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PLANE CURVES WITH SMALL LINEAR ORBITS, I

by P. ALUFFI[1] and C. FABER

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0. Introduction.

In this paper we study the "linear orbits" of certain singular plane curves. We have dealt with orbits of *smooth* plane curves in [AF1]; the results in this paper are the next natural step towards a treatment of *arbitrary* plane curves.

Here is the set-up. The group PGL(3) of projective transformations of the plane $\mathbb{P}^2$ acts naturally on the projective space $\mathbb{P}^N$ parametrizing plane curves of degree d (here $N = \frac{1}{2}d(d+3)$). The orbit of a curve C is a quasi-projective variety of dimension ≤ 8, which we call the "linear orbit" of C. Most curves have linear orbits of dimension 8; we say that C has a *small* linear orbit if the dimension of its orbit is 7 or less. This paper studies the enumerative geometry of most plane curves whose orbit is small.

It is natural to study the closures of these linear orbits in the projective space $\mathbb{P}^N$: questions arise as to *e.g.* the degrees of these projective varieties (on what features of a plane curve does the degree of its orbit closure depend?); the decomposition of their boundaries in smaller orbits; their singularities (which orbit closures are smooth?); and the behavior of orbit closures in families of plane curves.

In [AF1] we answer some of these questions in the case of a smooth plane curve. Our main tool is the construction through explicit blow-ups of a nonsingular projective variety dominating the orbit closure. The degree of the orbit closure can then be determined with the aid of standard

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intersection theory. The answer depends naturally on the degree of the plane curve and the order of its stabilizer, but also (somewhat surprisingly) on the types of its flexes: in fact, the structure of the blow-up sequence depends precisely on the number and type of the flexes on the curve.

Unfortunately, this natural approach seems inadequate for most singular curves: we do not know a sequence of blow-ups producing a nonsingular variety dominating the orbit closure for arbitrary singularities. In a different approach that we have developed for the study of orbit closures, the first step is to determine which orbits appear in the boundary of the orbit closure of a given curve; this was in essence carried out more than 60 years ago in [Ghi](2), and will be discussed elsewhere. The second step is to study these 'small' orbits in detail; the present paper contains such a study, for almost all small orbits. More precisely, we deal here with all curves whose orbit is small and which contain some non-linear component. Curves consisting entirely of lines require a different (and in some sense simpler) treatment; their orbits, and the classification of small orbits, are the subject matter of [AF4].

In §1 we describe the curves that we study in this paper, and state the main result: the computation of the degree of the orbit closures of these curves. These degrees (together with the related results of §4) will be the input necessary to treat arbitrary singular curves.

For curves with small orbits, the precise knowledge of the singularities that can arise allows us to carry through the approach used for smooth curves. The computation is again based upon the construction (§2) of a nonsingular projective variety admitting a dominant morphism to the orbit closure. The explicit blow-up sequence yielding this variety now mimics the embedded resolution of the singular curve in the plane (as mentioned above, this approach surprisingly does *not* seem to work for arbitrary singular curves).

In §3 we describe the actual degree computation, which is rather involved; the main tool is a refinement (Proposition 2.3) of a blow-up formula from [Alu]. The final answer (Theorem 1.1) has a remarkably simple form, considering the laborious procedure leading to it. For example, while the blow-up sequence we use relies in an essential way on the Dynkin diagrams of the singularities, only very coarse numerical information (such as the degree of the components of the curve, or their multiplicity at the singular points) enters in the formula for the degree of its orbit closure.

(2) We are grateful to the referee of [AF1] for pointing us to Ghizzetti's work.

In §4 we also discuss "predegree polynomials", which combine information concerning the enumerative geometry of the curves when certain natural constraints are introduced. Formulas for the degrees of loci of curves with these constraints are obtained by applying a suitable differential operator to the expression in Theorem 1.1. These results are included both because they are natural extensions of the other results in this paper, and because they will be ingredients in the computation of the degree of the orbit closure of an *arbitrary* plane curve, which we will describe elsewhere.

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1. Statement of the main result.

We work over an algebraically closed field of characteristic 0.

Let $m < n$ be coprime integers, with $m \geq 1$. The prototype irreducible curve we consider in this paper is the cuspidal plane curve C "of type (m, n)", *i.e.*, with projective equation

$$x^n = y^m z^{n-m}$$

for suitable coordinates $(x:y:z)$. We aim to studying the locus of all curves of type (m, n), which form the PGL(3)-orbit of a single such curve. In fact, we are interested in studying *all* curves whose PGL(3)-orbit has dimension < 8; so we will study here the orbit of a more general (possibly reducible) type of curve, specified below.

Note that type (m, n) and type $(n - m, n)$ only differ by a coordinate switch $y \leftrightarrow z$. The only two (possibly) singular points of C are located at $(0:0:1)$ and $(0:1:0)$; we will generally call these points "cusps", although they may in fact be nonsingular (for $m = 1$ or $m = n - 1$, respectively). Note also that C determines a triangle, formed by the line $\lambda = \{x = 0\}$, joining the two cusps, and by the tangent cones $\mu = \{y = 0\}$, $\bar{\mu} = \{z = 0\}$ to C at the cusps:

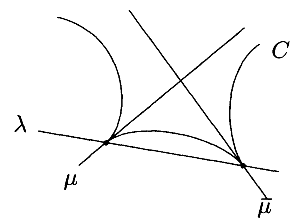


Figure 1

More generally, fix two coprime integers $n > m \geq 1$. The curves C we study in this paper consist of arbitrary unions of curves from the pencil

$$x^n = \alpha\, y^m z^{n-m}, \quad (\alpha \neq 0),$$

counted with arbitrary multiplicities s_i, and of the lines λ, μ, $\bar{\mu}$ of the basic triangle, taken with multiplicities r, q, $\bar{q}$ respectively. We denote

$$S = \sum s_i, \quad \overline{m} = n - m.$$

Now we act on the plane by the group PGL(3) of projective linear transformations. This action induces a (right) action on the projective space $\mathbb{P}^N$, $N = \frac{1}{2}d(d+3)$, parametrizing degree-d plane curves. A curve C as specified above has degree $d = Sn + r + q + \bar{q}$, and its orbit in $\mathbb{P}^N$ has dimension 7 for all but very special cases (for example, if $S = 0$ then C consists of lines from the basic triangle, and the dimension of its orbit is necessarily ≤ 6).

In case C contains, besides lines, at most one curve of type (m, n), the set of all curves of the same type and with the same multiplicities s_i, r, q, $\bar{q}$ is precisely the orbit of C; we study the closure of this orbit. If C contains two or more curves from the pencil, then the set of all curves of the same type and with the same multiplicities consists of infinitely many orbits. As explained in the introduction, we study the orbits of these curves rather than the set of all of them. We will find that the infinitely many orbits for a given set of data have essentially the same behavior; a special choice of the curves in the pencil may give rise to a bigger automorphism group, which affects the degree of the orbit closure only by a multiplicative factor.

Here is the main numerical result of the paper. First, working in the ring with

$$r^3 = q^3 = \bar{q}^3 = 0,$$

expand the expression

$$n^2m^2\overline{m}^2\Big(\Big(S+\frac{r}{n}+\frac{q}{m}+\frac{\bar{q}}{\overline{m}}\Big)^7+2\Big(S+\frac{r}{n}+\frac{q}{m}\Big)^7+2\Big(S+\frac{r}{n}+\frac{\bar{q}}{\overline{m}}\Big)^7 \\ +\Big(S+\frac{r}{n}\Big)^7-42\Big(S+\frac{r}{n}\Big)^5\Big(\frac{q^2}{m^2}-\frac{q}{m}\frac{\bar{q}}{\overline{m}}+\frac{\bar{q}^2}{\overline{m}^2}\Big)\Big)$$

obtaining a *polynomial* in all the variables (of degree ≤ 2 in r, q, $\bar{q}$); then, subtract

$$84\,(Sn+r+q+\bar{q})^2\sum s_i^5-252\,(Sn+r+q+\bar{q})\sum s_i^6+192\sum s_i^7.$$

The result is a polynomial expression $Q(n,m,s_i,r,q,\bar{q})$.

THEOREM 1.1. — *If 7-dimensional, the orbit closure of a curve with data n,m,s_i, $r,q,\bar{q}$ as above has degree*

$$\frac{1}{A}Q(n,m,s_i,r,q,\bar{q}),$$

where A is the number of components of the $\mathrm{PGL}(3)$*-stabilizer of the curve. If the orbit has dimension lower than* 7, *the expression evaluates to* 0.

The number A accounts for special automorphisms of the curve, due to extra symmetries in the position of the components in the pencil; *cf.* Lemma 3.1. The number A equals 1 for most choices of n,m, *etc.*

The enumerative meaning of the formula obtained in Theorem 1.1 rests on the fact that imposing C to contain a given point is a linear condition in $\mathbb{P}^N$. If all multiplicities are 0 or 1, it is easy to see that the number computed in the theorem equals the number of curves in the orbit which contain 7 general points.

The first of the two expressions building up to Q will be obtained by combining a "Bézout term" with contributions arising from the "local" part of the construction in §2, essentially aimed at resolving the singularities of the curve. Note that it only depends on the multiplicities s_i of the cuspidal components via their sum S. The second term will arise from the "global" stage of the construction, taking care of the curve after singularities have been resolved. The multiplicities enter here in a more interesting way, but note that this term depends otherwise only on the total degree d of the curve. We do not have conceptual explanations for these features, or for the remarkable shape of the first expression (indeed, our construction only yields a complicated raw expression, which we then recognize to equal the relatively simple one given above).

To our knowledge, there is minimal overlap of the results in this note with the existing literature in enumerative geometry. J.M. Miret and S. Xambó have computed hundreds of characteristic numbers for cuspidal plane *cubics,* in [MX]; our formulas allow us to reproduce 27 of the numbers in their lists. In fact, a particular case of our result yields closed formulas for these numbers for curves of *arbitrary* degree, in terms of the type (m, n) of the curve (see §4.3).

Notice that the dual of a curve of type (m, n) as above is a curve of type $(\overline{m}, n)$, hence again of type (m, n) after a coordinate switch. Therefore, the degrees computed here also compute *characteristic numbers*: that is, the number of curves of given type and tangent to 7 lines in general position in the plane.

The identity component of the stabilizer of a curve with 7-dimensional linear orbit is either $\mathbb{G}_m$ or $\mathbb{G}_a$. All curves containing some non-linear component and whose stabilizer contains $\mathbb{G}_m$ are of the kind considered above (see [AF4]). The curves with 7-dimensional orbit and whose stabilizer contains a $\mathbb{G}_a$ are not of this kind (one example of such curves is the union of two smooth conics touching at exactly one point). This case is briefly discussed in §4.1; the formula given above turns out to be correct for this case as well, with suitable choices of the variables.

We also include here (see §4.2) a few remarks that extend the result given above to cases in which the orbit has dimension < 7. Moreover, we discuss the degree of subsets of the orbit closures determined by imposing conditions on the lines of the basic triangle (see §4.3).

2. Local and global blow-ups.

Our goal in this section is the explicit construction of nonsingular varieties dominating the closure $\overline{\mathcal{O}}_C$ of the orbits of the curves discussed in §1. The general approach we take is a natural extension of the one in [AF1], and we summarize it here.

After choosing coordinates in $\mathbb{P}^2$, we consider the $\mathbb{P}^8$ of 3×3 matrices as a completion of PGL(3). The action of PGL(3) on a fixed curve C determines then a rational map

$$\mathbb{P}^8 \overset{c}{\dashrightarrow} \mathbb{P}^N,$$

by sending a matrix φ to the curve with equation $F(\varphi(x:y:z)) = 0$, where F is an equation for C. Our aim is to resolve the indeterminacies of this map, by a sequence of blow-ups at nonsingular centers, starting from $\mathbb{P}^8$. We will then obtain a *nonsingular* variety $\widetilde{V}$ surjecting onto the orbit closure:

$$\widetilde{V} \longrightarrow \overline{\mathcal{O}}_C.$$

The challenge is to perform the resolution explicitly enough to be able to keep track of the intersection theory and of other relevant information. If this is accomplished, then several invariants of $\overline{\mathcal{O}}_C$ (such as degree, Euler characteristic, multiplicity along components of the singular locus, *etc.*) can be computed in principle. This will be illustrated in §3 by the computation of the degree of $\overline{\mathcal{O}}_C$, with the result stated in §1. The computation of other invariants might be substantially more involved; for an example in which multiplicity computations can be carried out explicitly, see [AF2]. With this broader range of problems in mind, we insist on aiming to construct a *nonsingular* $\widetilde{V}$, although this forces us into a bit of extra work in this §2.

As is immediately checked, the base locus of the rational map c defined above is supported on the set of rank-1 matrices whose image is a point of C, union the set of rank-2 matrices whose image is a line contained in C.

The resolution of the indeterminacies of c will require two distinct stages. In a first stage we will deal with the fact that the curves we consider are *singular* (in general): this causes the base locus of c to be itself singular, and we employ a sequence of blow-ups to resolve its singularity. We call these blow-ups 'local', to remind ourselves that they deal with local features of the curves under exam. Once the singularities of the base locus are resolved, we need a second stage of 'global' blow-ups to eliminate the indeterminacies of the lifted rational map. This stage is considerably simpler, particularly because the situation is reduced to the case of nonsingular curves, which was examined in [AF1].

The details of the construction are rather technical; however, a rather explicit description of a variety $\widetilde{V}$ as above is necessary in order to perform the degree computations in §3, and would be essential to attack subtler problems such as the study of singularities of the orbit closure. We therefore feel that it would not be opportune to omit these details altogether. Here is a summary of how the section is organized; the hurried reader should feel free to skip the rest of this §2 at first reading.

- We consider a curve C = the union of finitely many curves of type (m, n), and of lines from the basic triangle, with arbitrary multiplicities (see §1).

• The action of PGL(3) extends to a dominant rational map from the $\mathbb{P}^8$ of 3×3 matrices to the orbit closure of C: $\mathbb{P}^8 \overset{c}{\dashrightarrow} \overline{\mathcal{O}}_C \subset \mathbb{P}^N$.

