FINITE PERIOD INVENTORY MODELS

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PREFACE

This dissertation is based on the assumption that certain items are kept as inventory for a finite period of time. The primary objective of this dissertation is to develop and present the application of dynamic programming for determining optimal decision rules for the finite period inventory problem in which the parameters involved may vary from period-to-period. Assumptions in this investigation are described as follows. The procurement lead time may be probabilistic or deterministic and may vary from period-to-period depending upon the period when the order is made, and the source of supply. The procurement system may involve several suppliers, each with different characteristics. Several types of items may be kept as inventory in a warehouse which has a limited space. Seasonal variations may affect the quantity available from each supplier. Demands, which may be deterministic or probabilistic, may vary over the study periods. Costs associated in this investigation may also vary from period-to-period.

At the beginning of each period, the optimal amount for each type of item to be ordered can be determined based upon a minimum expected total system cost for all remaining periods. For the case where the orders are instantly fulfilled or the case where both demands and procurement lead time are deterministic, the decision can be made based on inventory on hand at the time of making the decision. Otherwise, the decision is made based on the inventory on hand plus the outstanding order at that period, assuming that the demand is backlogged.

Chapter II discusses deterministic demand and deterministic procurement lead time systems. The case where demands are probabilistic and the item is immediately fulfilled is considered in Chapter III. Chapter IV presents the problem with probabilistic demands and deterministic lead time. The investigation is extended to the case in which procurement lead time is probabilistic for probabilistic demands in Chapter V.

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NOMENCLATURE

English Symbols

Ch_{ik} = holding cost per unit per period of item type No. i in period k.

Ci_{ik} = item cost per unit of item type No. i, when the order is made in period k.

$\tilde{C}i_{ik}$ = item cost per unit of item type No. i, where the order made previously will arrive at period k.

Co_{ik} = fixed ordering cost when the order for item type No. i is made at period k.

$\tilde{C}o_{ik}$ = fixed ordering cost when the order for item type No. i is made previously, and the order will arrive at period k.

Cs_{ik} = shortage cost per unit shortage per period for the shortage of item type No. i in period k.

L_k = procurement lead time when the order is made in period k.

$\underline{L}_k$ = minimum procurement lead time when the order is made in period k.

N = number of type of items in the system.

P = number of planning periods.

$P_{ik}(r_{ik})$ = probability that demand for item type No. i in period k will be r_{ik}.

$P(r_i:K,k)$ = probability that sum of demand for item type No. i from period K to period k will be r_i.

$\bar{P}_k(L)$ = probability that procurement lead time for the order made in period k will be L.

$\bar{\bar{P}}_k(L)$ = probability that the order made in period k will arrive after the order made in period k+1 by L periods.

r_{ik} = demand of item type No. i in the period k.

$\underline{r}_{ik}$ = minimum demand of item type No. i in the period k.

S_{ik} = available supply of item type No. i for the order made in period k.

$\tilde{S}_{ik}$ = available supply of item type i for the order made previously that will arrive in period k.

U_{ik} = inventory on hand plus the outstanding order of item type No. i at the beginning of period k.

v_i = a volume of an item type No. i.

W = warehouse space.

w_i = warehouse space available for the addition of item types No. 1 to No. i.

X_{ik} = inventory level of item type No. i at the beginning of period k.

Z_{ik} = amount of item type No. i to be ordered in the period k.

$\tilde{Z}_{ik}$ = amount of item type No. i ordered previously that will arrive in the period k.

Greek Symbols

$\varphi_{ik}(Z_{ik})$ = item cost plus fixed cost of ordering Z_{ik} in period k

$= Co_{ik} + Ci_{ik} \cdot Z_{ik}$

$\tilde{\varphi}_{ik}(\tilde{Z}_{ik})$ = item cost plus fixed cost of ordering $\tilde{Z}_{ik}$

$= \tilde{C}o_{ik} + \tilde{C}_{ik} \cdot \tilde{Z}_{ik}$.

Section Number

The first digit of the section number indicates the chapter number, the last digit corresponds to the number of section in the chapter.

Equation Number

The first digit of the equation number indicates the chapter number, the second digit corresponds to the number of section in the chapter, and the last number indicates the number of equation in the section.

CHAPTER I

INTRODUCTION

Many inventory models have been developed under a steady-state condition, where the parameters are assumed to be unchanged over an infinite period. These models elaborate on the static inventory problem. The decision criteria are based upon the minimization of a total system cost, relying on expected value in the long run. Optimal decision rules usually can be determined in a simple formula, such as the square root formula, often called the "Wilson formula".

Such work mentioned above is discussed in many texts which usually consider only the Single-Item Single-Source problem in which several assumptions are utilized. Banks (5) presents a solution to a static problem in which several types of items are stocked in limited warehouse space and in which several sources of supply are available. It can be said that inventory theory involving static problems has been nearly fully developed. However, not too many results have been obtained for the dynamic inventory problem.

An inventory problem is considered as "dynamic" when parameters change from period-to-period, or when the time-value of money, usually called the "discounted cost", is involved in the problem. More complicated situations exist when inventory is considered to be kept in only a given finite period in which decisions cannot be based upon the minimization of costs in a long run.

The finite period dynamic inventory problem is most likely to be ound in job shop situations. Here, a certain quantity of items may be inufactured or purchased and are retained as inventory in order to itisfy demand during a finite period of time. After a given period of .me, those items remaining in inventory may be considered valueless .nce orders for that particular job are not likely to be received in ie near future.

One of the pioneering works in dynamic inventory theory is by 'row, Harris, and Marschak (1). Models in which there is a discount ıst are considered. It is assumed that ordering cost includes fixed 'dering cost and linear item cost; holding cost is linear; and penalty st due to shortage is considered as a constant when demand exceeds e stock available. Optimal policy is based on the expected total scounted cost in the long run by assuming an infinite time period. eir results indicate that the optimal policy can be so defined that inventory on hand, x, is less than or equal to a given quantity, s, der S-x; otherwise do not order.

Bowman and Fetter (8) have introduced an application of linear ogramming to the simple dynamic inventory problem where inventory rrying charge and production costs are to be minimized for a firm cing a seasonal demand pattern. In the model, demands are considered terministic and lead time is assumed to be zero. No stockouts are lowed, and production costs are assumed to be linear without a set up st.

Wagner and Whitin (17) have presented an algorithm for solving a namic inventory problem which considers deterministic demands for a ngle item with assumptions that shortage cost is infinite and item cost per unit in N periods is constant. Allowing linear holding charges and set up costs to vary over N periods, they show that the optimal ordering policy is to allow stock to fall to zero in period of order. Their results indicate the possibility of eliminating the necessity of having data for the full N periods.

For a one-stage inventory model, Karlin (9) found that when the sum of expected holding cost and shortage cost is convex, increasing and vanishing at zero, a simple decision rule can be determined. For the model with linear order cost function assuming no set up cost, the optimal decision for a given inventory on hand, x, is given by y_0 so that if x is less than y_0, order up to y_0; otherwise do not order. For a model with assumed linear item cost and with a set up cost, the optimal decision is given by S,s so that if x is less than or equal to s, order up to S; otherwise do not order.

Karlin (10) discusses the case where the demands that arise in successive periods are independent and identically known distributions of demand occur in each period. Assuming ordering costs to be linear, holding costs and shortage costs to be convex, and there is a discount cost, if the marginal expected penalty exceeds the marginal cost of ordering the optimal policy for the infinite time horizontal is characterized by a single critical number, $\bar{\bar{x}}$: if x is less than $\bar{\bar{x}}$, order up to $\bar{\bar{x}}$; otherwise do not order. When the model includes set up cost and assumes linearity in holding costs and shortage costs providing demand distribution is a Pólya frequency function, the optimal policy for the period k is characterized by s_k, S_k so that if x_k is less than s_k, order up to S_k; otherwise do not order.

Scarf (14) considered Karlin's work and found that when the holding cost and shortage cost are convex, the optimal policy will always be of the S, s type without any additional conditions such as are required by Karlin. The policy in period k can be defined by S_k, s_k so that if x, inventory on hand, is less than s_k, order up to S_k; make no order otherwise.

An extended version of the classical dynamic inventory model with emphasis on the varying nature of the demand distribution has been considered by Karlin (11). The demand in each period is assumed to be an observation of a random variable with a known distribution function. These random variables are postulated to be independent but not necessarily identically distributed from period-to-period. Under the assumption that the purchase cost is linear and other cost functions are convex, it is proved that the optimal policy possesses a simple form such that in each period whether or not to place an order is determined by comparing the stock level with a single critical number. This critical number may vary in successive periods. A similar result can also be obtained for the backlogged problem with constant procurement lead time.

Iglehart and Karlin (16) have considered a dynamic inventory model with stochastic demands in which the distributions of demand in successive periods are not identical, but, in general, are correlated. It is assumed that at each period there is a finite number of demand states i = 1, 2, ..., k, and for each demand state there is a density function $D_i(x)$ such that the demand state in a given period indicates which demand density holds in that period. The demand state can change from period-to-period, obeying a Markov transition law. Assuming a linear purchasing cost, and that holding cost and shortage cost are convex-increasing and vanishing at the origin, the optimal policy is characterized by k critical numbers $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ corresponding to demand densities $D_1(x), D_2(x), \ldots, D_k(x)$ so that for a known demand state, i, at the beginning of decision period if inventory on hand, x, is less than $\bar{x}_i$, order up to $\bar{x}_i$; do not order otherwise.

Karlin and Scarf (12) investigate the constant time lag problem in a dynamic inventory model. It is proved that when a backogged condition is assumed, an optimal decision can be based upon the sum of inventory on hand and the outstanding orders at the time the decision is made. Furthermore, if shortage cost and holding cost are convex increasing and ordering cost is linear, the optimal policy for a backlogged problem can be characterized by a critical value $\bar{x}$ so that if the sum of inventory on hand and the outstanding orders, U, is less than $\bar{x}$, order $U - \bar{x}$; otherwise do not order. For the case where demand exceeds the available is considered as loss of sale, if all the cost functions are linear and there is a one period lag in delivery, the optimal policy $Z^*(x)$ has the following property: $Z^*(x) > 0$, if x is less than $\bar{x}$; otherwise $Z^*(x) = 0$.

Scarf (13) extended the single-item, single-source model to a stochastic procurement lead time problem in which excess demands are backlogged. With the assumption that at most one outstanding order is permitted, the recursive model can be simplified. An optimal ordering policy at each period would then be based on amount of stock on hand and at most one outstanding order to be delivered at some specific time in the future.

Iglehart (14) has considered a problem of a firm which produces a single commodity and which must make a production decision and a capital decision at periodic intervals of time. The firm is assumed to have the necessary capital required to produce a product which is to be held in inventory. The cost of product is assumed to be convex and is a function of both capital and the quantity to be produced at each period. Holding cost and shortage cost are assumed to be convex. Demands and capital depreciation are distributed independently from period-to-period. The optimal production and capital decision were obtained for an N period problem.

The prior works discussed above have generally involved a single-item, single-source dynamic inventory problem. The emphasis in most of the works is in defining the qualitative characteristics of optimal policy for particular assumptions. It may be pointed out that the characteristics are restricted under the particular assumptions and may not be easily determined quantitatively. However, these characteristics give an indication of the optimal point for each period which may be useful in reducing the amount of calculation. Some work has been extended to a probabilistic procurement lead time problem under the restriction that it is not possible to order whenever there is an outstanding order at the time of making a decision.

The contribution of this investigation may be considered in several ways. The case where lead times are probabilistic is considered and the procedure for determining optimal policy is presented under the assumption that an order can be made any time regardless of whether there is or is not an outstanding order. Algorithms presented for determining optimal policy are not restricted to the single-item, single-source problem. Throughout the investigation the problem in which there is a warehouse restriction is considered.

Progress in inventory theory is shown in Figure 1. In the figure, previous works and contributions of this investigation as well as works which have not been done are summarized in a simple fashion.

For the single-item, single-source system where procurement lead time is deterministic, the analysis presented here is likely to be the same as in prior works with the exception that a warehouse restriction is included. It is the purpose of this part to formulate the basic concepts necessary to an understanding of the more complicated problems. It should be noticed that the algorithms presented here are given quantitatively in general, and no attempt has been made to give qualitative characteristics of the optimal policies.

Dynamic programming is very useful when one is involved in a multi-stage decision process, as in the problems in this investigation. Dynamic programming, as pointed out by Bellman and Dreyfus (6), is based on the Principle of Optimality, which states that an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. Although dynamic programming and the principle of optimality will be used throughout this dissertation, it will not be discussed here since it is available in many texts. It should be noted that throughout this thesis the word "period k" means the period where there are k periods remaining.

The recurrence relation employed in the primary solution can be defined in a simple manner as follows:

$$X_{ik} = X_{i,k+1} + \widetilde{Z}_{ik} - r_{i,k+1},$$

Inventory Theory

Static

- Single-Item Single-Source — Several Texts
- Multi-Item Multi-Source — (5)

Dynamic

- Infinite Period with Discounted Cost — (1)

		Instantly Fulfill		Deterministic Lead Time		Infinite Period Lead Time	
		Deterministic Demand	Probabilistic Demand	Deterministic Demand	Probabilistic Demand	Deterministic Demand	Probabilistic Demand
Single-Item Single-Source	A	(7), (8), (17) Sec 2.1*	(7) Sec 3.1*	(10) Sec 2.1*	(12) Sec 4.1	(13)** Sec 5.1	(13)** Sec 5.1
	B	(10), (14) (11), (16)	(10), (14) (11), (16)	(12)	(11), (12)	None	None
Multi-Item Single-Source	A	Sec 2.2 and 2.3	Sec 3.2 and 3.3	Sec 2.2 and 2.3	Sec 4.2 and 4.3	Sec 5.2 and 5.3	Sec 5.2 and 5.3
	B	None	None	None	None	None	None
Single-Item Multi-Source	A	Sec 2.4	Sec 3.4	Sec 2.4	Sec 4.4	Sec 5.4	Sec 5.4
	B	None	None	None	None	None	None
Multi-Item Multi-Source	A	Sec 2.5 and 2.6	Sec 3.5 and 3.6	Sec 2.5 and 2.6	Sec 4.5	Sec 5.5	Sec 5.5
	B	None	None	None	None	None	None

A - Programming Algorithm

B - Characteristic Analysis

* - with warehouse restriction

** - only one outstanding order is allowed

Figure 1. Summary of Development in Inventory Theory

where X_{ik} is an inventory level at period k for item i, Z_{ik} is an amount to be arrived at period k, and r_{ik} is the demand in period k.

When U_{ik}, inventory on hand plus outstanding orders at period k, is to be used as the criterion for making a decision, the recurrence relation can be defined as follows:

$$U_{ik} = U_{i,k+1} + Z_{ik} - r_{i,k+1}$$

where Z_{ik} is an amount to be ordered at period k.

Four classifications of cost elements are considered in this thesis: a fixed ordering cost, to be charged when an order is made; an item cost depending on the number of items purchased or produced; a carrying cost depending on the stock on hand; and a shortage cost proportional to the excess of demand over available stock during the period. These four cost elements may vary from period-to-period. Throughout this thesis it is assumed that both fixed ordering cost and item cost are the costs at the period when the order is made. It is also assumed that carrying cost is proportional to the size of the stock of inventory at the beginning of each period.

Both deterministic and probabilistic demands are considered. For a probabilistic case, it is assumed that the demand distributions of each period are independent and not necessarily identical.

Usually when demand exceeds the available supply, two possibilities are considered. First, the excess demand is deferred to a later period and stock level is assumed to be either negative or positive. The second possibility is to consider excess demands as a loss or the extra supply can be immediately obtained from some other source with a penalty cost. In the latter, the stock level in the system always will be positive. Throughout this thesis the former condition, which is usually referred to as the "backlogged problem", is assumed.

Limited warehouse space may be considered as one of the serious limitations in practical inventory problems. Optimization methods for the case in which a warehouse restriction exists differ from those in cases in which no restriction is applied, especially for the cases where several types of items are considered. In this regard the analysis of particular inventory models is restricted under the assumption that there is no chance of inventory exceeding a warehouse restriction.

Availability of supply from each source in each period is considered here. This gives a restriction that the amount to be ordered at each period cannot be greater than the availability.

Procurement lead time can be considered as deterministic or probabilistic. In deterministic cases, it may be considered as constant throughout the planning period for a simple problem. However, in some practical problems, procurement lead time for each order made at each period may be considered deterministic but not necessarily constant.

In a probabilistic procurement lead time case, several assumptions may be assumed. This investigation will rely on the assumption that procurement lead time for an order being made in any period is independent of other orders regardless of whether ordering at other periods will be made or not. It is also assumed that the difference of arrival time for each two successive ordering period is distributed independently from arrival time. The important assumption in the investigation is that an order made at each period from a particular source will not arrive before those orders made previously from the source. For the probabilistic case, this means that if a probability for procurement lead time L_{k+1} at an immediately previous period is $\bar{P}_{k+1}(L_{k+1})$, a probability $\bar{P}_k(L_k)$ of the lead time when the order is made in period k being L_k will be less than or equal to $\sum_{L'=0}^{L_{k+1}} \bar{P}_{k+1}(L')$.

Let $\bar{P}_{k+1}(L_1)$ be a probability that lead time for the order made in period k+1 will be L_1, and $\bar{\bar{P}}_k(L_2)$ be a probability that the difference of lead time for the orders made in period k and k+1 will be L_2. Assume that $\bar{P}_{k+1}(L_1)$ and $\bar{\bar{P}}_k(L_2)$ are known. Then the probability that lead time for the order made in period k will be L, $\bar{P}_k(L)$, will be the sum of the joint probability of lead time in period k+1, L_1, and the difference of lead time, L_2, so that $L_1 + L_2 = L + 1$.

$$\text{Then, } \bar{P}_k(L) = \bar{P}_{k+1}(L+1) \cdot \bar{\bar{P}}_k(0) + \bar{P}_{k+1}(L) \cdot \bar{\bar{P}}_k(1) + \dots$$

$$+ \bar{P}_{k+1}(0) \cdot \bar{\bar{P}}_k(L+1)$$

$$= \sum_{L'=0}^{L+1} \bar{P}_{k+1}(L') \cdot \bar{\bar{P}}_k(L+1-L') \leq \sum_{L'=0}^{L+1} \bar{P}_{k+1}(L').$$

Thus, if $\bar{P}_P(L)$ and $\bar{\bar{P}}_k(L_2)$, for $1 \leq k \leq P-1$, are known, $\bar{P}_k(L)$ for all values of k can be determined.

CHAPTER II

DETERMINISTIC DEMAND AND DETERMINISTIC PROCUREMENT LEAD TIME SYSTEM

This chapter considers the deterministic demand and deterministic procurement lead time problem. Other general assumptions are as previously described in Chapter I. Deterministic lead time considered in this chapter includes either the case where an order is instantly fulfilled or where there is a finite value for lead time at each period.

At the beginning of each period, the inventory on hand can be determined and can be used as a basis for determining the optimal decision. By comparing the total system cost for the whole remaining periods for different amounts to be ordered, the optimal ordering policy can be determined. If a particular amount is ordered, the total system cost for any period will be the sum of an item cost, a fixed ordering cost, a shortage cost and carrying cost for the period, and the minimum total system cost in the next period, presuming the optimal decision will be made at that period.

Let Z_{ik} be an amount ordered which will arrive at the beginning of period k. Since demands and arrival times can be determined in advance, the analysis for both the immediately fulfilled and deterministic lead time cases will be the same. By letting lead time equal zero, the latter case will be reduced to the former case.

Let Z_{ik} be an amount ordered to arrive at the beginning of period

k. The order is made at the beginning of period k for the immediately fulfilled case, but for the case where there is a procurement lead time, L, the order for $\tilde{Z}_{ik}$ is made L periods in advance. Note also that the ordering cost, which is the sum of item cost and fixed ordering cost, is a cost incurred at the beginning of the period of ordering. Thus, letting $\tilde{C}o_{ik}$ and $\tilde{C}_{ik}$ be fixed ordering cost and item cost for the order which arrives at period k, the following relations are obtained:

$$\varphi_{ik}(Z_{ik}) = Co_{ik} + Ci_{ik} \cdot Z_{ik}$$

$$\tilde{\varphi}_{ik}(\tilde{Z}_{ik}) = \tilde{C}o_{ik} + \tilde{C}i_{ik} \cdot \tilde{Z}_{ik},$$

and

$$\tilde{\varphi}_{ik}(Z_{ik}) = \tilde{\varphi}_{i,k+Lk}(\tilde{Z}_{i,k+Lk}),$$

where

$$Co_{ik} = \tilde{C}o_{i,k+Lk}$$

$$Ci_{ik} = \tilde{C}i_{i,k+Lk},$$

and

$$S_{ik} = \tilde{S}_{i,k+Lk}.$$

Considering lead time as deterministic but not constant, at period k, there is some possibility that orders made previously at different periods will arrive at the same time as at this period; $\tilde{\varphi}_{ik}(\tilde{Z}_{ik})$ can be determined by employing the following dynamic programming technique so that:

$$\tilde{\varphi}_{ik}(\tilde{Z}_{ik}) = \text{Min } \varphi_{i,k+Lj}(Z_{i,k+Lj})$$

subject to

$$\sum_{j} Z_{i,k+Lj} = \tilde{Z}_{i,k},$$

where

$$j - L_j = k, \text{ for all } j.$$

The maximum available supply at period k, $\tilde{S}_{ik}$, is the sum of supply available in those previous periods such that, if ordered, will arrive at period k, or

$$\widetilde{S}_{ik} = \sum_{j} S_{i,k+L_j},$$

where $j - L_j = k$, for all j.

2.1 SINGLE-ITEM SINGLE-SOURCE SYSTEM

This section considers the case in which only one type of item and only one source of supply are available. The analysis is as follows:

Consider period 1, and for a given X_{11}, assume that an amount $\mathcal{Z}_{11}$ is ordered for this period. The decision made for this period affects the system cost only in this first period.

The total system cost is the sum of:

(1) item cost plus fixed cost of ordering $\mathcal{Z}_{11}$, which is $\mathcal{C}_{11}(\mathcal{Z}_{11})$,

(2) shortage cost in the period 1, which is $Cs_{11} \cdot \text{Max}(r_{11} - X_{11} - Z_{11}, 0)$, and

(3) carrying cost in period 1, which is $Ch_{11} \cdot \text{Max}(X_{11} + \mathcal{Z}_{11}, 0)$.

Thus, the total controllable system cost, $\mathcal{C}(X_{11}; Z_{11})$

$$= \mathcal{C}_{11}(\mathcal{Z}_{11}) + Cs_{11} \cdot \text{Max}(r_{11} - X_{11} - Z_{11}, 0) + Ch_{11} \cdot \text{Max}(X_{11} + \mathcal{Z}_{11}, 0). \quad (2\text{-}1\text{-}1)$$

Let $f^*_1(X_{11})$ be the minimum controllable system cost for period 1, resulting from ordering an optimal amount $\mathcal{Z}_{11} = \mathcal{Z}_{11}(X_{11})$ for a given X_{11}. Therefore,

$$f^*_1(X_{11}) = \underset{\mathcal{Z}_{11}}{\text{Min}} \left\{ \mathcal{C}(X_{11}; \mathcal{Z}_{11}) \right\}, \quad (2\text{-}1\text{-}2)$$

where

$$0 \leq \mathcal{Z}_{11} \leq \text{Min}\left\{ \widetilde{S}_{11}, \frac{W}{V_1} - X_{11} \right\}. \quad (2\text{-}1\text{-}3)$$

Consider period 2 and for a given value X_{12}, assume that an amount $\mathcal{Z}_{12}$ is ordered for this period. The decision made for this period affects the total system cost for periods 2 and 1.

The total system cost is the sum of:

(1) item cost plus fixed cost of ordering $\mathcal{Z}_{12}$, which is $\Phi_{12}(\mathcal{Z}_{12})$,

(2) shortage cost in period 2, which is $Cs_{12} \cdot \mathrm{Max}(r_{12} - X_{12} - \mathcal{Z}_{12}, 0)$,

(3) carrying cost in period 2, which is $Ch_{12} \cdot \mathrm{Max}(X_{12} + \mathcal{Z}_{12}$, and

(4) optimal controllable cost presuming an optimal decision is made for the period 1, which is $f^*_1(X_{12} + \mathcal{Z}_{12} - r_{12})$.

Thus, the total controllable system cost, $\mathcal{C}(X_{12};\mathcal{Z}_{12})$

$$= \Phi_{12}(\mathcal{Z}_{12}) + Cs_{12} \cdot \mathrm{Max}(r_{12} - X_{12} - \mathcal{Z}_{12}, 0)$$

$$+ Ch_{12} \cdot \mathrm{Max}(X_{12} + \mathcal{Z}_{12}, 0) + f^*_1(X_{12} + \mathcal{Z}_{12} - r_{12}). \quad (2\text{-}1\text{-}4)$$

Let $f^*_2(X_{12})$ be a minimum controllable system cost for period 2, resulting from ordering an optimal amount of $\mathcal{Z}_{12} = \mathcal{Z}^*_{12}(X_{12})$ for a given X_{12}. Therefore,

$$f^*_2(X_{12}) = \underset{\mathcal{Z}_{12}}{\mathrm{Min}} \left\{ \mathcal{C}(X_{12};\mathcal{Z}_{12}) \right\}, \quad (2\text{-}1\text{-}5)$$

where

$$0 \leq \mathcal{Z}_{12} \leq \mathrm{Min}\left\{ \mathcal{S}_{12}, \frac{W}{V_1} - X_{12} \right\}. \quad (2\text{-}1\text{-}6)$$

Next, consider in general period p, where $2 \leq p \leq P$.

$$\mathcal{C}(X_{1p};\mathcal{Z}_{1p}) = \Phi_{1p}(\mathcal{Z}_{1p}) + Cs_{1p} \cdot \mathrm{Max}(r_{1p} - X_{1p} - \mathcal{Z}_{1p}, 0)$$

$$+ Ch_{1p} \cdot \mathrm{Max}(X_{1p} + \mathcal{Z}_{1p}, 0) + f^*_{p-1}(X_{1p} + \mathcal{Z}_{1p} - r_{1p}). \quad (2\text{-}1\text{-}7)$$

It follows then that

$$f^*_p(X_{1p}) = \underset{Z_{1p}}{\text{Min}}\left\{\mathcal{C}(X_{1p};Z_{1p})\right\}, \tag{2-1-8}$$

where
$$0 \leq Z_{1p} \leq \text{Min}\left\{S_{1p}, \frac{W}{V_1} - X_{1p}\right\}. \tag{2-1-9}$$

Example

planning period, $P = 5$

warehouse space, $W = 5$ cubic units

volume of an item, $v_1 = 1$ cubic unit

procurement lead time, $L_k = 2$ (for all k)

initial inventory, $X_{15} = 5$ units

	k=1	k=2	k=3	k=4	k=5
S_{1k} - unit	4	5	5		
r_{1k} - unit	3	4	2	1*	2*
Co_{1k} - \$/order	0.00	0.00	0.00		
Ci_{1k} - \$/unit	0.50	0.60	0.50		
Cs_{1k} - \$/unit/period	2.00	2.00	1.50		
Ch_{1k} - \$/unit/period	1.00	0.90	1.00		

*difference between demand and arrival from order previous to planning period.

Solution:

Consider period 1. Note that the order made at period 3 arrives at this period. Using (2-1-1) to (2-1-3), for $X_{11} = 0$;

$f^*_1(0)$

$$= \underset{0 \leq Z_{11} \leq \text{Min}(4, \frac{5}{1}-0)}{\text{Min}}\left\{\Phi_{11}(Z_{11}) + (2)\text{Max}(3-0-Z_{11},0) + (1)\text{Max}(0+Z_{11},0)\right\}$$

$$= \text{Min}\begin{bmatrix} 0.0+6+0 \\ 0.5+4+1 \\ 1.0+2+2 \\ 1.5+0+3 \\ 2.0+0+4 \end{bmatrix} = \text{Min}\begin{bmatrix} 6.0 \\ 5.5 \\ 5.0 \\ 4.5 \\ 6.0 \end{bmatrix} = 4.5, \text{ where } Z_{11}(0) = 3.$$

For other values of X_{11}, using (2-1-1) to (2-1-3), $f^*_1(X_{11})$ and $Z^*_{11}(X_{11})$ can be determined. The results are summarized below:

$f^*_1(-4) = 8.0; Z^*_{11}(-4) = 4$ $f^*_1(1) = 4.0; Z^*_{11}(1) = 2$

$f^*_1(-3) = 7.0; Z^*_{11}(-3) = 4$ $f^*_1(2) = 3.5; Z^*_{11}(2) = 1$

$f^*_1(-2) = 6.0; Z^*_{11}(-2) = 4$ $f^*_1(3) = 3.0; Z^*_{11}(3) = 0$

$f^*_1(-1) = 5.0; Z^*_{11}(-1) = 4$ $f^*_1(4) = 4.0; Z^*_{11}(4) = 0$

$f^*_1(0) = 4.5; Z^*_{11}(0) = 3$ $f^*_1(5) = 5.0; Z^*_{11}(5) = 0.$

Consider period 2, and note that the order made at period 4 arrives at this period. Using (2-1-4) to (2-1-6), for $X_{12} = 0$;

$f^*_2(0)$

$$= \underset{0 \leq Z_{12} \leq \text{Min}(5,\underset{1}{\overset{}{5}}-0)}{\text{Min}} [\Phi_{12}(Z_{12}) + (2)\text{Max}(4-0-Z_{12},0) + (1)\text{Max}(0+Z_{12},0) + f^*_1(0+Z_{12}-4)]$$

$$= \text{Min}\begin{bmatrix} 0.0+8+0.0+8.0 \\ 0.6+6+0.9+7.0 \\ 1.2+4+1.8+6.0 \\ 1.8+2+2.7+5.0 \\ 2.4+0+3.6+4.5 \\ 3.0+0+4.5+4.0 \end{bmatrix} = \text{Min}\begin{bmatrix} 16.0 \\ 14.5 \\ 13.0 \\ 11.5 \\ 10.5 \\ 11.5 \end{bmatrix} = 10.5; \text{ where } Z^*_{12}(0) = 4.$$

For other values of X_{12}, using (2-1-4) to (2-1-6), $f^*_2(X_{12})$ and $Z^*_{12}(X_{12})$ can be determined. The results are summarized below:

$f^*_2(0) = 10.5;\ Z^*_{12}(0) = 4$ $f^*_2(3) = 8.7;\ Z^*_{12}(3) = 1$

$f^*_2(1) = 9.9;\ Z^*_{12}(1) = 3$ $f^*_2(4) = 8.1;\ Z^*_{12}(4) = 0$

$f^*_2(2) = 9.3;\ Z^*_{12}(2) = 2$ $f^*_2(5) = 8.5;\ Z^*_{12}(5) = 0.$

Consider period 3. The order made at period 5 arrives at this period; therefore, this period will be the last stage.

The inventory level at the beginning of period 3, $X_{13} = X_{15} - r_{14} - r_{15} = 5 - 1 - 2 = 2$.

Using (2-1-7) to (2-1-9), $f^*_3(2)$

$$= \underset{0 \le Z_{13} \le \text{Min}(5, \sum_1 5-2)}{\text{Min}} \{\Phi_{13}(Z_{13}) + (1.5)\,\text{Max}(2-2-Z_{13}, 0) + (1)\,\text{Max}(2-Z_{13}, 0) + f^*_2(2+Z_{13}-2)\}$$

$$= \text{Min}\begin{bmatrix} 0.0+0+2+10.5 \\ 0.5+0+3+\ 9.9 \\ 1.0+0+4+\ 9.3 \\ 1.5+0+5+\ 8.7 \end{bmatrix} = \text{Min}\begin{bmatrix} 12.5 \\ 13.4 \\ 14.3 \\ 15.2 \end{bmatrix} = 12.5; \text{ where } Z^*_{13}(2) = 0.$$

Therefore, the decision is: make no order at period 5. The result is $X_{12} = 0$; then order 4 units at period 4. This yields $X_{11} = 0$; then order 3 units at period 3. The minimum total system cost when an optimal decision is made at period 5 is \$12.50.

2.2 MULTI-ITEM SINGLE-SOURCE FOR MIXABLE ITEMS

This section is an extension of Section 2.1; several types of items are to be carried and they can be mixed together in the warehouse. There continues to be only one source of supply as in Section 2.1, and other assumptions remain the same as before. The analysis is as follows:

Assume that there are N types of items in the system, and consider period 1. For a given set of $X_{11}, X_{21}, \ldots, X_{N1}$; assume that an order of the amount Z_{11} is made only for item type No. 1 for this period. The total system cost when a decision is made for this period affects the system cost in period 1, which is the sum of:

(1) item cost plus fixed cost of ordering $\mathcal{Z}_{11}$, which is $\Phi_{11}(\mathcal{Z}_{11})$,

(2) shortage cost due to the shortage of item type No. 1 in period 1, which is $Cs_{11} \cdot Max(r_{11} - X_{11} - \mathcal{Z}_{11}, 0)$,

(3) carrying cost in carrying item type No. 1 in period 1, which is $Ch_{11} \cdot Max(X_{11} + \mathcal{Z}_{11}, 0)$,

(4) total shortage cost due to the shortage of item types No. 2 to No. N in period 1, which is $\sum_{i=2}^{N} Cs_{i1} \cdot Max(r_{i1} - X_{i1}, 0)$, and

(5) total carrying cost in carrying item types No. 2 to No. N in period 1, which is $\sum_{i=2}^{N} Ch_{i1} \cdot Max(X_{i1}, 0)$.

Thus, the total controllable cost, $\mathcal{C}(U_{11}, U_{21}, \ldots, U_{N1}; Z_{11})$

$$= \Phi_{11}(\mathcal{Z}_{11}) + Cs_{11} \cdot Max(r_{11} - X_{11} - \mathcal{Z}_{11}, 0)$$

$$+ Ch_{11} \cdot Max(X_{11} + \mathcal{Z}_{11}, 0) + K(X_{11}, X_{21}, \ldots, X_{N1}), \qquad (2\text{-}2\text{-}1)$$

where $K(X_{11}, X_{21}, \ldots, X_{N1})$

$$= \sum_{i=2}^{N} \left\{ Cs_{i1} \cdot Max(r_{i1} - X_{i1}) + Ch_{i1} Max(X_{i1}, 0) \right\}. \qquad (2\text{-}2\text{-}2)$$

Note that for a given set of $X_{11}, X_{21}, \ldots, X_{N1}$; the space available for the additional items to be ordered in the period K will be $W - \sum_{i=1}^{N} v_i \cdot Max(X_{i1}, 0)$.

In order to apply the principle of optimality to this problem, let w_1, the space available for the additional item type No. 1, increase in increments of v_1 from 0, $v_1, 2v_1, \ldots, Cv_1, \ldots,$ to $W - \sum_{i=1}^{N} v_i \cdot Max(X_{i1}, 0)$.

Then, let $f_{11}(X_{11},X_{21}, \ldots, X_{N1}/w_1)$ be the minimum total cost when a decision is made for period 1 when only item type No. 1 is being considered, resulting from ordering an optimal amount of $\mathcal{Z}_{11} = \mathcal{Z}^*_{11}(X_{11},X_{21}, \ldots, X_{N1})$ for a given set of $X_{11},X_{21}, \ldots,$ and w_1.

Therefore $f_{11}(X_{11},X_{21}, \ldots, X_{N1}/w_1) = \underset{\mathcal{Z}_{11}}{\text{Min}}\ \mathcal{C}(X_{11},X_{21}, \ldots, X_{N1};\mathcal{Z}_{11}),$

$$\tag{2-2-3}$$

where $$0 \leq \mathcal{Z}_{11} \leq \text{Min}(\mathcal{S}_{11}, \frac{w_1}{v_1} - \text{Min}(X_{11},0)). \tag{2-2-4}$$

For $w_1 = 0$, and $X_{11} < 0$; the restriction of $\mathcal{Z}_{11}$ in (2-2-4) becomes

$$0 \leq \mathcal{Z}_{11} \leq \text{Min}(\mathcal{S}_{11},|X_{11}|). \tag{2-2-5}$$

For $w_1 = 0$, and $X_{11} \geq 0$; (2-2-3) becomes

$$f_{11}(X_{11},X_{21}, \ldots, X_{N1}/0) = \mathcal{C}(X_{11},X_{21}, \ldots, X_{N1};0). \tag{2-2-6}$$

For $w_1 = v_1 \leq v_1(\mathcal{S}_{11} + \text{Min}(X_{11},0))$; (2-2-4) becomes

$0 \leq \mathcal{Z}_{11} \leq 1 - \text{Min}(X_{11},0)$.

Then, $f_{11}(X_{11},X_{21}, \ldots, X_{N1}/v_1)$

$$= \text{Min}\begin{Bmatrix} f_{11}(X_{11},X_{21}, \ldots, X_{N1}/0), \\ \mathcal{C}(X_{11},X_{21}, \ldots, X_{N1};1 - \text{Min}(X_{11},0) \end{Bmatrix}. \tag{2-2-7}$$

In general, for $w_1 = Cv_1 \leq v_1(\mathcal{S}_{11} + \text{Min}(X_{11},0))$;

$f_{11}(X_{11},X_{21}, \ldots, X_{N1}/Cv_1)$

$$= \text{Min}\begin{Bmatrix} f_{11}(X_{11},X_{21}, \ldots, X_{N1}/Cv_1), \\ \mathcal{C}(X_{11},X_{21}, \ldots, X_{N1};C-\text{Min}(X_{11},0)) \end{Bmatrix}. \tag{2-2-8}$$

For $w_1 = Cv_1 > v_1(\mathcal{S}_{11} + \text{Min}(X_{11},0))$

Let $\bar{C}v_1 \leq v_1(\mathcal{S}_{11} + \text{Min}(X_{11},0) < (\bar{C}+1)v_1$,

then $\mathcal{S}_{11} < (\bar{C}+1) - \text{Min}(X_{11},0)$,

and $\mathcal{S}_{11} \geq \bar{C} - \text{Min}(X_{11},0)$.

Therefore, using (2-2-3) and (2-2-4), $f_{11}(X_{11},X_{21}, \ldots, X_{N1}/Cv_1)$

$$= \text{Min}\begin{Bmatrix} f_{11}(X_{11},X_{21}, \ldots, X_{N1}/\bar{C}v_1), \\ \mathcal{C}(X_{11},X_{21}, \ldots, X_{N1};\mathcal{S}_{11}) \end{Bmatrix}. \tag{2-2-9}$$

For a given set of $X_{11}, X_{21}, \ldots, X_{N1}$, consider that orders are made for item types No. 1 and No. 2, and that item type No. 2 is ordered first in the amount of $\mathcal{Z}_{21}$. Let w_1, the space available for the additional item types No. 1 and No. 2, increase from 0 through the value of $w_1 + mv_2$ $(m = 0,1,..)$ until $W - \sum_{i=1}^{N} v_i \mathrm{Max}(X_{i1},0)$. After $\mathcal{Z}_{21}$ is ordered, an optimal amount of item type No. 1 is ordered for a given set of $X_{11}, X_{21} + Z_{21}, \ldots, X_{N1}$, and for an available space of $w_2 - v_2 \mathrm{Max}(\widetilde{Z}_{21} + \mathrm{Min}(X_{21},0),0)$. Therefore, the total system cost is the sum of:

(1) item cost plus fixed cost of ordering $\mathcal{Z}_{21}$, which is $\Phi_{21}(\mathcal{Z}_{21})$, and

(2) minimum total cost when a decision is made for period 1 when only item type No. 1 is considered, resulting from ordering an optimal amount of $\mathcal{Z}_{11}$ for a given set of $X_{11}, X_{21} + \mathcal{Z}_{21}, \ldots, X_{N1}$ and for the space available $w_2 - v_2 \mathrm{Max}(\mathcal{Z}_{21} + \mathrm{Min}(X_{21},0),0)$, which is $f_{11}(X_{11}, X_{21} + \mathcal{Z}_{21}, \ldots, X_{N1}/w_2 - v_2 \mathrm{Max}(\mathcal{Z}_{21} + \mathrm{Min}(X_{21},0),0)$.

Thus, the total cost, $\mathcal{C}(X_{11}, X_{21}, \ldots, X_{N1}; \mathcal{Z}_{21}) = \Phi_{21}(\mathcal{Z}_{21})$ $+ f_{11}(X_{11}, X_{21} + \mathcal{Z}_{21}, \ldots, X_{N1}/w_2 - v_2 \mathrm{Max}(\widetilde{Z}_{21} + \mathrm{Min}(X_{21},0),0)$. (2-2-10)

Then, let $f_{21}(X_{11}, X_{21}, \ldots, X_{N1}/w_2)$ be the minimum total cost when a decision is made for period 1 when item types No. 1 and No. 2 are considered and item type No. 2 is considered first, resulting from ordering an optimal amount of $\mathcal{Z}_{21} = \mathcal{Z}^*_{21}(X_{11}, X_{21}, \ldots, X_{N1})$ and presuming optimal amount of $\mathcal{Z}_{11}$ will be ordered later, for a given set of $X_{11}, X_{21}, \ldots, X_{N1}$ and w_2. Therefore,

$$f_{21}(X_{11}, X_{21}, \ldots, X_{N1}/w_2) = \underset{\mathcal{Z}_{21}}{\mathrm{Min}} \left\{ \mathcal{C}(X_{11}, X_{21}, \ldots, X_{N1}; \mathcal{Z}_{21}) \right\} \qquad (2\text{-}2\text{-}11)$$

where $$0 \leq \mathcal{Z}_{21} \leq \mathrm{Min}(\mathcal{S}_{21}, \frac{w_2}{v_2} - \mathrm{Min}(X_{21}, 0). \qquad (2\text{-}2\text{-}12)$$

In general, item types No. 1 to No. n $(2 \leq n \leq N)$ are considered and item type No. n being considered first. The space available for the additional items of types No. 1 to No. n, w_n, increase from 0 through the value of $w_{n-1} + mv_n (m = 0, 1, \ldots)$ until $W - \sum_{i=1}^{N} v_i \ \mathrm{Max}(X_{i1}, 0)$. Then, it follows that $\mathcal{C}(X_{11}, X_{21}, \ldots, X_{N1}; \mathcal{Z}_{n1}) = \widetilde{\varphi}_{n1}(\mathcal{Z}_{n1})$

$$+ f_{n-1,1}(X_{11}, \ldots, X_{n1} + \mathcal{Z}_{n1}, \ldots, X_{N1}/w_n - v_n \ \mathrm{Max}(\mathcal{Z}_{n1} + \mathrm{Min}(X_{n1}, 0),)), \qquad (2\text{-}2\text{-}13)$$

and $$f_{n1}(X_{11}, X_{21}, \ldots, X_{N1}/w_n) = \underset{\mathcal{Z}_{n1}}{\mathrm{Min}} \left\{ \mathcal{C}(X_{11}, X_{21}, \ldots, X_{N1}; \mathcal{Z}_{n1}) \right\}, \qquad (2\text{-}2\text{-}14)$$

where $$0 \leq \mathcal{Z}_{n1} \leq \mathrm{Min}(\mathcal{S}_{n1}, \frac{w_n}{v_n} - \mathrm{Min}(X_{n1}, 0)). \qquad (2\text{-}2\text{-}15)$$

By letting $n = N$, and let

$$f^*_1(X_{11}, X_{21}, \ldots, X_{N1}) = f_{N1}(X_{11}, X_{21}, \ldots, X_{N1}/W - \sum_{i=1}^{N} v_i \ \mathrm{Max}(X_{i1}, 0)).$$

$f^*_1(X_{11}, X_{21}, \ldots, X_{N1})$ is obtained as a partial-optimization for this stage.

