EXPONENTIAL REPRESENTATION THEORY FOR LINEAR PHYSICAL SYSTEMS

By

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NOMENCLATURE

$\underline{A}(t)$	The coefficient matrix
a_{11}, a_{12}, a_{21}, a_{22}	The time varying entries of the coefficient matrix
g_1, g_2, g_3, g_4	The scalar exponents in the exponential representation
$\underline{L}_1$, $\underline{L}_2$, $\underline{L}_3$, $\underline{L}_4$	The Lie algebraic basis vectors
$M(t)$	Amplitude modulation term
ω	Instantaneous frequency of the FM signal
ω_0	Constant carrier frequency
ω_m	Modulation function
$\Delta\omega$	Maximum carrier frequency deviation
Ω	Instantaneous phase of the FM signal
Φ	Fundamental solution matrix
θ	Total phase distortion varying in time
μ	Amplifier gain
ω_d	Total rate distortion frequency bound
*	Multiplication sign

CHAPTER I

INTRODUCTION

Historical Development and Survey of Literature

The linear differential equation which is commonly associated with the electronic generation of frequency modulation is

$$d^2x \,/\, dt^2 + \omega^2(t)\, x = 0 \qquad (1.1)$$

where $\omega(t)$ is the instantaneous frequency of the oscillator. The above equation was first studied with reference to FM by Carson [1] followed shortly thereafter by the works of Van der Pol [2] and Barrow [3]. In the domain of quantum mechanics, the approximate solution to (1.1) when $\omega^2(t)$ does not vary rapidly, is given by Wenzel-Kramers-Brillouin or the so-called W.K.B form [4]. The general mathematical analysis of the above differential equation probably dates from the work of Liouville [5] in 1837. The mathematical results discovered since then are many, including the important fact that the differential equation cannot be solved in terms of elementary functions or by a finite number of quadratures. This result plus many others, along with an excellent set of references are found in the book by Bellman [6].

Carson and Van der Pol correctly applied to the engineering problem of modulation distortion, the classical results known for the analysis of (1.1), including the special results known for the equations of Hill, Mathieu, and Riccati. The basic engineering problem lies in the fact that the generation of FM as described by the above differential equation is not pure. That is, its solution contains an amplitude modulation (AM) and rate distortion components along with the desired FM solution. Recently Gardner [7] discussed the analysis of modulation rate distortion in frequency modulators by studying the differential equation (1.1). All of these authors studied the FM differential equation using essentially perturbation or harmonic methods. In recent years, Lie algebras and Lie groups have become a powerful tool for studying differential equations, special functions, classical and quantum mechanics, perturbation theory, nuclear physics and solid state physics. [8], [9], [10], [11], [12]. And of late, the Lie algebraic approach has been applied to the study of linear matrix differential equations.

Wei and Norman in 1964 [13] gave certain theorems on global representations of the solutions of linear differential equations as a product of matrix exponentials with special reference to (2x2) real coefficient matrices. Mariani and Magnus show in their paper [14] that even for the case of (2x2) matrices, a general global representation cannot be obtained without severe restrictions on the coefficient matrix. As an application to engineering problems of interest, Mulholland employed matrix decomposition using the Lie algebra to study the generation of FM [15], wave motion on a non-uniform transmission line [16] and the Riccati equation [17]. Outside of the publications of Mulholland, not much literature seems to be available on Lie algebraic methods as applied to engineering problems.

The Lie algebraic approach, as shown by Wei and Norman, is strongly basis dependent and suffers from the disadvantage that the exponential state representation is local in nature. The investigation of additional global bases with special emphasis on engineering applications needs to be studied in greater detail.

Scope of Thesis

This thesis investigates several global bases for the exponential state representation with special reference to (2x2) continuous real matrix and the generation of rate distortionless orthogonal FM with and without the use of gyrators. The analysis of modulation rate distortion as a degradation of pure FM signals and the rate distortive effects of loading upon the RC modulators are studied by Lie algebraic methods using the state variable approach. Use is made of the Wei-Norman basis development for the representation of modulation and distortion in terms of finite matrix product. The outstanding result obtained is that the rate distortion is expressed as an implicit function of the instantaneous phase, from which bounds on the maximum rate distortion are derived. Using the Wei-Norman global representation for the (2x2) continuous real coefficient matrix, the stability properties of the general linear differential equation are obtained from the fundamental solution matrix.

Approach to the Problem

An outline of the approach to the four problems considered in the thesis is given below:

1. Global bases are those Lie algebraic bases which give rise to the global validity of exponential state representations of solutions of linear matrix differential equations. They need to satisfy the require-

ment that the exponents in the matrix product of exponentials can be expressed as analytic functions of the entries of the fundamental solution matrix. Several new global bases are presented for the first time in this thesis.

2. The generation of rate distortionless FM is obtained by the decomposition of the coefficient matrix, resulting for example from the state equations of a Wein-bridge RC network, into the sum of a scalar matrix and a skew-symmetric matrix. The decomposition is effected by means of a two parameter design technique given in terms of the amplification μ and a network parameter m depending upon the modulator element values of the oscillator. The skew-symmetric matrix gives rise to orthogonal FM while the scalar matrix generates amplitude distortion.

3. With regard to modulation rate distortion, an analysis is performed on the state equations of the linear differential equation that describes rate distortion. An algebraic basis development is presented for the exponential representation of the pure FM signal and the amplitude and rate distortion as a finite matrix product. This results in the rate distortion being expressed as an implicit function of the instantaneous phase from which bounds on the maximum rate distortion are derived.

4. With the Wei-Norman global representation of (2x2) continuous real matrices, the fundamental solution matrix is observed to yield the conditions for stability and asymptotic stability of linear second order matrix differential equations.

CHAPTER II

INTRODUCTION TO LIE ALGEBRA

Lie Algebraic Theory

An algebra is a triple $\{s, +, \circ\}$ consisting of a set s of elements, a binary operation (+) and a binary operation ($\circ$) both maping sxs into s. A Lie algebra is an algebra in which s is a vector space and in which the product relation defined by the commutator product $[\circ\ ,\ \circ]$ is bilinear. That is, for $\underline{x}$, $\underline{y}$ and $\underline{z}$ in s,

$$[(\underline{x} + \underline{y}), \underline{z}] = [\underline{x}, \underline{z}] + [\underline{y}, \underline{z}] \quad ;$$

$$[\underline{x}, (\underline{y} + \underline{z})] = [\underline{x}, \underline{y}] + [\underline{x}, \underline{z}] \quad ;$$

and

$$m\,[\underline{x}, \underline{y}] = [m\,\underline{x}, \underline{y}] = [\underline{x}, m\underline{y}] \quad .$$

In addition, the commutator product $[\circ\ ,\ \circ]$ is required to satisfy the conditions $[\underline{x}, \underline{x}] = 0$ and

$$[[\underline{x}, \underline{y}], \underline{z}] + [[\underline{y}, \underline{z}], \underline{x}] + [[\underline{z}, \underline{x}], \underline{y}] = 0 \ .$$

The latter condition is known as the Jacobi identity.

The Lie algebras considered in this thesis have for s the set of nxn matrices whose entries are real numbers. The Lie product is the usual matrix commutator $[\underline{x}, \underline{y}] = \underline{x}\,\underline{y} - \underline{y}\,\underline{x}$ which satisfies the above conditions. A subset of a Lie algebra L is called a subalgebra if it is closed under the operations of addition, scalar multiplication and commutation. Let $\{H_i\}$ be a set of nxn matrices. The enveloping Lie algebra, or the Lie algebra generated by $\{H_i\}$consists of $\{H_i\}$, all the elements obtained from H by repeated commutations and all the linear combinations of these. A subalgebra S_1 of a given algebra is called an ideal if $[S_1, L] \subset S_1$, that is, for all $\underline{x} \in S_1$ and $\underline{y} \in L$ the product $[\underline{x}, \underline{y}]$ belongs to S_1.

The set of all elements of L which are the result of commutation of some two elements form the derived algebra. This is denoted by L'. Clearly, L' is an ideal of L. The derived algebra of L' is denoted by L''. Thus,

$$L \supset L' \supset L'' \quad \ldots \quad L^{(h)} \supset L^{(h+1)} \quad .$$

A Lie algebra is said to be solvable if $L^{(h)} = \{0\}$ for some h. The union of two solvable ideals is again a solvable ideal. The radical of L is the union of all of its solvable ideals.

The Lie algebra is said to be semi-simple if its radical is $\{0\}$. It is called simple if it has no other ideal than L and $\{0\}$ and if L' is not equal to $\{0\}$.

The lower central series is constructed by relabelling $L^2 = L'$ and defining

$$L^{n+1} = [L, L^n]$$

one has $L \supset L^2 \supset L^3 \quad \ldots$ A Lie algebra is said to be nilpotent if

$L^k = \{0\}$ for some k.

For each $\underline{x} \in L$, the operator $ad_{\underline{x}}$ is defined by

$$ad_{\underline{x}}\, \underline{y} = [\underline{x}, \underline{y}], \text{ all } \underline{y} \in L .$$

Powers of operator $ad_{\underline{x}}$ are defined by

$$ad^n_{\underline{x}}\,(\underline{y}) = \underbrace{[\underline{x}, [\underline{x}, \ldots [\underline{x}}_{n \text{ times}}, \underline{y}] \ldots]$$

The relationships between these Lie algebraic properties are described by the following theorems.

