by van Kampen and we will refer the reader to (17) for the proofs. To avoid unnecessary complication, we replace a simple closed curve by the standard 1-sphere.

Proposition 2.8. Let h be a homeomorphism on S^1. If h has no periodic points, then the set E of all cluster points of $\mathcal{O}(x)$ is independent of the choice of x and it is either S^1 or a cantor subset of S^1.

Proposition 2.9. If h is a homeomorphism on S^1 such that $E = S^1$, then it is topologically equivalent to a rotation.

We are now ready to prove the following proposition.

Proposition 2.10. For any homeomorphism h on S^1, Irr(h) is nowhere dense in S^1.

Proof: In case that h has a periodic point, we may assume that h has a fixed point. If h has exactly one fixed point $x_o \in S^1$, then for each $x \in S^1$, $\sigma(x)$ is asymptotic to x_o. So if $x \neq x_o$, by an argument similar to that of Lemma 2.4, we can easily show that $x \in \text{Reg}(h)$. If h has more than one fixed point, then we can further assume that two arcs A_1 and A_2 on S^1 determined by any two fixed points are invariant under h. Since $\text{Irr}(h) \cap A_i$ is nowhere dense in A_i for $i = 1,2$, Irr(h) is nowhere dense in S^1.

If h has no periodic points and $E = S^1$, then h is topologically equivalent to a rotation. Consequently, $\text{Irr}(h) = \emptyset$.

If he has no periodic points and $E \neq S^1$, then we can find a minimal set which is a cantor set and which consists of the common cluster points of orbits (9). Let C denote this minimal set and $\{A_i\}_{i=1}^{\infty}$ denote the complementary arcs of C. Given $\varepsilon > 0$, there are only finitely many complementary arcs $A_{i_1}, \ldots, A_{i_k}$ with diameters greater than or equal to ε. Since h does not have any periodic point, $h(A_i) = A_j$ where $i \neq j$. Therefore, there is a positive integer N such that if $|n| > N$ then diameter of $h^n(A_i)$ is less than ε. By uniform continuity of h^i, $i = -N, -N+1, \ldots, N$, choose $\delta > 0$ such that if diam $A < \delta$ then diam $h^i(A) < \varepsilon$. Thus, for $x \in A_i$, if we choose $N_\delta(x) \subset A_i$, then we see that diam $(h^n(N_\delta(x))) < \varepsilon$ for all n. Consequently, $\text{Irr}(h) \subset C$ and it is nowhere dense in S^1.

<u>Corollary 2.11</u>. A finite graph does not support an irregular homeomorphism.

<u>Proof</u>: A finite graph X is a union of finite number of arcs and simple closed curves. Let h be a homeomorphism on X and let $\{V_i\}_{i=1}^{k}$ be the collection of vertices of orders different from 2. Then $h|_{\{V_i\}_{i=1}^{k}}$ is a permutation on $\{V_i\}_{i=1}^{k}$. Thus, there is an integer

$n \neq 0$ such that $h^n(V_i) = V_i$ for all $i = 1, \ldots, k$. Now, h^n permutes arcs with the common vertices and permutes simple closed curves with the common vertex. Therefore, for some integer $m \neq 0$, $h^{nm}(A) = A$ where A is an arc or a simple closed curve. Consequently, h^{nm} is not an irregular homeomorphism so that h is not an irregular homeomorphism.

CHAPTER III

IRREGULARITY IN COMPACT POLYHEDRA

By a compact polyhedron, we mean the underlying space of a finite simplicial complex or a finite cell complex. Since there is a subdivision of a cell complex into a simplicial complex, we simply refer to either one of them as a complex. An annulus is a space which is homeomorphic to the product space $S^1 \times I$.

By a pair (X,A), we mean a space X with a subset A. By a homeomorphism from a pair (X,A) onto a pair (Y,B), we mean a homeomorphism from X onto Y such that the image of A under the homeomorphism is B. In case that there is such a homeomorphism, we say that (X,A) and (Y,B) are homeomorphic.

A map is a continuous function and 1_X denotes the identity map on X. If f is a map with the domain X and $A \subset X$ then $f|_A$ denotes the restriction of f to A. A map $f:X \to Y$ is null homotopic if f is homotopic to a constant map from X into Y. X is said to be contractible if 1_X is null homotopic. A retraction from a space X onto a subset A is a map r such that $r|_A = 1_A$. A mapping cylinder M_f of a map $f:X \to Y$ is obtained by taking disjoint union of $X \times I$ and Y and identifying $(x,1)$ with $f(x)$.

A principal n-cell in a complex is an n-cell which intersects higher dimensional cells in a subset of its boundary.

Throughout this chapter we use some standard terminologies of topological dynamics and piecewise linear topology and refer the reader to (6) and (15) for definitions. If X is a manifold, the symbols $\mathring{X}$ and ∂X are used to denote the <u>combinatorial</u> <u>interior</u> of X and the <u>combinatorial</u> <u>boundary</u> of X respectively.

It is a well known fact that the homeomorphism h on S^1, in the complex plane, defined by $h(e^{i\theta}) = e^{i(\theta + 2\pi t)}$, $0 \leq t \leq 1$, is periodic with the period q if t is rational and $t = \frac{p}{q}$ in the lowest term, and each point of S^1 has dense orbit under h if t is irrational.

<u>Lemma 3.1</u>. Define $h: S^1 \times I \to S^1 \times I$ by $h(e^{i\theta},t) = (e^{i(\theta + 2\pi t)},t)$. Then h is an irregular homeomorphism.

<u>Proof</u>: In view of lemma 2.3, we may assume that the metric d on $S^1 \times I$ is the product metric. Let $x \in S^1 \times I$ and write $x = (e^{i\theta},t)$, $0 \leq \theta \leq 2\pi$, $t \in I$. For each $\delta < 0$, we can take t' such that $0 < |t-t'| < \min\{\delta, \frac{1}{4}\}$. Then $0 < d((e^{i\theta},t), (e^{i\theta},t')) < \delta$ and there is an integer n such that $\frac{1}{4} \leq n(t-t') \leq \frac{1}{2}$. Thus, $d(h^n(e^{i\theta},t), H^n(e^{i\theta},t')) = d((e^{i(\theta+2n\pi t)},t), (e^{i(\theta+2n\pi t')},t')) > d((e^{i(\theta+2n\pi t)},t), (e^{i(\theta+2n\pi t')},t)) = e^{i(\theta+2n\pi t)} - e^{i(\theta+2n\pi t')} = 2 \sin n\pi(t-t') \geq \sqrt{2}$ by the choice of n. Thus, with any $\varepsilon \leq \sqrt{2}$, $x \in Irr(h)$ for each $x \in X$.