Consider period 2 and for a given set of $X_{12}, X_{22}, \ldots, X_{N2}$, assume that an order is made only for item type No. 1 in the amount of $\mathcal{Z}_{12}$ at this period. The decision made for this period affects the total system cost for periods 2 and 1.

The total system cost is the sum of:

(1) item cost plus fixed cost of ordering $\mathcal{Z}_{12}$, which is $\varphi_{12}(\mathcal{Z}_{12})$,

(2) shortage cost due to the shortage of item type No. 1 in period 2; which is $Cs_{12} \cdot \mathrm{Max}(r_{12} - X_{12} - \mathcal{Z}_{12}, 0)$,

(3) carrying cost in carrying item type No. 1 in period 2, which is $Ch_{12} \ \mathrm{Max}(X_{12} + \mathcal{Z}_{12}, 0)$,

(4) shortage cost due to the shortage of items type No. 2

to No. N in period 2, which is

$\sum_{i=2}^{N} Cs_{i2}$ Max$(r_{i2} - X_{i2}, 0)$,

(5) total carrying cost of item types No. 2 to No. N in period 2, which is $\sum_{i=2}^{N} Ch_{i2}$ Max$(X_{i2}, 0)$, and

(6) minimum total cost, presuming an optimal decision is made in period 1, which is

$f^*_1(X_{12} + Z_{12} - r_{12},\ X_{22} - r_{22},\ \ldots,\ X_{N2} - r_{N2})$

$= G(X_{12} + Z_{12},\ X_{22},\ \ldots,\ X_{N2})$.

Thus, the total system cost, $\mathcal{C}(X_{12}, X_{22}, \ldots, X_{N2}; Z_{12}) = \Phi_{12}(Z_{12})$

$+ Cs_{12} \cdot \text{Max}(r_{12} - X_{12} - Z_{12}, 0) + Ch_{12} \cdot \text{Max}(X_{12} + Z_{12}, 0)$

$+ K(X_{22}, X_{32}, \ldots, X_{N2}) + G(X_{12} + Z_{12},\ X_{22},\ \ldots,\ X_{N2})$, (2-2-16)

where $K(X_{22}, X_{32}, \ldots, X_{N2})$

$$= \sum_{i=2}^{N} \left\{ Cs_{i2} \cdot \text{Max}(r_{i2} - X_{i2}, 0) + Ch_{i2} \cdot \text{Max}(X_{i2}, 0) \right\}. \quad (2\text{-}2\text{-}17)$$

Note that for a given set of $X_{12}, X_{22}, \ldots, X_{N2}$, the space available for the additional items ordered in the period 2 is

$W - \sum_{i=1}^{N} v_i \cdot \text{Max}(X_{i2}, 0)$.

As before, let w_1, the space available for the additional item type No. 1, increase in increments v_1 from $0, v_1, 2v_1, \ldots, Cv_1, \ldots$ to $W - \sum_{i=1}^{N} v_i \cdot \text{Max}(X_{i2}, 0)$. Then, let $f_{12}(X_{12}, X_{22}, \ldots, X_{N2}/w_1)$ be the minimum total cost when a decision is made for period 2 when only item type No. 1 is considered, resulting from ordering an optimal amount of $Z_{12} = Z^*_{12}(X_{12}, X_{22}, \ldots, X_{N2}/w_1)$, presuming an optimal decision is made in period 1, for a given set of $X_{12}, X_{22}, \ldots, X_{N2}$ and w_2.

Therefore $f_{12}(X_{12}, X_{22}, \ldots, X_{N2}/w_1)$

$$= \underset{Z_{12}}{\text{Min}} \left\{ \mathcal{C}(X_{12}, X_{22}, \ldots, X_{N2}; Z_{12}) \right\}, \quad (2\text{-}2\text{-}18)$$

where $$0 \leq \mathcal{Z}_{12} \leq \text{Min}(\mathcal{S}_{12}, \frac{w_1}{v_1} - \text{Min}(X_{12},0)). \quad (2\text{-}2\text{-}19)$$

For $w_1 = 0$, and $X_{12} < 0$; the restriction of $\mathcal{Z}_{12}$ in (3-2-19) becomes

$$0 \leq \mathcal{Z}_{12} \leq \text{Min}(\mathcal{S}_{12}, |X_{12}|). \quad (2\text{-}2\text{-}20)$$

for $w_1 = 0$, and $X_{12} \geq 0$; (2-2-18) becomes $f_{12}(X_{12},X_{22}, \ldots, X_{N2}/0)$

$$= \mathcal{C}(X_{12},X_{22}, \ldots, X_{N2};0). \quad (2\text{-}2\text{-}21)$$

For $w_1 = Cv_1 \leq v_1(\mathcal{S}_{12}, \text{Min}(X_{12},0))$, $f_{12}(X_{12},X_{22}, \ldots, X_{N2}/Cv_1)$

$$= \text{Min}\begin{Bmatrix} f_{12}(X_{12},X_{22}, \ldots, X_{N2}/(C-1)v_1), \\ \mathcal{C}(X_{12},X_{22}, \ldots, X_{N2};C - \text{Min}(X_{12},0)) \end{Bmatrix}. \quad (2\text{-}2\text{-}22)$$

For $w = Cv_1 > v_1(\mathcal{S}_{12} + \text{Min}(X_{12},0))$; let $\bar{C}v_1 < v_1(\mathcal{S}_{12} + \text{Min}(X_{12},0)) <$ $(\bar{C}+1)v_1$, then $f_{12}(X_{12},X_{22}, \ldots, X_{N2}/Cv_1)$

$$= \text{Min}\begin{Bmatrix} f_{12}(X_{12},X_{22}, \ldots, X_{N2}/\bar{C}v_1), \\ \mathcal{C}(X_{12},X_{22}, \ldots, X_{N2};\mathcal{S}_{12}) \end{Bmatrix}. \quad (2\text{-}2\text{-}23)$$

As in previous discussions, if item types No. 1 to No. n $(2 \leq n \leq N)$ are considered and item type No. n is considered first, for a given set of $X_{12},X_{22}, \ldots, X_{n2}, \ldots, X_{N2}$ and for w_n, it follows that

$$\mathcal{C}(X_{12}, \ldots, X_{n2}, \ldots, X_{N2};\mathcal{Z}_{n2}) = \Phi_{12}(\mathcal{Z}_{12})$$
$$+ f_{n-1,2}(X_{12}, \ldots, X_{n2}+\mathcal{Z}_{n2}, \ldots, X_{N2}/w_n - v_n \text{ Max}(\mathcal{Z}_{n2}+\text{Min}(X_{n2},0),0)), \quad (2\text{-}2\text{-}24)$$

and $f_{n2}(X_{12}, \ldots, X_{n2}, \ldots, X_{N2}/w_n)$

$$= \underset{\mathcal{Z}_{n2}}{\text{Min}}\left\{\mathcal{C}(X_{12}, \ldots, X_{n2}, \ldots, X_{N2};\mathcal{Z}_{n2})\right\} \quad (2\text{-}2\text{-}25)$$

where $$0 \leq \mathcal{Z}_{n2} \leq \text{Min}(\mathcal{S}_{n2}, \frac{w_n}{v_n} - \text{Min}(X_{n2},0)). \quad (2\text{-}2\text{-}26)$$

By letting $n = N$, and let $f^*_2(X_{12},X_{22}, \ldots, X_{N2})$

$$= f_{N2}(X_{12},X_{22}, \ldots, X_{N2}/W - \sum_{i=1}^{N} v_i \text{ Max}(X_{i2},0)).$$

$f^*_2(X_{12},X_{22}, \ldots, X_{N2})$ is obtained as a partial-optimization for this stage.

Consider in general period p, where $K+1 \leq p \leq P$.

Using previous developments it follows that

$$\mathcal{C}(X_{1p},X_{2p}, \ldots, X_{Np};\mathcal{Z}_{1p}) = \Phi_{1p}(\mathcal{Z}_{1p})$$

$$+ Cs_{1p} \cdot Max(r_{1p} - X_{1p} - \mathcal{Z}_{1p},0) + Ch_{1p} \cdot Max(X_{1p} + \mathcal{Z}_{1p},0)$$

$$+ K(X_{2p},X_{3p}, \ldots, X_{Np}) + G(X_{1p} + \mathcal{Z}_{1p}, X_{2p}, \ldots, X_{Np}), \qquad (2\text{-}2\text{-}27)$$

where $K(X_{2p},X_{3p}, \ldots, X_{Np})$

$$= \sum_{i=2}^{N}\left\{Cs_{ip} \cdot Max(r_{ip} - X_{ip},0) + Ch_{ip} \cdot Max(X_{ip},0)\right\},$$

and $G(X_{1p} + \mathcal{Z}_{1p}, X_{2p}, \ldots, X_{Np})$

$$= f^*_{p-1}(X_{1p} + \mathcal{Z}_{1p} - r_{1p}, X_{2p} - r_{2p}, \ldots, X_{Np} - r_{Np}).$$

Therefore, $f_{1p}(X_{1p},X_{2p}, \ldots, X_{Np}/w_p)$

$$= \underset{\mathcal{Z}_{1p}}{Min}\left\{\mathcal{C}(X_{1p},X_{2p}, \ldots, X_{Np};\mathcal{Z}_{Np})\right\}, \qquad (2\text{-}2\text{-}28)$$

where $$0 \leq \mathcal{Z}_{1p} \leq Min(\mathcal{S}_{1p}, \frac{w_1}{v_1} - Min(X_{1p},0)). \qquad (2\text{-}2\text{-}29)$$

For $w_1 = 0$, and $X_{1p} < 0$; the restriction of $\mathcal{Z}_{1p}$ in (2-2-29) becomes

$$0 \leq \mathcal{Z}_{1p} \leq Min(S_{1p}, |X_{1p}|). \qquad (2\text{-}2\text{-}30)$$

For $w = 0$, and $X \geq 0$; (2-2-28) becomes $f_{1p}(X_{1p},X_{2p}, \ldots, X_{Np}/0)$

$$= \mathcal{C}(X_{1p},X_{2p}, \ldots, X_{Np};0). \qquad (2\text{-}2\text{-}31)$$

For $w_1 = Cv_1 \leq v_1(\mathcal{S}_{1p} + Min(X_{1p},0))$, it follows that

$f_{1p}(X_{1p},X_{2p}, \ldots, X_{Np}/Cv_1)$

$$= Min\left\{\begin{matrix} f_{1p}(X_{1p},X_{2p}, \ldots, X_{Np}/(C-1)v_1) \\ \mathcal{C}(X_{1p},X_{2p}, \ldots, X_{Np};C - Min(X_{1p},0)) \end{matrix}\right\}, \qquad (2\text{-}2\text{-}32)$$

and for $w_1 = Cv_1 > v_1(\mathcal{S}_{1p} + Min(X_{1p},0))$, $f_{1p}(X_{1p},X_{2p}, \ldots, X_{Np}/Cv_1)$

$$= Min\left\{\begin{matrix} f_{1p}(X_{1p},X_{2p}, \ldots, X_{Np}/\bar{C}v_1), \\ \mathcal{C}(X_{1p},X_{2p}, \ldots, X_{Np};\mathcal{S}_{1p}) \end{matrix}\right\}, \qquad (2\text{-}2\text{-}33)$$

where $\bar{C}v_1 \leq v_1(\mathcal{S}_{1p} + Min(X_{1p},0) > (\bar{C}+1)v_1$. $\qquad$ (2-2-34)

Again, using previous development, if item types No. 1 to No. n

$(2 \leq n \leq N)$ are considered and item type No. n is considered first, it follows that $\mathcal{C}(X_{1p}, X_{2p}, \ldots, X_{Np}; Z_{np}) = \Phi_{np}(Z_{np})$

$$+ f_{n-1,}(X_{1p}, \ldots, X_{np} + Z_{np}, \ldots, X_{Np}/w_n - v_n \text{Max}(Z_{np} + \text{Min}(X_{np}, 0), 0)), \tag{2-2-35}$$

and $f_{np}(X_{1p}, \ldots, X_{np}, \ldots, X_{Np}/w_n)$

$$= \underset{Z_{np}}{\text{Min}} \left\{ \mathcal{C}(X_{1p}, \ldots, X_{np}, \ldots, X_{Np}; Z_{np}) \right\}, \tag{2-2-36}$$

where $$0 \leq Z_{np} \leq \text{Min}(S_{np}, \frac{w_n}{v_n} - \text{Min}(X_{np}, 0)). \tag{2-2-37}$$

By letting $n = N$, and let

$$f^*_p(X_{1p}, X_{2p}, \ldots, X_{Np}) = f_{Np}(X_{1p}, X_{2p}, \ldots, X_{Np}/W - \sum_{i=1}^{N} v_i \text{Max}(X_{ip}, 0)).$$

$f^*_p(X_{1p}, X_{2p}, \ldots, X_{Np})$ is obtained as a partial-optimization for this stage.

And if $p = P$, $f^*_P(X_{1P}, X_{2P}, \ldots, X_{NP})$ is the final optimization to the problem.

<u>Example</u>

planning period,	$P = 4$
warehouse space,	$W = 5$ cubic units
number of types of items,	$N = 3$
lead time,	$L = 2$ periods
a volume of an item	$v_1 = 1.5$ cubic units
	$v_2 = 1.0$ cubic unit
	$v_3 = 0.5$ cubic unit
initial inventory,	$X_{14} = 2$
	$X_{24} = 1$
	$X_{34} = 0.$

	i = 1				i = 2				i = 3			
	k=1	k=2	k=3	k=4	k=1	k=2	k=3	k=4	k=1	k=2	k=3	k=4
$\tilde{S}_{ik}$ -unit	3	2			1	1			1	0		
r_{ik} -unit	3	1	1*	1*	2	1	0*	1*	1	0	0*	1*
$\tilde{C}o_{ik}$-dollars/order	1.5	1.5			2.0	2.0			1.5	2.0		
$\tilde{C}i_{ik}$-dollars/unit	2.0	1.8			1.5	2.0			1.0	1.2		
Cs_{ik}-dollars/unit/period	4.0	4.0			3.0	3.2			3.0	3.0		
Ch_{ik}-dollars/unit/period	1.0	1.2			2.0	2.0			1.0	1.0		

*difference between demand and arrival from orders previous to planning period.

Solution:

Consider period 1 with the value of w_1 being 0, 1.5, 3.0, 4.5.

$$\text{Minimum of } X_{11} = X_{14} - r_{14} - r_{13} - r_{12}$$
$$= 2 - 1 - 1 - 1 = -1.$$
$$\text{Maximum of } X_{11} = X_{14} - r_{14} - r_{13} - r_{12} + \tilde{S}_{12}$$
$$= 2 - 1 - 1 - 1 + 2 = 1.$$
$$\therefore -1 \leq X_{11} \leq 1.$$
$$\text{Minimum of } X_{21} = X_{24} - r_{24} - r_{23} - r_{22}$$
$$= 1 - 1 - 0 - 1 = -1.$$
$$\text{Maximum of } X_{21} = X_{24} - r_{24} - r_{23} - r_{22} + \tilde{S}_{22} + \tilde{S}_{21}$$
$$= 1 - 1 - 0 - 1 + 1 + 1 = 1.$$
$$\therefore -1 \leq X_{21} \leq 1.$$
$$\text{Minimum of } X_{31} = X_{34} - r_{34} - r_{33} - r_{32}$$
$$= 0 - 1 - 0 - 0 = -1.$$
$$\text{Maximum of } X_{31} = X_{34} - r_{34} - r_{33} - r_{32} + \tilde{S}_{32} + \tilde{S}_{31}$$
$$= 0 - 1 - 0 - 0 + 0 + 1 = 0.$$
$$\therefore -1 \leq X_{31} \leq 0.$$

For $X_{11} = -1$, $X_{21} = -1$, $X_{31} = 0$;

Using (2-2-2), $K(-1,0) = (3)(2+1)+0+(3)(1+0)+0 = 12.$

$$\mathfrak{S}_{11} + \text{Min}(X_{11},0) = 3-1=2$$

$$\therefore \bar{\bar{C}} = 2.$$

Using (2-2-1) to (2-2-3), and (2-2-5), $f_{11}(-1,-1,0/0)$

$$= \underset{0 \leq \mathfrak{Z}_{11} \leq \text{Min}(3,1)}{\text{Min}} \Big[\Phi_{11}(\mathfrak{Z}_{11}) + (4)\text{Max}(r_{11}+1-\mathfrak{Z}_{11},0) + (1)\text{Max}(-1+\mathfrak{Z}_{11},0) + K(-1,0)\Big]$$

$$= \text{Min}\begin{bmatrix} 0.0 + (4)\text{Max}(3+1-0,0) + (1)\text{Max}(-1+0,0) + 12, \\ 3.5 + (4)\text{Max}(3+1-1,0) + (1)\text{Max}(-1+1,0) + 12. \end{bmatrix}$$

$$= \text{Min}\begin{bmatrix} 28.0 \\ 27.5 \end{bmatrix} = 27.5; \text{ where } \mathfrak{Z}^*_{11}(-1,-1,0/0) = 1.$$

Using (2-2-7); $f_{11}(-1,-1,0/1.5)$

$$= \text{Min}\begin{bmatrix} f_{11}(-1,-1,0/0), \\ \Phi_{11}(2) + (4)\text{Max}(3+1-2,0) + (1)\text{Max}(-1+2,0) + K(-1,0). \end{bmatrix}$$

$$= \text{Min}\begin{bmatrix} 27.5 \\ 26.5 \end{bmatrix} = 26.5; \text{ where } \mathfrak{Z}^*_{11}(-1,-1,0/1.5) = 2.$$

Using (2-2-8); $f_{11}(-1,-1,0/3)$

$$= \text{Min}\begin{bmatrix} f_{11}(-1,-1,0/1.5), \\ \Phi_{11}(3) + (4)\text{Max}(3+1-3,0) + (1)\text{Max}(-1+3,0) + K(-1,0) \end{bmatrix}$$

$$= \text{Min}\begin{bmatrix} 26.5 \\ 25.5 \end{bmatrix} = 25.5; \text{ where } \mathfrak{Z}^*_{11}(-1,-1,0) = 3.$$

Using (2-2-9); $f_{11}(-1,-1,0/4.5)$

$$= \text{Min}\begin{bmatrix} f_{11}(-1,-1,0/3), \\ \Phi_{11}(3) + (4)\text{Max}(3+1-3,0) + (1)\text{Max}(-1+3,0) + K(-1,0) \end{bmatrix}$$

$$= 25.5; \text{ where } \mathfrak{Z}^*_{11}(-1,-1,0/4.5) = 3.$$

For other sets of X_{11}, X_{21}, X_{31} and for a given w_1, the values of $f_{11}(X_{11},X_{21},X_{31}/w_1)$ and $\mathfrak{Z}^*_{11}(X_{11},X_{21},X_{31}/w_1)$ can be determined. The results are summarized in Table I.

The next step is to determine $f_{21}(X_{11}, X_{21}, X_{31}/w_1)$. The values of w_2 will be $w_1 + mv_2$ $(m = 0, 1, \ldots)$ which are 0,1,1.5,2,2.5,3,3.5,4,4.5, and 5.

$$\text{Minimum of } X_{21} = X_{24} - r_{24} - r_{23} - r_{22}$$

$$= -1.$$

$$\text{Maximum of } X_{21} = X_{24} - r_{24} - r_{23} - r_{22} + \mathcal{S}_{22}$$

$$= 0.$$

$$-1 \leq X_{21} \leq 0.$$

The regions of X_{11} and X_{31} will remain the same; they are:

$$-1 \leq X_{11} \leq 1$$

$$-1 \leq X_{31} \leq 0.$$

For $X_{11} = -1, X_{21} = -1, X_{31} = 0$;

Using (2-2-10) to (2-2-12), $f_{21}(-1,-1,0/0)$

$$= \min_{0 \leq \mathcal{Z}_{21} \leq \text{Min}(1,1)} \left[\Phi_{21}(\mathcal{Z}_{21}) + f_{11}(-1, -1 + \mathcal{Z}_{21}, 0/0 - (1)\text{Max}(-1 + \mathcal{Z}_{21}, 0)) \right]$$

$$= \text{Min} \begin{bmatrix} 0 + f_{11}(-1,-1,0/0), \\ 3.5 + f_{11}(-1,0,0/0) \end{bmatrix}$$

$$= \text{Min} \begin{bmatrix} 0 + 27.5, \\ 3.5 + 24.5 \end{bmatrix} = \text{Min} \begin{bmatrix} 27.5 \\ 28.0 \end{bmatrix} = 27.5; \text{ where } \mathcal{Z}^*_{21} = 0, \ \mathcal{Z}^*_{11} = 1.$$

For $X_{11} = -1$, $X_{21} = -1$, $X_{31} = 0$, and for the other values of w_2, as well as for the other set of X_{11}, X_{21}, X_{31} for a given w_2, the values of $f_{21}(X_{11}, X_{21}, X_{31})$ and $\mathcal{Z}^*_{21}(X_{11}, X_{21}, X_{31}/w_2)$ can be determined. The results are summarized in Table II.

The last calculation for the first stage is to determine the values of $f^*_1(X_{11}, X_{21}, X_{31})$. Since there are only three types of items in this system, it is not necessary to determine the value of

TABLE I

OPTIMAL POLICY AND MINIMUM COST WHEN ONLY ITEM TYPE NO. 1 IS CONSIDERED IN THE FIRST STAGE PERIOD, $f_{11}(X_{11}, X_{21}, X_{31}/w_1)$ AND $\tilde{Z}^*_{11}(X_{11}, X_{21}, X_{31}/w_1)$

w_1		$X_{11}=-1$						$X_{11}=0$						$X_{11}=1$					
		$X_{21}=-1$		$X_{21}=0$		$X_{21}=1$		$X_{21}=-1$		$X_{21}=0$		$X_{21}=1$		$X_{21}=-1$		$X_{21}=0$		$X_{21}=1$	
		$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$
0	f_{11}	30.5	27.5	27.5	24.5	25.5	22.5	27.0	24.0	24.0	21.0	22.0	19.0	24.0	21.0	21.0	18.0	19.0	16.0
	$\tilde{Z}^*_{11}$	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1.5	f_{11}	29.5	26.5	26.5	23.5	24.5	21.5	27.0	24.0	24.0	21.0	22.0	19.0	24.0	21.0	21.0	18.0	19.0	16.0
	$\tilde{Z}^*_{11}$	2	2	2	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0
3.0	f_{11}	28.5	25.5	25.5	22.5	23.5	20.5	26.5	23.5	23.5	20.5	21.5	18.5	23.5	20.5	20.5	17.5	*	*
	$\tilde{Z}^*_{11}$	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2		
4.5	f_{11}	28.5	25.5	25.5	22.5	*	*	25.5	22.5	22.5	19.5	*	*	*	*	*	*	*	*
	$\tilde{Z}^*_{11}$	3	3	3	3														

*Not feasible.

TABLE II

OPTIMAL POLICY AND MINIMUM COST WHEN ITEM TYPES NO. 1 AND 2 ARE CONSIDERED IN THE FIRST STAGE PERIOD, $f_{21}(X_{11}, X_{21}, X_{31}/w_2)$ AND $\tilde{Z}^*_{21}(X_{11}, X_{21}, X_{31}/w_2)$

w_2		$X_{11}=-1$				$X_{11}=0$				$X_{11}=1$			
		$X_{21}=-1$		$X_{21}=0$		$X_{21}=-1$		$X_{21}=0$		$X_{21}=-1$		$X_{21}=0$	
		$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$	$X_{31}=-1$	$X_{31}=0$
	f_{21}	30.5	27.5	27.5	24.5	27.0	24.0	24.0	21.0	24.0	21.0	21.0	18.0
0	$\tilde{Z}^*_{11}$	1	1	1	1	0	0	0	0	0	0	0	0
	$\tilde{Z}^*_{21}$	0	0	0	0	0	0	0	0	0	0	0	0
	f_{21}	30.5	27.5	27.5	24.5	27.0	24.0	24.0	21.0	24.0	21.0	21.0	18.0
1	$\tilde{Z}^*_{11}$	1	1	1	1	0	0	0	0	0	0	0	0
	$\tilde{Z}^*_{21}$	0	0	0	0	0	0	0	0	0	0	0	0
	f_{21}	29.5	26.5	26.5	23.5	27.0	24.0	24.0	21.0	24.0	21.0	21.0	18.0
1.5	$\tilde{Z}^*_{11}$	2	2	2	2	0	0	0	0	0	0	0	0
	$\tilde{Z}^*_{21}$	0	0	0	0	0	0	0	0	0	0	0	0
	f_{21}	29.5	26.5	26.5	23.5	27.0	24.0	24.0	21.0	24.0	21.0	21.0	18.0
2	$\tilde{Z}^*_{11}$	2	2	2	2	0	0	0	0	0	0	0	0
	$\tilde{Z}^*_{21}$	0	0	0	0	0	0	0	0	0	0	0	0
	f_{21}	29.5	26.5	26.5	23.5	27.0	24.0	24.0	21.0	24.0	21.0	21.0	18.0
2.5	$\tilde{Z}^*_{11}$	2	2	2	2	0	0	0	0	0	0	0	0
	$\tilde{Z}^*_{21}$	0	0	0	0	0	0	0	0	0	0	0	0
	f_{21}	28.5	25.5	25.5	22.5	26.5	23.5	23.5	20.5	23.5	20.5	20.5	17.5
3	$\tilde{Z}^*_{11}$	3	3	3	3	2	2	2	2	2	2	2	2
	$\tilde{Z}^*_{21}$	0	0	0	0	0	0	0	0	0	0	0	0
	f_{21}	28.5	25.5	25.5	22.5	26.5	23.5	23.5	20.5	23.5	20.5	20.5	17.5
3.5	$\tilde{Z}^*_{11}$	3	3	3	3	2	2	2	2	2	2	2	2
	$\tilde{Z}^*_{21}$	0	0	0	0	0	0	0	0	0	0	0	0
	f_{21}	28.5	25.5	25.5	22.5	26.5	23.5	23.5	20.5				
4	$\tilde{Z}^*_{11}$	3	3	3	3	2	2	2	2	*	*	*	*
	$\tilde{Z}^*_{21}$	0	0	0	0	0	0	0	0				
	f_{21}	28.5	25.5	25.5	22.5	25.5	22.5	22.5	19.5				
4.5	$\tilde{Z}^*_{11}$	3	3	3	3	3	3	3	3	*	*	*	*
	$\tilde{Z}^*_{21}$	0	0	0	0	0	0	0	0				
	f_{21}	28.5	25.5	25.5	22.5	25.5	22.5	22.5	19.5				
5	$\tilde{Z}^*_{11}$	3	3	3	3	3	3	3	3	*	*	*	*
	$\tilde{Z}^*_{21}$	0	0	0	0	0	0	0	0				

*Not feasible.

$f_{31}(X_{11},X_{21},X_{31}/w_3)$ for all values of w. For each set of X_{11},X_{21},X_{31} the value of $f^*_1(X_{11},X_{21},X_{31})$ can be determined directly from $f_{31}(X_{11},X_{21},X_{31}/W - \sum_{i=1}^{3} v_i \text{ Max}(X_{i1},0))$.

$$\text{Minimum value of } X_{31} = -1.$$

$$\text{Maximum value of } X_{31} = -1 + 0$$

$$= -1.$$

The regions of X_{11},X_{21} will remain the same; they are:

$$-1 \leq X_{11} \leq 1$$

$$-1 \leq X_{21} \leq 0.$$

For $X_{11} = -1$, $X_{21} = -1$, $X_{31} = -1$;

Using (2-2-13) to (2-2-15); $f^*_1(-1, -1, -1)$

$= f_{31}(-1, -1, -1,/5)$

$$= \min_{0 \leq Z_{31} \leq \text{Min}(1,11)} \Big[\phi_{31}(Z_{31}) + f_{21}(-1, -1, -1 + Z_{31}/5 - (.5) \text{ Max}(Z_{31} + \text{Min}(X_{31},0)))\Big]$$

$$= \text{Min}\begin{bmatrix} 0 + f_{21}(-1, -1, -1/5), \\ 2.5 + f_{21}(-1, -1, 0/5) \end{bmatrix}.$$

$$= \text{Min}\begin{bmatrix} 0 + 28.5 \\ 2.5 + 25.5 \end{bmatrix} = 28; \text{ where } Z^*_{31} = 1,\ Z^*_{21} = 0, \text{ and } Z^*_{11} = 3.$$

For other set of X_{11},X_{21},X_{31}; the values of $f^*_1(X_{11},X_{21},X_{31})$ can be determined. Results are summarized in Table III.

Consider period 2. This is the last stage of this problem. The regions necessary for the calculations are as follows:

$$X_{12} = 0,$$

$$0 \leq X_{22} \leq 1,$$

$$X_{32} = -1.$$

For $X_{12} = 0$, $X_{22} = 0$, $X_{32} = -1$;

Using (2-2-17),

TABLE III

OPTIMAL POLICY AND MINIMUM COST WHEN ONLY ITEM TYPE NO. 1 IS CONSIDERED IN THE SECOND STAGE PERIOD

	$X_{31}=-1$					
	$X_{11}=-1$		$X_{11}=0$		$X_{11}=1$	
	$X_{21}=-1$	$X_{21}=0$	$X_{21}=-1$	$X_{21}=0$	$X_{21}=-1$	$X_{21}=0$
f_1^*	28.0	25.0	25.0	22.0	23.0	20.0
$\tilde{Z}_{11}^*$	3	3	3	3	3	3
$\tilde{Z}_{21}^*$	0	0	0	0	0	0
$\tilde{Z}_{31}^*$	1	1	1	1	1	1

TABLE IV

OPTIMAL POLICY AND MINIMUM COST WHEN ITEM TYPES NO. 1 AND 2 ARE CONSIDERED IN THE SECOND STAGE PERIOD

w_1		$X_{12}=0,\ X_{22}=0,\ X_{32}=-1$	$X_{12}=0,\ X_{22}=1,\ X_{32}=-1$
0-less than 3	f_{12}	38.2	34.0
	$\tilde{Z}_{12}^*$	0	0
	$\tilde{Z}_{11}^*$	3	3
	$\tilde{Z}_{21}^*$	0	0
	$\tilde{Z}_{31}^*$	1	1
3-4.5	f_{12}	33.7	31.5
	$\tilde{Z}_{12}^*$	1	1
	$\tilde{Z}_{11}^*$	3	3
	$\tilde{Z}_{21}^*$	0	0
	$\tilde{Z}_{22}^*$	1	1

$K(0,-1) = (3.2)(1) + 0 + (3)(1) + 0 = 6.2.$

$$\tilde{S}_{12} + \text{Min}(X_{12},0) = 2 + 0 = 2$$

$$\bar{\bar{C}} = 2.$$

Using (2-2-16) and (2-2-21), $f_{12}(0,0,-1/0)$

$= (4)\text{Max}(1-0,0) + (1.2)\text{Max}(0+0,0) + 6.2 + f^*_1(-1, -1, -1)$

$= 4 + 6.2 + 28 = 38.2$; where $\tilde{Z}^*_{12} = 0$, $\tilde{Z}^*_{11} = 3$, $\tilde{Z}^*_{21} = 0$, and $\tilde{Z}^*_{31} = 1$.

Using (2-2-22), $f_{12}(0,0,-1/1.5)$

$$= \text{Min}\begin{bmatrix} f_{12}(0, 0, -1/0), \\ \Phi_{12}(1) + (4)\text{Max}(0,0) + (1.2)\text{Max}(1,0) + 6.2 + 25) \end{bmatrix}$$

$= \text{Min}\begin{bmatrix} 38.2 \\ 35.2 \end{bmatrix} = 35.7$; where $\tilde{Z}^*_{12} = 1$, $\tilde{Z}^*_{11} = 3$, $\tilde{Z}^*_{21} = 0$, and $\tilde{Z}^*_{31} = 1$,

and $f_{12}(0, 0, -1/3)$

$$= \text{Min}\begin{bmatrix} f_{12}(0, 0, -1/1.5), \\ \Phi_{12}(2) + (4)\text{Max}(-1, 0) + (1.2)\text{Max}(2, 0) + 6.2 + 20 \end{bmatrix}$$

$= \text{Min}\begin{bmatrix} 35.7 \\ 33.7 \end{bmatrix} = 33.7$; where $\tilde{Z}^*_{12} = 2$, $\tilde{Z}^*_{11} = 3$, $\tilde{Z}^*_{21} = 0$, and $\tilde{Z}^*_{31} = 1$.

Using (2-2-23), $f_{12}(0,0,-1/4.5)$

$= 33.7$; where $\tilde{Z}^*_{12} = 2$, $\tilde{Z}^*_{11} = 3$, $\tilde{Z}^*_{21} = 0$, and $\tilde{Z}^*_{31} = 1$.

For other sets of X_{12}, X_{22}, X_{32} and other values of w_1, the values of $f_{12}(X_{12}, X_{22}, X_{32})$ can be determined. The results are summarized in Table IV.

Since there are only two types of items, types No. 1 and No. 2, are available for this period, the next step, which is the last step, is to determine the value of $f^*_2(X_{12}, X_{22}, X_{32})$. The value of $f^*_2(X_{12}, X_{22}, X_{32})$ can be determined directly from $f_{22}(X_{12}, X_{22}, X_{32}/W - \sum_{i=1}^{3} v_i \text{Max}(X_{i2},0)$. The only set necessary for calculation in this stage is $X_{12} = 0$, $X_{22} = 0$, $X_{32} = -1$.

Using (2-2-24) to (2-2-26), $f^*_2(0, 0, -1) = f_{22}(0, 0, -1/5)$

$$= \min_{0 \leq \mathcal{Z}_{22} \leq \mathrm{Min}(1,5)} \left[\Phi_{22}(Z_{22}) + f_{12}(0, 0 + \mathcal{Z}_{22}, -1/5 - (1)\mathrm{Max}(\mathcal{Z}_{22} + \mathrm{Min}(X_{22}, 0))) \right]$$

$$= \mathrm{Min}\begin{bmatrix} 0 + 33.7 \\ 4 + 31.5 \end{bmatrix} = 33.7; \text{ where } \mathcal{Z}^*_{22} = 1,\ \mathcal{Z}^*_{12} = 1,\ \mathcal{Z}^*_{11} = 3,\ \mathcal{Z}^*_{21} = 0, \text{ and } \mathcal{Z}^*_{31} = 1.$$

Thus, the optimal policy for the problem is determined. By ordering $\mathcal{Z}^*_{22} = 1$, $\mathcal{Z}^*_{12} = 1$, $\mathcal{Z}^*_{11} = 3$, $\mathcal{Z}^*_{21} = 0$, and $\mathcal{Z}^*_{31} = 1$; the optimal total system cost is \$35.50.

2.3 MULTI-ITEM SINGLE-SOURCE FOR THE NON-MIXABLE ITEMS

This section considers the case of Section 2.2 in which several types of items cannot be mixed together and the space for each type is allocated at the beginning of the planning period. Inventorying chemical items would be a case in which this specific restriction is necessary.

The analysis begins by considering that each item type is kept in a specific given space as previously considered in the single-item single-source system. Thus, for item type No. 1, the analysis in Section 2.1 may be used to determine $f^*_p(0)$ for the selected value of w $(0 \leq w \leq W)$. Let $G_1(w) = f^*_p(0)$, for available space w.

Consider that only item type No. 1 is stored in the warehouse. For a given total space W, let w_1, the space which is to be allocated to item type No. 1, increase with an increment v_1 from 0 through the values v_1, $2v_1$, ..., W. $G_1(w)$ is the minimum cost for space w_1 occupied only by item type No. 1. Note $G_1(w) = G_1(Cv_1)$ for $Cv_1 \leq w < (C+1)v_1$.

Next, consider that only item types No. 1 and No. 2 are stored in the warehouse. Let w_2, the space which is to be allocated to item types No. 1 and No. 2, increase from 0 through the values $w_1 + mv_2 (m = 0, 1, \ldots)$.

Let $F_2(w_2)$ be the minimum expected cost when only item types No. 1 and No. 2 are stored in w_2 unit space, resulting from the optimal allocation of the given space to the two types of items.

$$F_2(w_2) = \underset{Z}{\mathrm{Min}}[G_2(v_2 Z) + G_1(w_2 - v_2 Z)] \qquad (2\text{-}3\text{-}1)$$

where

$$0 \le Z \le \frac{w_2}{v_2}. \qquad (2\text{-}3\text{-}2)$$

Again, for w between adjacent values $w'_2 \le w < w''_2$ of the set w_2, $F_2(w) = F_2(w'_2)$.

In general, consider only item types No. 1 to No. n in the warehouse. Let w_n, the space allocated to items types No. 1 to No. n, increase from 0 through the values $w_{n-1} + mv_n (m = 0, 1, \ldots)$. The following relation is obtained:

$$F_n(w_n) = \underset{Z}{\mathrm{Min}}[G_n(v_n Z) + F_{n-1}(w_n - v_n Z)], \qquad (2\text{-}3\text{-}3)$$

where

$$0 \le Z \le \frac{w_n}{v_n}. \qquad (2\text{-}3\text{-}4)$$

Letting $n = N$, $f_N(W)$ can be determined and is the final optimization to the problem.

<u>Example</u>

planning period,	$P = 2$ periods
warehouse space,	$W = 3$ cubic units
number of types of item,	$N = 2$
a volume of an item,	$v_1 = 1$ cubic unit
	$v_2 = 1$ cubic unit
procurement lead time,	$L_k = 0$ (for all k)

	i = 1		i = 2	
	k=1	k=2	k=1	k=2
S_{ik}-unit	3	3	3	3
r_{ik}-unit	1	2	2	1
Co_{ik}-dollars/order	2.00	3.00	3.00	2.00
Ci_{ik}-dollars/unit	1.00	1.50	1.50	1.00
Cs_{ik}-dollars/unit/period	5.00	6.00	8.00	7.00
Ch_{ik}-dollars/unit/period	0.50	0.50	0.50	0.50

Solution:

Employing the procedure of Section 2.1, for each type of item and for each given value of w_i, $G_i(w_i)$ can be determined. The results are summarized as follows:

$$G_1(0) = 21.00;\ Z^*_{12} = 0$$

$$G_1(1) = 14.50;\ Z^*_{12} = 1$$

$$G_1(2) = 10.50;\ Z^*_{12} = 2$$

$$G_1(3) = 9.50;\ Z^*_{12} = 3$$

$$G_2(0) = 27.50;\ Z^*_{22} = 0$$

$$G_2(1) = 16.50;\ Z^*_{22} = 1$$

$$G_2(2) = 10.00;\ Z^*_{22} = 2$$

$$G_2(3) = 7.50;\ Z^*_{22} = 3.$$

Using (2-3-1) and (2-3-2);

$$F_2(3) = \text{Min}\begin{bmatrix} G_2(0) + G_1(3), \\ G_2(1) + G_1(2), \\ G_2(2) + G_1(1), \\ G_2(3) + G_1(0). \end{bmatrix} = \text{Min}\begin{bmatrix} 27.50 + 9.50 \\ 16.50 + 10.50 \\ 10.00 + 14.50 \\ 7.50 + 21.00 \end{bmatrix} = \text{Min}\begin{bmatrix} 37.00 \\ 27.00 \\ 24.50 \\ 28.50 \end{bmatrix}.$$

The optimal decision is: allocate 1 cubic unit of space to item type No. 1 and 2 cubic units to item type No. 2. By ordering 1 and 2 units of types No. 1 and No. 2, respectively, the minimum total system cost is $24.50.

2.4 SINGLE-ITEM MULTI-SOURCE SYSTEM

This section considers the case in which there is only one type of item but several sources of supply available in the system. Other assumptions are the same as previously used in the chapter.

Assume that there are J sources of supply which can supply the items for the period k.

Let $\Phi_{1jk}(Z_{1jk})$ be the item cost plus fixed ordering cost when Z_{1jk} items of type No. 1 (in this model, it is assumed that only a single type of item in the system) are ordered from source No. j, with this amount arriving at the beginning of period k.

And let S_{1jk} be the available supply of item type No. 1, from source No. j, in which the order made from this source will arrive at period k.

The total amount available which will arrive at this period will be $\sum_{j=1}^{J} S_{1jk}$.

Let $\Phi_{1jk}(Z_{1k})$ be a minimum cost when Z_{1k} is ordered from sources No. 1 to No. j.

Then, $\Phi^*_{11k}(Z_{1k}) = \Phi_{11k}(Z_{1k})$; where $Z_{1k} \leq S_{11k}$

$$\Phi^*_{1jk}(Z_{1k}) = \underset{0 \leq Z_{1jk} \leq \text{Min}(S_{1jk}, Z_{1k})}{\text{Min}} \Big[\Phi_{1jk}(Z_{1jk}) + \Phi^*_{1,j-1,k}(Z_{1k} - Z_{1jk})\Big],$$

where $$Z_{1k} \leq \sum_{j'=1}^{j} S_{1j'k}.$$

Increasing j until $j = J$, and letting $\Phi_{1k}(Z_{1k}) = \Phi^*_{1Jk}(Z_{1k})$, $\Phi_{1k}(Z_{1k})$ becomes a minimum item cost plus fixed ordering cost function for a given value of Z_{1k}. Using this $\Phi_{1k}(Z_{1k})$ in Section 2.1, the system will be reduced to a simple Single-Item and Single-Source system.

2.5 MULTI-ITEM MULTI-SOURCE SYSTEM FOR THE MIXABLE ITEMS

It is assumed in this section that several types of items are to be carried, and several sources of supply are available in the system described previously in this chapter.

Assume that there are N types of items and J sources of supply which can supply the items in the period k.

For a particular item type No. i, following the discussion in Section 2.4:

$$\Phi^*_{i1k}(Z_{1k}) = \Phi_{i1k}(Z_{1k}),$$

where

$$Z_{ik} \leq S_{i1k},$$

and

$$\Phi^*_{ijk}(Z_{ik}) = \underset{0 \leq Z_{ijk} \leq \mathrm{Min}(S_{ijk},\, Z_{ik})}{\mathrm{Min}} \Big[\Phi_{ijk}(Z_{ijk}) + \Phi^*_{i,j-1,k}(Z_{ik} - Z_{ijk}) \Big],$$

where

$$Z_{ik} \leq \sum_{j'=1}^{J} S_{ij'k}.$$

As in Section 2.4, letting $\Phi_{ik}(Z_{ik}) = \Phi^*_{iJk}(Z_{ik})$, $\Phi_{ik}(Z_{ik})$ can be used in Section 2.2. Then, the system is reduced to the multi-item single-source system.

2.6 MULTI-ITEM MULTI-SOURCE SYSTEM FOR THE NON-MIXABLE ITEMS

Consider the case in Section 2.3 in which the several types of items cannot be mixed together for the multi-item multi-source problem.

