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EDGE CONJUGATION AND COLORATION IN CUBIC MAPS

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EDGE CONJUGATION AND COLORATION IN CUBIC MAPS

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CHAPTER I

INTRODUCTION

In nontechnical terms the four-color conjecture may be stated as follows: the regions of any map on a plane or on the surface of a sphere can be colored, using only four colors, so that no two linearly contiguous regions have the same color. During the last century this conjecture has acted as a primary catalyst in the development of graph theory and combinatorial topology; nevertheless, few current workers attempt outright to prove or disprove the conjecture. Concerning the writers on graph theory over the years, Ore [35] has said that, however practical were the problems they examined or however abstract their papers may appear, the authors more often than not seem to have had some thoughts about application of their results to the four-color problem.

It has been suggested by Ball [1] and others that the four-color property was familiar to practical mapmakers for a long time prior to the initial interest by mathematicians. May [34], however, finds little evidence to support this. His research reveals that books on cartography and the history of mapmaking do not mention the sufficiency of four colors, though they often discuss other problems relating to the coloring of maps, and his sampling of atlases indicates no tendency to minimize the number of colors used. In fact, maps utilizing only four colors are rare and those that do often require only three.

Instead of being the culmination of a series of individual efforts, the conjecture apparently first crossed the mind of Francis Guthrie while he was coloring a map of England sometime after receiving his mathematics degree in 1850 from University College, London. He attempted a proof but, as recorded by his brother Frederick [20], considered it unsatisfactory. Meanwhile, in October, 1852, Frederick communicated the conjecture, but not the attempted proof, to Professor A. DeMorgan under whom he was currently studying at University College. The latter gave it some thought and in a letter to W. R. Hamilton, dated 23 October 1852, tried to interest him in attempting a solution [34]. On 26 October, Hamilton replied that this was a quaternion which he did not wish to work on. Giving due credit to Guthrie, DeMorgan tried to interest his students and others, including A. Cayley, in this intriguing question.

Twenty-five years passed before Cayley revived the problem at the June 13, 1878, meeting of the London Mathematical Society. There he stimulated interest among mathematicians by asking whether the conjecture had been proved and by stating that he had been unable to devise a rigorous proof for it. The first printed reference to the problem appears in the proceedings of that meeting [9]. Interest in the question was immediate and pseudo-solutions were soon published by Kempe [28] in 1879 and by Tait [41] in 1880.

Kempe's argument appears to have been accepted as valid until 1890 when P. J. Heawood, Lecturer in Mathematics at Durham University, pointed out a hiatus in Kempe's logic [25]. His modestly written paper, "Map color theorems," is referred to by Dirac (one of today's foremost workers in graph theory) in [15] as "...undoubtedly the greatest contribution so far made to the mathematical theory of the colouring of maps." In it Heawood modified Kempe's argument to prove that any map drawn on a surface of genus 0, i.e., a plane or sphere, can be colored using only <u>five</u> colors and that for any map on an orientable surface of genus $p \geq 1$, $s = \frac{1}{2}[7 + (48p + 1)^{\frac{1}{2}}]$ colors will suffice. The necessity of s colors for maps on all such surfaces has very recently been demonstrated by Ringel and Youngs [39]. Heawood also considered maps in which a country may consist of several detached portions all of which require the same color; e.g., a country and its colonies should be colored the same and all oceans and lakes are usually colored blue. He proved that if m is the maximum number of detached portions of a country in a map drawn on an orientable surface of genus $p \geq 0$, then at most $\frac{1}{2}(6m + 1 + [24p + (6m+1)^2 - 48]^{\frac{1}{2}})$ colors are sufficient for its coloring except, of course, when $p = 0$

and $m = 1$. In fact, for $p = 0$ amd $m = 2$, Heawood produced an ingenious example of a map consisting of twelve countries, each in two portions, which needs twelve colors for its coloring.

"Map color theorems" was an epoch-making contribution to the theory because it introduced methods used until this day and because it settled many new and seemingly more difficult questions, while the four-color conjecture on surfaces of genus $p = 0$ emerged as the central unsolved problem. Heawood wrote other penetrating papers on the problem, the most significant of which [26] appeared in 1897, and conducted correspondence with other mathematicians who were considering it until shortly before his death in January, 1955.

Many mathematicians since Heawood have investigated the problem and have sought to bring a solution nearer by translating it into equivalent new forms or by casting it in more general settings. Although these efforts have succeeded in opening new avenues of mathematical research and have yielded numerous partial results, none has surmounted the difficulty and the conjecture remains unsolved.

CHAPTER II

CUBIC MAPS

Despite the age of the four-color conjecture, formal graph theory is relatively new and there is not yet total agreement on conventions and definitions. In this chapter we set forth certain definitions and record some results pertinent to the exposition. These results come from scattered sources and have been organized for consistent presentation herein. For the most part the unreferenced theorems are common knowledge in graph theory. Although the form may differ considerable, their statements may be found in the books by Ore [35,36] or Harary [21,22].

A <u>graph</u> G consists of a non-empty set V(G) of elements called <u>vertices</u>, a (perhaps empty) set E(G) of elements called <u>edges</u>, and an <u>incidence mapping</u> I(G) which associates with each edge an unordered pair of vertices called its <u>ends</u>. The two ends of a given edge need not be distinct; an edge with coincident ends is called a <u>loop</u>. If I(G) is not one-to-one, then G is said to possess <u>multiple edges</u>. Vertices v and w in V(G) are said to be <u>adjacent</u> in G iff (if and only if) they are ends of some edge in E(G), and edges e and f in E(G) are <u>adjacent</u> iff the intersection of their ends is non-empty.

For a graph G let V and E denote the cardinality of V(G) and E(G) respectively. Then G is said to be <u>finite</u> iff both V and E are finite, and <u>infinite</u> otherwise. In the present work the word "graph" should always be taken to mean finite graph unless the contrary is explicitly stated. If $V = 1$ and $E = 0$, then G is called the trivial graph.

As customary we represent a graph by a diagram on the plane (see Figure 2.1) in which each vertex is shown as a dot and each edge as a simple Jordan curve joining dots representing its ends. Two curves representing edges may cross at a point that does not represent a vertex. Clearly

(a) (b) (c) (d) (e)

Figure 2.1.

the diagrammatic representation of a graph is not unique. In fact, diagrams of the same graph G may look quite different but each such diagram is called a plane representation of G. Two graphs are said to be isomorphic iff there is a one-to-one correspondence between the unions of their edge and vertex sets which maps edges to edges and vertices to vertices and which preserves incidences. The diagrams in Figure 2.1(a) and (b) represent isomorphic graphs.

The valency or degree, $d(v)$, of a vertex v in $V(G)$ is the number of edges incident to v, loops being counted twice. If $d(v) = 0$, then v is called an isolated vertex. A graph G with n vertices is said to be the complete graph, K_n, iff every pair of distinct vertices are adjacent. Hence

$d(v) = n-1$ for every v in $V(K_n)$. Figure 2.1(b) and (d) show K_4 and K_5. A graph for which all vertices have the same degree is said to be regular, and a regular graph in which every vertex has valency three is called trivalent. It is well known that in any graph the number of vertices of odd degree is even. Hence every trivalent graph G has an even number of vertices and the natural number k for which $V = 2k$ and $E = 3k$ will be called the index of G. For convenience in exposition a trivalent graph without loops will be called cubic. Figure 2.1(a), (b), and (c) illustrate cubic graphs.

A finite sequence $s = [v_0,e_1,v_1,e_2,\ldots,e_n,v_n]$, $n \geq 1$, whose terms are alternately vertices v_i and edges e_i of a graph G, is said to be a walk of length n in G iff v_{i-1} and v_i are the ends of edge e_i. If G has no loops or multiple edges, then s is often written $s = [v_0,v_1,\ldots,v_n]$, the edges being evident by context. In discussing walks we use many expressions with obvious meanings, e.g., the walk passes through the vertex v_i, it joins its origin v_0 to its terminus v_n, it traverses edge e_i, and it leaves v_i along e_{i+1}, $0<i<n$. A trail is a walk in which all edges are distinct and a path is a walk in which all vertices (and hence all edges) are distinct. When the origin and terminus of a walk are coincident we call it a closed walk. A circuit (also called a cycle) is a closed walk of length n in which the vertices $v_0,v_1,\ldots,v_{n-1}$ are distinct. In a graph G a trail or closed trail is called Eulerian iff it contains each edge in E(G) and a path or circuit is called Hamiltonian iff

it contains each vertex in V(G). In Figure 2.2 the walk $[v_1,v_4,v_2,v_4,v_3]$ is not a trail, $[v_1,v_2,v_3,v_4,v_1]$ is a Hamiltonian circuit of length four, and $[v_2,v_1,v_4,v_2,v_3,v_4]$ is an Eulerian trail. Characterizations of graphs possessing Eulerian trails are well known but no elegant characterization of Hamiltonian graphs yet exists, although several necessary or sufficient conditions are known.

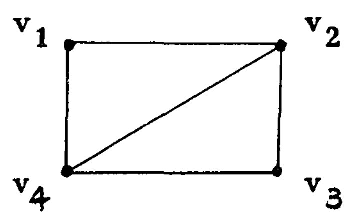


Figure 2.2.

A graph H is said to be a subgraph of a graph G iff $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and each edge of H has the same ends in H as in G; this is written $H \subseteq G$. If H is not identical with G it is called a proper subgraph and we write $H \subset G$. By the removal of a vertex v from a graph G is meant that graph $G - v$ which is the maximal (with respect to inclusion) subgraph of G not containing v. Similarly, the graph $G - e$ obtained by removing an edge e is the maximal subgraph of G not containing e. Thus $V(G-e) = V(G)$ and $E(G-e) = E(G) - \{e\}$ while $V(G-v) = V(G) - \{v\}$ and $E(G-v) = E(G) - \{\text{edges in G incident with v}\}$.

CONNECTIVITY

A graph G is said to be connected iff every pair of distinct vertices can be joined by a path in G. A maximal connected subgraph of G is called a component of G.

A cutpoint of a graph G is a vertex whose removal increases the number of components and a bridge is such an edge. In Figure 2.3, v is a cutpoint while w is not; e is a bridge but f is not. If a graph is connected and nontrivial, then it is said to be nonseparable iff it has no cutpoints. A block of a graph G is a maximal nonseparable subgraph of G.

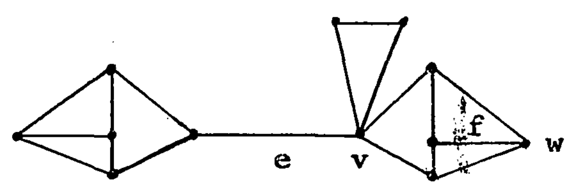


Figure 2.3.

The above example has four blocks. Observe that each edge of a graph resides in exactly one of its blocks, as does each vertex which is neither an isolated vertex nor a cutpoint. Also the edges of any circuit lie entirely in a single block.

Theorem 2.1. The following are equivalent for a connected graph G:

(i) Vertex v in V(G) is a cutpoint.

(ii) There exist u and w in V(G), distinct from v, such that every path joining them contains v - in fact, u and w can be chosen adjacent to v.

(iii) There exists a partition of V(G) - {v} into subsets U and W such that for any u in U and w in W, vertex v is on every path joining u and w.

Theorem 2.2. The following are equivalent for a connected graph G:

(i) Edge e in E(G) is a bridge.

(ii) There exist two vertices u and v in V(G) such that every path joining them contains e - in fact, u and v can be chosen as the ends of e.

(iii) There exists a partition of V(G) into subsets U and W such that for any u in U and w in W, the edge e is on every path joining u and w.

(iv) No circuit in G contains e.

Remark 2.3. Because of (iv) a bridge is sometimes called an acylic edge while non-bridges are called circuit edges.

Theorem 2.4. The following are equivalent for a connected graph G that contains three or more vertices and has no loops and no multiple edges:

(i) Graph G is nonseparable.

(ii) Given two vertices of G, there exists a circuit in G containing them.

(iii) Given a vertex and an edge of G, there exists a circuit in G containing them.

(iv) Given two edges of G, there exists a circuit in G containing them.

(v) Given two vertices and one edge of G, there is a path joining the vertices which contains the edge.

(vi) For every three vertices u, v, and w in V(G), there exists a path joining u and v which contains w.

(vii) For every three vertices u, v, and w in V(G), there is a path joining u and v which does not contain w.

Theorem 2.5. Every connected nontrivial graph without loops or multiple edges has at least two vertices which are not cutpoints.

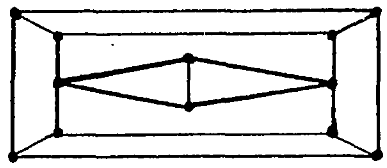
More generally, the <u>vertex-connectivity</u>, VC(G), of a graph G is the smallest number of vertices whose removal yields a disconnected or trivial graph and the <u>edge-connectivity</u>, EC(G), is the least number of edges whose removal results in a disconnected (i.e. not connected) graph. Thus the vertex and edge-connectivities of a disconnected graph are 0, while their value for a connected graph containing two or more edges and a bridge (and hence at least one cutpoint) is 1.

<u>Theorem 2.6</u> (Whitney [48]). For any graph G, if m(G) is the minimum valency of the vertices in V(G), then

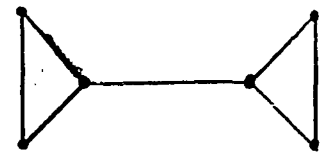
$$VC(G) \leq EC(G) \leq m(G).$$

In view of Theorem 2.6, for any natural number n, a graph G is called <u>n-connected</u> iff $n \leq VC(G)$ and it is called <u>n-edge-connected</u> iff $n \leq EC(G)$. In [48] Whitney proved that a graph is n-connected iff every pair of vertices are joined by at least n vertex-disjoint paths. Similarly, a graph is n-edge-connected iff every pair of vertices are joined by at least n edge-disjoint paths.

A graph G is said to be <u>cyclically n-connected</u> iff it cannot be disconnected into components, each of which contains a circuit, by the removal of fewer than n edges. Let the least number of edges whose removal results in such a disconnection be called the <u>cyclic-connectivity</u> of G, CC(G). Clearly $EC(G) \leq CC(G)$. Figure 2.4 illustrates these concepts.



$VC(G) = 2$, $EC(G) = 3$, $CC(G) = 4$



$VC(G) = EC(G) = CC(G) = 1$

Figure 2.4.

PLANARITY

A graph is said to be embedded in a surface S iff it is diagrammed on S in such a manner that two edges have, at most, end vertices in common. Although all graphs have plane representations, a graph is said to be planar iff it can be embedded in the plane. As shown in Figure 2.1(a) and (b), not all plane representations of a planar graph G are plane embeddings. The form of a plane embedding can be varied greatly, however, by virtue of the Jordan-Schönflies theorem: If ϕ is a homeomorphism between two closed Jordan curves in the plane, then ϕ can be extended to a homeomorphism of the entire plane.

There is considerable interplay between considering a planar graph as a combinatorial object and as a geometric figure. Given a plane embedding of a planar graph G, each point p of the plane not on the diagram of G belongs to a unique face, F(p), of G defined to be the set of all points in the plane which can be joined to p by Jordan curves disjoint from G. Thus the embedding partitions the point set complement of G into a finite number, F, of disjoint arcwise connected subsets. Henceforth the term plane graph will be used when referring to a specific plane embedding of a planar graph G together with its set of faces F(G). The following is often called Euler's formula:

Theorem 2.7. For any connected plane graph G, V - E + F = 2.

Theorem 2.8 (Stein [40]). Every planar graph without loops or multiple edges can be embedded in the plane in such a manner that all its edges are represented by straight line segments.

For a plane graph G the boundary of each face in F(G) is some subgraph of G, but unfortunately the boundary of a face in one plane embedding of a planar graph G does not always correspond to the boundary of a face in another such embedding of G. To discuss this consider a circuit C on a plane graph G. A walk lying interior to C with distinct ends on C is called an inner transversal for C. An outer transversal for C is a walk lying exterior to C with distince ends on C. A circuit C is said to be a minimal circuit iff it has no inner transversals and it is caller a maximal circuit iff there are no outer transversals for C. The interior of a minimal circuit is called a minimal inner domain; its exterior a maximal outer domain. Similarly, the exterior of a maximal circuit is a minimal outer domain, its interior a maximal inner domain. In Figure 2.5, $C = [v_1,v_2,\ldots,v_{15},v_1]$ is a minimal circuit but not a maximal one while $d = [w_1,w_2,\ldots w_7,w_1]$ is maximal but not minimal. In general, a minimal circuit may have edges lying in its minimal inner domain but if G is non-separable and without loops, then there are no edges of G within any minimal domain. A face F(p) is called a bounded

<u>face</u> of G iff p lies interior to some circuit of G; otherwise F(p) is called the <u>unbounded</u> or <u>exterior</u> face of G. The

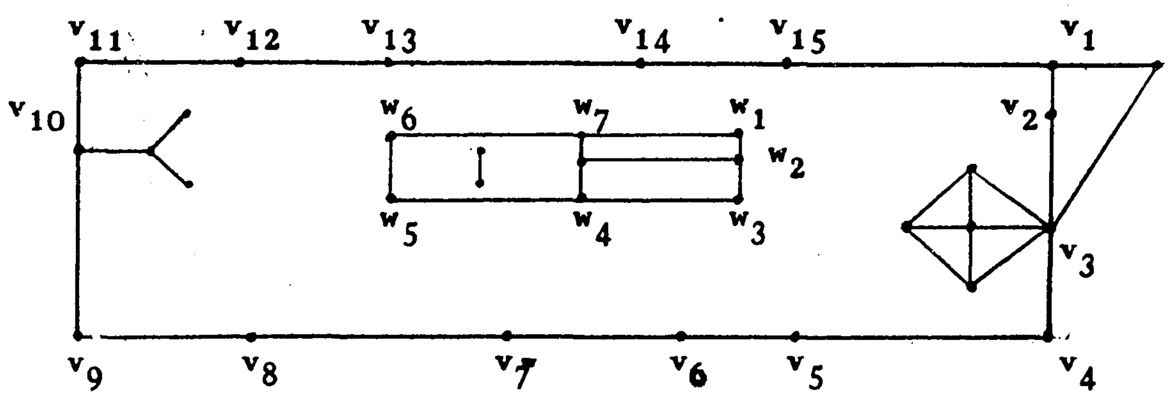


Figure 2.5.

<u>boundary</u> of the exterior face consists of the union of the maximal circuits, each of which lies interior to no other circuit, together with the bridges and isolated vertices lying interior to no circuits. The <u>boundary</u> of a bounded face F(p) is that subgraph of G consisting of (i) the minimal circuit C whose interior contains p and (ii) those maximal circuits, bridges, and isolated vertices lying interior to C which are not interior to any other circuits interior to C.

<u>Theorem 2.9</u>. In any plane graph every bridge is on the boundary of a single face and every circuit edge is on the boundary of exactly two faces.

Two plane embeddings of a graph G are said to be <u>plane equivalent</u> iff there is a correspondence λ between the face sets of the two embeddings with the property that each boundary edge of a face F(p) in one embedding is a boundary edge of $\lambda(F(p))$ in the other. For a planar graph G we say that its plane embedding is <u>essentially unique</u> iff all the plane embeddings of G are plane equivalent. In [35] Ore characterized the planar graphs with essentially unique planar embeddings in terms of the following concept: A nonseparable graph G is said to be <u>properly two-vertex separated</u> by vertices v and w iff there exist subgraphs H_1 and H_2 of G such that

(i) $E(H_1) \cap E(H_2)$ is empty,

((ii) $V(H_1) \cap V(H_2) = \{v,w\}$,

(iii) $V(G) = V(H_1) \cup V(H_2)$ and $E(G) = E(H_1) \cup E(H_2)$,

(iv) For $i = 1,2$, H_i is not a path (although it may be a circuit).

<u>Theorem 2.10</u>. A plane embedding of a planar nonseparable graph G is essentially unique iff no component of G can be properly two-vertex separated.

<u>Remark 2.11</u>. Figure 2.9(a) and (c) illustrate non-plane equivalent embeddings of the same planar graph. The faces of planar graphs for which no component is properly two-vertex separable are frequently specified by describing their boundaries.

<u>Remark 2.12</u>. If a connected graph G is properly two-vertex separable, then $VC(G) = 2$, but a counterexample to the converse is provided by the graph of Figure 2.2.

By stereographic projection any plane graph can be embedded on a sphere. Conversely, by a suitable choice of projection center any graph embedded on a sphere can be embedded in a plane. For a spherical embedding, the distinction between minimal and maximal circuits and bounded and unbounded faces disappears.

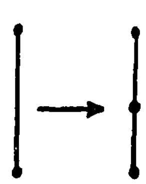
Theorem 2.13. Every plane graph can be embedded in the plane so that any specified face is the exterior face.

Corollary 2.13(a). Every planar graph can be embedded in the plane so that any specified edge is a boundary edge of the exterior face.