<u>Corollary 3.2</u>. Let X be a compact space. If there is a map $f: X \to I$ such that $int(f^{-1}(t))$ is empty for each $t \in I$, then $S^1 \times X$ admits an irregular homeomorphism.

<u>Proof</u>: Assume that $S^1 \times X$ has the product metric. Define $g: S^1 \times X \to S^1 \times X$ by $g((e^{i\theta},x)) = (e^{i(\theta+\pi f(x))},x)$. For each $(e^{i\theta},x) \in S^1 \times X$ and any neighborhood U of $(e^{i\theta},x)$, there is a $N_\delta(x)$ in $\pi_x(U)$, where π_x is the projection map of $S^1 \times X$ onto X. Thus, there is a point

$y \neq x$ in $N_\delta(x)$ such that $f(x) \neq f(y)$ since Int $(f^{-1}(t)) = \emptyset$ for each t. Therefore $0 < |f(x) - f(y)| \leq 1$ so that $\frac{1}{2} < n(f(x) - f(y)) \leq 1$ for some integer n. Then $d(h^n(e^{i\theta},x), h^n(e^{i\theta},y)) = d((e^{i(\theta+n\pi\cdot f(x))},x), (e^{i(\theta+n\pi\cdot f(y)},y)) > d((e^{i(\theta+n\pi\cdot f(x))},x), (e^{i(\theta+n\pi\cdot f(y))},x)) \geq \sqrt{2}$. Therefore, with any $\varepsilon \leq \sqrt{2}$, $(e^{i\theta},x) \in \text{Irr}(g)$ for any $(e^{i\theta},x) \in S^1 \text{x} X$.

Lemma 3.3. There is an irregular homeomorphism ζ_2 on B^2 such that $\zeta_2|\partial_B 2$ is equal to $1_{\partial B^2}$.

Proof: Let $f\colon S^1 \times 1 \to B^2$ be a map which satisfies the following conditions: $f|_{S^1 \times (0,1]}$ is a homeomorphism of $S^1 \times (0,1]$ onto $B^2 - \{(x,0)|- \frac{1}{2} \leq x \leq \frac{1}{2}\}$, $f((e^{i\theta},0)) = f((e^{i(2\pi-\theta)},0))$, $0 \leq \theta \leq 2\pi$ $f((e^{i\pi},0)) = (-\frac{1}{2},0)$, $f((e^{i0},0)) = (\frac{1}{2},0)$ and $f|_{\{(e^{i\theta},0)|0 \leq \theta \leq \pi\}}$ is a homeomorphism of $\{(e^{i\theta},0)|0 \leq \theta \leq \pi\}$ onto $\{(x,0)| - \frac{1}{2} \leq x \leq \frac{1}{2}\}$. Take $h\colon S^1 \times I \to S^1 \times I$ defined in lemma 3.1. Define $\zeta_2\colon B^2 \to B^2$ by $\zeta_2(x) = fhf^{-1}(x)$. Then it is easy to see that ζ_2 is a homeomorphism on B^2. If $p \in B^2 - \{(x,0)|- \frac{1}{2} \leq x \leq \frac{1}{2}\}$ then $p = f((e^{i\theta},t))$ for some $t \neq 0$. Since $f|_{S^1 \text{x} [\frac{t}{2},1]}$ is a homeomorphism of $S^1 \times [\frac{t}{2},1]$ onto an annulus $A \subset B^2-[-\frac{1}{2},\frac{1}{2}]$, both $f|_{S^1 \text{x}[\frac{t}{2},1]}$ and $f^{-1}|_A$ are uniformly continuous. Thus, if $\zeta_{2|A}$ were equicontinuous at p then $h_{|S^1 \text{x}[\frac{t}{2},1]}$ would be equicontinuous at $(e^{i\theta},t)$. Therefore $\zeta_{2|A}$ is not equicontinuous at p so that ζ_2 is not equicontinuous at p. If $p \in \{(x,0)|-\frac{1}{2} \leq x \leq \frac{1}{2}\}$, then $\zeta_2(p) = p$. Choose $\varepsilon > 0$ so that $N_{2\varepsilon}(p)$ does not contain $\{(x,0)|-\frac{1}{2} \leq x \leq \frac{1}{2}\}$. Then there is an η and a neighborhood U of $e^{i\eta}$ such that $U \times [0,t] \subset S^1 \times [0,t] - f^{-1}(N_\varepsilon(p))$ for each $t \in I$. For each $\delta > 0$, pick $(e^{i\theta},t) \in f^{-1}(N_\delta(p))$ which has dense orbit in $S^1 \times \{t\}$ under h. Then there is an integer n such that

$h^n((e^{i\theta},t)) \in U \times \{t\}$. Therefore, $\zeta_2^n(f((e^{i\theta},t))) \notin N_\varepsilon(p)$ and $f((e^{i\theta},t)) \in N_\delta(p)$ which shows that ζ_2 is not equicontinuous at p. It is clear that $\zeta_2|_{\partial B^2} = 1_{\partial B^2}$.

Lemma 3.4. For each $n \geq 2$, B^n admits an irregular homeomorphism ζ_n such that $\zeta_n|_{\partial B^n} = 1_{\partial B^n}$.

Proof: We prove this lemma by induction on n. By lemma 3.3, B^2 admits such a homeomorphism. Assume that there is such a homeomorphism ζ_{n-1} on B^{n-1}. For each θ, $0 \leq \theta < 2\pi$, let $B_\theta^{n-1} = (x_1, \ldots, x_{n-2}, x_{n-1}\cos\theta, x_{n-1}\sin\theta) | \sum_{i=1}^{n-1} x_i^2 \leq 1$ and $x_{n-1} \geq 0\}$. Then B_θ^{n-1} is the closed half of the unit ball sitting in the subvector space in R^n of dimension n-1 which is determined by R^{n-2} and the vector $(0, \ldots, 0, \cos\theta, \sin\theta) \in R^n$. Thus, it is easy to see that $B^n = \bigcup_{0 \leq \theta \leq 2} B_\theta^{n-1}$ and $B_\theta^{n-1} \cap B_\theta^{n-1} = B^{n-2}$ for $\theta \neq \theta'$. Since $(B^{n-1}, \partial B^{n-1})$ and $(B_o^{n-1}, \partial B_o^{n-1})$ are homeomorphic as compact pairs, there is an irregular homeomorphism $\psi_{n-1}: B_o^{n-1} \to B_o^{n-1}$ such that $\psi_{n-1}|_{\partial B_o^{n-1}} = 1_{\partial B_o^{n-1}}$. Define $\zeta_n: B^n \to B^n$ by $\zeta_n|_{B_\theta^{n-1}} = \rho_\theta \psi_{n-1} \rho_\theta^{-1}$ where $\rho_\theta: B_o^{n-1} \to B_\theta^{n-1}$ is the homeomorphism defined by $\rho_\theta((x_1, \ldots, x_{n-2}, x_{n-1}, 0)) = (x_1, \ldots, x_{n-1}\cos\theta, x_{n-1}\sin\theta)$. Then ζ_n is a well defined function since $\rho_\theta\psi_{n-1}\rho_\theta^{-1}|_{B^{n-2}} = 1|_{B^{n-2}}$ for any θ. Let $x \in B^n - B^{n-2}$. Then a sequence $\{x^i = (x_1^i, \ldots, x_{n-2}^i, x_{n-1}^i\cos\theta^i, x_{n-1}^i\sin\theta^i)\}_{i=1}^{\infty}$ converges to $x = (x_1, \ldots, x_{n-2}, x_{n-1}\cos\theta, x_{n-1}\sin\theta)$ if and only if $\{(x_1^i, \ldots, x_{n-2}^i, x_{n-1}^i)\}_{i=1}^{\infty}$ converges to $(x_1, \ldots, x_{n-2}, x_{n-1})$ and $\{\theta^i\}_{i=1}^{\infty}$ converges to θ up to module 2π. Thus, the continuity of