Employing the discussion in Section 2.3, the system may be reduced, first, to a single-item multi-source. Thus, for item type No. i, the procedure discussed in Section 2.3 may be used to determine $G_i(w)$ which is $f^*p(0)$ for a selected value of w. Then, the procedure to allocate space to each type of item is the same as in Section 2.3.

CHAPTER III

PROBABILISTIC DEMAND AND IMMEDIATE FULFILLED SYSTEM

Differing from the one in the previous chapter, the case when demands are probabilistic is considered in this chapter. The problems are restricted to the case of immediate fulfill or zero lead time.

As in Chapter II, the optimal decisions for this chapter are based on the amount of inventory on hand at the beginning of each period.

3.1 Single Item Single Source System

This section considers the case in which only one type of item and only one source of supply are available. The analysis is as follows:

Consider period 1, and for a given X_{11}, assume that an amount Z_{11} is ordered in this period. The decision made at this period affects the system cost only in this first period.

The expected total controllable cost is the sum of:

(1) item cost plus fixed cost of ordering Z_{11}, which is

$\varphi_{11}(Z_{11})$,

(2) expected shortage cost in the period 1, which is

$Cs_{11} \sum_{r_1 > X_{11} + Z_{11}} (r_1 - X_{11} - Z_{11}) P(r_1)$, and

(3) carrying cost in period 1, which is

$Ch_{11} \cdot \mathrm{Max}(X_{11} + Z_{11}, 0)$.

Thus, the expected total controllable cost, $\tilde{C}(X_{11}; Z_{11})$

$$= \varphi_{11}(Z_{11}) + Cs_{11} \sum_{r_{11} > X_{11} + Z_{11}} (r_{11} - X_{11} - Z_{11})P_{11}(r_{11})$$

$$+ Ch_{11} \cdot \text{Max}(X_{11} + Z_{11}, 0). \qquad (3\text{-}1\text{-}1)$$

Let $f^*_1(X_{11})$ be a minimum expected total controllable cost for period 1, resulting from ordering an optimal amount of $Z_{11} = Z^*_{11}(X_{11})$ for a given X_{11}.

Therefore, $$f^*_1(X_{11}) = \underset{Z_{11}}{\text{Min}}\left\{\mathcal{C}(X_{11}; Z_{11})\right\}, \qquad (3\text{-}1\text{-}2)$$

where $$0 \leq Z_{11} \leq \text{Min}\left\{S_{11}, \frac{W}{V_1} - X_{11}\right\}. \qquad (3\text{-}1\text{-}3)$$

Consider period 2 and for a given value X_{12}, assume that an amount Z_{12} is ordered in this period. The decision made at this period affects the system cost for periods 2 and 1.

The expected total controllable cost is the sum of:

(1) item cost plus fixed cost of ordering Z_{12}, which is $\varphi_{12}(Z_{12})$,

(2) expected shortage cost in period 2, which is $Cs_{12} \sum_{r_{12} > X_{12} + Z_{12}} (r_2 - X_{12} - Z_{12})P_{12}(r_{12})$,

(3) carrying cost in period 2, which is $Ch_{12} \cdot \text{Max}(X_{12} + Z_{12}, 0)$, and

(4) expected optimal controllable cost presuming an optimal decision is made in the period 1, which is $\sum_{r_{12} \geq 0} f^*_1(X_{12} + Z_{12} - r_{12})P_{12}(r_{12})$.

Thus, the expected total controllable cost, $\mathcal{C}(X_{12}; Z_{12})$

$$= \varphi_{12}(Z_{12}) + Cs_{12} \sum_{r_{12} > X_{12} + Z_{12}} (r_{12} - X_{12} - Z_{12}) P_{12}(r_{12})$$

$$+ Ch_{12} \cdot \text{Max}(X_{12} + Z_{12}, 0) + \sum_{r_{12} \geq 0} f^*_1(X_{12} + Z_{12} - r_{12})P_{12}(r_{12}). \qquad (3\text{-}1\text{-}4)$$

Let $f^*_2(X_{12})$ be a minimum expected controllable cost for period 2, resulting from ordering an optimal amount of $Z_{12} = Z^*_{12}(X_{12})$ for a given X_{12}.

Therefore, $$f^*_2(X_{12}) = \underset{Z_{12}}{\text{Min}}\left\{\mathcal{C}(X_{12};\, Z_{12})\right\}, \qquad (3\text{-}1\text{-}5)$$

where $$0 \leq Z_{12} \leq \text{Min}\left\{S_{12},\, \frac{W}{V_1} - X_{12}\right\}. \qquad (3\text{-}1\text{-}6)$$

Next, consider in general period p,

where $2 \leq p \leq P$.

$$\mathcal{C}(X_{1p};Z_{1p}) = \varphi_{1p}(Z_{1p}) + Cs_{1p}\sum_{r_{1p} > X_{1p} + Z_{1p}} (r_{1p} - X_{1p} - Z_{1p})P_{1p}(r_{1p})$$

$$+ Ch_{1p} \cdot \text{Max}(X_{1p} + Z_{1p}, 0)$$

$$+ \sum_{r_{1p} \geq 0} f^*_{p-1}(X_{1p} + Z_{1p} - r_{1p})P_{1p}(r_{1p}). \qquad (3\text{-}1\text{-}7)$$

It follows then that

$$f^*_p(X_{1p}) = \underset{Z_{1p}}{\text{Min}}\left\{\mathcal{C}(X_{1p};Z_{1p})\right\}, \qquad (3\text{-}1\text{-}8)$$

where $$0 \leq Z_{1p} \leq \text{Min}\left\{S_{1p},\, \frac{W}{V_1} - X_{1p}\right\}. \qquad (3\text{-}1\text{-}9)$$

<u>Example</u>

planning period, $P = 3$

warehouse space, $W = 5$ cubic units

volume of an item, $v_1 = 1$ cubic unit

initial inventory, $x_{13} = 4$ units

		k=1	k=2	k=3
available of supply, S_{1k}	- unit	3	5	4
item cost, Ci_{1k}	- $/unit	0.50	0.60	0.50
fixed ordering cost, Co_{1k}	- $/order	0.50	0.50	0.50
shortage cost, Cs_{1k}	- $/unit/period	6.00	6.00	6.00
carrying cost, Ch_{1k}	- $/unit/period	1.00	0.90	1.00

r_{1k}	0	1	2	3	4
$P_{11}(r_{11})$	.20	.25	.30	.25	.00
$P_{12}(r_{12})$	.10	.20	.35	.20	.15
$P_{13}(r_{13})$	.55	.45	.00	.00	.00

Solution:

Consider period 1. Using (3-1-1) to (3-1-3), for $X_{11} = -1$;

$f^*_1(-1)$

$$= \underset{0 \le Z_{11} \le \text{Min}(3,\underset{1}{\Sigma}+1)}{\text{Min}} \left[\varphi_{11}(Z_{11}) + (6) \cdot \sum_{r_{11} > -1 + Z_{11}} (r_{11} + 1 - Z_{11}) P_{11}(r_{11}) + (1) \cdot \text{Max}(-1 + Z_{11}, 0) \right]$$

$$= \text{Min} \begin{bmatrix} 0.0 + 6.\ \{(1)(.20) + (2)(.25) + (3)(.30) + (4)(.25)\} + (1)(0) \\ 1.0 + 6.\ \{(1)(.25) + (2)(.30) + (3)(.25)\} \qquad + (1)(0) \\ 1.5 + 6.\ \{(1)(.30) + (2)(.25)\} \qquad + (1)(1) \\ 2.0 + 6.\ \{(1)(.25)\} \qquad + (1)(2) \end{bmatrix}$$

$$= \text{Min} \begin{bmatrix} 16.60 \\ 10.60 \\ 7.30 \\ 5.50 \end{bmatrix} = 5.50, \text{ where } Z^*_{11}(-1) = 3.$$

For other values of X_{11}, using (3-1-1) to (3-1-3), $f^*_1(X_{11})$ and $Z^*_{11}(X_{11})$ can be determined. The results are summarized below:

$f^*_1(0) = 4.50$; $Z^*_{11}(0) = 2$

$f^*_1(1) = 4.50$; $Z^*_{11}(1) = 1$ or 2

$f^*_1(2) = 4.00$; $Z^*_{11}(2) = 1$

$f^*_1(3) = 3.00$; $Z^*_{11}(3) = 0$

$f^*_1(4) = 4.00$; $Z^*_{11}(4) = 0$

$f^*_1(5) = 5.00$; $Z^*_{11}(5) = 0.$

Consider period 2. Using (3-1-4) to (3-1-6), for $X_{12} = 3$,

$$f^*_2(3)$$

$$= \underset{0 \le Z_{12} \le \text{Min}(5,\underset{1}{5}-3)}{\text{Min}} \left\{ \varphi_{12}(Z_{12}) + (6) \cdot \sum_{r_{12} > 3 + Z_{12}} (r_{12} - 3 - Z_{12}) P_{12}(r_{12}) + (0.9)\text{Max}(3 + Z_{12}, 0) + \sum_{r_{12} > 0} f^*_1(3+Z_{12}-r_{12}) P_{12}(r_{12}) \right\}$$

$$= \text{Min} \begin{bmatrix} 0.0 + 6.(1)(.15) + (.9)(3) + (3.0)(.10) + (4.0)(.20) + (4.5)(.35) + (4.5)(.20) + (5.5)(.15), \\ 1.1 + 0 + (.9)(4) + (4.0)(.10) + (3.0)(.20) + (4.0)(.35) + (4.5)(.20) + (4.5)(.15), \\ 1.7 + 0 + (.9)(5) + (5.0)(.10) + (4.0)(.20) + (4.0)(.35) + (4.0)(.20) + (4.5)(.15). \end{bmatrix}$$

$$= \text{Min} \begin{bmatrix} 8.00 \\ 8.68 \\ 10.03 \end{bmatrix} = 8.00; \text{ where } Z^*_{12}(3) = 0.$$

For other values of X_{12}, using (3-1-4) to (3-1-6), $f^*_1(X_{12})$ and $Z^*_{11}(X_{12})$ can be determined. The results are summarized below:

$f^*_2(4) = 7.58$; $Z^*_{12}(4) = 0$

$f^*_2(5) = 8.33$; $Z^*_{12}(5) = 0.$

Consider period 3. This period is a last stage and the initial inventory is known equal 4. Therefore, only the values of $f^*_3(4)$ and $Z^*_{13}(4)$ will be determined. Using (3-1-7) to (3-1-9), for $X_{13} = 4$,

$f^*_3(4)$

$$= \underset{0 \le Z_{13} \le \mathrm{Min}(4, \sum_1^5 - 4)}{\mathrm{Min}} \Big[\varphi_{13}(Z_{13}) + 6. \sum_{r_{13} > 4 + Z_{13}} (r_{13} - 4 - Z_{13}) P_{13}(r_{13})$$

$$+ 1.\ \mathrm{Max}(4 + Z_{13}, 0) + \sum_{r_{13} > 0} f^*_2(4 + Z_{13} - r_{13}) P_{13}(r_{13}) \Big]$$

$$= \mathrm{Min}\begin{bmatrix} 0 + 0 + (1)(4) + (7.58)(.55) + (8.00)(.45), \\ 1 + 0 + (1)(5) + (8.33)(.55) + (7.58)(.45) \end{bmatrix}.$$

$$= \mathrm{Min}\begin{bmatrix} 11.77 \\ 13.99 \end{bmatrix} = 11.77, \text{ where } Z^*_{13}(4) = 0.$$

Thus, the optimal decision at the period 3 is do not order and the minimum expected total system cost is \$11.77.

3.2 MULTI-ITEM SINGLE-SOURCE SYSTEM FOR THE MIXABLE ITEMS

This section is an extension of Section 3.1; several types of items are to be carried and they can be mixed together in the warehouse. There continues to be only one source of supply as in Section 3.1, and other assumptions remain the same as before. The analysis is as follows: Assume that there are N types of items in the system, and consider period 1. For a given set of $X_{11}, X_{21}, \ldots, X_{N1}$; assume that an order of the amount Z_{11} is made only for item type No. 1 at this period. A decision made at this period affects the system cost in period 1, and the expected total system cost is the sum of:

(1) item cost plus fixed cost of ordering Z_{11}, which is $\varphi_{11}(Z_{11})$,

(2) expected shortage cost due to the shortage of item type No. 1 in period 1, which is

$$Cs_{11}\sum_{r_{11} > X_{11} + Z_{11}}(r_{11} - X_{11} - Z_{11})P_{11}(r_{11}),$$

(3) carrying cost in carrying item type No. 1 in period 1, which is $Ch_{11}\,\mathrm{Max}(X_{11} + Z_{11}, 0)$,

(4) expected total shortage cost due to the shortages of item types No. 2 to No. N in period 1, which is

$$\sum_{i=2}^{N} Cs_{i1}\sum_{r_{i1} > X_{i1}}(r_{i1} - X_{i1})P_{i1}(r_{i1}),$$ and

(5) total carrying cost in carrying item types No. 2 to No. N in period 1, which is

$$\sum_{i=2}^{N} Ch_{i1} \cdot \mathrm{Max}(X_{i1}, 0).$$

Thus, the expected total controllable cost, $\mathcal{C}(U_{11}, U_{21}, \ldots, U_{N1}; Z_{11})$

$$= \varphi_{11}(Z_{11}) + Cs_{11}\sum_{r_{11} > X_{11} + Z_{11}}(r_{11} - X_{11} - Z_{11})P_{11}(r_{11})$$

$$+ Ch_{11} \cdot \mathrm{Max}(X_{11} + Z_{11}, 0) + K(X_{11}, X_{21}, \ldots, X_{N1}), \qquad (3\text{-}2\text{-}1)$$

where $K(X_{11}, X_{21}, \ldots, X_{N1})$

$$= \sum_{i=2}^{N} Cs_{i1}\sum_{r_{i1} > X_{i1}}(r_{i1} - X_{i1})P_{i1}(r_{i1}) + Ch_{i1} \cdot \mathrm{Max}(X_{i1}, 0). \qquad (3\text{-}2\text{-}2)$$

Note that for a given set of $X_{11}, X_{21}, \ldots, X_{N1}$, the space available for the additional items to be ordered in the period K will be

$$W - \sum_{i=1}^{N} v_i \cdot \mathrm{Max}(X_{i1}, 0).$$

In order to apply the principal of optimality to this problem, let w_1, the space available for the additional items type No. 1, increase in increments of v_1 from 0,

$v_1, 2v_1, \ldots, Cv_1, \ldots,$ to $W - \sum_{i=1}^{N} v_i \cdot \text{Max}(X_{i1}, 0)$.

Then, let $f_{11}(X_{11}, X_{21}, \ldots, X_{N1}/w_1)$ be the minimum expected total cost when a decision is made in period 1 when only item type No. 1 is being considered, resulting from ordering an optimal amount of $Z_{11} = Z^*_{11}(X_{11}, X_{21}, \ldots, X_{N1})$ for a given set of $X_{11}, X_{21}, \ldots,$ and w_1.

Therefore, $f_{11}(X_{11}, X_{21}, \ldots, X_{N1}/w_1) = \underset{Z_{11}}{\text{Min}}\ C(X_{11}, X_{21}, \ldots, X_{N1}; Z_{11})$

$$\tag{3-2-3}$$

where $$0 \leq Z_{11} \leq \text{Min}(S_{11}, \frac{w_1}{v_1} - \text{Min}(X_{11}, 0)). \tag{3-2-4}$$

For $w_1 = 0$, and $X_{11} < 0$; the restriction of Z_{11} in (3-2-4) becomes

$$0 \leq Z_{11} \leq \text{Min}(S_{11}, |X_{11}|). \tag{3-2-5}$$

For $w_1 = 0$, and $X_{11} \geq 0$; (3-2-3) becomes

$$f_{11}(X_{11}, X_{21}, \ldots, X_{N1}/0)$$

$$= \mathcal{C}(X_{11}, X_{21}, \ldots, X_{N1}; 0). \tag{3-2-6}$$

For $w_1 = v_1 \leq v_1(S_{11} + \text{Min}(X_{11}, 0))$; (3-2-4) becomes

$$0 \leq Z_{11} \leq 1 - \text{Min}(X_{11}, 0).$$

Then, $f_{11}(X_{11}, X_{21}, \ldots, X_{N1}/v_1)$

$$= \text{Min}\left\{\begin{matrix} f_{11}(X_{11}, X_{21}, \ldots, X_{N1}/0), \\ \mathcal{C}(X_{11}, X_{21}, \ldots, X_{N1}; 1 - \text{Min}(X_{11}, 0)) \end{matrix}\right\}. \tag{3-2-7}$$

In general, for $w_1 = Cv_1 \leq v_1(S_{11} + \text{Min}(X_{11}, 0))$,

$f_{11}(X_{11}, X_{21}, \ldots, X_{N1}/Cv_1)$

$$= \text{Min}\left\{\begin{matrix} f_{11}(X_{11}, X_{21}, \ldots, X_{N1}/Cv_1), \\ \mathcal{C}(X_{11}, X_{21}, \ldots, X_{N1}; C - \text{Min}(X_{11}, 0)) \end{matrix}\right\}. \tag{3-2-8}$$

For $w_1 = Cv_1 > v_1(S_{11} + \text{Min}(X_{11}, 0))$;

let $\bar{\bar{C}}v_1 \leq v_1(S_{11} + \text{Min}(X_{11}, 0) < (\bar{\bar{C}} + 1)v_1$,

then $S_{11} < (\bar{\bar{C}} + 1) - \text{Min}(X_{11}, 0)$, and $S_{11} \geq \bar{C} - \text{Min}(X_{11},0)$.

Therefore, using (3-2-3) and (3-2-4),

$$f_{11}(X_{11},X_{21},\ \ldots,\ X_{N1}/Cv_1)$$

$$= \text{Min}\left\{\begin{array}{l} f_{11}(X_{11},X_{21},\ \ldots,\ X_{N1}/\bar{C}v_1), \\ \tilde{C}(X_{11},X_{21},\ \ldots,\ X_{N1};S_{11}) \end{array}\right\}. \qquad (3\text{-}2\text{-}9)$$

For a given set of $X_{11},X_{21},\ \ldots,\ X_{N1}$, consider that orders are made only for item types No. 1 and No. 2, and that item type No. 2 is ordered first in the amount of Z_{21}. Let w_1, the space available for the additional item types No. 1 and No. 2, increase from 0 through the values of $w_1 + mv_1 (m = 0,\ 1,\ ..)$ until $W - \sum_{i=1}^{N} v_i \text{Max}(X_{i1},0)$. After Z_{21} is ordered, an optimal amount of item type No. 1 is ordered for a given set of $X_{11},X_{21} + Z_{21},\ \ldots,\ X_{N1}$, and for an available space of $w_2 - v_2\text{Max}(Z_{21} + \text{Min}(X_{21},0),0)$. Therefore, the expected total cost is the sum of:

(1) item cost plus fixed cost of ordering Z_{21}, which is $\varphi_{21}(Z_{21})$, and

(2) minimum expected total cost when a decision is made in period 1 when only item type No. 1 is considered, resulting from ordering an optimal amount of Z_{11} for a given set of $X_{11},X_{21} + Z_{21},\ \ldots,\ X_{N1}$ and for the space available $w_2 - v_2\text{Max}(Z_{21} + \text{Min}(X_{21},0),0)$, which is $f_{11}(X_{11},X_{21} + Z_{21},\ \ldots,\ X_{N1}/w_2 - v_2\ \text{Max}(Z_{21} + \text{Min}(X_{21},0),0))$.

Thus, the expected total cost, $\tilde{C}(X_{11},X_{21},\ \ldots,\ X_{N1};Z_{21}$

$$= \varphi_{21}(Z_{21}) + f_{11}(X_{11},X_{21} + Z_{21},\ \ldots,\ X_{N1}/w_2 - v_2\text{Max}(Z_{21} + \text{Min}(X_{21},0),0)). \qquad (3\text{-}2\text{-}10)$$

Then, let $f_{21}(X_{11},X_{21},\ \ldots,\ X_{N1}/w_2)$ be the minimum expected total cost when a decision is made in period 1 when item types No. 1 and No. 2

are considered and item type No. 2 is considered first, resulting from ordering and optimal amount of $Z_{21} = Z^*_{21}(X_{11}, X_{21}, \ldots, X_{N1})$ and presuming optimal amount of Z_{11} will be ordered later, for a given set of $X_{11}, X_{21}, \ldots, X_{N1}$ and w_2. Therefore,

$$f_{21}(X_{11}, X_{21}, \ldots, X_{N1}/w_2) = \underset{Z_{21}}{\text{Min}}\left\{\mathcal{C}(X_{11}, X_{21}, \ldots, X_{N1}; Z_{21})\right\}, \quad (3\text{-}2\text{-}11)$$

where $$0 \leq Z_{21} \leq \text{Min}(S_{21}, \frac{w_2}{v_2} - \text{Min}(X_{21}, 0)). \quad (3\text{-}2\text{-}12)$$

In general, item types No. 1 to No. n $(2 \leq n \leq N)$ are considered and item type No. n being considered first. The space available for the additional items of types No. 1 to No. n, w_n, increase from 0 through the values of $w_{n-1} + mv_n (m = 0, 1, \ldots)$ until $W - \sum_{i=1}^{N} v_i \text{Max}(X_{i1}, 0)$. Then, it follows that $\mathcal{C}(X_{11}, X_{21}, \ldots, X_{N1}; Z_{n1})$

$$= \varphi_{n1}(Z_{n1})$$
$$+ f_{n-1,1}(X_{11}, \ldots, X_{n1} + Z_{n1}, \ldots, X_{N1}/w_n - v_n \text{ Max}(Z_{n1} + \text{Min}(X_{n1}, 0), 0)), \quad (3\text{-}2\text{-}13)$$

and $$f_{n1}(X_{11}, X_{21}, \ldots, X_{N1}/w_n) = \underset{Z_{n1}}{\text{Min}}\left\{\mathcal{C}(X_{11}, X_{21}, \ldots, X_{N1}; Z_{n1})\right\}, \quad (3\text{-}2\text{-}14)$$

where $$0 \leq Z_{n1} \leq \text{Min}(S_{n1}, \frac{w_n}{v_n} - \text{Min}(X_{n1}, 0)). \quad (3\text{-}2\text{-}15)$$

By letting n = N, and letting $f^*_1(X_{11}, X_{21}, \ldots, X_{N1})$

$= f_{N1}(X_{11}, X_{21}, \ldots, X_{N1}/W - \sum_{i=1}^{N} v_i \text{ Max}(X_{i1}, 0))$ $f^*_1(X_{11}, X_{21}, \ldots, X_{N1})$ is obtained as a partial-optimization for this stage.

Consider period 2 and for a given set of $X_{12}, X_{22}, \ldots, X_{N2}$, assume that an order is made only for item type No. 1 in the amount of Z_{12} at this period. The decision made in this period affects the expected system cost for periods 2 and 1.

The expected total system cost is the sum of:

(1) item cost plus fixed cost of ordering Z_{12}, which is $\varphi_{12}(Z_{12})$,

(2) expected shortage cost due to the shortage of item type No. 1 in period 2, which is
$$Cs_{12} \sum_{r_{12} > X_{12} + Z_{12}} (r_{12} - X_{12} - Z_{12}) P_{12}(r_{12}),$$

(3) carrying cost in carrying item type No. 1 in period 2, which is $Ch_{12}\,\mathrm{Max}(X_{12} + Z_{12}, 0)$,

(4) expected shortage cost due to the shortages of item types No. 2 to No. N in the period 2, which is
$$\sum_{i=2}^{N} Cs_{i2} \sum_{r_{i2} > X_{i2}} (r_{i2} - X_{i2}) P_{i2}(r_{i2}),$$

(5) total carrying cost of carrying item types No. 2 to No. N in the period 2, which is
$$\sum_{i=2}^{N} Ch_{i2}\,\mathrm{Max}(X_{i2}, 0), \text{ and}$$

(6) minimum expected total cost, presuming an optimal decision is made in period 1, which is
$$\sum_{r_{12} > 0} \cdots \sum_{r_{N2} > 0} \Big\{ f^*_1(X_{12} + Z_{12} - r_{12}, X_{22} - r_{22}, \ldots, X_{N2} - r_{N2}) \sum_{i=1}^{N} P_{i2}(r_{i2}) \Big\}$$
$$= G(X_{12} + Z_{12}, X_{22}, \ldots, X_{N2}).$$

Thus, the expected total system cost, $\mathcal{C}(X_{12}, X_{22}, \ldots, X_{N2}; Z_{12})$

$$= \varphi_{12}(Z_{12})$$
$$+ Cs_{12} \sum_{r_{12} > X_{12} + Z_{12}} (r_{12} - X_{12} - Z_{12}) P_{12}(r_{12}) + Ch_{12}\,\mathrm{Max}(X_{12} + Z_{12}, 0)$$
$$+ K(X_{22}, C_{32}, \ldots, X_{N2}) + G(X_{12} + Z_{12}, X_{22}, \ldots, X_{N2}) \qquad (3\text{-}2\text{-}16)$$

where, $K(X_{22}, X_{32}, \ldots, X_{N2})$
$$= \sum_{i=2}^{N} Cs_{i2} \sum_{r_{i2} > X_{i2}} (r_{i2} - X_{i2}) P_{i2}(r_{i2}) + Ch_{i2}\,\mathrm{Max}(X_{i2}, 0). \qquad (3\text{-}2\text{-}17)$$

Note that for a given set of $X_{12}, X_{22}, \ldots, X_{N2}$, the available space for the additional items to be ordered in the period 2 is $W - \sum_{i=1}^{N} v_i \cdot \text{Max}(X_{i2}, 0)$.

As before, let w_1, the space available for the additional item type No. 1, increase in increments v_1 from $0, v_1, 2v_1, \ldots Cv_1, \ldots,$ to $W - \sum_{i=1}^{N} v_i \cdot \text{Max}(X_{i2}, 0)$.

Then, let $f_{12}(X_{12}, X_{22}, \ldots, X_{N2}/w_1)$ be the minimum expected total cost when a decision is made in period 2 where only item type No. 1 is considered, resulting from ordering an optimal amount of $Z_{12} = Z^*_{12}(X_{12}, X_{22}, \ldots, X_{N2}/w_1)$, presuming an optimal decision is made in period 1, for a given set of $X_{12}, X_{22}, \ldots, X_{N2}$ and w_2.

Therefore, $f_{12}(X_{12}, X_{22}, \ldots, X_{N2}/w_1) = \underset{Z_{12}}{\text{Min}} \left\{ \mathcal{C}(X_{12}, X_{22}, \ldots, X_{N2}; Z_{12}) \right\},$

$$\tag{3-2-18}$$

where $$0 \leq Z_{12} \leq \text{Min}(S_{12}, \frac{w_1}{v_1} - \text{Min}(X_{12}, 0)). \tag{3-2-19}$$

For $w_1 = 0$, and $X_{12} < 0$; the restriction of Z_{12} in (3-2-19) becomes

$$0 \leq Z_{12} \leq \text{Min}(S_{12}, |X_{12}|). \tag{3-2-20}$$

For $w_1 = 0$, and $X_{12} \geq 0$; (3-2-18) becomes

$$f_{12}(X_{12}, X_{22}, \ldots, X_{N2}/0) = \mathcal{C}(X_{12}, X_{22}, \ldots, X_{N2}; 0). \tag{3-2-21}$$

For $w_1 = Cv_1 \leq v_1(S_{12}, \text{Min}(X_{12}, 0))$;

$f_{12}(X_{12}, X_{22}, \ldots, X_{N2}/Cv_1)$

$$= \text{Min} \left\{ \begin{array}{l} f_{12}(X_{12}, X_{22}, \ldots, X_{N2}/(C-1)v_1), \\ \mathcal{C}(X_{12}, X_{22}, \ldots, X_{N2}; C - \text{Min}(X_{12}, 0)) \end{array} \right\}. \tag{3-2-22}$$

For $w = Cv_1 > v_1(S_{12} + \text{Min}(X_{12}, 0))$;

let $\bar{C}v_1 \leq v_1(S_{12} + \text{Min}(X_{12}, 0)) < (\bar{C} + 1)v_1$,

then $f_{12}(X_{12}, X_{22}, \ldots, X_{N2}/Cv_1)$

$$= \operatorname{Min}\left\{\begin{matrix} f_{12}(X_{12}, X_{22}, \ldots, X_{N2}/\bar{C}v_1), \\ \tilde{C}(X_{12}, X_{22}, \ldots, X_{N2}; S_{12}) \end{matrix}\right\}, \qquad (3\text{-}2\text{-}23)$$

As in previous discussions, if item types No. 1 to No. n ($2 \leq n \leq N$) are considered and item type No. n is considered first, for a given set of $X_{12}, X_{22}, \ldots, X_{n2}, \ldots, X_{N2}$ and w_n, it follows that

$$\tilde{C}(X_{12}, \ldots, X_{n2}, \ldots, X_{N2}; Z_{n2})$$

$$= \varphi_{12}(Z_{12})$$

$$+ f_{n-1,2}(X_{12}, \ldots, X_{n2} + Z_{n2}, \ldots, X_{N2}/w_n - v_n \operatorname{Max}(Z_{n2} + \operatorname{Min}(X_{n2}, 0), 0)), \qquad (3\text{-}2\text{-}24)$$

and $f_{n2}(X_{12}, \ldots, X_{n2}, \ldots, X_{N2}/w_n)$

$$= \operatorname*{Min}_{Z_{n2}}\left\{\tilde{C}(X_{12}, \ldots, X_{n2}, \ldots, X_{N2}/Z_{n2})\right\}, \qquad (3\text{-}2\text{-}25)$$

where $$0 \leq Z_{n2} \leq \operatorname{Min}\left(S_{n2}, \frac{w_n}{v_n} - \operatorname{Min}(X_{n2}, 0)\right). \qquad (3\text{-}2\text{-}26)$$

By letting n = N, and letting

$$f^*_2(X_{12}, X_{22}, \ldots, X_{N2})$$

$$= f_{N2}(X_{12}, X_{22}, \ldots, X_{N2}/W - \sum_{i=1}^{N} v_i \operatorname{Max}(X_{i2}, 0)).$$

$f^*_2(X_{12}, X_{22}, \ldots, X_{N2})$ is obtained a partial-optimization for this stage.

Consider in general period p, where $K + 1 \leq p \leq P$.

Using previous developments, it follows that

$$\tilde{C}(X_{1p}, X_{2p}, \ldots, X_{Np}; Z_{1p})$$

$$= \varphi_{1p}(Z_{1p})$$

$$+ Cs_{1p} \sum_{r_{1p} > X_{1p} + Z_{1p}} (r_{1p} - X_{1p} - Z_{1p}) P_{1p}(r_{1p}) + Ch_{1p} \operatorname{Max}(X_{1p} + Z_{1p}, 0)$$

$$+ K(X_{2p}, X_{3p}, \ldots, X_{Np}) + G(X_{1p} + Z_{1p}, X_{2p}, \ldots, X_{Np}), \qquad (3\text{-}2\text{-}27)$$

where $K(X_{2p}, X_{3p}, \ldots, X_{Np})$

$$= \sum_{i=2}^{N} Cs_{ip} \sum_{r_{ip} > X_{ip}} (r_{ip} - X_{ip}) P_{ip}(r_{ip}) + Ch_{ip} \operatorname{Max}(X_{ip}, 0)$$

and, $G(X_{1p}+Z_{1p}, X_{2p}, \ldots, X_{Np})$

$$= \sum_{\overline{r_{1p}>0}} \cdots \sum_{\overline{r_{Np}>0}} f^*_{p-1}(X_{1p}+Z_{1p}-r_{1p}, X_{2p}-r_{2p}, \ldots, X_{Np}-r_{Np}) \sum_{i=1}^{N} P_{ip}(r_{ip}).$$

Therefore, $f_{1p}(X_{1p}, X_{2p}, \ldots, X_{Np}/w_p)$

$$= \underset{Z_{1p}}{\text{Min}}\left\{\mathcal{C}(X_{1p}, X_{2p}, \ldots, X_{Np}; Z_{Np})\right\} \qquad (3\text{-}2\text{-}28)$$

where $$0 \leq Z_{1p} \leq \text{Min}(S_{1p}, \frac{w_1}{v_1} - \text{Min}(X_{1p}, 0)). \qquad (3\text{-}2\text{-}29)$$

For $w_1 = 0$, and $X_{1p} < 0$; the restriction of Z_{1p} in (3-2-29) becomes

$$0 \leq Z_{1p} \leq \text{Min}(S_{1p}, |X_{1p}|). \qquad (3\text{-}2\text{-}30)$$

For $w_1 = 0$, and $X \geq 0$; (5-2-28) becomes

$$f_{1p}(X_{1p}, X_{2p}, \ldots, X_{Np}/0) = \mathcal{C}(X_{1p}, X_{2p}, \ldots, X_{Np}; 0). \qquad (3\text{-}2\text{-}31)$$

For $w_1 = Cv_1 \leq v_1(S_{1p} + \text{Min}(X_{1p}, 0))$; it follows that

$f_{1p}(X_{1p}, X_{2p}, \ldots, X_{Np}/Cv_1)$

$$= \text{Min}\begin{Bmatrix} f_{1p}(X_{1p}, X_{2p}, \ldots, X_{Np}/(C-1)v_1), \\ \mathcal{C}(X\quad, X\quad, \ldots, X\quad; C\text{-Min}(X\quad, 0)) \end{Bmatrix}, \qquad (3\text{-}2\text{-}32)$$

and for $w_1 = Cv_1 > v_1(S_{1p} + \text{Min}(X_{1p}, 0))$;

$f_{1p}(X_{1p}, X_{2p}, \ldots, X_{Np}/Cv_1)$

$$= \text{Min}\begin{Bmatrix} f_{1p}(X_{1p}, X_{2p}, \ldots, X_{Np}/\bar{C}v_1) \\ \mathcal{C}(X_{1p}, X_{2p}, \ldots, X_{Np}; S_{1p}). \end{Bmatrix} \qquad (3\text{-}2\text{-}33)$$

where $$\bar{C}v_1 \leq v_1(S_{1p} + \text{Min}(X_{1p}, 0) > (\bar{C}+1)v_1. \qquad (3\text{-}2\text{-}34)$$

Again, using previous development, if item types No. 1 to No. n $(2 \leq n \leq N)$ are considered and item type No. n is considered first, it follows that

$$\mathcal{C}(X_{1p}, X_{2p}, \ldots, X_{Np}; Z_{np}) = \varphi_{np}(Z_{np})$$
$$+ f_{n-1,p}(X_{1p}, \ldots, X_{np}+Z_{np}, \ldots, X_{Np}/w_n - v_n \text{Max}(Z_{np} + \text{Min}(X_{np}, 0), 0)) \qquad (3\text{-}2\text{-}35)$$

and, $f_{np}(X_{1p}, \ldots, X_{np}, \ldots, X_{Np}/w_n)$

$$= \underset{Z_{np}}{\mathrm{Min}}\left\{\mathcal{C}(X_{1p}, \ldots, X_{np}, \ldots, X_{Np}; Z_{np})\right\} \qquad (3\text{-}2\text{-}36)$$

where $$0 \leq Z_{np} \leq \mathrm{Min}(S_{np}, \frac{w_n}{v_n} - \mathrm{Min}(X_{np}, 0)). \qquad (3\text{-}2\text{-}37)$$

By letting $n = N$, and let

$$f^*_p(X_{1p}, X_{2p}, \ldots, X_{Np}) = f_{Np}(X_{1p}, X_{2p}, \ldots, X_{Np}/W - \sum_{i=1}^{N} v_i \mathrm{Max}(X_{ip}, 0))$$

$f^*_p(X_{1p}, X_{2p}, \ldots, X_{Np})$ is obtained as a partial-optimization for this stage.

And if $p = P$, $f^*_p(X_{1P}, X_{2P}, \ldots, X_{NP})$ is the final optimization to the problem.

3.3 MULTI-ITEM SINGLE-SOURCE SYSTEM FOR THE NON-MIXABLE ITEMS

This section considers the case for the specific assumption in Section 2.3 when demands are probabilistic and orders are immediately fulfilled. Employing the discussion in Section 2.3, the system can be reduced, first, to single-item single-source. Thus, for item type No. i, one can use the development in Section 3.1 to determine $G_i(w)$, which is $f^*_p(0)$ for the selected value of w. And then the procedure to allocate space to each type of item will be the same as in Section 2.3.

3.4 SINGLE-ITEM MULTI-SOURCE SYSTEM

This section considers the case in which there is only one type of item but several sources of supply available in the system. Other assumptions are the same as previously used in the chapter.

Assume that there are J sources of supply which can supply the item for the period k. Let $\varphi_{1jk}(Z_{1jk})$ be the item cost plus fixed cost of ordering item type No. 1 in the amount of Z_{1jk} from source No. j in period k. And let S_{1jk} be the available supply of item type No. 1, from source No. j, in period k.

Thus, the total amount available in period k will be $\sum_{j=1}^{J} S_{1jk}$.

Let $\varphi^*_{1jk}(Z_{1k})$ be a minimum cost when Z_{1k} is ordered from sources No. 1 to No. j. Then, it follows that

$$\varphi^*_{11k}(Z_{1k}) = \varphi_{11k}(Z_{1k}),$$

where $Z_{1k} \leq S_{11k}$.

And that

$$\varphi^*_{1jk}(Z_{1k}) = \underset{0 \leq Z_{1jk} \leq \text{Min}\,(S_{1jk}, Z_{1k})}{\text{Min}} [\varphi_{1jk}(Z_{1jk}) + \varphi_{1,j-1,k}(Z_{1k} - Z_{1jk})],$$

where $Z_{1k} \leq \sum_{j'=1}^{j} S_{1j'k}$.

Increasing j until j = J, and letting $\varphi_{1k}(Z_{1k})$ to be $\varphi^*_{1Jk}(Z_{1k})$, $\varphi_{1k}(Z_{1k})$ becomes a minimum ordering cost function for a given value of Z_{1k}. Using this $\varphi_{1k}(Z_{1k})$ in Section 3.1, the system is reduced to a simple Single-Item Single-Source system.

3.5 MULTI-ITEM MULTI-SOURCE SYSTEM FOR THE MIXABLE ITEMS

It is assumed in this section that several types of items are to be carried, and several sources of supply are available in the system described previously in this chapter.

Assume that there are N types of items and J sources of supply which can supply the items in the period k.

For a particular item type No. i, following the discussion in Section 3.4:

$$\varphi^*_{i1k}(Z_{ik}) = \varphi_{i1k}(Z_{ik}),$$

where

$$Z_{ik} \leq S_{i1k}.$$

And $$\varphi^*_{ijk}(Z_{ik}) = \underset{0 \leq Z_{ijk} < \text{Min }(S_{ijk},\, Z_{ik})}{\text{Min}} [\varphi_{ijk}(Z_{ijk}) + \varphi^*_{i,j-1,k}(Z_{1k} - Z_{ijk})],$$

where

$$Z_{ik} \leq \sum_{j'=1}^{J} S_{ij'k}.$$

As in Section 3.4, let $\varphi_{ik}(Z_{ik}) = \varphi^*_{iJk}(Z_{ik})$, $\varphi_{ik}(Z_{ik})$ can be used in Section 3.2. Then the system is reduced to the Multi-Item Single-Source system.

3.6 MULTI-ITEM MULTI-SOURCE SYSTEM FOR NON-MIXABLE ITEMS

Different from Section 3.3, this section considers the case of multi-item multi-source. Employing the development in Section 2.3, the system can be reduced, first, to single-item multi-source. Thus, for item type No. i, one can use the development in Section 3.4 to determine $G_i(w)$, which is $f^*p(0)$ for the selected value of w. And then the procedure to allocate space to each type of item is the same as in Section 2.3.

CHAPTER IV

PROBABILISTIC DEMAND AND DETERMINISTIC PROCUREMENT LEAD TIME SYSTEM

In this chapter, extension of Chapter III, the case in which there is procurement lead time for the order made in each period will be considered. Procurement lead time being considered in this chapter is assumed to be deterministic, but not necessarily constant.

As already mentioned in Chapter I, assume that the excess demands in any period are allowed for deferring to a later period, and that an order made in any period will not arrive before those orders made previously. In this case, the decision being made at each period will be based on the amount of inventory at the beginning of the period plus the outstanding orders. A minimum expected total controllable cost for each period can also be determined by employing a minimum expected total controllable cost pre-determined in a previously calculated stage, using the following recurrence relation:

$$U_{ik} = U_{i,k+1} + Z_{i,k+1} - r_{i,k+1}.$$

4.1 SINGLE-ITEM SINGLE-SOURCE SYSTEM

This section considers the case wherein only a single type of item and only a single source of supply are available. The analysis is as follows:

Consider period K,

where $$K - L_K \geq 1$$

and $$K - 1 - L_{K-1} < 1.$$

This means that any order made after period K will arrive after the beginning of period 1.

For a given U_{1K}, assume that an amount Z_{1K} is ordered in this period and arrives L_K periods later. Therefore, a decision made at this period affects the system cost in periods $K - L_K$, $K - L_K - 1$, ..., 1. The system cost from period K to period $K - L_K + 1$ will be "uncontrollable cost", the system cost which is not affected by the decision made in this period.

The expected total controllable cost is the sum of:

(1) item cost plus fixed cost of ordering Z_{1K}, which is $\varphi_{1K}(Z_{1K})$,

(2) total expected controllable shortage cost in periods $K - L_K$, $K - L_K - 1$, ..., 1, which is

$$\sum_{k=1}^{K-L_K} \left\{ Cs_{1k} \cdot \sum_{r_1 > U_{1K} + Z_{1K}} (r_1 - U_{1K} - Z_{1K}) P(r_1 : K, k) \right\}, \text{ and}$$

(3) total expected controllable carrying cost in periods $K - L_K$, $K - L_K - 1$, ..., 1, which is

$$\sum_{k=1}^{K-L_K} \left\{ Ch_{1k} \cdot \sum_{r_1 < U_{1K} + Z_{1K}} (U_{1K} + Z_{1K} - r_1) P(r_1 : K, k+1) \right\}.$$

Note that sum of demands in determining the carrying cost will not include the demand in period k. Thus, the expected total controllable cost, $\mathcal{C}(U_{1K}; Z_{1K})$

$$= \varphi_{1K}(Z_{1K}) + \sum_{k=1}^{K-L_K} \left\{ Cs_{1k} \sum_{r_1 > U_{1K} + Z_{1K}} (r_1 - U_{1K} - Z_{1K}) P(r_1 : K, k) \right.$$

$$+ \, Ch_{1K} \sum_{r_1 < U_{1K} + Z_{1K}} (U_{1K} + Z_{1K} - r_1) P(r_1 : K, k+1) \Big\} .$$

(4-1-1)

Let $f^*_K(U_{1K})$ be a minimum expected total controllable cost for period K, resulting from ordering an optimal amount $Z_{1K} = Z^*_{1K}(U_{1K})$ for a given U_{1K}. Therefore,

$$f^*_K(U_{1K}) = \underset{Z_{1K}}{\mathrm{Min}} \Big\{ \mathcal{C}(U_{1K}; Z_{1K}) \Big\}, \qquad (4\text{-}1\text{-}2)$$

where $$0 \leq Z_{1K} \leq \mathrm{Min} \Big\{ S_{1K}, \; \frac{W}{v_1} - U_{1K} + \sum_{k=K-L_K+1}^{K} r_{1k} \Big\} . \qquad (4\text{-}1\text{-}3)$$

Consider period $K+1$ and for a given $U_{1,K+1}$, assume that an amount $Z_{1,K+1}$ is ordered at this period and arrives L_{K+1} periods later. Then, "controllable" periods are the periods $K+1-L_{K+1}$, $K-L_{K+1}$, ..., and 1. Note that, in this assumption, the order at period $K+1$ could not arrive after the order made at period K arrives.