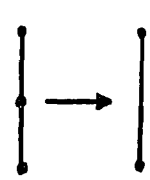
A graph G is said to be bipartite iff V(G) can be partitioned into two non-empty subsets V_1 and V_2 such that there is no edge of G with both its ends in V_i, i = 1,2. Suppose G contains no multiple edges and that the cardinality of V_1 and V_2 is m and n respectively. Then G is called the complete bipartite graph $K_{m,n}$ iff every pair of vertices $\{v_1,v_2\}$ such that $v_1 \in V_1$ and $v_2 \in V_2$ are adjacent.

Theorem 2.14. The graphs $K_{3,3}$ and K_5 (pictured in Figure 2.1(c) and (d)) are non-planar.

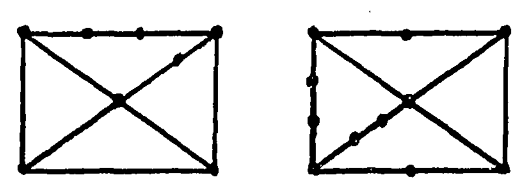
The operation of replacing an edge of a graph by two edges and a divalent vertex as shown in Figure 2.6 will be referred to as vertex insertion and that of replacing two edges and a divalent vertex incident to both by a single edge as vertex suppression (not to be confused with vertex removal). Two graphs are said to be isomorphic within divalent vertices



vertex insertion



vertex suppression



a pair of graphs isomorphic within divalent vertices

Figure 2.6.

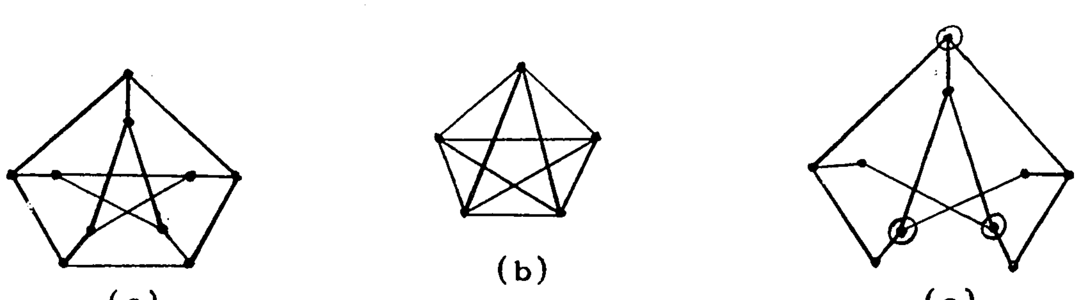
iff they are isomorphic or can be transformed into isomorphic graphs by vertex insertions or suppressions.

Theorem 2.15 (Kuratowski [31]). A graph is planar iff it contains no subgraph isomorphic within divalent vertices to K_5 or $K_{3,3}$.

An elementary edge-contraction of a graph G is the graph obtained by identifying two adjacent vertices v and w, i.e., by the removal of v and w and the addition of a new vertex u adjacent to those vertices which were adjacent to v and w. A graph G is said to be edge-contractible to a graph H iff H can be obtained from G by a sequence of elementary edge-contractions. Harary and Tutte [24] have proved

Theorem 2.16. A graph is planar iff it does not have a subgraph edge-contractible to K_5 or $K_{3,3}$.

The Petersen graph, Figure 2.7(a), is edge-contractible to K_5 and it has a subgraph, Figure 2.7(c), isomorphic within divalent vertices to $K_{3,3}$, but no subgraph isomorphic within divalent vertices to K_5. Of course the latter could not be the case since G is cubic.

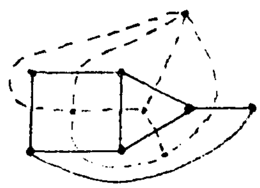


(a) (b) (c)

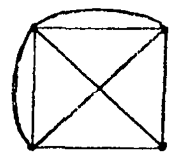
Figure 2.7.

The geometric dual G^* of a plane graph G is the planar graph constructed as follows: For each face $F_i \in$

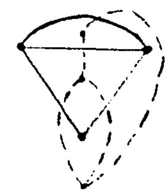
F(G) select a point p_i in F_i and set $V(G^*) = \{p_i: i = 1,2,...,F\}$. Then for every edge e in E(G) let e^* be an edge of G^* with ends corresponding to the face(s) for which e is a boundary edge (see Theorem 2.9). Figure 2.8 illustrates this concept.



G and G^*



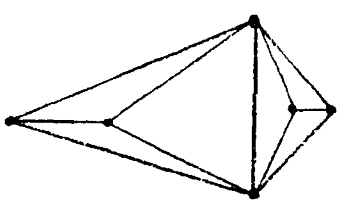
G^*



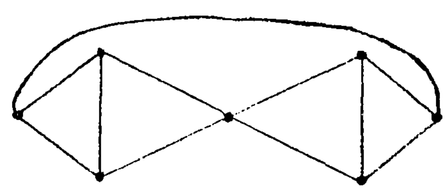
H and H^*

Figure 2.8.

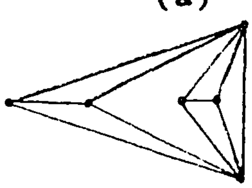
Clearly G^* has a loop iff G has a bridge and G^* has multiple edges iff there are two faces of G having at least two edges in common. It is to be emphasized that the geometric dual is defined in terms of a <u>plane</u> graph and that different embeddings of a planar graph may give rise to different geometric duals. Figure 2.9 shows two embeddings of a planar graph and plane representations of their non-isomorphic geometric duals.



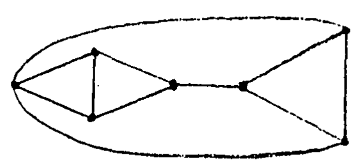
(a)



(b)



(c)



(d)

Figure 2.9.

For a plane graph G the valency or degree of a face F_i in $F(G)$ is defined to be the number, $d(F_i)$, of boundary edges for F_i. The girth of G is the least integer n such that $F(G)$ contains a face of valency n. Using subscripts to specify the graph being referred to, the following relations hold between a graph and its dual.

Theorem 2.17. If G is a connected plane graph, then

(i) $G^{**} = (G^*)^* = G$,

(ii) $E_{G^*} = E_G$ and $V_{G^*} = F_G$

(iii) $V_G = F_{G^*}$ and the vertices v_i, $i = 1,2,\ldots,V_G$, of G lie in distinct faces F_i^* of G^*,

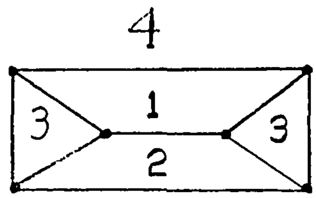
(iv) $d_G(F_i) = d_{G^*}(p_i)$ for $i = 1,2,\ldots,F_G$,

(v) $d_G(v_i) = d_{G^*}(F_i^*)$ for $i = 1,2,\ldots,V_G$,

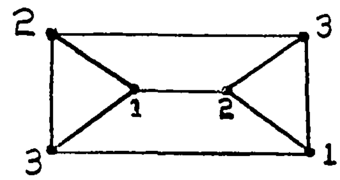
(vi) Unless G is the self-dual graph shown in Figure 2.8(c), G and G^* are essentially unique plane embeddings iff G is nonseparable and has no proper two-vertex separation.

COLORATION

A face-coloring of a plane graph is an assignment of colors to its faces so that no two faces with a common boundary edge are assigned the same color. A vertex-coloring of a graph without loops is an assignment of colors to its vertices so that no two adjacent vertices have the same color. (For an illustration of these concepts see Figure 2.10.) Since Whitney introduced the concept of a dual graph in [49] many workers have obtained results on the vertex colorability of various classes of graphs.



face-colored



vertex-colored

Figure 2.10.

Clearly it suffices to consider connected plane graphs when discussing face or vertex-colorability. To facilitate exposition we make the following definition.

Definition 2.18. A plane graph without loops or multiple edges is said to be a map iff it is 2-edge-connected and the valency of every vertex is greater than or equal to three.

A necessary condition for the face-colorability of a connected plane graph G is that G be 2-edge-connected. However, if G is a map and can be face-colored using $k \geq 3$ colors, then any map obtained from G by one or more of the following can also be colored using k colors:

(i) addition of a loop or a multiple edge,

(ii) vertex insertion,

(iii) addition of an isolated vertex v and an edge joining v to some vertex on the boundary of F(v).

We adopt the convention that coloring a map always refers to coloring its faces while coloring a graph indicates that it is the vertices which are colored. In view of Theorem 2.8 it is convenient to refer to the faces of a map as polygons.

An n-coloring of a graph or map uses n colors and therefore partitions the colored objects into n disjoint color classes; and two objects belong to the same class iff they have been assigned the same color. The minimum n for which a graph or map G has an n-coloring is called its chromatic number, K(G). A graph or map is n-colorable iff $K(G) \leq n$ and is n-chromatic iff $K(G) = n$.

Theorem 2.19 (Heawood [25]). Every map is 5-colorable.

Conjecture 2.20. Every map is 4-colorable.

Theorem 2.21. Every map is 4-colorable iff every planar graph is 4-colorable.

Remark 2.22. In what follows the word colorable (coloring) should be taken to mean 4-colorable (4-coloring) unless explicitly stated to the contrary.

Definition 2.23. A map is said to be reducible iff its coloring can be made to depend on the coloring of a map with fewer faces. Any face, edge, vertex or collection of such whose occurrence in a map renders it reducible is called a reducible configuration.

If every map contained a reducible configuration, then one could prove the four-color conjecture inductively. Although this and other direct attacks have generated much interesting mathematics, indirect approaches appear to be more fruitful.

Definition 2.24. A map is said to be minimal iff it is 5-chromatic and every map with fewer faces is colorable.

Numerous investigations of minimal maps have been made in the hope of finding mutually contradictory properties, and thus proving such cannot exist, or of finding information which would aid in their construction. Our finiteness restriction is no real limitation, for DeBruijn and Erdős [13] have proved

Theorem 2.25. Every infinite n-chromatic map contains a finite n-chromatic submap.

Recalling that the definition of map requires every vertex to have valency greater than two, one can easily prove

Theorem 2.26. A minimal map is cubic. In face, the n-coloring of any map can be reduced to the case of cubic maps.

On the strength of Theorem 2.26, the remainder of this work will deal almost exclusively with cubic graphs.

The following connectivity conditions were not found in the literature; hence I have formulated their proofs.

Lemms 2.27. A connected nontrivial cubic graph G without cutpoints has VC(G) = 2 iff it is properly two-vertex separable.

Proof: If G can be properly two-vertex separated by vertices v and w, then their removal will disconnect G, so VC(G) = 2.

If removal of vertices v and w disconnects G, then, since G is cubic and without cutpoints, their removal yields a graph with two components H_1' and H_2'. But at v, as well as at w, two of the three edges are incident with one component while the remaining one is incident with the other component; for otherwise either v or w would be a cutpoint or removal of v and w would not disconnect G. Augment H_1' to form the subgraph H_1 by adding the vertex v, the edge(s) incident to v and H_1', vertex w and the edge(s) incident to w and H_1'. Similarly form H_2 to obtain edge disjoint subgraphs for which $E(G) = E(H_1) \cup E(H_2)$, $V(G) = V(H_1) \cup V(H_2)$, and $V(H_1) \cap V(H_2) = \{v,w\}$. Neither H_1 nor H_2 can be paths without contradicting the cubic nature of G or the fact that removal of v and w disconnects G. Hence G is properly two-vertex separable. Q.E.D.

Theorem 2.28. For a cubic graph G, VC(G) = EC(G).

Proof: If G is not connected, then VC(G) = EC(G) = 0. Suppose G is connected. If EC(G) = 1, then $0 < VC(G) \leq EC(G)$ implies VC(G) = 1. If VC(G) = 1, then removal of a cutpoint v will disconnect G into two or three components. In either case there is one component incident with v via only one edge. Thus, by Theorem 2.2, that edge is a bridge so EC(G) = 1.

Suppose VC(G) = 2 and removal of vertices $v,w \in V(G)$ disconnects G into two components H_1 and H_2. Two of the three edges at v are incident with one component while the remaining one, e_v, is incident with the other. Similarly, at w there is an edge e_w through which every v-avoiding path joining vertices in H_1 to vertices in H_2 must pass. Thus removal of e_v and e_w will disconnect G and since $VC(G) \leq EC(G)$, EC(G) = 2. Suppose EC(G) = 2 and removal of edges $e,f \in E(G)$ disconnects G. Clearly e and f are not adjacent, else the other edge at their common vertex would be a bridge. Since G is cubic, some end v of e and some end w of f are not joined by a double edge. Thus removal of v and w will disconnect G and since we have $1 \neq VC(G) \leq EC(G) = 2$, then VC(G) = 2.

If $VC(G) = 3$, then $VC(G) \leq EC(G) \leq 3$ implies $EC(G) = 3$. On the other hand, if $EC(G) = 3$, then the two preceeding paragraphs imply $VC(G) = 3$. Q.E.D.

Corollary 2.28(a). A connected planar cubic graph has an essentially unique plane embedding iff it is 3-edge-connected. This is a direct consequence of Lemma 2.27, Theorem 2.10 and Theorem 2.28.

An edge-coloring of a graph G without loops is an assignment of colors to its edges so that no two adjacent edges have the same color. An edge n-coloring uses n colors and partitions E(G) into n color classes. The proof of the following theorem, essentially due to Tait [41] and discussed in [2] and [33], is included for later reference and because of its significance to this work.

Theorem 2.29. A 2-connected cubic plane graph G can be face 4-colored iff it can be edge 3-colored.

Proof: If G has been face 4-colored with colors $\{a,b,c,d\}$, assign to each edge of G the color 1, 2, or 3, according to the colors of the two regions it bounds, as prescribed by the table:

*	a	b	c	d
a	-	1	2	3
b	1	-	3	2
c	2	3	-	1
d	3	2	1	-

Examination of the table shows that if two adjacent edges have the same color, then (since G is cubic) two adjacent faces have the same color, a contradiction.

Assume now that G has been edge 3-colored. Since G is cubic, the plane subgraph H_{12}, determined by the edges colored 1 and 2, is either a circuit or a union of disjoint circuits. Hence its faces can be colored with two colors A and B. Similarly, the faces of the plane subgraph H_{13}, determined by the edges colored 1 and 3, can be colored with colors C and D. When H_{12} and H_{13} are superimposed, every face of G has one of the four color pairs AC,BD,BC,AD associated with it. It is easy to verify that association of the colors a,b,c,d with the color pairs AC,BD,BC,AD, respectively, results in a face 4-coloring of G consistent with table *. Q.E.D.

In Tait's pseudo-solution of the four-color problem he assumed that all cubic graphs were edge 3-colorable, but Petersen [38] produced the nonplanar counterexample of Figure 2.7(a). Analogous to Heawood's Five-Color Theorem, we have the result recently proved by Johnson [27], every cubic graph can be edge 4-colored.

Of course it is unknown whether all cubic maps are edge 3-colorable. However, if such a map G possesses a Hamiltonian circuit C, then C is of even length and G can be edge 3-colored by alternately assigning the colors 1 and 2 to the edges of C and assigning color 3 to the remaining edges of G. Tutte [44] produced the first known example of a cubic map without a Hamiltonian circuit, but this map is readily seen to be colorable.

Consider an edge 3-colored cubic map G. Then the union of any two of its color classes is the edge set of a finite family of disjoint circuits $\{C_i\}$ where each C_i is of even length and each vertex lies on exactly one of the C_i. A different edge 3-coloration of G can be obtained by interchanging the colors assigned to the edges of one or more of the C_i. However, this just amounts to a permutation of colors in the case when the family is singleton and C_1 is a Hamiltonian circuit. In what follows the circuits of the family $\{C_i\}$ formed by removal of the edges in color class 3 will be referred to as <u>(1,2)-circuits</u>. Similarly for <u>(1,3)-circuits</u> and <u>(2,3)-circuits</u>.

<u>Definition 2.30</u>. Calling two faces <u>adjacent</u> iff they have a common boundary edge, a sequence of distinct faces $F_1,\ldots,F_n$ is said to be an <u>n-ring</u> in a cubic plane graph G iff

(i) face F_i is adjacent to F_j iff $j = i \pm 1$ modulo n,

(ii) if, whenever e_i, $i=1,2,\ldots,n$, is an edge common to the boundary of F_i and F_{i+1} (subscripts taken mod n), then removal of $\{e_i: i=1,2,\ldots,n\}$ from G disconnects it into two components each of which contains a circuit.

The sequence F_1,F_2,F_3,F_4 in Figure 2.11 represents a 4-ring in (a) but not in (b).

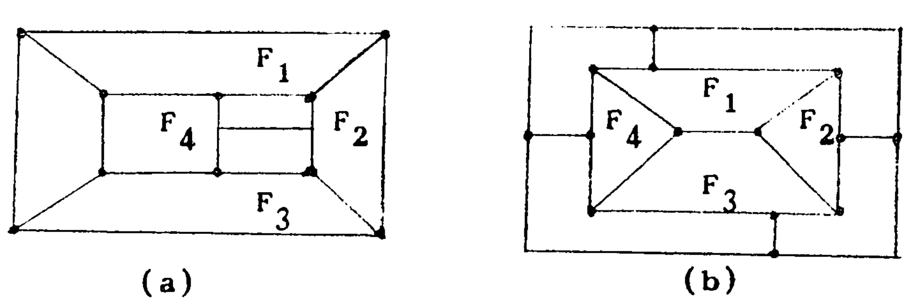


Figure 2.11.

Theorem 2.31. A minimal map contains no n-rings for $n=1,2,3,4$.

Corollary 2.31(a). No two faces of a minimal map have more than one boundary edge in common.

Corollary 2.31(b). A minimal map is 3-edge-connected and thus it has an essentially unique plane embedding.

Corollary 2.31(c). The set of faces adjacent to a given face of a minimal map form a ring called a neighbor ring.

Corollary 2.31(d). A minimal map has girth ≥ 5.

Theorem 2.32 (Heawood [25]). Every cubic map with girth greater than four contains at least twelve pentagons, hence has girth five.

Corollary 2.32(a). A minimal map is cyclically 5-connected.

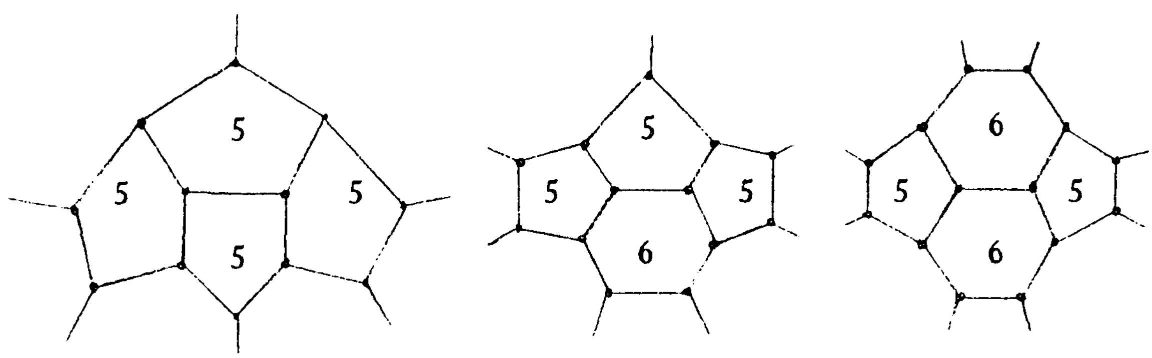
Corollary 2.32(b). The geometric dual of a minimal map is a connected triangulated map containing at least twelve vertices of valency five and no circuits of length one, two, or three (other than face boundaries).

Corollary 2.32(c) (Bernhart [5]). Excluding twelve pentagons from a cubic map of girth 5, the average valency of its F-12 other faces is exactly six.

Calling pentagons and hexagons minor faces, this last observation puts no limit on the number of hexagons, but the occurrence of major (i.e., non-minor) faces in a minimal map implies the existence of additional pentagons. On the other hand, too high a concentration of pentagons is excluded by the following beautiful result achieved by a team of workers led by C. E. Winn [51,52].

Theorem 2.33. In a minimal map, the neighbor ring of no minor polygon consists entirely of minor polygons.

Winn's theorem involves configurations of six or seven minor polygons. However, the simpler configurations of minor polygons shown in Figure 2.12(a), (b), and (c) were proved to be reducible by Birkhoff [6], Franklin [17], and Bernhart [3], respectively. Errera [16] examined some higher cases and proved that in a minimal map, an n-gon, n even and greater than 6, can have at most (n-3) consecutive pentagon neighbors while for n odd and greater than 5 there can be at most (n-2) consecutive pentagon neighbors. In addition to Theorem 2.33, Winn proved that a minimal map must contain at least six major faces.



(a) 5-555 (b) 6-555 (c) 6-565

Figure 2.12.