THEOREM 2-1 (Baker-Hausdorff): If $\underline{x}, \underline{y} \in L$, then $e^{\underline{x}}\, \underline{y}\, e^{-\underline{x}} = (e^{ad\, \underline{x}})\, \underline{y}$ where the exponential is defined by the usual power series

$$(e^{ad\, \underline{x}})\, \underline{y} = \underline{y} + [\underline{x}, \underline{y}] / 1! + [\underline{x} [\underline{x}\, \underline{y}]] / 2! + [\underline{x} [\underline{x} [\underline{x}\, \underline{y}]]] / 3! + .$$

Let $\underline{L}_1, \ldots \underline{L}_n$ be a basis for L. Then $[\underline{L}_i, \underline{L}_j]$ can be expressed in this basis and is given by

$$[\underline{L}_i, \underline{L}_j] = \sum_{k=1}^{n} \gamma^k_{ij}\, \underline{L}_k, \; i=1, \; \ldots n .$$

The numbers γ^k_{ij} are called structural constants of L with respect to chosen basis.

THEOREM 2-2 (Wei-Norman): Let $\underline{L}_1, \ldots \underline{L}_n$ be a basis for L. Then

$$\prod_{j=1}^{r} \exp\,(g_j \underline{L}_j)\; \underline{L}_i \; \prod_{j=r}^{1} \exp\,(-g_j \underline{L}_j) = \sum_{k=1}^{n} \xi_{ki}\, \underline{L}_k .$$

Let $\underline{A}(t)$ the linear operator be expressed in the form

$$\underline{A}(t) = \sum_{i=1}^{m} a_i(t)\, \underline{L}_i ,$$

m finite, where $a_i(t)$'s are scalar functions of time and $\underline{L}_1, \ldots \underline{L}_n$ are time independent bases. If m is chosen as small as possible, the $\underline{L}_i$'s will be linearly independent.

THEOREM 2-3: Let $\underline{A}(t)$ be expressed as above and let the Lie algebra $\underline{L}$ generated by $\underline{A}(t)$ be of finite dimensional l. Then there exists a neighborhood of t=0 in which the solution of the equation $d\underline{x} / dt = \underline{A}(t)\, \underline{x}$, $\underline{X}(0) = \underline{I}$ may be expressed in the form

$$\underline{X}(t) = \exp(g_1 \underline{L}_1) \exp(g_2 \underline{L}_2) \cdots \exp(g_l \underline{L}_l) ,$$

where g_i's are scalar functions of time. Moreover, the g_i's satisfy a set of differential equations which depend only on the Lie algebra L and $a_i(t)$'s.

THEOREM 2-4 (Wei-Norman): If $d\underline{x} / dt = \underline{A}(t)\, \underline{x}$, where $\underline{A}(t)$ is any real continuous (2x2) matrix, then $\underline{X}(t)$ has the form

$$\underline{X}(t) = \exp(g_1 \underline{L}_1) \exp(g_2 \underline{L}_2) \exp(g_3 \underline{L}_3) \exp(g_4 \underline{L}_4)$$

where

$$\underline{L}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \underline{L}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \underline{L}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \underline{L}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

This representation is global.

These theorems provide the mathematical framework for the analysis and applications which follow. An excellent introduction to Lie algebras has been provided by Belin fante et al [18] in which these theorems and definitions are discussed in more detail.

Lie Algebraic Aspects of Wei-Norman Basis

The basis vector expansion for the (2x2) continuous real matrix $\underline{A}(t)$ is given as

$$\underline{A}(t) = a_1(t)\,\underline{L}_1 + a_2(t)\,\underline{L}_2 + a_3(t)\,\underline{L}_3 + a_4(t)\,\underline{L}_4 \qquad (2.1)$$

where the coefficients $a_i(t)$'s in terms of the entries of the coefficient matrix are given as

$$a_1(t) = -\,a_{21}(t)$$

$$a_2(t) = \tfrac{1}{2}\,[a_{11}(t) - a_{22}(t)]$$

$$a_3(t) = a_{12}(t) + a_{21}(t)$$

$$a_4(t) = \tfrac{1}{2}\,[a_{11}(t) + a_{22}(t)] \qquad (2.2)$$

and the $\underline{L}$-matrices are as given in theorem 2-4.

The Wei-Norman basis vectors yield

$$[\underline{L}_1\ \underline{L}_2] = 2\underline{L}_1 - 4\underline{L}_3$$

$$[\underline{L}_1\ \underline{L}_3] = \underline{L}_2$$

$$[\underline{L}_2\ \underline{L}_3] = 2\underline{L}_3$$

$$[\underline{L}_1\ \underline{L}_4] = [\underline{L}_2\ \underline{L}_4] = [\underline{L}_3\ \underline{L}_4] = \underline{0}\ . \qquad (2.3)$$

The relationship for the functions g_i's can be derived from the following calculation:

$$\dot{\underline{X}}\,\underline{X}^{-1} = a_1(t)\,\underline{L}_1 + a_2(t)\,\underline{L}_2 + a_3(t)\,\underline{L}_3 + a_4(t)\,\underline{L}_4$$

$$= \dot{g}_1\,\underline{L}_1 + \dot{g}_2\,e^{\mathrm{ad}\ g_1\underline{L}_1}\,\underline{L}_2 + \dot{g}_3\,e^{\mathrm{ad}\ g_1\underline{L}_1}\,e^{\mathrm{ad}\ g_2\underline{L}_2}\,\underline{L}_3$$

$$+ \dot{g}_4\,e^{\mathrm{ad}\ g_1\underline{L}_1}\,e^{\mathrm{ad}\ g_2\underline{L}_2}\,e^{\mathrm{ad}\ g_3\underline{L}_3}\,\underline{L}_4\,, \tag{2.4}$$

where the Ado operator is defined by

$$e^{\mathrm{ad}\ \underline{M}}\,\underline{N} = e^{\underline{M}}\,\underline{N}\,e^{-\underline{M}}\,, \tag{2.5}$$

for any two matrices $\underline{M}$ and $\underline{N}$ in L_B. The Ado operator can be explicitly evaluated by the Baker-Hausdorff theorem given earlier. Note that since $\underline{X}$ is a fundamental solution matrix, $\underline{X}^{-1}$ always exists [19]. Thus, a formal calculation produces

$$e^{\mathrm{ad}\ g_1\underline{L}_1}\,\underline{L}_2 = \underline{L}_2 \cos 2g_1 + \underline{L}_1 \sin 2g_1 - 2\underline{L}_3 \sin 2g_1\,, \tag{2.6}$$

$$e^{\mathrm{ad}\ g_2\underline{L}_2}\,\underline{L}_3 = \underline{L}_3\,e^{2g_2}\,, \tag{2.7}$$

$$e^{\mathrm{ad}\ g_1\underline{L}_1}\,e^{\mathrm{ad}\ g_2\underline{L}_2}\,\underline{L}_3 = e^{2g_2}\,[\underline{L}_3 \cos 2g_1 + \tfrac{1}{2}\,\underline{L}_2 \sin 2g_1$$

$$+ \tfrac{1}{2}\,\underline{L}_1\,(1 - \cos 2g_1)]\,, \tag{2.8}$$

$$e^{\mathrm{ad}\ g_3\underline{L}_3}\,\underline{L}_4 = \underline{L}_4\,, \tag{2.9}$$

and

$$e^{\mathrm{ad}\ g_1\underline{L}_1}\ e^{\mathrm{ad}\ g_2\underline{L}_2}\ e^{\mathrm{ad}\ g_3\underline{L}_3}\ \underline{L}_4 = \underline{L}_4 \ . \qquad (2.10)$$

Therefore, by expanding the Ado operators in terms of the basis for L_B, substituting these results into (2.4), and equating like coefficients for the unique basis vector expansion for $\underline{A}(t)$, equation (2.4) yields

$$a_1(t) = \dot{g}_1 + \dot{g}_2 \sin 2g_1 + \tfrac{1}{2}\, \dot{g}_3\, e^{2g_2} (1 - \cos 2g_1) \ ,$$

$$a_2(t) = \dot{g}_2 \cos 2g_1 + \tfrac{1}{2}\, \dot{g}_3\, e^{2g_2} \sin 2g_1 \ ,$$

$$a_3(t) = -\, 2\dot{g}_2 \sin 2g_1 + \dot{g}_3\, e^{2g_2} \cos 2g_1 \ ,$$

$$a_4(t) = \dot{g}_4 \ . \qquad (2.11)$$

Since $\underline{X}(0) = \underline{I}$, a particular solution of (2.11) is desired so that

$$\underline{I} = \prod_{k=1}^{4} \exp g_k(0)\ \underline{L}_k \ . \qquad (2.12)$$

An interesting solution is provided by the initial values $g_k(0) = 0$ ($k = 1, 2, 3, 4$) which clearly satisfy (2.12).

The determinant associated with (2.11) is e^{2g_4}; hence, the system is globally invertible to give the normal form of differential equation system:

$$\dot{g}_1 = a_1(t) - a_2(t) \sin 2g_1 + \tfrac{1}{2}\, a_3(t)\ (1 - \cos 2g_1)$$

$$\dot{g}_2 = a_2(t) \cos 2g_1 - \tfrac{1}{2} a_3(t) \sin 2g_1 ,$$

$$\dot{g}_3 = 2a_2(t)\, e^{-2g_2} \sin 2g_1 + a_3(t)\, e^{-2g_2} \cos 2g_1 ,$$

$$\dot{g}_4 = a_4(t). \qquad (2.13)$$

The solution of (2.13) for the exponents g_k as a time series starting at intial values prescribed by (2.12) provide the exponential solution representation as defined in theorem 2-4. In most cases, equation (2.13) cannot be solved in closed form, however this does not invalidate the result presented as analysis can now be performed on (2.13) which will yield useful information about the nature of the solution of the linear differential equation associated with $\underline{A}(t)$.