For the first case, when the order made at period $K+1$ arrives before the order made at period K, L_{K+1} is less than or equal to L_K. The total expected controllable cost, presuming the optimal policy at period K, is the sum of:

(1) item cost plus fixed cost of ordering $Z_{1,K+1}$, which is $\varphi_{1,K+1}(Z_{1,K+1})$,

(2) total expected shortage cost from the period when an order made at period $K+1$ arrives to one period before an order made at period K arrives, which is

$$\sum_{k=K-L_{K+1}}^{K+1-L_{K+1}} \Big\{ Cs_{1k} \sum_{r_1 > U_{1,K+1} + Z_{1,K+1}} (r_1 - U_{1,K+1} - Z_{1,K+1}) P(r_1 : K+1, k) \Big\},$$

(3) total expected carrying cost from the period when

an order made at period K + 1 arrives to one period before an order made at period K arrives, which is

$$\sum_{k=K-L_{K+1}}^{K+1-L_{K+1}} \left\{ Ch_{1k} \sum_{r_1 < U_{1,K+1}+Z_{1,K+1}} (U_{1,K+1}+Z_{1,K+1} - r_1) \cdot P(r_1 : K+1, k+1) \right\},$$

and

(4) expected optimal controllable cost presuming an optimal decision is made at period K, which is

$$\sum_{r_{1,K+1}} f^*_K(U_{1,K+1} + Z_{1,K+1} - r_{1,K+1}) P(r_{1,K+1})$$

For the second case, when the order made at the period K + 1 arrives at the same time the order made at the period K arrives, L_{K+1} is equal to $L_K + 1$. The total expected controllable cost, presuming the optimal policy at period K, is the sum of:

(1) item cost plus fixed cost of ordering $Z_{1,K+1}$, which is $\varphi_{1,K+1}(Z_{1,K+1})$, and

(2) expected optimal controllable cost presuming an optimal decision is made at period K, which is

$$\sum_{r_{1,K+1}} f^*_K(U_{1,K+1} + Z_{1,K+1} - r_{1,K+1}) P(r_{1,K+1})$$

Thus, for both cases, expected total controllable cost,

$$\begin{aligned} \mathcal{C}(U_{1,K+1}; Z_{1,K+1}) &= \varphi_{1,K+1}(Z_{1,K+1}) \\ &+ \delta \sum_{k=K-L_K+1}^{K+1-L_{K+1}} \left\{ Cs_{1k} \sum_{r_1 > U_{1,K+1}+Z_{1,K+1}} (r_1 - U_{1,K+1} + Z_{1,K+1}) P(r_1 : K+1, k) \right. \\ &+ \left. Ch_{1k} \sum_{r_1 < U_{1,K+1} + Z_{1,K+1}} (U_{1,K+1} + Z_{1,K+1} - r_1) P(r_1 : K+1, k+1) \right\} \\ &+ \sum_{r_{1,K+1}} f^*_K(U_{1,K+1} + Z_{1,K+1} - r_{1,K+1}) P(r_{1,K+1}) \end{aligned} \quad (4\text{-}1\text{-}4)$$

where $\delta = 1$; for $L_{k+1} \le L_K$

$$= 0; \text{ for } L_{K+1} = L_K + 1.$$

Let $f^*_{K+1}(U_{1,K+1})$ be a minimum expected total controllable cost for period K+1, resulting from ordering an optimal amount of $Z_{1,K+1} = Z^*_{1,K+1}(U_{1,K+1})$ for a given $U_{1,K+1}$. Therefore,

$$f^*_{K+1}(U_{1,K+1}) = \min_{Z_{1,K+1}} \left\{ \mathcal{C}(U_{1,K+1}; Z_{1,K+1}) \right\},$$

where $$0 \leq Z_{1,K+1} \leq \text{Min}\left\{ S_{1,K+1},\ \frac{W}{v_1} - U_{1,K+1} + \sum_{k=K+L_{K+1}+2}^{K+1} \underline{r}_{1k} \right\}. \qquad (4\text{-}1\text{-}5)$$

Next, consider, in general, period p, where $K + 1 \leq p \leq P$.

$$\mathcal{C}(U_{1p}; Z_{1p}) = \varphi_{1p}(Z_{1p})$$
$$+ \delta \sum_{k=p-L_{p-1}}^{p-L_p} \left\{ Cs_{1k} \sum_{r_1 > U_{1p} + Z_{1p}} (r_1 - U_{1p} - Z_{1p}) P(r_1 : p, k) \right.$$
$$\left. + Ch_{1k} \sum_{r_1 < U_{1p} + Z_{1p}} (U_{1p} + Z_{1p} - r_1) P(r_1 : p, k+1) \right\}$$
$$+ \sum_{r_{1p}} f^*_{p-1}(U_{1p} + Z_{1p} - r_{1p}) P(r_{1p}), \qquad (4\text{-}1\text{-}6)$$

where $$\delta = 1; \text{ for } L_p \leq L_{p-1}$$
$$0; \text{ for } L_p = L_{p-1} + 1.$$

It follows then that

$$f^*_p(U_{1p}) = \min_{Z_{1p}} \left\{ \mathcal{C}(U_{1p}; Z_{1p}) \right\}, \qquad (4\text{-}1\text{-}7)$$

where $$0 \leq Z_{1p} \leq \text{Min}\left\{ S_{1p},\ \frac{W}{v_1} - U_{1p} + \sum_{k=p-L_p+1}^{p} \underline{r}_{1k} \right\}. \qquad (4\text{-}1\text{-}8)$$

<u>Example</u>

planning period,	$P = 5$
warehouse space,	$W = 5$ cubic units
a volume of an item,	$v_1 = 1$ cubic unit

procurement lead time, L_k = 2 periods (for all k)

initial inventory, U_{15} = 4 units

	k=1	k=2	k=3	k=4	k=5
S_{1k}-unit			3	5	4
Co_{1k}-dollars/order			0.50	0.50	0.50
Ci_{1k}-dollars/unit			0.50	0.60	0.50
Cs_{1k}-dollars/unit/period	6.00	6.00	6.00		
Ch_{1k}-dollars/unit/period	1.00	0.90	1.00		

r_{1k}	0	1	2	3	4
$P_{11}(r_{11})$	.20	.25	.30	.25	.00
$P_{12}(r_{12})$	.10	.20	.35	.20	.15
$P_{13}(r_{13})$	.55	.45	.00	.00	.00
$P_{14}(r_{14})$	.30	.40	.30	.00	.00
$P_{15}(r_{15})$	.50	.50	.00	.00	.00

Solution:

Using the data given above, the necessary values of $P(r_1:K,k)$ can be determined as shown in Table V.

Consider period 3. Using (4-1-1) to (4-1-3), for $U_{13} = 1$;

$f^*_3(1)$

$$= \underset{0 \le Z_{13} \le \text{Min}(3,\underline{5}-1+0)}{\text{Min}} \Big[\varphi_{13}(Z_{13}) + \sum_{k=1}^{1} \Big\{ Cs_{1k} \sum_{r_1 > 1 + Z_{13}} (r_1 - 1 - Z_{13}) \cdot P(r_1;3,k) + Ch_{1k} \sum_{r_1 < 1 + Z_{13}} (1 + Z_{13} - r_1) \cdot P(r_1;3,k+1) \Big\} \Big].$$

TABLE V

CUMULATIVE PROBABILITY OF DEMANDS FROM PERIODS k TO K, $P(r_1:K,k)$

r_1	0	1	2	3	4	5	6	7	8
$P(r_1:3,2)$	.055	.155	.283	.267	.173	.067	.000	.000	.000
$P(r_1:3,1)$	.011	.045	.112	.184	.225	.208	.136	.063	.016
$P(r_1:4,3)$	.165	.355	.345	.135	.000	.000	.000	.000	.000
$P(r_1:4,2)$	.017	.069	.163	.240	.244	.169	.078	.020	.000
$P(r_1:5,4)$	.150	.350	.350	.150	.000	.000	.000	.000	.000
$P(r_1:5,3)$	.083	.260	.350	.240	.067	.000	.000	.000	.000

$$= \underset{0 \le Z_{13} \le 3}{\text{Min}} \left[\varphi_{13}(Z_{13}) + Cs_{11} \sum_{r_1 > 1 + Z_{13}} (r_1 - 1 - Z_{13}) P(r_1 : 3, 1) + Ch_{11} \sum_{r_1 < 1 + Z_{13}} (1 + Z_{13} - r_1) P(r_1 : 3, 2) \right].$$

$$= \text{Min} \begin{bmatrix} 0.0 + (6)\{(1)(.112) + (2)(.184) + (3)(.225) + (4)(.208) + (5)(.136) \\ \quad + (6)(.063) + (7)(.016)\} + (1)\{(1)(.055)\}, \\ 1.0 + (6)\{(1)(.184) + (2)(.225) + (3)(.208) + (4)(.136) + (5)(.063) \\ \quad + (6)(.016)\} + (1)\{(1)(.155) + (2)(.055)\}, \\ 1.5 + (6)\{(1)(.225) + (2)(.208) + (3)(.136) + (4)(.063) + (5)(.016)\} \\ \quad + (1)\{(1)(.283) + (2)(.155) + (3)(.055)\}, \\ 2.0 + (6)\{(1)(.225) + (2)(.208) + (3)(.136) + (4)(.016)\} \\ \quad + (1)\{(1)(.267) + (2)(.283) + (3)(.155) + (4)(.055)\}. \end{bmatrix}$$

$$= \text{Min} \begin{bmatrix} 18.997 \\ 14.543 \\ 10.544 \\ 7.916 \end{bmatrix} = 7.916; \text{ where } Z^*_{13}(1) = 3.$$

For other values of U_{13}, $f^*_3(U_{13})$ and $Z^*_{13}(U_{13})$ can be determined. The results are summarized below:

$f^*_3(2) = 6.311;\ Z^*_{13}(2) = 3$ $\qquad$ $f^*_3(4) = 5.311;\ Z^*_{13}(4) = 1$

$f^*_3(3) = 5.811;\ Z^*_{13}(3) = 2$ $\qquad$ $f^*_3(5) = 4.311;\ Z^*_{13}(5) = 0.$

Consider period 4. Using (4-1-6) to (4-1-8), for $U_{14} = 3$;

$f^*_4(3)$

$$= \underset{0 \le Z_{14} \le \text{Min}(5, \underset{1}{5} - 3 + 0)}{\text{Min}} \left[\varphi_{14}(Z_{14}) + \sum_{k=2}^{2} Cs_{1k} \sum_{r_1 > 3 + Z_{14}} (r_1 - 3 - Z_{14}) P(r_1 : 4, k) + Ch_{1k} \sum_{r_1 < 3 + Z_{14}} (3 + Z_{14} - r_1) P(r_1 : 4, k+1) + \sum_{r_{14} > 0} f^*_3(3 + Z_{14} - r_{14}) P_{14}(r_{14}) \right].$$

$$= \min_{0 \le Z_{14} \le (5,2)} \Big[\varphi_{14}(Z_{14}) + Cs_{12} \sum_{r_1 > 3 + Z_{14}} (r_1 - 3 - Z_{14}) P(r_1 : 4,2)$$

$$+ Ch_{12} \sum_{r_1 > 3 + Z_{14}} (3 + Z_{14} - r_1) P(r_1 : 4,3)$$

$$+ \sum_{r_{14} > 0} f^*_3(3 + Z_{14} - r_{14}) P_{14}(r_{14}) \Big].$$

$$= \text{Min} \begin{bmatrix} 0.0 + (6)\{(1)(.244) + (2)(.169) + (3)(.078) + (4)(.020)\} \\ \quad + (.9)\{(1)(.345) + (2)(.355) + (3)(.165)\} + (.30)(5.811) \\ \quad + (.4)(6.311) + (.3)(7.916), \\ 1.1 + (6)\{(1)(.169) + (2)(.078) + (3)(.020)\} + (.9)\{(1)(.135) \\ \quad + (2)(.345) + (3)(.355) + (4)(.165)\} + (.3)(5.311) \\ \quad + (.4)(5.811) + (.3)(6.311), \\ 1.7 + (6)\{(1)(.078) + (2)(.020)\} + (.9)\{(2)(.135) + (3)(.345) \\ \quad + (4)(.355) + (5)(.165)\} + (.3)(4.311) + (.4)(5.311) \\ \quad + (.3)(5.811). \end{bmatrix}.$$

$$= \text{Min} \begin{bmatrix} 13.413 \\ 11.516 \\ 8.184 \end{bmatrix} = 8.184; \text{ where } Z^*_{14}(4) = 2.$$

For other values of U_{14}, $f^*_4(U_{14})$ and $Z^*_{14}(U_{14})$ can be determined. The results are shown below:

$$f^*_4(4) = 7.584 \quad ; \quad Z^*_{14}(4) = 1$$

$$f^*_4(5) = 6.484 \quad ; \quad Z^*_{14}(5) = 0.$$

Consider period 5, which is the last stage. Using (4-1-6) to (4-1-8); $f^*_5(4)$

$$= \min_{0 \le Z_{15} \le \text{Min}(4,1)} \Big[\varphi_{15}(Z_{15}) + Cs_{13} \sum_{r_1 > 4 + Z_{15}} (r_1 - 4 - Z_{15}) P(r_1 : 5,3)$$

$$+ Ch_{13} \sum_{r_1 < 4 + Z_{15}} (4 + Z_{15} - r_1) P(r_1 : 5,4)$$

$$+ \sum_{r_{15} > 0} f^*_4(4 + Z_{15} - r_{15})P_{15}(r_{15}) \Big].$$

$$= \text{Min}\begin{bmatrix} 0 + (6)(0) + (1)\{(1)(.15) + (2)(.35) + (3)(.35) + (4)(.15)\} \\ + (.5)(7.584) + (.5)(8.184), \\ 1 + (6)(0) + (1)\{(2)(.15) + (3)(.35) + (4)(.35) + (5)(.15)\} \\ + (.5)(6.485) + (.5)(7.584). \end{bmatrix}$$

$$= \text{Min}\begin{bmatrix} 10.384 \\ 11.534 \end{bmatrix} = 10.384; \text{ where } Z^*_{15}(4) = 0.$$

Therefore, the optimal policy in period 5 is do not make an order. The minimum expected total cost is \$10.384.

4.2 MULTI-ITEM SINGLE-SOURCE SYSTEM FOR THE MIXABLE ITEMS

In this section, an extension of Section 4.1, several types of items are to be carried and they can be mixed together in the warehouse. There continues to be only one source of supply as in Section 4.1, and other assumptions remain the same as before. The analysis is as follows.

Assume that there are N types of items in the system, and consider period K,

$$\text{where } K - L_K \geq 1,$$
$$\text{and } K - 1 - L_{K-1} > 1.$$

This means that if an order is made at period K, the order will arrive before the beginning of period 1. But if an order is made at the period $K-1$, the order will not arrive before the beginning of period 1.

For a given set of $U_{1K}, U_{2K}, \ldots, U_{NK}$, assume that an order of the amount Z_{1K} is made only for item type No. 1 at this period. The expected total controllable cost will include those expected system costs in periods $K-L_K$, $K-L_K-1$, ..., 1, which is the sum of:

(1) item cost plus fixed cost of ordering Z_{1K}, which is $\varphi_{1K}(Z_{1K})$,

(2) total expected controllable cost due to shortage of item type No. 1, which is

$$\sum_{k=1}^{K-L_K} Cs_{1k} \sum_{r_1 > U_{1K} + Z_{1K}} (r_1 - U_{1K} - Z_{1K}) P(r_1; K, k),$$

(3) total expected controllable cost in carrying item type No. 1, which is

$$\sum_{k=1}^{K-L_K} Ch_{1k} \sum_{r_1 < U_{1K} + Z_{1K}} (U_{1K} + Z_{1K} - r_1) P(r_1; K, k+1),$$

(4) total expected controllable shortage cost due to shortages of item types No. 2 to No. N, which is

$$\sum_{i=2}^{N} \sum_{k=1}^{K-L} Cs_{ik} \sum_{r_i > U_{iK}} (r_i - U_{iK}) P(r_i; K, k), \text{ and}$$

(5) total expected controllable carrying cost in carrying item types No. 2 to No. N, which is

$$\sum_{i=2}^{N} \sum_{k=1}^{K-L} Ch_{ik} \sum_{r_i < U_{iK}} (U_{iK} - r_i) P(r_i; K, k+1).$$

Thus, the expected total controllable cost, $\mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}; Z_{1K})$

$$= \varphi_{1K}(Z_{1K}(Z_{1K})$$

$$+ \sum_{k=1}^{K-L_K} \Big\{ Cs_{1k} \sum_{r_1 > U_{1K} + Z_{1K}} (r_1 - U_{1K} - Z_{1K}) P(r_1; K, k)$$

$$+ Ch_1 \sum_{r_1 < U_{1K} + Z_{1K}} (U_{1K} + Z_{1K} - r_1) P(r_1; K, k+1) \Big\}$$

$$+ K_K(U_{2K}, U_{3K}, \ldots, U_{NK}), \qquad (4\text{-}2\text{-}1)$$

where $K_K(U_{2K}, U_{3K}, \ldots, U_{NK})$

$$= \sum_{i=2}^{N} \sum_{k=1}^{K-L_K} \Big\{ Cs_{ik} \sum_{r_i > U_{iK}} (r_i - U_{iK}) P(r_i; K, k)$$

$$+ Ch_{ik} \sum_{r_i < U_{iK}} (U_{iK} - r_i) P(r_i; K, k+1). \qquad (4\text{-}2\text{-}2)$$

Note that for a given set of U_{1K}, U_{2K}, ..., U_{NK}, the space available for the addition items to be ordered in period K will be

$$W - \sum_{i=1}^{N} v_i \cdot \text{Max}(U_{iK} - \sum_{k=K-L_K+1}^{K} r_{ik}, 0).$$

In order to apply the principle of optimality to this problem, let w_1, the space available for the additional items type No. 1, increase in increments of v_1 from 0, v_1, $2v_1$, ..., Cv_1, ... to

$$W - \sum_{i=1}^{N} v_i \ \text{Max}(U_{iK} - \sum_{k=K-L_K+1}^{K} r_{ik}, 0).$$

Then let $f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/w_1)$ be the minimum expected total controllable cost when a decision is made in period K where only item type No. 1 is being considered, resulting from ordering an optimal amount of $Z_{1K} = Z^*_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK})$ for a given set of U_{1K}, U_{2K}, ..., U_{NK} and w_1. Therefore,

$$f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/w_1) = \text{Min}_{Z_{1K}} \mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}; Z_{1K}), \qquad (4\text{-}2\text{-}3)$$

$$\text{where} \quad 0 \leq Z_{1K} \leq \text{Min}(S_{1K}, \frac{w_1}{v_1} - \text{Min}(U_{1K} - \sum_{k=K-L_K+1}^{K} r_{1k}, 0)). \qquad (4\text{-}2\text{-}4)$$

For $w_1 = 0$, and $U_{1K} - \sum_{k=K-L_K+1}^{K} r_{1k} < 0$; the restriction of Z_{1K} in (4-2-4) becomes

$$0 \leq Z_{1K} \leq \text{Min}(S_{1K}, \sum_{k=K-L_K+1}^{K} r_{1k} - U_{1k}). \qquad (4\text{-}2\text{-}5)$$

For $w_1 = 0$, and $U_{1K} - \sum_{k=K-L_K+1}^{K} r_{1k} \geq 0$; (4-2-3) becomes $f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/0) = \mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK};0)$. (4-2-6)

For $w_1 = v_1 \leq v_1(S_{1K} + \mathrm{Min}(U_{1K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{1k}, 0))$;

(4-2-4) becomes

$$0 \leq Z_{1K} \leq 1 - \mathrm{Min}(U_{1K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{1k}, 0)).$$

Then, $f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/v_1$

$$= \mathrm{Min}\left\{\begin{matrix} f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/0), \\ \mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}; 1 - \mathrm{Min}(U_{1K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{1k}, 0)) \end{matrix}\right\}. \quad (4\text{-}2\text{-}7)$$

In general, for $w_1 = Cv_1 \leq v_1(S_{1K} + \mathrm{Min}(U_{1K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{1k}, 0))$,

$f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/Cv_1)$

$$= \mathrm{Min}\left\{\begin{matrix} f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/Cv_1), \\ \mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}; C - \mathrm{Min}(U_{1K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{1k}, 0)) \end{matrix}\right\}. \quad (4\text{-}2\text{-}8)$$

For $w_1 = Cv_1 \geq v_1(S_{1K} + \mathrm{Min}(U_{1K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{1k}, 0))$;

Let $\bar{C}v_1 \leq v_1(S_{1K} + \mathrm{Min}(U_{1K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{1k}, 0)) < (\bar{C}+1)v_1$

then, $S_{1K} < (\bar{C} + 1) - \mathrm{Min}(U_{1K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{1k}, 0)$,

and $S_{1K} \geq \bar{C} - \mathrm{Min}(U_{1K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{1k}, 0)$.

Therefore, using (4-2-3) and (4-2-4); $f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/Cv_1)$

$$= \mathrm{Min}\left\{\begin{matrix} f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/\bar{C}v_1), \\ \mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}; S_{1K}) \end{matrix}\right\}. \quad (4\text{-}2\text{-}9)$$

For a given set of $U_{1K}, U_{2K}, \ldots, U_{NK}$, consider that orders are made for item types No. 1 and No. 2, and that item type No. 2 is ordered first in the amount of Z_{2K}. Let w_2, the space available for the additional item types No. 1 and No. 2, increase from 0 through the values of $w_1 + mv_2 (m = 0, 1, \ldots)$ until $W - \sum_{i=1}^{N} v_i \; \mathrm{Max}(U_{1K} - \sum_{k=K-L_K+1}^{K} r_{1k}, 0)$.

After Z_{2K} is ordered, an optimal amount of item type No. 1 is ordered for a given set of U_{1K}, $U_{2K} + Z_{2K}$, ..., U_{NK}, and for an available space of $w_2 - v_2 \; \mathrm{Max}(Z_{2K} + \mathrm{Min}(U_{2K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{2k}, 0, 0)$. Therefore, the expected total controllable cost is the sum of:

(1) item cost plus fixed cost of ordering Z_{2K}, which is $\varphi_{2K}(Z_{2K})$,

(2) minimum expected total controllable cost when a decision is made in period K when only item type No. 1 is considered, resulting from ordering an optimal amount of Z_{1K} for a given set of U_{1K}, $U_{2K} + Z_{2K}$, ..., U_{NK}, and for a space $w_2 - v_2 \; \mathrm{Max}(Z_{2K} + \mathrm{Min}(U_{2K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{2k}, 0, 0)$, which is $f_{1K}(U_{1K}, U_{2K} + Z_{2K}, \ldots, U_{NK}/w_2 - v_2 \; \mathrm{Max}(Z_{2K} + \mathrm{Min}(U_{2K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{2k}, 0, 0)$.

Thus, the expected total controllable cost, $\mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}/Z_{2K})$

$$= \varphi_{2K}(Z_{2K})$$

$$+ f_{1K}(U_{1K}, U_{2K} + Z_{2K}, \ldots, U_{NK}/w_2 - v_2 \mathrm{Max}(Z_{2K} + \mathrm{Min}(U_{2K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{2k}, 0, 0). \tag{4-2-10}$$

Then, let $f_{2K}(U_{1K}, U_{2K}, \ldots, U_{NK}/w_2)$ be the minimum expected total controllable cost when a decision is made in period K when item types No. 1 and No. 2 are considered and item type No. 2 is considered first, resulting from ordering an optimal amount of $Z_{2K} = Z^*_{2K}(U_{1K}, U_{2K}, \ldots, U_{NK})$ presuming optimal amount of Z_{1K} will be ordered later, for a given set of U_{1K}, U_{2K}, ..., U_{NK} and w_2.

Therefore, $f_{2K}(U_{1K}, U_{2K}, \ldots, U_{NK}/w_2) = \underset{Z_{2K}}{\text{Min}}\left\{\mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}/Z_{2K})\right\}$, (4-2-11)

$$\text{where } 0 \leq Z_{2K} \leq \text{Min}(S_{2K}, \frac{w_2}{v_2} - \text{Min}(U_{2K} - \sum_{k=K-L_K+1}^{K} \underline{r}_{2k}, 0). \quad (4\text{-}2\text{-}12)$$

In general, item types No. 1 to No. n $(2 \leq n \leq N)$ are considered and item type No. n being considered first. The space available for the additional item types No. 1 to No. n, increase from 0 through the values of $w_{n-1} + mv_n$ $(m = 0, 1, \ldots)$ until $W - \sum_{i=1}^{N} v_i \text{Max}(U_{iK} - \sum_{k=K-L_K+1}^{K} \underline{r}_{ik}, 0)$.

Then, it follows that $\mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}/Z_{nK}) = \varphi_{nK}(Z_{nK})$

$$+ f_{n-1,K}(U_{1K}, \ldots, U_{nK} + Z_{nK}, \ldots, U_{NK}/w_n - v_n$$

$$\text{Max}(Z_{nK} + \text{Min}(U_{nK} - \sum_{k=K-L_K+1}^{K} \underline{r}_{nK}, 0), 0), \quad (4\text{-}2\text{-}13)$$

and $f_{2K}(U_{1K}, U_{2K}, \ldots, U_{NK}/w_n) = \underset{Z_{nK}}{\text{Min}}\left\{\mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}/Z_{nK})\right\}$, (4-2-14)

$$\text{where } 0 \leq Z_{nK} \leq \text{Min}(S_{nK}, \frac{w_n}{v_n} - \text{Min}(U_{nK} - \sum_{k=K-L_K+1}^{K} \underline{r}_{nk}, 0). \quad (4\text{-}2\text{-}15)$$

By letting $n = N$, and let $f^*_K(U_{1K}, U_{2K}, \ldots, U_{NK}) =$

$$f_{NK}(U_{1K}, U_{2K}, \ldots, U_{NK}/W - \sum_{i=1}^{N} v_i \ \text{Max}(U_{ik} - \sum_{k=K-L_K+1}^{K} \underline{r}_{ik}, 0)$$

$f^*_K(U_{1K}, U_{2K}, \ldots, U_{NK})$ is obtained as a partial-optimization for this stage.

Consider period $K + 1$, for a given set of $U_{1,K+1}$, $U_{2,K+1}$, ..., $U_{N,K+1}$, assume that an order is made only for item type No. 1 in the amount of $Z_{1,K+1}$ at this period. The decision made in this period affects those expected system costs in periods $K+L-L_{K+1}$, $K-L_{K+1}$, ..., and 1.

The expected total controllable cost in the sum of:

(1) item cost plus fixed cost of ordering $Z_{1,K+1}$, which

is $\phi_{1,K+1}(Z_{1,K+1})$,

(2) total expected controllable shortage cost due to the shortage of item type No. 1, during the periods from the period when the order made in period K + 1 arrives to one period before the order made in period K arrives, which is

$$\delta \sum_{k=K-L_K+1}^{K-L_{K+1}} Cs_{1k} \sum_{r_1 > U_{1,K+1} + Z_{1,K+1}} (r_1 - U_{1,K+1} - Z_{1,K+1})$$

$P(r_1;K+1,k)$, where $\delta = 1$; for $L_{K+1} + 1 < L_K = 0$, otherwise.

(3) total expected controllable carrying cost in carrying item type No. 1, during the periods from the period when the order made in period K+1 arrives to one period before the order made in period K arrives, which is

$$\delta \sum_{k=K-L_{K+1}}^{K-L_{K+1}} Ch_{1k} \sum_{r_1 > U_{1,K+1} + Z_{1,K+1}} (r_1 - U_{1,K+1} - Z_{1,K+1})\cdot$$

$P(r_1;K+1, k+1)$,

(4) total expected controllable shortage cost due to the shortages of item types No. 2 to No. N, during the periods from the period when the order made in the period K+1 arrives to one period before the order made in period K arrives, which is

$$\delta \sum_{i=2}^{N} \sum_{k=K-L_K+1}^{K-L_{K+1}} Cs_{ik} \sum_{r_i > U_{i,K+1}} (r_i - U_{i,K+1})P(r_i;K+1,k),$$

(5) total expected controllable carrying cost in carrying item types No. 2 to No. n, during the periods from the period where the order made in period K+1 arrives to one period before the order made in the period K

arrives, which is

$$\delta \sum_{i=2}^{N} \sum_{k=K-L_K+1}^{K-L_{K+1}} Ch_{ik} \sum_{r_i < U_{i,K+1}} (U_{i,K+1} - r_i)\cdot$$

$P(r_i;K+1, k+1)$, and

(6) minimum expected total controllable cost, presuming an optimal decision is made at period K, which is

$$\sum_{r_{1,K+1} \geq 0} \cdots \sum_{r_{N,K+1} \geq 0} f^*_K(U_{1,K+1} + Z_{1,K+1} - r_{1,K+1}, U_{2,K+1} - r_{2,K+1}, \ldots, U_{N,K+1} - r_{N,K+1}) \prod_{i=1}^{N} P(r_{i,K+1}).$$

$$= G(U_{1,K+1}, \ldots, U_{N,K+1}).$$

Thus, the expected total controllable cost,

$$\mathcal{C}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}; Z_{1,K+1}) = \varphi_{1,K+1}(Z_{1,K+1})$$

$$+ \delta \sum_{k=K-L_K+1}^{K-L_{K+1}} \Big\{ Cs_{1k} \sum_{r_1 > U_{1,K+1} + Z_{1,K+1}} (r_1 - U_{1,K+1} - Z_{1,K+1}) P(r_1;K+1,k)$$

$$+ Ch_{1k} \sum_{r_1 < U_{1,K+1} + Z_{1,K+1}} (U_{1,K+1} + Z_{1,K+1} - r_1) P(r_1;K+1,k+1)$$

$$+ K(U_{2,K+1}, U_{3,K+1}, \ldots, U_{N,K+1}) + G(U_{1,K+1}, \ldots, U_{N,K+1}) \qquad (4\text{-}2\text{-}16)$$

where $K(U_{2,K+1}, U_{3,K+1}, \ldots, U_{N,K+1})$

$$= \delta \sum_{i=2}^{N} \sum_{k=K-L_K+1}^{K-L_{K+1}} \Big\{ Cs_{ik} \sum_{r_i > U_{i,K+1}} (r_i - U_{i,K+1}) P(r_i;K+1,k)$$

$$+ Ch_{ik} \sum_{r_i < U_{i,K+1}} (U_{i,K+1} - r_i) P(r_i;K+1,k+1) \Big\}. \qquad (4\text{-}2\text{-}17)$$

Note that for a given set of $U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}$, the space available for the additional items to be ordered in period $K+1$ is $W - \sum_{i=1}^{N} v_i \cdot \text{Max}(U_{i,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} r_{ik}, 0)$.

As before, let w_1, the space available for the additional item type No. 1, increase in increments v_1 from 0

$v_1, 2v_1, \ldots, Cv_1, \ldots$ to $W = \sum_{i=1}^{N} v_i \ \text{Max}(U_{i,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} r_{ik}, 0)$.

Then, let $f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/w_1)$ be the minimum expected total controllable cost when a decision is made in period $K+1$ when only item type No. 1 is considered, resulting from ordering an optimal amount of $Z_{1,K+1} = Z^*_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1})$, presuming an optimal decision is made in period K, for a given set of $U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}$ and w_1.

Therefore, $f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/w_1)$

$$= \underset{Z_{1,K+1}}{\text{Min}} \left\{ \mathcal{C}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}; Z_{1,K+1}) \right\}, \qquad (4\text{-}2\text{-}18)$$

where $0 \leq Z_{1,K+1} \leq \text{Min}(S_{1,K+1}, \frac{w_1}{v_1} - \text{Min}(U_{1,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} r_{1k}, 0))$.

$$(4\text{-}2\text{-}19)$$

For $w_1 = 0$, and $U_{1,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} r_{1k} < 0$: the restriction $Z_{1,K+1}$ in (4-2-19) becomes

$$0 \leq Z_{1,K+1} \leq \text{Min}(S_{1,K+1}) \sum_{k=K-L_{K+1}+2}^{K+1} r_{1k} - U_{1,K+1}). \qquad (4\text{-}2\text{-}20)$$

For $w_1 = 0$, and $U_{1,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} r_{1k} \geq 0$, (4-2-18) becomes

$f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/0)$

$$= \mathcal{C}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}; 0 \qquad (4\text{-}2\text{-}21)$$

For $w_1 = Cv_1 \leq v_1(S_{1,K+1} + \text{Min}(U_{1,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} r_{1k}, 0))$,

$f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/Cv_1)$

$$= \text{Min} \begin{bmatrix} f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/(C-1)v_1), \\ \mathcal{C}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}; C - \text{Min}(U_{1,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} r_{1k}, 0)) \end{bmatrix}. \qquad (4\text{-}2\text{-}22)$$

For $w_1 = Cv_1 > v_1(S_{1,K+1} + \text{Min}(U_{1,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} \underline{r}_{1k}, 0))$;

let $\bar{C}v_1 \leq v_1(S_{1,K+1} + \text{Min}(U_{1,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} \underline{r}_{1k}, 0)) < (\bar{C}+1)v_1$,

then $f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/Cv_1)$

$$= \text{Min}\begin{Bmatrix} f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/\bar{C}v_1), \\ \tilde{C}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}; S_{1,K+1}) \end{Bmatrix}. \tag{4-2-23}$$

As in previous discussions, if item types No. 1 to No. n $(2 \leq n \leq N)$ are considered and item type No. n is considered first, for a given set of $U_{1,K+1}, \ldots, U_{n,K+1}, \ldots, U_{N,K+1}$ and for w_n, it follows that

$$\tilde{C}(U_{1,K+1}, \ldots, U_{n,K+1}, \ldots, U_{N,K+1}; Z_{n,K+1}) = \varphi_{n,K+1}(Z_{n,K+1})$$
$$+ f_{n-1,K+1}(U_{1,K+1}, \ldots, U_{n,K+1} + Z_{n,K+1}, \ldots, U_{N,K+1}/w_n - v_n \text{ Max}(Z_{n,K+1}$$
$$+ \text{Min}(U_{n,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} \underline{r}_{nk}, 0), 0)), \tag{4-2-24}$$

and $f_{n,K+1}(U_{1,K+1}, \ldots, U_{n,K+1}, \ldots, U_{N,K+1}/w_n)$

$$= \underset{Z_{n,K+1}}{\text{Min}} \left\{\tilde{C}(U_{1,K+1}, \ldots, U_{n,K+1}, \ldots, U_{N,K+1}; Z_{n,K+1})\right\}, \tag{4-2-25}$$

where $0 \leq Z_{n,K+1} \leq \text{Min}(S_{n,K+1}, \frac{w_n}{v_n} - \text{Min}(U_{n,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} \underline{r}_{nk}, 0))$

(4-2-26)

By letting $n = N$, and let $f^*_{K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1})$

$$= f_{N,K+1}(U_{1,K+1}, \ldots, U_{N,K+1}/W - \sum_{i=1}^{N} v_i \text{ Max}(U_{i,K+1} - \sum_{k=K-L_{K+1}+2}^{K+1} \underline{r}_{ik}, 0))$$

$f^*_{K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1})$ is obtained as a partial-optimization for this stage.

Consider, in general, period p, where

$$K + 1 \leq p < P.$$

Using the previous developments, it follows that

$$\tilde{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; Z_{1p}) = \varphi_{1p}(Z_{1p})$$

$$+ \delta \sum_{k=K-L_{K-1}}^{K-L_p} \Big\{ Cs_{1k} \sum_{r_1 > U_{1p} + Z_{1p}} (r_1 - U_{1p} - Z_{1p}) P(r_1; p, k)$$

$$+ Ch_{1k} \sum_{r_1 < U_{1p} + Z_{1p}} (U_{1p} + Z_{1p} - r_1) P(r_1; p, k+1) \Big\}$$

$$+ K(U_{2p}, U_{3p}, \ldots, U_{Np}) + G(U_{1p} + Z_{1p}, U_{2p}, \ldots, U_{Np}), \qquad (4\text{-}2\text{-}27)$$

where $\delta = 1$; for $L_p + 1 < L_{p-1}$

$= 0$; otherwise

and $K(U_{2p}, U_{3p}, \ldots, U_{Np})$

$$= \sum_{i=2}^{N} \sum_{k=p-L_{p-1}}^{p-L_p} \Big\{ Cs_{ik} \sum_{r_i > U_{ip}} (r_i - U_{ip}) P(r_i; p, k)$$

$$+ Ch_{ik} \sum_{r_i < U_{ip}} (U_{ip} - r_i) P(r_i; p, k+1) \Big\},$$

and $G(U_{1p} + Z_{1p}, U_{2p}, \ldots, U_{Np})$

$$= \sum_{r_{1p} > 0} \cdots \sum_{r_{Np} > 0} \Big\{ f^*_{p-1}(U_{1p} + Z_{1p} - r_{1p}, \ldots, U_{Np} - r_{Np}) \prod_{i=1}^{N} P(r_{ip}) \Big\}.$$

Therefore $f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/w_1)$

$$= \underset{Z_{1p}}{\text{Min}} \Big\{ \tilde{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; Z_{1p}) \Big\}, \qquad (4\text{-}2\text{-}28)$$

where $0 \leq Z_{1p} \leq \text{Min}(S_{1p}, \frac{w_1}{v_1} - \text{Min}(U_{1p} - \sum_{k=p-L_p+1}^{p} \underline{r}_{1k}, 0))$. (4-2-29)

For $w_1 = 0$, and $U_{1p} - \sum_{k=p-L_p+1}^{p} \underline{r}_{1k} < 0$; the restriction of Z_{1p} in (4-2-29) becomes

$$0 \leq Z_{1p} \leq \text{Min}(S_{1p}, \sum_{k=p-L_p+1}^{p} \underline{r}_{1k} - U_{1p}). \qquad (4\text{-}2\text{-}30)$$

For $w_1 = 0$, and $U_{1p} - \sum_{k=p-L_p+1}^{p} \underline{r}_{1k} \geq 0$; (4-2-28) becomes

$$f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/0) = \mathcal{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; 0). \tag{4-2-31}$$

For $w_1 = Cv_1 \leq v_1(S_{1p} + \text{Min}(U_{1p} - \sum_{k=p-L_p+1}^{p} \underline{r}_{1k}, 0))$; it follows that

$$f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/Cv_1)$$

$$= \text{Min}\left\{\begin{matrix} f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/(C-1)v_1), \\ \mathcal{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; C - \text{Min}(U_{1p} - \sum_{k=p-L_p+1}^{p} \underline{r}_{1k}, 0)) \end{matrix}\right\}. \tag{4-2-32}$$

and for $w_1 = Cv_1 > v_1(S_{1p} + \text{Min}(U_{1p} - \sum_{k=p-L_p+1}^{p} \underline{r}_{1k}, 0))$;

$$f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/Cv_1$$

$$= \text{Min}\left\{\begin{matrix} f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/\bar{C}v_1) \\ \mathcal{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; S_{1p}) \end{matrix}\right\}. \tag{4-2-33}$$

$$\text{where } \bar{C}v_1 \leq v_1(S_{1p} + \text{Min}(U_{1p} - \sum_{k=p-L_p+1}^{p} \underline{r}_{1k}, 0)) < (\bar{C} + 1)v_1. \tag{4-2-34}$$

Again, using previous developments, if item types No. 1 to No. n ($2 \leq n \leq N$) are considered and item type No. n is considered first, it follows that $\mathcal{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; Z_{np}) = \varphi_{np}(Z_{np})$

$$+ f_{n-1,p}(U_{1p}, \ldots, U_{np} + Z_{np}, \ldots, U_{Np}/w_n - v_n \text{ Max}(Z_{np} + \text{Min}(U_{np} - \sum_{k=p-L_p+1}^{p} \underline{r}_{nk}, 0), 0)) \tag{4-2-35}$$

and $f_{np}(U_{1p}, \ldots, U_{np}, \ldots, U_{Np}/w_n)$

$$= \underset{Z_{np}}{\text{Min}}\left\{C(U_{1p}, \ldots, U_{np}, \ldots, U_{Np}; Z_{np})\right\} \tag{4-2-36}$$

$$\text{where } 0 \leq Z_{np} \leq \text{Min}(S_{np}, \frac{w_n}{v_n} - \text{Min}(U_{np} - \sum_{k=p-L_p+1}^{p} \underline{r}_{nk}, 0)). \tag{4-2-37}$$

By letting $n = N$, and let $f^*_p(U_{1p}, U_{2p}, \ldots, U_{Np}) =$

$$f_{Np}(U_{1p}, U_{2p}, \ldots, U_{Np}/W - \sum_{i=1}^{N} v_i \text{ Max}(U_{ip} - \sum_{k=p-L_p+1}^{p} \underline{r}_{ik}, 0)$$

$f^*_p(U_{1p}, U_{2p}, \ldots, U_{Np})$ is obtained as a partial-optimization for this stage.

And if $p = P$, $f^*_P(U_1P, U_2P, \ldots, U_NP)$ is the final optimization to the problem.

4.3 MULTI-ITEM SINGLE-SOURCE SYSTEM FOR THE NON-MIXABLE ITEMS

This section considers the case for the specific assumption in Section 2.3 when demands are probabilistic and procurement lead times are deterministic. Employing the discussion in Section 2.3, the system can be reduced, first, to single-item single-source. Thus, for item type No. i one can use the development in Section 4.1 to determine $G_i(w)$, which is $f^*_P(0)$ for the selected value of w. And then the procedure to allocate space to each type of item will be the same as in Section 2.3.

4.4 SINGLE-ITEM MULTI-SOURCE SYSTEM

This section considers the case in which several sources of supply are available for a single type of item. It is assumed that for any particular source, the order made at any period will not arrive before orders made in any previous period from that same source, and that each period order must be made from only one source.

For simplicity of discussion, assume that there are two sources of supply available at each period. Assume also that for the periods after P-2, orders made from these two sources arrive after the beginning of period 1. The system is shown in Figure 2.