Each time another reducible configuration is found, new restrictions are imposed in the number of polygons of various types which can occur in a minimal map. As the number of restrictions increases, progressively more faces are required; the last count being at least forty as reportedly [21] shown by Ore and Stemple [37].

Such bounds on the number of faces give an idea of the complexity of five-chromatic maps but say little about their structure. Birkhoff [6] considered rings as a natural generalization of the neighbors of a single face and provided structural insight by proving

Theorem 2.34. The neighbor rings surrounding pentagons are the only 5-rings in a minimal map.

The solution of an n-ring R for a given value of n means the determination of which interior structures of R are reducible. Birkhoff gave the complete solution for a 5-ring in Theorem 2.34. In [5] Bernhart completed the solution of the 6-ring and developed a method which may be extended to any n-ring. Also his paper introduces the concept of an irreducible configuration, viz., one which survives the reducibility tests developed therein. While it takes only one contradiction to show reducibility; irreducibility guarantees that thousands of conditions are simultaneously satisfied. Irreducible configurations may thus form the building units essential for the construction of a five-chromatic map. In comparison with the illustrations of Figure 2.12, Goldbeck [18] has shown the configuration 5-566 to be irreducible. The analysis of rings provides a systematic procedure for studying clusters of faces and are thus being programmed for electronic computation by M. Rill and A. Bernhart.

After giving due credit to the originators Dirac [14] translates the results of Winn, Birkhoff, and Bernhart to the geometric dual of a minimal map and shows that they can be independently derived in that context.

In conclusion of this chapter we mention that the construction of examples satisfying various properties of minimal maps has not kept up with the discovery of reducible configurations. Recently H. Walther [46] found the following example of a cyclically 5-connected non-Hamiltonian cubic map. Although it satisfies Corollary 2.32(c), Birkhoff's Theorem 2.34 is clearly violated.

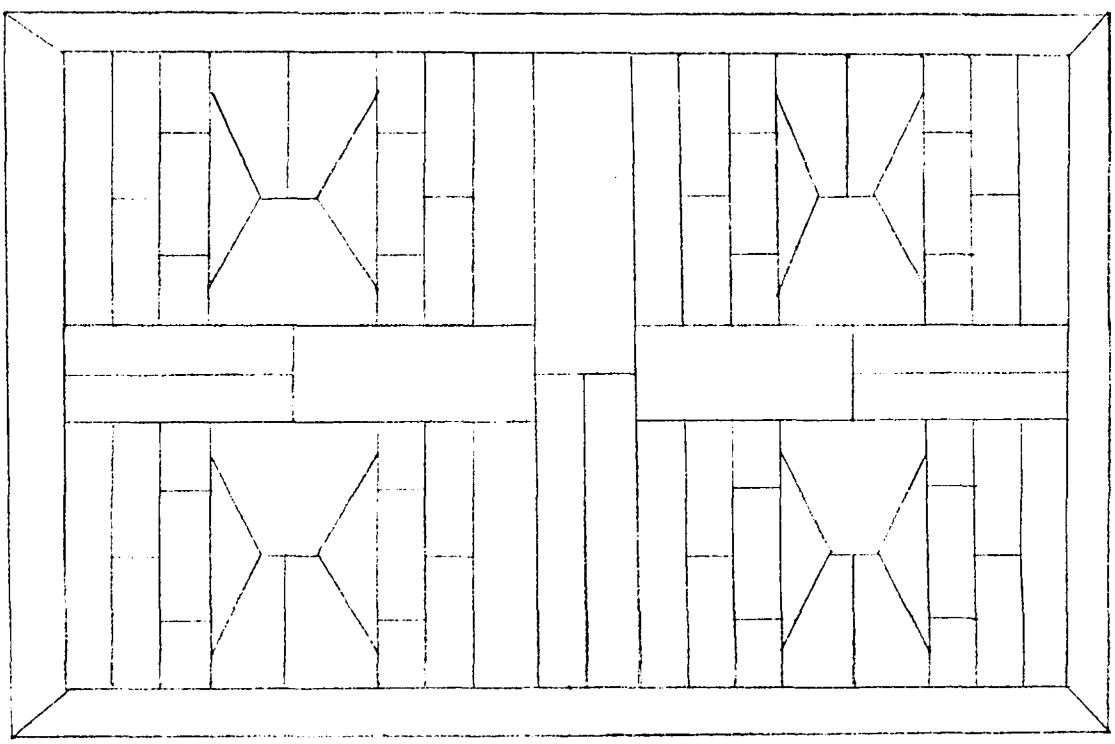


Figure 2.13.

CHAPTER III

EDGE CONJUGATION

Most studies of chromatic graphs or maps depend on certain contractions or other methods of reducing the number of objects to be colored. This will still be somewhat true in the following but here the interest is in retaining the number of objects to be colored and studying the effect of certain of their rearrangements on colorability. In view of Theorem 2.26 and Theorem 2.29, it is natural to consider rearrangements of edges in cubic maps. Since the index of a cubic plane graph is that natural number k such that $V = 2k$, $E = 3k$, and $F = k + 2$, all minimal maps have the same index.

<u>Definition 3.1</u>. Let e be a regular (i.e. nonmultiple) edge in a cubic plane graph G. If its ends are v and w, suppose the four edges incident to e are labelled e_1, e_2, e_3, e_4 in a manner such that e_1 and e_2 are incident to e at v and e_3 and e_3 and e_4 are incident to e at w, and the e_i, $i = 1,2,3,4$, occur in clockwise order; Figure 3.1(a). By <u>edge-con-</u>

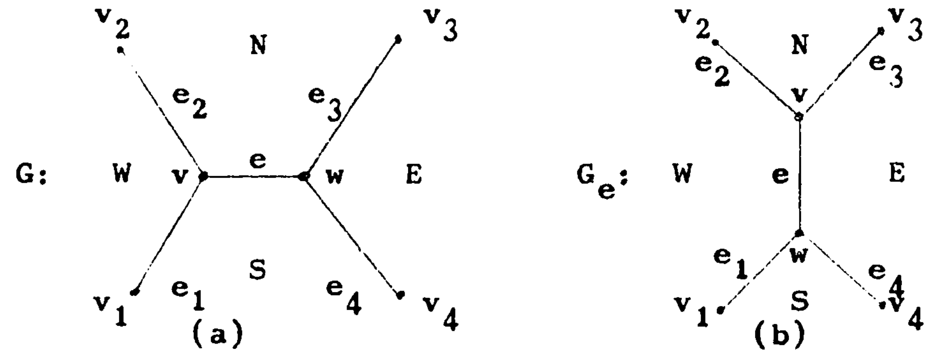


Figure 3.1.

jugation of e is meant the rearrangement such that e_2 and e_3 are incident with e at v, e_1 and e_4 are incident with e at w, and the incidence relations at all other vertices of G are unchanged; Figure 3.1(b). A graph obtained from G in this way will be denoted by G_e; thus G_e is a cubic graph with the same index as G.

Observe that the v_i, $i = 1,2,3,4$, may not all be distinct, but they cannot all be identical. If $v_2 = v_3$, for example, then face N is a triangle and conjugation reduces it to a lune, i.e., a face of valency two bounded by the double edges e_2 and e_3. When all the v_i, $i = 1,2,3,4$, are distinct, then the configuration of Figure 3.1(a) will be called the H-subgraph determined by edge e. Conjugation of edge e removes it from the boundary of faces N and S and inserts it as a boundary edge for faces E and W. Of course, if N and E are identical, then conjugation leaves the valency of N unaltered; but if E and W are identical or if N and S are identical, then conjugation alters their valency by two.

It is to be emphasized that conjugation is defined in terms of a plane graph G and that conjugation of the same edge in different plane embeddings of G may give rise to different graphs. Figure 3.2 illustrates two embeddings of a planar cubic graph and nonisomorphic graphs resulting from conjugation of edge e.

Clearly the graph G_e constructed in Definition 3.1 is planar. In what follows we shall be interested in performing sequences of edge conjugations; thus care must be taken when embedding G_e in the plane, else there will be an undesired ambiguity. Hence we adopt the convention that G_e is a plane graph such that the boundary edges of each face, with the exception of e, are the same as they were in G. Having done this we insure that edge conjugation is a self-inverse operation on cubic plane graphs.

G: (a) G_e: (b)

H: (c) H_e: (d)

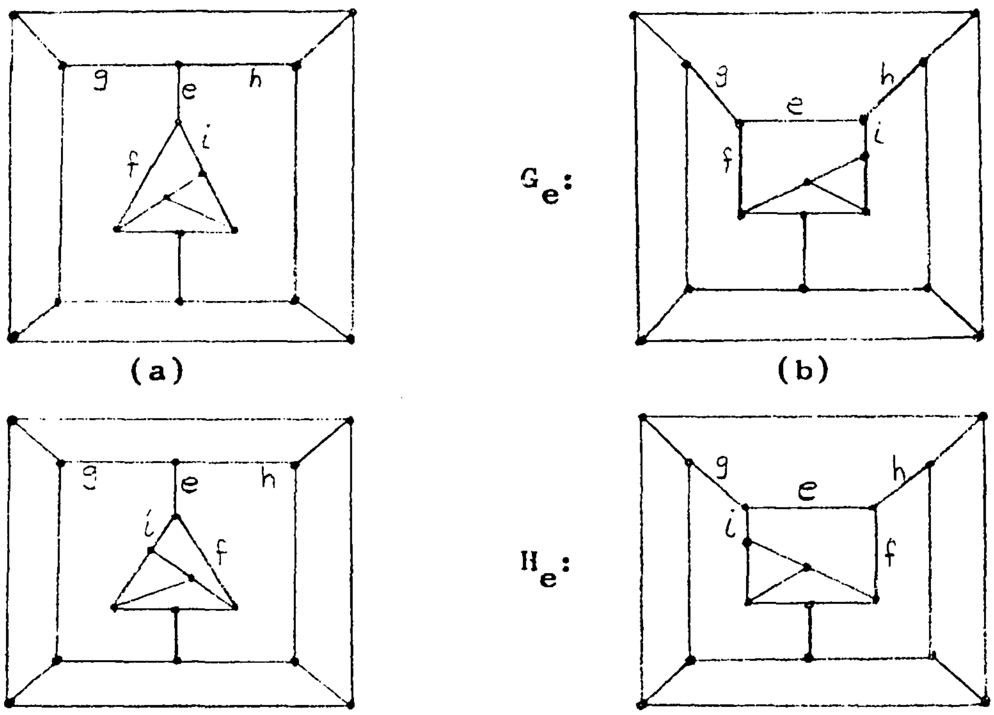


Figure 3.2.

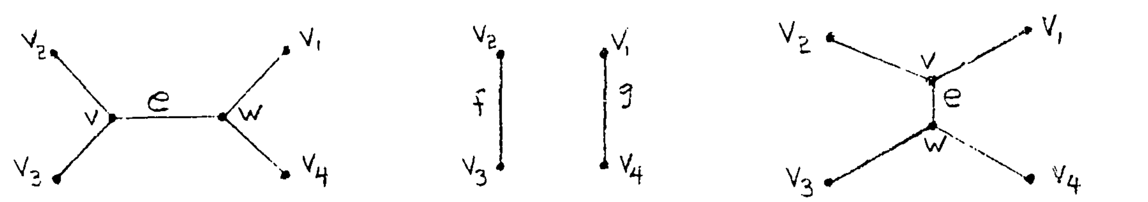
The following theorem is an immediate consequence of Corollary 2.28(a) and the definition of edge conjugation.

Theorem 3.2. For a planar cubic graph, edge conjugation is independent of the plane embedding iff the graph is 3-edge-connected.

Motivation for the study of edge conjugation is provided by Bernhart who first introduced the concept in [5]. The primary thrust of that paper was the analysis of 6-rings but it contained Theorem 3.3 as an incidental remark. Although Bernhart's complete solution of the 6-ring has been acknowledged by other workers, see [14] and [35], an extensive examination of the literature suggests that his remark on edge conjugation was overlooked. The following proof appeals to edge coloring and simplifies the original one which used the techniques of ring analysis.

Theorem 3.3. Conjugation of an arbitrary edge of a minimal map renders it colorable.

Proof: Let M be a minimal map of index k. Suppose e is an edge of M with ends v and w and let v_i, $i = 1,2,3,4$, be defined as in Figure 3.2(a). From M, form the cubic plane graph M' by removal of edge e followed by suppression of



(a) edge e in M (b) M' (c) edge e in M_e

Figure 3.2.

vertices v and w. Since M possesses no triangles or quadrilaterals, $v_i \neq v_j$ and v_i is not adjacent to v_j for every $i,j \in \{1,2,3,4\}$ such that $i \neq j$. Therefore M' and M_e are cubic plane graphs without multiple edges or triangles. In M' denote the edges with ends $\{v_2,v_3\}$ and $\{v_1,v_4\}$ by f and g, respectively (Figure 3.2(b)).

To verify that M' is a map, suppose first that it is 1-edge-connected. Then removal of one of its bridges and e in M would disconnect M, contradicting the fact that EC(M) = 3 (Corollary 2.31(b)). Similarly, since M contains no 3-rings, it follows that M_e is not 1-edge-connected and therefore is a map.

Map M' is of index k-1, so we can assume it to be edge 3-colored with colors 1, 2 and 3.

Case 1: Edges f and g are colored differently. In M_e assign the color of f to (v,v_2) and to (w,v_3); assign the color of g to (v,v_1) and (w,v_4) and assign the third color to e. Then, if the remaining edges of M_e inherit their coloration from M', M_e will be properly edge 3-colored.

Case 2: Edges f and g are colored the same color, say 1. If the (1,2)-circuit C containing f does not contain g, then interchange of 1 and 2 on the edges of C established an edge 3-coloration of M' with f and g colored differently; this situation is taken care of by Case 1. Suppose that the (1,2)-circuit C containing f also contains g. It follows from the Jordan curve theorem that v_1 and v_2 occur in order on C, illustrated by the dashed curve of Figure 3.3(a). Then assign-

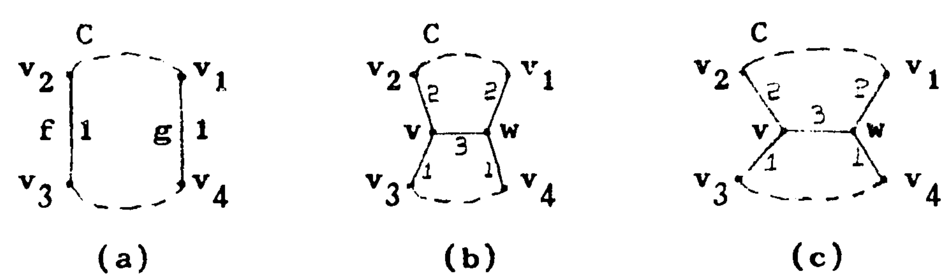


Figure 3.3.

ment of color 2 to edges (v,v_2) and (w,v_1) of M, color 1 to edges (v,v_3) and (w,v_4), color 3 to (v,w), and interchange of colors 1 and 2 on the arc of C from v_1 to v_2 not containing v_3 (Figure 3.3(b)) implies an edge 3-coloration of M, a contradiction. Q.E.D.

<u>Corollary 3.3(a)</u>. For any edge e of a minimal map M, using the labels of Figure 3.2, every edge 3-coloring of M_e must have (v,v_2) and (w,v_3) colored the same.

<u>Proof</u>: If some edge 3-coloring of M_e had (v,v_2) and (w,v_3) colored differently, say with colors 1 and 2, respectively, then (v,w) must be colored 3 and (v,v_1) and (w,v_4) would be colored 1. But, since edge conjugation is a self-inverse operation, this situation implies the existence of an edge 3-coloration of M, a contradiction. Q.E.D.

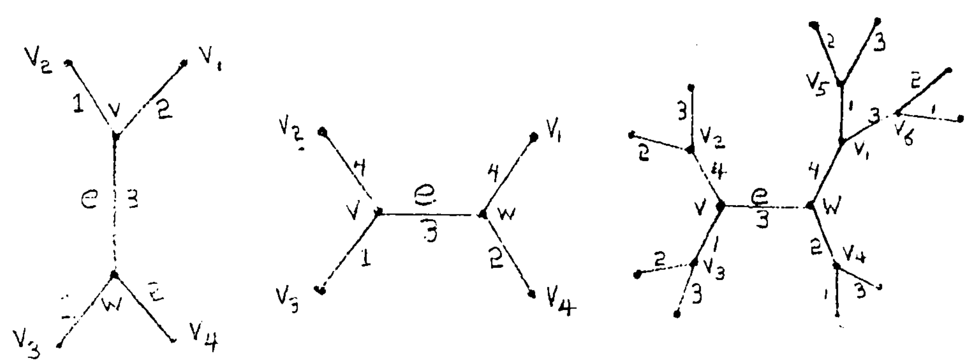
<u>Corollary 3.3(b)</u>. Let e be an edge of a minimal map M. If, using the labels of Figure 3.2, edge e in M_e is colored 3, then the (1,2)-circuit C containing v must also contain w; Figure 3.3(c).

<u>Proof</u>: If the (1,2)-circuit C which contains v does <u>not</u> contain w, then interchanging 1 and 2 on C produces a coloration of M_e for which (v,v_2) and (w,v_3) are colored differently, contradicting Corollary 3.3(a). Q.E.D.

<u>Corollary 3.3(c)</u>. There is a face 5-coloration of a minimal map M for which only one face need be assigned color 5 and that face may be chosen arbitrarily. Analogously, there exists an edge 4-coloration of M for which only two edges need be assigned color 4 and these edges may be cho-

sen as any nonadjacent pair.

Proof: The first statement follows immediately from the fact that $M - \{e\}$ is colorable for any edge e in $E(M)$. By Corollary 3.3(a) edge e in M_e and the edges adjacent to it may be assumed colored as shown in Figure 3.4(a). Thus M can be edge 4-colored as shown in Figure 3.4(b), with only edges (v,v_2) and (w,v_1) assigned color 4. As a sample



(a) (b) (c)

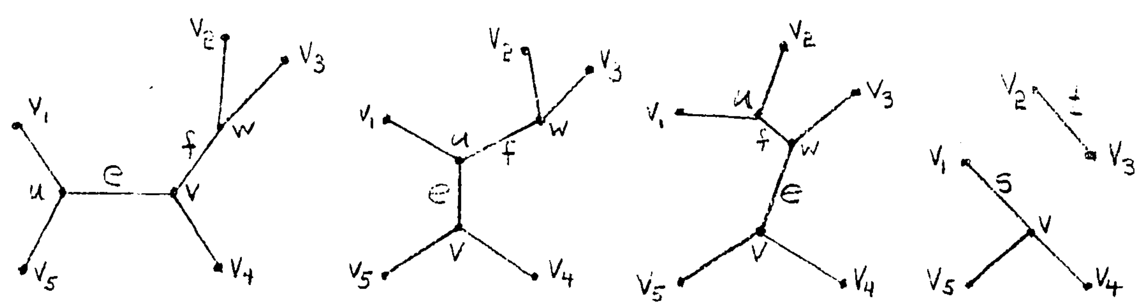
Figure 3.4.

case suppose the other edges at the v_i, $i = 1,2,3,4$, are colored as shown in Figure 3.4(c). Notice that there are edges of all other colors incident with those colored 4. The color 4 on edge (w,v_1) could be moved to edge (v_1,v_4) by interchanging it with the 2 on (w,v_4) or it could be moved to (v_1,v_5) by interchanging it with the 1 on (v_1,v_5). After interchanging the 1's and 3's on the (1,3)-path through v_1, the color 4 could be moved to (v_1,v_6) in the same manner. A similar argument shows that whenever an edge f is colored 4, this color can be moved to any of the four edges adjacent to f, providing the receiving edge is not already adjacent to an edge colored 4. Hence M can be edge 4-colored so that any pair of nonadjacent edges are assigned the color 4. Q.E.D.

Remark 3.4. Let G be a cubic plane graph without multiple edges and triangles. If, for edges $e,f,g \in E(G)$, we denote by G_{ef} the graph obtained from G by conjugation of e followed by conjugation of f, then it is easy to verify that G_{ef} is isomorphic to G_{fe} if e is not adjacent to f. Similarly, denote $(G_{ef})_g$ by G_{efg} and so on.

Theorem 3.5. If two adjacent edges e and f of a minimal map M are conjugated to form the map M_{ef}, then M_{ef} is colorable.

Proof: Suppose M is of index k and let e, f, and their neighbors be labelled as in Figure 3.5(a). From M, form the cubic plane graph M' of index k-1 by removing the vertices u, v and w and substituting the configuration shown in Figure 3.5(d). Let s and t denote the edges (v,v_1) and (v_2,v_3) in M'; Figure 3.5(d).