Investigation of New Global Bases

The following are some of the global bases studied by the author for the (2x2) continuous real matrix.

Symmetric Basis: The Lie algebra L_B is represented by the matrices $\underline{L}_1$, $\underline{L}_2$, $\underline{L}_3$, $\underline{L}_4$ as given by

$$\underline{L}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \underline{L}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \underline{L}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \underline{L}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

The fundamental solution matrix is given as

$$\underline{X}(t) = \exp(g_1\underline{L}_1) \exp(g_2\underline{L}_2) \exp(g_3\underline{L}_3) \exp(g_4\underline{L}_4) ,$$

that is,

$$\underline{X}(t) = e^{g_4} \begin{bmatrix} \cos g_1 & \sin g_1 \\ -\sin g_1 & \cos g_1 \end{bmatrix} \begin{bmatrix} \cosh g_2 & \sinh g_2 \\ \sinh g_2 & \cosh g_2 \end{bmatrix} \begin{bmatrix} e^{g_3} & 0 \\ 0 & e^{-g_3} \end{bmatrix} .$$

where

$$\underline{X}(t) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} .$$

Solving for the g_1's in terms of x_{ij}'s gives

$$g_1(t) = \tfrac{1}{2} \cos^{-1} \left[1 + \left(\frac{x_{11}\, x_{12} - x_{21}\, x_{22}}{x_{11}\, x_{22} + x_{12}\, x_{21}} \right)^2 \right]^{-\frac{1}{2}} ,$$

$$g_2(t) = \tfrac{1}{2} \cosh^{-1} \left[1 + \left(\frac{x_{11}\, x_{12} + x_{21}\, x_{22}}{x_{11}\, x_{22} - x_{12}\, x_{21}} \right)^2 \right]^{\frac{1}{2}} ,$$

$$g_3(t) = \tfrac{1}{4} \log_e \, (x_{11}^2 + x_{21}^2) \,/\, (x_{22}^2 + x_{21}^2) ,$$

$$g_4(t) = \tfrac{1}{2} \log_e \, (x_{11}\, x_{22} - x_{12}\, x_{21}) .$$

Since $\underline{X}(t)$ is non-singular for all time, x_{22} and x_{21} cannot vanish simultaneously. Hence g_i's are analytic functions of x_{ij}(x_{ij} assumed real). Thus, symmetric basis gives rise to global representation. The g_i's satisfy the following non-linear differential equations given below (see Appendix):

$$\dot{g}_1 = a_1(t) - a_2(t) \tanh 2g_2 \sin 2g_1 + a_3(t) \tanh 2g_2 \cos 2g_1 ,$$

$$\dot{g}_2 = a_2(t) \cos 2g_1 + a_3(t) \sin 2g_1 ,$$

$$\dot{g}_3 = - a_2(t) \frac{\sin 2g_1}{\cosh 2g_2} + a_3(t) \frac{\cos 2g_1}{\cosh 2g_2} ,$$

$$\dot{g}_4 = a_4(t) .$$

Wei-Norman 2 Basis: With the vector bases given by

$$\underline{L}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \underline{L}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \underline{L}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \underline{L}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ,$$

the fundamental solution matrix is

$$\underline{X}(t) = \begin{bmatrix} \cos g_1 & \sin g_1 \\ -\sin g_1 & \cos g_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{g_2} \end{bmatrix} \begin{bmatrix} 1 & g_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{g_4} & 0 \\ 0 & 1 \end{bmatrix} ,$$

that is,

$$\underline{X}(t) = \begin{bmatrix} e^{g_4} \cos g_1 & g_3 \cos g_1 + e^{g_2} \sin g_1 \\ -e^{g_4} \sin g_1 & -g_3 \sin g_1 + e^{g_2} \cos g_1 \end{bmatrix} .$$

Solving for g_i's yields

$$g_1(t) = \tan^{-1} (-x_{21} / x_{11}) ,$$

$$g_2(t) = \log_e (x_{11} x_{22} - x_{12} x_{21}) / (x_{11}^2 + x_{21}^2)^{\frac{1}{2}} ,$$

$$g_3(t) = (x_{21} x_{22} + x_{12} x_{11}) / (x_{11}^2 + x_{21}^2)^{\frac{1}{2}} ,$$

$$g_4(t) = \frac{1}{2} \log_e (x_{11}^2 + x_{21}^2) \ .$$

With the reasoning as given for the symmetric basis, it is easily seenthat the g_i's are analytic functions of the x_{ij}'s. Hence, the representation is global with the g_i's given as (see Appendix):

$$\dot{g}_1 = a_1(t) + \frac{1}{2} a_2(t) \sin 2g_1 + \frac{1}{2} a_3(t) (1 - \cos 2g_1) - \frac{1}{2} a_4(t) \sin 2g_1,$$

$$\dot{g}_2 = \frac{1}{2} a_2(t) (1+\cos 2g_1) + \frac{1}{2} a_3(t) \sin 2g_1 + \frac{1}{2} a_4(t) (1-\cos 2g_1),$$

$$\dot{g}_3 = a_2(t) [\tfrac{1}{2} g_3 (1-\cos 2g_1) - e^{g_2} \sin 2g_1]$$

$$+ a_3(t) [e^{g_2} \cos 2g_1 - \tfrac{1}{2} g_3 \sin 2g_1]$$

$$+ a_4(t) [e^{g_2} \sin 2g_1 + \tfrac{1}{2} g_3 (1+\cos 2g_1)],$$

$$\dot{g}_4 = \frac{1}{2} a_2(t) (1-\cos 2g_1) - \frac{1}{2} a_3(t) \sin 2g_1 + \frac{1}{2} a_4(t) (1+\cos 2g_1).$$

<u>Wei-Norman 3 Basis</u>: The Lie algebra L_B which is spanned by

$$\underline{L}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \underline{L}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ \underline{L}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \underline{L}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

gives rise to the following fundamental solution matrix:

$$\underline{X}(t) = \begin{bmatrix} e^{g_4} \cos g_1 & e^{g_4} (g_3 \cos g_1 + e^{g_2} \sin g_1) \\ -e^{g_4} \sin g_1 & e^{g_4} (-g_3 \sin g_1 + e^{g_2} \cos g_1) \end{bmatrix} .$$

The g_i's are given by

$$g_1(t) = \tan^{-1}(-x_{21} / x_{11}) ,$$

$$g_2(t) = \log_e (x_{11} x_{22} - x_{12} x_{21}) / (x_{11}^2 + x_{21}^2) ,$$

$$g_3(t) = (x_{21} x_{22} + x_{12} x_{11}) / (x_{11}^2 + x_{21}^2) ,$$

and

$$g_4(t) = \tfrac{1}{2} \log_e (x_{11}^2 + x_{21}^2) .$$

Once again the g_i's are analytic functions of the x_{ij}'s and the representation is global with the exponents satisfying the following nonlinear differential equations (see Appendix):

$$\dot{g}_1 = a_1(t) + \tfrac{1}{2} a_2(t) \sin 2g_1 + \tfrac{1}{2} a_3(t) (1-\cos 2g_1) ,$$

$$\dot{g}_2 = a_2(t) \cos 2g_1 + a_3(t) \sin 2g_1 ,$$

$$\dot{g}_3 = e^{g_2} [- a_2(t) \sin 2g_1 + a_3(t) \cos 2g_1] ,$$

$$\dot{g}_4 = \tfrac{1}{2} a_2(t) (1-\cos 2g_1) - \tfrac{1}{2} a_3(t) \sin 2g_1 + \tfrac{1}{2} a_4(t) .$$

The application of these various basis developments to engineering problems is discussed in the following chapters of this thesis.

CHAPTER III

RATE DISTORTIONLESS ORTHOGONAL FREQUENCY MODULATORS

Introduction

A frequency modulator can be modelled as a two port with a modulating signal v_m as the input, and a modulated signal v_{fm} as the output, as shown in Figure 1. Ideally, the output signal would be given by the following expression:

$$v_{fm} = M(t) \cos [(\int kv_m \, dt + \theta)] \tag{3.1}$$

where M(t) represents the amplitude modulation and kv_m the frequency modulation with k as constant parameter. If the phase parameter θ is constant, then the signal is called <u>rate distortionless</u>. For otherwise, the frequency of the modulated signal (3.1)

$$\omega(t) = d/dt \, [(\int kv_m \, dt + \theta)] = kv_m + d\theta/dt \tag{3.2}$$

is distorted by the rate $d\theta/dt$. Unlike rate distortion, the amplitude distortion M(t) is in most cases easily removed in the generation or detection process by limiting and filtering [20]. Thus the analysis and construction of rate distortionless frequency modulators are of practical interest.