The analysis starts from period P-2. Since an order can be made from either source No. 1 or No. 2 at each period, in order to apply the principle of optimality, one would consider each particular given

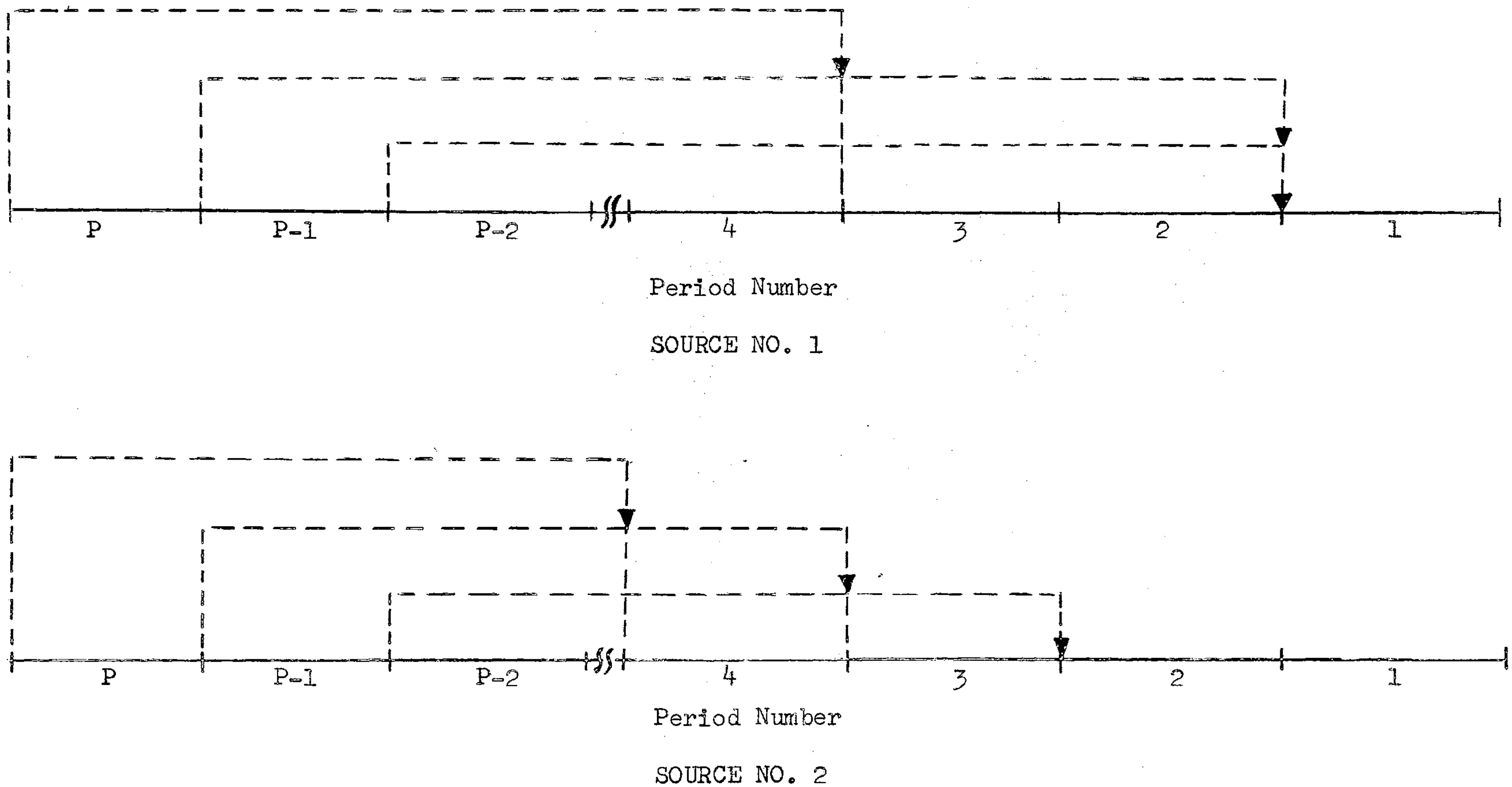


Figure 2. Single-Item Multi-Source System Probabilistic Demands and Deterministic Procurement Lead Times

situation in which the sources will be chosen at period P-2 and P-1.

For illustration, consider the alternative that at period P-1, source No. 1 is chosen; and at period P-2 source, No. 2 is chosen.

Let $\hat{U}_{1k}$ be the inventory on hand plus outstanding orders (from both sources) at period k less the order made at period k+1, the only previous order that may overlap to the order made at this period. As before, Z_{1kj} is the amount ordered from source j at period k. Then, $\hat{U}_{1k} = \hat{U}_{1,k+1} + Z_{1,k+2,j} - r_{1,k+1}$ (where j is either 1 or 2) and $U_{1k} = \hat{U}_{1k} + Z_{1,k+1}$.

For the case being considered, for a given set of $\hat{U}_{1,P-2}$ and $Z_{1,P-1,1}$ if an order of the amount $Z_{1,P-2,2}$ is made the total expected controllable cost is the sum of:

(1) item cost plus fixed cost for ordering $Z_{1,P-2,2}$, which is $\varphi_{1,P-2,2}(Z_{1,P-2,2})$,

(2) expected shortage cost in periods 1 and 2, which is $\sum_{k=1}^{2} Cs_{1k} \sum_{r_1} \text{Max}(r_1 - \hat{U}_{1,P-2} - \delta \cdot Z_{1,P-1,1} - Z_{1,P-2,2}, 0) \cdot$ $P(r_1 : P-2, k)$, and

(3) expected carrying cost in periods 1 and 2, which is $\sum_{k=1}^{2} Ch_{1k} \sum_{r_1} \text{Max}(\hat{U}_{1,P-2} + \delta \cdot Z_{1,P-1,1} + Z_{1,P-2,2} - r_1, 0) \cdot$ $P(r_1 : P-2, k+1)$,

where $\delta = 0$, for $k = 2$

$= 1$, for $k = 1$.

Therefore, total expected controllable cost, $\mathcal{C}(\hat{U}_{1,P-2}, Z_{1,P-1,1} : Z_{1,P-2,2}) = \varphi_{1,P-2,2}(Z_{1,P-2,2})$

$$+ \sum_{k=1}^{2} \Big\{ Cs_{1k} \cdot \sum_{r_1} \text{Max}(r_1 - \hat{U}_{1,P-2} - \delta \cdot Z_{1,P-1,1} - Z_{1,P-2,2}, 0) P(r_1 : P-2, k)$$

$$+ Ch_{1k}\sum_{r_1} \text{Max}(\hat{U}_{1,P-2} + \delta Z_{1,P-1,1} + Z_{1,P-2,2} - r_1, 0)P(r_1 : P-2, k+1)\Big\}$$

(4-4-1)

Let $f_{P-2,2}(\hat{U}_{1,P-1},\ Z_{1,P-1,1})$ be the minimum expected total controllable cost when a decision is made in period P-2 where the order is made from source No. 2, assuming an order was made from source No. 1 in period P-1, resulting from ordering an optimal amount of $Z_{1,P-2,2}$ for a given set of $\hat{U}_{1,P-2},\ Z_{1,P-1,1}$. Therefore

$$f_{P-2,2}(\hat{U}_{1,P-2},\ Z_{1,P-1,1}) = \underset{Z_{1,P-2,2}}{\text{Min}} \left[\mathcal{C}(\hat{U}_{1,P-2},\ Z_{1,P-1,1} : Z_{1,P-2,2})\right],$$

(4-4-2)

$$\text{where } 0 \leq Z_{1,P-2,2} \leq \text{Min}\Big(S_{1,P-2,2},\ \frac{W}{V_1} - \hat{U}_{1,p-2} - Z_{1,P-1,1} + \sum_{k=3}^{P-2} \underline{r}_{1k}\Big).$$

(4-4-3)

For other combinations of sources that could be chosen in periods P-1 and P-2, for each given set of $\hat{U}_{1,P-2},\ Z_{1,P-1,j}$, $f_{P-2,j}(\hat{U}_{1,P-2},\ Z_{1,P-1,j})$ can be determined.

Let $f^*_{P-2/j}(\hat{U}_{1,P-2},\ Z_{1,P-1,j})$ be the minimum expected total controllable cost when a decision is made in period P-2, assuming the order was made from source No. j in period P-1, resulting from ordering an optimal amount of $Z_{1,P-2,j}$ from the optimal source for a given set of $\hat{U}_{1,P-2},\ Z_{1,P-1,j}$. Therefore

$$f^*_{P-2/j}(\hat{U}_{1,P-2},\ Z_{1,P-1,j}) = \underset{j'}{\text{Min}}\left[f_{P-2,j'}(\hat{U}_{1,P-2},\ Z_{1,P-1,j})\right] \qquad (4\text{-}4\text{-}4)$$

where j' is source to be considered in period P-2.

Then $f^*_{P-2/j}(\hat{U}_{1,P-2},\ Z_{1,P-1,j})$ can be used in determining the optimal policy in the next stage.

Consider period P-1; assume again for illustration purposes that at period P source No. 2 is chosen and at period P-1 source No. 1 is chosen.

For a given set of $\hat{U}_{1,P-1}$, Z_{1P2}, if an order of the amount $Z_{1,P-1,1}$ is made from source No. 1 the total controllable cost is the sum of:

(1) item cost plus fixed cost for ordering $Z_{1,P-1,1}$, which is $\varphi_{1,P-1,1}(Z_{1,P-1,1})$,

(2) expected shortage cost in periods 3 and 4, which is
$$\sum_{k=3}^{4} Cs_{1k} \sum_{r_1} Max(r_1 - \hat{U}_{1,P-1} - Z_{1P2}, 0) P(r_1 : P-1, k),$$

(3) expected carrying cost in periods 3 and 4, which is
$$\sum_{k=3}^{4} Ch_{1k} \sum_{r_1} Max(\hat{U}_{1,P-1} + Z_{1P2} - r_1, 0) P(r_1 : P-1, k+1), \text{ and}$$

(4) minimum expected total controllable cost presuming an optimal decision is made at period P-2, which is
$$\sum_{r_{1,P-1}} f^*_{P-2/1} \left(\hat{U}_{1,P-1} + Z_{1P2} - r_{1,P-1},\ Z_{1,P-1,1}\right) P_{1,P-1}(r_{1,P-1}).$$

Thus, the total expected controllable cost, $\mathcal{C}(\hat{U}_{1,P-1},\ Z_{1P2};\ Z_{1,P-1,1})$

$$= \varphi_{1,P-1,1}(Z_{1,P-1,1}) + \sum_{k=3}^{4}\Big\{Cs_{1k} \sum_{r_1} Max(r_1 - \hat{U}_{1,P-1} - Z_{1P2}, 0) P(r_1, P-1, k)$$
$$+ Ch_{1k} \sum_{r_1} Max(\hat{U}_{1,P-1} + Z_{1P2} - r_1, 0) P(r_1 : P-1, k+1)\Big\}$$
$$+ \sum_{r_{1,P-1}} f^*_{P-2/1} \left(U_{1,P-1} + Z_{1P2} - r_{1,P-1},\ Z_{1,P-1,1}\right) P_{1,P-1}(r_{1,P-1}). \qquad (4\text{-}4\text{-}5)$$

$$\text{And } f_{P-1,2}(\hat{U}_{1,P-1}, Z_{1P2}) = \underset{Z_{1,P-1,1}}{Min} \left[\mathcal{C}(\hat{U}_{1,P-1},\ Z_{1P2};\ Z_{1,P-1,1})\right], \qquad (4\text{-}4\text{-}6)$$

$$\text{where } 0 \leq Z_{1,P-1,1} \leq Min\left(S_{1,P-1,1},\ \frac{W}{v_1} - \hat{U}_{1,P-1} - Z_{1P2} + \sum_{k=5}^{P-1} \bar{r}_{1k}\right). \qquad (4\text{-}4\text{-}7)$$

For other combinations of sources that could be chosen in period P and P-1, for a given set of $\hat{U}_{1,P-1}$, Z_{1Pj}, $f_{P-1,j}(\hat{U}_{1,P-1}, Z_{1,P-1,j})$ can be determined.

And for a given source to be chosen in period P, it follows that

$$f^*_{P-1/j}(\hat{U}_{1,P-1},\ Z_{1Pj}) = \underset{j'}{\mathrm{Min}}\left[f_{P-1,j'}(\hat{U}_{1,P-1},\ Z_{1,P-1,j})\right]$$

where j' are the sources to be considered in period P-1. (4-4-8)

Consider period P for a given source to be chosen in this period, let $f_{Pj}(U_{1P})$ be a minimum expected controllable cost when a decision is made in period P where the order is made from source j for a given value of U_{1P}, it follows that

$$f_{Pj}(U_{1P}) = \underset{Z_{1Pj}}{\mathrm{Min}}\Big[\varphi_{1Pj}(Z_{1Pj}) + \sum_{r_{1P}} f^*_{P-1/j}\ (U_{1P} - r_{1P},\ Z_{1Pj})P_{1P}(r_{1P})\Big] \qquad (4\text{-}4\text{-}9)$$

where $0 \leq Z_{1Pj} \leq \mathrm{Min}(S_{1Pj},\ \frac{W}{v_1} - U_{1P} + \sum_{k=5}^{P} \underline{r}_{1k})$. (4-4-10)

Let EC_j be the expected lost during period $P - \underline{\underline{L}}_P$ to $P - L_{jP}+1$ for the source that $L_{jP} > \underline{\underline{L}}_P$, where

$$\underline{\underline{L}}_P = \underset{j}{\mathrm{Min}}\left[L_{jP}\right].$$

Then $f^*_P(U_{1P})$, the minimum expected cost when a decision is made in period P where all sources are considered for a given value of U_{1P}, becomes

$$f^*_P(U_{1P}) = \underset{j}{\mathrm{Min}}\left[f_{Pj}(U_{1P}) + EC_j\right]. \qquad (4\text{-}4\text{-}11)$$

Employing procedure developed above, for the case of more than two sources, at each period for $k < P$ $f^*_{k/j}(\hat{U}_{1k},\ Z_{1,k-1,j})$ can be determined.

For $k = P$, employing the procedure from (4-4-9) to (4-4-11), the final optimization for the system can be found.

<u>Example</u>

planning period, $P = 7$

warehouse space, $W = 3$ cubic units

number of sources, $J = 2$

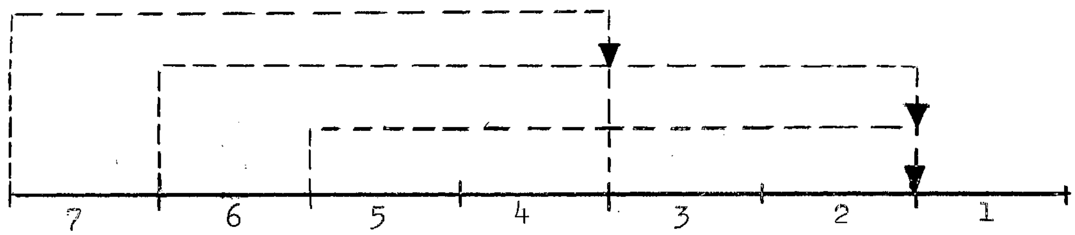
a volume of an item, $v_1 = 1$ cubic unit

initial inventory, $U_{17} = 2$ units

	j = 1			j = 2		
	k=5	k=6	k=7	k=5	k=6	k=7
S_{jk}-unit	3	2	2	1	2	1
Co_{jk}-dollars/order	0.50	0.50	0.50	0.60	0.60	0.60
Ci_{jk}-dollars/unit	2.00	3.00	2.00	3.00	3.00	2.00

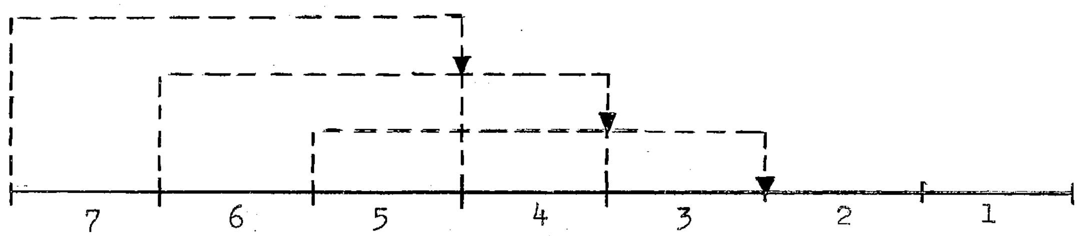
	k=1	k=2	k=3	k=4
Cs_{1k}-dollars/unit/period	5.00	6.00	6.00	5.00
Ch_{1k}-dollars/unit/period	2.00	2.00	2.00	2.00

	$P_{1k}(r_{1k})$						
r_{1k}	k=1	k=2	k=3	k=4	k=5	k=6	k=7
0	.5	.6	.2	.3	.5	.3	.4
1	.5	.4	.5	.3	.5	.7	.6
2	.0	.0	.3	.4	.0	.0	.0



Period Number

SOURCE NO. 1



Period Number

SOURCE NO. 2

Solution:

Using the given data, the necessary values of $P(r_1 : K, k)$ can be determined as shown in Table VI.

Consider period 5, for the alternative that at period 6 source No. 1 is chosen and at period 5 source No. 2 is chosen.

For $\hat{U}_{15} = 0$, $Z_{161} = 0$; using (4-4-1) to (4-4-3)

$$0 \le Z_{152} \le \text{Min}(1, \underset{1}{\underline{3}}-0-0+0) = 1, \text{ and}$$

$$f_{52}(0,0) = \text{Min}\left[\begin{array}{l} 0.0 + [(5)\{(1)(.055) + (2)(.153) + (3)(.251) + (4)(.267) \\ \quad + (5)(.182) + (6)(.071) + (7)(.012)] \\ \quad + [(6)\{(1)(.093) + (2)(.213) + (3)(.289) + (4)(.245) + (5)(.118) \\ \quad + (6)(.024)], \\ 3.6 + [(5)\{(1)(.153) + (2)(.251) + (3)(.267) + (4)(.182) \\ \quad + (5)(.071) + (6)(.012)\} + (2)\{(1)(.018)\}] \\ \quad + [(6)\{(1)(.213) + (2)(.289) + (3)(.245) + (4)(.118) \\ \quad + (5)(.24)\} + (2)\{(1)(.030)\}] \end{array}\right]$$

TABLE VI

CUMULATIVE PROBABILITY OF DEMANDS FROM PERIOD k TO K, $P(r_1:K,k)$

	$P(r_1:K,k)$							
	K=5			K=6			K=7	
r_1	k=1	k=2	k=3	k=3	k=4	k=5	k=4	k=5
0	.009	.018	.030	.009	.045	.150	.018	.060
1	.055	.093	.135	.061	.195	.500	.105	.290
2	.153	.213	.265	.175	.315	.350	.243	.440
3	.251	.289	.305	.276	.305	.000	.311	.210
4	.267	.245	.205	.276	.140	.000	.239	.000
5	.182	.118	.060	.161	.000	.000	.084	.000
6	.071	.024	.000	.042	.000	.000	.000	.000
7	.012	.000	.000	.000	.000	.000	.000	.000

$$= \text{Min}\begin{bmatrix} 36.610 \\ 29.159 \end{bmatrix} = 29.159; \text{ where, } Z^*_{152} = 1.$$

For other sets of $\hat{U}_{15}$ and Z_{16j}, and for other alternatives, $f_{5j}(\hat{U}_{15}, Z_{16j})$ can be determined as above. The results are summarized in Table VII.

For the given alternative that source No. 1 is chosen at period 6 and for $\hat{U}_{15} = 0$, $Z_{161} = 0$; using (4-4-4) $f^*_{5/1}(0,0) = \underset{j'}{\text{Min}}\left[f_{5j'}(0,)\right]$

$$= \text{Min}\begin{bmatrix} 31.733 \\ 29.159 \end{bmatrix} = 29.159;$$ where the best policy is to order 1 unit from source No. 2.

For other sets of $\hat{U}_{15}$ and Z_{161}, and for other alternatives that source No. 2 is chosen at period 6, $f^*_{5/j}(\hat{U}_{15}, Z_{16j})$ can be determined. The results are summarized in Table VIII.

Consider period 6, for the alternative that at period 7 source No. 2 is chosen and at period 6 source No. 1 is chosen. For $U_{16} = 1$, $Z_{172} = 0$; using (4-4-5) to (4-4-7), $0 \leq Z_{161} \leq \text{Min}(2, \underset{1}{\overset{3}{\sum}}\text{-}1\text{-}0\text{+}0)$, and $f_{61}(0,0)$

$$= \text{Min}\begin{bmatrix}
0.0 + [(6)\{(1)(.175) + (2)(.276) + (3)(.276) + (4)(.161) \\
\quad + (5)(.042\} + (2)\{(1)(.045)\}] \\
\quad + (5)\{(1)(.315) + (2)(.305) + (3)(.140)\} + (2)\{(1)(.15)\}] \\
\quad + (.3).f^*_{5/1}(1,0) + (.7).f^*_{5/1}(0,0), \\
3.5 + [(6)\{(1)(.175) + (2)(.276) + (3)(.276) + (4)(.161) \\
\quad + (5)(.042)\} + (2)\{(1)(.045)\}] \\
\quad + [(5)\{(1)(.315) + (2)(.305) + (3)(.140)\} + (2)\{(1)(.15)\}] \\
\quad + (.3).f^*_{5/1}(1,1) + (.7).f^*_{5/1}(0,1), \\
6.5 + [(6)\{(1)(.175) + (2)(.276) + (3)(.276) + (4)(.161) \\
\quad + (5)(.042)\} + (2)\{(1)(.045)\}] \\
\quad + [(5)\{(1)(.315) + (2)(.305) + (3)(.140)\} + (2)\{(1)(.15)\}] \\
\quad + (.3).f^*_{5/1}(1,2) + (.7).f^*_{5/1}(0,2)
\end{bmatrix}$$

TABLE VII

OPTIMAL POLICY AND MINIMUM COST FOR THE FIRST STAGE DECISION IN FIRST STAGE PERIOD

Source No. 1 at period 6
Source No. 1 at period 5

U_{15}	Z_{161}	$f_{51}(U,Z)$	Order
0	0	31.733	2
	1	28.312	2
	2	26.312	1
1	0	22.580	2
	1	20.580	1
	2	18.080	0
2	0	15.576	1
	1	13.076	0
3	0	9.790	0

Source No. 1 at period 6
Source No. 2 at period 5

U_{15}	Z_{161}	$f_{51}(U,Z)$	Order
0	0	29.159	1
	1	25.001	1
	2	18.080	1
1	0	19.997	1
	1	16.676	1
	2	18.080	0
2	0	13.390	1
	1	13.076	0
3	0	9.790	0

Source No. 2 at period 6
Source No. 1 at period 5

U_{15}	Z_{162}	$f_{51}(U,Z)$	Order
0	0	52.308	2
	1	37.299	2
	2	25.030	1
1	0	37.299	2
	1	25.020	1
	2	15.814	0
2	0	25.020	1
	1	15.814	0
3	0	15.814	0

Source No. 2 at period 6
Soutce No. 2 at period 5

U_{15}	Z_{162}	$f_{51}(U,Z)$	Order
0	0	50.034	1
	1	34.716	1
	2	22.834	1
1	0	34.716	1
	1	22.834	1
	2	15.814	0
2	0	22.834	1
	1	15.814	0
3	0	15.814	0

TABLE VIII

OPTIMAL POLICY AND MINIMUM COST FOR THE SECOND STAGE DECISION IN FIRST STAGE PERIOD

Source No. 1 at period 6

$\hat{U}_{15}$	Z_{161}	$f_{5/1}(\hat{U},Z)$	Source#	Order
0	0	29.159	2	1
	1	25.001	2	1
	2	18.080	2	1
1	0	19.997	2	1
	1	16.676	2	1
	2	18.080	–	0
2	0	13.390	2	1
	1	13.676	–	0
3	0	9.790	–	0

Source No. 2 at period 6

$\hat{U}_{15}$	Z_{162}	$f_{5/2}(\hat{U},Z)$	Source#	Order
0	0	50.034	2	1
	1	34.716	2	1
	2	28.834	2	1
1	0	34.716	2	1
	1	22.834	2	1
	2	15.814	–	0
2	0	22.834	2	1
	1	15.814	–	0
3	0	15.814	–	0

Source to be chosen.

$$= \text{Min}\begin{bmatrix}48.146\\47.748\\46.324\end{bmatrix} = 46.324;\ \text{order 2 units.}$$

For other sets of $\hat{U}_{16}$ and Z_{17j}, and for other alternatives, $f_{6j}(\hat{U}_{16},Z_{17j})$ can be determined. The results are summarized in Table IX.

For the given alternative that source No. 1 is chosen at period 7 and for $\hat{U}_{16} = 1, Z_{171} = 0$; using (4-4-8),

$$f_{6/1}(1,0) = \underset{j'}{\text{Min}}\left[f_{6j'}(1,0)\right]$$

$$= \text{Min}\begin{bmatrix}39.299\\31.928\end{bmatrix} = 31.928.$$

The decision is to choose source No. 2 in period 6 and order 2 units.

For other sets of $\hat{U}_{16}$ and Z_{17j}, and for the other alternatives that source No. 2 is chosen at period 7; $f^*_{6/j}(\hat{U}_{16},Z_{16j})$ can be determined. The results are summarized in Table X.

Consider the last stage, period 7. Employing (4-4-9) and (4-4-10),

$f_{71}(2)$

$$= \text{Min}\begin{bmatrix}0.0 + (.4)f^*_{6/1}(2,0) + (.6)f^*_{6/1}(1,0),\\2.5 + (.4)f^*_{6/1}(2,1) + (.6)f^*_{6/1}(1,1)\end{bmatrix} = \text{Min}\begin{bmatrix}30.365\\26.246\end{bmatrix}$$

$= 26.246$; order 1 unit.

The same manner, $f_{72}(2)$ can be determined which is equal to 28.181, by ordering 1 unit from source No. 2 at period 7.

The final optimization, then, can be determined by employing (4-4-11);

$$EC_1 = (5)\{(1)(.311) + (2)(.239) + (3)(.084)\} + (2)\{(1)(.290) + (2)(.060)\}$$

$$= 6.025.$$

TABLE IX

OPTIMAL POLICY AND MINIMUM COST FOR THE FIRST STAGE DECISION IN THE SECOND STAGE PERIOD

Source No. 1 at period 7
Source No. 1 at period 6

$\hat{U}_{16}$	Z_{171}	$f_{61}(\hat{U},Z)$	Order
1	0	39.299	2
	1	28.020	1
2	0	28.020	1
	1	17.334	0

Source No. 1 at period 7
Source No. 2 at period 6

$\hat{U}_{16}$	Z_{171}	$f_{62}(\hat{U},Z)$	Order
1	0	31.928	2
	1	28.528	1
2	0	28.528	1
	1	24.928	0

Source No. 2 at period 7
Source No. 1 at period 6

$\hat{U}_{16}$	Z_{172}	$f_{61}(\hat{U},Z)$	Order
1	0	46.324	2
	1	32.545	1
2	0	32.545	1
	1	21.634	0

Source No. 2 at period 7
Source No. 2 at period 6

$\hat{U}_{16}$	Z_{172}	$f_{62}(\hat{U},Z)$	Order
1	0	38.953	2
	1	33.053	1
2	0	33.053	1
	1	29.228	0

TABLE X

OPTIMAL POLICY AND MINIMUM COST FOR THE FIRST STAGE DECISION IN THE SECOND STAGE PERIOD

Source No. 1 at period 7

$\hat{U}_{16}$	Z_{171}	$f_{6/1}(\hat{U},Z)$	Source#	Order
1	0	31.928	2	2
	1	28.020	2	1
2	0	28.020	2	1
	1	17.334	–	0

Source No. 2 at period 7

$\hat{U}_{16}$	Z_{172}	$f_{6/2}(\hat{U},Z)$	Source#	Order
1	0	38.953	2	2
	1	32.545	2	1
2	0	32.545	2	1
	1	21.634	–	0

Therefore, $f^*_7(0) = \begin{bmatrix} 26.246 + 6.025 \\ 28.181 \end{bmatrix} = 28.181.$

Then, the optimal policy in period 7 is to choose source No. 2 and order 1 unit.

4.5 MULTI-ITEM MULTI-SOURCE SYSTEM FOR THE NON-MIXABLE ITEMS

This section considers the case of multi-item multi-source. Employing the development in Section 2.3, the system can be reduced, first, to the single-item multi-source. Thus, for item type No. i, one can use the development in Section 4.4 to determine $G_i(w)$, which is $f^*_p(0)$ for the selected value of w. And then the procedure to allocate space to each type of item is the same as in Section 2.3.

CHAPTER V

DETERMINISTIC OR PROBABILISTIC DEMAND AND PROBABILISTIC PROCUREMENT LEAD TIME SYSTEM

This chapter considers the problem in which demands are either deterministic or probabilistic but the procurement lead times are probabilistic. Other assumptions remain the same as in previous chapters.

The analysis in this chapter is based on probabilistic problem. However, for the deterministic demands case, this analysis can also be applied by substituting the probability of demands for those deterministic values by one and for those remaining by zero. Models in this chapter can be considered as the general models for those in the previous chapters.

The assumption for procurement lead time as discussed on pages 10 and 11 in Chapter I is used in this chapter. A principle of dynamic programming can be applied and optimal policy for each period can be determined by employing the recurrence relation and basic ideas discussed in Chapter IV.

5.1 SINGLE-ITEM SINGLE-SOURCE SYSTEM

This section concerns the system in which only one type of item is carried and only one source of supply is available.

Consider period K,

where $$K - \underline{L}_K \geq 1,$$

and $$K - 1 - \underline{L}_{K-1} \geq 1.$$

This means that there is a chance that the order made before or at period K will arrive before or at the beginning of period 1, but there is no chance that the order made after period K will arrive before the beginning of period 1.

For a given U_{1K}, assume that an amount Z_{1K} is ordered at this period. This amount will arrive next $\underline{L}_K$ periods or later. Therefore, a decision made at this period affects the expected total system cost in periods $K - \underline{L}_K$, $K - \underline{L}_K - 1$, ..., and 1. Then, the expected total "controllable cost" is the sum of the expected cost in periods $K - \underline{L}_K$, $K - \underline{L}_K - 1$, ..., and 1.

The expected total "controllable cost" is the sum of:

(1) item cost plus fixed ordering cost of ordering Z_{1K}, which is $\varphi_{1K}(Z_{1K})$,

(2) total expected controllable shortage cost, which is

$$\sum_{L=0}^{K-1} \bar{P}_K(L) \left\{ \sum_{k=1}^{K-L} Cs_{1k} \sum_{r_1 > U_{1K} + Z_{1K}} (r_1 - U_{1K} - Z_{1K}) P(r_1; K, k+1) \right\},$$

and

(3) total expected controllable carrying cost, which is

$$\sum_{L=0}^{K-1} \bar{P}_K(L) \sum_{k=1}^{K-L} Ch_{1k} \cdot \sum_{r_1 < U_{1K} + Z_{1K}} (U_{1K} + Z_{1K} - r_1) P(r_1; K, k+1).$$

Thus, the expected total controllable cost, $\mathcal{C}(U_{1K}; Z_{1K})$

$$= \varphi_{1K}(Z_{1K}) + \sum_{L=0}^{K-1} \bar{P}_K(1) \sum_{k=1}^{K-L} \left\{ Cs_{1K} \cdot \sum_{r_1 > U_{1K} + Z_{1K}} (r_1 - U_{1K} - Z_{1K}) P(r_1; K, k) \right.$$

$$+ Ch_{1K} \cdot \overline{\sum_{r_1 < U_{1K} + Z_{1K}}} (U_{1K} + Z_{1K} - r_1) P(r_1; K, k+1) \Big\}. \qquad (5\text{-}1\text{-}1)$$

Let $f^*_K(U_{1K})$ be a minimum expected total controllable cost for period K, resulting from ordering an optimal amount of $Z_{1K} = Z^*_{1K}(U_{1K})$ for a given U_{1K}. Therefore,

$$f^*_K(U_{1K}) = \underset{Z_{1K}}{\text{Min}} \Big\{ \mathcal{C}(U_{1K}; Z_{1K}) \Big\} \qquad (5\text{-}1\text{-}2)$$

where $0 \leq Z_{1K} \leq \text{Min}(S_{1K}, \frac{W}{v_1} - U_{1K} + \delta \overline{\sum_{k=K-\underline{L}_K+1}^{K}} r_{\min_{1k}}),$

and

$$\delta = 0, \text{ for } \underline{L}_K = 0$$
$$= 1 \text{ otherwise.} \qquad (5\text{-}1\text{-}3)$$

Consider period $K+1$ and for a given $U_{1,K+1}$, assume that an amount $Z_{1,K+1}$ is ordered in this period and arrives next $\underline{L}_{K+1}$ periods or later. Note that the order at period $K+1$ cannot arrive after the order made at period K arrives. The expected total controllable cost when a decision is made at this period affects the expected cost in periods $K+1-L_{K+1}$, $K-L_{K+1}$, ..., and 1.

The expected total "controllable cost" is the sum of

(1) item cost plus fixed cost of ordering $Z_{1,K+1}$, which is $\varphi_{1,K+1}(Z_{1,K+1})$,

(2) total expected controllable shortage cost, during periods from the period when $Z_{1,K+1}$ arrives to a period before Z_{1K} arrives, which is

$$\sum_{L=0}^{K} \bar{P}_{K+1}(L) \sum_{L' \geq 0} \bar{\bar{P}}_K(L'+1) \overline{\sum_{k=\text{Max}[K-L-L',0]}^{K-L}} Cs_{1k} \overline{\sum_{r_1 > U_{1,K+1} + Z_{1,K+1}}} (r_1 - U_{1,K+1} - Z_{1,K+1}) P(r_1; K+1, k+1),$$

(3) total expected controllable carrying cost, during

periods from the period where $Z_{1,K+1}$ arrives to a period before Z_{1K} arrives, which is

$$\sum_{L=0}^{K} P_{K+1}(L) \sum_{L' \geq 0} \bar{\bar{P}}_K(L'+1) \sum_{k=\mathrm{Max}[K-L-L',0]}^{K-L} Ch_{1k} \sum_{r_1 < U_{1,K+1}+Z_{1,K+1}} (U_{1,K+1} + Z_{1,K+1} - r_1) P(r_1;K+1k+2),$$

(4) minimum expected total controllable cost, presuming an optimal decision is made at period K, which is

$$\sum_{r_{1,K+1} \geq 0} f^*_K(U_{1,K+1} + Z_{1,K+1} - r_{1,K+1}) P(r_{1,K+1}).$$

Thus, the expected total controllable cost, $\mathcal{C}(U_{1,K+1};Z_{1,K+1})$

$$= \varphi_{1,K+1}(Z_{1,K+1})$$
$$+ \sum_{L=0}^{K} P_{K+1}(L) \sum_{L' \geq 0} \bar{\bar{P}}_K(L'+1) \sum_{k=\mathrm{Max}[K-L-L',0]}^{K-L} \Big\{ Cs_{1k} \sum_{r_1 > U_{1,K+1}+Z_{1,K+1}} (r_1 - U_{1,K+1} - Z_{1,K+1}) P(r_1;K+1,k+1)$$
$$+ Ch_{1k} \sum_{r_1 < U_{1,K+1}+Z_{1,K+1}} (U_{1,K+1} + Z_{1,K+1} - r_1) P(r_1;K+1,k+2) \Big\}$$
$$+ \sum_{r_{1,K+1} \geq 0} f^*_K(U_{1,K+1} + Z_{1,K+1} - r_{1,K+1}) P(r_{1,K+1}). \qquad (5\text{-}1\text{-}4)$$

Let $f^*_{K+1}(U_{1,K+1})$ be the minimum expected total controllable cost for period K+1, resulting from ordering an optimal amount of $Z_{1,K+1} = Z^*_{1,K+1}(U_{1,K+1})$ for a given $U_{1,K+1}$. Therefore,

$$f^*_{K+1}(U_{1,K+1}) = \underset{Z_{1,K+1}}{\mathrm{Min}} \left\{ \mathcal{C}(U_{1,K+1};Z_{1,K+1}) \right\} \qquad (5\text{-}1\text{-}5)$$

where $$0 \leq Z_{1,K+1} \leq \mathrm{Min}\Big(S_{1,K+1}, \frac{W}{v_1} - U_{1,K+1} + \sum_{k=K+2-\underline{L}_{K+1}}^{K+1} \underline{r}_{1k}\Big). \qquad (5\text{-}1\text{-}6)$$

Next, consider, in general, period p where $K+1 \leq p \leq P$.

$$\mathcal{C}(U_{1p};Z_{1p}) = \varphi_{1p}(Z_{1p})$$

$$+ \sum_{L=0}^{p-1} \bar{P}_p(L) \sum_{L' \geq 0} \bar{\bar{P}}_{p-1}(L'+1) \sum_{k=Max(p-L-L',0)}^{p-L} \Big\{ Cs_{1k} \sum_{r_1 > U_{1p}+Z_{1p}} (r_1 - U_{1p} - Z_{1p}) \, P(r_1;p,k+1)$$

$$+ Ch_{1k} \sum_{r_1 < U_{1p}+Z_{1p}} (U_{1p} - Z_{1p} - r_1) \, P(r_1;p,k+2) \Big\}$$

$$+ \sum_{r_{1p} \geq 0} f^*_{p-1}(U_{1p} + Z_{1p} - r_{1p}) P(r_{1p}). \qquad (5\text{-}1\text{-}7)$$

It follows that $f^*_p(U_{1p}) = \underset{Z_{1p}}{Min} \Big\{ \mathcal{C}(U_{1p};Z_{1p}) \Big\}$, (5-1-8)

where $0 \leq Z_{1p} \leq Min(S_{1p}, \frac{W}{v_1} - U_{1p} + \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{1k})$. (5-1-9)

<u>Example</u>

planning period, $P = 5$

warehouse space, $W = 5$ cubic units

a volume of an item, $v_1 = 1$ cubic unit

initial inventory, $U_{15} = 4$ units

	k=1	k=2	k=3	k=4	k=5
S_{1k}-unit			3	5	4
Co_{1k}-dollars/order			0.50	0.50	0.50
Ci_{1k}-dollars/unit			0.50	0.60	0.50
Cs_{1k}-dollars/unit/period	6.00	6.00	6.00		
Ch_{1k}-dollars/unit/period	1.00	0.90	1.00		

r_{1k}	0	1	2	3	4
$P_{11}(r_{11})$	.20	.25	.30	.25	.00
$P_{12}(r_{12})$	.10	.20	.35	.20	.15
$P_{13}(r_{13})$	.55	.45	.00	.00	.00
$P_{14}(r_{14})$	.30	.40	.30	.00	.00
$P_{15}(r_{15})$	.50	.50	.00	.00	.00

L	0	1	2	3
$\bar{P}_5(L)$	.00	.00	.60	.40
$\bar{\bar{P}}_4(L)$	.50	.50	.00	.00
$\bar{\bar{P}}_3(L)$	.00	.40	.60	.00
$\bar{\bar{P}}_2(L)$	.00	.00	.60	.40

Solution:

Using the data given above, the necessary values of $P(r_1{:}K,k)$ and $\bar{P}_k(L)$ can be determined as shown in Table XI and Table XII, respectively.