(a) $e,f \in E(M)$ (b) $e,f \in E(M_e)$ (c) $e,f \in E(M_{ef})$ (d) M'

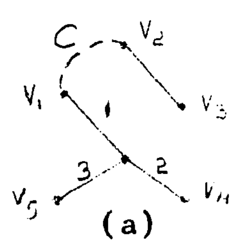
Figure 3.5.

Since M contains no triangles or quadrilaterals, the v_i, $i = 1,2,3,4,5$, are distinct, but v_1 and v_2 may be adjacent in M. From the fact that M contains no n-rings for $n = 1,2,3,4$, it follows that neither M_{ef} nor M' are 1-edge-connected. Therefore they are cubic maps.

Assume M' is edge 3-colored with colors 1,2 and 3.

Case 1: Edges s and t are colored differently. In M_{ef}, assign the color of s to (u,v_1) and (v,w), assign the color of t to (u,v_2) and (w,v_3), and assign the third color to (u,w). Then, if the remaining edges of M_{ef} inherit their coloration from M', M_{ef} will be properly edge 3-colored.

Case 2: Edges s and t are colored the same, say with color 1. If the (1,2)-circuit C containing t does not contain s, then interchange of 1 and 2 on the edges of C establishes an edge 3-coloration of M' with s and t colored differently, and this situation is cared for by Case 1. Suppose the (1,2)-circuit C containing s also contains t. It follows from the Jordan curve theorem that vertices v_1 and v_2 occur in order on C, illustrated by the dashed curve in Figure 3.6(a).



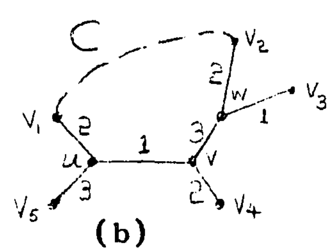
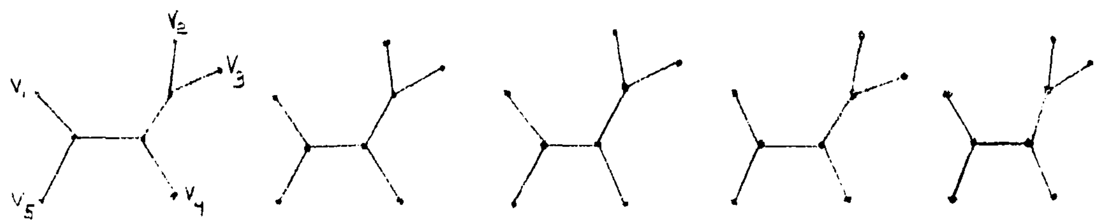


Figure 3.6.

Without loss of generality assume that colors 2 and 3 have been assigned to (v,v_4) and (v,v_5), respectively; for we could have initiated the argument by assigning the colors at vertex v in this manner. Then interchange of colors on the arc of C from v_1 to v_2 not containing v_3, followed by the color assignments pictured in Figure 3.6(b), implies an edge 3-coloration of M, a contradiction. Q.E.D.

As just seen, two adjacent edges in a cubic map G of girth 5 determine five distinct vertices $v_1, \ldots, v_5$. The five illustrations of Figure 3.7 exhibit the possible configurations incident to $v_1, \ldots, v_5$ which preserve the cubic nature, index and planarity of G.



(a) (b) (c) (d) (e)

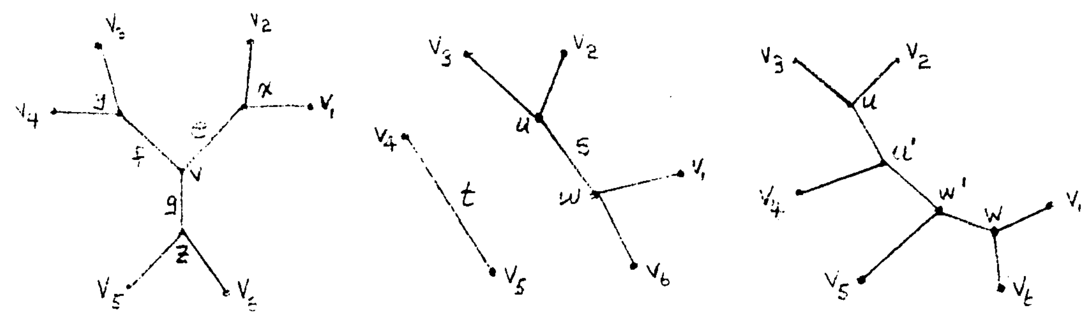
Figure 3.7.

<u>Corollary 3.5(a)</u>. When one of the above configurations occurs in a minimal map M, then rearrangement of its edges to produce any of the other four configurations yields a colorable map.

<u>Proof</u>: There are two edges of the given configuration which are not incident with $v_1, \ldots, v_5$. Call them e and f. Then M_e, M_f, M_{ef} and M_{fe} represent the maps containing each of the other four configurations. Hence the desired conclusion follows from Theorems 3.3 and 3.5. Q.E.D.

<u>Theorem 3.6</u>. Let v be a vertex of a minimal map M and label the edges incident with v as e, f and g in counter-clockwise order. Then the map $M_{efg}(\cong M_{egf})$ is colorable.

Proof: Suppose M is of index k and let v, e, f, g and their neighbors be labelled as in Figure 3.8(a). From M, form the



(a) $e,f,g \in E(M)$ (b) M' (c) M_{efg}

Figure 3.8.

cubic plane graph M' of index $k-1$ by removing the vertices v, x, y and z and substituting the configuration shown in Figure 3.8(b). Since M contains no triangles or quadrilaterals the v_i, $i = 1,\ldots,6$, are distinct, but there may be a double edge with ends $\{v_4,v_5\}$ in M'. From the fact that M contains no n-rings, $n = 1,2,3,4$, it follows that neither M' nor M_{efg} is 1-edge-connected.

Assume M' is edge 3-colored with colors 1, 2 and 3.

Case 1: Edges s and t in M' are colored differently. In M_{efg}, assign the color of s to edges (u,u') and (w,w'), assign the color of t to (u',v_4) and (w',v_5) and assign the third color to (u',w'). Then, if the remaining edges of M_{efg} inherit their coloration from M', M_{efg} will be properly edge 3-colored.

Case 2: Edges s and t are colored the same color, say 1. If there is a (1,2)-circuit or a (1,3)-circuit containing t but not s, then interchanging colors on such a circuit reduces the problem to that of Case 1. Suppose then that the (1,2)-circuit and the (1,3)-circuit containing t also contain s. Without loss of generality assume (w,v_6) and (w,v_1) are colored 2 and 3, respectively.

If edges (u,v_2) and (u,v_3) are colored 2 and 3 (Figure 3.9(a)), then the color assignments pictured in Figure 3.9(b) imply an edge 3-coloration of M, a contradiction. If edges (u,v_2) and (u,v_3) are colored 3 and 2, respectively,

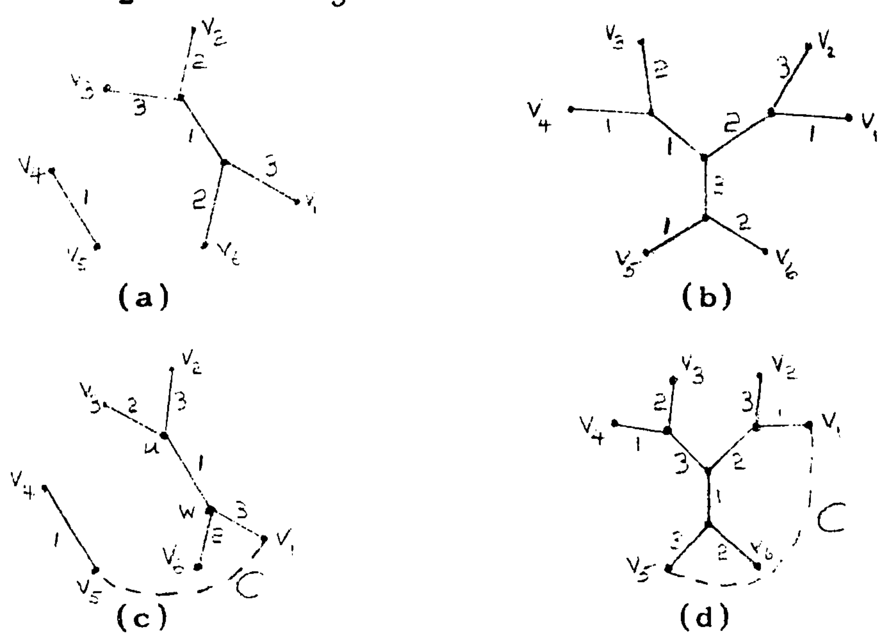
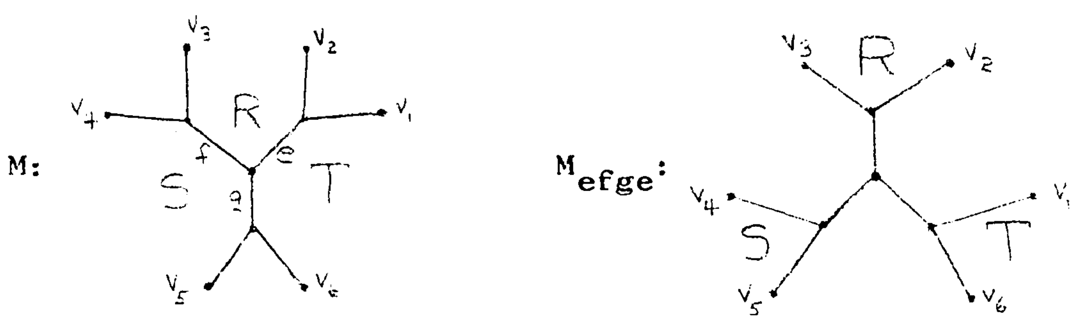


Figure 3.9.

it follows from the Jordan Curve Theorem that vertices v_1 and v_5 occur in order on the (1,3)-circuit C containing w; see Figure 3.9(c). Then interchange of colors on the arc of C from v_1 to v_5 not containing u, followed by the color assignments pictured in Figure 3.9(d), implies an edge 3-coloration of M, a contradiction. Q.E.D.

As just seen the three edges incident to a vertex v in a cubic map G without triangles or quadrilaterals determine six distinct vertices $v_1,\ldots,v_6$. In a manner analogous to Figure 3.7, there are fourteen possible configurations incident to $v_1,\dots,v_6$ which preserve the cubic nature, index and planarity of G. If the configuration of Figure 3.10(a) occurs in M, then an exhaustive investigation shows that twelve of the remaining thirteen configurations are of the form M_x, M_{xy}, or M_{xyz} where x, y and z are distinct elements of $\{e,f,g\}$. The exceptional case is that shown in Figure 3.10(b). It is isomorphic to M_{efge} but it is unknown whether this configuration is colorable for all choices of v. Since



(a) (b)

Figure 3.10.

M_{efge} is of the same index as M, if it is noncolorable, this implies that it is a minimal map and thus contains no triangles or quadrilaterals. Because this can happen only if all of R, S and T are major faces, there are vertices v of M such that M_{efge} is colorable.

The colorability of maps obtained from M by conjugation of two or more <u>nonadjacent</u> edges is not known. To facilitate discussion of this problem we introduce the following result essentially due to Heawood [26].

<u>Theorem 3.7</u>. Let G be a 2-edge-connected cubic plane graph.

If G can be edge 3-colored, then a coefficient $\emptyset^*(v)$, equal to +1 or - 1, can be assigned to each vertex $v \in V(G)$ such that for any face $R \in F(G)$

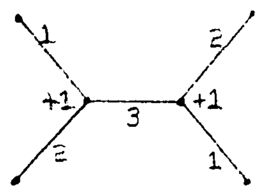
$$\sum_{v \in B(R)} \emptyset^*(v) = 0 \pmod 3,$$

where B(R) is the boundary of R.

Proof: Let the mapping $\emptyset: E(G) \longrightarrow \{1,2,3\}$ be an edge 3-coloring of G. Define $\emptyset^*: V(G) \longrightarrow \{+1,-1\}$ by $\emptyset^*(v) = +1$ if the three edges incident with v have their colors occurring in counterclockwise order; otherwise put $\emptyset^*(v) = -1$. Let $R \in F(G)$ and, starting with an arbitrary edge $e \in B(R)$, traverse B(R) in the counterclockwise direction. By definition of $\emptyset^*$, each time we encounter a +1, the color of the edge being followed is changed according to the negative cyclic permutation (3,2,1); and each time we encounter a -1 it is changed according to the positive cyclic permutation (1,2,3). Therefore, in order for the color ultimately determined for e to agree with its actual starting color, the congruence condition must be satisfied. Q.E.D.

The converse of Theorem 3.7 is also true, but it will not be needed here. For a proof see [2].

Remark 3.8. Let $\emptyset$ be an edge 3-coloring of a cubic map G and $\emptyset^*$ its corresponding vertex valuation as defined in Theorem 3.7. Call an edge of G with ends u and v <u>neutral</u> iff $\emptyset^*(u) = \emptyset^*(v)$; otherwise call it <u>polar</u>. It is easy to verify that if e is a neutral edge of G, then G_e is a colorable map in which e has become a neutral edge with ends of opposite parity from those of e; Figure 3.11 illustrates this with an example. Conjugation of a polar edge may, however, yield a noncolorable map.



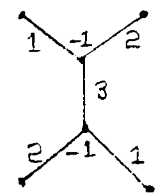


Figure 3.11.

As can be seen from Corollary 3.3(a), the edge e in M_e must be polar with respect to all its edge 3-colorings; for otherwise M would be colorable. Of course any number of the neutral edges of M_e can be conjugated to yield colorable maps.

Let G be a cubic map of index k which is face 4-colored with colors a,b,c and d and let $\emptyset : E(G) \longrightarrow \{1,2,3\}$ be the edge 3-coloring of G determined by table * of Theorem 2.29, page 24. Also, let $\emptyset^*$ be the corresponding valuation of the vertices V(G). For any two distinct colors $x,y \in \{a,b,c,d\}$, define E(x,y) to be the subset of E(G) whose elements are simultaneously incident with faces colored x and y, and define V(x,y) as the set of ends of the edges in E(x,y). Put $S(x,y) = \sum_{v \in V(x,y)} \emptyset^*(v)$. We shall be interested in deducing corollaries to the following theorem recently proved by Kotzig [30].

<u>Lemma 3.9</u>. S(a,b) = S(c,d), S(a,c) = S(b,d), S(a,d) = S(b,c).

<u>Proof</u>: The color class of edges assigned color 1, E(1), equals $E(a,b) \cup E(c,d)$. Define $H_{2,3}$ to be the subgraph

$G - E(1)$ of disjoint $(2,3)$-circuits and let K be one of its components. If K is traversed counterclockwise, partition its vertices into two classes V_{32} and V_{23} in a manner such that after passing a vertex in V_{32} (a vertex in V_{23}) one moves through an edge colored 2 (an edge colored 3). Evidently K has four types of vertices, viz., the type T_1 of those vertices v in V_{32} for which $\phi^*(v) = -1$, the type T_2 of those vertices in V_{32} with $\phi^*(v) = +1$, the type T_3 of those vertices in V_{23} with $\phi^*(v) = -1$ and the type T_4 of those vertices in V_{23} with $\phi^*(v) = +1$. Let t_i denote the cardinal number of the set of vertices of type T_i, $i = 1,2,3,4$, and let L (R) denote the set of vertices of K for which the edge colored 1 lies in the interior (exterior) of K. Since K is of even length, the cardinality of V_{23} equals that of V_{32} so that $t_1 + t_2 = t_3 + t_4$, and furthermore $\sum_{v \in L} \phi^*(v) = t_4 - t_1$, $\sum_{v \in R} \phi^*(v) = t_2 - t_3$.

Hence $\sum_{v \in L} \phi^*(v) = \sum_{v \in R} \phi^*(v)$. Since this result holds for each component of $H_{2,3}$, it holds for $H_{2,3}$ itself. Since each vertex of G belongs to both $E(1)$ and $H_{2,3}$, it follows that $S(a,b) = S(c,d)$. Similarly $S(a,c) = S(b,d)$ and $S(a,d) = S(b,c)$. Q.E.D.

Using the preceding notation, for every $x \in \{a,b,c,d\}$, let $V(x)$ denote the set of all vertices of G incident with a face of color x and put $S_x = \sum_{v \in V(x)} \phi^*(v)$.

<u>Theorem 3.10</u>. There exists an integer S, divisible by three, such that $S = S_a = S_b = S_c = S_d$.

Proof: Let $T = \sum_{v\in V(G)} \phi^*(v)$. Then $S_x + S_y = T + S(x,y)$ for any two colors $x,y \in \{a,b,c,d\}$, $x \neq y$; for the left hand side counts the value of a vertex not incident with an edge in $E(x,y)$ exactly once while it counts those in $E(x,y)$ twice. From Lemma 2.9 it, therefore, follows that $S_a+S_b = S_c+S_d$, $S_a+S_c = S_b+S_d$ and $S_a+S_d = S_b+S_c$. Hence $S_a = S_b = S_c = S_d$. The fact that their common value, S, is a multiple of three follows from Theorem 3.7. Q.E.D.

Corollary 3.10(a). The sum $T = \sum_{v\in V(G)} \phi^*(v) = 0 (\text{mod } 4)$.

Proof: In the sum $S_a+S_b+S_c+S_d = 4S$, the value of each vertex in $V(G)$ is counted exactly three times. Thus $4S = 3T$ and the desired result follows from the theorem. Q.E.D.

Corollary 3.10(b). If n is the number of neutral edges determined by ϕ^*, then $n = 0(\text{mod } 6)$.

Proof: Partition $E(G)$ into three disjoint subsets K_o, K_+ and K_-, where K_o is the set of polar edges, K_+ is the set of neutral edges with ends having value +1, and $K_- = E(G) - (K_o \cup K_+)$. Define $\phi^{**}: E(G) \longrightarrow \{-2,0,2\}$ by $\phi^{**}(e) = \phi^*(u) + \phi^*(v)$ where u and v are the ends of $e \in E(G)$.

From Corollary 3.10(a) and the fact that each vertex is an end of three edges, we have

$$\sum_{e\in K_o} \phi^{**}(e) + \sum_{e\in K_+} \phi^{**}(e) + \sum_{e\in K_-} \phi^{**}(e) = 3T = 0(\text{mod } 12).$$

Thus, letting k_+ and k_- be the cardinality of K_+ and K_-, respectively, it follows that $2k_+ + 2k_- = 0(\text{mod } 12)$, and hence $n = k_+ + k_- = 0(\text{mod } 6)$. Q.E.D.

Corollary 3.10(c). If p is the number of polar edges determined by $\emptyset^{*}$, then $p = 0 \pmod 3$.

Proof: From Corollary 3.10(b) and the fact that $p + n = k_o + (k_+ + k_-) = 3k$, where k_o is the cardinal number of K_o, we have $p = 3(k - 2m)$ for some natural number m. Q.E.D.

Corollary 3.10(d). If n_i is the number of neutral edges of color i, $i = 1,2,3$, then n_i is even.

Proof: Let $K_{i_o} = E(i)\cap K_o$, $K_{i_+} = E(i)\cap K_+$ and $K_{i_-} = E(i)\cap K_-$. From Corollary 3.10(a) and the fact that each vertex is an end of exactly one edge in E(i) we have

$$\sum_{e\in K_{i_o}} \emptyset^{**}(e) + \sum_{e\in K_{i_+}} \emptyset^{**}(e) + \sum_{e\in K_{i_-}} \emptyset^{**}(e) = T = 0(\mathrm{mod} 4).$$

Thus, letting k_{i+} and k_{i-} denote the cardinal numbers of K_{i_+} and K_{i_-}, respectively, $2k_{i+} + 2k_{i-} = 0 \pmod 4$, and hence $n_i = k_{i+} + k_{i-} = 0 \pmod 2$. Q.E.D.

Corollary 3.10(e). If G is of odd index k, then G has at least three polar edges.

Proof: This follows from Corollaries 3.10(c) and (d) and the fact that $k = k_{io} + k_{i+} + k_{i-} = k_{io} + n_i$ for each $i = 1,2,3$, where k_{io} is the cardinal number of K_{i_o}. Q.E.D.

Definition 3.11. A connected cubic plane graph of index k is said to be a k-gonal prism, P_k, iff k of its faces are quadrilaterals and its other two faces are k-gons without a common boundary edge. The quadrilaterals are called lateral faces, and the k-gons base faces. An edge common to the boundary of two lateral faces is called a lateral edge while all other edges are called base edges.

<u>Definition 3.12</u>. Let G and G' be cubic plane graphs of index k. Then G is said to be <u>conjugate equivalent</u> to G' iff there exists a sequence $G_1, G_2, \ldots, G_n$ such that $G = G_1$, G_n is plane equivalent to G', and $G_{i+1} = (G_i)_{e_i}$ for some $e_i \in E(G_i)$, $i = 1,2,\ldots,n-1$.

Since edge conjugation is self-inverse, it follows that conjugate equivalence is an equivalence relation.