ζ_n at $x \in B^n - B^{n-2}$ is clear. Suppose $x \in B^{n-2}$. Then a sequence $\{x^i\}_{i=1}^{\infty}$ converges to x if and only if $\{\rho_{\theta^i}^{-1}(x^i)\}_{i=1}^{\infty}$ converges to x, since $d(x, \rho_{\theta i}^{-1}(x^i)) = d(x,x^i)$ for each i. Therefore ζ_n is continuous at x. Since the map $\zeta_n': B^n \to B^n$ defined by $\zeta_n' \mid B_\theta^{n-1} = \rho_\theta \psi_{n-1}^{-1} \rho_\theta^{-1}$ is the inverse of ζ_n, ζ_n is a homeomorphism. Furthermore, $\zeta_n|_{B_\theta^{n-1}}$ is the identity on ∂B^{n-1} for each θ so that $\zeta_n|_{\partial B^n} = 1_{B^n}$ since $\partial B^n \subset \bigcup_{0<\theta 2\pi} \partial B_\theta^{n-1}$. ζ_n is an irregular homeomorphism since $\zeta_n|_{B_\theta^{n-1}}$ is an irregular homeomorphism for each θ.

Theorem 3.5. A compact polydedron P admits an irregular homeomorphism if and only if P contains no principal 1-cells.

Proof: To prove the necessity, suppose that P contains a principal 1-cell and suppose that there is an irregular homeomorphism h on P. Let K be a triangulation of P, K_1 be the collection of prinicpal 1-cells in K and write $|K_1| = P_1$. Then $h(P_1) = P_1$. Since $P_1 \cap |K - K_1|$ is finite, the regular set of $h|_{P_1}$ is at most finite. But using the fact that the irregular set of a homeomorphism on either a simple closed curve or a 1-cell is nowhere dense, we can show that the irregular set of a homeomorphism on P_1 is nowhere dense. Therefore, h cannot be an irregular homeomorphism on P.

If P does not contain any principal 1-cell then we can write $P = \cup\{\sigma_j\}_{j=1}^{k}$ where σ_j is a principal n-simplex with $n \geq 2$ in some triangulation $\{\sigma_i\}_{i=1}^{m}$. Therefore, since $\zeta_n|_{\partial B^n} = 1_{\partial B^n}$, we can define an irregular h on P by taking h to be $g_n\zeta_n g_n^{-1}$ on each principal cell σ_j of dimension n where $g_n\colon B^n \to \sigma_j$ is a homeomorphism of B^n onto σ_j.

Lemma 3.6. Let C be a locally connected contractible continuum in int B^2, where $B^2 \subset R^2$. If C is nowhere dense in R^2, then there is a map f from S^1 onto C such that the pair (M_f,C) is homeomorphic to (B^2,C).

Proof: Since C is strongly cellular (16), there is a circle S and a homotopy H of S in R^2 such that

(1) H_o is the identity.

(2) H_t is an embedding for $t < 1$.

(3) $H_t(S) \cap H_u(S) = \emptyset$ for $t \neq u$, and

(4) $h_1(S) = C$ (7)

By the Schöenflies theorem, S bounds a 2-cell. Therefore, we may assume that $S = S^1$. It is clear, from the properties of H, that $H|_{S^1 \times [0,1)}$ is an imbedding and $\mathrm{Im}(H) \subset B^2$. To prove that $\mathrm{Im}(H) = B^2$, suppose that there is $x \in \mathrm{Int}(B^2) - C$ such that $\mathrm{Im}(H) \subset B^2-\{x\}$. Then there is a retraction $\gamma\colon B^2-\{x\} \to S^1$. Now, γH_1 is homotopic to $\gamma H_o = 1_{S^1}$. But, since C is contractible, γH_1 is null homotopic. Thus, we obtain a contradiction. By taking $f = H_1$, we see that (M_f,C) is homeomorphic to (B^2,C).

Theorem 3.7. For each nondegenerate locally connected contractible continuum C which is nowhere dense in intB^2, there is an irregular homeomorphism h_C on B^2 such that $Fix(h_C) = \{x \in B^2 | h_C(x) = x\}$ is C.

<u>Proof</u>: Let f be the map in lemma 3.6. Since f is a closed map from S^1 onto C, C has the identification topology with respect to f. Thus, (M_f, C) is homeomorphic to $(\frac{S^1 \times I}{\sim}, \{[e^{i\theta},1] | e^{i\theta} \in S^1\})$ where ~ is the equivalence relation on $S^1 \times I$ induced by the map H which is defined in lemma 3.6 and [x,t] denotes the equivalence class of (x,t). Write $\{[e^{i\theta},1] | e^{i\theta} \in S^1\} = C'$. Then it suffices to show the existence of an irregular homeomorphism h* on $\frac{S^1 \times I}{\sim}$ with Fix(h*) = C'. Let $p: S^1 \times I \to \frac{S^1 \times I}{\sim}$ be the projection and $h: S^1 \times I \to S^1 \times I$ be defined by $h(e^{i\theta},t) = (e^{i(\theta+\pi(1-t))},t)$. Then, by the argument used in lemma 3.1, h is an irregular homeomorphism on $S^1 \times I$ and Fix(h) = $S^1 \times \{1\}$. Define $h^*: \frac{S^1 \times I}{\sim} \to \frac{S^1 \times I}{\sim}$ by $h^*([e^{i\theta},t]) = ph(e^{i\theta},t)$. Then, since h* is well defined one to one correspondence, it is a homeomorphism. Since $p|_{S^1 \times [0,t]}$ is a homeomorphism and $S^1 \times [0,t]$ is compact for each t < 1, it is clear that $\{[e^{i\theta},s] \in \frac{S^1 \times I}{\sim} | s < 1\} \subset Irr(h^*)$. To show that $[e^{i\theta},1] \in Irr(h^*)$, note first that Fix(h*) = C' and diam C' > 0. For each neighborhood U of $[e^{i\theta},1]$, U contains $[e^{i\theta},t]$ for some irrational t. Since the orbit of $(e^{i\theta},t)$ under h is dense in $S^1 \times \{t\}$, the orbit of $[e^{i\theta},t]$ under h* is dense in $\{[e^{i\theta},t] | 0 \leq \theta \leq 2\pi\}$. Now, if we take $\delta = \frac{1}{3}$ diam (C'), then we can find $n \in Z$ such that $d(h^{*n}[e^{i\theta},t], h^{*n}[e^{i\theta},1]) > \delta$.