It is obvious that:

$$3 - \underline{\underline{L}}_3 > 1,$$

and

$$2 - \underline{\underline{L}}_2 < 1.$$

Therefore, the first period to be considered is period 3. Using (5-1-1) to (5-1-3) for $u_{13} = 1$;

$f^*_3(1)$

$$= \underset{0 \leq Z_{13} \leq \mathrm{Min}(3, \underset{1}{\underline{5}}-1+0)}{\mathrm{Min}} \Big[\varphi_{13}(Z_{13}) + \sum_{L=0}^{2} \bar{P}_3(L) \sum_{k=1}^{3-L} \Big\{ Cs_{1k} \sum_{r_1 > 1+Z_{13}} (r_1 - 1 - Z_{13}) P(r_1{:}3,k)$$

TABLE XI

CUMULATIVE PROBABILITY OF DEMANDS FROM PERIOD k TO K, $P(r_1:K,k)$

r_1	0	1	2	3	4	5	6	7	8	9	10
$P(r_1:3,2)$	.055	.155	.283	.267	.173	.067	.000	.000	.000	.000	.000
$P(r_1:3,1)$	.011	.045	.112	.184	.225	.208	.136	.063	.016	.000	.000
$P(r_1:4,3)$	.165	.355	.345	.135	.000	.000	.000	.000	.000	.000	.000
$P(r_1:4,2)$	.017	.069	.163	.240	.244	.169	.078	.020	.000	.000	.000
$P(r_1:4,1)$	.003	.018	.055	.114	.175	.207	.191	.136	.071	.025	.005
$P(r_1:5,4)$	.150	.350	.350	.150	.000	.000	.000	.000	.000	.000	.000
$P(r_1:5,3)$	.083	.260	.350	.240	.067	.000	.000	.000	.000	.000	.000
$P(r_1:5,2)$	.008	.043	.116	.202	.242	.206	.124	.049	.010	.000	.000

TABLE XII

PROBABILITY OF PROCUREMENT LEAD TIME, $\bar{P}_k(L)$

L	0	1	2	3
$\bar{P}_2(L)$	.00	.00		
$\bar{P}_3(L)$	.00	.12	.38	
$\bar{P}_4(L)$	.00	.30	.50	.20
$\bar{P}_5(L)$	.00	.00	.60	.40

$$+ \; Ch_{1k} \sum_{r_1 < 1+Z_{13}} (1+Z_{13}-r_1) P(r_1:3,k+1) \Big\} \Big]$$

$$= \min_{0 \le Z_{13} \le 3} \Big[\varphi_{13}(Z_{13})$$

$$+ \left\{ \overline{P}_3(1) + \overline{P}_3(2) \right\} \Big\{ Cs_{11} \sum_{r_1 > 1+Z_{13}} (r_1-1-Z_{13}) P(r_1:3,1)$$

$$+ \; Ch_{11} \sum_{r_1 < 1+Z_{13}} (1+Z_{13}-r_1) P(r_1:3,2) \Big\}$$

$$+ \; \overline{P}_3(1) \Big\{ Cs_{12} \sum_{r_1 > 1+Z_{13}} (r_1-1-Z_{13}) P(r_1:3,2)$$

$$+ \; Ch_{12} \sum_{r_1 < 1+Z_{13}} (1+Z_{13}-r_1) P(r_1:3,3) \Big\} \Big].$$

$$= \text{Min} \begin{bmatrix} 0.0+(.5)\{(6)\{(1)(.112)+(2)(.184)+(3)(.225)+(4)(.208) \\ \quad +(5)(.136)+(6)(.063)+(7)(.016)\}+(1)\{(1)(.055)\}\} \\ +(.12)\{(6)\{(1)(.283)+(2)(.267)+(3)(.173)+(4)(.067)\} \\ \quad +(.9)\{(1)(.55)\}\}, \\ 1.0+(.5)\{(6)\{(1)(.184)+(2)(.225)+(3)(.208)+(4)(.136) \\ \quad +(5)(.063)+(6)(.016)\}+(1)\{(1)(.155)+(2)(.055)\}\} \\ +(.12)\{(6)\{(1)(.267)+(2)(.173)+(3)(.067)\} \\ \quad +(.9)\{(1)(.45)+(2)(.55)\}\}, \\ 1.5+(.5)\{(6)\{(1)(.225)+(2)(.208)+(3)(.136)+(4)(.063) \\ \quad +(5)(.016)\}+(1)\{(1)(.283)+(2)(.155)+(3)(.055)\}\} \\ +(.12)\{(6)\{(1)(.173)+(2)(.067)\}+(.9)\{(2)(.45)+(3)(.55\}\}, \\ 2.0+(.5)\{(6)\{(1)(.208)+(2)(.136)+(3)(.063)+(4)(.016)\} \\ \quad +(1)\{(1)(.267)+(2)(.283)+(3)(.155)+(4)(.055)\}\} \\ +(.12)\{(6)\{(6)\{(1)(.067)\}+(.9)\{(3)(.45)+(4)(.55)\}\}. \end{bmatrix}$$

$$= \text{Min}\begin{bmatrix} 10.727 \\ 8.525 \\ 6.518 \\ 5.292 \end{bmatrix} = 5.292 \text{ ; where } Z^*_{13}(1) = 3.$$

For other values of U_{13}, $f^*_3(U_{13})$ and $Z^*_{13}(U_{13})$ can be determined. The results are summarized below:

$f^*_3(2) = 4.647$; $Z^*_{13}(2) = 3$ $\quad$ $f^*_3(4) = 3.147$; $Z^*_{13}(4) = 1$

$f^*_3(3) = 4.147$; $Z^*_{13}(3) = 2$ $\quad$ $f^*_3(5) = 2.147$; $Z^*_{13}(5) = 0.$

Consider period 4. Using (5-1-6) to (5-1-8), for $U_{14} = 3$;

$$f^*_4(3)$$

$$= \underset{0 \le Z_{14} \le \text{Min}(5, \underset{1}{5}-3+0)}{\text{Min}} \Big[\varphi_{14}(Z_{14}) + \sum_{L=0}^{3} \bar{P}_4(L) \sum_{L' \ge 0} \bar{\bar{P}}_3(L'+1) \sum_{k=\text{Max}[4-L-L',1]}^{4-L} \Big\{ Cs_{1k} \sum_{r_1 > 3+Z_{14}} (r_1 - 3 - Z_{14}) P(r_1 : 4, k) + Ch_{1k} \sum_{r_1 < 3+Z_{14}} (3 + Z_{14} - r_1) P(r_1 : 4, k+1) \Big\} + \sum_{r_{14} \ge 0} f^*_3(3 + Z_{14} - r_{14}) P_{14}(r_{14}) \Big].$$

$$= \underset{0 \le Z_{14} \le 2}{\text{Min}} \Big[\varphi_{14}(Z_{14}) + \bar{P}_4(1) \Big\{ \bar{\bar{P}}_3(1) \sum_{k=3}^{3} \Big\{ Cs_{1k} \sum_{r_1 > 3+Z_{14}} (r_1 - 3 - Z_{14}) P(r_1 : 4, k) + Ch_{1k} \sum_{r_1 > 3+Z_{14}} (3 + Z_{14} - r_1) P(r_1 : 4, k+1) \Big\}$$

$$+ \bar{\bar{P}}_3(2) \sum_{k=2}^{3} \Big\{ Cs_{1k} \sum_{r_1 > 3+Z_{14}} (r_1 - 3 - Z_{14}) P(r_1 : 4, k) + Ch_{1k} \sum_{r_1 < 3+Z_{14}} (3 + Z_{14} - r_1) P(r_1 : 4, k+1) \Big\} \Big\}$$

$$+ \bar{P}_4(2)\Big\{\bar{\bar{P}}_3(2)\sum_{k=2}^{2}\Big\{Cs_{1k}\sum_{r_1>3+Z_{14}}(r_1-3-Z_{14})P(r_1:4,k)$$

$$+ Ch_{1k}\sum_{r_1<3+Z_{14}}(3+Z_{14}-r_1)P(r_1:4,k+1)\Big\}$$

$$+ \bar{\bar{P}}_3(2)\sum_{k=1}^{2}\Big\{Cs_{1k}\sum_{r_1>3+Z_{14}}(r_1-3-Z_{14})P(r_1:4,k)$$

$$+ Ch_{1k}\sum_{r_1<3+Z_{14}}(3+Z_{14}-r_1)P(r_1:4,k+1)\Big\}\Big\}$$

$$+ \bar{P}_4(3)\Big\{\bar{\bar{P}}_3(1)\sum_{k=1}^{1}\Big\{Cs_{1k}\sum_{r_1>3+Z_{14}}(r_1-3-Z_{14})P(r_1:4,k)$$

$$+ Ch_{1k}\sum_{r_1<3+Z_{14}}(3+Z_{14}-r_1)P(r_1:4,k+1)\Big\}$$

$$+ \bar{\bar{P}}_3(2)\sum_{k=1}^{1}\Big\{Cs_{1k}\sum_{r_1>3+Z_{14}}(r_1-3+Z_{14})P(r_1:4,k)$$

$$+ Ch_{1k}\sum_{r_1<3+Z_{14}}(3+Z_{14}-r_1)P(r_1:4,k+1)\Big\}\Big\}$$

$$+ \sum_{r_{14}\geq 0} f^*_3(3+Z_{14}-r_{14})P_{14}(r_{14})\Big].$$

$$= \min_{0\leq Z_{14}\leq 2}\Big[\varphi_{14}(Z_{14})$$

$$+ \bar{P}_4(2)\bar{\bar{P}}_3(2)+\bar{P}_4(3)\bar{\bar{P}}_3(1)+\bar{P}_4(3)\bar{\bar{P}}_3(2)\Big\{Cs_{11}\sum_{r_1>3+Z_{14}}(r_1-3+Z_{14})P(r_1:4,1)$$

$$+ Ch_{11}\sum_{r_1<3+Z_{14}}(3+Z_{14}-r_1)P(r_1:4,2)\Big\}$$

$$+ \bar{P}_4(1)\bar{\bar{P}}_3(2)+\bar{P}_4(2)\bar{\bar{P}}_3(1)+\bar{P}_4(2)\bar{\bar{P}}_3(2)\Big\{Cs_{12}\sum_{r_1>3+Z_{14}}(r_1-3+Z_{14})P(r_1:4,2)$$

$$+ Ch_{12}\sum_{r_1<3+Z_{14}}(3+Z_{14}-r_1)P(r_1:4,3)\Big\}$$

$$+ \overline{P}_4(1)\overline{\overline{P}}_3(1)+\overline{P}_4(1)\overline{\overline{P}}_3(2)\left\{Cs_{13}\sum_{r_1>3+Z_{14}}(r_1-3+Z_{14})P(r_1:4,3)\right.$$

$$\left. + Ch_{13}\sum_{r_1<3+Z_{14}}(3+Z_{14}-r_1)P(r_1:4,4)\right\}$$

$$+\sum_{r_{14}\geq 0} f^*{}_3(3+Z_{14}-r_{14})P_{14}(r_{14})\Big].$$

$$= \text{Min}\left[\begin{array}{l}
0.0+(.50)\{(6)\{(1)(.175)+(2)(.207)+(3)(.191)+(4)(.136) \\
\quad +(5)(.071)+(6)(.025)+(7)(.005)\} \\
\quad +(1)\{(1)(.163)+(2)(.069)+(3)(.017)\}\} \\
\quad +(.68)\{(6)\{(1)(.244)+(2)(.169)+(3)(.078)+(4)(.020)\} \\
\quad +(.9)\{(1)(.345)+(2)(.355)+(3)(.165)\}\} \\
\quad +(.30)\{(6)(0)+(1)\{(1)(.3)+(2)(.4)+(2)(.3)\}\} \\
\quad +(.3)(4.147)+(.4)(4.647)+(.3)(5.292), \\
\\
1.1+(.50)\{(6)\{(1)(.207)+(2)(.191)+(3)(.136)+(4)(.071) \\
\quad +(5)(.025)+(6)(.005)\} \\
\quad +(1)\{(1)(.240)+(2)(.163)+(3)(.069)+(4)(.017)\}\} \\
\quad +(.68)\{(6)\{(1)(.169)+(2)(.078)+(3)(.020)\} \\
\quad +(.9)\{(1)(.135)+(2)(.345)+(3)(.355)+(4)(.165)\}\} \\
\quad +(.30)\{(6)(0)+(1)\{(2)(.3)+(3)(.4)+(4)(.3)\}\} \\
\quad +(.3)(3.147)+(.4)(4.147)+(.3)(4.647), \\
\\
1.7+(.50)\{(6)\{(1)(.191)+(2)(.136)+(3)(.071)+(4)(.025) \\
\quad +(5)(.005)\}+(1)\{(1)(.244)+(2)(.240) \\
\quad +(3)(.163)+(4)(.069)+(5)(.017)\}\} \\
\quad +(.68)\{(6)\{(1)(.078)+(2)(.020)\} \\
\quad +(.9)\{(2)(.135)+(3)(.345)+(4)(.355)+(5)(.165)\}\} \\
\quad +(.30)\{(6)(0)+(1)\{(3)(.3)+(4)(.4)+(5)(.3)\}\} \\
\quad +(.3)(2.147)+(.4)(3.147)+(.3)(4.147).
\end{array}\right].$$

$$= \text{Min}\begin{bmatrix}16.687\\12.511\\11.621\end{bmatrix} = 11.621; \text{ where } Z^*_{14}(3) = 2.$$

For other values of U_{14}, the values of $f^*_3(U_{14})$ and $Z^*_{14}(U_{14})$ can be determined. The results are summarized below:

$f^*_4(4) = 11.021; \qquad Z^*_{14}(4) = 1$

$f^*_4(5) = \ 9.921: \qquad Z^*_{14}(5) = 0.$

Consider period 5, which is the last stage. Using (5-2-6) to (5-2-8),

$$f^*_5(4)$$

$$= \underset{0\leq Z_{15}\leq \text{Min}(4,1)}{\text{Min}}\Big[\varphi_{15}(Z_{15})$$

$$+ \sum_{L=0}^{4}\bar{P}_5(L)\sum_{L'\geq 0}\bar{\bar{P}}_4(L'+1)\sum_{k=\text{Max}[5-L-L',1]}^{5-L}\Big\{Cs_{1k}\sum_{r_1>4+Z_{15}}(r_1-4-Z_{15})P(r_1:5,k)$$

$$+ Ch_{1k}\sum_{r_1<4+Z_{15}}(4+Z_{15}-r_1)P(r_1:5,k+1)\Big\}$$

$$+ \sum_{r_{15}\geq 0} f^*_4(4+Z_{15}-r_{15})P_{15}(r_{15})\Big]$$

$$= \underset{0\leq Z_{15}\leq 1}{\text{Min}}\Big[\varphi_{15}(Z_{15})$$

$$+ \bar{P}_5(2)\Big\{\bar{\bar{P}}_4(1)\sum_{k=3}^{3}\Big\{Cs_{1k}\sum_{r_1>4+Z_{15}}(r_1-4-Z_{15})P(r_1:5,k)$$

$$+ Ch_{1k}\sum_{r_1<4+Z_{15}}(4+Z_{15}-r_1)P(r_1:5,k+1)\Big\}\Big\}$$

$$+ \bar{P}_5(3)\Big\{\bar{\bar{P}}_4(1)\sum_{k=2}^{2}\Big\{Cs_{1k}\sum_{r_1>4+Z_{15}}(r_1-4-Z_{15})P(r_1:5,k)$$

$$+ Ch_{1k}\sum_{r_1<4+Z_{15}}(4+Z_{15}-r_1)P(r_1:5,k+1)\Big\}\Big\}$$

$$+ \sum_{r_{15}\geq 0} f^*_4(4+Z_{15}-r_{15})P_{15}(r_{15})\Big].$$

$$= \text{Min}\begin{bmatrix} 0+(.3)\{(6)(0)+(1)\{(1)(.5)+(2)(.35)+(3)(.35)+(4)(.15)\}\} \\ +(.2)\{(6)\{(1)(.206)+(2)(.124)+(3)(.049)+(4)(.010)\} \\ +(.9)\{(1)(.240)+(2)(.350)+(3)(.260)+(4)(.083)\}\} \\ +(.5)(11.021)+(.5)(11.621), \\ \\ 1+(.3)\{(6)(0)+(1)\{(2)(.15)+(3)(.35)+(4)(.35)+(5)(.15)\}\} \\ (.2)\{(6)\{(1)(.124)+(2)(.049)+(3)(.010)\}+(.9)\{(1)(.067) \\ +(2)(.240)+(3)(.350)+(4)(.260)+(5)(.083)\}\} \\ +(.5)(9.921)+(.5)(11.021). \end{bmatrix}$$

$$= \text{Min}\begin{bmatrix} 13.315 \\ 13.363 \end{bmatrix} = 13.315; \quad Z^*_{15}(4) = 0.$$

Therefore, the optimal policy in period 5 is do not make an order. The minimum expected total controllable cost is \$13.315.

5.2 MULTI-ITEM SINGLE-SOURCE SYSTEM FOR THE MIXABLE ITEMS

This section is an extension of Section 5.1; several types of items are to be carried and they can be mixed together in the warehouse. There continues to be only one source of supply as in Section 5.1, and other assumptions remain the same as before. The analysis is as follows.

Assume that there are N types of items in the system, and consider period K,

where $$K - \underline{L}_K \geq 1,$$

and $$K - 1 - \underline{L}_K \geq 1.$$

This means that there is a chance that the order made before or at period K will arrive before or at the beginning of period 1, but there is no chance that the order made after period K will arrive before the beginning of period 1.

For a given set of U_{1K}, U_{2K}, ..., U_{NK}; assume that an order of the amount Z_{1k} is made only for item type No. 1 at this period. The expected total controllable cost will include those expected system costs in periods $K-\underline{L}_K$, $K-\underline{L}_K-1$, ..., and 1, which is the sum of:

(1) item cost plus fixed cost of ordering Z_{1K}, which is $\varphi_{1K}(Z_{1K})$,

(2) total expected controllable cost due to shortage of item type No. 1, which is

$$\sum_{L=0}^{K-1}\overline{P}_K(L)\left\{\sum_{k=1}^{K-L}Cs_{1k}\sum_{r_1>U_{1K}+Z_{1K}}(r_1-U_{1K}-Z_{1K})P(r_1;K,k+1)\right\}$$

(3) total expected controllable cost in carrying item type No. 1, which is

$$\sum_{L=0}^{K-1}\overline{P}_K(L)\sum_{k=1}^{K-L}Ch_{1k}\cdot\sum_{r_1<U_{1K}+Z_{1K}}(U_{1K}+Z_{1K}-r_1)P(r_1;K,k+1)$$

(4) total expected controllable shortage cost due to shortages of item types No. 2 to No. N, which is

$$\sum_{i=2}^{N}\sum_{L=0}^{K-1}\overline{P}_K(L)\sum_{k=1}^{K-L}Cs_{ik}\cdot\sum_{r_i>U_{iK}}(r_i-U_{iK})P(r_i;K,k),\text{ and}$$

(5) total expected controllable carrying cost in carrying item types No. 2 to No. N, which is

$$\sum_{i=2}^{N}\sum_{L=0}^{K-1}\overline{P}_K(L)\sum_{k=1}^{K-L}Ch_{ik}\cdot\sum_{r_i<U_{iK}}(U_{iK}-r_i)P(r_i;K,k+1).$$

Thus, the expected total controllable cost, $\mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK};Z_{1K})$

$$= \varphi_{1K}(Z_{1K})$$

$$+ \sum_{L=0}^{K-1} \overline{P}_K(L) \sum_{k=1}^{K-L} \Big\{ Cs_{1k} \cdot \sum_{r_1 > U_{1K}+Z_{1K}} (r_1 - U_{1K} - Z_{1K}) P(r_1; K, k)$$

$$+ Ch_{1k} \cdot \sum_{r_1 < U_{1K}+Z_{1K}} (U_{1K} + Z_{1K} - r_1) P(r_1; K, k+1)$$

$$+ K_K(U_{2K}, U_{3K}, \ldots, U_{NK}), \qquad (5\text{-}2\text{-}1)$$

where $K_K(U_{2K}, U_{3K}, \ldots, U_{NK})$

$$= \sum_{i=2}^{N} \sum_{L=0}^{K-1} \overline{P}_K(L) \sum_{k=1}^{K-L} \Big\{ Cs_{ik} \cdot \sum_{r_i > U_{iK}} (r_i - U_{iK}) P(r_i; K, k)$$

$$+ Ch_{ik} \sum_{r_i < U_{iK}} (U_{iK} - r_i) P(r_i; K, k+1). \qquad (5\text{-}2\text{-}2)$$

Note that for a given set of $U_{1K}, U_{2K}, \ldots, U_{NK}$; the space available for the additional items to be ordered in period K will be

$$W - \sum_{i=1}^{N} v_i \cdot \text{Max}(U_{iK} - \sum_{k=K-\underline{\underline{L}}_K+1}^{K} \underline{r}_{ik}, 0).$$

In order to apply the principle of optimality to this problem, let w_1, the space available for the additional item type No. 1, increase in increments of v_1 from 0, v_1, $2v_1$, ..., Cv_1, ... to $W - \sum_{i=1}^{N} v_i$.

$$\text{Max}(U_{iK} - \sum_{k=K-\underline{\underline{L}}_K+1}^{K} \underline{r}_{ik}, 0).$$

Let $f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/w_1)$ be the minimum expected total controllable cost when a decision is made in period K where only item type No. 1 is being considered, resulting from ordering an optimal amount of $Z_{1K} = Z^*_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK})$ for a given set of $U_{1K}, U_{2K}, \ldots, U_{NK}$ and w_1.

Therefore, $f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/w_1) = \underset{Z_{1K}}{\text{Min}}\ \mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}; Z_{1K})$, (5-2-3)

where $0 \leq Z_{1K} \leq \text{Min}(S_{1K}, \frac{w_1}{v_1} - \text{Min}(U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k}, 0))$. (5-2-4)

For $w_1 = 0$, and $U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k} < 0$; the restriction of Z_{1K} in (5-2-4) becomes

$$0 \leq Z_{1K} \leq \text{Min}(S_{1K}, \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k} - U_{1K}). \quad (5\text{-}2\text{-}5)$$

For $w_1 = 0$, and $U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k} \geq 0$; (5-2-3) becomes

$$f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/0) = \mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}; 0). \quad (5\text{-}2\text{-}6)$$

For $w_1 = v_1 \leq v_1(S_{1K} + \text{Min}(U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k}, 0))$; (5-2-4) becomes

$$0 \leq Z_{1K} \leq 1 - \text{Min}(U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k}, 0)).$$

Then, $f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/v_1)$

$$= \text{Min}\left\{\begin{array}{l} f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/0), \\ \mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}; 1 - \text{Min}(U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k}, 0)) \end{array}\right\}. \quad (5\text{-}2\text{-}7)$$

In general, for $w_1 = Cv_1 \leq v_1(S_{1K} + \text{Min}(U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k}, 0))$;

$f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/Cv_1)$

$$= \text{Min}\left\{\begin{array}{l} f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/Cv_1), \\ \mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}; C - \text{Min}(U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k}, 0)) \end{array}\right\} \quad (5\text{-}2\text{-}8)$$

For $w_1 = Cv_1 \geq v_1(S_{1K} + \text{Min}(U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k}, 0))$; let

$$\bar{C}v_1 \leq v_1(S_{1K} + \text{Min}(U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k}, 0)) < (\bar{C}+1)v_1, \text{ then}$$

$$S_{1K} < (\bar{C}+1) - \text{Min}(U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k}, 0), \text{ and}$$

$$S_{1K} \geq \bar{C} - \text{Min}(U_{1K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{1k}, 0).$$

Therefore, using (5-2-3) and (5-2-4),

$$f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/Cv_1)$$

$$= \text{Min}\left\{ \begin{matrix} f_{1K}(U_{1K}, U_{2K}, \ldots, U_{NK}/\bar{C}v_1) \\ \mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}; S_{1K}) \end{matrix} \right\}. \quad (5\text{-}2\text{-}9)$$

For a given set of $U_{1K}, U_{2K}, \ldots, U_{NK}$; consider that orders are made for item types No. 1 and No. 2, and item type No. 2 is ordered first in the amount of Z_{2K}. Let w_2, the space available for the additional item types No. 1 and No. 2, increase from 0 through the values of $w_1 + mv_2 (m = 0, 1, \ldots)$ until $W - \sum_{i=1}^{N} v_i \text{ Max}(U_{iK} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{ik}, 0)$.

After Z_{2K} is ordered, an optimal amount of item type No. 1 is ordered for a given set of $U_{1K}, U_{2K} + Z_{2K}, \ldots, U_{NK}$, and for an available space of $w_2 - v_2\text{Max}(Z_{2K} + \text{Min}(U_{2K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{2k}, 0), 0)$. Therefore, the expected total controllable cost is the sum of:

(1) item cost plus fixed cost of ordering Z_{2K}, which is $\varphi_{2K}(Z_{2K})$,

(2) minimum expected total controllable cost when a decision is made in period K when only item type

No. 1 is considered, resulting from ordering an optimal amount of Z_{1K} for a given set of U_{1K}, $U_{2K} + Z_{2K}$, ..., U_{NK}, and for a space $w_2 - v_2 \text{Max}(Z_{2K} + \text{Min}(U_{2K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{2k}, 0), 0)$, which is $f_{1K}(U_{1K}, U_{2K} + Z_{2K}, \ldots, U_{NK}/w_2 - v_2 \text{Max}(Z_{2K} + \text{Min}(U_{2K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{2k}, 0), 0)$.

Thus, the expected total controllable cost, $\mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}/Z_{2K})$

$$= \varphi_{2K}(Z_{2K}) + f_{1K}(U_{1K}, U_{2K} + Z_{2K}, \ldots, U_{NK}/w_2 - v_2 \text{Max}(Z_{2K} + \text{Min}(U_{2K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{2k}, 0), 0). \quad (5\text{-}2\text{-}10)$$

Then, let $f_{2K}(U_{1K}, U_{2K}, \ldots, U_{NK}/w_2)$ be the minimum expected total controllable cost when a decision is made in period K where item types No. 1 and No. 2 are considered and item type No. 2 is considered first, resulting from ordering an optimal amount of $Z_{2K} = Z^*_{2K}(U_{1K}, U_{2K}, \ldots, U_{NK})$ presuming optimal amount of Z_{1K} is ordered later, for a given set of U_{1K}, U_{2K}, ..., U_{NK} and w_2. Therefore,

$$f_{2K}(U_{1K}, U_{2K}, \ldots, U_{NK}/w_2) = \underset{Z_{2K}}{\text{Min}}\left\{\mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}/Z_{2K})\right\}, \quad (5\text{-}2\text{-}11)$$

where $$0 \leq Z_{2K} \leq \text{Min}(S_{2K}, \frac{w_2}{v_2} - \text{Min}(U_{2K} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{2k}, 0)). \quad (5\text{-}2\text{-}12)$$

In general, item types No. 1 to No. n $(2 \leq n \leq N)$ are considered and item type No. n is considered first. The space available for the additional item types No. 1 to No. n, increase from 0 through the values of $w_{n-1} + m v_n$ $(m = 0, 1, \ldots)$ until $W - \sum_{i=1}^{N} v_i \text{Max}(U_{iK} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{ik}, 0)$. Then, it follows that $\mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}/Z_{nK})$

$= \varphi_{nK}(Z_{nK})$

$+ f_{n-1,K}(U_{1K}, \ldots, U_{nK} + Z_{nK}, \ldots, U_{NK}/w_n - v_n \mathrm{Max}(Z_{nK} +$

$$\mathrm{Min}(U_{nK} - \sum_{k=K-\underline{L}_K+1}^{K} r_{nK}, 0), 0). \qquad (5\text{-}2\text{-}13)$$

And $f_{2K}(U_{1K}, U_{2K}, \ldots, U_{NK}/w_n) = \underset{Z_{nK}}{\mathrm{Min}}\left\{\mathcal{C}(U_{1K}, U_{2K}, \ldots, U_{NK}/Z_{nK})\right\},$

(5-2-14)

where $0 \leq Z_{nK} \leq \mathrm{Min}(S_{nK}, \frac{w_n}{v_n} - \mathrm{Min}(U_{nK} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{nk}, 0)$ (5-2-15)

By letting $n = N$, and let $f^*_K(U_{1K}, U_{2K}, \ldots, U_{NK}) =$

$f_{NK}(U_{1K}, U_{2K}, \ldots, U_{NK}/W - \sum_{i=1}^{N} v_i \mathrm{Max}(U_{iK} - \sum_{k=K-\underline{L}_K+1}^{K} \underline{r}_{ik}, 0))$

$f^*_K(U_{1K}, U_{2K}, \ldots, U_{NK})$ is obtained as a partial-optimization for this stage.

Consider period K+1, for a given set of $U_{1,K+1}, U_{2,K+1}, \ldots,$ $U_{N,K+1}$; assume that an order is made only for item type No. 1 in the amount of $Z_{1,K+1}$ at this period. The decision made in this period affects those expected system costs in periods $K + L - \underline{L}_{K+1}$, $K - \underline{L}_{K+1}$, ..., and 1.

The expected total controllable cost is the sum of:

(1) item cost plus fixed cost of ordering $Z_{1,K+1}$, which is $\varphi_{1,K+1}(Z_{1,K+1})$,

(2) total expected controllable shortage cost due to the shortage of item type No. 1, during periods from the period when the order made in period K + 1 arrives to one period before the order made in period K arrives, which is

$$\sum_{L=0}^{K}\bar{P}_{K+1}(L)\sum_{L'0}\bar{\bar{P}}_K(L'+1)\sum_{k=Max[K-L-L',0]}^{K-L} Cs_{1k}\sum_{r_1>U_{1,K+1}+Z_{1,K+1}}$$

$$(r_1-U_{1,K+1}-Z_{1,K+1})P(r_1;K,k+1),$$

(3) total expected controllable carrying cost in carrying item type No. 1, during periods from when the order made in period K+1 arrives to one period before the order made in period K arrives, which is

$$\sum_{L=0}^{K}\bar{P}_{K+1}(L)\sum_{L'0}\bar{\bar{P}}_K(L'+1)\sum_{k=Max[K-L-L',0]}^{K-L} Ch_{1k}\sum_{r_1<U_{1,K+1}+Z_{1,K+1}}$$

$$(U_{1,K+1}+Z_{1,K+1}-r_1)P(r_1;K,k+2),$$

(4) total expected controllable shortage cost due to the shortage of item types No. 2 to No. N, during periods from when the order made in period K+1 arrives to one period before the order made in period K arrives, which is

$$\sum_{i=2}^{N}\sum_{L=0}^{K}\bar{P}_{K+1}(L)\sum_{L'\geq 0}\bar{\bar{P}}_K(L'+1)\sum_{k=Max[K-L-L',0]}$$

$$Cs_{ik}\sum_{r_i>U_{i,K+1}}(r_i-U_{i,K+1})P(r_i;K,k+1),$$

(5) total expected controllable carrying cost in carrying item types No. 2 to No. N, during periods from the period where the order made in period K+1 arrives to one period before the order made in period K arrives, which is

$$\sum_{i=2}^{N}\sum_{L=0}^{K}\bar{P}_{K+1}(L)\sum_{L'0}\bar{\bar{P}}_K(L'+1\sum_{k=Max[K-L-L',0]}^{K-L}$$

$$Ch_{ik}\sum_{r_i>U_{i,K+1}}(U_{i,K+1}-r_i)P(r_i;K,k+2),$$

(6) minimum expected total controllable cost, presuming an optimal decision is made at period K; which is

$$\sum_{r_1>0}\sum_{r_2>0}\cdots\sum_{r_N>0}\Big\{f^*(U_{1,K+1}+Z_{1,K+1}-r_{1,K+1},\ \ldots,$$

$$U_{N,K+1}-r_{N,K+1})\prod_{i=1}^{N}P(r_{i,K+1})\Big\} = G\,(U_{1,K+1}+Z_{1,K+1},$$

$$U_{2,K+1},\ \ldots,\ U_{N,K+1}).$$

Thus, the expected total controllable cost,

$$\mathcal{C}(U_{1,K+1},\ U_{2,K+1},\ \ldots,\ U_{N,K+1};Z_{1,K+1}) = \varphi_{1,K+1}(Z_{1,K+1})$$

$$+\sum_{L=0}^{K}\overline{P}_{K+1}(L)\sum_{L'0}\overline{\overline{P}}_{K}(L'+1)\sum_{k=\mathrm{Max}[K-L-L',0]}^{K-L}\Big\{Cs_{1k}\sum_{r_1>U_{1K+1}+Z_{1K+1}}(r_1-U_{1K+1}-Z_{1K+1})P(r_1;K,k+1)$$

$$+\ Ch_{1k}\sum_{r_1<U_{1K+1}+Z_{1K+1}}(U_{1K+1}Z_{1K+1}-r_1)P(r_1;K,k+2)\Big\}$$

$$+\ K_{K+1}(U_{2,K+1},\ U_{3,K+1},\ \ldots,\ U_{N,K+1})$$

$$+\ G(U_{1,K+1}+Z_{1,K+1},\ U_{1,K+1},\ \ldots,\ U_{1,K+1}), \qquad (5\text{-}2\text{-}16)$$

where $K_{K+1}(U_{2,K+1},\ U_{3,K+1},\ \ldots,\ U_{N,K+1})$

$$=\sum_{i=2}^{N}\sum_{L=0}^{K}\overline{P}_{K+1}(L)\sum_{L'0}\overline{\overline{P}}_{K}(L'+1)\sum_{k=\mathrm{Max}[K-L-L',0]}^{K-L}\Big\{Cs_{ik}\sum_{r_i>U_{i,K+1}}(r_i-U_{i,K+1})$$

$$P(r_i;K,k+1) + Ch_{1k}\sum_{r_i<U_{i,K+1}}(U_{i,K+1}-r_i)P(r_i;K,k+2)\Big\}. \qquad (5\text{-}2\text{-}17)$$

Note that for a given set of $U_{1,K+1},\ U_{2,K+1},\ \ldots,\ U_{N,K+1}$; the space available for the additional items ordered in the period K+1 is

$$W-\sum_{i=1}^{N}v_i\cdot \mathrm{Max}(U_{i,K+1}-\sum_{k=K-\underline{L}_{K+1}+2}^{K+1}\underline{r}_{ik},0).$$

As before, let w_1, the space available for the additional item type No. 1, increase in increments v_1 from 0, v_1, $2v_1$, ..., Cv_1, ... to $W - \sum_{i=1}^{N} v_i \text{Max}(U_{i,K+1} - \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{ik}, 0)$. Let $f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/w_1)$ be the minimum expected total controllable cost when a decision is made in period K+1 when only item type No. 1 is considered, resulting from ordering an optimal amount of $Z_{1,K+1} = Z^*_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1})$, presuming an optimal decision is made in period K, for a given set of $U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}$ and w_1. Therefore, $f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/w_1)$

$$= \underset{Z_{1,K+1}}{\text{Min}} \left\{ \mathcal{C}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}; Z_{1,K+1}) \right\} \tag{5-2-18}$$

where $0 \leq Z_{1,K+1} \leq \text{Min}(S_{1,K+1}, \frac{w_1}{v_1} - \text{Min}(U_{1,K+1} \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{1k}, 0))$.

(5-2-19)

For $w_1 = 0$, and $U_{1,K+1} - \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{1k} < 0$; the restriction of $Z_{1,K+1}$ in (5-2-19) becomes

$$0 \leq Z_{1,K+1} \leq \text{Min}(S_{1,K+1}, \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{1k} - U_{1,K+1}). \tag{5-2-20}$$

For $w_1 = 0$, and $U_{1,K+1} - \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{1k} \geq 0$; (5-2-18) becomes

$f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/0)$

$$= \mathcal{C}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}; 0). \tag{5-2-21}$$

For $w_1 = Cv_1 \leq v_1(S_{1,K+1} + \text{Min}(U_{1,K+1} - \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{1k}, 0)$;

$f_{1,K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1}/Cv_1)$

$$= \text{Min}\left\{\begin{matrix} f_{1,K+1}(U_{1,K+1},\ U_{2,K+1},\ \ldots,\ U_{N,K+1}/(C-1)v_1), \\ C(U_{1,K+1},\ U_{2,K+1},\ \ldots,\ U_{N,K+1};C-\text{Min}(U_{1,K+1} - \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{1k},0)) \end{matrix}\right\}.$$

(5-2-22)

For $w_1 = Cv_1 > v_1(S_{1,K+1} + \text{Min}(U_{1,K+1} - \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{1k},0)$;

let $\bar{C}v_1 \leq v_1(S_{1,K+1} + \text{Min}(U_{1,K+1} - \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{1k},0)) < (\bar{C}+1)v_1$,

then $f_{1,K+1}(U_{1,K+1},\ U_{2,K+1},\ \ldots,\ U_{N,K+1}/Cv_1)$

$$= \text{Min}\left\{\begin{matrix} f_{1,K+1}(U_{1,K+1},\ U_{2,K+1},\ \ldots,\ U_{N,K+1}/\bar{C}v_1), \\ \mathcal{C}(U_{1,K+1},\ U_{2,K+1},\ \ldots,\ U_{N,K+1};S_{1,K+1}) \end{matrix}\right\}. \qquad (5\text{-}2\text{-}23)$$

As in previous discussions, if item types No. 1 to No. n $(2 \leq n \leq N)$ are considered and item type No. n is considered first, for a given set of $U_{1,K+1}, \ldots, U_{n,K+1}, \ldots, U_{N,K+1}$ and for w_n, it follows that

$$\mathcal{C}(U_{1,K+1},\ \ldots,\ U_{n,K+1},\ \ldots,\ U_{N,K+1};Z_{n,K+1}) = \varphi_{n,K+1}(Z_{n,K+1})$$
$$+ f_{n-1,K+1}(U_{1,K+1},\ \ldots,\ U_{n,K+1} + Z_{n,K+1},\ \ldots,\ U_{N,K+1}/w_n - v_n\text{Max}(Z_{n,K+1}$$
$$+ \text{Min}(U_{n,K+1} - \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{nk},0),0)), \qquad (5\text{-}2\text{-}24)$$

and $f_{n,K+1}(U_{1,K+1},\ \ldots,\ U_{n,K+1},\ \ldots,\ U_{N,K+1}/w_n)$

$$= \underset{Z_{n,K+1}}{\text{Min}} \left\{\mathcal{C}(U_{1,K+1},\ \ldots,\ U_{n,K+1},\ \ldots,\ U_{N,K+1};Z_{n,K+1})\right\}, \qquad (5\text{-}2\text{-}25)$$

where $0 \leq Z_{n,K+1} \leq \text{Min}(S_{n,K+1},\ \frac{w_n}{v_n} - \text{Min}(U_{n,K+1} - \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{nk},0))$.

(5-2-26)

By letting n = N, and let $f^*_{K+1}(U_{1,K+1},\ U_{2,K+1},\ \ldots,\ U_{N,K+1})$

$$= f_{N,K+1}(U_{1,K+1},\ \ldots,\ U_{N,K+1}/W - \sum_{i=1}^{N} v_i\text{Max}(U_{i,K+1} - \sum_{k=K-\underline{L}_{K+1}+2}^{K+1} \underline{r}_{ik},0)).$$

$f^*_{K+1}(U_{1,K+1}, U_{2,K+1}, \ldots, U_{N,K+1})$ is obtained as a partial-optimization for this stage.

Consider in general period p, where $K+1 \leq p \leq P$.

Using the previous developments, it follows $\mathcal{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; Z_{1p}) = \varphi_{1p}(Z_{1p})$

$$+ \sum_{L=0}^{p-1} \bar{P}_p(L) \sum_{L'>0} \bar{\bar{P}}_{p-1}(L'+1) \sum_{k=Max[p-L-L',0]}^{p-L-1} \Big\{ Cs_{1k} \sum_{r_1 > U_{1p}+Z_{1p}} (r_1 - U_{1p} - Z_{1p})$$

$$P(r_1; p, k+1) + Ch_{1k} \sum_{r_1 < U_{1p}+Z_{1p}} (U_{1p} + Z_{1p} - r_1) P(r_1; p, k+2)$$

$$+ K_p(U_{2p}, U_{3p}, \ldots, U_{Np}) + G(U_{1p} + Z_{1p}, U_{2p}, \ldots, U_{Np}), \qquad (5\text{-}2\text{-}27)$$

where $K_p(U_{2p}, U_{3p}, \ldots, U_{Np})$

$$= \sum_{i=2}^{N} \Big\{ Cs_{ik} \sum_{r_i > U_{ip}} (r_i - U_{ip}) P(r_i; p, k+1) + Ch_{ik} \sum_{r_i < U_{ip}} (U_{ip} - r_i) P(r_i; p, k+2) \Big\},$$

and $G(U_{1p} + Z_{1p}, U_{2p}, \ldots, U_{Np})$

$$= \sum_{r_{1p}>0} \cdots \sum_{r_{Np}>0} \Big\{ f^*_{p-1}(U_{1p} + Z_{1p} - r_{1p}, \ldots, U_{1p} - r_{1p}) \prod_{i=1}^{N} P(r_{ip}) \Big\}.$$

Therefore, $f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/w_1) = \underset{Z_{1p}}{\mathrm{Min}} \Big\{ \mathcal{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; Z_{1p}) \Big\},$

$$(5\text{-}2\text{-}28)$$

where $0 \leq Z_{1p} \leq \mathrm{Min}(S_{1p} - \frac{w_1}{v_1} - \mathrm{Min}(U_{1p} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{1k}, 0)).$ (5-2-29)

For $w_1 = 0$, and $U_{1p} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{1k} < 0$; the restriction of Z_{1p} in (5-2-29) becomes

$$0 \leq Z_{1p} \leq \mathrm{Min}(S_{1p}, \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{1k} - U_{1p}). \qquad (5\text{-}2\text{-}30)$$

For $w_1 = 0$, and $U_{1p} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{1k} \geq 0$; (5-2-28) becomes

$$f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/0) = \mathcal{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; 0). \qquad (5\text{-}2\text{-}31)$$

For $w_1 = Cv_1 \leq v_1(S_{1p} + \mathrm{Min}(U_{1p} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{1k}, 0))$, it follows that $f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/Cv_1)$

$$= \mathrm{Min}\left\{\begin{array}{l} f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/(C-1)v_1), \\ \mathcal{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; C-\mathrm{Min}(U_{1p} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{1k}, 0)) \end{array}\right\}. \qquad (5\text{-}2\text{-}32)$$

And for $w_1 = Cv_1 > v_1(S_{1p} + \mathrm{Min}(U_{1p} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{1k}, 0))$;

$f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/Cv_1)$

$$= \mathrm{Min}\left\{\begin{array}{l} f_{1p}(U_{1p}, U_{2p}, \ldots, U_{Np}/\bar{C}v_1), \\ \mathcal{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; S_{1p}) \end{array}\right\}, \qquad (5\text{-}2\text{-}33)$$

$$\text{where } \bar{C}v_1 \leq v_1(S_{1p} + \mathrm{Min}(U_{1p} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{1k}, 0)) < (\bar{C}+1)v_1. \qquad (5\text{-}2\text{-}34)$$

Again using previous developments, if item types No. 1 to No. n $(2 \leq n \leq N)$ are considered and item type No. n is considered first, it follows that $\mathcal{C}(U_{1p}, U_{2p}, \ldots, U_{Np}; Z_{np}) = \varphi_{np}(Z_{np})$

$$+ f_{n-1,p}(U_{1p}, \ldots, U_{np} + Z_{np}, \ldots, U_{Np}/w_n - v_n \mathrm{Max}(Z_{np} + \mathrm{Min}(U_{np} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{nk}, 0), 0)), \qquad (5\text{-}2\text{-}35)$$

and $f_{np}(U_{1p}, \ldots, U_{np}, \ldots, U_{Np}/w_n)$

$$= \underset{Z_{np}}{\mathrm{Min}}\left\{C(U_{1p}, \ldots, U_{np}, \ldots, U_{Np}; Z_{np})\right\}, \qquad (5\text{-}2\text{-}36)$$

$$\text{where } 0 \leq Z_{np} \leq \mathrm{Min}(S_{np}, \frac{w_n}{v_n} - \mathrm{Min}(U_{np} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{nk}, 0)). \qquad (5\text{-}2\text{-}37)$$

By letting $n = N$, and let $f^*_p(U_{1p}, U_{2p}, \ldots, U_{Np})$

$$= f_{Np}(U_{1p}, U_{2p}, \ldots, U_{Np}/W - \sum_{i=1}^{N} v_i \ \mathrm{Max}(U_{ip} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{ik}, 0))$$

$$f^*_p(U_{1p}, U_{2p}, \ldots, U_{Np}) = f_{Np}(U_{1p}, U_{2p}, \ldots, U_{Np}/W - \sum_{i=1}^{N} v_i$$

$\text{Max}(U_{ip} - \sum_{k=p-\underline{L}_p+1}^{p} \underline{r}_{ik}, 0))\ f^*_p(U_{1p}, U_{2p}, \ldots, U_{Np})$ is obtained as a partial-optimization of this stage.

And if p=P, $f^*p(U_{1P}, U_{2P}, \ldots, U_{NP})$ is the final optimization of the problem.

Example

planning period,	$P = 4$
warehouse space,	$W = 5$ cubic units
number of type of items,	$N = 2$
a volume of an item,	$v_1 = 1$ cubic unit
	$v_2 = 1$ cubic unit
initial inventory,	$U_{14} = 3$ units
	$U_{24} = 0$ unit

	i = 1				i = 2			
	k=1	k=2	k=3	k=4	k=1	k=2	k=3	k=4
S_{ik}-unit			3	5			2	1
Co_{ik}-dollars/order			0.50	0.50			0.50	0.50
Ci_{ik}-dollars/unit			0.50	0.60			0.70	0.70
Cs_{ik}-dollars/unit/period	6.00	6.00	6.00		10.0	9.00	9.00	
Ch_{ik}-dollars/unit/period	1.00	0.90	1.00		1.00	1.00	1.00	

	$P_{ik}(r_{ik})$							
	i = 1				i = 2			
r	k=1	k=2	k=3	k=4	k=1	k=2	k=3	k=4
0	.20	.10	.55	.30	.00	.40	.30	.00
1	.25	.20	.45	.40	.50	.60	.70	.60
2	.30	.35	.00	.30	.50	.00	.00	.40
3	.25	.20	.00	.00	.00	.00	.00	.00
4	.00	.15	.00	.00	.00	.00	.00	.00

L	0	1	2	3
$\bar{P}_5(L)$	.00	.00	.60	.40
$\bar{\bar{P}}_4(L)$	.50	.50	.00	.00
$\bar{\bar{P}}_3(L)$	.00	.40	.60	.00
$\bar{\bar{P}}_2(L)$	.00	.00	.60	.40

Solution:

Using the given data, the necessary values of $P(r_i:K,k)$ and $\bar{\bar{P}}_k(L)$ can be determined as shown in Table XIII and Table XIV, respectively.

It is obvious that:

$$3 - \underline{L}_3 > 1$$

and,

$$2 - \underline{L}_2 < 1.$$

Therefore, the first period to be considered is period 3.