• The indeterminacies of this map are removed by a sequence of blow-ups, and more precisely:

(i) a sequence of "local" blow-ups of $\mathbb{P}^8$ along nonsingular centers, corresponding to the two cusps of the components of C. These blow-ups mirror the sequence of blow-ups yielding the embedded resolution of C. The sequence corresponding to one cusp is described in Proposition 2.6. This produces a variety $\widetilde{V}^{\mathrm{loc}}$;

(ii) two "global" blow-ups with nonsingular centers of dimension 3, 4, over $\widetilde{V}^{\mathrm{loc}}$; these are discussed in Theorem 2.4;

(iii) a blow-up along a $\mathbb{P}^2$ obtained as the (isomorphic) inverse image of the set of matrices whose image is $\mu \cap \bar{\mu}$; and

(iv) blow-ups along three 5-dimensional nonsingular varieties. These are the proper transforms of the set of matrices whose image is one of the lines of the basic triangle, see the discussion in §2.3.

We denote by $\widetilde{V}$ the variety obtained at the end of this process. By pasting together the pieces of our discussion, we will have:

THEOREM 2.1. — *The procedure described above produces a nonsingular variety $\widetilde{V}$ mapping to $\mathbb{P}^8$, such that c lifts to a regular map $\tilde{c} : \widetilde{V} \to \mathbb{P}^N$. The image of the map $\tilde{c}$ is the orbit closure $\overline{\mathcal{O}}_C$:*

$$\begin{tikzcd} \widetilde{V} \arrow[d] \arrow[dr, "\tilde{c}"] & \\ \mathbb{P}^8 \arrow[r, dashed, "c"] & \overline{\mathcal{O}}_C \subset \mathbb{P}^N \end{tikzcd}$$

This section is devoted to the construction of $\widetilde{V}$ and the proof of Theorem 2.1.

2.1. Directed blow-ups.

As mentioned above, the local blow-ups will essentially mirror the blow-ups needed to obtain an embedded resolution of the cuspidal curve of type (m, n) presented in §1. We start by recalling how this resolution is accomplished, and introduce a device ("directed blow-ups") which streamlines the construction considerably.

Consider the affine portion of a cuspidal curve C of type (m, n), centered at one of the cusps:

$$x^n = \alpha\, y^m \quad (\alpha \neq 0),$$

together with the tangent cone μ to C at the cusp:

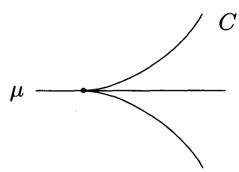


Figure 2

To get an embedded resolution of the union $C \cup \mu$, start by blowing up the plane at the cusp $C \cap \mu$, then successively blow-up at the point of intersection of the proper transform of C with the latest exceptional divisor, until the resolution is achieved. A more refined description of this sequence is controlled by the steps of the Euclidean algorithm for $m = m_1$, n:

$$\begin{aligned} n &= m_1\ell_1 + m_2, \\ m_1 &= m_2\ell_2 + m_3, \\ &\dots \\ m_{e-2} &= m_{e-1}\ell_{e-1} + m_e, \\ m_{e-1} &= m_e\ell_e, \end{aligned}$$

with all ℓ_i, m_i positive integers, $m_i < m_{i-1}$, and $m_e = 1$.

The center of each of the first ℓ_1 blow-ups is the intersection of the proper transforms of C and μ; if $m_1 = 1$, these ℓ_1 blow-ups produce the resolution. If $m_1 > 1$, after this sequence the proper transform of C is

(i) transversal to the proper transform of μ; and

(ii) an affine cuspidal curve of type (m_2, m_1) (in a suitable chart of the blow-up), with tangent cone at the cusp equal to the latest exceptional divisor E_1.

That is, at the end of the first ℓ_1 blow-ups we are left with the same problem with which we had started, but related to a "simpler" curve, of type (m_2, m_1).

Similarly, the second line of the Euclidean algorithm corresponds to a sequence of ℓ_2 blow-ups, at the end of which the proper transform of C will be a type-(m_3, m_2) curve with the latest exceptional divisor, E_2, as tangent cone (if $m_2 \neq 1$). Proceeding in this fashion, the situation simplifies until the last ℓ_e blow-ups, which yield a curve "of type $(0,1)$"–that is, a curve transversal to the last exceptional divisor E_e. At this point the embedded resolution is achieved.

Note that this subdivision of the resolution process in e steps, according to the lines of the Euclidean algorithm, is natural from the point of view of the multiplicity of the curve at the successive centers of blow-ups: this is m_1 for the ℓ_1 blow-ups corresponding to the first line, then m_2 for the next ℓ_2 blow-ups, *etc.*

In view of these considerations, and of how they will be mirrored by the "local" blow-ups over $\mathbb{P}^8$, we define a notion of "directed" blow-up. Let

$$B \subset P \subset V$$

be three nonsingular varieties, with $\dim B < \dim P < \dim V$, and let $\ell > 0$. We define a *nonsingular* variety $V^{(\ell)}$ birational to V, and dominating the blow-ups of V along the "j-th thickening of B in the direction of P" for all $0 \leq j \leq \ell$ (see the example following Lemma 2.2).

DEFINITION. — *With $B \subset P \subset V$ as above, we let $V^{(1)}$ be the blow-up of V along B; for $\ell \geq 2$, we let $V^{(\ell)}$ be the blow-up of $V^{(\ell-1)}$ along the intersection of the proper transform of P with the exceptional divisor $E^{(\ell-1)}$ in $V^{(\ell-1)}$. We call $V^{(\ell)}$ the ℓ-directed blow-up of V along B, in the direction of P. The exceptional divisor of the directed blow-up is the last exceptional divisor, $E^{(\ell)}$, produced in the sequence. Also, the exceptional divisor of the directed blow-up contains a distinguished subvariety, namely its intersection with the proper transform of P.*

With this terminology, each stage of the resolution described above (corresponding to one line of the Euclidean algorithm) is simply one directed blow-up at the cusp, in the direction of the tangent cone.

Directed blow-ups satisfy a few simple properties, whose proof we leave to the reader:

LEMMA 2.2.

1) *For $j \geq 1$, the proper transform $\widetilde{P} \subset V^{(j)}$ of P is isomorphic to the blow-up $\widetilde{P} = B\ell_B P$ of P along B. The centers of the blow-ups in $V^{(j)}$ are all isomorphic to the projectivization $\widetilde{B}$ of the normal bundle $N_B P$ of B in P.*

Let $\mathcal{O}(-1)$ denote the universal line subbundle on $\widetilde{B} = \mathbb{P}(N_B P)$.

2) *For all $j \geq 1$, $\widetilde{P}$ and $\widetilde{B} \subset E^{(j)}$ are disjoint from the proper transforms of 'previous' exceptional divisors $E^{(i)}$, $i < j$. Also,*

$$E^{(j)} \cdot \widetilde{B} = c_1\big(\mathcal{O}(-1)\big) \cap [\widetilde{B}].$$

3) *For all $j \geq 1$,*

$$c\big(N_{\widetilde{B}} V^{(j)}\big) = c\big(\mathcal{O}(-1)\big) \cdot c\big(N_P V \otimes \mathcal{O}(-j)\big).$$

4) *Let $(x_1, \ldots, x_n)$ be local parameters for V, such that P and B are locally given respectively by the ideals $(x_1, \ldots, x_p)$ and $(x_1, \ldots, x_b)$ $(p < b)$. Then near $\widetilde{B} \subset V^{(\ell)}$ we may give covering charts $U_{p+1}, \ldots, U_b$ and local parameters $(y_1, \ldots, y_n)$ on U_j so that the composition of the blow-up maps*

$$U_j \longrightarrow V$$

is given by

$$x_i = \begin{cases} y_i y_j^{\ell} & i = 1, \ldots, p; \\ y_i y_j & i = p+1, \ldots, \widehat{\jmath}, \ldots, b; \\ y_i & i = j, i = b+1, \ldots, n. \end{cases}$$

The ideal of $E^{(\ell)}$ in this chart is (y_j); the ideal of $\widetilde{P}$ is

$$(y_1, \ldots, y_p).$$

5) *Let $\mathcal{J} \subset \mathcal{I}$ be the ideal sheaves of P, B respectively. For all $j \geq 1$, let $B^{(j)}$ be the subscheme of V defined by the ideal $(\mathcal{I}^j + \mathcal{J})$. Then $B = B^{(1)}, \ldots, B^{(\ell)}$ all pull-back to Cartier divisors on $V^{(\ell)}$.*

Example. — To clarify the construction, let us compare the directed blow-up corresponding to the first line of the Euclidean algorithm for the curve $x^n = y^m$ (with $n = \ell_1 m + m_2$, $B =$ the origin, and $P =$ the x-axis;

so, $\mathcal{I} = (x, y)$ and $\mathcal{J} = (y)$) with blowing-up the plane directly along the fat point with ideal $\mathcal{I}^{\ell_1} + \mathcal{J} = (x^{\ell_1}, y)$. The ℓ_1-directed blow-up produces ℓ_1 exceptional divisors, and (as we observed already) the proper transform of the curve is the curve $t^m - x^{m_2}$ in a suitable chart of the resulting nonsingular surface. The blow-up along $\mathcal{I}^{\ell_1} + \mathcal{J}$ is covered by charts

$$\operatorname{Spec} \frac{k[x,y,t]}{(y - tx^{\ell_1})}, \quad \operatorname{Spec} \frac{k[x,y,s]}{(sy - x^{\ell_1})};$$

so the total transform of the curve $x^n = y^m$ is covered by

$$\operatorname{Spec} \frac{k[x,y,t]}{(y - tx^{\ell_1}, x^n - y^m)} \cong \operatorname{Spec} \frac{k[x,t]}{(x^{\ell_1 m}(t^m - x^{m_2}))},$$
$$\operatorname{Spec} \frac{k[x,y,s]}{(sy - x^{\ell_1}, x^n - y^m)} \cong \operatorname{Spec} \frac{k[x,y,s]}{(sy - x^{\ell_1}, y^m(1 - s^m x^{m_2}))}.$$

From this we see that the proper transform of the curve sits in the nonsingular part of the blow-up, and there it behaves just as in the ℓ_1-directed blow-up of the plane along the origin, in the direction of the tangent cone.

The ℓ-directed blow-up of V along B in the direction of P is simply a resolution of singularities of the blow-up along the subscheme $\mathcal{I}^\ell + \mathcal{J}$. The few extra exceptional divisors introduced in the process seem a price worth paying for the benefit of obtaining a *nonsingular* variety dominating the orbit closure.

The "local blow-ups" in the resolution of the map c introduced at the beginning of this section will be a sequence of directed blow-ups, also controlled by the Euclidean algorithm on m, n. The application in §3 (yielding the degree of the closure of the image of c) will rely on keeping track of the intersection of several divisors in these blow-ups. We will make use of the following formula, which compares the intersection number of a collection of divisors with the intersection number of their proper transforms after a directed blow-up.

To state this formula, we will use the notation $c(\mathcal{E}^{(\ell)})$ for the Adams operation on the Chern class of a bundle $\mathcal{E}$:

$$c_r(\mathcal{E}^{(\ell)}) = \ell^r c_r(\mathcal{E}), \quad r \geq 0.$$

PROPOSITION 2.3. — *Denote by* $V^{(\ell)} \xrightarrow{\pi} V$ *the* ℓ*-directed blow-up of* V *along* B *in the direction of* P, *as above. Let* X_i, $i = 1, \dots, j$ *and* Y_i, $i = 1, \dots, k$ *be effective divisors in* V, *such that*

- *each* X_i *and its proper transforms contain the centers* B, $\widetilde{B}$ *of blow-ups with the same multiplicity* m_i, *and*

- *each* Y_i *has multiplicity* r_i *along* B, *and its proper transforms do not contain the other centers* $\widetilde{B}$ *of blow-ups.*

Further, assume that the number k *of divisors* Y_i *be less than the codimension of* B *in* P. *Denote by* $\widetilde{X}_i$, $\widetilde{Y}_i$ *the proper transforms of the divisors in* $V^{(\ell)}$. *Then*

$$\prod_{i=1}^{k} Y_i \cdot \prod_{i=1}^{j} X_i \cap [V] - \pi_* \Big(\prod_{i=1}^{k} \widetilde{Y}_i \cdot \prod_{i=1}^{j} \widetilde{X}_i \cap [V^{(\ell)}] \Big)$$

equals the push-forward from B *of the term of dimension* $\dim V - j - k$ *in*

$$\ell^{\dim P - \dim B - k} \cdot \frac{\prod_{i=1}^{k} (r_i + \ell Y_i) \prod_{i=1}^{j} (m_i + X_i)}{c(N_B^{(\ell)} P) c(N_P V)} \cap [B].$$

For $\ell = 1$, that is for the ordinary blow-up of V along B, this says that the intersection of the divisors changes under proper transforms by the term of expected dimension in

$$\frac{\prod (r_i + Y_i) \prod (m_i + X_i)}{c(N_B V)} \cap [B].$$

This is a restatement of a particular case of Theorem II in [Alu]. The formula for directed blow-ups can be deduced from the $\ell = 1$ case; we leave the details to the reader.

2.2. Blow-ups for one curve.

We first consider the case of a single irreducible curve C of type (m, n), and focus our attention on one of the cusps. So choose affine coordinates and write the equation of C

$$x^n = \alpha\, y^m$$

with $\alpha \neq 0$, $1 \leq m < n$ and $(m, n) = 1$. We may and will in fact assume $\alpha = 1$, by rescaling y. Note that the base locus of the corresponding rational map $\mathbb{P}^8 \overset{c}{\dashrightarrow} \mathbb{P}^N$ contains the set $B \cong \mathbb{P}^2$ of rank-1 matrices whose image is the cusp of C. In fact, the base locus of c consists of an isomorphic copy of $\mathbb{P}^2 \times C$: the set of rank-1 matrices with arbitrary kernel, and image a point on C. For $m > 1$, this locus is singular along B.