If h_1 and h_2 are topologically equivalent homeomorphisms, then $Fix(h_1)$ is homeomorphic to $Fix(h_2)$. Consequently, theorem 7 implies the existence of uncountable many conjugacy classes of irregular homeomorphisms on B^2.

Definition 3.8. Let (X,T,π) be a transformation group. Then for each $t \in T$, $\pi^t: X \to X$ defined by $\pi^t(x) = \pi(x,t)$ is called a t-transition. We say that a homeomorphism h on X can be embedded in a continuous flow if there is a transformation group (X, R, π) such that the 1-transition π^1 coincides with h. A discrete flow (X, Z, ρ) embeds in a continuous flow (X, R, π) if $\rho = \pi|_{X \times Z}$.

It is clear that a discrete flow (X, Z, ρ) embeds in a continuous flow (X, R, π) if, and only if, the 1-transition ρ^1 is embedded in (X, R, π).

Remark A: Let $f: X \to Y$ be an onto map where X and Y are compact metric spaces. Then the relation on X defined by $x \sim x'$ if, and only if, $f(x) = f(x')$ partitions X into subsets each of which is an inverse image of a point (point inverse) in Y under f. Define $G(f) = \{h \in \mathcal{H}(X) |$ for each $p \in Y$, $h(f^{-1}(p)) = f^{-1}(q)$ for some $q \in Y\}$ where $\mathcal{H}(X)$ denotes the group of all homeomorphisms on X. Then it is easy to see that $G(f)$ forms a subgroup of $\mathcal{H}(X)$. Since X and Y are compact metric spaces, $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are topological groups with compact-open topology (1). Define $\alpha: G(f) \to \mathcal{H}(Y)$ by $(\alpha(h))(y) = fh(f^{-1}(y))$, for each $h \in G(f)$ and each $y \in Y$. Then α is a continuous homomorphism from the topological group $G(f)$ into the topological group $\mathcal{H}(Y)$.

<u>Theorem 3.9</u>. Let $f: X \to Y$ be an onto map where X and Y are compact metric spaces. If $h \in H(X)$ can be embedded in a continuous flow (X, R, π) such that $\pi^t \in G(f)$ for each t, then $h' \in H(Y)$ defined by $h'(y) = (\alpha(h))(y)$ can be embedded in a continuous flow (Y, R, λ).

<u>Proof</u>: Define $\rho_\pi: R \to H(X)$ by $\rho_\pi(t) = \pi^t$ for each $t \in R$. Then the continuity of π gives the continuity of ρ_π. Since the map $\alpha: G(f) \to H(y)$ is continuous and $\rho_\pi(R) \subset G(f)$, we have a map $\rho_\lambda: R \to H(Y)$ defined by $\rho_\lambda(t) = (\alpha \rho_\pi)(t) = \alpha(\pi^t)$. Define $\lambda: Y \times R \to Y$ by $\lambda(y,t) = (\alpha(\pi^t))(y)$. Then λ is continuous. It is clear that $\lambda(y,0) = y$ and $\lambda(y,1) = \lambda^1(y) = fh(f^{-1}(y))$ for all $y \in Y$.
$\lambda(\lambda(y,t),t') = \lambda((\alpha(\pi^t))(y),t') = \lambda(f\pi^t f^{-1}(y),t') = (f\pi^t f^{-1})$
$(f\pi^{t'} f^{-1}(y)) = f\pi^{t'}\pi^t f^{-1}(y) = f\pi^{t'+t} f^{-1}(y) = f\pi^{t'+t} f^{-1}(y) =$
$\lambda(y,t'+t)$.

<u>Remark B</u>. Define a homeomorphism h on $S^1 \times I$ by

$$h(e^{i\theta},t) = \begin{cases} (e^{i(\theta+2\pi t)},t), & 0 \le t \le \frac{1}{2} \\ (e^{i(\theta+2\pi(1-t)},t), & \frac{1}{2} \le t \le 1. \end{cases}$$

Then h is embedded in a continuous flow $(S^1 \times I, R, \pi)$ where $\pi: (S^1 \times I) \times R \to S^1 \times I$ is defined by

$$\pi((e^{i\theta},t),r) = \begin{cases} (e^{i(\theta+2\pi t\cdot r)},t), & 0 \le t \le \frac{1}{2} \\ (e^{i(\theta+ 2\pi(1-t)\cdot r)},t), & \frac{1}{2} \le t \le 1. \end{cases}$$

The map $f: S^1 \times I \to B^2$ defined in the proof of lemma 3.4 is an onto map such that for each $y \in B^2$, $\pi^t(f^{-1}(y)) = f^{-1}(y')$ for some $y' \in B^2$. Consequently, by theorem 3.10, ξ_2 defined by $\xi_2(y) = fh(f^{-1}(y))$ for each $y \in B^2$ can be embedded in a continuous flow (B^2, R, λ_2). Note that $\lambda_2(y,r) = y$ for all $y \in \partial B^2$ and all $r \in R$. Thus, if we define ξ_n inductively as in lemma 3.4 then it is clear that ξ_n can be embedded in a continuous flow (B^n, R, λ_n) such that $\lambda_n(x,r) = x$ for all $x \in \partial B^n$ and all $r \in R$ and $Irr(\lambda_n^1) = B^n$.

Definition 3.10. Given a transformation group (X, T, π) and $S \subset T$, we say that S is regular at $x \in X$ if $\{\pi^s | s \in S\}$ is an equicontinuous family at x. If S is not regular at x then we say that S is irregular at x. Reg(X,S) and Irr(X,S) denote the set of all regular points of S and the set of all irregular points of S, respectively. If Irr(X,T) = X then (S, T, π) is called an irregular transformation group.

Lemma 3.11. Let (X, T, π) be a transformation group and $S \subset T$. If X is compact and $\{\pi^s \mid s \in S\}$ is dense in $\{\pi^t \mid t \in T\}$, with compact-open topologies then Reg(X,T) = Reg(X,S).