In a paper recently published by Gardner [7] it is shown that rate distortion increases from zero as the ratio of the rate at which the

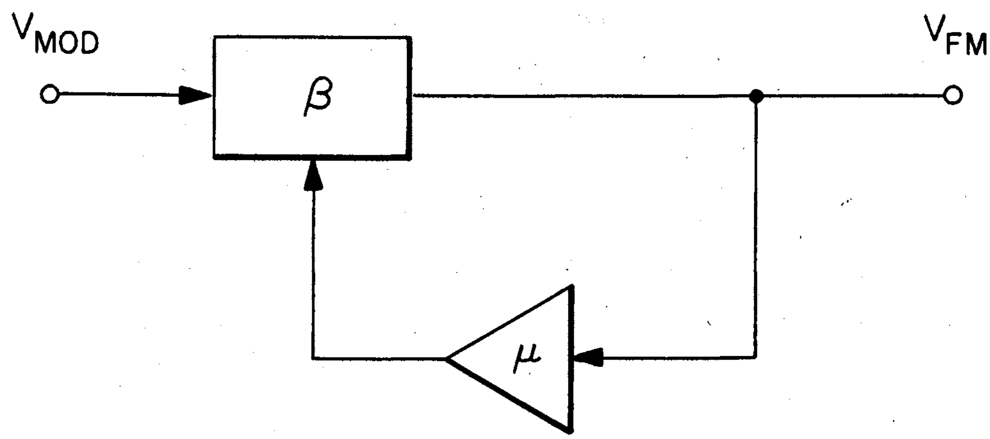


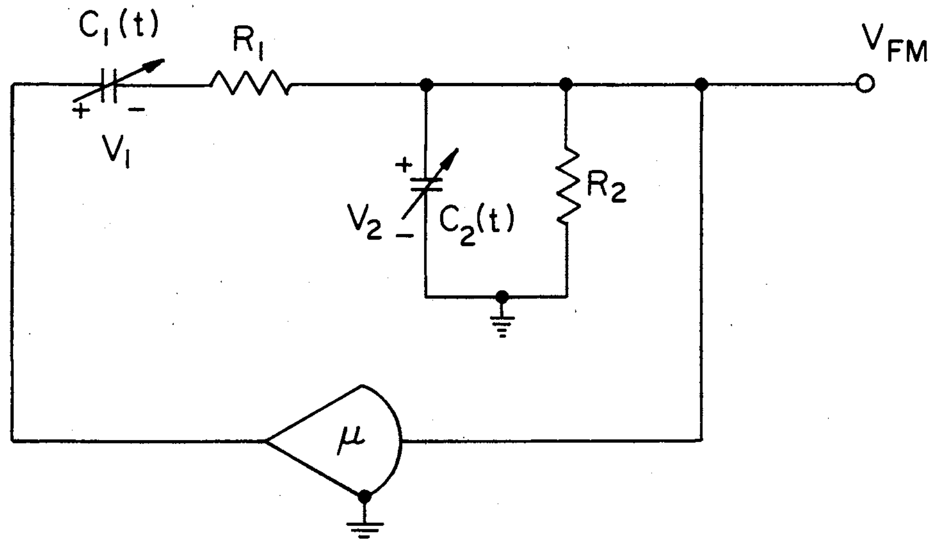
Figure 1. A Frequency Modulator

frequency of oscillation is deviated to the magnitude of the frequency of oscillation, increases from zero. But the rate at which the frequency of oscillation is deviated is proportional to the rate at which the energy storage capacity of one element varies relative to the storage capacity of the other, and the magnitude of the frequency of oscillation is proportional to the rate at which energy is exchanged between the two storage elements. Therefore, rate distortion in modulators operating under equilibrium conditions depends upon the ratio of the change of energy storage capacity and that of stored energy. Thus, it is clear that a general method of analysis apparently requires a complete energy description for the modulator. This is clearly provided by the state-variable approach which is used in this thesis.

The generation of rate distortionless FM as discussed by Gardner is based upon the solutions of second-order linear differential equations for the frequency modulators and upon the adjustment of the amplifier gain μ to cancel the losses in the frequency selective network of the modulator. A new two-parameter design technique given in terms of the amplifier gain and a network parameter, will be developed in this chapter to reduce the state coefficient matrix to a fundamental formulation that yields an orthogonal pure FM solution matrix and a scalar AM distortion factor.

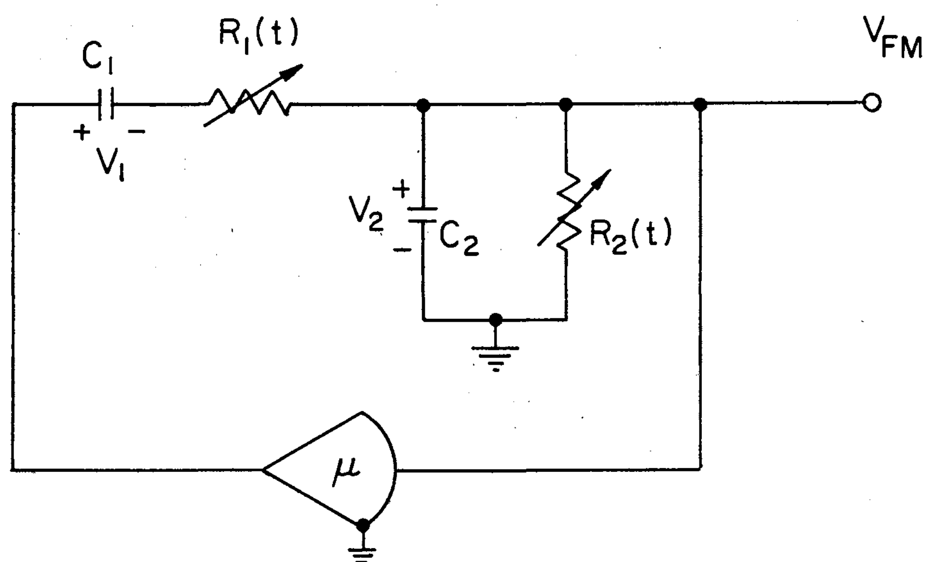
Rate Distortionless Modulation

The two types of frequency modulators considered contain amplifiers of voltage amplification μ in conjunction with two fundamental forms of frequency selective networks: the Wein-bridge RC network shown



(a)

Figure 2a. Wein-bridge RC(t) Modulator



(b)

Figure 2b. Wein-bridge R(t)C Modulator

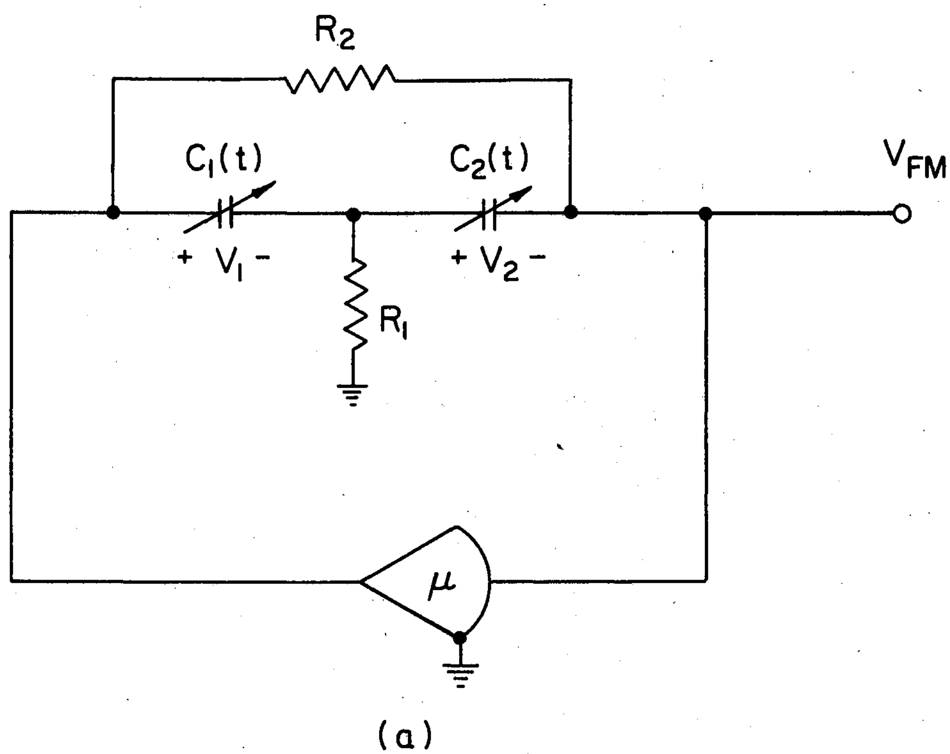


Figure 3a. Bridged-T RC(t) Modulator

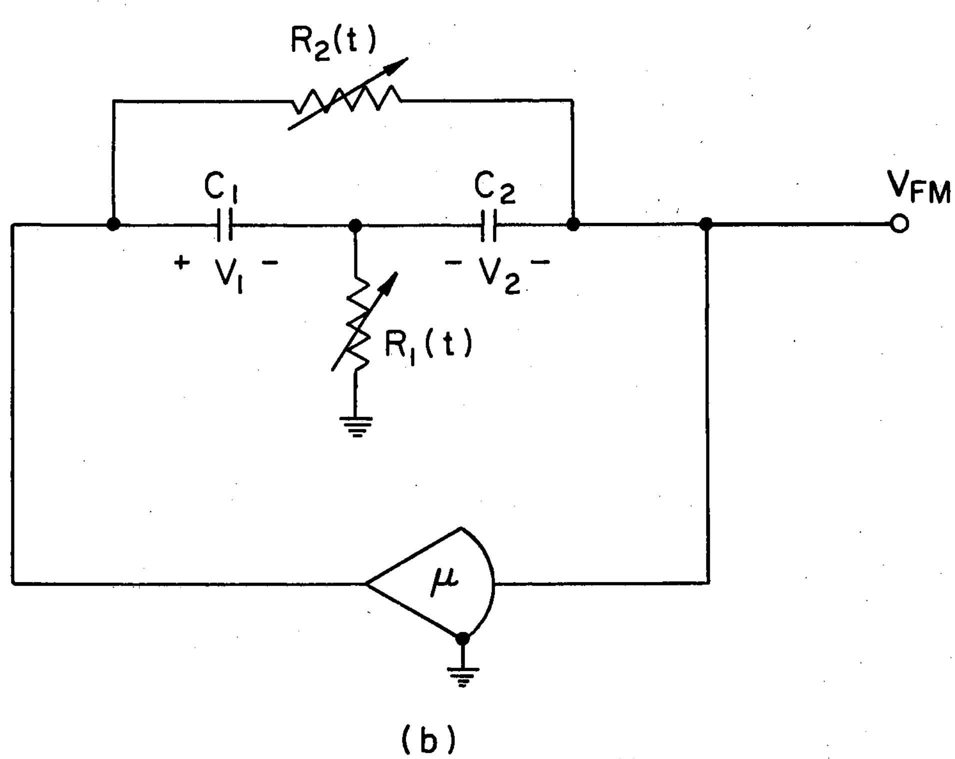


Figure 3b. Bridged-T R(t)C Modulator

in Figure 2 and the bridged-T RC network in Figure 3. The state variables v_1 and v_2, which are the capacitor voltages for each modulator, determine the network energy function and satisfy a vector matrix linear differential equation of the form