As mentioned in the summary preceding the statement of Theorem 2.1, we construct our resolution $\widetilde{V}$ by a two-stage process. The first stage consists of a sequence of directed blow-ups, mirroring the sequence giving the resolution of the union of the curve and its tangent cone at the cusp. More precisely, assume that the Euclidean algorithm for m, n consists of e lines, as in §2.1:

$$\begin{aligned} n &= m_1\ell_1 + m_2, \\ m_1 &= m_2\ell_2 + m_3, \\ &\dots \\ m_{e-2} &= m_{e-1}\ell_{e-1} + m_e, \\ m_{e-1} &= m_e\ell_e. \end{aligned}$$

Also, the base locus of c is a copy of $\mathbb{P}^2 \times C$; the cuspidal point $(0,0)$ of C determines a distinguished $B = \mathbb{P}^2 \times \{(0,0)\}$ in the base locus. The tangent cone to C at $(0,0)$ is the line $y = 0$, which determines a distinguished $P = \mathbb{P}^5 \subset \mathbb{P}^8$, that is the set of matrices whose image is contained in this line.

DEFINITION. — *We define a variety $\widetilde{V}^{\mathrm{loc}}$ by the following sequence of e directed blow-ups :*

- *first, perform the ℓ_1-directed blow-up of $\mathbb{P}^8$ along B in the direction of P; this produces a variety V_1^{loc}, with an exceptional divisor E_1^{loc} and a distinguished 4-fold $B_1^{\mathrm{loc}} \subset E_1^{\mathrm{loc}}$;*

- *next, for $i > 1$ perform inductively the ℓ_i-directed blow-up of V_{i-1}^{loc} along B_{i-1}^{loc} in the direction of E_{i-1}^{loc}; this produces a variety V_i^{loc}, with an exceptional divisor E_i^{loc} and a distinguished 6-fold $B_i^{\mathrm{loc}} \subset E_i^{\mathrm{loc}}$.*

With these notations, we let $\widetilde{V}^{\mathrm{loc}} = V_e^{\mathrm{loc}}$.

In order to study $\widetilde{V}^{\mathrm{loc}}$, and to describe the second stage of the process, we introduce affine coordinates

$$\begin{pmatrix} 1 & p_1 & p_2 \\ p_3 & p_4 & p_5 \\ p_6 & p_7 & p_8 \end{pmatrix}$$

for $\mathbb{P}^8$ near $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, that is a rank-one matrix with image the origin $(1:0:0)$, and remark that this choice of coordinates is irrelevant in the sense that we can move (by multiplying on the right by a constant matrix) any such 3×3 matrix to one in the chosen $\mathbb{A}^8$, so that we must be able to detect in this $\mathbb{A}^8$ every phenomenon relevant to our computation.

The sequence of "local" directed blow-ups specified above produces a variety $\widetilde{V}^{\mathrm{loc}} \xrightarrow{\pi^{\mathrm{loc}}} \mathbb{P}^8$; we give coordinates on a chart in $\widetilde{V}^{\mathrm{loc}}$:

$$\begin{pmatrix} \boxed{\widetilde{V}^{\mathrm{loc}}} & s_1 & s_2 \\ s_3 & s_4 & s_5 \\ s_6 & s_7 & s_8 \end{pmatrix}$$

(the boxed entry reminds us of the variety where the coordinates are given) and we claim that again the choice of this particular chart will be irrelevant for what follows. The expression of $\pi^{\mathrm{loc}} : \widetilde{V}^{\mathrm{loc}} \to \mathbb{P}^8$ in these coordinates depends on the parity of the number of steps e in the Euclidean algorithm displayed above:

- if e is *odd*, we will have

$$\begin{pmatrix} 1 & p_1 & p_2 \\ p_3 & p_4 & p_5 \\ p_6 & p_7 & p_8 \end{pmatrix} = \begin{pmatrix} 1 & s_1 & s_2 \\ s_3^m s_6^A & s_3^m s_6^A s_4 & s_3^m s_6^A s_5 \\ s_3^n s_6^B & s_3^n s_6^B s_7 & s_3^n s_6^B s_8 \end{pmatrix},$$

with $Bm - An = 1$ (the actual values of A, B can be obtained in terms of the Euclidean algorithm, but are not important here);

- if e is *even*, we will have

$$\begin{pmatrix} 1 & p_1 & p_2 \\ p_3 & p_4 & p_5 \\ p_6 & p_7 & p_8 \end{pmatrix} = \begin{pmatrix} 1 & s_1 & s_2 \\ s_3^A s_6^m & s_3^A s_6^m s_4 & s_3^A s_6^m s_5 \\ s_3^B s_6^n & s_3^B s_6^n s_7 & s_3^B s_6^n s_8 \end{pmatrix},$$

with $An - Bm = 1$.

Remark. — These coordinate expressions are slightly different in the case $m = 1$, *i.e.*, $e = 1$. Other details of the construction require minor modifications in this case; we leave these to the reader.

In Proposition 2.6 we will prove that coordinates can be given on $\widetilde{V}^{\mathrm{loc}}$ so that these expressions hold. First, we claim that if we show that this coordinate description holds (and that the choice of the chart is indeed irrelevant), then we are essentially done:

THEOREM 2.4. — *Two blow-ups at smooth centers remove the indeterminacies of the lifted map* $c^{\mathrm{loc}} : \widetilde{V}^{\mathrm{loc}} \dashrightarrow \mathbb{P}^N$.

COROLLARY 2.5. — *For the curve C with equation $x^n = \alpha\, y^m z^{n-m}$, $\alpha \neq 0$, the indeterminacies of the corresponding rational map c can be removed by performing the sequence of "local" blow-ups for the two cusps, followed by two "global" blow-ups at smooth centers.*

We prove the theorem right away, and concentrate on the more involved details of the coordinate description in Proposition 2.6. By a "point-condition" we mean the hypersurface in $\mathbb{P}^8$ formed by all matrices which send the chosen curve C (say with equation $F = 0$) to contain a fixed point p. More precisely, the point-condition in $\mathbb{P}^8$ corresponding to $p \in \mathbb{P}^2$ has equation (in $\varphi \in \mathbb{P}^8$)

$$F\big(\varphi(p)\big) = 0.$$

Further, we call "point-conditions" the proper transforms of point-conditions in any variety mapping birationally to $\mathbb{P}^8$. The point-conditions in $\mathbb{P}^8$ generate the linear system corresponding to c; hence, showing that a lift of c to a variety $\widetilde{V}$ removes the indeterminacies of c amounts to showing that the (proper transforms of the) point-conditions in $\widetilde{V}$ do not have a common intersection.

Proof of Theorem 2.4. — The point-condition corresponding to (ξ_0, ξ_1, ξ_2) in $\mathbb{A}^8 \subset \mathbb{P}^8$ has equation

$$(\xi_0 + p_1\xi_1 + p_2\xi_2)^{n-m}(p_6\xi_0 + p_7\xi_1 + p_8\xi_2)^m = (p_3\xi_0 + p_4\xi_1 + p_5\xi_2)^n$$

(taking $(1 : x : y)$ for coordinates in $\mathbb{P}^2$). Pulling back through π^{loc} (in the first case written above; the second case is analogous) gives

$$(\xi_0+s_1\xi_1+s_2\xi_2)^{n-m}(\xi_0+s_7\xi_1+s_8\xi_2)^m s_3^{mn} s_6^{Bm} = (\xi_0+s_4\xi_1+s_5\xi_2)^n s_3^{mn} s_6^{An}.$$

Using $Bm - An = 1$, we see that the proper transform of this point-condition in $\widetilde{V}^{\mathrm{loc}}$ has equation

$$(\xi_0 + s_1\xi_1 + s_2\xi_2)^{n-m}(\xi_0 + s_7\xi_1 + s_8\xi_2)^m s_6 = (\xi_0 + s_4\xi_1 + s_5\xi_2)^n.$$

Note that s_3 does not appear in this equation. Next, recall that the support of the base locus of c in $\mathbb{P}^8$ is $\mathbb{P}^2 \times C$; as C is parametrized by (t^m, t^n), we may parametrize the (affine part of the) support of the base locus by

$$(k_1, k_2, t) \longmapsto \begin{pmatrix} 1 & k_1 & k_2 \\ t^m & t^m k_1 & t^m k_2 \\ t^n & t^n k_1 & t^n k_2 \end{pmatrix}.$$

Using $Bm - An = 1$ again we lift this to

$$(k_1, k_2, t) \longmapsto \begin{pmatrix} \boxed{\widetilde{V}^{\mathrm{loc}}} & k_1 & k_2 \\ t & k_1 & k_2 \\ 1 & k_1 & k_2 \end{pmatrix}$$

and observe that a point of this subvariety of $\widetilde{V}^{\mathrm{loc}}$ lies above the special point of $C \iff t = 0 \iff s_3 = 0$. Since the equations of the point-conditions do not involve s_3, their behavior over such a point is the same as over any point with nonzero t. As no point with $t \neq 0$ is a flex on C, we know from [AF1], Proposition 2.7, that two "global" blow-ups resolve the indeterminacies of c at such points. □

As stated in the proof of the theorem, the two blow-ups needed to remove the indeterminacies of c^{loc} are the two blow-ups, discussed in [AF1], resolving the map over nonsingular non-flex points of C. We refer the reader to [AF1] for a more thorough description of the centers of these blow-ups, and freely use that information in §3. Here we will just recall that the first "global" blow-up will have a nonsingular irreducible 3-dimensional center (the proper transform of $\mathbb{P}^2 \times C$ in $\widetilde{V}^{\mathrm{loc}}$); after blowing up this locus, the point-conditions meet along a 4-dimensional locus, in fact a $\mathbb{P}^1$-bundle over the preceding center. The point-conditions are separated from each other by blowing up this last locus. This $\mathbb{P}^1$ bundle is described in the discussion preceding Proposition 2.2 in [AF1].

Now we move to the coordinate description of $\widetilde{V}^{\mathrm{loc}}$ used above. All the varieties we consider are obtained by a sequence of blow-ups over $\mathbb{P}^8$, and inherit a right action of PGL(3) since the centers of the blow-ups are invariant. We say that a chart in any such variety is *essential* if every point of the variety can be moved to that chart by this action.

PROPOSITION 2.6. — *The variety $\widetilde{V}^{\mathrm{loc}}$ admits an essential chart with the coordinate description specified above.*

Proof. — Let V_1^{loc} be the ℓ_1-directed blow-up of $\mathbb{P}^8$ along B in the direction of P. In order to study this variety, we first obtain local coordinates for the base locus of c. Writing out the matrix with kernel on $x_0 + k_1 x_1 + k_2 x_2 = 0$ and image $(1 : t^m : t^n)$ gives a local parametrization

$$(k_1, k_2, t) \longmapsto \begin{pmatrix} 1 & k_1 & k_2 \\ t^m & t^m k_1 & t^m k_2 \\ t^n & t^n k_1 & t^n k_2 \end{pmatrix}$$

for $\mathbb{P}^2 \times C$. Setting $t = 0$ selects the distinguished $\mathbb{P}^2$, locally parametrized by

$$(k_1, k_2, 0) \longmapsto \begin{pmatrix} 1 & k_1 & k_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So B has equations

$$p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = 0$$

in our chart. As for P, the matrices with image contained in the line $y = 0$ are in the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}$$

so P has equations

$$p_6 = p_7 = p_8 = 0.$$

The distinguished 4-fold in V_1^{loc} is the intersection B_1^{loc} of the proper transform of P and the last exceptional divisor E_1^{loc} of the sequence producing the directed blow-up. In fact we can also consider the j-directed blow-up for all $1 \leq j < \ell_1$, and a simple inductive computation shows that at each stage the proper transforms of the point-conditions meet along the proper transform of $\mathbb{P}^2 \times C$, and along the locus B_i obtained by intersecting the proper transform of P with the last exceptional divisor. If $e > 1$, the same holds for the V_1^{loc} (as we will see below), so we only need to examine V_1^{loc} near B_1^{loc}. Now, by part (4) of Lemma 2.2, a neighborhood of B_1^{loc} is covered by charts U_3, U_4, U_5 with local parameters $(q_1, \ldots, q_8)$ so that the map $U_j \to \mathbb{P}^8$ is given by

$$p_i = \begin{cases} q_i & i = j,\ i = 1, 2; \\ q_i q_j & i = 3, 4, 5 \ \text{ but } i \neq j; \\ q_i q_j^{\ell_1} & i = 6, 7, 8. \end{cases}$$

Claim. — *The chart U_3 is essential.*

That is, we claim that we can use the action of PGL(3) to move points from the other charts to this chart. The proof of this fact is a simple but tedious coordinate computation, which we leave to the reader.

The consequence of the claim is that it is not restrictive to choose local coordinates q_i on V_1^{loc} so that the blow-up map $V_1^{\mathrm{loc}} \to \mathbb{P}^8$ is given by

$$\begin{pmatrix} 1 & p_1 & p_2 \\ p_3 & p_4 & p_5 \\ p_6 & p_7 & p_8 \end{pmatrix} = \begin{pmatrix} \boxed{V_1^{\mathrm{loc}}} & q_1 & q_2 \\ q_3 & q_3q_4 & q_3q_5 \\ q_3^{\ell_1}q_6 & q_3^{\ell_1}q_7 & q_3^{\ell_1}q_8 \end{pmatrix}.$$

The equation of the exceptional divisor E_1^{loc} is $q_3 = 0$ in these coordinates. The equation of the point-condition corresponding to $(\xi_0 : \xi_1 : \xi_2)$ in $\mathbb{P}^8$ is

$$(\xi_0 + p_1\xi_1 + p_2\xi_2)^{n-m}(p_6\xi_0 + p_7\xi_1 + p_8\xi_2)^m = (p_3\xi_0 + p_4\xi_1 + p_5\xi_2)^n,$$

which pulled back via the above map gives

$$(\xi_0 + q_1\xi_1 + q_2\xi_2)^{n-m}(q_6\xi_0 + q_7\xi_1 + q_8\xi_2)^m q_3^{\ell_1 m} = (\xi_0 + q_4\xi_1 + q_5\xi_2)^n q_3^n;$$

clearing a common factor of $q_3^{\ell_1 m}$ (notice $n - \ell_1 m = m_2 \geq 0$), we obtain the equation of the point-condition in V_1^{loc}:

$$(\xi_0 + q_1\xi_1 + q_2\xi_2)^{n-m}(q_6\xi_0 + q_7\xi_1 + q_8\xi_2)^m = (\xi_0 + q_4\xi_1 + q_5\xi_2)^n q_3^{m_2}.$$

If $m_2 = 0$, that is $e = 1$, then $\widetilde{V}^{\mathrm{loc}} = V_1^{\mathrm{loc}}$; setting $s_i = q_i$, $A = 0$, and $B = 1$ gives the prescribed coordinate description, and we are done in this case.