Proof: It is clear that Reg(T) $\subset$ Reg(S). Suppose $x \in$ Reg(S). Then given $\varepsilon > 0$, there is $\delta > 0$ such that if $d(x,y) < \delta$ then $d(\pi^s(x), \pi^s(y)) < \varepsilon/3$ for all $s \in S$. Since X is a compact metric space, the compact-open topology on $\{\pi^t \mid t \in T\}$ coincides with the topology induced by the metric $\rho(\pi^t,\pi^{t'}) = \sup_{x \in X} \{d(\pi^t(x),\pi^{t'}(x)\}$ (3). So for each $t \in T$, we can pick an $s \in S$ such that $\rho(\pi^t,\pi^s) < \varepsilon/3$. Then $d(\pi^t(x),\pi^t(y) \leq d(\pi^t(x),\pi^s(x)) + d(\pi^s(x),\pi^s(y)) + d(\pi^s(y),\pi^t(y)) < \varepsilon$.

Lemma 3.12. Let (X, R, π) be a continuous flow with X = I or $X = S^1$. Then Irr(X,R) is nowhere dense in X.

Proof: Irr(X,Z) is nowhere dense in X by propositions 2.6 and 2.10. Q, the set of all rational numbers, is dense in R. Thus, $\{\pi^s \mid s\varepsilon Q\}$ is dense in $\{\pi^t \mid t\varepsilon R\}$ with the compact-open topology. Then, by lemma 2.4, Irr(X,Q) = Irr(X,Z). Now, by lemma 3.11, Irr(X,Q) = Irr(X,R). Since Irr(X,Z) is nowhere dense, we are done.

Theoren 3.13. A compact polydedron P admits an irregular continuous flow if, and only if, P contains no principal 1-cells.

Proof: If P contains a principal 1-cell and (P,R,π) is a continuous flow there is a principal 1-cell C which is R-invariant. Thus, $(C,R,\pi/_{CxR})$ is a continuous flow so that Irr(C,R) is nowhere dense in C. Consequently, if we choose X ε ∮C such that xεReg(C,R) then xεReg(p,R).

Conversely, if P has no principal 1-cells, then by remark B and theorem 3.5, we can define, piecewise on each principal n-simplex, $n \leq 2$, a continuous flow (P,R,π) with $Irr(\pi^1) = P$ so that Irr(P,R) = P.

CHAPTER IV

LIFTS AND PROJECTIONS OF IRREGULAR HOMEOMORPHISMS

A <u>bisequence</u> in a set S is a function from Z into S. For convenience, a bisequence in S is written as $\langle x_i \rangle$. The <u>diagonal</u> of a product space $X \times X$ is denoted by $\Delta(X)$ and the <u>deleted</u> product X^* is defined to be $(X \times X) - \Delta(X)$. For each complex K, we let $|K|$ denote the <u>carrier</u> of K.

<u>Definition 4.1</u>. A homeomorphism h on X is said to be <u>uniformly irregular</u> if there is a $\delta > 0$ such that for each $\varepsilon > 0$ and for each $x \in X$, there exists $y \in X$, which is ε-close to x but $h^n(y)$ is not δ-close to $h^n(x)$ for some $n \in Z$. The number δ is called an <u>uniform irregularity constant</u>.

<u>Definition 4.2.</u> A homeomorphism h on X is <u>expansive</u> if there is a $\delta > 0$ such that for each pair of distinct points x and y in X, $h^n(x)$ and $h^n(y)$ are not δ-close to each other for some $n \in Z$. The number δ is called an <u>expansive constant</u>.

It is obvious that if δ is an uniform irregularity (expansive) constant then any positive number δ' such that $\delta' < \delta$ is also an uniform irregularity (expansive) constant. It is also obvious that an expansive homeomorphism is uniformly irregular and an uniformly irregular homeomorphism is irregular.

We can prove, with little adjustments in the arguments, that all irregular homeomorphisms constructed in Chapter III are uniformly irregular. However, following two examples show that an irregular homeomorphism on a continuum need not be uniformly irregular. In fact, example 4.4 shows that there is a Peano continuum on which an irregular homeomorphism can be defined but it supports no uniformly irregular homeomorphisms.

Let $f_1, f_2, \ldots$ be maps such that domain$(f_i) = A_i$. Then by the union $\cup f_i$ of f_i's, we mean a function f defined on $\cup A_i$ by $f(x) = f_i(x)$, $x \in A_i$ whenever it is well defined.

Example 4.3. Let K be a 2-simplex in R^2 with the barycenter c_o and let $K^{(1)}$ denote the barycentric subdivision of K. Choose a 2-simplex $\alpha^{(1)}$ in $K^{(1)}$. For each 2-simplex $\beta_i^{(1)} \varepsilon K^{(1)} - \alpha^{(1)}$, $1 \leq i \leq 5$, we define an irregular homeomorphism $h_i^{(1)}$ such that $h_i^{(1)} | \partial\beta_i^{(1)} = 1_{\partial\beta_i^{(1)}}$, |. Define h_1: $|K^{(1)} - \alpha^{(1)}| \to |K^{(1)} - \alpha^{(1)}$ by $h_1 = \bigcup_{i=1}^{5} h_i^{(1)}$. Let $K^{(2)}$ be the barycentric subdivision of $\alpha^{(1)}$. Choose a 2-simplex $\alpha^{(2)}$ in $K^{(2)}$ such that c_o is a vertex of $\alpha^{(2)}$. For each simplex $\beta_i^{(2)} \varepsilon K^{(2)} - \alpha^{(2)}$, $1 \leq i \leq 5$, define an irregular homeomorphism $h_i^{(2)}$ such that $h_i^{(2)}|_{\partial\beta_i^{(2)}} = 1_{\partial\beta_i^{(2)}}$. Define

$h_2 : |K^{(2)} - \alpha^{(2)}| \to |K^{(2)} - \alpha^{(2)}|$ by $h_2 = \bigcup_{i=1}^{5} h_i^{(2)}$. Now, the inductive process to define h_n, for each $n > 0$, is clear (see Figure 1).