Using (5-2-1), (5-2-2), and (5-2-6), for $U_{13} = 1$, $U_{23} = -2$;

$f_{13}(1,-2/0)$

$$= \sum_{L=0}^{2} \bar{P}_3(L) \sum_{k=1}^{3-L} \Big\{ Cs_{1k} \sum_{r_1 > 1+Z_{13}} (r_1 - 1 - Z_{13}) P(r_1:3,k)$$

TABLE XIII

CUMULATIVE PROBABILITY OF DEMANDS FROM PERIOD k TO K, $P(r_i:K,k)$

	i=1					i=2				
	K=3		K=4			K=3		K=4		
r_i	k=1	k=2	k=1	k=2	k=3	k=1	k=2	k=1	k=2	k=3
0	.011	.055	.003	.017	.165	.000	.120	.000	.000	.000
1	.045	.155	.018	.069	.355	.060	.460	.000	.072	.180
2	.112	.283	.055	.163	.345	.290	.420	.036	.324	.540
3	.184	.267	.114	.240	.135	.440	.000	.198	.436	.280
4	.225	.173	.175	.244	.000	.210	.000	.380	.168	.000
5	.208	.067	.207	.169	.000	.000	.000	.302	.000	.000
6	.136	.000	.191	.078	.000	.000	.000	.084	.000	.000
7	.063	.000	.136	.020	.000	.000	.000	.000	.000	.000
8	.016	.000	.071	.000	.000	.000	.000	.000	.000	.000
9	.000	.000	.025	.000	.000	.000	.000	.000	.000	.000
10	.000	.000	.005	.000	.000	.000	.000	.000	.000	.000

TABLE XIV

PROBABILITY OF PROCUREMENT LEAD TIME, $\bar{P}_k(L)$

L	0	1	2	3
$\bar{P}_2(L)$	.00	.00		
$\bar{P}_3(L)$	.00	.12	.38	
$\bar{P}_4(L)$	.00	.30	.50	.20
$\bar{P}_5(L)$	.00	.00	.60	.40

$$+ Ch_{1k} \sum_{r_1<1+Z_{13}} (1+Z_{13}-r_1)P(r_1:3,k+1)\Big\}$$

$$+ K(-2)$$

$$= 0.0+(.5)\{(6)\{(1)(.112)+(2)(.184)+(3)(.225)+(4)(.208)$$

$$+(5)(.136)+(6)(.063)+(7)(.016)\}+(1)\{(1)(.055)\}\}$$

$$+(.12)\{(6)\{(1)(.283)+(2)(.267)+(3)(.173)+(4)(.067)\}$$

$$+(.9)\{(1)(.55)\}\}+27.564$$

$$= 38.289.$$

For the values of $w_1 = 1$, 2, and 3, by using (5-2-8), $f_{13}(1,-2/w_1)$ can be determined. The results, determined by the computer, are as shown in Annex I-3.

Since $w_1 = 4 > v_1(S_{13} + \text{Min} \sum_{k=3-\underline{L}_3+1}^{3} \underline{r}_{1k}, 0))$, applying (5-2-9), then

$f_{13}(1,-2/4) = f_{13}(1,-2/3)$.

For the other sets of U_{13}, U_{23}; $f_{13}(U_{13}, U_{23}/w_1)$ can be determined. The results, determined by the computer, are as shown in Annex I-3.

The last calculation for the first stage is to determine $f^*_3(U_{13}, U_{23})$. Since there are only two types of items in the system $f_{23}(U_{13}, U_{23}/w_2)$ for all values of w_2 are not necessary. For each set of U_{13}, U_{23}, $f^*_3(U_{13}, U_{23})$ can be determined directly from $f_{23}(U_{13}, U_{23}/w_2)$, where $w_2 = W - \sum_{i=1}^{2} v_i \cdot \text{Max}(U_{i3} - \sum_{k=K-\underline{L}_3+1}^{3} \underline{r}_{ik}, 0)$.

Using (5-2-10) to (5-2-12), for $U_{13} = 1$, $U_{23} = -2$;

$f^*_3(1,-2) = f_{23}(1,-2/4)$

$$= \underset{0\le Z_{23}\le \text{Min}(2,\frac{4}{1}-\text{Min}(-2-0,0))}{\text{Min}} \Big[\varphi_{23}(Z_{23}) + f_{13}(1,-2+Z_{23}/4-(1)\text{Max}(Z_{23}+ \text{Min}(-2-0,0))) \Big]$$

$$= \text{Min}\begin{bmatrix} 0.0 + f_{13}(1,-2/4), \\ 1.2 + f_{13}(1,-1/4), \\ 1.9 + f_{13}(1,0/4) \end{bmatrix} = \text{Min}\begin{bmatrix} 32.963 \\ 28.083 \\ 22.703 \end{bmatrix}$$

$= 22.703$; where $Z^*_{13}(1,-2)=3$, and $Z^*_{23}(1,-2)=2$.

For other sets of U_{13}, U_{23}; $f^*_3(U_{13}, U_{23})$ can be determined. The results, determined by the computer, are as shown in Annex I-3.

Consider period 4. The first calculation for this stage is to determine $f_{14}(U_{14}, U_{24}/w_2)$.

Using (6-2-16), (6-2-17), and (6-2-21), for $U_{14} = 3$, $U_{24} = 0$;

$f_{14}(3,0/0)$

$$= \sum_{L=0}^{3} \bar{P}_4(L) \sum_{L' \geq 0} \bar{\bar{P}}_3(L'+1) \sum_{k=\text{Max}(4-L-L',1)}^{4-L} \Big\{ Cs_{1k} \sum_{r_1 > 3} (r_1-3)P(r_1:4,k)$$

$$+ Ch_{1k} \sum_{r_1 < 3} (3-r_1)P(r_1:4,k+1) \Big\}$$

$+ K(0) + G(3,0) = 73.961$; where $Z^*_{14}(3,0) = 0$.

For other values of w_1, by using (5-2-22), $f_{14}(3,0/w_1)$ can be determined. The results, determined by the computer, are as shown in Annex I-3.

For other sets of U_{14}, U_{24}; $f_{14}(U_{14}, U_{24}/w_1)$ can be determined. The results, determined by the computer, are as shown in Annex I-3.

The last calculation for this problem is to determine $f^*_4(3,0)$ which can be determined directly from $f_{24}(3,0/2)$.

Using (5-2-25) and (5-2-26); $f^*_4(3,0) = f_{24}(3,0/2)$

$$= \underset{0 \leq Z_{24} \leq \text{Min}(1,\frac{2}{1}+1)}{\text{Min}} \Big[\varphi_{24}(Z_{24}) + f_{14}(3,0 + Z_{24}/2-(1)\text{Max}(Z_{24}+\text{Min}(0-1,0),0)) \Big]$$

$$= \text{Min}\begin{bmatrix} \varphi_{24}(0)\ ) + f_{14}(3,0/2), \\ \varphi_{24}(1) + f_{14}(3/1/2) \end{bmatrix}$$

$$= \text{Min}\begin{bmatrix} 0.0 + 70.114 \\ 1.2 + 53.189 \end{bmatrix} = 54.389; \text{ where } Z^*_{14}(3,0) = 1, \text{ and } Z^*_{24}(3,0) = 1.$$

The optimal policy in period 4 is then order 1 unit for both items No. 1 and No. 2.

5.3 MULTI-ITEM SINGLE-SOURCE SYSTEM FOR THE NON-MIXABLE ITEMS

This section considers the case for the specific assumption in Section 2.3 when demands and procurement lead times are probabilistic. Employing the discussion in Section 2.3, the system can be reduced to single-item single-source. Thus, for item type No. i, one can use the development in Section 5.1 to determine $G_i(w)$, which is $f^*_P(0)$ for the selected value of w. And then the procedure to allocate space to each type of items will be the same as in Section 2.3.

5.4 SINGLE-ITEM MULTI-SOURCE SYSTEM

This section considers the problem in Section 4.4 when lead time is probabilistic. For simplicity purposes, the case that two sources are available at each period and the system shown in Figure 3 are considered.

As in Section 4.4, for illustration, consider the alternative that at period P-1 source No. 1 is chosen and at period P-2 source No. 2 is chosen. For a given set of $\hat{U}_{1,P-2}, Z_{1,P-1,1}$ if an order of the amount $Z_{1,P-2,2}$ is made, the total expected controllable cost is the sum of:

(1) item cost plus fixed cost for ordering $Z_{1,P-2,2}$, which is $\varphi_{1,P-2,2}, (Z_{1,P-2,2})$,

(2) total expected shortage cost during the period 1

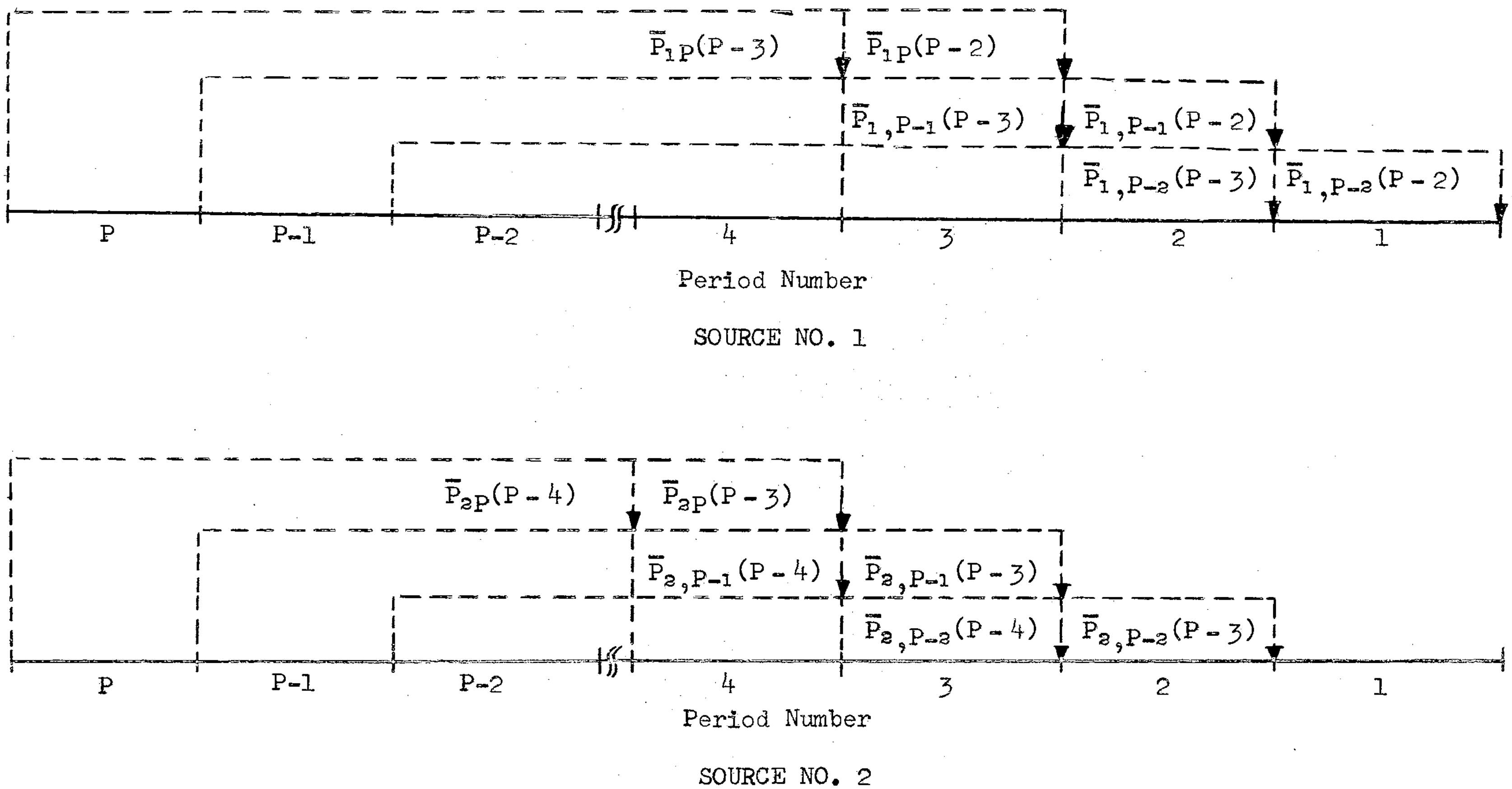


Figure 3. Single-Item Multi-Source System, Probabilistic Demands and Probabilistic Procurement Lead Times

to $P-1-\underline{L}_{1,P-1}$, which is

$$\sum_{k=1}^{P-1-\underset{j}{Max}(\underline{L}_{j,P-1})} \left\{ Cs_{1k} \cdot \sum_{r_1} Max(r_1 - \hat{U}_{1,P-2} - \delta_1 Z_{1,P-1,1} - \delta_2 Z_{1,P-2,2}, 0) P(r_1 : P-2, k) \cdot \delta_3 \delta_4 \right\},$$

where values of both δ_1 and δ_2 are varied for 0 and 1, and

$$\delta_3 = \delta_1 \cdot \sum_{L=0}^{P-1-k} \overline{P}_{1,P-1}(L) + (1 - \delta_1)(1 - \sum_{L=0}^{P-1-k} \overline{P}_{1,P-1}(L)),$$

$$\delta_4 = \delta_2 \cdot \sum_{L=0}^{P-2-k} \overline{P}_{2,P-2}(L) + (1 - \delta_2)(1 - \sum_{L=0}^{P-2-k} \overline{P}_{2,P-2}(L)),$$

(3) total expected carrying cost during the period 1 to $P-1-\underline{L}_{1,P-1}$, which is

$$\sum_{k=1}^{P-1-\underset{j}{Max}(\underline{L}_{j,P-1})} \left\{ Ch_{1k} \cdot \sum_{r_1} Max(\hat{U}_{1,P-2} + Z_{1,P-1,1} - Z_{1,P-2,2} - r_1, 0) P(r_1 : P-2, k+1) \cdot \delta_3 \cdot \delta_4 \right\}.$$

Thus, the total expected system cost, $\mathcal{C}(\hat{U}_{1,P-2}, Z_{1,P-1,1} : Z_{1,P-2,2})$

$$= \varphi_{1,P-2,2}(Z_{1,P-2,2}) + \sum_{k=1}^{P-1-\underset{j}{Max}(\underline{L}_{j,P-1})} \Big[\Big\{ Cs_{1k} \cdot \sum_{r_1} Max(r_1 - \hat{U}_{1,P-2} - \delta_1 Z_{1,P-1,1} - \delta_2 Z_{1,P-2,2}, 0) P(r_1 : P-2, k) + Ch_{1k} \cdot \sum_{r_1} Max(\hat{U}_{1,P-2} + \delta_1 Z_{1,P-1,1} + \delta_2 Z_{1,P-2,2} - r_1, 0) P(r_1 : P-2, k+1) \Big\} \delta_3 \; \delta_4 \Big]. \qquad (5\text{-}4\text{-}1)$$

Let $f_{P-2,2}(\hat{U}_{1,P-1}, Z_{1,P-1,1})$ be the minimum expected total system cost when a decision is made in period P-2 where the order is made from source No. 2, assuming the order was made from source No. 1 in period P-1, resulting from ordering an optimal amount of $Z_{1,P-2,2}$ for a given set of $\hat{U}_{1,P-2}, Z_{1,P-1,1}$. Therefore,

$$f_{P-2,2}(\hat{U}_{1,P-2}, Z_{1,P-1,1}) = \underset{Z_{1,P-2,2}}{\text{Min}}\left[\mathcal{C}(\hat{U}_{1,P-2}, Z_{1,P-1,1} : Z_{1,P-2,2})\right], \tag{5-4-2}$$

$$\text{where } 0 \leq Z_{1,P-2,2} \leq \text{Min}(S_{1,P-2,2}, \frac{W}{V_1} - \hat{U}_{1,p-2} - Z_{1,P-1,1} + \sum_{k=P-\underset{j}{\text{Max}}(\underline{L}_j, P-1)}^{P-2} r_{1k}. \tag{5-4-3}$$

For other combinations of sources that could be chosen in period P-1 and P-2, for each given set of $\hat{U}_{1,P-2}$, $Z_{1,P-1,j}$, $f_{P-2,j}(\hat{U}_{1,P-2}, Z_{1,P-1,j})$ can be determined.

Let $f^*_{P-2/j}(\hat{U}_{1,P-2}, Z_{1,P-1,j})$ be the minimum expected total system cost when a decision is made in period P-2, assuming the order was made from source No. j in period P-1, resulting from ordering an optimal amount of $Z_{1,P-2,j}$ from the optimal source for a given set of $\hat{U}_{1,P-2}$, $Z_{1,P-1,j}$, $f^*_{P-2/j}(\hat{U}_{1,P-2}, Z_{1,P-1,j}) =$

$$\underset{j'}{\text{Min}}\left[f_{P-2,j'}(\hat{U}_{1,P-2}, Z_{1,P-1,j})\right], \tag{5-4-4}$$

where j' is source to be considered in period P-2.

Then, $f^*_{P-2/j}(\hat{U}_{1,P-2}, Z_{1,P-1,j})$ can be used in determining the optimal policy in next stage.

Consider period P-1, assume again for illustration purposes that at period P source No. 2 is chosen and at period P-1 source No. 1 is chosen.

For a given set of $\hat{U}_{1,P-1}$, Z_{1P2}, if an order of the amount $Z_{1,P-1,1}$ is made from source No. 1 the total controllable system cost is the sum of:

(1) item cost plus fixed cost for ordering $Z_{1,P-1,1}$, which is $\varphi_{1,P-1,1}(Z_{1,P-1,1})$,

(2) total expected shortage cost during periods $P - \underset{j}{Max}(\underline{L}_{j,P-1})$ to $P - \underline{L}_{2P}$, which is

$$\sum_{k=P-Max(\underline{L}_{j,P-1})}^{P-\underline{L}_{2P}} \left\{ Cs_{1k} \cdot \sum_{r_1} Max(r_1 - \hat{U}_{1,P-1} - \delta_1 \cdot Z_{1P2} - \delta_2 Z_{1,P-1,1}, 0) P(r_1 : P-1, k) \right\}.$$

where values of both δ_1 and δ_2 are varied for 0 and 1, and

$$\delta_3 = \delta_1 \cdot \sum_{L=0}^{P-k} \overline{P}_{2,P}(L) + (1 - \delta_1)(1 - \sum_{L=0}^{P-k} \overline{P}_{2,P}(L))$$

$$\delta_4 = \delta_2 \circ \sum_{L=0}^{P-1-k} \overline{P}_{1,P-1}(L) + (1-\delta_2)(1 - \sum_{L=0}^{P-1-k} \overline{P}_{1,P-1}(L)),$$

(3) total expected carrying cost during periods $P-1-\underline{L}_{P-1}$ to $P-\underline{L}_{2P}$, which is

$$\sum_{k=P-1-\underline{L}_{1,P-1}}^{P-\underline{L}_{2P}} \left\{ Ch_{1k} \cdot \sum_{r_1} Max(U_{1,P-1} + \delta_1 Z_{1P2} + \delta_2 Z_{1,P-1,1} - r_1, 0) P(r_1 : P-1, k+1) \delta_3 \delta_4 \right\},$$ and

(4) total minimum expected total controllable cost presuming an optimal decision is made at period P-2, which is

$$\sum_{r_{1,P-1}} f^*_{P-2/1}(\hat{U}_{1,P-1} + Z_{1P2} - r_{1,P-1}, Z_{1,P-1,1}) P_{1,P-1}(r_{1,P-1}).$$

Thus, the total expected system cost, $\mathcal{T}(U_{1,P-1}, Z_{1P2} : Z_{1,P-1,1})$

$= \varphi_{1,P-1,1}(Z_{1,P-1,1})$

$$+ \sum_{k=P-1-\underline{L}_{1,P-1}}^{P-\underline{L}_2 P} \Big[\Big\{Cs_{1k} \cdot \sum_{r_1} \text{Max}(r_1 - \hat{U}_{1,P-1} - \delta_1 Z_{1P2} - \delta_2 Z_{1,P-1,1}, 0)$$

$$P(r_1 : P-1, k)$$

$$+ Ch_{1k} \cdot \sum_{r_1} \text{Max}(\hat{U}_{1,P-1} + \delta_1 Z_{1P2} + \delta_2 Z_{1,P-1,1} - r_1, 0)$$

$$P(r_1 : P-1, k+1)\Big\}\delta_3 \delta_4$$

$$+ \sum_{r_{1,P-1}} f^*_{P-2/1}(\hat{U}_{1,P-1} + Z_{1P2} - r_{1,P-1},\ Z_{1,P-1,1}) P_{1,P-1}(r_{1,P-1}).$$

(5-4-5)

And $f_{P-1,2}(\hat{U}_{1,P-1},\ Z_{1P2}) = \underset{Z_{1,P-1,1}}{\text{Min}} \Big[\mathcal{T}(\hat{U}_{1,P-1}, Z_{1P2} : Z_{1,P-1,1})\Big],$ (5-4-6)

where $0 \leq Z_{1,P-1,1} \leq \text{Min}(S_{1,P-1,1},\ \frac{W}{v_1} - \hat{U}_{1,P-1} - Z_{1P2} + \sum_{k=P-\underline{L}_2 P+1}^{P-1} \underline{r}_{1k}).$

(5-4-7)

For other combinations of sources that could be chosen in period P and P-1, for a given set of $\hat{U}_{1,P-1}, Z_{1P_j}$, $f_{P-1,j}(\hat{U}_{1,P-1}, Z_{1,P-1,j})$ can be determined.

And for a given source to be chosen in period P, it follows that $f^*_{P-1/j}(\hat{U}_{1,P-1},\ Z_{1P_j}) = \underset{j'}{\text{Min}}\Big[f_{P-1,j'}(\hat{U}_{1,P-1},\ Z_{1,P-1,j})\Big]$, where j' are the sources to be considered in period P-1. (5-4-8)

Consider period P, with a given source to be chosen in this period, if one lets $f_{P_j}(U_{1P})$ be a minimum expected controllable cost when a decision is made in period P where the order is made from source j for a given value of U_{1P}, $f_{P_j}(U_{1P}) = \underset{Z_{1P_j}}{\text{Min}} \Big[\varphi_{1P_j}(Z_{1P_j})$

$$+ \sum_{r_{1P}} f^*_{P-1/j}(U_{1P} - r_{1P},\ Z_{1P_j}) P_{1P}(r_{1P})\Big]$$

(5-4-9)

where $$0 \leq Z_{1Pj} \leq \mathrm{Min}(S_{1Pj}, \frac{W}{v_1} - U_{1P} + \sum_{k=5}^{P} \underline{r}_{1k}). \qquad (5\text{-}4\text{-}10)$$

Let EC_j be the expected lost during periods $P-\underline{L}_P$ to $P-\underline{L}_{jP}+1$ for the source that $\underline{L}_{jP} > \underline{\underline{L}}_P$, where

$$\underline{\underline{L}}_P = \mathrm{Min}_j \left[\underline{L}_{jP} \right].$$

Then, $f^*_P(U_{1P})$, the minimum expected cost when a decision is made in period P where all sources are considered for a given value of U_{1P}, becomes

$$f^*_P(U_{1P}) = \mathrm{Min}_j \left[f_{Pj}(U_{1P}) + EC_j \right]. \qquad (5\text{-}4\text{-}11)$$

Employing procedure developed above, for the case of more than two sources, at each period for $k < P$ one can determine $f^*_{k/j}(\hat{U}_{1k}, Z_{1,k-1,j})$, which is the minimum expected total controllable cost, assuming the order made from source j in period k, when an optimal source and amount is chosen in period k for a given set of $\hat{U}_{1k}$, $Z_{1,k-1,j}$.

For $k = P$, employing the procedure from (5-4-9) to (5-4-11), the final optimization of the system can be found.

Example

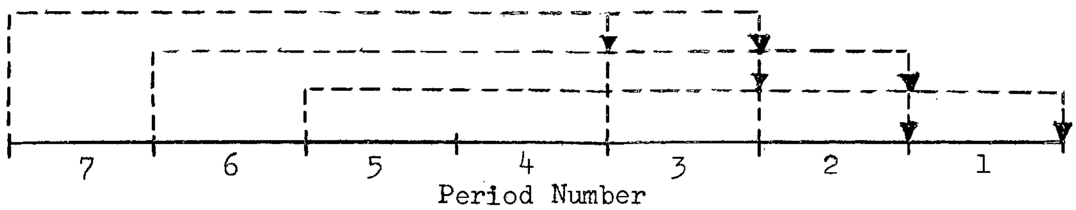
planning period,	$P = 7$
warehouse space,	$W = 3$ cubic units
number of sources,	$J = 2$
a volume of an item,	$v_1 = 1$ cubic unit
initial inventory,	$U_{17} = 2$ units

	j = 1			j = 2		
	k=5	k=6	k=7	k=5	k=6	k=7
S_{jk}-unit	3	2	2	1	2	1
Co_{jk}-dollars/order	0.50	0.50	0.50	0.60	0.60	0.60
Ci_{jk}-dollars/order	2.00	3.00	2.00	3.00	3.00	2.00

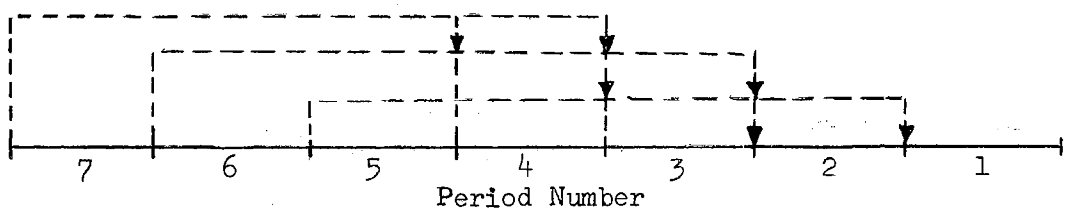
	k=1	k=2	k=3	k=4
Cs_{1k}-dollars/unit/period	5.00	6.00	6.00	5.00
Ch_{1k}-dollars/unit/period	2.00	2.00	2.00	2.00

	j = 1		j = 2	
	L=4	L=5	L=3	L=4
$\overline{P}_{j5}(L)$	.3	.7	.2	.8
$\overline{P}_{j6}(L)$	.5	.5	.4	.6
$\overline{P}_{j7}(L)$	.6	.4	.5	.5

	$P_{1k}(r_{1k})$						
r_{1k}	k=1	k=2	k=3	k=4	k=5	k=6	k=7
0	.5	.6	.2	.3	.5	.3	.4
1	.5	.4	.5	.3	.5	.7	.6
2	.0	.0	.3	.4	.0	.0	.0



SOURCE NO. 1



SOURCE NO. 2

Solution:

Using the given data, the necessary values of $P(r_1:K,k)$ can be determined as shown in Table XV.

Consider period 5. For the alternative that at period 6 source No. 1 is chosen and at period 5 source No. 2 is chosen.

Using (5-4-1) to (5-4-3), for $\hat{U}_{16} = 0$, $Z_{161} = 0$;

$$0 \leq Z_{152} \leq \mathrm{Min}(1, \tfrac{3}{1}-0-0+0) = 1, \text{ and}$$

$$f_{52}(0,0)$$

$$= \mathrm{Min}\left[\begin{array}{l} 0.0 + [(5)\{(1)(.055)+(2)(.153)+(3)(.251)+(4)(.267) \\ \qquad +(5)(.182)+(6)(.071)+(7)(.012)\}(1)(1)] \\ \quad + [(6)\{(1)(.093) + (2)(.213)+(3)(.289)+(4)(.245)+(5)(.118) \\ \qquad +(6)(.024)\}\{(.5)(.2)+(.5)(.8)+(.5)(.2)+(.5)(.8)\}], \\ 3.6 + [(5)\{(1)(.153)+(2)(.251)+(3)(.267)+(4)(.182) \\ \qquad +(5)(.071)+(6)(.012)\}+(2)\{(1)(.018)\}\{(1)(1)\}] \\ \quad + [(6)\{(1)(.213)+(2)(.289)+(3)(.245)+(4)(.118) \\ \qquad +(5)(.024)\}+(2)\{(1)(.030)\}\{(.5)(.2)+(.5)(.2)\} \\ \quad + [(6)\{(1)(.093)+(2)(.213)+(3)(.289)+(4)(.245) \\ \qquad +(5)(.118)+(6)(.024)\}\{(.5)(.8)+(.5)(.8)\} \end{array}\right]$$

TABLE XV

CUMULATIVE PROBABILITY OF DEMANDS FROM PERIOD k TO K, $P(r_1:K,k)$

r_1	$P(r_1:K,k)$							
	K=5			K=6			K=7	
	k=1	k=2	k=3	k=3	k=4	k=5	k=4	k=5
0	.009	.018	.030	.009	.045	.150	.018	.060
1	.055	.093	.135	.061	.195	.500	.105	.290
2	.153	.213	.265	.175	.315	.350	.243	.440
3	.251	.289	.305	.276	.305	.000	.311	.210
4	.267	.245	.205	.276	.140	.000	.239	.000
5	.182	.118	.060	.161	.000	.000	.084	.000
6	.071	.024	.000	.042	.000	.000	.000	.000
7	.012	.000	.000	.000	.000	.000	.000	.000

$$= \text{Min}\begin{bmatrix} 36.610 \\ 34.115 \end{bmatrix} = 34.115; \text{ where } Z^*_{152} = 1.$$

For other sets of $\hat{U}_{15}$, Z_{16j} and for other alternatives, $f_{5j}(\hat{U}_{15}, Z_{16})$ can be determined. The results, determined by the computer, are as shown in Annex II-3.

For the given alternative that source No. 1 is chosen at period 6 and for $\hat{U}_{15} = 0$, $Z_{161} = 0$ using (5-4-4);

$$f^*_{5/1}(0,0) = \underset{j'}{\text{Min}}\left[f_{5j'}(0,0)\right] = \text{Min}\begin{bmatrix} 36.600 \\ 34.115 \end{bmatrix}$$

= 34.115; where the best policy is to order 1 unit from source No. 2.

For other sets of $\hat{U}_{15}$, Z_{161} and for other alternatives that source No. 2 will be chosen at period 6, the values of $f^*_{5/j}(\hat{U}_{15}, Z_{16j})$ can be determined. The results, determined by the computer, are as shown in Annex II-3.

Consider period 6 and for the alternative that at period 7 source No. 2 is chosen and at period 6 source No. 1 is chosen. Using (5-4-5) to (5-4-7), for $U_{16} = 1$, $Z_{172} = 0$, it follows that

$$0 \le Z_{161} \le \text{Min}(2, \underset{1}{\underline{3}}-1-0+0), \text{ and}$$

$f_{61}(0,0)$

$$= \text{Min}\begin{bmatrix} 0.0 + [(6)\{(1)(.175)+(2)(.276)+(3)(.276)+(4)(.161) \\ \quad +(5)(.042)\}+(2)\{(1)(.045)\}](1) \\ \quad +(5)\{(1)(.315)+(2)(.305)+(3)(.140)\} \\ \quad +(2)\{(1)(.15)\}]\{(.5)+(.5)\} \\ \quad +(.3).f^*_{5/1}(1,0)+(.7).f^*_{5/1}(0,0), \\ 3.5 + [(6)\{(1)(.175)+(2)(.276)+(3)(.276)+(4)(.161) \\ \quad +(5)(.042)\}+(2)\{(1)(.045)\}](1) \\ \quad + [(5)\{(1)(.315)+(2)(.305)+(3)(.140)\} \\ \quad +(2)\{(1)(.15)\}]\{(.5)+(.5)\} \\ \quad + (.3).f^*_{5/1}(1,1) + (.7).f^*_{5/1}(0,1) \\ 6.5 + [(6)\{(1)(.175)+(2)(.276)+(3)(.276)+(4)(.161) \\ \quad +(5)(.042)\} + (2)\{(1)(.045)\}](1) \\ \quad + [(5)\{(1)(.315)+(2)(.305)+(3)(.140)\} \\ \quad +(2)\{(1)(.15)\}]\{(.5)(.5)\} \\ \quad + (.3).f^*_{5/1}(1,2) + (.7).f^*_{5/1}(0.2) \end{bmatrix}$$

$$= \text{Min}\begin{bmatrix} 52.831 \\ 49.532 \\ 47.146 \end{bmatrix} = 47.146; \text{ order 2 units.}$$

For other sets of $\hat{U}_{16}$, Z_{17j} and for other alternatives; $f_{6j}(\hat{U}_{16}, Z_{17j})$ can be determined. The results, determined by the computer, are as shown in Annex II-3.

For the given alternative that source No. 1 is chosen at period 7 and for $\hat{U}_{16} = 1$, $Z_{171} = 0$, using (5-4-8);

$$f_{6/1}(1,0) = \left[\underset{j'}{\text{Min}}\ f_{5j'}(1,0)\right]$$

$$= \text{Min}\begin{bmatrix} 40.121 \\ 31.975 \end{bmatrix} = 31.975.$$

The decision is to choose the source No. 2 in period 6 and order 2 units.

For other sets of $\hat{U}_{16}$, Z_{17j} and for the other alternatives that source No. 2 is chosen at period 7, $f^*_{6/j}(\hat{U}_{16}, Z_{16j})$ can be determined. The results, determined by the computer, are as shown in Annex II-3.

Consider the last stage, period 7. Employing (6-3-9) and (6-3-10);

$f_{71}(0)$

$$= \text{Min}\begin{bmatrix} 0.0 + (.4)f^*_{6/1}(2,0) + (.6)f^*_{6/1}(1,0), \\ 2.5 + (.4)f^*_{6/1}(2,1) + (.6)f^*_{6/1}(1,1) \end{bmatrix} = \text{Min}\begin{bmatrix} 29.525 \\ 27.700 \end{bmatrix}$$

= 27.700; order 2 units.

The same manner, $f_{72}(0)$ is 31.485, by ordering 2 units from source No. 2 at period 7.

The final optimization, then, can be determined by employing (5-4-11);

$$\begin{aligned} EC_1 &= (5)\{(1)(.311)+(2)(.239)+(3)(.084)\} \\ &\quad + (2)\{(1)(.290)+(2)(.060)\} \\ &= 6.025. \end{aligned}$$

$$\text{Therefore } f^*_7(0) = \begin{bmatrix} 27.700 + 6.025 \\ 31.485 \end{bmatrix} = 31.485.$$

Then, the optimal policy in period 7 is to choose source No. 2 and order 2 units. The minimum expected cost is 31.485.

5.5 MULTI-ITEM MULTI-SOURCE SYSTEM FOR THE NON-MIXABLE ITEMS

Different from Section 5.3, this section considers the case of multi-item multi-source. Employing the development in Section 2.3, the system can be reduced, first, to the single item multi-source. Thus, for item type No. i, one can use the development in section 5.4 to determine $G_i(w)$, which is $f^*_p(0)$ for the selected value of w. And then the procedure to allocate space to each type of item is the same as in Section 2.3.

CHAPTER VI

SUMMARY AND CONCLUSION

The procedure for choosing optimal decisions for finite period inventory problems have been obtained through the application of dynamic programming and the principle of optimality. Single-item single-source, multi-item single-source, single-item multi-source, and multi-item multi-source systems have been considered in the various chapters.

In Chapter II, cases concerning deterministic demands and deterministic procurement lead time were considered. The analysis in the chapter provided a basis for the chapters that followed. Multi-item single-source and multi-item multi-source models were developed for the two special cases of mixing and non-mixing inventory.

Chapter III was devoted to the case of probabilistic demands with zero lead time. In both Chapter II and III the decision could be made based on the inventory on hand at each decision stage.

In Chapter IV, the case of probabilistic demands and deterministic lead time was introduced. The demands were considered as being independent and not necessarily identical with excess demands being deferred to a later period. Orders made in any period from a particular source were assumed not to arrive before those orders made previously from the same source.

In Chapter V a probabilistic lead time case was developed for the probabilistic demands problem. It was assumed that probability of lead time for the order made in any period was independent from other periods regardless of whether the order is made at other periods. It was found that the decision for the problems in Chapter IV and V was based on the amount of inventory on hand plus outstanding orders at that decision stage.

Examples were given for illustrative purposes for the key basic sections. Examples for other sections which were not given can be illustrated by following the key basic sections, substituting the proper cost functions developed for the particular model as necessary.

In the appendixes, there are computer programmings for those algorithms in Chapter V. Since the multi-item multi-source system is the most general for the others, the programs developed may be applied to the remaining chapters.

A general conclusion from this dissertation is that dynamic programming provides a feasible means for solutions of finite period inventory problems under the warehouse restriction. To determine a partial optimization at each stage, the partial optimization at the previous stage must be employed through the recurrence relation. It should be stated that a recurrence relation is one of the most important keys for solving multi-stage decision problems such as finite period inventory problems.

Much effort must be put forth in determining a proper basis for making a decision. A proper basis means the basic parameters on which the recurrence relation for that particular problem may be based. A basis for one problem may not be applied to the others. Not only must a proper basis be chosen and a recurrence relation be developed, but the proper cost function for the problem must also be determined.

Dynamic programming, applied to multi-stage decision problems such as in this dissertation, is not a means that will reduce the calculation to nothing. But the procedure does eliminate much unnecessary computation by employing the partial optimization at each stage. Availability of high speed electronic computers will continue to make this technique applicable to large problems.

Thus, this investigation presents a unified hierarchy of finite period inventory systems together with decision algorithms for variations of each system. The techniques developed in this dissertation may involve much initial effort in solving real world problems, but it is believed that the additional effort will yield a high return for some problems, especially for those that consider high total inventory value.

The following recommendations are suggested for further studies and investigation:

a. Derive models representing the theoretical distributions for demands and procurement lead time. This may lead to a simpler calculation.

b. Determine optimal policies for systems subject to other restrictions, i.e., limited capital, or the combination of restrictions such as the restricted warehouse and limited capital.

c. Study the sensitivity of optimal policies related to parameter changes such as cost coefficients.

d. Study and sensitivity of using a finite period model rather than an infinite period model for the medium interval planning period.

e. Extend Chapter IV and V to the case in which the items can be mixed for the multi-item multi-source system.

f. Determine the qualitative characteristics of the decision policies for the models developed here, similar to the characteristics determined for the single-item single-source not restricted models discussed in several publications.

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APPENDIX A

SOLUTION OF PROBABILISTIC DEMANDS AND LEAD TIME AND MULTI-ITEM SINGLE SOURCE PROBLEM BY IBM 7040

The computer program presented in this appendix will process the probabilistic demand and lead time model for the Multi-Item Single-Source problem discussed in Section 5.2. The maximum dimension for this program is provided for the example at the end of Section 5.2. The program may be applied to the larger problems by changing the limiting dimension statements and rewriting some of the format statements along with appropriate modifications of input data. The expected costs are computed and the optimal policies are determined by utilizing the analysis in Section 5.2. Written in FORTRAN IV, the program is as in Annex I-1.

The program can be applied to the deterministic problems as well by replacing the appropriate probabilities with zero or one. For those Single-Item Single-Source problems, by changing the number of items (N) to 1, this program can also be applied.

Input Data

Input is via standard punch cards. For the illustrated problem there are 21 input cards, each of which is explained below:

Card No. 1 : N. The symbol N refers to the number of items. The value is placed in column 2.

Card No. 2 : IP. The symbol IP is analogous to P as used in Section 5.2. The value is placed in column 2.

Card No. 3 : W. The symbol W refers to warehouse space as used in Section 5.2. The value is placed in columns 1 to 6.

Card No. 4 : V(I). The symbol V(I) is analogous to v_i as used in Section 5.2. The first value is v_1 the last one is v_2.

Card No. 5 : IUINI(I). The symbol IUINI(I) is analogous to U_{iP} as used in Section 5.2. The first value is U_{14} and the last one is U_{24}.

Card No. 6 : IS(K,I). The symbol IS(K,I) is analogous to S_{ik} as used in Section 5.2. Each value occupies columns space. Starting from column 2, first three values are the values for i=1 and k=3 to 4, respectively. The last three values are the values for i=2.

Card No. 7 : CO(K,I). The symbol CO(K,I) is analogous to Co_{ik} as used in Section 5.2. Each value occupies 6 column spaces. Starting from column 1, first two values are the values for i=1 and k=3 to 4, respectively. The last two values are the values for i=2.

Card No. 8 : CI(K,I). The symbol CI(K,I) is analogous to Ci_{ik} as used in Section 5.2. Each value occupies 6 column spaces. Starting from column 1, first two values are the values for i=1 and k=3 to 4, respectively. The last two values are the values for i=2.

Card No. 9 : CS(K,I). The symbol CS(K,I) is analogous to Cs_{ik} as used in Section 5.2. Each value occupies 6 column spaces. Starting from column 1, first three values are the values for i=1 and k=2 to 3, respectively. The last three values are the values for i=2.

Card No. 10 : CH(K,I). The symbol CH(K,I) is analogous to Ch_{ik} as used in Section 5.2. Each value occupies 6 column spaces. Starting from column 1, first three values are the values for i=1 and k=1 to 3, respectively. The last three values are the values for i=2.

Card No. 11-18 : PP(IR,K,I). The symbol PP(IR,K,I) is analogous to $P_{ik}(r_{ik})$ as used in Section 5.2. Card No. 11 to 14 represent the values for i=1 and k=1 to 4, respectively. Card No. 15 to 18 represent i=2. Each card has 5 values for 4_{ik} = 0 to 4 and each value occupies 6 column spaces.

Card No. 19 : PL(L,4). The symbol PL(L,4) is analogous to $\bar{P}_4(L)$ as used in Section 5.2. Each value occupies 6 column spaces. This card has four values for L=0 to 3, respectively.

Card No. 20-21 : PDL(L,K). The symbol PDL(L,K) is analogous to $\bar{\bar{P}}_k(L)$ as used in Section 5.2. Each value occupies 6 column spaces. First card is for k=2 and the second card for k=3. Each card has four values for L=0 to 3.

The input data are displayed in Annex I-2 as they appeared on the data card.

Output

Output is via the standard print feature of the computer. The output massage symbols, heading the columns in Annex I-3, are explained below:

U(I,K) : The symbol U(I,K) is analogous to U_{ik} as used in Section 5.2.

W(1) : The symbol W(1) is analogous to w_1 as used in Section 5.2.

MIN.COST : The symbol MIN.COST is analogous to $f_{1k}(U_{1k},U_{2k}/w_1)$ as used in Section 5.2, for the table under the policy when only item 1 is considered. The symbol is analogous to $f_k^*(U_{1k},U_{2k})$ as used in Section 5.2, for the table under the policy when item 1 and 2 are considered.

ORDER : The symbol ORDER is analogous to $Z_{ik}^*(U_{1k},U_{2k})$ as used in Section 5.2.

PRE.SPACE : The symbol PRE.SPACE, as appeared on the head of last column in the table under the policy when items 1 and 2 are considered, refers to the space which is left for item 1 after item 2 has been ordered. The optimal order for item 1 can be found by using the table for the policy when only item 1 is considered for the given:

$$U_{1k} = U_{1k}$$

$$U_{2k} = U_{2k} + \text{ORDER}$$

and, $$w_1 = \text{PRE.SPACE} .$$

The values at the left hand of the above equations are the values to be used for reading the values in the table under the policy when only item 1 is considered. The values on the right hand of above equations can be read from the table under the policy when items 1 and 2 are considered.