Otherwise, this shows that along E_1^{loc} (*i.e.*, setting $q_3 = 0$) the point-conditions meet along the locus with equations

$$q_6 = q_7 = q_8 = 0$$

that is, the intersection of the proper transform of P with E_1^{loc}. This is the distinguished 4-fold, B_1^{loc}.

Next, we perform the ℓ_2-directed blow-up of V_1^{loc} along B_1^{loc}, in the direction of E_1^{loc}, obtaining V_2^{loc}, with exceptional divisor E_2^{loc}. The discussion is very similar to the discussion of the first step; now $B_1^{\mathrm{loc}} \subset E_1^{\mathrm{loc}}$ are given by the ideals $(q_3, q_6, q_7, q_8) \supset (q_3)$, so by Lemma 2.2 we can choose a chart in V_2^{loc} with coordinates r_i, such that the map $V_2^{\mathrm{loc}} \to V_1^{\mathrm{loc}}$ is given by

$$\begin{pmatrix} \boxed{V_1^{\mathrm{loc}}} & q_1 & q_2 \\ q_3 & q_4 & q_5 \\ q_6 & q_7 & q_8 \end{pmatrix} = \begin{pmatrix} \boxed{V_2^{\mathrm{loc}}} & r_1 & r_2 \\ r_3r_6^{\ell_2} & r_4 & r_5 \\ r_6 & r_6r_7 & r_6r_8 \end{pmatrix}.$$

Again, the reader should have no difficulties checking that this chart is essential.

The new exceptional divisor E_2^{loc} is given by $r_6 = 0$ in these coordinates. Pulling back the point-conditions from V_1^{loc} and clearing a common factor of $r_6^{\ell_2 m_2}$ shows that the equation of the point-condition corresponding to $(\xi_0 : \xi_1 : \xi_2)$ in $V_2{}^{\mathrm{loc}}$ is

$$(\xi_0 + r_1\xi_1 + r_2\xi_2)^{n-m}(\xi_0 + r_7\xi_1 + r_8\xi_2)^m r_6^{m_3} = (\xi_0 + r_4\xi_1 + r_5\xi_2)^n r_3^{m_2}.$$

If $m_3 = 0$, that is $e = 2$, then $\widetilde{V}^{\mathrm{loc}} = V_2^{\mathrm{loc}}$, and we have reached the desired coordinate expression.

Otherwise, we see that the intersection of all point-conditions along E_2^{loc} is the locus with equations

$$r_3 = r_6 = 0,$$

giving the distinguished 6-fold B_2^{loc}.

Having reached this stage, the expression for the point-condition

$$(\xi_0 + r_1\xi_1 + r_2\xi_2)^{n-m}(\xi_0 + r_7\xi_1 + r_8\xi_2)^m r_6^{m_3} = (\xi_0 + r_4\xi_1 + r_5\xi_2)^n r_3^{m_2}$$

is so symmetric that the remaining blow-ups can be all understood together. Assuming that we have defined V_i^{loc}, we will have

• either i odd, equation of E_i^{loc}: $r_3 = 0$; and equation of the point-conditions

$$(\xi_0 + r_1\xi_1 + r_2\xi_2)^{n-m}(\xi_0 + r_7\xi_1 + r_8\xi_2)^m r_6^{m_i} = (\xi_0 + r_4\xi_1 + r_5\xi_2)^n r_3^{m_{i+1}};$$

• or i even, equation of E_i^{loc}: $r_6 = 0$; and equation of the point-conditions

$$(\xi_0 + r_1\xi_1 + r_2\xi_2)^{n-m}(\xi_0 + r_7\xi_1 + r_8\xi_2)^m r_6^{m_{i+1}} = (\xi_0 + r_4\xi_1 + r_5\xi_2)^n r_3^{m_i}.$$

As long as $m_{i+1} > 0$, the point-conditions meet on E_i^{loc} along the 6-fold B_i^{loc} defined by $r_3 = r_6 = 0$ (in both cases). Applying again part 4) of Lemma 2.2, we see that the ℓ_{i+1}-directed blow-up of V_i^{loc} along B_i^{loc} in the direction of E_i^{loc} produces a V_{i+1}^{loc} with the data prescribed above; in particular, we see that this automatically chooses the essential chart in each successive blow-up. Notice in passing that at each stage B_i^{loc} is the intersection of E_i^{loc} with the proper transform $\widetilde{E}_{i-1}^{\mathrm{loc}}$; and the two divisors are swapped from one stage to the next. In particular, the restriction of $\widetilde{E}_{i-1}^{\mathrm{loc}}$ to $B_i^{\mathrm{loc}} \cong B_{i+1}^{\mathrm{loc}}$ equals the restriction of E_{i+1}^{loc}. This fact will be used in §3.

At the e-th stage we will have $m_e = \gcd(m,n) = 1$ and $m_{e+1} = 0$, so the point-conditions will have equation

$$(\xi_0 + r_1\xi_1 + r_2\xi_2)^{n-m}(\xi_0 + r_7\xi_1 + r_8\xi_2)^m r_6 = (\xi_0 + r_4\xi_1 + r_5\xi_2)^n$$

for odd e, and

$$(\xi_0 + r_1\xi_1 + r_2\xi_2)^{n-m}(\xi_0 + r_7\xi_1 + r_8\xi_2)^m = (\xi_0 + r_4\xi_1 + r_5\xi_2)^n r_3$$

for even e. Writing the map to $\mathbb{P}^8$ explicitly shows that this gives the claimed coordinate description of $V_e^{\mathrm{loc}} = \widetilde{V}^{\mathrm{loc}}$, as needed. □

It follows from the explicit equations obtained in this proof that the multiplicity of the point-conditions along the various centers of blow-up also mirrors the multiplicity of C at the centers of the blow-ups resolving it. So this multiplicity is m_1 for the ℓ_1 blow-ups giving the first directed blow-up, m_2 for the second batch, *etc.* This information will be used in §3.

We will also need the multiplicities of the $\mathbb{P}^7$'s obtained as point-conditions for the lines of the basic triangle, so we note here that these also mirror the corresponding multiplicities of the lines in the blow-ups resolving the curve. Explicitly, for the blow-ups examined here

- a $\mathbb{P}^7$ corresponding to the line connecting the two cusps of C (λ in the notation of §1) has multiplicity 1 along the first center B of the first directed blow-up, and multiplicity 0 at all other centers;
- a $\mathbb{P}^7$ corresponding to the line μ supporting the tangent cone to C at the cusp under consideration has multiplicity 1 along all the centers of the blow-ups giving the first directed blow-up; multiplicity 1 along the first center B_1^{loc} of the second directed blow-up; and multiplicity 0 at all other centers;
- a $\mathbb{P}^7$ corresponding to the line $\overline{\mu}$ at infinity has multiplicity 0 along all centers.

2.3. Blow-ups for the general case.

It is now a simple matter to go from the case of one curve, treated in §2.2, to the case of many. Again, the more general curves C we consider in this paper are arbitrary unions (with multiplicities) of elements of the pencil

$$x^n = \alpha\, y^m z^{n-m} \quad (\alpha \neq 0)$$

together with multiples of the lines $\lambda, \mu, \overline{\mu}$ of the basic triangle. As pointed out in §2.2, removing the indeterminacies of the corresponding rational map c amounts to separating the point-conditions; so we have to understand what the point-conditions of C look like, and how they behave under the blow-ups described in §2.2.

PROPOSITION 2.7. — *For a curve C as above, the point-condition in $\mathbb{P}^8$ corresponding to a point $p \in \mathbb{P}^2$ consists of the union of the point-conditions of each component, each appearing with multiplicity equal to the multiplicity of the corresponding component.*

This should be clear: if $F = 0$ is an equation for C, then the equation of the point-condition corresponding to p is the vanishing of

$$F\big(\varphi(p)\big) = 0.$$

This polynomial (in φ) factors according to how F factors.

The supports of the point-conditions of C are therefore unions of point-conditions considered in §2.2 (for different α's), and of copies of the three $\mathbb{P}^7$'s corresponding to λ, μ, $\overline{\mu}$ mentioned at the end of §2.2.

Disregarding the lines of the basic triangles for a moment, note that different irreducible curves from the same pencil as above have the same history through this blow-up sequence. The situation in the plane mirrors precisely the situation at the level of point-conditions: different curves determine the same centers B, B_i^{loc}, and the corresponding point-conditions have the same multiplicities along these loci. Further, the curves are separated at the very last stage, and correspondingly the base locus of the lifted map $\widetilde{V}^{\text{loc}} \dashrightarrow \mathbb{P}^N$ consists of the disjoint union of copies of $\widetilde{C} \times \mathbb{P}^2$ (where $\widetilde{C}$ denotes the normalization of a single curve of type (m, n)). For each of these, Theorem 2.4 shows that two ("global") blow-ups will suffice to remove the indeterminacies.

In other words, the same sequence of local blow-ups used for one curve of type (m, n), followed by two global blow-ups for each component, removes the indeterminacies for any finite union of such curves. The multiplicities with which these appear are irrelevant to this discussion.

To account for the lines in the basic triangle, we need to keep track of the three pencils of hyperplanes of $\mathbb{P}^8$ corresponding to the points on these three lines. The relevant data is implicit in the multiplicity statement at the end of §2.2:

- the $\mathbb{P}^7$'s corresponding to λ are separated from the point-conditions corresponding to curves of type (m,n) after the first blow-up of the sequence giving the first directed blow-up;

- the $\mathbb{P}^7$'s corresponding to μ are separated from the point-conditions corresponding to curves of type (m,n) at the first blow-up of the sequence giving the second directed blow-up over the cusp $\lambda \cap \mu$;

- the $\mathbb{P}^7$'s corresponding to $\overline{\mu}$ are separated from the point-conditions corresponding to curves of type (m,n) at the first blow-up of the sequence giving the second directed blow-up over the cusp $\lambda \cap \overline{\mu}$.

That is, after the local blow-ups of §2.2 have been performed over both cusps, and after the two global blow-ups of §2.2 have removed indeterminacies arising from the type-(m,n) components of C, we still have three groups of hypersurfaces, corresponding to the three lines of the triangle. Again, it is easily checked that the incidence of these groups of hypersurfaces reflects the incidence of the corresponding proper transforms of the lines in the plane (see Fig. 3).

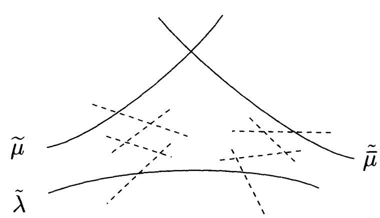


Figure 3

Finally we deal with these hypersurfaces. The intersection of the hypersurfaces in each group is a 5-dimensional variety (the proper transform of the $\mathbb{P}^5$ of rank-2 matrices with image the corresponding line of the triangle); further, the two 5-dimensional varieties corresponding to μ and $\overline{\mu}$ still meet along a $\mathbb{P}^2$, corresponding to rank-1 matrices whose image is the point of intersection $\mu \cap \overline{\mu}$.

By our good luck (as the reader can see by performing the relevant computation, using the coordinates given in §2.2), the "obvious" strategy works: blowing up along the $\mathbb{P}^2$ corresponding to $\mu \cap \bar{\mu}$, and then along the proper transforms of the three $\mathbb{P}^5$ described above, finally produces a variety $\widetilde{V}$ satisfying the condition in the statement of Theorem 2.1.

3. Degree computations.

With the coordinate analysis of §2 behind us, we are ready to set up the intersection theoretic part of the computation. The discussion leading to Proposition 3.2 below reduces the computation of the degree of an orbit closure to the computation of an intersection product of divisors on the variety $\widetilde{V}$ we constructed in §2. Our main tool will then be Proposition 2.3, by which we keep track of the intersection of divisors under directed blow-ups.

The information needed to apply this formula consists of the multiplicity of the divisors at the center of blow-up, together with the Chern classes of the relevant normal bundles. The first piece of information is listed at the end of §2.2; the second will be obtained along the way, mostly by using part (3) of Lemma 2.2.

Here is the main reduction. We want to compute the degree of the orbit closure $\overline{\mathcal{O}}_C \subset \mathbb{P}^N$, assuming this has dimension 7 (which is the case for most choices of the parameters m, n, *etc.*). In the set-up of §2, we have obtained a completion $\widetilde{V}$ of PGL(3) over which the action extends to a regular map to $\mathbb{P}^N$; $\widetilde{V}$ was obtained by suitably blowing up $\mathbb{P}^8$:

$$\begin{tikzcd} \mathrm{PGL}(3) \subset \widetilde{V} \arrow[d, "\pi"'] \arrow[dr, "\tilde{c}"] & \\ \mathbb{P}^8 \arrow[r, dashed, "c"] & \mathbb{P}^N \end{tikzcd}$$

This realizes $\overline{\mathcal{O}}_C$ as the image of $\widetilde{V}$ by $\tilde{c}$.

Now, seven general hyperplanes intersect $\overline{\mathcal{O}}_C$ transversally at $\deg \overline{\mathcal{O}}_C$ points of $\mathcal{O}_C$. The inverse image of these points in PGL(3) will be $\deg \overline{\mathcal{O}}_C$ translated copies of the stabilizer of C. It follows that

$$(*) \qquad (\tilde{c}^* h)^7 = \deg \overline{\mathcal{O}}_C \, [Z]$$

where h is the class of a hyperplane in $\mathbb{P}^N$, and Z is the cycle obtained by closing up in $\widetilde{V}$ the stabilizer of C.