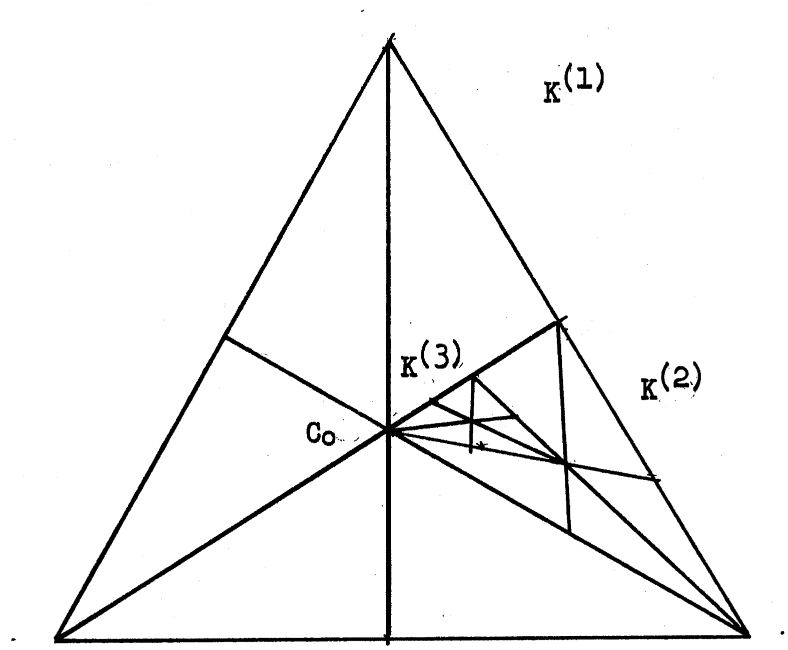

Figure 1. $K^{(3)}$

Then $\bigcup_{j=1}^{\infty} |K^{(j)}| = K$. By the construction of h_n, the function $h: K \to K$ defined by $h = \bigcup_{i=1}^{\infty} h_i$ is a well defined irregular homeomorphism. However, h cannot be uniformly irregular since, for each $\delta > 0$, we can find an open subset U_δ of $|K|$, with diam $(U_\delta) < \delta$, which is invariant under h.

<u>Example 4.4.</u> Let E be a bouquet of circles S_0, S_1, . . . , in R^2 with the common point p such that S_{n+1} lies in the bounded domain of S_n for each n and $\text{diam}(S_n) = \frac{1}{n+1}$ (Hawaiian ear ring). Let F (Figure 2) be a subset of R^2 obtained, from E, by attaching n disjoint 1-cells C_n^i, i = 1, 2, . . . , n, to S_n such that each C_n^i lies in the pinched annulus bounded by S_n and S_{n-1}, $C_n^i \cap S_n$ is a point and $C_n^i \cap S_{n-1}$ is empty. Let $D_n = S_n \cup (\bigcup_{i=1}^{n} C_n^i)$.

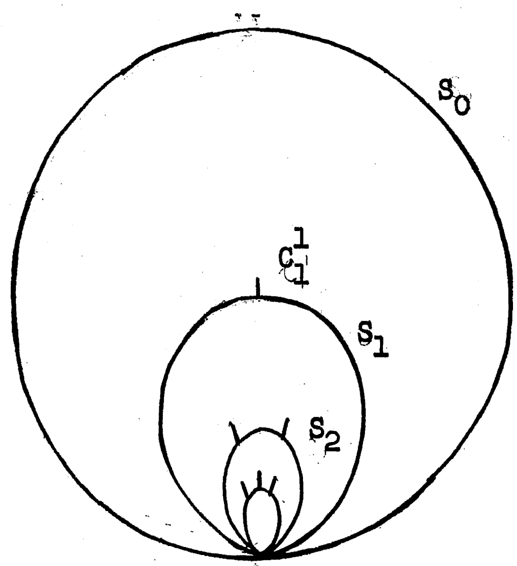

Figure 2. The Set F in R^2

Define a quotient space S of $F \times I$ by "smashing" $p \times I$ to a point $*$. We can consider X as a subset of R^3. It is clear that $D_i \times I / p \times I$ is not homeomorphic to $D_j \times I / p \times I$ for $i \neq j$ and $\lim_{i \to \infty} (\operatorname{diam} D_i \times I / p \times I) = 0$ (see Figure 3).

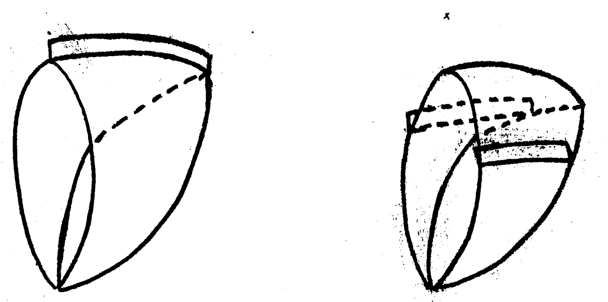

Figure 3. $D_i \times I/p \times I$, $i = 1,2$

We can define an irregular homeomorphism h_i on $D_i \times I / p \times I$ such that $* \in \operatorname{Fix}(h_i)$. Then $h = \bigcup_{i=0}^{\infty} h_i$ is an irregular homeomorphism on X. However, we cannot define any uniformly irregular homeomorphism on X for if g is a homeomorphism on X then $g(*) = *$ and the restriction of g to $D_i \times I / p \times I$ is a homeomorphism on $D_i \times I / p \times I$. Consequently, we can find, for any $\delta > 0$, an open set U_δ, with $\operatorname{diam}(U_\delta) < \delta$, such that U_δ is invariant under g.

Perhaps the most significant difference between the concepts of expansiveness and uniform irregularity is that a space X cannot support an expansive homeomorphism if it has a subset which is invariant under any homeomorphism on X and which itself cannot support an expansive homeomorphism while such a space X may as well support an uniformly irregular homeomorphism. This fact can be illustrated by pointing out that B^2 cannot support an expansive homeomorphism since it has a subset S^1 which cannot support an expansive homeomorphism (9) and is invariant under any homeomorphism on B^2 whereas B^2 can support an uniformly irregular homeomorphism.

Despite such a difference, both notions enjoy somewhat similar properties. For instance, we can give a characterization of an uniformly irregular homeomorphism quite similar to that of an expansive homeomorphism given by Keynes and Robertson (11). Furthermore, as for the case of expansiveness (8), lifting and projecting uniformly irregular homeomorphisms, via covering maps, yield uniformly irregular homeomorphisms.

We now state the theorem of Keynes and Robertson mentioned above and prove an analogeous theorem for uniformly irregular homeomorphisms.

<u>Theorem 4.5</u>. (11) A homeomorphism h on a compact space X is expansive if, and only if, there is an open cover U of X such that for each bisequence $\langle A_i \rangle$ in U, $\bigcap_{-\infty}^{\infty} h^{-i}(A_i)$ is at most a point.

<u>Theorem 4.6.</u> A homeomorphism h on a compact space X is uniformly irregular if, and only if, there is an open cover U of X

such that for each bisequence $\langle A_i \rangle$ in U, $\mathrm{int}(\bigcap_{-\infty}^{\infty} h^{-i}(A_i)) = \emptyset$.