<u>Lemma 3.13</u>. Consider a connected cubic plane graph G of index $k \geq 2$. Then G can be transformed into a graph containing a lune, i.e., a double edge, by successive conjugation of edges.

<u>Proof</u>: If G contains a lune there is nothing to prove. When G is of girth n, then conjugation of any boundary edge of a face of valency n produces one of valency n-1 or n-2. The lemma follows by induction. Q.E.D.

<u>Remark 3.14</u>. Given a cubic plane graph G of index k and girth two, we can obtain a cubic plane graph H of index k-1 by suppressing the lune as indicated in Figure 3.12.

(a) (b)

Figure 3.12.

Then conjugation of edge m in H followed by restoration of the lune yields a graph isomorphic to and plane equivalent to the graph obtained from G by successively conjugating edges e,f,g and h.

<u>Theorem 3.15</u>. Every connected cubic plane graph G of index

$k \geq 2$ is conjugate equivalent to the k-gonal prism P_k.

Proof: By induction on k. The three essentially different connected cubic plane graphs of index 2 and the conjugations to produce P_2 are shown in Figure 3.13.

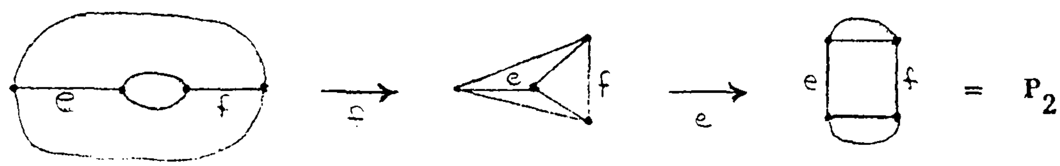
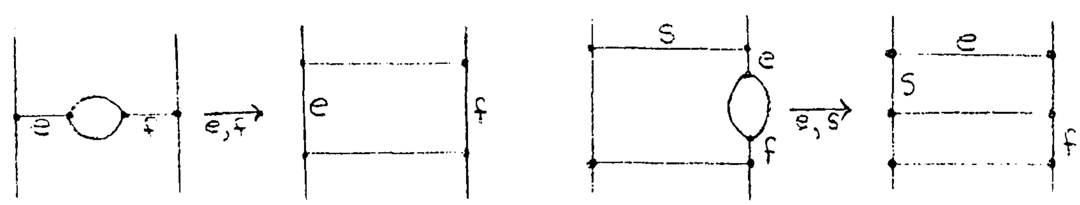


Figure 3.13.

Assume the theorem to be true for all connected cubic plane graphs of index k-1 and suppose G is such a graph of index k. Then by Lemma 3.13 G is conjugate equivalent to a graph possessing a lune. Let H be the graph of index k-1 formed by suppressing the lune as in Figure 3.12 and employ the inductive hypothesis to produce P_{k-1}. If m is a lateral edge of P_{k-1}, then restoration of the lune followed by conjugation of edges e and f yields P_k (Figure 3.14(a)). If m is a base edge, then as shown in Figure 3.14(b)), restoration of the lune followed by conjugation



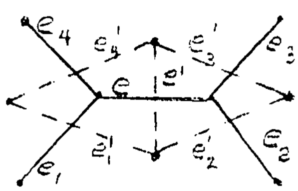
(a) (b)

Figure 3.14.

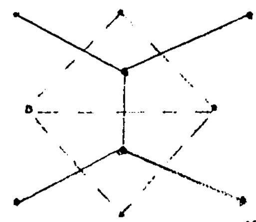
of edges e and s yields P_k. Q.E.D.

Corollary 3.15(a). Any two connected cubic plane graphs of index k are conjugate equivalent.

The geometric dual of a cubic plane graph G without multiple edges is a triangulated plane graph, i.e., a plane graph all of whose faces are triangles. If $e \in E(G)$, then the edges e_1', e_2', e_3' and e_4' in $E(G^*)$ (Figure 3.15(a)) corresponding to the edges e_1, e_2, e_3 and e_4 in $E(G)$ form a quadrilateral with diagonal e' corresponding to e. The operation of removing e' from G^* and inserting the other diagonal of the quadrilateral with sides e_1', e_2', e_3' and e_4'



(a) $e \in G$, $e' \in G^*$



(b) $e \in G_e$, $e' \in G_e^*$

Figure 3.15.

(Figure 3.15(b)) is called a diagonal transformation and corresponds to the process of conjugating edge e in G.

Wagner [45] proved that any two triangulated plane graphs with n vertices are "equivalent under diagonal transformations." The dual formulation of Wagner's result yields a weakened version of Corollary 3.15(a), viz., two cubic plane graphs of index n-2, without bridges and multiple edges, and with the property that no two faces have more than one boundary edge in common are conjugate equivalent. The presentation in this work is more detailed and, as we shall see later (Theorem 3.18), generalizes to nonplanar cubic graphs as well.

In order to discuss edge conjugation for plane representations of non-planar cubic graphs, Definition 3.1 can still be employed even though there may be other edges crossing e, e_1, e_2, e_3 and e_4 in the companion Figure 3.1. Since the concept of plane equivalence is meaningless for nonplanar graphs, more care must be taken in associating a plane representation with G_e. Let us stipulate that it be constructed from the plane representation of G by removing the existing edges e_1 and e_3 and inserting edges $e_1 = (v_1, w)$ and $e_3 = (v_3, w)$ in such a manner that e_1, e_2, e_3 and e_4 remain in clockwise order around e. This will insure that edge conjugation remains a self-inverse operation on plane representations of cubic graphs.

<u>Definition 3.16</u>. Let G and G' be connected cubic graphs of index k. Then G is said to be <u>conjugate similar</u> to G' iff whenever G_0 is a plane representation of G there exists a sequence $G_0, G_1, \ldots, G_n$ such that G_n is a plane representation of G' and $G_{i+1} = (G_i)_{e_i}$ for some $e_i \in E(G_i)$, $i = 0, 1, \ldots, n-1$.

It is easy to verify that conjugation of an arbitrary edge in any plane representation of $K_{3,3}$ produces a planar graph, although the resulting diagram may not be a plane embedding. This observation suggests the theorems which follow.

<u>Lemma 3.17</u>. Every connected cubic graph is conjugate similar to a graph containing a double edge.

<u>Proof</u>: Let m be the least integer such that the graph G contains a circuit of length m. If $m = 2$, there is nothing to prove. Suppose then that $m > 2$ and that the lemma is true for $m-1$.

Let C be a circuit of length m in G. As C is traversed in a plane representation of G the edges incident with C, but not on C, lie either to the left or to the right. If two consecutive such edges lie to the same side, then conjugation of the edge on C joining them produces a circuit of length m-1. If the edges incident to C alternate then, as shown in Figure 3.16, conjugation of any edge on C produces

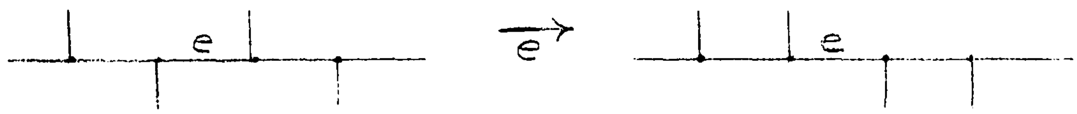


Figure 3.16.

a circuit of length m with two consecutive incident edges lying on the same side; thus another conjugation yields a circuit of length m-1. Hence the lemma follows from the inductive hypothesis. Q.E.D.

Theorem 3.18. Every connected nonplanar cubic graph is conjugate similar to a planar cubic graph.

Proof: The nonplanar graph of smallest index is $K_{3,3}$. It is the only such graph of index 3 and, as mentioned earlier, is conjugate similar to a planar graph. Assume the theorem to be true for every connected nonplanar cubic graph of index less than k and let G be such a graph of index k. By Lemma 3.17, G is conjugate similar to a graph H containing a double edge and it follows as in the proof of Theorem 3.15 that G is so related to a connected cubic planar graph. Q.E.D.

Corollary 3.18(a). Every connected cubic graph of index k is conjugate similar to P_k.

For a cubic plane graph G, consider a nonmultiple edge e and its neighbors as labelled in Figure 3.17(a). In addition to conjugation there is essentially only one other way to rearrange these objects and retain a cubic graph, viz., that shown in Figure 3.17(b). Call this operation <u>edge diagonalization</u> and denote the graph thus obtained by G^e. In what follows we adopt the convention that G^e will be represented in the plane with exactly one pair (e_i, e_{i+1}), $i = 1,2,3$, of the edges crossing while the remainder of its plane representation will be inherited from G.

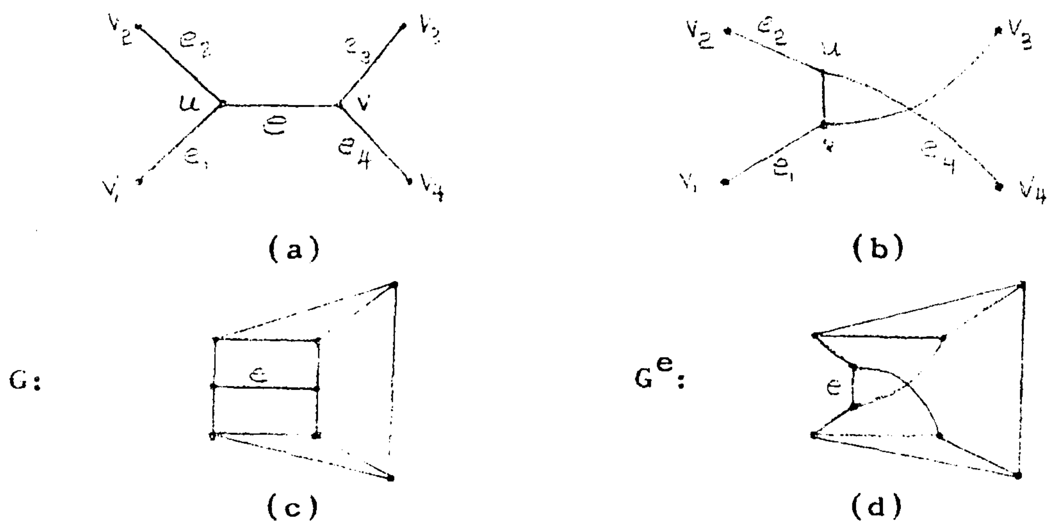


Figure 3.17.

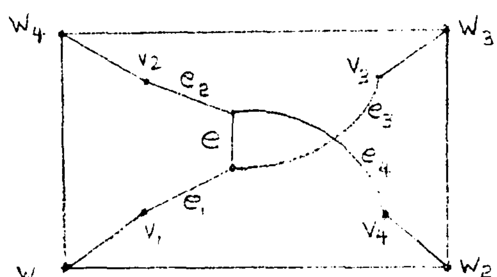
Figure 3.17(b) suggests that G^e will often be nonplanar, but the illustrations of Figure 3.17(c) and (d) show that 3-edge-connectivity is <u>not</u> a sufficient condition for G^e to be nonplanar. The conditions under which G^e is nonplanar are given by

<u>Theorem 3.19</u>. Using the above notation, the graph G^e is nonplanar iff (i) there exist paths in G from v_1 to v_2 and from v_3 to v_4 which are edge disjoint and do not contain e and (ii) there exist paths in G from v_2 to v_3 and from v_1

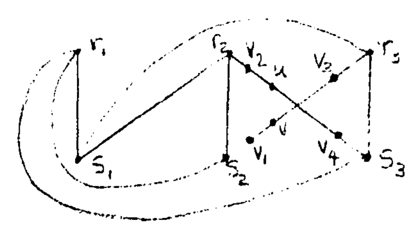
to v_4 which are edge disjoint and do not contain u or v.

Proof: It suffices to consider the case when the v_i, i = 1, 2,3,4, are all distinct, for otherwise G^e is always planar and condition (i) or (ii) is violated in all cases.

Suppose conditions (i) and (ii) are satisfied and consider the subgraph G' of G^e formed by removing all edges and vertices, other than u and v, which are not on at least one of the four shortest (i.e., containing the fewest edges) paths described in (i) and (ii). Then G' is isomorphic within divalent vertices to the graph shown in Figure 3.18(a), where perhaps $w_i = v_i$ for some i = 1,2,3 or 4. This graph is inturn isomorphic within divalent vertices to $K_{3,3}$ and hence G^e is nonplanar.



(a)



(b)

Figure 3.18.

Suppose G^e is nonplanar; then it contains a subgraph isomorphic within divalent vertices to $K_{3,3}$ and each edge e_i, i = 1,2,3,4, must lie on the subgraph -- for removal of an e_i produces a planar graph $G^e - e_i$. There are two cases to be considered.

Case 1. Edge e is contained in some subgraph H of G^e which is isomorphic within divalent vertices to $K_{3,3}$. Since each e_i is in H, d(u) = d(v) = 3. Thus H can be represented, within divalent vertices, by the diagram of Figure 3.18(a), where perhaps $w_i = v_i$ for some $i = 1,2,3$ or 4. Clearly conditions (i) and (ii) are satisfied in this case.

Case 2. No subgraph isomorphic within divalent vertices to $K_{3,3}$ contains e. If H is such a subgraph, let the six vertices of degree three in H be denoted by r_i and s_i, $i = 1, 2, 3$, where each r_i is joined to each s_i by a path in H containing no trivalent vertices other than its origin and terminus. Call these nine paths principal arcs of H. Now vertices v_1, v and v_3 must occur consecutively along some principal arc of H as must v_2, u and v_4. Since we have agreed to draw G^e with exactly one pair of the edges e_i crossing, only two principal arcs of H cross and not all six vertices u, v, v_i, $i = 1,2,3,4$, lie on the same principal arc.

If v_1, v and v_3 lie on a principal arc adjacent to the one containing v_2, u and v_4, then it would be possible to embed H in the plane, a contradiction.

Thus v_1, v and v_3 lie on some principal arc of H joining r_i and s_j, $i,j \in \{1,2,3\}$, while v_2, u and v_4 lie on a principal arc joining r_k and s_m, $k,m \in \{1,2,3\}$, $k \neq i$, $m \neq j$; Figure 3.18(b). Then the principal arcs joining r_i to s_m and r_k to s_j are subpaths of paths satisfying (i) and the principal arcs joining r_i to r_k and s_j to s_m are subpaths of paths satisfying (ii). Q.E.D.

CHAPTER IV

PLANTS AND TOURS

Many recent applications of graph theory to other disciplines or branches of the mathematical sciences demonstrate the utility and importance of a certain type graph called a tree. Heretofore, however, there is no known published material relating trees to the four-color problem.

A finite connected graph G is said to be a <u>tree</u> iff $V \neq 1$ and G contains no circuits. Useful and well known characterizations of trees are given in

<u>Theorem 4.1</u>. The following are equivalent for a graph G:

(i) G is a tree;

(ii) G contains no circuits and $E = V - 1$;

(iii) G is connected and $E = V - 1$;

(iv) G contains no circuits and $G + e$ contains exactly one circuit whenever e is an additional edge joining two nonadjacent vertices of G;

(v) G is connected but every edge of G is a bridge;

(vi) Every two vertices of G are joined by a unique path.

If, for any two vertices u and v of a connected graph G, we define the <u>distance</u>, $d(u,v)$, between u and v to be the length of the shortest path joining u and v, then the function d is a metric on $V(G)$. The <u>eccentricity</u>, $e(v)$, of a vertex v is the length of the longest nonclosed path in G with origin v; the <u>diameter</u>, $d(G)$, is the maximum eccentricity, and the <u>radius</u>, $r(G)$, is the minimum eccentricity. The <u>center</u> of G is the subset of vertices with the property that their eccentricity is equal to the radius. The following two theorems are part of the general knowledge among graph theorists concerning trees:

<u>Theorem 4.2</u>. Let T be a tree. Then

(i) If $d(T)$ is even, then the center of T is a singleton and $r(T) = \frac{1}{2}d(T)$.

(ii) If $d(T)$ is odd, then the center of T consists of exactly two adjacent vertices in T and $r(T) = \frac{1}{2}(d(T) + 1)$.

(iii) All diametral paths of T pass through the vertices in the center of T. In fact, when $d(T)$ is odd they all pass through the edge joining the central vertices.

Calling the vertices of a tree T with valency one <u>terminal vertices</u>, a <u>branch at a vertex v</u> of T is a maximal subtree of T containing v as a terminal vertex. Thus the number of branches at v equals its valency, $d(v)$. The <u>weight of a branch</u> is the number of edges that it contains; the <u>weight of a vertex</u> is the maximum number of its branch weights; and the <u>centroid</u> of T consists of those vertices of T which have minimum weight. Although the centroid is not, in general, the same as the center, we have

<u>Theorem 4.3</u>. The centroid of every tree is either singleton or consists of two adjacent vertices.

A <u>rooted tree</u> is a tree in which one vertex, called the <u>root</u>, is distinguished from the others. When the root is a terminal vertex, such a tree is said to be a <u>planted</u> <u>tree</u>. As before, a <u>plane</u> <u>tree</u> is one which is embedded in the Euclidean plane. Two planted plane trees will be called <u>equivalent</u> iff each can be transformed into the other by an orientation-preserving homeomorphism of the plane onto itself which maps vertices onto vertices and which preserves the root. Henceforth no distinction will be made among equivalent planted plane trees.

Among planted plane trees, those of special interest in what follows are described in

<u>Definition 4.4</u>. A planted plane tree is said to be a <u>plant</u> iff each of its nonterminal vertices is of degree three.

Now a digression is necessary before relating these objects to the four-color conjecture.

In [50] Whitney presented a lengthy but rather straightforward proof of the following lemma:

<u>Lemma 4.5</u>. Let G be a plane graph all of whose faces are triangles and with the property that it contains no circuits of length ≤ 3 other than face boundaries. For a circuit C in G let u and v be two of its distinct vertices and call the two arcs of C with origin u and terminus v C_1 and C_2. Suppose

(i) No pair of vertices of C_1 are joined by an edge lying interior to C, and

(ii) Either no pair of vertices of C_2 are joined by such an edge, or else there is a vertex w in C_2, distinct from u and v, which disects C_2 into two subarcs C_3 and C_4 with origin w and terminating in u and v, respectively, such that no pair of vertices of C_3 and no pair of vertices of C_4 are joined by an edge lying interior to C.

Then there exists a path with origin u and terminus v which passes through each vertex on and interior to C and contains no edges lying exterior to C.

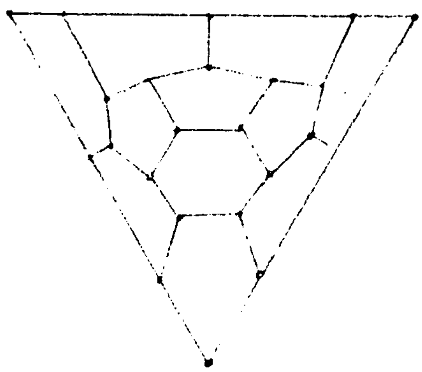
Taking C as a face boundary of G Whitney immediately obtained

Theorem 4.6. Every triangulated plane graph containing no circuits of length ≤ 3 other than face boundaries possesses a Hamiltonian circuit.

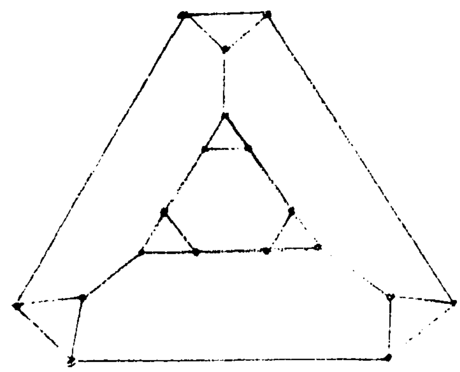
Definition 4.7. A *Whitney tour* of a plane graph G is a simple closed plane curve W with the properties that $W \cap V(G)$ is empty and $W \cap R$ is connected and nonempty for every face R in F(G).

Thus a Whitney tour passes through each face of G exactly once and every Hamiltonian circuit of a plane embedding of G^* corresponds to a Whitney tour of G and conversely.

From Corollary 2.32(b) and Theorem 4.6 it follows that a minimal map possesses a Whitney tour. The importance of maps of girth 5 and an examination of the proof of Lemma 4.5 suggested the conjecture that all 3-edge-connected cubic maps with girth 5 possess Whitney tours. An attempted proof was nearly successful; however, its ultimate failure led to a counterexample involving twelve copies of the graph shown in Figure 4.1(a). As a result of further investigation, it is believed that the smallest counterexample is that of Figure 4.1(b) where each triangle represents a copy of the graph of Figure 4.1(a). It can easily be verified that this configuration is indeed a 3-edge-connected cubic map of girth 5 without a Whitney tour.