By construction, the class $\tilde{c}^*h$ is represented by a "point-condition" $\widetilde{W} \subset \widetilde{V}$; that is, by the proper transform of the hypersurface $W \subset \mathbb{P}^8$ consisting of matrices ϕ mapping a fixed point $p \in \mathbb{P}^2$ to a point of C. Note that $\pi(Z)$ consists of the closure in $\mathbb{P}^8$ of the stabilizer of C. As mentioned after the statement of Theorem 1.1, the number of components of the stabilizer of C depends on symmetries of the specific S-tuple of points in $\mathbb{A}^1$ corresponding to the non-linear components of C. Explicitly, assume that C is given by the equation

$$x^r y^q z^{\bar{q}} \prod_i (x^n - \alpha_i y^m z^{\overline{m}})^{s_i} = 0,$$

so that it corresponds to the S-tuple in $\mathbb{A}^1_\alpha$ given by the equation

$$\prod_i (\alpha - \alpha_i)^{s_i} = 0.$$

The components of the stabilizer of C depend on automorphisms $\mathbb{A}^1 \to \mathbb{A}^1$ fixing this S-tuple. The precise statement (whose proof is left to the reader) is

LEMMA 3.1. — *With notations as above, assume that the orbit of C has dimension 7. Then if $n \neq 2$ or $q \neq \bar{q}$ the number A of components of the stabilizer of C equals the number of automorphisms $\mathbb{A}^1 \to \mathbb{A}^1$, $\alpha \mapsto u\alpha$ (with u a root of unity) preserving the S-tuple corresponding to C; when $n = 2$ and $q = \bar{q}$, A equals twice this number.*

The extra automorphisms in the latter case come from the switch $y \leftrightarrow z$.

Examples.

1) C given by $(x^3 - yz^2)(x^3 - 2yz^2) = 0$. The corresponding S-tuple is given by $(\alpha - 1)(\alpha - 2) = 0$, that is, by the pair of points $\alpha = 1$, $\alpha = 2$ in $\mathbb{A}^1$. The only automorphism of $\mathbb{A}^1$ of the kind specified in the statement and preserving this pair of points is the identity, so $A = 1$ in this case (as in most others).

2) C given by $(x^2 - yz)(x^2 + yz) = 0$. The corresponding S-tuple is $\alpha = \pm 1$. Two automorphisms preserve this pair: the identity and $\alpha \mapsto -\alpha$. Since $n = 2$ and $q = \bar{q} = 1$, we have $A = 4$ in this case.

It is easily checked that each component of the closure of the stabilizer is a copy of the curve $x^n = y^m z^{\overline{m}}$. For example, the identity component of the stabilizer of C consists of the diagonal matrices with entries $(1, t^{\overline{m}}, t^{-m})$; since $(m, \overline{m}) = 1$, its closure has equation $1 = p_4^m p_8^{\overline{m}}$ in the coordinates of §2. In particular, the degree of $\pi(Z)$ equals An.

Pushing forward (*) to $\mathbb{P}^8$ and intersecting by a general hyperplane H, we see then that

$$\int H \cdot \pi_*(\widetilde{W}^7) = A\,n \deg \overline{\mathcal{O}}_C.$$

Observing that the inverse image $\pi^{-1}H$ of a general hyperplane equals its proper transform $\widetilde{H}$, and applying the projection formula, we conclude:

PROPOSITION 3.2. — *If* $\dim \overline{\mathcal{O}}_C = 7$, *then*

$$\deg \overline{\mathcal{O}}_C = \frac{1}{A\,n} \int_{\mathbb{P}^8} \pi_*(\widetilde{H} \cdot \widetilde{W}^7).$$

Our goal is therefore to perform the intersection product on the right-hand-side of this formula.

The reader now sees why we stated the formula in Proposition 2.3, comparing intersection products of divisors and of their proper transforms under directed blow-ups. The role of the different divisors considered in that formula might not be immediately apparent, however, and the next lemma should clarify it. We denote by H the general hyperplane in $\mathbb{P}^8$; by H_λ, H_μ, $H_{\overline{\mu}}$ respectively hyperplanes obtained as point-conditions relative to the lines λ, μ, $\overline{\mu}$. Further, we denote by X a point-condition in $\mathbb{P}^8$ relative to the part of C consisting of the union of type-(m,n) curves.

LEMMA 3.3. — *With these notations,*

$$An \deg \overline{\mathcal{O}}_C = 7! \sum \frac{q^{j_\mu}}{j_\mu!} \frac{\overline{q}^{j_{\overline{\mu}}}}{j_{\overline{\mu}}!} \frac{r^{j_\lambda}}{j_\lambda!} \frac{1}{j_c!} \int \pi_*\big(\widetilde{H} \cdot \widetilde{H}_\mu^{j_\mu} \cdot \widetilde{H}_{\overline{\mu}}^{j_{\overline{\mu}}} \cdot \widetilde{H}_\lambda^{j_\lambda} \cdot \widetilde{X}^{j_c}\big),$$

where the summation runs over all

$$0 \le j_\mu \le 2, \quad 0 \le j_{\overline{\mu}} \le 2, \quad 0 \le j_\lambda \le 2, \quad 0 \le j_c \le 7$$

such that $j_\mu + j_{\overline{\mu}} + j_\lambda + j_c = 7$.

Proof. — As observed in Proposition 2.7, point-conditions of a reducible curve split into the point-conditions of its irreducible components.

This implies

$$\widetilde{W} = q\widetilde{H}_\mu + \bar{q}\widetilde{H}_{\bar{\mu}} + r\widetilde{H}_\lambda + \widetilde{X}.$$

The formula follows then from Proposition 3.2, once one observes that if ℓ is any of λ, μ, $\bar{\mu}$, then $\widetilde{H}_\ell^3 = 0$: indeed, a line does not contain three general points, so the intersection of three point-conditions of a line must be empty. □

By this lemma, we are reduced to computing intersection products

$$\widetilde{H} \cdot \widetilde{H}_\mu^{j_\mu} \cdot \widetilde{H}_{\bar{\mu}}^{j_{\bar{\mu}}} \cdot \widetilde{H}_\lambda^{j_\lambda} \cdot \widetilde{X}^{j_c},$$

for $0 \le j_\mu \le 2$, $0 \le j_{\bar{\mu}} \le 2$, $0 \le j_\lambda \le 2$, $0 \le j_c \le 7$ such that $j_\mu + j_{\bar{\mu}} + j_\lambda + j_c = 7$. The divisors H_λ, H_μ, $H_{\bar{\mu}}$ will play the role of the divisors "of type Y_i" in Proposition 2.3.

3.1. Local blow-ups.

Now we move to the core of the computation. Proposition 2.3 will be used iteratively to evaluate the intersection product listed above on successively higher and higher level blow-ups. At each directed blow-up, the formula evaluates a correction term measuring by how much the intersection product changes upon taking proper transforms. The starting point is the intersection product in $\mathbb{P}^8$,

$$H \cdot H_\mu^{j_\mu} \cdot H_{\bar{\mu}}^{j_{\bar{\mu}}} \cdot H_\lambda^{j_\lambda} \cdot X^{j_c}:$$

since (with the notation of §1) X has degree Sn, this is simply

$$(Sn)^{j_c}$$

by Bézout's Theorem. Summing up as in Lemma 3.3, we get:

$$7!\Big(\sum_{\substack{0\le j_\mu\le 2\\ 0\le j_{\bar{\mu}}\le 2\\ 0\le j_\lambda\le 2\\ j_c=7-j_\mu-j_{\bar{\mu}}-j_\lambda}} \frac{q^{j_\mu}}{j_\mu!}\,\frac{\bar{q}^{j_{\bar{\mu}}}}{j_{\bar{\mu}}!}\,\frac{r^{j_\lambda}}{j_\lambda!}\,\frac{(Sn)^{j_c}}{j_c!}\Big).$$

This unpleasant expression prompts us to establish the following:

Convention. — We are going to treat the multiplicities q, $\bar{q}$, r, S as variables, and impose that

$$q^3 = \bar{q}^3 = r^3 = 0.$$

This takes care automatically of the bounds for the j's in the summation, so that the Bézout term simply becomes

$$(Sn + r + q + \bar{q})^7.$$

The geometric reason behind the convention is that the self-intersection of three or more point-conditions in $\widetilde{V}$ corresponding to lines must vanish, as was mentioned above. Imposing this from the start saves us some computational time: in practice, all the terms that we discard at this stage would be cancelled anyway along the blow-up process, so we can ignore them. The important caveat to keep in mind is that one *may not substitute the multiplicities for their value before expanding expressions in which they appear.* All such expressions must be expanded, and the relations $q^3 = \bar{q}^3 = r^3 = 0$ must be applied, before any substitution can be made.

Next, we deal with the correction term due to the e directed blow-ups over the cusp at $\lambda \cap \mu$ (where $e =$ number of lines in the Euclidean algorithm for (m, n)). By symmetry, we will get a similar contribution for the cusp $\lambda \cap \bar{\mu}$. In the next subsection we will evaluate analogous contributions due to the other ("global") blow-ups.

Recall our notation: the Euclidean algorithm performed on $m = m_1$ and n gives

$$\begin{aligned} n &= \ell_1 m_1 + m_2, \\ m_1 &= \ell_2 m_2 + m_3, \\ &\dots \\ m_{e-1} &= \ell_e m_e, \end{aligned}$$

with all m_i, ℓ_i positive integers, $0 < m_{i+1} < m_i$, and $m_e = \gcd(m, n)$; we are in fact assuming $m_e = 1$. The first blow-up from Proposition 2.6 is the ℓ_1-directed blow-up of $\mathbb{P}^8$ along $\mathbb{P}^2$ in the direction of $\mathbb{P}^5$, where these subspaces are defined immediately preceding the statement of Proposition 2.6. The multiplicities of the (proper transforms of the) divisors we need to intersect were discussed at the end of §2.2, and are as follows:

- for the general hyperplane: 0 for all centers of the ℓ_1 blow-ups;
- for the j_μ hyperplanes corresponding to the tangent cone to C at the point: 1 for all centers;
- for the $j_{\bar{\mu}}$ hyperplanes corresponding to the tangent cone to C at the other point: 0 for all centers;

- for the j_λ hyperplanes corresponding to the line joining the two distinguished points of C: 1 for the first center, 0 for the remaining $\ell_1 - 1$;
- for the j_c point-conditions X: Sm_1 at all centers.

(indeed, each support of a component of X has multiplicity m_1, and S =the sum of the multiplicities of the components). Further, X has degree Sn. Also observe that we have $j_\lambda \leq 2 < \operatorname{codim}_{\mathbb{P}^2} \mathbb{P}^5$ terms with "mixed multiplicities", as is necessary in order to apply Proposition 2.3.

Finally, denoting by k the hyperplane class in $\mathbb{P}^2$, we have

$$c(N_{\mathbb{P}^2}\mathbb{P}^5) = (1+k)^3, \quad c(N_{\mathbb{P}^5}\mathbb{P}^8) = (1+k)^3$$

and therefore

$$c\big(N_{\mathbb{P}^2}^{(\ell_1)}\mathbb{P}^5\big) = (1+\ell_1 k)^3$$

and Proposition 2.3 evaluates the correction term due to the first directed blow-up:

$$\ell_1^{3-j_\lambda} \int_{\mathbb{P}^2} \frac{k(1+k)^{j_\mu}(1+\ell_1 k)^{j_\lambda} k^{j_{\overline{\mu}}}(Sm_1 + Snk)^{j_c}}{(1+\ell_1 k)^3(1+k)^3} \cap [\mathbb{P}^2]$$

that is, with minimal manipulations:

$$\boxed{S^{j_c}\ell_1^{3-j_\lambda} \int_{\mathbb{P}^2} \frac{k^{j_{\overline{\mu}}+1}(m_1+nk)^{j_c}}{(1+\ell_1 k)^{3-j_\lambda}(1+k)^{3-j_\mu}} \cap [\mathbb{P}^2].}$$

We will see that, remarkably, *this formula contains all the information necessary to compute the "local" contributions.* However, to understand this we have to write similar terms for the other directed blow-ups.

For the ℓ_2-directed blow-up of V_1^{loc} along B_1^{loc} in the direction of E_1^{loc}, the formula will evaluate the term as a degree over the 4-fold B_1^{loc}. As B_1^{loc} was obtained as the distinguished $\widetilde{B}$ in the ℓ_1-directed blow-up considered at the first stage, points (1) and (3) in Lemma 2.2 give that B_1^{loc} is the projectivization of the normal bundle of $\mathbb{P}^2$ in $\mathbb{P}^5$, and that

$$c(N_{B_1^{\mathrm{loc}}}E_1^{\mathrm{loc}}) = c\big(N_{\mathbb{P}^5}\mathbb{P}^8 \otimes \mathcal{O}(-\ell_1)\big) = (1+k-\ell_1 e_1)^3$$

where k, e_1 are respectively the pull-back of k from $\mathbb{P}^2$ and the restriction of the class of E_1^{loc} to B_1^{loc}. The description of B_1^{loc} as $\mathbb{P}(N_{\mathbb{P}^2}\mathbb{P}^5)$ gives easily all the information needed to perform computations in the intersection ring of $B_1^{\rm loc}$: to evaluate explicitly the term we are going to write in a moment we would push-forward to $\mathbb{P}^2$, then use

$$\begin{aligned} e_1^4\cdot[B_1^{\rm loc}] &\longmapsto 6k^2\cdot[\mathbb{P}^2], & e_1^3\cdot[B_1^{\rm loc}] &\longmapsto 3k\cdot[\mathbb{P}^2],\\ e_1^2\cdot[B_1^{\rm loc}] &\longmapsto [\mathbb{P}^2], & e_1\cdot[B_1^{\rm loc}] &\longmapsto 0 \end{aligned}$$

(this follows immediately from

$$\frac{[\mathbb{P}(N_{\mathbb{P}^2}\mathbb{P}^5)]}{(1+e_1)}\longmapsto c(N_{\mathbb{P}^2}\mathbb{P}^5)^{-1}\cap[\mathbb{P}^2]=\frac{[\mathbb{P}^2]}{(1+k)^3},$$

cf. [Ful]).