ANNEX I-1

IBM 7040 PROGRAM

```
$ID          B-0001 T.RAENGKHUM                           2523-40031
$JOB                T.RAENGKHUM                           2523-40031
$IBJOB NAMEPR  MAP
$IBFTC
      DIMENSION CO(5,2),CI(5,2),CS(5,2),CH(5,2),IUMAX(5,2),
     1IUMAB(5,2),IUMIB(5,2),IRMIN(5,2),IS(5,2),LMIN(5),
     2PL(5,5),PDL(5,5),PP(5,5,2),P(15,5,2),LMAX(5),V(2),
     3IUINI(2),ISS(2),IRMAX(5,2),IW(2),WAM(5,5),F(5,5,5),
     4FFOP(5,5),IUMIN(5,2),LLMAX(5),COST(5),IUX(2),FOP(5,5)
    1 FORMAT(I2)
    2 FORMAT(F6.3)
    3 FORMAT(2F6.3)
    4 FORMAT(4I2)
    5 FORMAT(2I2)
    6 FORMAT(4F6.3)
    7 FORMAT(6F6.3)
    8 FORMAT(5F6.3)
    9 FORMAT(X,6HRESULT,X,I2,X,F6.3,X,I2)
   11 FORMAT(1H1,10X,18HRESULTS FOR PERIOD,I2//)
   12 FORMAT(10X,37HPOLICY WHEN ONLY ITEM 1 IS CONSIDERED///
     14X,8HMIN.COST,3X,5HORDER)
   13 FORMAT(10X,4HU(1,,I1,2H) ,2X,4HU(2,,I1,2H) ,3X,4HW(1),
   14 FORMAT(12X,I2,7X,I2,5X,F6.3,4X,F6.3,5X,I2)
   15 FORMAT(21X,I2,5X,F6.3,4X,F6.3,5X,I2)
   16 FORMAT(28X,F6.3,4X,F6.3,5X,I2)
   17 FORMAT(1H1,10X,32HWHEN ITEM 1 AND 2 ARE CONSIDERED//)
   18 FORMAT(10X,4HU(1,,I1,2H) ,2X,4HU(2,,I1,2H) ,X,
     18HMIN.COST,4X,5HORDER,2X,9HPRE.SPACE)
   19 FORMAT(12X,I2,7X,I2,5X,F6.3,6X,I2,5X,F6.3)
   20 FORMAT(21X,I2,5X,F6.3,6X,I2,5X,F6.3)
      READ(5,1) N
      READ(5,1) IP
      READ(5,2) W
      READ(5,3) (V(I),I=1,2)
      READ(5,5)(IUINI(I),I=1,2)
      READ(5,4) ((IS(K,I),K=3,4),I=1,2)
      READ(5,6) ((CO(K,I),K=3,4),I=1,2)
      READ(5,6) ((CI(K,I),K=3,4),I=1,2)
      READ(5,7) ((CS(K,I),K=1,3),I=1,2)
      READ(5,7) ((CH(K,I),K=1,3),I=1,2)
      READ(5,8) (((PP(IR,K,I),IR=1,5),K=1,4),I=1,2)
      READ(5,6) (PL(L,4),L=1,4)
      READ(5,6) ((PDL(L,K),L=1,4),K=2,3)
      DO 26 ITEM=1,N
      DO 25 KX=1,IP
   22 SUM=0.
      IR=1
```

```
 23 IN=IR-1
    SUM=SUM+PP(IR,KX,ITEM)
    IF(SUM.GT.0.)GO TO 24
    IR=IR+1
    GO TO 23
 24 IR= IR+1
    SUM=SUM+PP(IR,KX,ITEM)
    IF(SUM.LT.0.999) GO TO 24
 27 IRMIN(KX,ITEM)=IN
    IRMAX(KX,ITEM)=IR-1
 25 CONTINUE
 26 CONTINUE
905 IXX=IP
907 IXX=IXX-1
    I2=IXX+1
    SUM2=0.
    DO 911 IL=1,IXX
908 IX=1
    SUM1=0.
909 I1=IL+2-IX
    PL(IL,IXX)=PL(IL,IXX)+PL(I1,I2)*PDL(IX,IXX)
    SUM1=SUM1+PDL(IX,IXX)
    IF(SUM1.EQ.1.) GO TO 910
    IX=IX+1
    GO TO 909
910 SUM2=SUM2+PL(IL,IXX)
911 CONTINUE
    IF(SUM2.GT.0.) GO TO 907
    KMIN=IXX+1
    KI=KMIN
912 SUMPL=0.
    LX=1
913 LN=LX-1
    SUMPL=SUMPL+PL(LX,KI)
    IF(SUMPL.GT.0.) GO TO 914
    LX=LX+1
    GO TO 913
914 LX=LX+1
    SUMPL=SUMPL+PL(LX,KI)
    IF(SUMPL.GE.0.999) GO TO 916
    IF(LX.LT.KI) GO TO 914
916 LMAX(KI)=LX-1
    LMIN(KI)=LN
    IF(KI.EQ.IP) GO TO 919
    SUMDL=0.
    LX=1
917 SUMDL=SUMDL+PDL(LX,KI)
    IF(SUMDL.GE.0.999) GO TO 918
    LX=LX+1
    GO TO 917
918 LLMAX(KI)=LX-1
```

```
      KI=KI+1
      GO TO 912
919   K=IP
      IUMAX(K,1)=IUINI(1)
      IUMIN(K,1)=IUINI(1)
      IF(N.EQ.1) GO TO 922
      IUMAX(K,2)=IUINI(2)
      IUMIN(K,2)=IUINI(2)
      IUMIB(K,2)=IUMIN(K,2)
922   WW=0.
      KX=K-LMIN(K)+1
      KXX=MINO(KX,K)
      DO 921 IX=1,N
      ISUM=0
      DO 920 IK=KXX,K
      ISUM=ISUM+IRMIN(IK,IX)
920   ISS(IX)=IUMIN(K,IX)-ISUM
      ID=ISS(IX)
      WW=V(IX)*AMAXO(ID,0)+WW
921   CONTINUE
      WX=W-WW
      IF(N.EQ.1) GO TO 725
      UMIN=AMINO(ISS(2),0)
      IF(WX.NE.0.) GO TO 923
      WA=0.-UMIN
      GO TO 924
923   WA=WX/V(2)-UMIN
924   IWA=INT(WA)
      IUMAB(K,2)=IUMAX(K,2)+MINO(IS(K,2),IWA)
725   IF(K.EQ.KMIN) GO TO 927
      K=K-1
      IUMIN(K,1)=IUMIN(K+1,1)-IRMAX(K+1,1)
      IF(N.EQ.1) GO TO 726
      IUMIN(K,2)=IUMIB(K+1,2)-IRMAX(K+1,2)
      IUMAX(K,2)=IUMAB(K+1,2)-IRMIN(K+1,2)
      IUMIB(K,2)=IUMIN(K,2)
726   UMIN=AMINO(ISS(1),0)
      IF(WX.NE.0.) GO TO 925
      WA=0.-UMIN
      GO TO 926
925   WA=WX/V(1)-UMIN
926   IWA=INT(WA)
      IB=IUMAX(K+1,1)+MINO(IS(K+1,1),IWA)
      IUMAX(K,1)=MINO(INT(W),IB)
      GO TO 922
927   K=KMIN
800   IRSX=MAXO(IRSX,1)
      DO 809 INDEX=1,N
      DO 808 KI=1,K
      DO 807 IR=1,IRSX
      P(IR,KI,INDEX)=0.
```

```
807 CONTINUE
808 CONTINUE
809 CONTINUE
    IRSX=0
810 DO 805 INDEX=1,N
    IRX=IRMAX(K,INDEX)+1
    DO 801 IR=1,IRX
    P(IR,K,INDEX)=PP(IR,K,INDEX)
801 CONTINUE
    IRM  =IRMAX(K,INDEX)+1
    KY=K-1
    DO  804 KI=1,KY
    KII=K-KI
    IRX=IRMAX(KII,INDEX)+1
    DO 803 IR1=1,IRX
    DO 802 IR2=1,IRM
    IRSUM=IR1+IR2-1
    CONST=P(IRSUM,KII,INDEX)
    PROD=PP(IR1,KII,INDEX)*P(IR2,KII+1,INDEX)
    P(IRSUM,KII,INDEX)=CONST+PROD
802 CONTINUE
803 CONTINUE
    IRX=IRMAX(KII,INDEX)+1
    IRM  =IRM  +IRX-1
804 CONTINUE
    IF(IRM.LT.IRSX)  GO TO 806
    IRSX=IRM
806 IRSMX=IRSX-1
805 CONTINUE
    IF(N.EQ.1) GO TO 102
    WRITE(6,11) K
    WRITE(6,12)
    WRITE(6,13) K,K
100 ITEM=2
    IU=IUMIB(K,ITEM)
    IZ=0
108 TESH=0.
    L=LMIN(K)
    IF(K.NE.KMIN) GO TO 205
109 KX=1
    SSHC=0.
110 IR=0
    ES=0.
    EH=0.
    HC=0.
    IF((IU+IZ).LE.0) GO TO 112
111 EH=EH+(FLOAT(IU)+FLOAT(IZ)-FLOAT(IR))*P(IR+1,KX+1,ITEM)
    IR=IR+1
    IF(IR.GT.IRSMX) GO TO 113
    IF(IR.NE.(IU+IZ)) GO TO   111
    HC=CH(KX,ITEM)*EH
```

```
116 IR=IR+1
112 ES=ES+(FLOAT(IR)-FLOAT(IU)-FLOAT(IZ))*P(IR+1,KX,ITEM)
    IR=IR+1
    IF(IR.LE.IRSMX) GO TO 112
113 SC=CS(KX,ITEM)*ES
    SSHC=SSHC+HC+SC
    IF(KX.EQ.(K-L)) GO TO 114
    KX=KX+1
    GO TO 110
114 IF(K.NE.KMIN) GO TO 207
    ESSHC=SSHC
118 TESH=TESH+ESSHC*PL(L+1,K)
    IF(L.EQ.(K-1)) GO TO 115
    IF(L.EQ.LMAX(K)) GO TO 115
    L=L+1
    IF(K.NE.KMIN) GO TO 205
    GO TO 109
205 LL=0
    ESSHC=0.
206 KX=K-L-LL
    SSHC=0.
    IF(KX.GE.1) GO TO 110
    KX=1
    GO TO 110
207 ESSHC=ESSHC+SSHC*PDL(LL+2,K-1)
    LL=LL+1
    IF(LL.GE.LLMAX(K-1)) GO TO 118
    GO TO 206
  5 IF(ITEM.EQ. ) GO TO    9
101 IUY=IU+3
    COST(IUY)=TESH
    IU=IU+1
    IF(IU.LE.IUMAB(K,ITEM)) GO TO 108
102 IUX(1)=IUMIN(K,1)
103 IUX(2)=0
    IF(N.EQ.1) GO TO 104
    IUX(2)=IUMIB(K,2)
104 IU=IUX(1)
    FMIN=0.
    IZ=0
    SW=0.
    DO 107 ITEM=1,N
    IF(LMIN(K).EQ.0) GO TO 106
    K1=K-LMIN(K)+1
    ISR=0
    DO 105 KK=K1,K
    ISR=ISR+IRMIN(KK,ITEM)
105 CONTINUE
106 IW(ITEM)=IUX(ITEM)-ISR
    SW=SW+V(ITEM)*AMAXO(IW(ITEM),0)
107 CONTINUE
```

```
      WA=W-SW
      IF(N.EQ.1) GO TO 402
      IY=IUX(2)+3
      IYY=IUX(1)
      WAM(IYY,IY)=WA
  402 IF(WA.LT.0.) GO TO 124
      ITEM=1
      IF(N.EQ.1) GO TO 400
      WX=0.
      GO TO 108
  400 WX=WA
      GO TO 108
  129 IF(IZ.NE.0) GO TO 127
      OC=0.
      GO TO  117
  127 OC=CO(K,ITEM)
  117 TOC=OC+FLOAT(IZ)*CI(K,ITEM)
      IUY=IUX(2)+3
      EF=0.
      IF(K.EQ.KMIN) GO TO 602
      I1=IUX(1)+IZ
      IF(N.EQ.1) GO TO 404
      IX2=IUX(2)+3
      IJ1=IRMIN(K,2)+1
      IJ2=IRMAX(K,2)+1
      DO 601 IR2=IJ1,IJ2
      SUMF=0.
  404 IJ3=IRMIN(K,1)+1
      IJ4=IRMAX(K,1)+1
      DO 600 IR1=IJ3,IJ4
      IY1=I1-IR1+1
      IF(N.EQ.1) GO TO 405
      IY2=IX2-IR2+1
      SUMF=SUMF+FOP(IY1,IY2)*PP(IR1,K,1)
      GO TO 600
  405 EF=EF+FFOP(IY1,K-1)*PP(IR1,K,1)
  600 CONTINUE
      IF(N.EQ.1) GO TO 602
      EF=EF+SUMF*PP(IR2,K,2)
  601 CONTINUE
  602 ECOST=0.
      IF(N.EQ.1) GOTO  401
      ECOST=COST(IUY)
  401 TC=TOC+ECOST+EF+TESH
  604 IF(IZ.EQ.0) GO TO 718
      IF(TC.GE.FMIN) GO TO 120
      IZOP=IZ
      GO TO 119
  718 IZOP=0
  119 FMIN=TC
  120 IF((IZ+1).GT.IS(K,ITEM)) GO TO 125
```

```
      IF(WX.NE.0.) GO TO 121
      WV=0.
      GO TO122
  121 WV=WX/V(ITEM)
  122 WXY=WV-AMINO(IW(ITEM),0)
      FIZ=FLOAT(IZ)+1.
      IF(FIZ.GT.WXY) GO TO 125
      IZ=IZ+1
      GO TO 108
  403 WRITE(6,9) IUX(1),FMIN,IZOP
      IYY=IUX(1)
      FFOP(IYY,K)=FMIN
      GO TO 124
  125 IF(N.EQ.1) GO TO 403
      IY=IUX(2)+3
      IWXX=INT(WX)+1
      IYY=IUX(1)
      F(IYY,IY,IWXX)=FMIN
      IF(IUX(2).NE.IUMIB(K,2)) GO TO 450
      IF(WX.NE.0.) GO TO 451
      WRITE(6,14)IUX(1),IUX(2),WX,F(IYY,IY,IWXX),IZOP
      GO TO 128
  450 IF(WX.NE.0.) GO TO 451
      WRITE(6,15)IUX(2),WX,F(IYY,IY,IWXX),IZOP
      GO TO 128
  451 WRITE(6,16)WX,F(IYY,IY,IWXX),IZOP
  128 WX=WX+1.
      IF(WX.LE.WA) GO TO 120
  126 IF(IUX(2).EQ.IUMAB(K,2)) GO TO 124
      IUX(2)=IUX(2)+1
      GO TO 104
  124 IF(IUX(1).EQ.IUMAX(K,1))GO TO 130
      IUX(1)=IUX(1)+1
      GO TO 103
  130 IF(N.EQ.1) GO TO 150
      WRITE(6,17)
      WRITE(6,18) K,K
      ITEM=2
      ISR=0
      IF(LMIN(K).EQ.0) GO TO 151
      K1=K-LMIN(K)+1
      DO 132 KK=K1,K
      ISR=ISR+IRMIN(KK,ITEM)
  132 CONTINUE
  151 IUX(1)=IUMIN(K,1)
  131 IUX(2)=IUMIN(K,2)
  133 IW2=IUX(2)-ISR
      IY=IUX(2)+3
      IYY=IUX(1)
      WA=WAM(IYY,IY)
      IF(WA.LT.0.) GO TO 141
```

```
      IF (WA.NE.0.) GO TO134
      WV=0.
      GO TO 135
  134 WV=WA/V(ITEM)
  135 WXY=WV-AMINO(IW2,0)
      IZMAX=MINO(IS(K,ITEM),INT(WXY))
      FMIN=0.
      IZ=0
      OC=0.
      GO TO 137
  136 OC=CO(K,ITEM)
  137 TOC=OC+FLOAT(IZ)*CI(K,ITEM)
      IU=IZ+IUX(2)+3
      IZS=IZ+MINO(IW2,0)
      WX=WA-V(ITEM)*AMAXO(IZS,0)
      WXX=WX+1.
      IWXX=INT(WXX)
      TC=F(IYY,IU,IWXX)+TOC
      IF(IZ.EQ.0) GO TO 138
      IF(TC.GT.FMIN) GO TO 139
  138 FOP(IYY,IY) =TC
      FMIN=TC
      IZOP=IZ
      PREW=WX
  139 IF(IZ.GE.IZMAX) GO TO 140
      IZ=IZ+1
      GO TO 136
  140 IF(IUX(2).NE.IUMIN(K,2))   GO TO 142
      WRITE(6,19) IUX(1),IUX(2),FOP(IYY,IY),IZOP,PREW
      GO TO 143
  142 WRITE(6,20) IUX(2),FOP(IYY,IY),IZOP,PREW
  143 IF(IUX(2).EQ.IUMAX(K,2)) GO TO 141
      IUX(2)=IUX(2)+1
      GO TO 133
  141 IF(IUX(1).EQ.IUMAX(K,1)) GO TO 150
      IUX(1)=IUX(1)+1
      GO TO 131
  150 IF(K.EQ.IP) GO TO 999
      K=K+1
      GO TO 800
  999 CALL EXIT
      END
```

ANNEX I-2

INPUT DATA

```
2
4
5.000
1.000 1.000
3 0
3 5 2 1
0.500 0.500 0.500 0.500
0.500 0.600 0.700 0.700
6.000 6.000 6.00010.000 9.000 9.000
1.000 0.900 1.000 1.000 1.000 1.000
0.200 0.250 0.300 0.250 0.000
0.100 0.200 0.350 0.200 0.150
0.550 0.450 0.000 0.000 0.000
0.300 0.400 0.300 0.000 0.000
0.000 0.500 0.500 0.000 0.000
0.400 0.600 0.000 0.000 0.000
0.300 0.700 0.000 0.000 0.000
0.000 0.600 0.400 0.000 0.000
0.000 0.300 0.500 0.200
0.000 0.000 0.600 0.400
0.000 0.400 0.600 0.000
```

ANNEX I-3

OUTPUT

RESULTS FOR PERIOD 3

POLICY WHEN ONLY ITEM 1 IS CONSIDERED

U(1,3)	U(2,3)	W(1)	MIN.COST	ORDER
1	-2	0.000	38.289	0
		1.000	36.101	1
		2.000	34.092	2
		3.000	32.963	3
		4.000	32.963	3
	-1	0.000	32.209	0
		1.000	30.021	1
		2.000	28.012	2
		3.000	26.883	3
		4.000	26.883	3
	-0	0.000	26.129	0
		1.000	23.941	1
		2.000	21.932	2
		3.000	20.803	3
		4.000	20.803	3
	1	0.000	20.275	0
		1.000	18.087	1
		2.000	16.078	2
		3.000	14.949	3
	2	0.000	15.531	0
		1.000	13.343	1
		2.000	11.334	2
2	-2	0.000	35.101	0
		1.000	33.592	1
		2.000	32.463	2
		3.000	32.219	3
	-1	0.000	29.021	0
		1.000	27.512	1
		2.000	26.383	2
		3.000	26.139	3
	-0	0.000	22.941	0
		1.000	21.432	1
		2.000	20.303	2
		3.000	20.059	3
	1	0.000	17.087	0
		1.000	15.578	1
		2.000	14.449	2
	2	0.000	12.343	0
		1.000	10.834	1
3	-2	0.000	32.592	0
		1.000	31.963	1
		2.000	31.719	2
	-1	0.000	26.512	0

		1.000	25.883	1
		2.000	25.639	2
	-0	0.000	20.432	0
		1.000	19.803	1
		2.000	19.559	2
	1	0.000	14.578	0
		1.000	13.949	1
	2	0.000	9.834	0
4	-2	0.000	30.963	0
		1.000	30.963	0
	-1	0.000	24.883	0
		1.000	24.883	0
	-0	0.000	18.803	0
		1.000	18.803	0
	1	0.000	12.949	0
5	-2	0.000	30.219	0
	-1	0.000	24.139	0
	-0	0.000	18.059	0

POLICY WHEN ITEM 1 AND 2 ARE CONSIDERED

U(1,3)	U(2,3)	MIN.COST	ORDER	PRE.SPACE
1	-2	22.703	2	4.000
	-1	16.849	2	3.000
	-0	13.234	2	2.000
2	-2	21.959	2	3.000
	-1	16.349	2	2.000
	-0	12.734	2	1.000
3	-2	21.459	2	2.000
	-1	15.849	2	1.000
	-0	11.734	2	-0.000
4	-2	20.703	2	1.000
	-1	14.849	2	-0.000
	-0	14.149	1	-0.000
5	-2	19.959	2	-0.000
	-1	19.259	1	-0.000
	-0	18.059	0	-0.000

RESULTS FOR PERIOD 4

POLICY WHEN ONLY ITEM 1 IS CONSIDERED

U(1,4)	U(2,4)	W(1)	MIN.COST	ORDER
3	0	0.000	73.961	0
		1.000	71.046	1
		2.000	70.114	2
	1	0.000	55.609	0
		1.000	53.189	1
		2.000	53.189	1

POLICY WHEN ITEM 1 AND 2 ARE CONSIDERED

U(1,4)	U(2,4)	MIN.COST	ORDER	PRE.SPACE
3	0	54.389	1	2.000

APPENDIX B

SOLUTION OF PROBABILISTIC DEMANDS AND LEAD TIME AND SINGLE ITEM MULTI-SOURCE PROBLEM BY IBM 7040

The computer program presented in this appendix will process the probabilistic demands and lead time model for the Single-Item Multi-Source problem which was discussed in Section 5.4. The maximum dimension for this program is provided for the example at the end of Section 5.4. The program may be applied to the larger problems by changing the limiting dimension statements and rewriting some of the format statements along with appropriate modifications of input data. The expected costs are computed and the optimal policies are determined by utilizing the analysis in Section 5.4. Written in FORTRAN IV, the program is as in Annex II-1.

Input Data

Input is via standard punch cards. For the illustrated problem there are 24 input cards, each of which is explained below:

Card No. 1 : IP. The symbol IP is analogous to P as used in Section 5.4. The value is placed in column 2.

Card No. 2 : W. The symbol W refers to warehouse space as used in Section 5.4. The value is placed in columns 1 to 6.

Card No. 3 : V. The symbol V is analogous to v_1 as used in Section 5.4. The value is placed in columns 1 to 6.

Card No. 4 : IUINI. The symbol IUINI is analogous to U_{17} as used in Section 5.4. The value is placed in column 2.

Card No. 5 : IS(K,J). The symbol IS(K,J) is analogous to S_{jk} as used in Section 5.4. Each value occupies two columns. Starting from column 2, first three values are the values for j=1 and k=5 to 7, respectively. The last three values are the values for j=2.

Card No. 6 : CO(K,J). The symbol CO(K,J) is analogous to Co_{jk} as used in Section 5.4. Each value occupies three columns. Starting from column 1, first three values are the values for j=1 and k=5 to 7, respectively. The last three values are the values for j=2.

Card No. 7 : CI(K,J). The symbol CI(K,J) is analogous to Ci_{jk} as used in Section 5.4. Each value occupies six columns. Starting from column 1, first three values are the values for j=1 and k=5 to 7, respectively. The last three values are the values for j=2.

Card No. 8 : CS(K). The symbol CS(K) is analogous to Cs_{1k} as used in Section 5.4. Each value occupies six columns. The values are for k=1 to 4, respectively.

Card No. 9 : CH(K). The symbol CH(K) is analogous to Ch_{1k} as used in Section 5.4. Each value occupies six columns. The values are for k=1 to 4, respectively.

Card No. 10-16 : PP(IR,K). The symbol PP(IR,K) is analogous to $P_{1k}(r_{1k})$ as used in Section 5.4. The cards will represent the values for k=1 to 7, respectively. Each card has three values for r_{1k}=0 to 2.

Card No. 17-22 : PL(L,K,J). The symbol PL(L,K,J) is analogous to $\bar{P}_{jk}(L)$ as used in Section 5.4. The first three cards represent the values for j=1 and k=5 to 7, respectively. The last three cards represent the values for j=2. Each card has three values for L=3 to 5.

The input data are displayed in Annex II-2 as they appeared on data cards.

Output

Output is via the standard print feature of the computer. The output massage symbols, heading the columns in Annex II-3, are explained below:

U(1,K) : The symbol U(1,K) is analogous to $\hat{U}_{1k}$ as used in Section 5.4.

Z(1,K,J) : The symbol Z(1,K,J) is analogous to Z_{1kj} used in Section 5.4.

MIN.COST : The symbol MIN.COST is analogous to $f_{kj}(\hat{U}_{1k}, Z_{1kj})$ as used in Section 5.4 for the policy which is based on the combination of sources to be chosen at period before and at the considering period. The symbol is analogous to $f^*_{k/j}(\hat{U}_{1k}, Z_{1kj})$ for the policy which is based on the source to be chosen at the period before the considering period.

ORDER : The symbol ORDER is analogous to $Z^*_{1kj}(\hat{U}_{1k}, Z_{1kj})$ as used in Section 5.4.

SOURCE : The symbol SOURCE to be chosen for that particular optimal policy.

ANNEX II-1

IBM 7040 PROGRAM

```
$ID          B-0001 T.RAENGKHUM                          2523-40031
$JOB                T.RAENGKHUM                          2523-40031
$IBJOB NAMEPR  MAP
$IBFTC
      DIMENSION CPL(7,7,2),PL(7,7,2),IRMAX(7),PP(10,7),
     1LMIN(7,2),LMAX(7,2),IUMIN(7,2),IZMIN(7,2),IRMIN(7),
     2CS(7),CO(7,2),CI(7,2),F(5,5,2),IZOP(5,5,2),FF(5,5,2),
     3IUMAX(7,2),P(10,7),IS(7,2),CH(7),IZMAX(7,2)
    1 FORMAT(I2)
    2 FORMAT(F6.3)
    3 FORMAT(6I2)
    4 FORMAT(6F6.3)
    5 FORMAT(4F6.3)
    6 FORMAT(3F6.3)
   10 FORMAT(1H1,15X,18HRESULTS FOR PERIOD,I2)
   11 FORMAT(2(15X,  17HORDER FROM SOURCE,I2,X,9HAT PERIOD,
     1I2,X/))
   12 FORMAT(15X,4HU(1,,I1,1H),3X,4HZ(1,,I1,1H,,I1,1H),3X,
     18HMIN.COST,3X,5HORDER)
   13 FORMAT(16X,I2,9X,I2,7X,F6.3,5X,I2)
   14 FORMAT(15X,22HPOLICY WHEN SOURCE NO.,I2,X,
     119HIS CHOSEN AT PERIOD,I2)
   15 FORMAT(15X,4HU(1,,I1,1H),3X,4HZ(1,,I1,1H,,I1,1H),3X,
     18HMIN.COST,3X,5HORDER,3X,6HSOURCE)
   17 FORMAT(15X,28HTHE FINAL POLICY IS TO ORDER,I2,X,
     117HITEMS FROM SOURCE,I2/15X,
     233HTHE MINIMUM EXPECTED COST WILL BE,X,F6.3)
      READ(5,1) IP
      READ(5,2) W
      READ(5,2) V
      READ(5,1) IUINI
      READ(5,3) ((IS(K,J),K=5,7),J=1,2)
      READ(5,4) ((CO(K,J),K=5,7),J=1,2)
      READ(5,4) ((CI(K,J),K=5,7),J=1,2)
      READ(5,5) (CS(K),K=1,4)
      READ(5,5) (CH(K),K=1,4)
      READ(5,6) ((PP(IR,K),IR=1,3),K=1,7)
      READ(5,6) (((PL(L,K,J),L=3,5),K=5,7),J=1,2)
      DO 425 KX=1,IP
      SUM=0.
      IR=1
  423 IN=IR-1
      SUM=SUM+PP(IR,KX)
      IF(SUM.GT.0.) GO TO 424
      IR=IR+1
      GO TO 423
  424 IR=IR+1
```

```
      SUM=SUM+PP(IR,KX)
      IF(SUM.LT.0.999) GO TO 424
      IRMIN(KX)=IN
      IRMAX(KX)=IR-1
425   CONTINUE
      DO 454 IX=1,2
      KI=5
450   SUMPL=0.
      LX=1
451   LN=LX
      SUMPL=SUMPL+PL(LX,KI,IX)
      IF(SUMPL.GT.0.) GO TO 452
      LX=LX+1
      GO TO 451
452   LX=LX+1
      SUMPL=SUMPL+PL(LX,KI,IX)
      IF(SUMPL.GE.0.999) GO TO 453
      IF(LX.LT.KI) GO TO 452
453   LMAX(KI,IX)=LX
      LMIN(KI,IX)=LN
      IF(KI.EQ.IP) GO TO 454
      KI=KI+1
      GO TO 450
454   CONTINUE
      DO 475 IX=1,2
      IUMIN(6,IX)=IUINI-IRMAX(7)
      IUMAX(6,IX)=IUINI-IRMIN(7)
      IZMIN(6,IX)=0
      WX=W/V
      IWA=INT(WX)-IUINI
      IZMAX(6,IX)=MIN0(IS(7,IX),IWA)
475   CONTINUE
      DO 476 IX=1,2
      IUMIN(5,IX)=IUMIN(6,IX)+IZMIN(6,IX)-IRMAX(6)
      IUMAX(5,IX)=IUMAX(6,IX)+IZMAX(6,IX)-IRMIN(6)
      IZMIN(5,IX)=0
      WX=W/V
      IWA=INT(WX)-IUMIN(6,IX)
      IZMAX(5,IX)=MIN0(IS(6,IX),IWA)
476   CONTINUE
      DO  902 K=5,7
      DO 901 IX=1,2
      SUM=0.
      DO 900 L=1,7
      CPL(L,K,IX)=SUM+PL(L,K,IX)
      SUM=CPL(L,K,IX)
900   CONTINUE
901   CONTINUE
902   CONTINUE
      K=5
800   IRSX=MAX0(IRSX,1)
```

```
      DO 808 KI=1,K
      DO 807 IR=1,IRSX
      P(IR,KI)=0.
  807 CONTINUE
  808 CONTINUE
      IRSX=0
      IRX=IRMAX(K)+1
      DO 801 IR=1,IRX
      P(IR,K)=PP(IR,K)
  801 CONTINUE
      IRM=IRMAX(K)+1
      KY=K-1
      DO  804 KI=1,KY
      KII=K-KI
      IRX=IRMAX(KII)+1
      DO 803 IR1=1,IRX
      DO 802 IR2=1,IRM
      IRSUM=IR1+IR2-1
      CONST=P(IRSUM,KII)
      P(IRSUM,KII)=CONST+PP(IR1,KII)*P(IR2,KII+1)
  802 CONTINUE
  803 CONTINUE
      IRX=IRMAX(KII)+1
      IRM  =IRM  +IRX-1
  804 CONTINUE
      IF(IRM.LT.IRSX)  GO TO 806
      IRSX=IRM
  806 IRSMX=IRSX-1
      WRITE(6,10) K
      IF(K.EQ.IP) GO TO 350
      KXY=1
  510 IX1=1
  515 IF(K.EQ.5) GO TO 50
      IF(IX1.NE.1) GO TO 51
      KXY=KXX+1
      IF(K.NE.(IP-1)) GO TO 50
   51 KXX=IP-LMIN(IP,IX1)
      GO TO511
   50 KXX=K+2-MAX0(LMAX(K+2,1),LMAX(K+2,2))
  511 IX2=1
  512 KT=K+1
      WRITE(6,11) IX1,KT,IX2,K
      WRITE(6,12) K,KT,IX1
      IUX=IUMIN(K,IX1)
  513 IZ1=IZMIN(K,IX1)
  514 IZ2=0
      IF((IUX+IZ1).GT.INT(W)) GO TO 230
  517 IJ=K-LMIN(K,IX2)+1
      ISUMJ=0
      DO516 IJX=IJ,K
      ISUMJ=ISUMJ+IRMIN(IJX)
```

```
516 CONTINUE
    WA=W/V-FLOAT(IUX)-FLOAT(IZ1)+FLOAT(ISUMJ)
    IZM=MINO(IS(K,IX2),INT(WA))
    IF(K.EQ.IP) GO TO 214
100 PETC=0.
    KX=KXY
651 I=1
652 IND1=2-I
    IJ=K-KX+1
    CPL1=CPL(IJ,K+1,IX1)
    II=1
650 IND2=2-II
    IK=K-KX
    CPL2=CPL(IK,K,IX2)
    IU=IUX+IND1*IZ1+IND2*IZ2
110 IR=0
    ES=0.
    EH=0.
    HC=0.
    IF((IU+IZ).LE.0) GO TO 112
111 EH=EH+(FLOAT(IU)+FLOAT(IZ)-FLOAT(IR))*P(IR+1,KX+1)
    IR=IR+1
    IF(IR.GT.IRSMX) GO TO 113
    IF(IR.NE.(IU+IZ)) GO TO   111
    HC=CH(KX)*EH
116 IR=IR+1
112 ES=ES+(FLOAT(IR)-FLOAT(IU)-FLOAT(IZ))*P(IR+1,KX)
    IR=IR+1
    IF(IR.LE.IRSMX) GO TO 112
113 SC=CS(KX)*ES
    SSHC=HC+SC
    IF(K.EQ.IP) GO TO 353
     ULT1=FLOAT(IND1)*CPL1+(1.-FLOAT(IND1))*(1.-CPL1)
     ULT2=FLOAT(IND2)*CPL2+(1.-FLOAT(IND2))*(1.-CPL2)
    PETC=PETC+SSHC* ULT1* ULT2
    II=II+1
    IF(II.LE.2) GO TO 650
    I=I+1
    IF(I.LE.2) GO TO 652
    KX=KX+1
    IF(KX.LE.KXX) GO TO 651
214 IF(IZ2.NE.0) GO TO 204
    OC=0.
    GO TO 205
204 OC=CO(K,IX2)
205 TOC=OC+CI(K,IX2)*FLOAT(IZ2)
    EF=0.
    IF(K.EQ.5) GO TO 212
    IRX=IRMIN(K)+1
    IRY=IRMAX(K)+1
    DO 300 IR=IRX,IRY
```

```
    IUY=IUX+IZ1-IR+1
    EF=EF+FF(IUY+1,IZ2+1,IX2)*PP(IR,K)
300 CONTINUE
    IF(K.NE.IP) GO TO 212
    TEC=TOC+EC+EF
    GO TO 213
212 TEC=TOC+PETC+EF
213 IF(IZ2.EQ.0) GO TO 206
    IF(TEC.GE.FMIN) GO TO 208
    IZO=IZ2
    GO TO 207
206 IZO=0
207 FMIN=TEC
208 IZ2=IZ2+1
    IF(IZ2.GT.IZM) GO TO 215
    IF(K.NE.IP) GO TO 100
    GO TC 214
215 IF(K.EQ.IP) GO TO 216
217 WRITE(6,13) IUX,IZ1,FMIN,IZO
    F(IUX+1,IZ1+1,IX2)=FMIN
    IZOP (IUX+1,IZ1+1,IX2)=IZO
211 IZ1=IZ1+1
    IF(IZ1.LE.IZMAX(K,IX1))GO TO514
230 IUX=IUX+1
    IF(IUX.LE.IUMAX(K,IX1)) GO TO513
    IX2=IX2+1
    IF(IX2.LE.2) GO TO512
    KT=K+1
    WRITE(6,14) IX1,KT
    WRITE(6,15) K,KT,IX1
    IUX=IUMIN(K,IX1)
755 IXX=INT(W)-IUX
    IZ1=IZMIN(K,IX1)
752 IF(F(IUX+1,IZ1+1,1).GT.F(IUX+1,IZ1+1,2)) GO TO 750
    FF(IUX+1,IZ1+1,IX1)=F(IUX+1,IZ1+1,1)
    IZB=IZOP(IUX+1,IZ1+1,1)
    ISSO=1
    GO TO 751
750 FF(IUX+1,IZ1+1,IX1)=F(IUX+1,IZ1+1,2)
    IZB=IZOP(IUX+1,IZ1+1,2)
    ISSO=2
751 WRITE(6,16) IUX,IZ1,FF(IUX+1,IZ1+1,IX1),IZB,ISSO
    IXM=MIN0(IZMAX(K,IX1),IXX)
    IF(IZ1.EQ.IXM) GO TO 753
    IZ1=IZ1+1
    GO TO 752
753 IF(IUX.EQ.IUMAX(K,IX1))GO TO 754
    IUX=IUX+1
    GO TO 755
754 IX1=IX1+1
    IF(IX1.LE.2) GO TO515
```

```
      K=K+1
      GO TO 800
 350  LM=MIN0(LMIN(IP,1),LMIN(IP,2))
      IX2=1
 220  EC=0.
      IZ2=0
      IF(LMIN(IP,IX2).EQ.LM) GO TO 352
      KXX=IP-LMIN(IP,IX2)+1
      KXY=IP-LM
      IU=IUINI
      DO 351 KX=KXX,KXY
      GO TO 110
 353  EC=EC+SSHC
 351  CONTINUE
 352  IUX=IUINI
      IZ1=0
      GO TO517
 216  IF(IX2.EQ.1) GO TO 218
      IF(FMIN.GE.FOP) GO TO 219
 218  FOP=FMIN
      IOR=IX2
      IZB=IZO
 219  IX2=IX2+1
      IF(IX2.LE.2) GO TO 220
      WRITE(6,17) IZB,IOR,FOP
 999  CALL EXIT
      END
```

ANNEX II-2

INPUT DATA

```
7
3.000
1.000
2
3 2 2 1 2 1
0.500 0.500 0.500 0.600 0.600 0.600
2.000 3.000 2.000 3.000 3.000 2.000
5.000 6.000 6.000 5.000
2.000 2.000 2.000 2.000
0.500 0.500 0.000
0.600 0.400 0.000
0.200 0.500 0.300
0.300 0.300 0.400
0.500 0.500 0.000
0.300 0.700 0.000
0.400 0.600 0.000
0.000 0.300 0.700
0.000 0.500 0.500
0.000 0.600 0.400
0.200 0.800 0.000
0.400 0.600 0.000
0.500 0.500 0.000
```

ANNEX II-3

OUTPUT

RESULTS FOR PERIOD 5

ORDER FROM SOURCE 1 AT PERIOD 6
ORDER FROM SOURCE 1 AT PERIOD 5

U(1,5)	Z(1,6,1)	MIN.COST	ORDER
0	0	36.600	0
0	1	28.765	0
0	2	21.807	0
1	0	25.849	0
1	1	18.891	0
1	2	14.029	0
2	0	16.389	0
2	1	11.527	0
3	0	9.929	0

ORDER FROM SOURCE 1 AT PERIOD 6
ORDER FROM SOURCE 2 AT PERIOD 5

U(1,5)	Z(1,6,1)	MIN.COST	ORDER
0	0	34.115	1
0	1	26.826	1
0	2	21.240	1
1	0	23.993	1
1	1	18.407	1
1	2	14.029	0
2	0	16.086	1
2	1	11.527	0
3	0	9.929	0

POLICY WHEN SOURCE NO. 1 IS CHOSEN AT PERIOD 6

U(1,5)	Z(1,6,1)	MIN.COST	ORDER	SOURCE
0	0	34.115	1	2
0	1	26.826	1	2
0	2	21.240	1	2
1	0	23.993	1	2
1	1	18.407	1	2
1	2	14.029	0	1
2	0	16.086	1	2
2	1	11.527	0	2
3	0	9.929	0	1

ORDER FROM SOURCE 2 AT PERIOD 6
ORDER FROM SOURCE 1 AT PERIOD 5

U(1,5)	Z(1,6,2)	MIN.COST	ORDER
0	0	36.600	0
0	1	25.849	0
0	2	16.389	0
1	0	25.849	0
1	1	16.389	0

1	2	9.929	0
2	0	16.389	0
2	1	9.929	0
3	0	9.929	0

ORDER FROM SOURCE 2 AT PERIOD 6
ORDER FROM SOURCE 2 AT PERIOD 5

U(1,5)	Z(1,6,2)	MIN.COST	ORDER
0	0	34.115	1
0	1	23.993	1
0	2	16.086	1
1	0	23.993	1
1	1	16.086	1
1	2	9.929	0
2	0	16.086	1
2	1	9.929	0
3	0	9.929	0

POLICY WHEN SOURCE NO. 2 IS CHOSEN AT PERIOD 6

U(1,5)	Z(1,6,2)	MIN.COST	ORDER	SOURCE
0	0	34.115	1	2
0	1	23.993	1	2
0	2	16.086	1	2
1	0	23.993	1	2
1	1	16.086	1	2
1	2	9.929	0	1
2	0	16.086	1	2
2	1	9.929	0	1
3	0	9.929	0	1

RESULTS FOR PERIOD 6

ORDER FROM SOURCE 1 AT PERIOD 7
ORDER FROM SOURCE 1 AT PERIOD 6

U(1,6)	Z(1,7,1)	MIN.COST	ORDER
1	0	40.121	2
1	1	31.329	1
2	0	29.290	1
2	1	21.632	0

ORDER FROM SOURCE 1 AT PERIOD 7
ORDER FROM SOURCE 2 AT PERIOD 6

U(1,6)	Z(1,7,1)	MIN.COST	ORDER
1	0	31.975	2
1	1	27.688	1
2	0	25.917	1
2	1	21.632	0

POLICY WHEN SOURCE NO. 1 IS CHOSEN AT PERIOD 7

U(1,6)	Z(1,7,1)	MIN.COST	ORDER	SOURCE
1	0	31.975	2	2
1	1	27.688	1	2
2	0	25.917	1	2
2	1	21.632	0	1

ORDER FROM SOURCE 2 AT PERIOD 7
ORDER FROM SOURCE 1 AT PERIOD 6

U(1,6)	Z(1,7,2)	MIN.COST	ORDER
1	0	47.146	2
1	1	44.593	1
2	0	43.343	1
2	1	31.557	0

ORDER FROM SOURCE 2 AT PERIOD 7
ORDER FROM SOURCE 2 AT PERIOD 6

U(1,6)	Z(1,7,2)	MIN.COST	ORDER
1	0	39.000	2
1	1	31.692	1
2	0	30.442	1
2	1	24.675	0

POLICY WHEN SOURCE NO. 2 IS CHOSEN AT PERIOD 7

U(1,6)	Z(1,7,2)	MIN.COST	ORDER	SOURCE
1	0	39.000	2	2
1	1	31.692	1	2
2	0	30.442	1	2
2	1	24.675	0	2

RESULTS FOR PERIOD 7

THE FINAL POLICY IS TO ORDER 1 ITEMS FROM SOURCE 2
THE MINIMUM EXPECTED COST WILL BE 31.485

VITA

Techapun Raengkhum

Candidate for the Degree of

Doctor of Philosophy

Thesis: FINITE PERIOD INVENTORY MODELS

Major Field: Engineering

Biographical:

Personal Data: Born May 30, 1939, in Lampang, Thailand, the son of Suwan and Tab Raengkhum, of Bangkok, Thailand.

Education: Attended Bangkok Christian School in Bangkok, Thailand and graduated in 1954. Entered Chulalongkorn Pre-university School and graduated in 1956. Entered Chulalongkorn University in Bangkok, Thailand, in 1956; received the Bachelor of Engineering degree with second class honor in Electrical Engineering in 1960. Entered Syracuse University at Syracuse, New York, in 1963; received the Master of Science degree in Engineering Administration in 1964. Completed requirements for the Degree of Doctor of Philosophy in January, 1968.

Professional Experience: Employed by National Energy Authority, Thailand, as an Electrical Engineer from April, 1960 to present.

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