$$\dot{\underline{v}} = \underline{A}(t)\ \underline{v} \tag{3.3}$$

where $\underline{A}(t)$ is a (2x2) time varying matrix. The solution of (3.3) is determined by the initial state $\underline{v}(0)$ and the fundamental solution matrix $\underline{V}$ which satisfies

$$\dot{\underline{V}} = \underline{A}(t)\ \underline{V}\ ;\ \underline{V}(0) = \underline{I} \tag{3.4}$$

where $\underline{I}$ is the (2x2) identity matrix. Thus,

$$\underline{v}(t) = \underline{V}\ \underline{v}(0).$$

The analysis of equation (3.4) proceeds by showing that amplitude modulated rate distortionless FM is obtained by decomposing the coefficient matrix $\underline{A}(t)$ into the sum of a scalar matrix $\underline{A}_1(t)$ and a skew-symmetric matrix $\underline{A}_2(t)$. This decomposition is effected by the adjustment of the amplifier gain μ and another network parameter m which depends upon the modulator element values. Such a decomposition for $\underline{A}(t)$ appears to be basic to the understanding of the modulation process discussed, as the skew-symmetric matrix gives rise to the pure orthogonal frequency modulation, while the scalar matrix generates amplitude distortion. Indeed, it is known that a skew-symmetric coefficient matrix is a necessary and sufficient condition for an orthogonal fundamental solution matrix [21].

For specific parameter values (μ, m), the coefficient matrix of equation (3.4) takes the form

$$\underline{A}(t) = \underline{A}_1(t) + \underline{A}_2(t) \tag{3.5}$$

where

$$\underline{A}_1(t) = \dot{M}/M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\underline{A}_2(t) = kv_m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

The parameter values involved in the decomposition (3.5) constrain the networks in question, and thereupon enable the design of rate distortionless modulators. In particular, the values μ and m are to be chosen to give the symmetry conditions imposed upon the matrices $\underline{A}_1(t)$ and $\underline{A}_2(t)$. The transfer function procedure is to choose the amplifier gain μ to cancel the losses in the frequency selective networks. Using state variable methods this is equivalent to requiring the characteristic roots of $\underline{A}_2(t)$ to lie on the imaginary axis. The method presented herein further requires that parameter m be chosen so that $\underline{A}_2(t)$ is skew-symmetric, thus avoiding a generally difficult characteristic root calculation, or for the transfer function approach an equivalent calculation of imaginary pole locations.

The solution of equation (3.4) given in product form is

$$\underline{V} = \underline{V}_1 \underline{V}_2 \tag{3.6}$$

where

$$\dot{\underline{V}}_1 = \underline{A}_1 \, \underline{V}_1 \; , \tag{3.7}$$

and

$$\dot{\underline{V}}_2 = \underline{V}_1^{-1} \, \underline{A}_2(t) \, \underline{V}_1 \, \underline{V}_2 \; . \tag{3.8}$$

The solution of equation (3.7) is

$$\underline{V}_1(t) = M(t) \, \underline{I} \tag{3.9}$$

which commutes with $\underline{A}_2(t)$. Thus equation (3.8) becomes

$$\dot{\underline{V}}_2 = \underline{A}_2(t) \, \underline{V}_2 \tag{3.10}$$

with solution

$$\underline{V}_2(t) = \begin{pmatrix} \cos \varphi(t) & \sin \varphi(t) \\ -\sin \varphi(t) & \cos \varphi(t) \end{pmatrix} \tag{3.11}$$

where $\varphi(t) = \int k v_m \, dt + \theta$ and θ is a constant chosen to give $\underline{V}(0) = \underline{I}$.

As indicated, the matrix $\underline{V}(t)$ is the product of a scalar amplitude distortion factor $M(t)$ and an orthogonal solution matrix which represents the distortionless FM. Other pure FM solutions are congruent transformations of that presented in equation (3.11). Equation (3.6) with factors defined in (3.9) and (3.11) generalizes the scalar results of (3.1) in which the modulator output v_{fm} may be considered as a linear combination of the state-variables v_1 and v_2. For example, the Wein-bridge oscillators of Figure 2 yield

$$v_{fm} = v_2 \tag{3.12}$$

and the bridged T oscillators of Figure 3 give

$$v_{fm} = (v_1 - v_2)/(1 - \mu) \tag{3.13}$$

as output equations.

Modulator Equations

The voltage across the series branch of the Wein-bridge RC(t) modulator of Figure 2a is

$$v_1 + R_1\ i_1 = (\mu-1)\ v_{fm} = (\mu-1)\ v_2 \tag{3.14}$$

and with $i_1 = d/dt\ (C_1\ v_1)$, equation (3.14) becomes

$$\dot{v}_1 = -\ (1/R_1C_1 + \dot{C}_1/C_1)\ v_1 + (\mu-1)\ v_2/R_1C_1 \tag{3.15}$$

The Kirchoff's currently law for the parallel branch yields

$$d/dt\ (C_2\ v_2) + v_2/R_2 = i_1 \tag{3.16}$$

and with $i_1 = (\mu-1)\ v_2/R_1 - v_1/R_1$ from (3.14), (3.16) becomes

$$\dot{v}_2 = -\ v_1/R_1C_2 + [(\mu-1)\ /\ R_1C_2 - 1/R_2C_2 - \dot{C}_2/C_2]\ v_2\ . \tag{3.17}$$

With $C_1 = C_2 = C$ and $R_1 = m\ R_2 = R$, equations (3.15) and (3.17) yield the following state equations:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -1/RC - \dot{C}/C & (\mu-1)/RC \\ -1/RC & (\mu-m-1)/RC - \dot{C}/C \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} . \qquad (3.18)$$

Similarly, the Wein-bridge R(t)C modulator of Figure 2b gives for the series branch

$$\dot{v}_1 = - v_1/R_1C_1 + (\mu-1)\, v_2/R_1C_1 \qquad (3.19)$$

while the parallel branch yields

$$C_2\, dv_2/dt + v_2/R_2 = i_1 = (\mu-1)\, v_2/R_1 - v_1/R_1 \,. \qquad (3.20)$$

Rearrangement of equation (3.20) becomes

$$\dot{v}_2 = - v_1/R_1C_2 + [(\mu-1)/R_1C_2 - 1/R_2C_2]\, v_2 \,. \qquad (3.21)$$

Again with $C_1 = C_2 = C$, $R_1 = m\, R_2 = R$, equations (3.19) and (3.21) yield the following state equations:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -1/RC & (\mu-1)/RC \\ -1/RC & (\mu-m-1)/RC \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} . \qquad (3.22)$$

For $\mu = 2$ and $m = 2$ the coefficient matrices of (3.18) and (3.22) have the form of (3.5) with $kv_m = 1/RC$, and $\dot{M}/M = - 1/RC - \dot{C}/C$ for Figure 2a and $\dot{M}/M = - 1/RC$ for Figure 2b. Fundamental solution matrices in the product form of equation (3.6) follow by direct substitution.

The voltage across R_2 of the Bridged T RC(t) modulator is given as

$$R_2\, i_2 = v_1 - v_2 \,. \tag{3.23}$$

With $i_2 = d/dt\ (C_2\, v_2)$, equation (3.23) yields

$$\dot{v}_2 = v_1/R_2C_2 - (1/R_2C_2 + \dot{C}_2/C_2)\, v_2 \,. \tag{3.24}$$

The voltage across R_1 is given as

$$v_0 = \mu\, v_{fm} - v_1$$

and in terms of v_1 and v_2 is given as

$$v_0 = v_1/(\mu-1) - v_2/(\mu-1) \,. \tag{3.25}$$

But the current through R_1 yields

$$i_1 + i_2 = v_1/(\mu-1)\, R_1 - v_2/(\mu-1)\, R_1 \tag{3.26}$$

and with $i_1 = d/dt\ (C_1\, v_1)$, equation (3.26) yields

$$\dot{v}_1 = [-\dot{C}_1/C_1 + 1/(\mu-1)\, R_1C_1 - 1/R_2C_1]\, v_1$$

$$+ [1/R_2C_1 - \mu/(\mu-1)\, R_1C_1]\, v_2 \,. \tag{3.27}$$

With $m\, C_1 = C_2 = C$ and $R_1 = R_2 = R$, equations (3.24) and (3.27) become

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} m(2-\mu)/(\mu-1)RC - \dot{C}/C & -m/(\mu-1)RC \\ 1/RC & -1/RC - \dot{C}/C \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} . \tag{3.28}$$

Similarly for the Bridged R(t)C modulator of Figure 3b

$$R_2\, i_2 = v_1 - v_2 \; , \tag{3.29}$$

which with $i_2 = d/dt\, (C_2\, v_2)$ yields the following state equation:

$$\dot{v}_2 = v_1/R_2C_2 - v_2/R_2C_2 \; . \tag{3.30}$$

Similarly the expression for the current in R_1 yields the second state equation as follows:

$$\dot{v}_1 = [1/(\mu-1)R_1C_1 - 1/R_2C_1]\, v_1$$

$$+ [1/R_2C_1 - \mu/(\mu-1)R_1C_1]\, v_2 \; . \tag{3.31}$$

With $m\, C_1 = C_2 = C$ and $R_1 = R_2 = R$, equations (3.30) and (3.31) yield

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} m(2-\mu)/(\mu-1)RC & -m/(\mu-1)RC \\ 1/RC & -1/RC \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \; . \tag{3.32}$$

For $\mu = 3$ and $m = 2$, the coefficient matrices of (3.28) and (3.32) have the form of (3.5) with $kv_m = 1/RC$, and $\dot{M}/M = -1/RC - \dot{C}/C$ for Figure 3a and $\dot{M}/M = -1/RC$ for Figure 3b. Again, fundamental solution matrices in the product form of (3.6) result.