<u>Proof</u>: Suppose h is uniformly irregular and let δ be a uniform irregularity constant. Let U be a finite open cover for X such that diam (A) $< \delta$ for each $A \in U$. Support that there is a bisequence $\langle A_i \rangle$ in U such that $\mathrm{int}(\bigcap_{-\infty}^{\infty} h^{-i}(A_i)) \neq \emptyset$. Then there is a point $p \in X$ and $\varepsilon > 0$ such that $N_\varepsilon(p) \subset \bigcap_{-\infty}^{\infty} h^{-i}(A_i)$. Thus $N_\varepsilon(p) \subset h^{-i}(A_i)$ for each i. This means that $h^i(N_\varepsilon(p)) \subset A_i$ for each i. Therefore, for each $x \in N_\varepsilon(p)$, $h^i(x)$ is δ-close to $h^i(p)$ for all $i \in Z$. This contradicts the choice of δ and proves that $\mathrm{int}(\bigcap_{-\infty}^{\infty} h^{-i}(A_i)) = \emptyset$.

Conversely, suppose there is an open cover U of X such that for any bisequence $\langle A_i \rangle$ in U, $\mathrm{int}(\bigcap_{-\infty}^{\infty} h^{-i}(A_i)) = \emptyset$. Let δ be a Lebesque number for this cover. We claim that $\delta/2$ is an uniform irregularity constant. For if not, then there is a point $p \in X$ and $\varepsilon > 0$ such that for each $x \in N_\varepsilon(p)$, $h^i(x)$ is $\delta/2$ - close to $h^i(p)$ for all $i \in Z$. This means that for each i, there is a set $A_i \in U$ such that $h^i(N_\varepsilon(p)) \subset A_i$ so that $N_\varepsilon(p) \subset h^{-i}(A_i)$. Therefore, $N_\varepsilon(x) \subset \bigcap_{-\infty}^{\infty} h^{-i}(A_i)$. This contradicts the choice of U and proves that $\delta/2$ is a uniform irregularity constant for h.

The cover U in theorem 4.5 (theorem 4.6) is called a <u>generator</u> of an expansive (uniformly irregular) homeomorphism. If an open cover U' of X is a refinement of a generator U, either for expansiveness or uniform irregularity, then U' itself is a generator.

Definition 4.7. Let $\rho\colon \tilde{X} \to X$ be a covering map. If $\tilde{h}$ and h are homeomorphisms on $\tilde{X}$ and X respectively, such that $\rho\tilde{h} = h\rho$ then $\tilde{h}$ is called a lift of h and h is called a projection of $\tilde{h}$.

Theorem 4.8. (8) Let $\rho\colon \tilde{X} \to X$ be a covering map where $\tilde{X}$ and X are compact spaces. Suppose a homeomorphism $\tilde{h}$ on $\tilde{X}$ is a lift of a homeomorphism h on X. Then $\tilde{h}$ is expansive if, and only if, h is expansive.

Theorem 4.9. Let $\rho\colon \tilde{X} \to X$ be a covering map where $\tilde{X}$ and X are compact spaces. Suppose the homeomorphism $\tilde{h}$ on $\tilde{X}$ is a lift of a homeomorphism h on X. Then $\tilde{h}$ is uniformly irregular if, and only if, h is uniformly irregular.

Proof: Suppose h is uniformly irregular and let U be a generator for h. Let $\tilde{U} = \{\rho^{-1}(A) \mid A \in U\}$. Then $\tilde{U}$ is an open cover for X and if $\langle\tilde{A}_i\rangle$ is bisequence in $\tilde{U}$ then $\tilde{A}_i = \rho^{-1}(A_i)$ for some bisequence $\langle A_i\rangle$ in U. Therefore, $\rho[\text{int}(\bigcap_{-\infty}^{\infty} \tilde{h}^{-i}(\tilde{A}_i))] \subset \text{int}[\rho(\bigcap_{-\infty}^{\infty} \tilde{h}^{-i}(\tilde{A}_i))] \subset \text{int}[\bigcap_{-\infty}^{\infty} \rho(\tilde{h}^{-i}(\tilde{A}_i))]$ $= \text{int}(\bigcap_{-\infty}^{\infty} h^{-i}\rho\ (\tilde{A}_i)) = \text{int}(\bigcap_{-\infty}^{\infty} h^{-i}(A_i)) = \emptyset$. Thus, $\text{int}(\bigcap_{-\infty}^{\infty} \tilde{h}^{-i}(\tilde{A}_i)) = \emptyset$.

Conversely, suppose $\tilde{h}$ is uniformly irregular and let δ be a uniform irregularity constant for $\tilde{h}$. Since $\tilde{X}$ is compact, ρ is k to 1 map for some positive integer k. Thus, for each $x \in X$, there is a neighborhood V_x of x such that $\rho^{-1}(V_x)$ is the union of disjoint sets $\{U_{x,i}\}_{i=1}^{k}$ with diam $U_{x,i} < \delta$ for all $i = 1, 2, \ldots, k$. Let β be a Lebesque number for the open cover $\{V_x\}$ of X. If h is not uniformly irregular, then for each $\varepsilon > 0$, there is an $\eta > 0$ and $x \in X$ such that for each $y \in N_\eta(x)$, $h^n(y) \in N_\varepsilon(h^n(x))$ for all n. Without loss of generality, we can assume that $\varepsilon < \beta/2$. Thus, for each n, $h^n(N_\eta(x)) \subset V_{x_n}$ for some $\tilde{x}_n \in X$. Pick $x \in \rho^{-1}(x)$ and choose $N_\lambda(\tilde{x}) \subset \rho^{-1}(N_\eta(x))$ such that $N_\lambda(\tilde{x})$ is connected by local connectivity of $\tilde{X}$. Then $\tilde{h}^n(N_\lambda(\tilde{x}))$ is connected for each n. Therefore, for each n, $\tilde{h}^n(N_\lambda(\tilde{x})) \subset U^j_{x_n}$ for some j, $1 \leq j \leq k$. Since $\operatorname{diam}(U^j_{x_n}) < \delta$, this is contrary to the choice of δ and proves that h is uniformly irregular.

We point out that similar argument can be used to show that the lifts and the projections of irregular homeomorphisms are irregular.

If h is a homeomorphism on X, it induces a homeomorphism h* on X* given by $h^*(x,y) = (h(x),h(y))$. In case that X is compact, we can give the following characterization of an expansive homeomorphism h on X.

<u>Theorem 4.10</u>. A homeomorphism h on compact space X is expansive if, and only if, $X^*/_{h^*}$, the orbit space of h*, is compact.