(a)



(b)

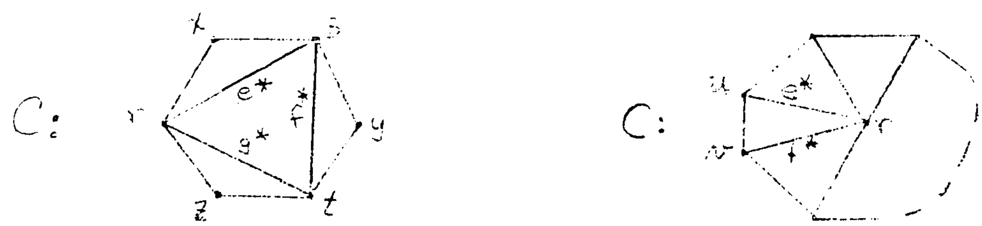
Figure 4.1.

A Whitney tour W of a plane graph G is said to <u>avoid</u> an edge e in E(G) iff the intersection of e and W is empty and it is said to <u>pass through</u> e iff this intersection is nonempty. We now prove

<u>Theorem 4.8</u>. Let G be a cubic map without n-rings, n = 1,2, 3,4. If e, f and g are three edges incident with a vertex v in V(G), then there exists a Whitney tour W of G which avoids e, f and g.

<u>Proof</u>: Since G is cubic, the edges e^*, f^*, and g^* form a triangle in G^*. The set $A = \{r,s,t\}$ of the vertices of the triangle alternate around a circuit C of length six, as

shown in Figure 4.2(a). Let $B = \{x,y,z\}$ be the set of the remaining vertices on C; since G has no 3-rings it follows that no vertex in A is joined to a vertex in B by an edge lying exterior to C. The fact that G has no 4-rings implies that G^* has no 4-gons and thus no pair of vertices in A and no pair in B are joined by an edge exterior to C. By stereographic



(a) (b)

Figure 4.2.

projection the interior and exterior of C can be interchanged. Taking $u = r$, $v = x$, and $w = y$, it follows from Lemma 4.5 that there is a Hamiltonian path P with origin r and terminus y and not containing e^*, f^* and g^*. Therefore $P + (x,r)$ is a Hamiltonian circuit in G^* and hence G possesses a Whitney tour which avoids e, f and g. Q.E.D.

As an immediate consequence of the theorem we have

<u>Corollary 4.8(a)</u>. Let e, f and g be the three edges incident with a given vertex of a cubic map G without n-rings, $n = 1,2,3,4$. Then there exists a Whitney tour W of G which will also suffice for G_e, G_{ef}, and G_{efg}.

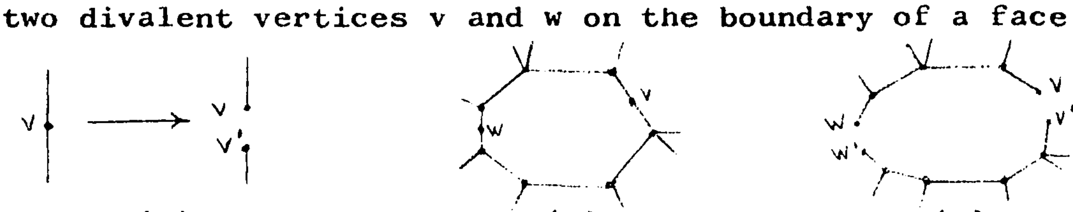
<u>Remark 4.9</u>. Inspection of the hypothesis of Corollary 4.8(a) shows that the conclusion holds for minimal maps. If the hypothesis of Theorem 4.8 is weakened to permit 4-rings, then a similar argument will demonstrate the existence of a

Whitney tour which avoids any pair of adjacent edges.

Theorem 4.10. If R is an arbitrary face of a cubic map G without n-rings, n = 1,2,3, then there exists a Whitney tour W passing through any two adjacent boundary edges of R.

Proof: Let e and f be adjacent boundary edges of R. The hypothesis implies that the neighbor ring of R is indeed a ring and thus it appears in G^* as a circuit C surrounding the vertex r representing R - Figure 4.2(b). From the definition of ring it follows that no two vertices of C are joined by an edge lying exterior to C. If we take u and v as the distinct ends of e^* and f^*, then the hypotheses of Lemma 4.5 are satisfied. Hence there exists a path P with origin u and terminus v which passes through each vertex on and exterior to C and contains no edges interior to C. Augmenting P with e^*, r and f^* produces a Hamiltonian circuit in G^* containing e^* and f^* and thus there exists a Whitney tour of G passing through e and f. Q.E.D.

Suppose v is a divalent vertex in a plane graph G. Then by splitting v is meant the operation of replacing v by two monovalent vertices in the natural way shown in Figure 4.3(a). Aware of possible ambiguity, we will call the monovalent vertices introduced in this way v and v'. Splitting two divalent vertices v and w on the boundary of a face of



(a) (b) (c)

Figure 4.3.

G disconnects the boundary circuit into two distinct paths; see Figure 4.3(b) and (c).

The thought of decomposing a cubic map possessing a Whitney tour into plants, as presented in the next paragraph, was conceived by A. Bernhart and has since been considered by his son, F. Bernhart, and myself. Although the proofs have been worked out independently, the material on the following four pages has also been developed by F. Bernhart. The terms <u>trace</u> and <u>spectrum</u> are his.

Let G be a cubic map of index k possessing a Whitney tour W. As W is traversed, one passes through each face and therefore through k+2 edges. If vertices are inserted on each of these edges and labelled $v_1, v_2, \ldots, v_{k+2}$, in order of their occurrence along W, then observe that splitting these vertices disconnects the plane graph into two trees. Designating v_1 and v_1' as roots qualifies each tree as a plant with k+2 terminal vertices. Figure 4.4 illustrates a decomposition of a cubic map of index 5 into two plants -- their symmetry is coincidental.

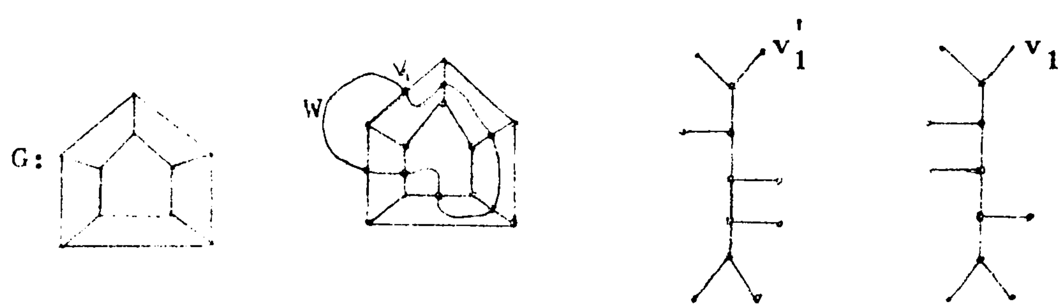


Figure 4.4.

It is convenient to call the number of nonterminal vertices the <u>index</u> of a plant, since a Whitney tour in a map of index k induces plants with k+2 terminal vertices and therefore k nonterminal vertices. Edges incident with terminal vertices will be called terminal edges or simply terminals, while the remaining vertices and edges will be described as interior. If two terminals of a plant are adjacent edges, they will be called an outer pair. Every plant has at least two outer pairs and, in fact, a plant of index $k > 1$ can have at most $[k/2] + 1$ outerpairs, for each outer pair accounts for two of the k+2 terminals.

A useful technique in inductive proofs is to think of a plant as having "grown" from a plant of lesser index, in that the latter plant's terminal vertices have "sprouted" outer pairs.

Definition 4.11. We define a plant to be a spine iff it has exactly two outer pairs.

The plants of Figure 4.4 illustrate spines of index 5. If M is a minimal map, then Theorem 4.8 implies that there exists a Whitney tour of M one of whose induced plants is not a spine, while Theorem 4.10 guarantees the existence of a Whitney tour of M whose decomposition into plants includes a spine.

Suppose G is an edge 3-colored cubic map which possesses a Whitney tour W. Then the "corresponding" terminal edges of the plants induced by W must agree in color. This suggests an equivalent formulation of the four-color conjecture in terms of the following concept.

Definition 4.12. Let T be a plant of index k. A sequence

$t = [t_1, t_2, \ldots, t_{k+2}]$ with $t_i \in \{1,2,3\}$, $i = 1,2,\ldots,k+2$, is said to be a trace for T iff

(i) $t_1 = 1$,

(ii) the least i such that $t_i = 2$ is less than the least j such that $t_j = 3$, $i,j \in \{2,3,\ldots,k+2\}$,

(iii) there exists an edge 3-coloring of T such that if the terminal edges of T are labelled e_1, $e_2, \ldots$, e_{k+2} in counterclockwise order starting with the root edge, then the color assigned to edge e_i is t_i, for $i = 1,2,\ldots,k+2$.

The set of all traces for T, S(T), is called the spectrum of T.

Theorem 4.13. If t is a trace for a plant T, then there exists a unique edge 3-coloring of T satisfying property (iii) of Definition 4.12.

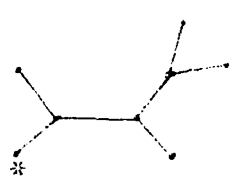
Proof: The theorem is trivially true for the plant of index 1. Suppose it is true for all plants of index k-1, $k > 1$. Let $t = [t_1, t_2, \ldots, t_{k+2}]$ be a trace for a plant T of index k. Since T contains at least two outer pairs, there exists an outer pair of T not containing the root. Let $i > 1$ be the least integer such that t_i and t_{i+1} are the elements of t which correspond to such an outer pair. Clearly $t_i \neq t_{i+1}$. If we form the sequence $s' = [t_1, \ldots, t_{i-1}, s_i, t_{i+2}, \ldots, t_{k+2}]$, where $s_i \in \{1,2,3\} - \{t_i, t_{i+1}\}$, then s' gives the colors of the terminals of the tree T' obtained by removal of the selected outer pair from T. However, it may happen that s' is not a trace for T' because of condition (ii). If this is so, then interchanging 2's and 3's in s' produces a trace s" for T'. Setting t' = s' or s" according as s' is or is not a trace for T', the induction hypothesis implies that there exists a unique edge 3-coloring of T' compatible with trace t'. Restoration of the outer pair, preceded by interchange of 2's and 3's if necessary, establishes the theorem. Q.E.D.

In view of earlier discussion we can state

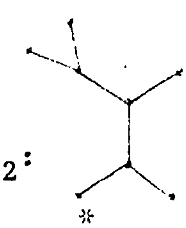
Theorem 4.14. The four-color conjecture is true iff the spectra of every pair of plants of the same index have a non-empty intersection.

Figure 4.5 gives two plants of index 3 whose spectra have a singleton intersection. The asterisk denotes the root.

T_1:



T_2:



$$S(T_1) = \left\{\begin{matrix}[11123]\\ [11213]\\ [12232]\\ [12322]\end{matrix}\right\} \quad S(T_2) = \left\{\begin{matrix}[12113]\\ [12131]\\ [12223]\\ [12232]\end{matrix}\right\} \quad S(T_1)\cap S(T_2) = \{[12232]\}$$

Figure 4.5.

Theorem 4.15. For every natural number k, a sequence $t = [t_1,t_2,\ldots,t_{k+2}]$ with $t_i \in \{1,2,3\}$, $i = 1,2,\ldots,k+2$, satisfying conditions (i) and (ii) of Definition 4.12 is a trace for some plant of index k iff $n_1 = n_2 = n_3 = k \pmod 2$, and at least two of n_1, n_2, n_3 are nonzero, where n_i is the number of elements of t equal to i.

Proof: By induction on k. If k = 1 the theorem is trivially true since [1,2,3] is the only sequence satisfying (i),

(ii) and the congruence condition. Suppose $k > 1$ and the theorem is true for all sequences $[t_1,t_2,\ldots,t_{k+1}]$ satisfying conditions (i) and (ii).

Let $t = [t_1,t_2,\ldots,t_{k+2}]$ be a trace for some plant T of index k and let $i > 1$ be the least integer such that t_i and t_{i+1} are elements of t corresponding to an outer pair of T not containing the root. If T' is the plant of index k-1 obtained by removing that outer pair then, as in the proof of Theorem 4.13, let t' be the trace of T' induced by t. By the induction hypothesis, t' satisfies the congruence relation. Thus restoration of the outer pair, preceeded by interchanging 2's and 3's if necessary, establishes the congruence condition for t since this process alters the parity of each number involved.

Conversely, let $t = [t_1,t_2,\ldots,t_{k+2}]$, $t_i \in \{1,2,3\}$ be a sequence satisfying (i), (ii) and the congruence condition. Not all of $t_2,\ldots,t_{k+2}$ are equal to 2, for otherwise the congruence condition would be violated. Let i be the least integer greater than 1 for which $t_i \neq t_{i+1}$. Clearly $t_i \neq 3$. Form the sequence $s' = [t_1,\ldots,t_{i-1},s_i,t_{i+2},\ldots,t_{k+2}]$ where $s_i \in \{1,2,3\} - \{t_i,t_{i+1}\}$. If s' does not satisfy (ii), then interchange of 2's and 3's in s' produces a sequence t' which does satisfy (ii). Clearly both s' and t' satisfy the congruence condition as well as (i). Thus, by the induction hypothesis, there exists a plane tree T' of index k-1 for which exactly one member of {s',t'} is a trace. Adjoining an outer pair to the ith terminal vertex of T', preceeded by interchange of 2's and 3's if necessary, produces a plant of index k with trace t. Q.E.D.

Theorem 4.16. Every plant of index k has exactly 2^{k-1} traces.

Proof: The theorem is trivially true for the plant of index 1 and the inductive step follows from the fact that adjoining an outer pair to a plant increases the number of traces by a factor of 2. Q.E.D.

An apparently more difficult question is that of finding the number of plants with a given trace. In fact, even significant upper bounds are not known. The analogous problem for vertex 2-coloring planted plane trees was recently solved by Tutte [43], but his methods do not seem to apply to traces and edge colorings. Using $2^{(k)}$ to represent k repetitions of 2, it is easily seen that only one plant of index k admits the trace $[1,2^{(k)},1]$ for k even, or the trace $[1,2^{(k)},3]$ for k odd. On the other hand, using edge conjugation of the interior edges of a plant, one can readily demonstrate at least k plants of index k which admit a suitably chosen trace -- and an exponential relation is suspected.

Among the many enumerative problems on plane trees solved by Harary, Prins and Tutte [23] is the following formula for the number P_k of plants of index k:

$$P_k = \frac{1}{k+2} \binom{2k+2}{k+1} = \frac{1\cdot 3\cdot 5\cdots(2k+1)}{(k+3)!}\, 2^{k+2}$$

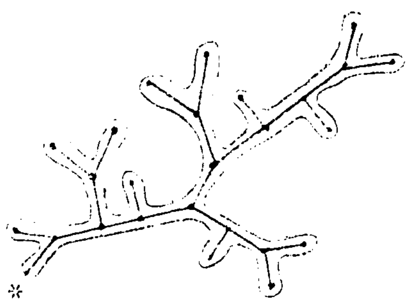
Theorem 4.17. The number S_k of spines of index k is given by $S_k = (k+2)\, 2^{k-3}$.

Proof: The number of spines with an outer pair terminal as root is 2^{k-1}, since each of the k-1 terminals excluding the root and outer pair not containing the root can be oriented in two ways. Also there are k-2 non-outer pair terminals which can be chosen as root and for each such choice there are k-3 remaining terminals which can be oriented in two distinct ways. Hence $S_k = 2^{k-1} + (k-2)2^{k-3} = (k+2)2^{k-3}$. Q.E.D.

For purposes of computation there are numerous codes available for representing trees. Of course, these apply to plants as well but, depending on what information is to be extracted, certain codes are more desirable than others. The plant code derived below was motivated by the Polish or parenthesis-free notation used in logic. It is not a minimal representation for a plant, but it is brief, versatile, and of greater utility than that presented by De Bruijn and Morselt in [12].

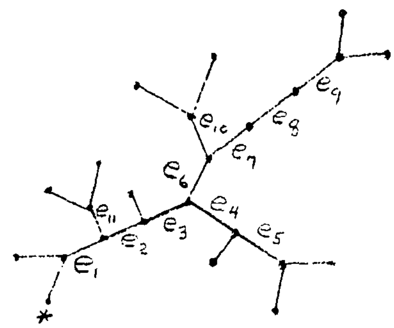
To every plant of index k we assign a binary code word consisting of k zeros, representing the interior vertices, and k+2 ones, representing its terminal vertices. It is obtained by travelling in the plane around the plant, as indicated in Figure 4.6(a), starting at the root on the "right bank" of the root edge. During this trip each vertex is passed on the left; in fact, each interior vertex is passed three times while each terminal vertex is passed once. The code is obtained by recording a 1 or 0 the first time each vertex is passed during the trip.

The following theorem presents a simple rule for deciding whether a binary sequence of length $k + (k+2) = 2(k+1)$ is the code for a plant. By a <u>segment</u> of a sequence $[s_1,s_2,\ldots,s_n]$ we mean a subsequence of the form $[s_i,s_{i+1},\ldots,s_{i+j}]$ where $i \in \{1,2,\ldots,n-1\}$ and $j \in \{1,2,\ldots,n-2\}$ and $i + j \leq n$.



1000000101100010111011101111

(a)



1000|001011|00|01|011101110111

(b)

Figure 4.6.

<u>Theorem 4.18</u>. Let $c = [c_0,c_1,\ldots,c_{2k+1}]$ be a binary sequence consisting of k zeros and k+2 ones. Then c is a code for a plant of index k iff $c_0 = 1$, $c_1 = 0$, and the ones are not more numerous than the zeros in every segment of c of the form $[c_1,\ldots,c_j]$, $1 < j \leq 2k$.

<u>Proof</u>: The code for the plant of index 1 is 1011 and is easily seen to be the only binary sequence of length 4 satisfying the conditions of the theorem. Suppose $k > 1$ and that the theorem is true for all sequences of length $2(n+1)$, $1 \leq n < k$, which contain n zeros and n+2 ones.

Let T be a plant of index k with code $c = [c_0,c_1,\ldots,c_{2k+1}]$. From the construction of c we see that $c_0 = 1$, $c_1 = 0$, there are k zeros and k+2 ones , and outer pairs not containing the root correspond to segments of c consisting of

011. Since T has at least two outer pairs there is an outer pair, not containing the root, whose removal produces a plant T' of index k-1. If c_i, $1 < i < 2k$, represents the zero of the 011 segment of c corresponding to the removed outer pair, then the sequence $c' = [c_0,\ldots,c_{i-1},1,c_{i+3},\ldots,c_{2k+1}]$ is the code for T'. By the induction hypotheses it satisfies the condition of the theorem and thus the original sequence did also.

Let $c = [c_0,c_1,\ldots,c_{2k+1}]$ be a binary sequence consisting of k zeros and k+2 ones satisfying the conditions of the theorem. Then $c_{2k+1} = 1$; for otherwise $[c_1,\ldots,c_{2k}]$ consists of (k-1) zeros and (k+1) ones. Also $c_{2k} = 1$; for otherwise $[c_1,\ldots,c_{2k}]$ consists of (k-1) zeros and k ones. Let i, $k \leq i < 2k$, be the greatest integer such that $c_i = 0$. Then $c_{i+1} = c_{i+2} = 1$ and the sequence $c' = [c_0,\ldots,c_{i-1},1,$ $c_{i+3},\ldots,c_{2k+1}]$ satisfies the conditions of the theorem so there exists a plant T' for which c' is its code. Then by "growing" and outer pair at the terminal vertex of T' corresponding to the (i+1)st entry of c' we produce a plant T for which c is its code. Q.E.D.

It will often be convenient to label the vertices of a plant T using $u_0,u_1,\ldots,u_{2k+1}$ where a vertex has label u_i iff it is represented by the (i+1)st digit in the code for T. As just seen $c_0=c_{2k}=c_{2k+1}=1$ and $c_1=0$ for all codes of plants of index k. Thus to distinguish different plants of index k it suffices to consider the 2(k-1) code digits $c_2,\ldots,c_{2k-1}$ called the <u>center</u> of the code. Since a code contains k zeros and k+2 ones the zeros and ones in the center are equinumerous. It is natural to order the set of plants of index k by placing their codes in order as binary numbers. In fact, this ordering is the same as that obtained by ordering the code centers in the same way. The first center consists of k-1 zeros followed by k-1 ones, while the last center consists of k-1 repetitions of 10.

Since the zeros and ones are equinumerous in the center, a procedure for obtaining the (n+1)st code center, $n < P_k$, from the nth can be thought of as a rearrangement of its digits in which at least one "zero" is moved to the right and hence at least one "one" is moved to the left. If the nth code center ends in 1, then it can easily be seen that interchange of the code's rightmost 0 with the 1 following it produces the next larger binary number satisfying Theorem 4.18. On the other hand suppose the nth code center ends in 0. In any final segment of the code center for a plant, the condition of Theorem 4.18 requires that the number of zeros not exceed the number of ones. Thus, unless the center consists of k-1 repetitions of 10 and represents the last plant in the ordering, the rightmost zero which can be moved to the right precedes a pair of consecutive ones. Exchanging this 0 with the 1 following it produces a valid code center, but to obtain the <u>next larger</u> binary code number, all other ones following this selected zero must be moved to the end of the code center.