The parameter values for each modulator are calculated so that the respective coefficient matrices agree with the decomposition of (3.5). Thus, the diagonal elements of $\underline{A}(t)$ are set equal to one another, while the off-diagonal elements are equated to the negative of one another. The simultaneous solution of these two constraints yields the desired parameter values.

Proportionate time variation of the modulator energy storage elements and adjustment of the gain μ to exactly cancel the losses in the frequency selective network are sufficient conditions for the generation of rate distortionless FM. These conditions are met by the state space analysis presented, which in addition gives necessary and sufficient conditions for the generation of rate distortionless orthogonal FM.

Illustrative Example

In order to contrast the two-parameter design technique presented with that of Gardner, consider in more detail the modulator of Figure 2a. Gardner chooses $C_1 = C_2 = C$ and $R_1 = R_2 = R$ and then adjusts the parameter μ to cancel the losses in the frequency selective network. A state interpretation of this technique gives, for m = 1 in (3.18),

$$\underline{A}_1 = -\dot{C}/C \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\underline{A}_2 = 1/RC \begin{pmatrix} -1 & \mu-1 \\ -1 & \mu-2 \end{pmatrix} ,$$

where $\underline{A} = \underline{A}_1 + \underline{A}_2$. The parameter μ is then adjusted to give pure imaginary eigen values for the constant matrix RC $\underline{A}_2$. It is found that $\mu = 3$ suffices and

$$\underline{A}_2 = 1/RC \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} ,$$

$$\underline{V}(t) = C^{-1} \begin{pmatrix} \cos\varphi(t) - \sin\varphi(t) & 2\sin\varphi(t) \\ -\sin\varphi(t) & \cos\varphi(t) + \sin\varphi(t) \end{pmatrix} .$$

By comparison, the two-parameter method with $\mu = 2$ and $m = 2$ yields

$$\Psi(t) = C^{-1} e^{-\varphi(t)} \begin{pmatrix} \cos \varphi(t) & \sin \varphi(t) \\ -\sin \varphi(t) & \cos \varphi(t) \end{pmatrix}$$

as the corresponding fundamental solution matrix. Thus, the two parameter design technique leads to an elementary orthogonal pure FM matrix with an additional amplitude factor which in most cases is easily suppressed.

Distortionless FM with Gyrators

The generation of distortionless FM with gyrators under ideal conditions is investigated with the inductances modulated as in Figure 4a and with the capacitances modulated as in Figure 4b. The loop equations for Figure 4a are

$$d/dt\ (L\ i_1) + v_1 = 0 \tag{3.33}$$

and

$$d/dt\ (L\ i_2) + v_2 = 0\ . \tag{3.34}$$

The gyrator which has the property of inverting the secondary impedance has the terminal equations [22] under ideal conditions given by

$$v_1 = -R\ i_2\ ,$$

$$v_2 = R\ i_1\ .$$

So, equations (3.33) and (3.34) yield

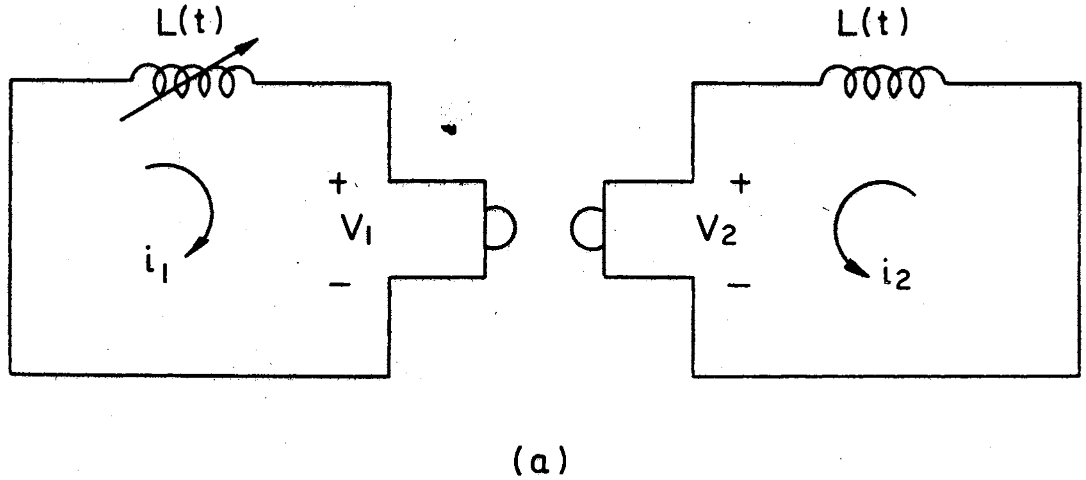


Figure 4a. Inductance Modulated Gyrator

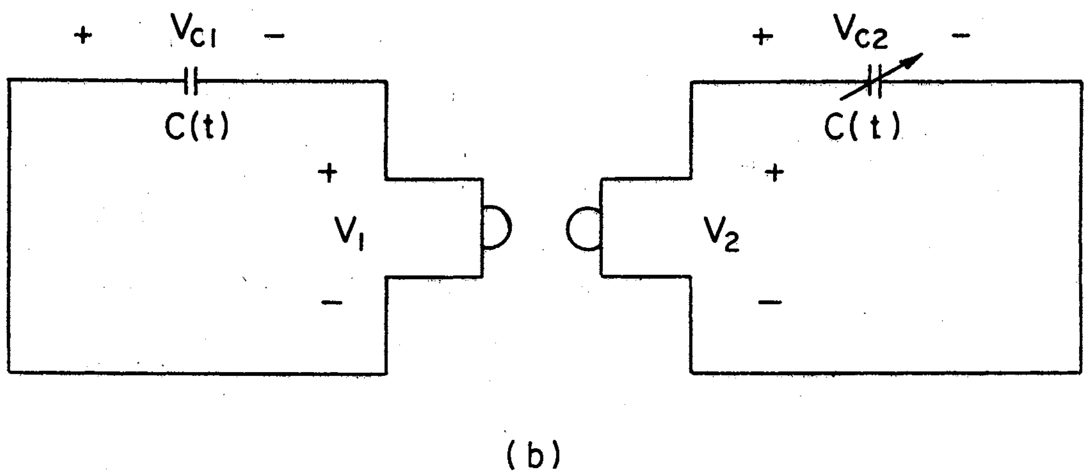


Figure 4b. Capacitance Modulated Gyrator

$$\begin{bmatrix} di_1/dt \\ di_2/dt \end{bmatrix} = \begin{bmatrix} -\dot{L}/L & R/L \\ -R/L & -\dot{L}/L \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{3.35}$$

Similarly, the state equations for Figure 4b are

$$\begin{bmatrix} \dot{v}_{c1} \\ \dot{v}_{c2} \end{bmatrix} = \begin{bmatrix} -\dot{C}/C & -1/RC \\ 1/RC & -\dot{C}/C \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} . \tag{3.36}$$

The symmetry of the coefficient matrices of equations (3.35) and (3.36) clearly prescribes the rate distortionless nature of the modulators.

The state space analysis of rate distortion as a degradation of pure FM could proceed along Lie algebraic lines. In the next chapter, the Wei-Norman basis development is made use of for the exponential representation of modulation and distortion in terms of a finite matrix product.

CHAPTER IV

STATE VARIABLE APPROACH TO MODULATION RATE DISTORTION

Introduction

For many frequency modulation applications, the undesired concurrent amplitude modulation component is of less concern than the inherent frequency distortion, as under certain conditions the amplitude modulation can be more easily removed by limiting and filtering. Of prime importance is the frequency distortion, because in most cases, it cannot be separated from the desired frequency modulated signal [20].

This chapter deals with the state variable analysis of rate distortion as a degradation of a frequency modulated signal. The Lie algebraic basis development of Wei-Norman discussed in the second chapter, is employed to represent the modulator state in terms of a finite matrix exponential product, the exponents of which are then identified as sources of distortion. The analysis begins with a decomposition of the state coefficient matrix into the sum of a primary skew-symmetric matrix and a residual matrix. Such a decomposition is believed to be fundamental to the understanding of rate distortion in view of the fact that the residual matrix, when analyzed with the aid of the Wei-Norman basis, is shown to produce rate distortion superimposed upon pure orthogonal frequency modulation as generated by the skew-symmetric matrix component.