<u>Proof</u>: Suppose that h is expansive and let δ be an expansive constant. Let U be an open converning of $X^*/_{h^*}$ and η be the quotient map. For each $(x,y) \in X^*$, there is $n \in Z$ such that $(h^*)^n(x,y) \in X \times X - N_\delta(\Delta)$ where $N_\delta(\Delta) = \{(a,b) \in X \times X \mid d((a,b),\Delta(x)) < \delta\}$. $\{\eta^{-1}(A) \mid A \in U\}$ is an open cover for X*. Thus $\{\eta^{-1}(A) \cap (X \times X - N_\delta(\Delta)) \mid A \in U\}$ is an open cover for $X \times X - N_\delta(\Delta)$. Since $X \times X - N_\delta(\Delta)$ is compact, there is a finite subcover $\{\eta^{-1}(A_i) \cap (X \times X - N_\delta(\Delta)) \mid i = 1, 2, \ldots, n\}$. Then $\{A_i \mid i = 1, 2, \ldots, n\}$ covers $X^*/_{h^*}$.

Conversely, suppose $X^*/_{h^*}$ is compact. The map η is an open map. Take an open cover $\mathcal{A} = \{X \times X - \overline{N_{\frac{1}{n}}(\Delta)} \mid n \text{ is a positive integer}\}$ of X^*. Then $\{\eta(A) \mid A \in \mathcal{A}\}$ is an open cover for $X^*/_{h^*}$ so that there is a finite subcover $\{\eta(A_i)\}_{i=1}^{k}$ for some k. Since $A_i \subset A_{i+1}$, $\eta(A_k) = X^*/_{h^*}$. Consequently, for each $(x,y) \in X^*$ there is $n \in Z$ such that $(h^*)^n(x,y) = (h^n(x), h^n(y)) \notin N_{\frac{1}{k}}(\Delta)$. This means that $\frac{1}{k}$ is an expansive constant for h and h is an expansive homeomorphism.

CHAPTER V

SOME OPEN QUESTIONS

Let X be a compact polyhedron with no principle 1-cells, $\mathcal{H}(X)$ be the set of all homeomorphisms with compact-open topology and $\mathcal{IH}(X)$ be the set $\{h \in \mathcal{H}(X) \mid \mathrm{Irr}(h) = X\}$. The immediate problem is to determine the "size" of $\overline{\mathcal{IH}(X)}$, the closure of $\mathcal{IH}(X)$ in $\mathcal{H}(X)$. For instance, we have mentioned, in Chapter III, that there are uncountably many conjugacy classes of irregular homeomorphisms on B^2. The following remark shows that $\overline{\mathcal{IH}(B^2)}$ contains all homeomorphisms which are conjugate to rotations about the origin.

Remark 5.1. Let ρ_τ denote the rotation of B^2, in the complex plane, about the origin with an angle τ and let ε be any positive number. Let $B^2(\varepsilon/2)$ be the set $\{x \in B^2 \mid \|x\| \leq \varepsilon/2\}$. We can define an irregular homeomorphism h_1 on $B^2(\varepsilon/2)$ such that $h|_{\partial B^2(\varepsilon/2)} = \rho_\tau|_{\partial B^2(\varepsilon/2)}$ by the same technique used in Chapter III. Let λ be a positive number such that $d(e^{i\theta}, e^{i(\theta+\lambda)}) < \varepsilon$ and define h_2 on $\overline{B^2 - B^2(\varepsilon/2)}$ by $h_2(re^{i\theta}) = re^{i(\theta+\tau+\lambda(r-\varepsilon/2))}$. Then by an argument similar to that used in lemma 3.1, h_2 is an irregular homeomorphism on $\overline{B^2 - B^2(\varepsilon/2)}$. Since $h_1|_{\partial B^2(\varepsilon/2)} = h_2|_{\partial B^2(\varepsilon/2)}$, $h = h_1 \cup h_2$ is a homeomorphism on B^2 with $\mathrm{Irr}(h) = B^2$. Furthermore, if $x \in \mathrm{int}(B^2(\varepsilon/2))$

then $\rho_\tau(x)$ and $h(x)$ are both in $int(B^2(\varepsilon/2))$ and if $x \in B^2 - int(B^2(\varepsilon/2))$ then $d(h(x), \rho_\tau(x)) < \varepsilon$ since $d(e^{i\theta}, \overline{e^{i(\theta+\lambda)}}) < \varepsilon$. Therefore, $d(h, \rho_\tau) \leq \varepsilon$ and shows that $\rho_\tau \in \overline{IH(B^2)}$. Suppose that $g = f\rho_\tau f^{-1}$ for some τ and a homeomorphism f on B^2 and let $\varepsilon > 0$ be given. By the uniform continuity of f, there is a positive number δ such that if $d(x,y) < \delta$ then $d(f(x),f(y,)) < \varepsilon$. But we can find $h \in IH(B^2)$ such that h is δ-close to ρ_τ. Thus, fhf^{-1} is ε-close to g and is irregular by lemma 2.3.

In Chapter II, we have used van Kampen's results in (17) to show that Irr(h) is nowhere dense in S^1 for each homeomorphism h on S^1. The key notion in (17) is the <u>rotation number</u> associated with an orientation preserving homeomorphism. His definition can easily be modified to obtain a function associated with an isotopy h of S^1 such that h_0 is an orientation preserving homeomorphism on S^1. The process is described in the following remark. We emphasize that the process is sketched roughly since it is totally analogous to the process used by van Kampen (17).

<u>Remark 5.2</u>. Let h be an isotopy of S^1 (a level preserving homeomorphism on $S^1 \times I$) with h_0 an orientation preserving homeomorphism. Then for each t, h_t is an orientation preserving homeomorphism on S^1. Let H be a lift of h to $R^1 \times I$ through the covering map $\pi \times 1_I: R^1 \times I \to S^1 \times I$ defined by $(\pi \times 1_I)(r,t) = (e^{i2\pi r},t)$ for each $(r,t) \in R^1 \times I$. Then H_t is a lift of h_t to R^1 through π. Thus, $\lambda_t = \lim_{n\to\infty} \frac{H_t^n(x)}{n}$ is independent of the choice of $x \in R^1$ and λ_t is

independent of the lift H_t of h_t module Z (17). Define $\lambda: I \to [0,1)$ by $\lambda(t) = \lambda_t - [(\lambda_t)]$ where $[(x)]$ denotes the greatest integer which does not exceed x. Then λ is a well defined function associated with h.

A few questions can be asked about the function defined in remark 5.2:

1. Is λ continuous?
2. Is λ conjugacy invariant in the set of all isotopies on S^1?
3. If λ is not continuous in general, can we "select" a lift G_t of h_t separately, rather than lifting h as a whole, in such a way that $\rho: I \to R$ defined by $(t) = \lim_{n \to \infty} \frac{G_t^n(x)}{n}$ is continuous?
4. Can one modify above process to define such a function, say M, for any homeomorphism on $S^1 \times 1$ so that M is invariant under conjugation in $H(S^1 \times I)$?

The answer "yes" to these questions would provide a beginning tool for the classification problem of homeomorphisms and 2-dimensional manifolds.

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