Next, we list here the multiplicities of the divisors and the classes of their pull-backs to $B_1^{\rm loc}$:

- for the general hyperplane: multiplicity 0, class k;

- for the j_λ hyperplanes corresponding to the line joining the two distinguished points of C: 0 for all centers; class $= k-e_1$;

- for the j_μ hyperplanes corresponding to the tangent cone: 1 along the first center, 0 along the remaining ℓ_2-1; class $=k-\ell_1e_1$;

- for the $j_{\overline{\mu}}$ hyperplanes corresponding to the tangent cone to C at the other point: 0 for all centers; class $=k$;

- for the j_c point-conditions: Sm_2 along all centers; class $=$ $Snk-S\ell_1m_1e_1$.

Note that this time the number of divisors with mixed multiplicity is $j_\mu\le 2<\operatorname{codim}_{B_1^{\rm loc}}E_1^{\rm loc}$, as needed to apply Proposition 2.3.

Finally, from the above:

$$c(N^{(\ell_2)}_{B_1^{\rm loc}}E_1^{\rm loc})=(1+\ell_2k-\ell_1\ell_2e_1)^3,\quad c(N_{E_1^{\rm loc}}V_1^{\rm loc})=(1+e_1)$$

and we are ready to apply Proposition 2.3, which gives

$$\boxed{\ell_2^{3-j_\mu}\int_{B_1^{\rm loc}}\frac{k(1+\ell_2k-\ell_1\ell_2e_1)^{j_\mu}(k-e_1)^{j_\lambda}k^{j_{\overline{\mu}}}(Sm_2+Snk-S\ell_1m_1e_1)^{j_c}}{(1+\ell_2k-\ell_1\ell_2e_1)^3(1+e_1)}\cap[B_1^{\rm loc}].}$$

Again, this could be somewhat simplified.

The remaining blow-ups all have isomorphic centers, so we can describe the corresponding terms uniformly. Recall from Proposition 2.6 that at the i-th stage ($i > 2$) we are performing the ℓ_i-directed blow-up of $V^{\rm loc}_{i-1}$ along $B^{\rm loc}_{i-1}$ in the direction of $E^{\rm loc}_{i-1}$; here $B^{\rm loc}_{i-1}$ is a 6-fold, and again we can describe it concretely by using Lemma 2.2, point 1): it is the projectivization of the normal bundle to $B^{\rm loc}_1$ in $E^{\rm loc}_1$. An alternative description is as the intersection of the proper transform of $E^{\rm loc}_{i-2}$ with $E^{\rm loc}_{i-1}$; denoting by e_j the restriction of $E^{\rm loc}_j$, the proper transform of $E^{\rm loc}_{i-2}$ will restrict to $e_{i-2} - \ell_{i-1}e_{i-1}$, so

$$c(N_{B^{\rm loc}_{i-1}}E^{\rm loc}_{i-1}) = (1 + e_{i-2} - \ell_{i-1}e_{i-1}).$$

Also notice that by the interchange of exceptional divisors we observed toward the end of the proof of Proposition 2.6 we have

$$e_i = e_{i-2} - \ell_{i-1}e_{i-1}.$$

Multiplicities and class of the divisors:

- for the general hyperplane: multiplicity 0, class k;

- for the j_λ hyperplanes corresponding to the line joining the two distinguished points of C: multiplicity 0, class $= k - e_1$;

- for the j_μ hyperplanes corresponding to the tangent cone: multiplicity 0, class $= k - \ell_1 e_1 - e_2$;

- for the $j_{\overline{\mu}}$ hyperplanes corresponding to the tangent cone to C at the other point: multiplicity 0; class $= k$;

- for the j_c point-conditions $\widetilde{X}$: Sm_i along all centers; class $= Snk - S\ell_1 m_1 e_1 - \cdots - S\ell_{i-1}m_{i-1}e_{i-1}$.

From the above, the classes of the relevant bundles are

$$c(N^{(\ell_i)}_{B^{\rm loc}_{i-1}}E^{\rm loc}_{i-1}) = (1 + \ell_i e_{i-2} - \ell_{i-1}\ell_i e_{i-1}), \quad c(N_{E^{\rm loc}_{i-1}}V^{\rm loc}_{i-1}) = (1 + e_{i-1})$$

and Proposition 2.3 evaluates the i-th term:

$$\ell_i \int_{B^{\rm loc}_{i-1}} \frac{k(k - \ell_1 e_1 - e_2)^{j_\mu}(k - e_1)^{j_\lambda} k^{j_{\overline{\mu}}} (S \cdot \text{p.c.})^{j_c}}{(1 + \ell_i e_{i-2} - \ell_{i-1}\ell_i e_{i-1})(1 + e_{i-1})} \cap [B^{\rm loc}_{i-1}]$$

where

$$\text{p.c.} = m_i + nk - \ell_1 m_1 e_1 - \cdots - \ell_{i-1}m_{i-1}e_{i-1}.$$

Recalling that $e_i = e_{i-2} - \ell_{i-1}e_{i-1}$, this simplifies to

$$\boxed{S^{j_c}\ell_i \int_{B^{\mathrm{loc}}_{i-1}} \frac{k^{j_{\bar\mu}}(k-\ell_1 e_1 - e_2)^{j_\mu}(k-e_1)^{j_\lambda}(\mathrm{p.c.})^{j_c}}{(1+\ell_i e_i)(1+e_{i-1})} \cap [B^{\mathrm{loc}}_{i-1}].}$$

The total "local" contribution from the cusp $\lambda\cap\mu$ is the sum of the first two boxes listed above, plus the sum of the last box over $i = 3,\ldots,e$. Each of the terms can be evaluated as a polynomial in n, the m_i's, and the ℓ_i's, which we take as indeterminates for a moment. With this understood, we let

$$Q_i(n, m_1, \ell_1, m_2, \ell_2, \ldots, m_i, \ell_i)$$

be the contribution coming from the i-th batch, and let

$$P_r(n, m_1, \ell_1, m_2, \ell_2, \ldots, m_r, \ell_r) = \sum_{i=1}^{r} Q_i(n, m_1, \ell_1, m_2, \ell_2, \ldots, m_i, \ell_i).$$

The total local contribution from the point $\lambda \cap \mu$ is then P_e, where the ℓ_i's are replaced with their values prescribed by the (e-line) Euclidean algorithm, that is

$$P_e\Big(n, m_1, \frac{n-m_2}{m_1}, m_2, \frac{m_1-m_3}{m_2}, \ldots, m_{e-1}, \frac{m_{e-2}-m_e}{m_{e-1}}, m_e, \frac{m_{e-1}}{m_e}\Big).$$

Again, we can leave the m_i's undetermined for a moment and treat this as an expression in the variables $n, m = m_1, m_2, \ldots, m_e$.

LEMMA 3.4. — *As an expression in the variables* $n, m = m_1, m_2, \ldots, m_e$,

$$P_e\Big(n, m_1, \ldots, \frac{m_{e-1}}{m_e}\Big) = S^{j_c} n^{3-j_\lambda} \int_{\mathbb{P}^2} \frac{k^{j_{\bar\mu}+1}(m+nk)^{j_c+j_\lambda-3}}{(1+k)^{3-j_\mu}} \cap [\mathbb{P}^2].$$

In particular, we are claiming that the above expression $P_e(\ldots)$ does not depend on the intermediate multiplicities $m_2, \ldots, m_e$, and in fact it is independent on the number e of lines taken by the Euclidean algorithm on m, n. In a sense, all the needed information is therefore contained *in nuce* in the contribution from the *first* directed blow-up!

Proof. — The right-hand-side is $P_1(n, m, n/m)$, that is the first box listed above but with $\ell_1 = n/m$. So the statement is correct for $e = 1$; and the reader can check it is correct for $e = 2$. It can be proved for all $e > 2$ by induction. For this, it suffices to show that, for $e > 2$, the expression

$$P_e\Big(n, m_1, \frac{n - m_2}{m_1}, \dots, m_{e-1}, \frac{m_{e-2} - m_e}{m_{e-1}}, m_e, \frac{m_{e-1}}{m_e}\Big)$$

agrees with the expression

$$P_{e-1}\Big(n, m_1, \frac{n - m_2}{m_1}, \dots, m_{e-1}, \frac{m_{e-2}}{m_{e-1}}\Big).$$

By definition:

$$\begin{aligned} P_e\Big(n, m_1, \frac{n - m_2}{m_1}, \dots, m_{e-1}, \frac{m_{e-2} - m_e}{m_{e-1}}, m_e, \frac{m_{e-1}}{m_e}\Big) \\ = P_{e-1}\Big(n, m_1, \frac{n - m_2}{m_1}, \dots, m_{e-1}, \frac{m_{e-2} - m_e}{m_{e-1}}\Big) \\ + Q_e\Big(n, m_1, \dots, m_e, \frac{m_{e-1}}{m_e}\Big) \end{aligned}$$

so it is enough to show that

1) $Q_e\Big(n, m_1, \dots, m_e, m_{e-1}/m_e\Big)$ is a polynomial in m_e, and vanishes for $m_e = 0$; and

2) $P_e\Big(n, m_1, \dots, m_e, m_{e-1}/m_e\Big)$ does *not* depend on m_e.

Indeed, by 2) we may then assume $m_e = 0$ in evaluating P_e; and by 1) this will give

$$P_{e-1}\Big(n, m_1, \frac{n - m_2}{m_1}, \dots, m_{e-1}, \frac{m_{e-2} - 0}{m_{e-1}}\Big) + 0$$

which is what is needed for the induction step.

Next, observe that the only summands in $P_e = \sum_{i=1}^{e} Q_i$ which involve m_e are Q_{e-1} and Q_e; therefore in order to prove 2) it is enough to prove

2′) $Q_{e-1}\Big(n, m_1, \dots, m_{e-1}, \frac{m_{e-2}-m_e}{m_{e-1}}\Big) + Q_e\Big(n, m_1, \dots, m_e, \frac{m_{e-1}}{m_e}\Big)$ does not depend on m_e.

Now we are only interested in the terms in Q_{e-1}, Q_e involving m_e, so we can neglect the homogeneous terms in Q_i and absorb most of the rest into a divisor class D. Then we are left with the term of codimension $j_c - 2$ in

$$\ell_{e-1}\frac{(m_{e-1}+D)^{j_c}}{(1+\ell_{e-1}e_{e-1})(1+e_{e-2})}+\ell_e\frac{(m_e+D-\ell_{e-1}m_{e-1}e_{e-1})^{j_c}}{(1+\ell_e e_e)(1+e_{e-1})}$$

(where $\ell_{e-1} = m_{e-2} - m_e/m_{e-1}$ and $\ell_e = m_{e-1}/m_e$). It is clear that the second summand is a polynomial in m_e and vanishes for $m_e = 0$, as there are no terms of codimension $j_c - 2$ in this summand if $m_e = 0$. Using [AF3] and $e_e = e_{e-2} - \ell_{e-1}e_{e-1}$, the reader can check that the sum equals the term in codimension $j_c - 2$ in

$$\frac{m_{e-2}m_{e-1}(1+D)^{j_c}}{(1+m_{e-1}e_{e-2})(1+m_{e-2}e_{e-1})},$$

so indeed it does not depend on m_e. □

A simple substitution now computes the contribution due to the other cusp, $\lambda \cap \bar{\mu}$. This amounts to replacing m by $n - m$, and of course reversing the roles of μ and $\bar{\mu}$. Putting the two contributions together, combining with the multiplicity data, and adding over j_μ, *etc.* (*cf.* Lemma 3.3) we obtain the total "local" correction term:

$$7! \sum_{j_\mu+j_{\bar{\mu}}+j_\lambda+j_c=7} n^{j_\mu+j_{\bar{\mu}}-4} \int_{\mathbb{P}^2} \Bigg(\frac{k^{j_{\bar{\mu}}+1}(m+nk)^{j_c+j_\lambda-3}}{(1+k)^{3-j_\mu}} + \frac{k^{j_\mu+1}(n-m+nk)^{j_c+j_\lambda-3}}{(1+k)^{3-j_{\bar{\mu}}}} \Bigg) \times \frac{q^{j_\mu}}{j_\mu!}\,\frac{\bar{q}^{j_{\bar{\mu}}}}{j_{\bar{\mu}}!}\,\frac{r^{j_\lambda}}{j_\lambda!}\,\frac{(Sn)^{j_c}}{j_c!} \cap [\mathbb{P}^2]$$

where we are maintaining the convention that $q^3 = \bar{q}^3 = r^3 = 0$.