Gardner's method for the analysis of rate distortion [7] is based essentially upon a perturbation technique whereby the harmonic content of the distorted signal is disclosed. This differs greatly from the method used in this chapter in which the rate distortion is derived as an implicit function of the instantaneous phase. While closed-form solutions for the rate distortion are not available, uniform bounds are easily computed.

With regard to specific applications, the bounds for distortion frequency are computed for the parallel antiresonant and series resonant LC networks when only one energy storage element is modulated, and the state equations for a Wein-bridge RC modulator are analyzed under the influence of loading.

State Equations

The linear differential equation which is commonly associated with the electronic generation of frequency modulation is

$$d^2x \,/\, dt^2 + \omega^2(t)\, x = 0 \tag{4.1}$$

where the variable $x(t)$ represents the frequency modulated signal and

$$\omega(t) = \omega_0 + \Delta\omega\, \omega_m(t) \tag{4.2}$$

in which $\omega(t)$ is the instantaneous frequency of the oscillator, ω_0 is the constant carrier frequency, $\Delta\omega$ is the constant maximum carrier frequency deviation, and $\omega_m(t)$ is the modulation function. It is assumed that the function $\omega_m(t)$ is at least continuously differentiable for all t and for convenience $|\omega_m(t)| \le 1$.

The differential equation (4.1) has the first order formulation [15] given by

$$\begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dot{\omega}/\omega & -\omega(t) \\ \\ \omega(t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix}, \qquad (4.3)$$

where $x_2 = x(t)$ and $x_1 = \dot{x}_2/\omega$. The state equations for the Wein-bridge RC modulator of Figure 5 are

$$\begin{bmatrix} \dot{v}_2 \\ \\ \dot{v}_1 \end{bmatrix} = \begin{bmatrix} -(\dot{C}_1/C_1 + 1/R_1C_1) - 1/RC_1 & -1/R_1C_1 \\ \\ 1/R_1C_1 & -(\dot{C}_1/C_1 + 1/R_1C_1) \end{bmatrix} \begin{bmatrix} v_2 \\ \\ v_1 \end{bmatrix} \qquad (4.4)$$

where the amplifier gain $\mu = 2$, the parameters $C_1 = C_2$ and $R_1 = 2R_2$ are chosen to give this symmetry. The resistor R represents the load placed upon this modulator.

In general, the state equations (4.3) and (4.4) can be compactly expressed as the vector-matrix differential equation

$$\dot{\underline{x}} = \underline{A}(t)\ \underline{x} \qquad (4.5)$$

where the time-dependent matrix $\underline{A}(t)$ defines the modulator element and the vector $\underline{x}$ the network variables. The solution of (4.5) is determined by the initial state $\underline{x}(0)$ and the fundamental solution matrix $\underline{X}$ which satisfies

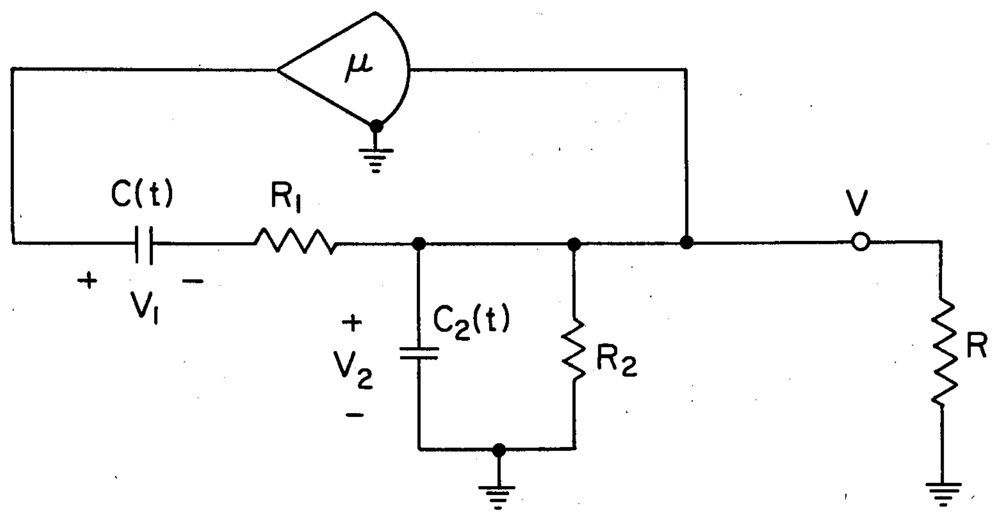


Figure 5. Wein-bridge RC(t) Modulator with Resistance Loading

$$\dot{\underline{X}} = \underline{A}(t)\ \underline{X}\ ;\ \underline{X}(0) = \underline{I} \tag{4.6}$$

where $\underline{I}$ is the identity matrix. Thus

$$x(t) = \underline{X}\ \underline{x}(0)$$

follows from (4.5) and (4.6). The method of analysis proceeds to the decomposition of $\underline{A}(t)$ into the sum of a skew-symmetric matrix $\underline{A}_s$ and a residual matrix $\underline{A}_r$:

$$\underline{A}(t) = \underline{A}_s(t) + \underline{A}_r(t) \tag{4.7}$$

where $\underline{A}_s(t)$ gives rise to the pure FM component and $\underline{A}_r(t)$ the amplitude and rate distortions.

The class of problems considered in this chapter can be expressed by

$$\underline{A}_s(t) = \begin{pmatrix} 0 & -\omega(t) \\ \omega(t) & 0 \end{pmatrix} , \ \underline{A}_r(t) = \begin{pmatrix} -\dot{\omega}/\omega & 0 \\ 0 & 0 \end{pmatrix} \tag{4.8}$$

where $\omega(t)$ represents the instantaneous modulation frequency. The multiplicative factor $(\dot{C}_1/C_1 + 1/R_1C_1)$ in the equation (4.4) is easily removed by scaling, and Wein-bridge RC modulator of (4.4) then yields $\omega(t) = 1/R_1C_1$, $-\dot{\omega}/\omega = -1/RC_1$. These points are pursued later in the chapter in more detail.

With regard to equation (4.7), the solution of (4.6) in product form is

$$\underline{X} = \underline{X}_1\ \underline{X}_2 \quad , \tag{4.9}$$

where

$$\dot{\underline{X}}_1 = \underline{A}_s(t)\ \underline{X}_1 \tag{4.10}$$

and

$$\dot{\underline{X}}_2 = \underline{X}_1^{-1}\ \underline{A}_r(t)\ \underline{X}_1\ \underline{X}_2 = \underline{B}(t)\ \underline{X}_2\ . \tag{4.11}$$

The solution of (4.10) is

$$\underline{X}_1(t) = \begin{pmatrix} \cos\Omega(t) & \sin\Omega(t) \\ -\sin\Omega(t) & \cos\Omega(t) \end{pmatrix} \tag{4.12}$$

where $\dot{\Omega} = \omega(t)$. Thus, the solution $\underline{X}_1(t)$ represents the pure FM component of the general solution $\underline{X}(t)$. The amplitude and rate distortion are defined by the solution $\underline{X}_2(t)$ of equation (4.11) with

$$\underline{B}(t) = \begin{pmatrix} -\dot{\omega}/\omega\ \cos^2\Omega & -\dot{\omega}/2\omega\ \sin 2\Omega \\ -\dot{\omega}/2\omega\ \sin 2\Omega & -\dot{\omega}/\omega\ \sin^2\Omega \end{pmatrix}, \tag{4.13}$$

as the state coefficient matrix.

Exponential Representation

In order to obtain a representation for the component $\underline{X}_2$, it is necessary to consider the Lie algebraic aspects of (4.11). Using Wei-Norman basis representation for (4.11), $\underline{B}(t)$ is written as

$$\underline{B}(t) = b_1(t)\ \underline{L}_1 + b_2(t)\ \underline{L}_2 + b_3(t)\ \underline{L}_3 + b_4(t)\ \underline{L}_4\ , \tag{4.14}$$

where $\underline{L}_1$, $\underline{L}_2$, $\underline{L}_3$, $\underline{L}_4$ are defined in the Wei-Norman theorem 2-4 given earlier. The scalar coefficients $b_i(t)'s$ are determined by $\underline{B}(t)$ matrix of equation (4.13). The basis expansion of equation (4.14) for the $\underline{B}(t)$ matrix is, of course, unique within the prescribed Lie algebra which is closed with respect to the usual commutator product $[\underline{C}, \underline{D}] = \underline{C}\,\underline{D} - \underline{D}\,\underline{C}$ for all matrices $\underline{C}$ and $\underline{D}$ in L_B.