<u>Definition 4.19</u>. Let $\bar{c}$ be a plant code center not consisting

of k-1 repetitions of 10. The following procedure will be called the <u>zero-exchange operation</u> on $\bar{c}$:

(i) If $c_{2k-1} = 1$, let i, $k \leq i < 2k-1$, be the greatest integer such that $c_i = 0$ and interchange c_i with c_{i+1}.

(ii) If $c_{2k-1} = 0$, let $k < i < 2(k-1)$ be the greatest integer such that $c_i = 0$ and $c_{i+1} = c_{i+2} = 1$. Then place c_i and all zeros of $\bar{c}$ to the right of c_{i+1} between c_{i+1} and c_{i+2}.

<u>Example 4.20</u>. Performing the zero-exchange operation on the following code centers for plants of index 5 produces the indicated results.

$$0\ 0\ 0\ 0\ 1\ 1\ 1\ 1 \longrightarrow 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1$$

$$0\ 1\ 0\ 1\ 1\ 0\ 1\ 0 \longrightarrow 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1$$

<u>Theorem 4.21</u>. For every natural number k, the set of code centers for plants of index k can be recursively generated from the sequence consisting of k-1 zeros followed by k-1 ones by the zero-exchange operation.

We omit the proof of the above theorem. In view of the preceding paragraphs it is easy to convince oneself of its truth and a formal proof is straightforward but tedious.

<u>Conjugation</u> of an interior edge of a plant can be defined as in Definition 3.1, but conjugation of terminal edges must remain undefined. As before, two plants T and T' of index $k \geq 2$ will be called <u>conjugate equivalent</u> iff there exists a sequence $T_1, T_2, \ldots, T_n$ such that $T = T_1$,

T' is equivalent to T_n, and $T_{i+1} = (T_i)_{e_i}$ for some interior edge $e_i \in E(T_i)$, $i = 1,2,\ldots,n-1$.

Lemma 4.22. If T is a plant of index $k \geq 2$, then T is conjugate equivalent to the spine $S_{(k)}$ whose code is first in the binary ordering.

Proof: If T is a plant of index 2, then either T is $S_{(2)}$ (with code 100111) or conjugation of the only interior edge of T will produce $S_{(2)}$. Suppose the theorem is true for all plants of index k-1 and let T be a plant of index k. If T is not $S_{(k)}$, then removal of an outer pair not containing the root produces a plant T' of index k-1. Letting e denote the terminal edge of T' incident with the outer pair removed, the inductive hypothesis implies that there are interior edges of T' which could be conjugated to produce $S_{(k-1)}$. Thus, as indicated in Figure 4.7, conjugation of these same edges in t followed, if necessary, by conjugation of e will produce $S_{(k)}$. Q.E.D.

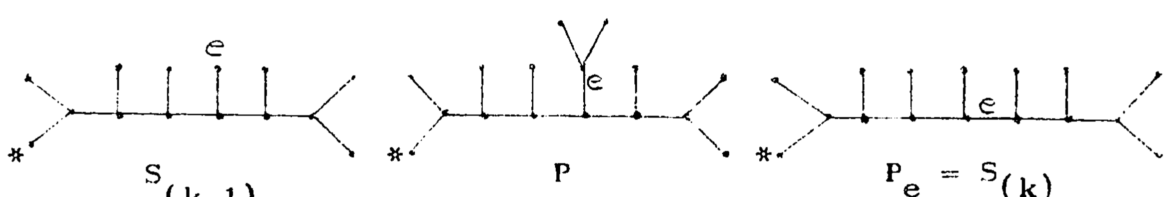


Figure 4.7.

Corollary 4.22. Any two plants of index k are conjugate equivalent.

As an immediate consequence we have

Theorem 4.23. If G and G' are any two cubic maps of index k possessing Whitney tours, then there exists a sequence G_1, $G_2,\ldots,G_n$ of cubic maps of index k all possessing Whitney tours such that $G = G_1$, G_n is plane equivalent to G', and $G_{i+1} = (G_i)_{e_i}$ for some $e_i \in E(G_i)$, $i = 1,2,\ldots,n-1$.

We indicate below an easily performed, but difficult to describe, procedure for altering the code of a plant corresponding to edge conjugation in the plant.

For a plant of index k, certain pairs of zeros in its code represent the ends of its interior edges, labelled $e_1,\ldots,e_{k-1}$ in order of their encounter when travelling around the plant as indicated in Figure 4.6(b). In that example observe that the first pair of zeros represents the ends of e_1, the fourth and seventh zeros those of e_6, and the seventh and eleventh zeros those of e_{10}, while the sixth and seventh zeros do not represent ends of the same edge.

Inspection of the coding procedure shows that the jth zero of the code c, $1 < j \leq k$, corresponds to that end u_s of e_{j-1} which is farthest from the root. Before identifying the zero of c representing the other end of e_{j-1}, we define the principal branch at an interior vertex u_i to be that branch of T at u_i corresponding to the shortest segment $[c_i,\ldots,c_{i+j}]$ of c for which zeros and ones are equinumerous. For the earlier example, the segments corresponding to the principal branches at u_4 and u_{12} are indicated by the first and last pairs of vertical bars in Figure 4.6(b).

Suppose the ith zero, $1 \leq i < j$, represents the end u_r of e_{j-1} nearest the root. Then e_{j-1} is the root edge of a branch of T at u_r. In view of the preceding paragraph, either $i = j-1$ or i is such that $[c_n,\ldots,c_s]$ is the shortest segment of c ending in c_s for which zeros and ones are equinumerous.

Thus in determining the ends of edge e_j in the code for T the $(j+1)$st zero represents the end of e_j farthest from the root. If the preceding digit is a zero, it denotes the other end of e_j; otherwise, starting with the one preceding the $(j+1)$st zero, count back until passing an equal number of zeros and ones.

If u_r is the end of e_j nearest the root, then e_j is the root edge of a branch of T at u_r and, as can be seen in Figure 4.8, if e_j is initially on the principal branch at u_r, conjugation of e_j removes it from that branch, and con-

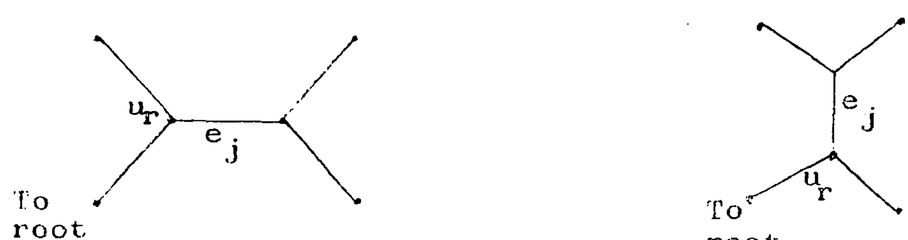


Figure 4.8.

versely. Thus we have

Theorem 4.23. If the interior edge e_j, $1 \leq j < k$, of a plant T of index k is conjugated to produce plant T', then the code c' for T' can be obtained from the code c of T as follows:

(i) If the ends of e_j are represented in c by nonadjacent zeros c_r and c_s, produce c' by extracting c_s and inserting it immediately following c_r.

(ii) If the ends of e_j are represented in c by adjacent zeros c_r and c_{r+1}, produce c' by extracting c_r and inserting it to the right in the first position arrived at by passing an equal number of zeros and ones.

Figure 4.9 illustrates Theorem 4.23. Of course, to perform a second conjugation, one must first relabel the interior edges of the plant.

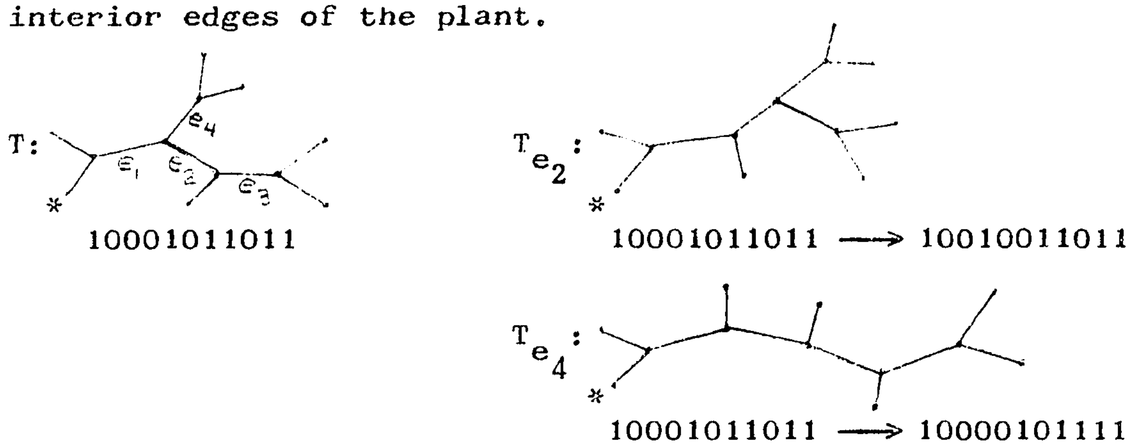


Figure 4.9.

Motivation for the development of the following algorithm is provided by the observation that the valency of an arbitrary face in a cubic map possessing a Whitney tour W can be obtained from a knowledge of the distances between adjacent terminal vertices on the plants induced by W; see Figure 4.4.

For the code c of a plant of index k and with vertices labelled u_i, $i = 0,1,\ldots,2k+1$, let h_i denote the <u>height</u> of u_i, i.e., the distance of the ith vertex from the root. Clearly $h_0 = 0$, $h_1 = 1$, and $h_n > 1$ for every n, $1 < n \leq 2k+1$.

If u_i is an <u>interior</u> vertex, then its neighbors are u_{i-1}, u_{i+1} and u_{i+m}, where m is the number of vertices in the principal branch at u_i.

If u_i is a <u>terminal</u> vertex other than the root, then its only neighbor is u_{i-m} where $m = i-1$ if u_{i-1} is an interior vertex and m is the number of vertices in the principal branch at u_{i-m}. Then $h_i - 1 = h_{i-m}$.

<u>Theorem 4.24</u>. Let $c = [c_0, c_1, \ldots, c_{2k+1}]$ be the code for a plant of index k. If the sequence $p = [p_0, p_1, \ldots, p_{2k+1}]$ is defined by $p_i = c_i$, $i = 0,1,\ldots,2k+1$, then the sequence $d = [h_0, h_1, \ldots, h_{2k+1}]$ can be conputed successively by setting $h_0 = 0$, $h_1 = 1$ and for $n = 1,2,\ldots,2k+1$,

$$h_{n+1} = \begin{cases} h_n + 1, & \text{if } c_n = 0, \\ h_m + 1 \text{ and setting } p_m = 1, & \text{if } c_n = 1, \end{cases}$$

where m is the greatest integer less than n such that $p_m = 0$.

Using the usual technique of "pruning" an outer pair, the proof follows by induction on k, after observing that the 011 corresponding to the outer pair removed does not affect the algorithm's computation of the other heights. A feeling for the mechanics of the algorithm can be obtained by working through

<u>Example 4.25</u>. Figure 4.10 shows a plant of index six and the corresponding computation of heights.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
c_i	1	0	0	1	0	0	0	1	1	1	0	1	1	1
p_i	1	Ø 1	Ø 1	1	Ø 1	Ø 1	Ø 1	1	1	1	Ø 1	1	1	1
h_i	0	1	2	3	3	4	5	6	6	5	4	5	5	2

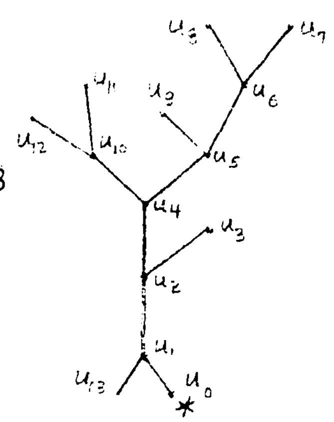


Figure 4.10.

Remark 4.26. At the end of application of the algorithm, we see that $p_i = 1$ for every $i = 0,1,\ldots,2k+1$, that there are an even number of vertices at each height > 1, and that if u_i and u_j are consecutive terminal vertices, then the distance between u_i and u_j is given by $d(u_i,u_j) = 2 + \sum_{k=1}^{j-1} |d_{k+1}-d_k|$.

Before developing an algorithm to find the distance between two arbitrary vertices u_i and u_j, $i < j$, of a plant, we make two definitions.

A down sequence for u_i and u_j is a sequence $l_1, l_2, \ldots, l_m$, defined inductively by l_1 = the largest integer less than j such that $d_{l_1} = d_j - 1$, l_{k+1} = the largest integer less than l_k such that $d_{l_{k+1}} = d_{l_k} - 1$ unless $l_k \leq i$, in which case the sequence terminates with $l_k = l_m$. An up sequence for u_i and u_j is a sequence $r_1, r_2, \ldots, r_n$ defined by r_1 = the largest integer greater than l_m and less than or equal to i such that $d_{r_1} = d_{l_m} + 1$, r_{k+1} = the largest integer greater than r_k and less than or equal to i such that $d_{r_{k+1}} = d_{r_k} + 1$, and the sequence terminates with $r_n = i$. If $l_m = i$, we say that the up sequence has zero length and write $n = 0$.

Theorem 4.27. Let u_i and u_j, $i < j$ be two vertices of a plant of index k. If the up and down sequences for u_i and u_j are of length n and m, respectively, then $d(u_i,u_j) = n+m$.

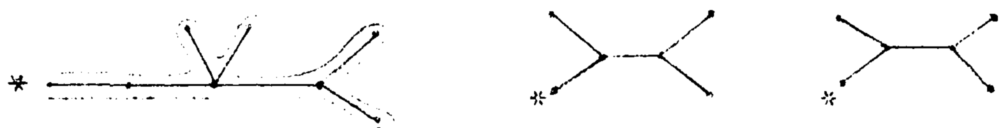
The proof follows from the fact that the unique path from u_j to u_i is $u_j, u_{l_1}, \ldots, u_{l_m}, u_{r_1}, \ldots, u_{r_n} = u_i$. In the following we tabulate some up and down sequences for Example 4.25.

u_1, $d_1 = 1$ u_5, $d_5 = 4$	Down sequence: 4,2,1 Up sequence: -	$d(u_1,u_5) = 3 + 0 = 3$
u_7, $d_7 = 6$ u_{11}, $d_{11} = 5$	Down sequence: 10,4 Up sequence: 5,6,7	$d(u_7,u_{11}) = 2 + 3 = 5$
u_8, $d_8 = 6$ u_{13}, $d_{13} = 2$	Down sequence: 1 Up sequence: 2,4,5,6,8	$d(u_8,u_{13}) = 1 + 5 = 6$

In a recent paper [23], Harary, Prins and Tutte succeeded in establishing a one-to-one correspondence between the family of all planted plane trees and the set of all plants. The description of this correspondence was quite complicated, but it has since been improved on by De Bruijn and Morselt [12] who used different methods. Knowledge of this correspondence suggests the possible utility of the collection of binary sequences satisfying the conditions of Theorem 4.18 as codes for planted plane trees. Not only is this possible but, in fact, the manner of encoding given below seems as simple as could be desired and provides for a still more straightforward demonstration of the abovementioned correspondence.

Before proceeding we mention that the planted plane tree consisting of a single edge together with its ends can be considered as a plant of index 0. Its plant code is then 11, and this will be the only planted plane tree having the same representation with respect to both codings.

To every planted plane tree T with k vertices we assign a binary code, called its <u>ppt-code</u>, of length 2(k-1) and consisting of k ones and k-2 zeros. As before, it is obtained by traveling in the plane around T as indicated in Figure 4.11. The code is written by recording a string of $d(v) - 1$ zeros followed by a one the first time each vertex v of T is encountered during the trip. That this procedure



1(01)(0001)(001)1111 ppt-code — 100111 plant code — 1001001111 ppt-code

Figure 4.11.

produces a code of length 2(k-1) follows from the fact that the sum of all valencies of T equals twice the number of edges which, in turn, is twice k-1.

Theorem 4.28. Let $c = [c_0, c_1, \ldots, c_{2k-3}]$ be a binary sequence consisting of k-2 zeros and k ones. Then c is the ppt-code for a unique planted plane tree T with k vertices iff $c_0 = c_{2k-3} = 1$ and the ones are not more numerous than the zeros in every segment of c of the form $[c_1, \ldots, c_j]$, $1 < j < 2k-3$.

Proof: The ppt-code for the planted plane tree with just two vertices is 11 and it is easily seen to be the only binary sequence of length 2 satisfying the conditions of the theorem. Suppose $k > 1$ and the theorem is true for all sequences of length 2(n-2), $1 < n < k$, which contain n-2 zeros and n ones.

Let T be a planted plane tree with k vertices and code c. Then removal of a terminal vertex v, other than the root, from T produces a planted plane tree T' with k-1 vertices. If e was the terminal edge of T with end v, then the code c' of T' can be obtained from c by deleting the 1 corresponding to v and by deleting a zero corresponding to the other end w of e. Since w must have been encountered before v in the trip defining c, the zero removed must have preceded the one removed. By the induction hypothesis, c' satisfied the conditions of the theorem and the preceding sentence implies that c does also.

Now let c be any binary sequence of length 2(k-1) consisting of k-2 zeros and k ones and satisfying the conditions of the theorem. Then zeros and ones are equinumerous in the segment $[c_1,\ldots,c_{2k-4}]$ and hence $c_{2k-4}= 1$. Let i, $k-2 \leq i < 2k-4$, be the greatest integer such that $c_i = 0$ and $c_{i+1}= c_{i+2}= 1$. Then the sequence $c' = [c_0,\ldots,c_{i-1}, c_{i+1},c_{i+3},\ldots,c_{2k-3}]$ of length 2k-4 satisfies the conditions of the theorem so, by the inductive hypothesis, there exists a unique planted plane tree T' with k-1 vertices and code c'. Let v be the vertex in T' corresponding to the digit $c_{i+1}= 1$ in c' and form a planted plane tree T by attaching a terminal edge e to v such that in a trip around T the terminal vertex of e would be encountered immediately following the first encounter with v. Then T is a planted plane tree with code c and its uniqueness follows immediately. Q.E.D.

From Theorems 4.18 and 4.28 we have

<u>Theorem 4.29</u>. For every natural number k there is a one-to-one correspondence between the plants of index k and the class of all planted plane trees with k+2 vertices.

In conclusion we mention the dual of this chapter's formulation of the four-color problem. If W is a Whitney tour of a cubic map G of index k, then W corresponds to a Hamiltonian circuit of G^*. The edges of H together with those in its interior form a triangulated $(k+2)$-gon which corresponds to the tour W and one of the plants it determines. Similarly the tour and the other plant determined by W correspond to the edges on and outside H. This correspondence between plants of index k and triangulations of a $(k+2)$-gon enables us to state

Theorem 4.30. The four-color conjecture is true iff for every pair of triangulated n-gons there exists a vertex coloring which properly 4-colors both simultaneously.

It is of interest to notice that, as is easily shown by mathematical induction, any triangulated n-gon is vertex 3-colorable. The four-color problem is thus the problem of finding, for each two triangulations, a coloring which is valid for both - using only one additional color.

CHAPTER V

THREE RELATED TOPICS

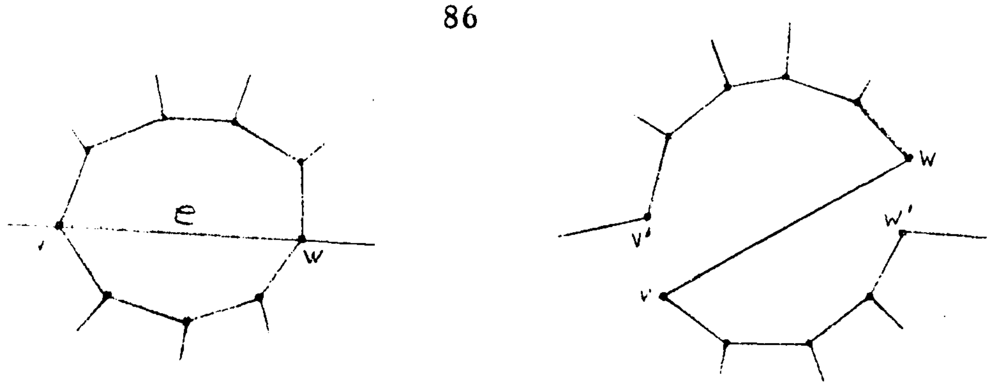
INSERTING CHORDS AND DIAGONALS IN MINIMAL MAPS

A minimal map cannot be modified by reducing the number of faces without making it colorable and, as shown in Chapter III, certain modifications with the same number of faces also render it colorable. This section presents some alterations which increase the number of faces and still lead to colorability.