This term can now be expanded with relative ease, yielding

$$\begin{aligned} n\,S^2\big(&630\,m\,q^2\,\bar{q}\,r^2 - 630\,m\,q\,\bar{q}^2\,r^2 + 630\,n\,q\,\bar{q}^2\,r^2\\ &+ 420\,m\,n\,q^2\,\bar{q}\,r\,S - 420\,m\,n\,q\,\bar{q}^2\,r\,S + 420\,n^2\,q\,\bar{q}^2\,r\,S\\ &- 210\,m^2\,q^2\,r^2\,S + 420\,m\,n\,q^2\,r^2\,S + 840\,m^2\,q\,\bar{q}\,r^2\,S\\ &- 840\,m\,n\,q\,\bar{q}\,r^2\,S + 420\,n^2\,q\,\bar{q}\,r^2\,S - 210\,m^2\,\bar{q}^2\,r^2\,S\\ &+ 210\,n^2\,\bar{q}^2\,r^2\,S + 105\,m\,n^2\,q^2\,\bar{q}\,S^2 - 105\,m\,n^2\,q\,\bar{q}^2\,S^2\\ &+ 105\,n^3\,q\,\bar{q}^2\,S^2 - 105\,m^2\,n\,q^2\,r\,S^2 + 210\,m\,n^2\,q^2\,r\,S^2 \end{aligned}$$

$$
\begin{aligned}
&+ 420\, m^2\, n\, q\, \bar q\, r\, S^2 - 420\, m\, n^2\, q\, \bar q\, r\, S^2 + 210\, n^3\, q\, \bar q\, r\, S^2 \\
&- 105\, m^2\, n\, \bar q^2\, r\, S^2 + 105\, n^3\, \bar q^2\, r\, S^2 - 315\, m^3\, q\, r^2\, S^2 \\
&+ 630\, m^2\, n\, q\, r^2\, S^2 - 315\, m\, n^2\, q\, r^2\, S^2 + 105\, n^3\, q\, r^2\, S^2 \\
&+ 315\, m^3\, \bar q\, r^2\, S^2 - 315\, m^2\, n\, \bar q\, r^2\, S^2 + 105\, n^3\, \bar q\, r^2\, S^2 \\
&- 21\, m^2\, n^2\, q^2\, S^3 + 42\, m\, n^3\, q^2\, S^3 + 84\, m^2\, n^2\, q\, \bar q\, S^3 \\
&- 84\, m\, n^3\, q\, \bar q\, S^3 + 42\, n^4\, q\, \bar q\, S^3 - 21\, m^2\, n^2\, \bar q^2\, S^3 \\
&+ 21\, n^4\, \bar q^2\, S^3 - 126\, m^3\, n\, q\, r\, S^3 + 252\, m^2\, n^2\, q\, r\, S^3 \\
&- 126\, m\, n^3\, q\, r\, S^3 + 42\, n^4\, q\, r\, S^3 + 126\, m^3\, n\, \bar q\, r\, S^3 \\
&- 126\, m^2\, n^2\, \bar q\, r\, S^3 + 42\, n^4\, \bar q\, r\, S^3 - 126\, m^4\, r^2\, S^3 \\
&+ 252\, m^3\, n\, r^2\, S^3 - 126\, m^2\, n^2\, r^2\, S^3 + 21\, n^4\, r^2\, S^3 \\
&- 21\, m^3\, n^2\, q\, S^4 + 42\, m^2\, n^3\, q\, S^4 - 21\, m\, n^4\, q\, S^4 \\
&+ 7\, n^5\, q\, S^4 + 21\, m^3\, n^2\, \bar q\, S^4 - 21\, m^2\, n^3\, \bar q\, S^4 \\
&+ 7\, n^5\, \bar q\, S^4 - 42\, m^4\, n\, r\, S^4 + 84\, m^3\, n^2\, r\, S^4 \\
&- 42\, m^2\, n^3\, r\, S^4 + 7\, n^5\, r\, S^4 - 6\, m^4\, n^2\, S^5 \\
&+ 12\, m^3\, n^3\, S^5 - 6\, m^2\, n^4\, S^5 + n^6\, S^5\big).
\end{aligned}
$$

This expression is much more structured than it appears at first sight. Using our convention ($q^3 = \bar q^3 = r^3 = 0$) we can rewrite it as

$$
\begin{aligned}
(Sn + r + q + \bar q)^7 &- n^3 m^2 \overline{m}^2 \Big(\Big(S + \frac{r}{n} + \frac{q}{m} + \frac{\bar q}{\overline{m}}\Big)^7 + 2\Big(S + \frac{r}{n} + \frac{q}{m}\Big)^7 \\
&+ 2\Big(S + \frac{r}{n} + \frac{\bar q}{\overline{m}}\Big)^7 + \Big(S + \frac{r}{n}\Big)^7 - 42\Big(S + \frac{r}{n}\Big)^5 \Big(\frac{q^2}{m^2} - \frac{q}{m}\frac{\bar q}{\overline{m}} + \frac{\bar q^2}{\overline{m}^2}\Big)\Big)
\end{aligned}
$$

(where $\overline{m} = n - m$). Subtracting from the Bézout term given in the beginning of this subsection, we get the first expression listed in Theorem 1.1:

$$
\begin{aligned}
n^3 m^2 \overline{m}^2 \Big(\Big(S + \frac{r}{n} + \frac{q}{m} + \frac{\bar q}{\overline{m}}\Big)^7 &+ 2\Big(S + \frac{r}{n} + \frac{q}{m}\Big)^7 + 2\Big(S + \frac{r}{n} + \frac{\bar q}{\overline{m}}\Big)^7 \\
&+ \Big(S + \frac{r}{n}\Big)^7 - 42\Big(S + \frac{r}{n}\Big)^5 \Big(\frac{q^2}{m^2} - \frac{q}{m}\frac{\bar q}{\overline{m}} + \frac{\bar q^2}{\overline{m}^2}\Big)\Big)
\end{aligned}
$$

(up to the multiplicative factor $1/(An)$, *cf.* Lemma 3.3).

This is the intersection product of the relevant divisors, after the sequence of local blow-ups is completed. The correction term for the global blow-ups is computed in the next subsection.

3.2. Global blow-ups.

Recall from §2.2 and §2.3 that after the local stages are completed, the base locus of the rational map c consists of several disjoint 3-dimensional components, each isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$, where the $\mathbb{P}^1$-factor represents the normalization of a curve of type (m, n), and of other 5-dimensional components due to the lines λ, μ, $\overline{\mu}$ of the basic triangle.

We deal here with the 3-dimensional components. By Theorem 2.4, two blow-ups will resolve the indeterminacies of c over such components; we evaluate the relevant intersection products (as in Lemma 3.3) by subtracting from the result of §3.1 a correction term due to these components. The correction term will again be obtained by applying Proposition 2.3, for which we need to compile the usual information of multiplicity and class of normal bundles. We will see that an interesting phenomenon rules these "global" contributions: they are independent of m.

First, we have to compute the Chern classes of the normal bundle to each component $\mathbb{P}^2 \times \mathbb{P}^1$; these will be

$$\frac{c(T\widetilde{\mathbb{P}}^8)}{c(T\mathbb{P}^2 \times \mathbb{P}^1)}$$

where $\widetilde{\mathbb{P}}^8$ is the blown-up $\mathbb{P}^8$ at this stage, and we omit the obvious pull-back to $\mathbb{P}^2 \times \mathbb{P}^1$. Now let k, p denote the pull-back of the class of a line from the $\mathbb{P}^2$ factor and of a point from the $\mathbb{P}^1$ factor; so

$$c(T\mathbb{P}^2 \times \mathbb{P}^1) = (1+k)^3(1+2p).$$

As for $c(T\widetilde{\mathbb{P}}^8)$, this can be obtained by repeated application of [Ful], §15.4: the reader will check that, after the sequence of blow-ups over $\lambda \cap \mu$, this pulls back to

$$\begin{aligned}&(1+k+np)^9 + (1+k+np)^8\\ &\qquad \times \big((m_1+m_2)(k-2)p - 3\ell_1 m_1 p - (\ell_2 m_2 + \ell_3 m_3 + \ldots)(k+1)p\big).\end{aligned}$$

From the Euclidean algorithm, we see that

$$\ell_2 m_2 + \ell_3 m_3 + \cdots = (m_1 - m_3) + (m_2 - m_4) + \cdots + (m_{e-2} - m_e) + m_{e-1}$$

telescopes to $m_1 + m_2 - 1$ since $m_e = \gcd(m, n) = 1$; and $\ell_1 m_1 = n - m_2$. Hence the class simplifies to

$$\begin{aligned}&(1+k+np)^9 + (1+k+np)^8\\ &\qquad \times \big((m_1+m_2)(k-2)p - 3(n-m_2)p - (m_1+m_2-1)(k+1)p\big)\\ &\qquad\qquad = c(T\mathbb{P}^8) + (1+k+np)^8\big((1-3n-3m)p + kp\big).\end{aligned}$$

The second summand evaluates the change in the total class of the tangent bundle due to the blow-ups over the point $\lambda \cap \mu$. Simply substituting $m \mapsto n - m$ evaluates the change due to the other point, $\lambda \cap \bar{\mu}$:

$$(1 + k + np)^8 \big((1 - 3n - 3(n - m))p + kp\big)$$

so that in total

$$\begin{aligned} c(T\widetilde{\mathbb{P}}^8) &= (1 + k + np)^9 + (1 + k + np)^8 \\ &\qquad \times \big(((1 - 3n - 3m) + (1 - 3n - 3(n - m)))p + 2kp\big) \\ &= (1 + k + np)^9 + (1 + k + np)^8 \big((2 - 9n)p + 2kp\big). \end{aligned}$$

Expanding and applying $k^3 = 0$, $p^2 = 0$ (as we are pulling back to $\mathbb{P}^2 \times \mathbb{P}^1$) gives

$$c(T\widetilde{\mathbb{P}}^8) = 1 + 9k + 2p + 36k^2 + 18kp + 72k^2p.$$

Remarkably, this expression is independent of m, n. In fact, the normal bundle to one 3-dimensional component is computed by

$$\begin{aligned} \frac{c(T\widetilde{\mathbb{P}}^8)}{c(T\mathbb{P}^2 \times \mathbb{P}^1)} &= \frac{1 + 9k + 2p + 36k^2 + 18kp + 72k^2p}{(1 + k)^3(1 + 2p)} \\ &= 1 + 6k + 15k^2 = (1 + k)^6 \end{aligned}$$

and is therefore particularly simple. This is the key ingredient in evaluating the correction term due to the first global blow-up.

As for the second, we use the analysis of the similar situation in [AF1] for smooth curves: indeed, the point of the proof of Theorem 2.4 is that after the local blow-ups and the first global blow-up, the geometry over one of these components is entirely analogous to the geometry for smooth curves, over non-flex points. In particular, we know that the center of the second global blow-up is the union of a $\mathbb{P}^1$-bundle over each 3-dimensional component of the center of the first blow-up, with $c_1(\mathcal{O}(-1)) = f$ =restriction of the exceptional divisor, and standard computations will show that, after push-forward to the underlying $\mathbb{P}^2 \times \mathbb{P}^1$,

$$f^4 \longmapsto 0, \quad f^3 \longmapsto -6k^2, \quad f^2 \longmapsto -3k, \quad f \longmapsto -1.$$

The normal bundle has class

$$(1 + f)(1 + k - f)^3.$$

The last ingredient necessary to perform the computation of the global contribution is the restriction to the centers of the classes of the proper transforms of the divisors. To find these, keep in mind that e_i restricts to $m_i p$; then

- the proper transform of a general hyperplane $\widetilde{H}$ restricts to $(k+np)$;
- the proper transform of H_λ restricts to $k + np - e_1 - \bar{e}_1$ (where e_1, $\bar{e}_1$ are the first exceptional divisors at $\lambda \cap \mu$, $\lambda \cap \bar{\mu}$, resp.) This equals $k + np - mp - (n-m)p = k$;
- the proper transform of H_μ restricts to $k + np - \ell_1 e_1 - e_2 = k + np - (\ell_1 m_1 + m_2) = k + np - np = k$; similarly, the proper transform of $H_{\bar{\mu}}$ must restrict to k;
- the proper transform of X restricts to

$$Sn(k+np) - m_1 S\ell_1 e_1 - m_2 S\ell_2 e_2 - \cdots$$

after the sequence over the point $\lambda \cap \mu$, that is to

$$\begin{aligned} Sn(k+np) &- S(\ell_1 m_1^2 - \ell_2 m_2^2 - \cdots)p \\ &= Sn(k+np) - S\big(m(n-m_2) + m_2(m-m_3) + \cdots\big)p \\ &= Sn(k+np) - Smnp. \end{aligned}$$

The second summand is the change due to the sequence of blow-ups over $\lambda \cap \mu$; to obtain the change due to $\lambda \cap \bar{\mu}$, just substitute $m \mapsto n - m$.

The conclusion is that the proper transform of X at the first global blow-up restricts to

$$Sn(k+np) - Smnp - S(n-m)np = Snk.$$

Also, note that X has multiplicity s_i along the i-th component, 0 along all others.

Note that none of the classes depend on m: as we claimed above, *the global contributions do not depend on m.*

Putting the above together and using Proposition 2.3, we see that the global contributions are obtained by evaluating

$$\sum_i \int \frac{(k+np)k^{7-j_c}(s_i + Snk)^{j_c}}{(1+k)^6} \quad \text{and}$$
$$\sum_i \int \frac{(k+np)k^{7-j_c}(s_i + Snk - s_i f)^{j_c}}{(1+f)(1+k-f)^3}.$$

Adding these two terms and inserting in Lemma 3.3 gives a relatively simple expression:

$$n\Big(84(Sn+r+q+\bar{q})^2\sum s_i^5 - 252(Sn+r+q+\bar{q})\sum s_i^6 + 192\sum s_i^7\Big)$$

which reproduces the one given in §1, again up to the multiplicative factor $1/(An)$.

4. End of the computation, and variations.

The careful reader knows that we are not quite done, since after the pair of global blow-ups we are still left with base loci corresponding to the lines of the basic triangle (*cf.* §2.3).

Here we reap the benefit of having shown that we only need to compute the relevant intersection products for $j_\mu, j_{\bar{\mu}}, j_\lambda \le 2$ (Lemma 3.3). Indeed, with these constraints the corrections due to these base loci are *zero*. For example, consider the correction term coming from the $\mathbb{P}^2$ of matrices whose image is the point $\mu \cap \bar{\mu}$: denoting by k the class of a line in $\mathbb{P}^2$ and applying once more the formula in Proposition 2.3, this term is evaluated as

$$\int_{\mathbb{P}^2} \frac{k(q+qk)^{j_\mu}(\bar{q}+\bar{q}k)^{j_{\bar{\mu}}}(rk)^{j_\lambda}(Snk)^{j_c}}{(1+k)^6}\,.$$

And since $1 + j_c + j_\lambda = 8 - (j_\mu + j_{\bar{\mu}}) \ge 4$, this term is automatically 0.

The same discussion applies to the remaining three 5-dimensional base loci; we leave the details to the reader. Theorem 1.1 then follows, since this shows that the expressions obtained in §3.1 and §3.2 combine to give the intersection product in Lemma 3.3.

4.1. Quadritangent conics.

There is (in characteristic zero) only one other class of curves C whose components are not all lines and whose orbit is small: C consists of two or more conics from a pencil through a conic and a double tangent line (Fig. 4); it may also contain that tangent line. The multiplicities of components are arbitrary. For this class of curves, the stabilizer is 1-dimensional; its identity component is the additive group $\mathbb{G}_a$.

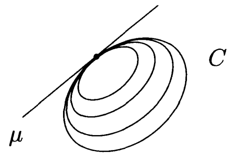


Figure 4

These curves are not of the type considered in previous sections. A variety $\widetilde{V}$ dominating the orbit closure of such a curve can however be constructed by a strategy very similar to the one followed in §2: again, the sequence of blow-ups giving an embedded resolution of the curve can be mirrored to produce a variety $\widetilde{V}^{\mathrm{loc}}$, from which a variety $\widetilde{V}$ is produced by the technique of Theorem 2.4. Intersection-theoretic computations similar to those in §3 allow us then to compute the degree of the orbit closure of these curves.