Consider a face R in a minimal map M. By inserting a <u>diagonal</u> in R is meant the insertion of an edge lying interior to B(R) and joining any two vertices on its boundary. This operation increases by one the cardinality of F(M) but it destroys its regularity by introducing two vertices of degree four.

<u>Theorem 5.1</u>. A map obtained by inserting any diagonal in any face of a minimal map is colorable.

<u>Proof</u>: Let R be a face of a minimal map M and let v and w be two vertices on B(R). Denote by G the map M + e where e is a diagonal joining v and w. Figure 5.1(a) represents G = M + e and is intended to be general in that there may be any number (greater than two) of vertices besides v and w in B(R). If we "split" vertices v and w of G as shown in Figure 5.1(b) we obtain a 2-connected plane graph G'; for otherwise G and therefore M would contain a 3-ring. Clearly G' has one less face than M and is thus face 4-colorable. But then, after



(a) G = M+e (b) G'

Figure 5.1.

restoration of vertices v and w, we have a face 4-coloring of G. Q.E.D.

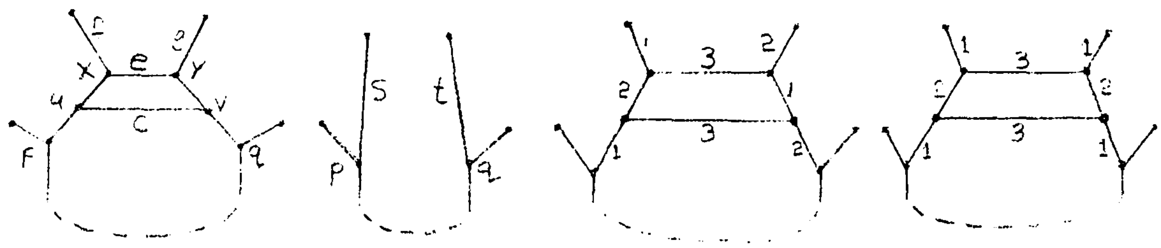
Again consider a face R of a minimal map M. By inserting a <u>chord</u> in R is meant the insertion of two vertices u and v on distinct boundary edges of R followed by insertion of an edge with ends u and v lying interior to B(R). This operation preserves the cubic nature of M and increases its index by one. If the valency of R is n, then the chord dissects B(R) into two paths joining u and v one of which contains, say, m of the original vertices in B(R) while the other contains n-m such vertices. The minimum of m and n-m will be called the <u>length</u> of the chord.

<u>Remark 5.2</u>. Insertion of any chord of length one in a cubic five-chromatic map yields a five-chromatic map and thus the existence of a minimal map implies the existence of five-chromatic maps of arbitrarily large index. This follows indirectly from the fact that in any edge 3-coloring of a cubic map containing a triangle, no two of the three edges incident with the triangle can be assigned the same color.

<u>Theorem 5.3</u>. Insertion of any chord of length two in a

minimal map yields a colorable map.

Proof: Suppose c is a chord of length two in a face R $\in$ F(M) with ends u and v and let x and y be the two vertices in B(R) which define its length; Figure 5.2(a). If the edge



(a) (b) (c) (d)

Figure 5.2.

e $\in$ E(M) joining x and y is removed from M and the vertices x and y are suppressed, then the resulting map M' is colorable; Figure 5.2(b).

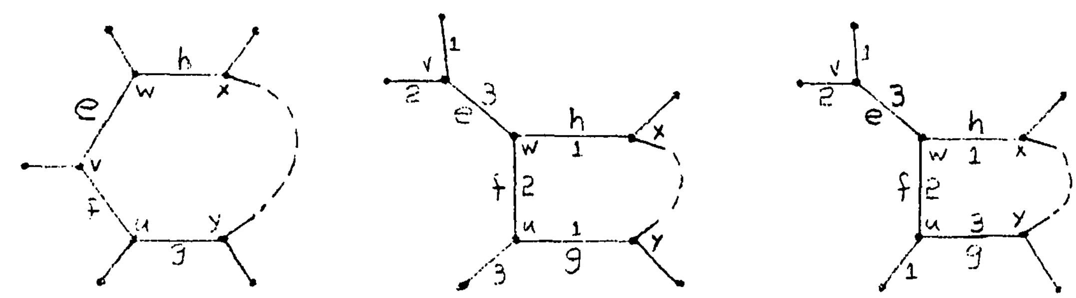
Case 1: Edges s and t of M' are colored differently. Then M+c can be edge 3-colored by assigning the color of s to f, (p,u), and (y,v), the color of t to g, (v,q), and (u,x), and assigning the third color to e and c; Figure 5.2(c).

Case 2: Edges s and t of M' are colored the same, say 1. Then M+c can be edge 3-colored by assigning color 1 to edges f, g, (p,u) and (q,v), color 2 to (u,x) and (v,y), and the third color to e and c; Figure 5.2(d). Q. E. D.

Theorem 5.4. Insertion of any chord of length three in a minimal map yields a colorable map.

Proof: Suppose R is a face of a minimal map M and let g and h be boundary edges of R for which a chord with ends on g and h is of length three; Figure 5.3(a). From Corollary

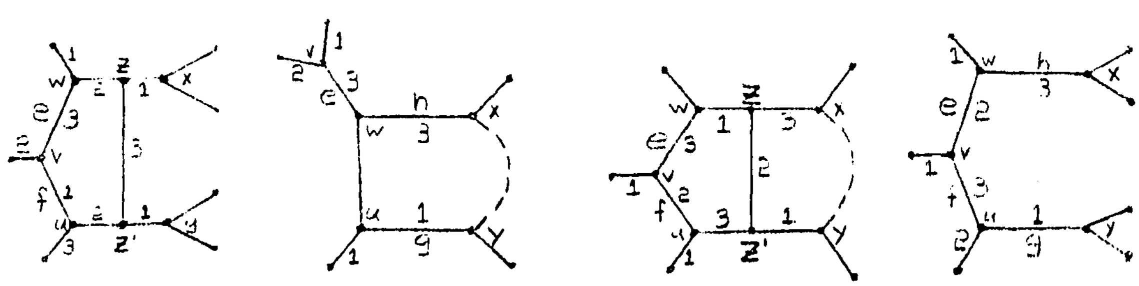
3.3(a) conjugation of e in M results in a map M_e which, without loss of generality, may be edge 3-colored with colors {1, 2,3} as shown in Figure 5.3(b) with edge g assigned color 1, or as shown in Figure 5.3(c) with edge g colored 3.



(a) (b) (c)

Figure 5.3.

In the first instance, M+c can be edge 3-colored by assigning the color of h to edges f, (z,x), and (z',y), assigning the color of e to e and c, and the third color to (w,z) and (u,z') with the remainder of M+c inheriting its coloration from that of M_e; Figure 5.4(a).



(a) (b) (c) (d)

Figure 5.4.

If, in the second instance, the (1,3)-circuit C passing through u does not pass through w, then by interchanging colors on C we can revert to the earlier case. On the other hand if C also passes through w, then it follows from the Jordan curve theorem that the arc C' of C from u to w containing y also contains x. By interchanging colors of C' we arrive at the pseudo-coloring shown in Figure 5.4(b) where vertices u and w are the only vertices of M_e not incident with edges of all three colors. There are two cases with respect to this pseudo-coloring.

<u>Case 1</u>. The (1,2)-path P_v through v does not pass through u. Then by interchanging colors on P_v we obtain another pseudo-coloring for M_e in which the only conflicts occur at vertices u and w. Then $M+c$ can be edge 3-colored by assigning the color of h to edges e, (z,x), and (z',u), assign the color of g to edges (w,z) and (z',y), and the third color to edges c and f; Figure 5.4(c).

<u>Case 2</u>. The (1,2)-path P_v through v passes through u. Then it follows from the Jordan curve theorem and the fact that the only conflicts occur at u and w that there is an arc P_v' of P_v joining v to u which does not contain y. As shown in Figure 5.4(d), by interchanging colors on P_v' we can edge 3-color map M by assigning the color of h to h and f, assigning the color of (u,w) to e, and the third color to g; a contradiction. Q.E.D.

Calling a chord <u>trivial</u> iff it is of length one, we can state

<u>Corollary 5.4(a)</u>. Insertion of a nontrivial chord in any n-gon, $n=5,6,7$, of a minimal map renders it colorable.

To conclude this section we make

<u>Conjecture 5.5</u>. Insertion of a nontrivial chord in any

face of a minimal map renders it colorable.

UNIQUE COLORABILITY OF MAPS

Let G be a plane graph and recall that any face n-coloring of G partitions F(G) into n color classes. If every pair of face n-colorings of G induce the same partition of F(G), then G is said to be uniquely face n-colorable and in the case n = 4 we simply say G is uniquely face colorable. Motivation for the material of this section was provided by the contents of a very recent paper by G. Chartrand and D. Geller [10]. Translating their results on vertex colorings yields:

Theorem 5.6. Every uniquely face colorable map is cubic.

Theorem 5.7. No map is uniquely face 5-colorable.

Definition 5.8. A cubic map G is said to be uniquely edge colorable iff it is edge 3-colorable and every such coloring induces the same partition of E(G).

As expected, the next theorem follows from the unique constructibility of table * on page 24.

Theorem 5.9. A cubic map G is uniquely face colorable iff it is uniquely edge colorable.

Proof: Soppose G is uniquely edge colorable but not uniquely face colorable. Let f_i: F(G) —— {a,b,c,d}, i=1,2, be face colorings of G which induce different partitions of F(G) and let e_i, i=1,2, be the corresponding edge colorings of G as determined by table *. Since G is uniquely edge colorable, there exists a permutation $\emptyset$ of $\{1,2,3\}$ such that $e_1 = \emptyset \circ e_2$. Hence there exists a permutation $\emptyset'$ of $\{a,b,c,d\}$ (which can be computed from knowledge of $\emptyset$ and table *) such that $f_1 = \emptyset' \circ f_2$ contradicting the fact that f_1 and f_2 are distinct colorings of G. Proof of the converse is similar. Q.E.D.

In view of Theorems 5.6 and 5.9, we simply say that a uniquely face colorable map is <u>uniquely colorable</u>.

<u>Theorem 5.10</u>. If a cubic map G is uniquely colorable, then for any edge 3-coloring of G the subgraph obtained by removing the edges of any one color class is connected.

<u>Proof</u>: Consider an edge 3-coloring of G and suppose there exist two color classes E_1 and E_2 such that the subgraph H obtained by deleting the edges in the remaining color class is disconnected. If C_1 and C_2 are two components of H, each of them must contain edges of both E_1 and E_2 and thus a different edge 3-coloration of G can be produced by interchanging the colors of the edges in $E_1 \cap C_1$ with those in $E_2 \cap C_1$. Hence G is not uniquely edge colorable, a contradiction. Q.E.D.

<u>Corollary 5.10(a)</u>. If a cubic map G is uniquely colorable, then G possesses exactly three Hamiltonian circuits.

<u>Proof</u>: An immediate consequence of the theorem is that G has at least three Hamiltonian circuits, viz., those obtained by deleting each of the three color classes. Suppose a fourth Hamiltonian circuit H exists. Since G is cubic, H is of even length and a different coloring can be obtained by alternately coloring the edges of H with two of the colors and assigning

the third color to the edges of G not in H -- contradicting the unique colorability of G. Q.E.D.

We shall be interested in utilizing the following theorem which was the primary result presented by Tutte in [44].

<u>Theorem 5.11</u>. If G is a cubic Hamiltonian map, then the number of its Hamiltonian circuits which contain any given edge is even.

<u>Proof</u>: Since G is Hamiltonian it is colorable. Let e be an arbitrary edge of G. If $\emptyset$: $E(G) \longrightarrow \{1,2,3\}$ is an edge 3-coloring of G, denote the subgraphs resulting from removal of the edges in color class i by K_i, $i = 1,2,3$, and define $k_i(e)$ to be 1 or 0 according as $e \in K_i$ or $e \notin K_i$. Then

$$k_1(e) + k_2(e) + k_3(e) = 0 \pmod 2). \qquad (*)$$

If $n(K_i)$ is the non-negative integer such that there are $n(K_i)$ +1 components of K_i, then there exist $2^{n(K_i)}$ edge 3-colorings of G which induce distinct partitions of E(G) with the property that removal of the edges of a suitably chosen color class yields K_i.

Hence if we delete subscripts for brevity and sum (*) over all edge 3-colorings of G we obtain

$$\sum_K 2^{n(K)} k(e) = 0 \pmod 2).$$

The Hamiltonian circuits of G correspond to the case $n(K) = 0$ and $2^{n(K)} = 1$ and hence the number of Hamiltonian circuits which contain e is even. Q.E.D.

<u>Corollary 5.11(a)</u>. A cubic Hamiltonian map possesses at

least three Hamiltonian circuits.

In order to characterize the uniquely colorable maps as those which possess exactly three Hamiltonian circuits one needs to prove the converse of Corollary 5.10(a). An intensive effort to do this was partially successful as indicated by the following results:

Theorem 5.12. If a cubic map G possesses exactly three Hamiltonian circuits H_1, H_2, and H_3 and if $\emptyset_i\colon E(G) \longrightarrow \{1,2,3\}$, $i = 1,2,3$, is the natural edge 3-coloring of G associated with each, then $\emptyset_1$, $\emptyset_2$, $\emptyset_3$ induce the same partition of E(G).

Proof: From Theorem 5.11 it follows that every edge of G lies on exactly two of the three Hamiltonian circuits. Hence every pair of adjacent edges lies on exactly one of the circuits H_1, H_2, or H_3.

If we assign to each edge of G the label 1, 2, or 3 according as it lies on H_2 and H_3, H_1 and H_3, or H_1 and H_2, then we see from the above paragraph that this is an edge 3-coloring of G. In fact, the edges on H_1 are alternately labelled 2 and 3, those on H_2 are alternately labelled 1 and 3, and those on H_3 are alternately labelled 1 and 2. Thus the colorings $\emptyset_1$, $\emptyset_2$, and $\emptyset_3$ induce the same partition of E(G). Q.E.D.

Let X be the family of all cubic plane graphs that can be obtained from the cubic plane graph of index 1 by successive replacing of vertices by triangles; see Figure 5.5. Using an inductive argument it can easily be shown that every map in X possesses exactly three Hamiltonian circuits,

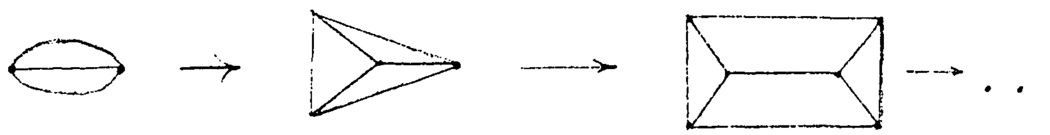


Figure 5.5.

for replacing a vertex v of a cubic map G by a triangle leaves the number of Hamiltonian circuits unaltered. It is unknown, however, if X contains all cubic maps with exactly three Hamiltonian circuits; in fact, the classification of those cubic maps possessing exactly n Hamiltonian circuits for any $n \neq 1,2$, is still an unsolved problem . . graph theory, [7]. The following theorem provides a converse to Corollary 5.10(a) for all known cubic maps with exactly three Hamiltonian circuits as well as showing that there exist uniquely colorable maps of arbitrarily large index.

<u>Theorem 5.13</u>. Every cubic map in X is uniquely colorable.

<u>Proof</u>: It is clearly true for the map in X of index 2. Assume the theorem to be true for every map in Z of index k-1 and suppose G is a map in X of index k. Since G is in X, it is colorable and contains at least one triangle. If G is not uniquely colorable, contraction of a triangle to a vertex produces a map in X of index k-1 which is also not uniquely colorable; a contradiction of the inductive hypotheses. Q.E.D.

We conjecture that the set of uniquely colorable maps is the set of cubic maps with exactly three Hamiltonian circuits. This would be consistent with

<u>Theorem 5.14</u>. The number of Hamiltonian circuits in a cubic map G is even iff the number of edge 3-colorings of G which

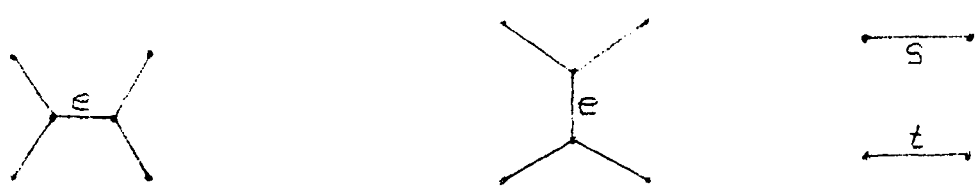
induce different partitions of E(G) is even.

<u>Proof</u>: Let c denote the number of edge 3-colorings of G which induce different partitions of E(G) and let h denote the number of Hamiltonian circuits in G. If I is a subgraph of G consisting of $i > 1$ disjoint circuits of even length whose union contains all vertices of G, then I corresponds to an even number (2^{i-1}) of distinct color partitions of E(G). Each Hamiltonian circuit, however, corresponds to exactly one color partition of E(G). Hence the parity of h is the same as that of c. Q.E.D.

To conclude this section we state

<u>Theorem 5.15</u>. If e is any edge of a minimal map M, then M_e is not uniquely colorable.

<u>Proof</u>: Let M' be the cubic map formed by removal of edge e from M_e. In any edge 3-coloration of M' edges s and t shown in Figure 5.6 must be assigned the same color, say 1; for



e ∈ M　　　　e ∈ M_e　　　　M'

Figure 5.6.

otherwise M would be colorable. For the same reason the (1,2)-circuit C_{12} containing s must contain t; as must the (1,3)-circuit C_{13} containing s.

If C_{12} and C_{13} are not both Hamiltonian circuits in M', then there exists an edge 3-coloring of M_e for which removal of the edges in a suitably chosen color class produces a disconnected subgraph of M_e and thus M_e is not uniquely colorable.

If C_{12} and C_{13} are both Hamiltonian circuits in M', then they can obviously be extended to Hamiltonian circuits of M_e not containing e. Then M_e cannot have exactly three Hamiltonian circuits; for if it did, e would not be contained in an even number of them contradicting Theorem 5.11. Hence, by Corollary 5.10(a), M_e is not uniquely colorable. Q.E.D.

ANALYSIS OF CIRCUITS

Although not all details will be given the purpose of this section is to report failure of an attempt to develop an analysis of circuits analogous to A. Bernhart's analysis of n-rings. In an unpublished manuscript Bernhart began a systematic enumeration of the possible configurations interior to a circuit of length n on a minimal map. Although an exhaustive analysis of circuits requires consideration of more circuit configurations than does that of rings, it was thought that one less color would accelerate computation for the individual circuit configurations.

Let G be a colorable cubic map and let $C = [v_0, v_1, \ldots, v_{n-1}, v_0]$ be a circuit of length n in G. Incident to each vertex v_i of C there is an edge s_i of G not on C called the <u>spoke</u> of C at v_i, $0 \leq i < n$. If s_i lies interior to C, it is called an <u>interior spoke</u>; otherwise it is called an <u>exterior spoke</u>. For an edge 3-coloring of G, let a, b, and c be the number of spokes of C which have been assigned color 1, 2, or 3, respectively, and let a_i, b_i, c_i and a_o, b_o, c_o be similarly defined for the interior and exterior spokes of C. Analogous to Theorem 4.15 we have

Theorem 5.16.

(i) $a_i \equiv b_i \equiv c_i \pmod 2$,

(ii) $a_o \equiv b_o \equiv c_o \pmod 2$,

(iii) $a \equiv b \equiv c \equiv n \pmod 2$.

Proof: Clearly $a_i + b_i$ must be even; for otherwise, there exists a (1,2)-path with an interior spoke of C as origin which terminates interior to C, a contradiction. Similarly $a_i + c_i$ and $b_i + c_i$ are even and hence (i). The proof of (ii) is the same and (i) and (ii) imply (iii). Q.E.D.

Given a face 4-coloring of an n-ring R and its interior faces, the success of ring analysis is triggered by the ability to insure essentially different colorings for the same configuration. This is achieved by interchanging color assignments in regions determined by certain "chains" of faces lying exterior to R with origin and terminus on R. A corresponding statement cannot be made for circuits, however, as indicated in Figure 5.7. Here the (1,3)-circuit contains all the exterior spokes of C.

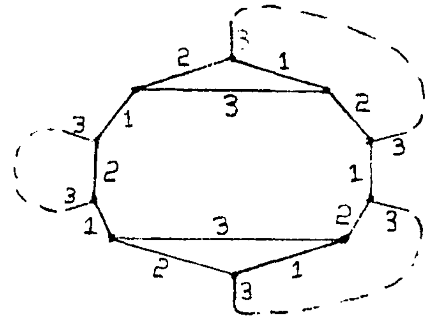


Figure 5.7.

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