It is perhaps a little surprising that the formula given in Theorem 1.1 turns out to be correct for this case as well: taking $n = 2$, $m = \overline{m} = 1$, and $\bar{q} = r = 0$ computes the degree of the orbit closure of a union of quadritangent conics appearing with multiplicity s_i, together with the tangent line μ at the point of contact, taken with multiplicity q. In other words, the polynomial Q for such a curve is the same as the polynomial for a union of "bitangent conics". Again, A equals the number of components of the stabilizer of C; the analogue of Lemma 3.1 is the fact that for C given by

$$y^q \prod_i (x^2 + yz + \alpha_i y^2)^{s_i} = 0,$$

the number A equals twice the maximum order of an automorphism $\alpha \mapsto u\alpha + v$ preserving the S-tuple given by $\prod(\alpha - \alpha_i)^{s_i} = 0$. Hence $A = 2$ in most cases, and it is bounded by twice the number of conics in C; for two conics with equal multiplicities, $A = 4$.

With notations as above, the polynomial Q is in this case

$$24S^7 + 84S^6 q + 84S^5 q^2 - 84(2S+q)^2 \sum s_i^5 + 252(2S+q) \sum s_i^6 - 192 \sum s_i^7.$$

For example, there are 504 ($= 2016/4$) pairs of quadritangent conics through seven general points.

Expressions for the degree of loci corresponding to curves with fixed tangent line, or tangent line constrained to contain a given point, can be obtained by differentiating this expression with respect to q (*cf.* §4.3).

4.2. Predegree polynomials.

Simple adjustments in the computations described in this paper allow us to compute the degrees of suitable subsets of the orbit closure, obtained by imposing general linear conditions on the matrices used to act on C. In a sense this note deals precisely with one such computation: we computed the degree of $\overline{\mathcal{O}}_C$ by imposing a general linear condition on $\mathbb{P}^8$ (and arguing that this would intersect the fibers over a point of $\mathcal{O}_C$ in An points, *cf.* Proposition 3.2).

We call the *predegree* of $\overline{\mathcal{O}}_C$ the product of the degree of $\mathcal{O}_C$ with the degree in $\mathbb{P}^8$ of the closure of the stabilizer of C.

Arguing as in the discussion leading to Proposition 3.2 (and using the same notations), we see that if $\mathcal{O}_C$ has dimension k then

$$(\text{predegree of } \mathcal{O}_C) = \int \widetilde{H}^{8-k}\widetilde{W}^k,$$

where the intersection product is taken in any variety resolving the indeterminacies of the relevant rational map. (Note: this notion of predegree agrees with the one used in [AF1], where $k = 8$.)

We find it in fact useful to introduce an *adjusted predegree polynomial* defined by

$$\sum_{j\geq 0}\left(\int \widetilde{H}^{8-j}\widetilde{W}^j\right)\frac{t^j}{j!}.$$

If $\dim \mathcal{O}_C = k$, then the coefficient of $t^k/k!$ in the adjusted predegree polynomial gives the predegree of $\mathcal{O}_C$, while for $j > k$ the coefficient of $t^j/j!$ is 0.

The introduction of denominators reflects some extra structure of these polynomials, which we will not discuss here. As an example, note the factorization of the polynomial for the degenerate case of a curve supported on the basic triangle, indicated below: such factorizations would not occur "without denominators".

Suitable variations of the computations in §3 yield the whole adjusted predegree polynomial for the curves considered in this paper; the result is as follows.

To obtain the polynomial for a curve with data m, n, $\overline{m} = n - m$, s_i, $S = \sum s_i$, $r, q, \bar{q}$, as above: *imposing* $q^3 = \bar{q}^3 = r^3 = 0$ *in all computations,* truncate to t^7 the expansion of $e^{(Sn+r+q+\bar{q})t}$, (the "Bézout" term), then subtract a global contribution (independent of m)

$$\begin{gathered}12\,n\Big(\sum s_i^5\Big)\frac{t^5}{5!}\\+n\left(48\,(Sn+r+q+\bar{q})\sum s_i^5-72\sum s_i^6\right)\frac{t^6}{6!}\\+n\left(84\,(Sn+r+q+\bar{q})^2\sum s_i^5-252\,(Sn+r+q+\bar{q})\sum s_i^6+192\sum s_i^7\right)\frac{t^7}{7!}\end{gathered}$$

and a local contribution given in degree 6 by

$$n^3\Big(m^3\Big(S+\frac{r}{n}+\frac{q}{m}\Big)^6+\overline{m}^3\Big(S+\frac{r}{n}+\frac{\bar{q}}{\overline{m}}\Big)^6\Big)\frac{t^6}{6!}$$

and in degree 7 by

$$\begin{aligned}\Big((Sn+r+q+\bar{q})^7&\\-n^3m^2\overline{m}^2\Big(\Big(S+\frac{r}{n}+\frac{q}{m}+\frac{\bar{q}}{\overline{m}}\Big)^7&+2\Big(S+\frac{r}{n}+\frac{q}{m}\Big)^7\\+2\Big(S+\frac{r}{n}+\frac{\bar{q}}{\overline{m}}\Big)^7&+\Big(S+\frac{r}{n}\Big)^7\\-42\Big(S+\frac{r}{n}\Big)^5&\Big(\frac{q^2}{m^2}-\frac{q}{m}\frac{\bar{q}}{\overline{m}}+\frac{\bar{q}^2}{\overline{m}^2}\Big)\Big)\Big)\frac{t^7}{7!}.\end{aligned}$$

The advantage of looking at the whole polynomial is that it carries degree information for all orbits, regardless of their dimension (while Theorem 1.1 does assume that the orbit of the curve under exam has dimension 7). For example, setting $m=1$, $n=2$, $S=s_1=1$, and $r=q=\bar{q}=0$ gives a polynomial

$$1+2t+\frac{4t^2}{2}+\frac{8t^3}{3!}+\frac{16t^4}{4!}+\frac{8t^5}{5!};$$

as the orbit closure of a conic is clearly the whole of $\mathbb{P}^5$, this correctly detects that the stabilizer of a conic is a threefold of degree 8. In fact, the second Veronese embedding of the $\mathbb{P}^3$ of 2×2 matrices in the $\mathbb{P}^9$ of space quadrics projects isomorphically to this threefold in $\mathbb{P}^8$; with suitable identifications, the center of the projection is the determinant quadric.

For another example, take all $s_i=0$: that is, consider a curve consisting solely of lines supported on the sides μ, $\bar{\mu}$, λ of the basic triangle, with multiplicities q, $\bar{q}$, r. This yields a degree-6 polynomial, which in fact factors

$$\left(1+qt+\frac{q^2t^2}{2}\right)\left(1+\bar{q}t+\frac{\bar{q}^2t^2}{2}\right)\left(1+rt+\frac{r^2t^2}{2}\right):$$

the orbit has dimension 6 and predegree $90q^2\bar{q}^2r^2$, as the reader could check independently by observing that this orbit closure can also be realized as the image of the evident map $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^N$ ($N = \frac{1}{2}d(d+3)$ for $d = q + \bar{q} + r$). The degree of the closure of the stabilizer depends on the multiplicities of the lines: it is 3! if $q = \bar{q} = r$, 2 if exactly two multiplicities agree, and 1 if the multiplicities are distinct.

All such computations are very particular cases of the general expression for the adjusted predegree polynomial given above. This covers then almost all small orbits of plane curves, with few exceptions such as curves consisting of a star of lines through a point p, union a line not containing p. Note that these are curves "of type $(1,1)$" according to the terminology used in this paper; but the blow-up construction of §2 assumes that the only linear components of the curve are the lines of the basic triangle, so the construction fails in this case. We will consider these curves in [AF4].

4.3. Curves with constraints.

The polynomial given in Theorem 1.1 also contains enumerative information on the subsets of the orbits parametrizing curves with specified constraints on the lines of the basic triangle (such as: containing a given point). Let $\overline{\mathcal{O}}_C(j_\mu, j_{\overline{\mu}}, j_\lambda)$ be the closure of the set of translations $C\circ\varphi$ such that $\mu\circ\varphi$ contains j_μ given points, $\overline{\mu}\circ\varphi$ contains $j_{\overline{\mu}}$ given points, and $\lambda\circ\varphi$ contains j_λ given points (all choices of the points being general).

PROPOSITION 4.1. — *Let $Q(n,m,s_i,r,q,\bar{q})/A$ be the polynomial giving the degree of $\overline{\mathcal{O}}_C$, as in Theorem 1.1. Then the degree of $\overline{\mathcal{O}}_C(j_\mu, j_{\overline{\mu}}, j_\lambda)$ is*

$$\frac{(7-j_\mu-j_{\overline{\mu}}-j_\lambda)!}{7!\,A}\frac{\partial^{j_\mu}}{\partial q^{j_\mu}}\frac{\partial^{j_{\overline{\mu}}}}{\partial \bar{q}^{j_{\overline{\mu}}}}\frac{\partial^{j_\lambda}}{\partial r^{j_\lambda}}Q(n,m,s_i,r,q,\bar{q}).$$

Proof. — Arguing as in Proposition 3.2,

$$An\deg\overline{\mathcal{O}}_C(j_\mu, j_{\overline{\mu}}, j_\lambda) = \int \widetilde{H}\cdot\widetilde{W}^{(7-j_\mu-j_{\overline{\mu}}-j_\lambda)}\cdot\widetilde{H}_\mu^{j_\mu}\cdot\widetilde{H}_{\overline{\mu}}^{j_{\overline{\mu}}}\cdot\widetilde{H}_\lambda^{j_\lambda},$$

where $\widetilde{W} = q\widetilde{H}_\mu + \bar{q}\widetilde{H}_{\bar{\mu}} + r\widetilde{H}_\lambda + \widetilde{X}$ is the class of a point-condition in a variety resolving the rational map corresponding to C. Now a direct computation shows that this equals

$$\frac{(7-j_\mu-j_{\bar{\mu}}-j_\lambda)!}{7!}\frac{\partial^{j_\mu}}{\partial q^{j_\mu}}\frac{\partial^{j_{\bar{\mu}}}}{\partial \bar{q}^{j_{\bar{\mu}}}}\frac{\partial^{j_\lambda}}{\partial r^{j_\lambda}}\int \widetilde{H}\cdot(q\widetilde{H}_\mu + \bar{q}\widetilde{H}_{\bar{\mu}} + r\widetilde{H}_\lambda + \widetilde{X})^7,$$

which gives the statement. □

This result has a clear enumerative meaning for example when $q=\bar{q}=r=0$, that is when C does not contain the lines in the triangle, and when all $s_i = 1$ (so that $S =$ the number of components of C). Then the formula of Proposition 4.1 gives the number of PGL(3)-translations of C which satisfy the given constraints on the lines of the basic triangle, and contain the appropriate number ($= 7 - j_\mu - j_{\bar{\mu}} - j_\lambda$) of general points.

From the formula in Theorem 1.1 and Proposition 4.1 it is easy to obtain closed formulas for these numbers. For example, the number of unconstrained curves of fixed type (that is, S given distinct components from the pencil of type (m,n)-curves) through seven points is

$$\frac{6S}{A}\left(m^2\overline{m}^2n^2S^6 - 14n^2S^2 + 42nS - 32\right);$$

for the simplest example, take $n = 3$, $m = 2$ and $S = 1$: there are 24 cuspidal plane cubics through seven general points. The number of curves such that the line λ contains a given point is

$$\frac{6S}{A}\left(m^2\overline{m}^2nS^5 - 4nS + 6\right);$$

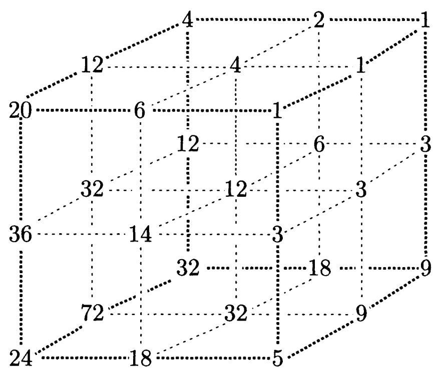
for example, there are 36 cuspidal cubics through six general points, and such that the line connecting the flex and the cusp contains a given point.

Analogously, the number of curves such that λ contains two given points is, according to Proposition 4.1:

$$\frac{2S}{A}\left(3m^2\overline{m}^2S^4 - 2\right);$$

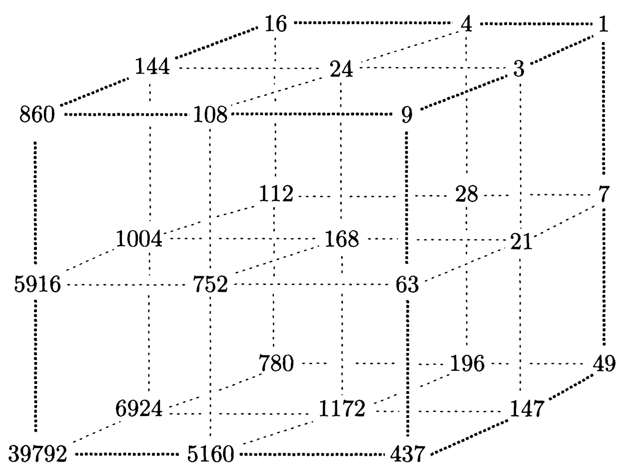
hence, there are 20 cuspidal cubics through five points, with fixed line through cusp and flex.

The reader who so wishes will have no difficulty deriving the other 24 closed formulas for assortments of conditions on the lines of the basic triangle. For cuspidal cubics, the 27 numbers so obtained reproduce results in [MX] (where hundreds more are computed). Here they are, where the number at the (i, j, k) spot denotes the number of curves with i points through μ, j points through $\bar{\mu}$, and k points through λ:



Curve $x^3 = y^2$

For higher degree curves, the results are, to our knowledge, new. Here are the numbers for the curve of (randomly chosen) type $(7, 4)$:



Curve $x^7 = y^4$

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