The Wei-Norman basis development has the very distinct advantage of yielding a global exponential representation for the solution of equation (4.11):

$$\underline{X}_2(t) = \prod_{i=1}^{4} \exp g_i(t)\, \underline{L}_i \ , \qquad (4.15)$$

where the functions $g_i(t)'s$ are to be determined. If equation (4.15) is substituted into (4.11), then by equating coefficients the following system results:

$$\dot{g}_1 = b_1(t) - b_2(t)\ \sin 2g_1 + \tfrac{1}{2}\, b_3(t)\ (1-\cos 2g_1) \ ,$$

$$\dot{g}_2 = b_2(t)\ \cos 2g_1 - \tfrac{1}{2}\, b_3(t)\ \sin 2g_1 \ ,$$

$$\dot{g}_3 = e^{-2g_2}\,[2b_2(t)\ \sin 2g_1 + b_3(t)\ \cos 2g_1] \ ,$$

$$\dot{g}_4 = b_4(t) \ , \qquad (4.16)$$

where

$$b_1(t) = (\dot{\omega}/2\omega)\ \sin 2\,\Omega$$

$$b_2(t) = -(\dot{\omega}/2\omega)\cos 2\,\Omega$$

$$b_3(t) = -(\dot{\omega}/\omega)\sin 2\,\Omega$$

$$b_4(t) = -(\dot{\omega}/2\omega). \qquad (4.17)$$

Upon simplification, equation (4.16) yields

$$\dot{g}_1 = (\dot{\omega}/2\omega)\sin 2(\Omega+g_1)\ ,$$

$$\dot{g}_2 = -(\dot{\omega}/2\omega)\cos 2(\Omega+g_1)\ ,$$

$$\dot{g}_3 = -(\dot{\omega}/\omega)\, e^{-2g_2}\sin 2(\Omega+g_1)\ ,$$

$$\dot{g}_4 = (\dot{\omega}/2\omega)\ . \qquad (4.18)$$

The analysis of distortion has now led to the solution of equation (4.18). Thus the general solution of the linear system (4.11) has been replaced by a particular solution of a non-linear one. However, the advantage of this approach lies not in determining solutions of (4.18), which in any event are not readily available, but rather in the analysis that can be performed upon (4.18). Before proceeding further in the analysis, it is noted that

$$\underline{X}_2(t) = e^{g_4}\begin{bmatrix} \cos g_1 & \sin g_1 \\ -\sin g_1 & \cos g_1 \end{bmatrix}\begin{bmatrix} e^{g_2} & 0 \\ 0 & e^{-g_2} \end{bmatrix}\begin{bmatrix} 1 & g_3 \\ 0 & 1 \end{bmatrix} \qquad (4.19)$$

and

$$\underline{X}(t) = e^{g_4} \begin{bmatrix} \cos(\Omega+g_1) & \sin(\Omega+g_1) \\ -\sin(\Omega+g_1) & \cos(\Omega+g_1) \end{bmatrix} \begin{bmatrix} e^{g_2} & 0 \\ 0 & e^{-g_2} \end{bmatrix} \begin{bmatrix} 1 & g_3 \\ 0 & 1 \end{bmatrix} \qquad (4.20)$$

are the general solutions of the problem in question. From equations (3.1) and (4.20) it follows that $\dot{g}_1$ can be associated with the physical source of rate distortion which is superimposed upon the pure orthogonal FM signal (4.12). Actually, $\dot{g}_1$ represents what may be called the principal rate distortion, since the exponents g_2, g_3 also contribute to the total rate distortion.

Three comments appear to be appropriate at this point. First, the principal rate distortion $\dot{g}_1$ is given by the first equation of (4.18)

$$\dot{g}_1 = (\dot{\omega}/2\omega) \sin 2(\Omega+g_1) \qquad (4.21)$$

as an implicit function of the phase distortion. Second, given the solution of (4.21) for the phase distortion, the other distortion terms present in (4.20) are determined by simple quadratures from (4.18). Finally, it should be noted that the Wei-Norman basis development could be applied directly to the coefficient matrix $\underline{A}(t)$ of (4.5), thus bypassing the initial product decomposition (4.9) as given above. However the decomposition of (4.7) is believed to be the more intuitive approach in that it initially places in evidence the pure FM component.

The relationship between the exponent g_1 and the modulation rate distortion is of interest. From equation (4.20), the solution of (4.5) with initial conditions is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{g_4} \begin{bmatrix} (x_{10}+x_{20}\,g_3)\,e^{g_2}\cos(\Omega+g_1)+x_{20}\,e^{-g_2}\sin(\Omega+g_1) \\ -(x_{10}+x_{20}\,g_3)\,e^{g_2}\sin(\Omega+g_1)+x_{20}\,e^{-g_2}\cos(\Omega+g_1) \end{bmatrix} \tag{4.22}$$

With the substitution $\xi = (x_{10}+x_{20}\,g_3)\,e^{g_2}$, $\zeta = x_{20}\,e^{-g_2}$,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (\xi^2 + \zeta^2)^{\frac{1}{2}}\,e^{g_4(t)} \begin{bmatrix} \sin\,(\Omega+g_1+\varphi) \\ \cos\,(\Omega+g_1+\varphi) \end{bmatrix} \tag{4.23}$$

where

$$\tan\varphi = \xi/\zeta = (x_{10}/x_{20} + g_3)\,e^{2g_2}\,. \tag{4.24}$$

Thus, from equations (3.1) and (4.23) the total rate distortion is given by

$$\dot{\theta} = \dot{g}_1 + \dot{\varphi}\,. \tag{4.25}$$

The angle φ is clearly determined by g_2 and g_3 which in turn are determined by g_1. Therefore, given the solution for $g_1(t)$, the amplitude distortion and total modulation rate distortion are computed by simple quadratures from equation (4.18)

Bounds for Rate Distortion

The analysis of rate distortion has now led to the representation of the solution of the nonlinear differential equation (4.21). While closed-form solutions are not available, uniform bounds can be easily computed by observing that the principal rate distortion is bounded by

$$\left|\dot{g}_1\right| \le \left|\dot{\omega}/2\omega\right| \tag{4.26}$$

and that the total rate distortion is bounded by

$$\left|\dot{\theta}\right| \le \left|\dot{g}_1\right| + \left|\dot{\varphi}\right| . \tag{4.27}$$

Differentiation of equation (4.24) yields for $\dot{\varphi}$

$$\dot{\varphi} = 1/x_{20}\ [2(x_{10}+x_{20}\ g_3)\ e^{2g_2}\ \dot{g}_2+x_{20}\ e^{2g_2}\ \dot{g}_3]\ \cos^2\varphi \tag{4.28}$$

Noticing that $\xi = (x_{10}+x_{20}\ g_3)\ e^{g_2} = 0$ when $\cos\varphi = 1$, $\dot{\varphi}$ has its largest value given by

$$\dot{\varphi} = e^{2g_2}\ \dot{g}_3 = -(\dot{\omega}/\omega)\ \sin 2(\Omega+g_1)$$

and is therefore bounded by

$$\left|\dot{\varphi}\right| \le \left|\dot{\omega}/\omega\right| . \tag{4.29}$$

Thus, the total rate distortion from equation (4.27) has the uniform bound given by

$$\left|\dot{\theta}\right| \le 3/2\ \left|\dot{\omega}/\omega\right| . \tag{4.30}$$

For the special case of single tone modulation, it follows that $\omega_m(t)$ =sin pt, for which case $\dot{\omega}_m(\max) = p$ is the audio signal frequency.

Thus, equation (4.30) becomes

$$\left| \omega_d \right| = \left| \dot{\theta} \right| \leq 3/2 \; p \; \Delta\omega/\omega_0 = 3/2 \; mp^2/\omega_0 \tag{4.31}$$

where m is the modulation index. The above bound for ω_d is useful for narrowband FM.

From equation (4.23), it follows that amplitude distortion is given as

$$M(t) = (\xi^2+\zeta^2)^{\frac{1}{2}} \; e^{g_4}$$

$$= [(x_{10}+x_{20} \; g_3)^2 \; e^{2g_2} + \tilde{x}_{20}^2 \; e^{-2g_2}]^{\frac{1}{2}} \; e^{g_4} \tag{4.32}$$

From equation (4.32) it follows that the amplitude distortion is maximum when $\xi = \zeta$, that is, when $\partial M/\partial g_2 = 0$. Hence, the maximum amplitude distortion is given by

$$M(t) = \sqrt{2} \; x_{20} \; e^{-g_2} \; e^{g_4}. \tag{4.33}$$

Equation (4.18) gives

$$\dot{g}_4 = - \dot{\omega}/2\omega$$

which, upon integration, becomes

$$g_4 = - \tfrac{1}{2} \log_e \omega(t) = - \tfrac{1}{2} \log_e [\omega_0+\Delta\omega \; \omega_m(t)]. \tag{4.34}$$

Again equation (4.18) gives

$$\left| \dot{g}_2 \right| \le \left| \dot{g}_4 \right| . \qquad (4.35)$$

The largest value of e^{-g_2} occurs when $g_2 = g_4$, so that

$$M(t) \le \sqrt{2} \; x_{20} . \qquad (4.36)$$

Thus, the Wei-Norman representation identifies $-\dot{\omega}/\omega$ in equation (4.3) as the primary source of distortion giving rise to undesired amplitude modulation in (4.32) and rate distortion in frequency modulation in (4.30).

With regard to specific applications, the parallel LC(t) and the series LC(t) modulators of Figures 6 and 7, satisfy the FM differential equation (4.1) with $x = q$, the charge of capacitor and $\omega^2(t) = 1/LC(t)$, and the rate distortion bounds $\left| \omega_d \right| = 3/2 \left| \dot{C}/C \right|$. Similarly, the parallel L(t)C and the series L(t)C modulators of Figures 8 and 9, satisfy the FM differential equation with $x = \lambda$, the flux linkages of the inductor, $\omega^2(t) = 1/L(t)C$, and the rate distortion bounds $\left| \omega_d \right| = 3/2 \left| \dot{L}/L \right|$. Thus, rate distortion is sensitive to the type of energy storage element being deviated in a particular configuration.

Rate Distortive Effects of Loading

For the RC modulator of Figure 5, the instantaneous principal rate distortion is from (4.21)

$$\dot{g}_1 = -\left(\tfrac{1}{2}\, \epsilon\, e^{-2\varphi(t)} / R_1 C_1\right) \sin 2(\Omega + g_1) \qquad (4.37)$$

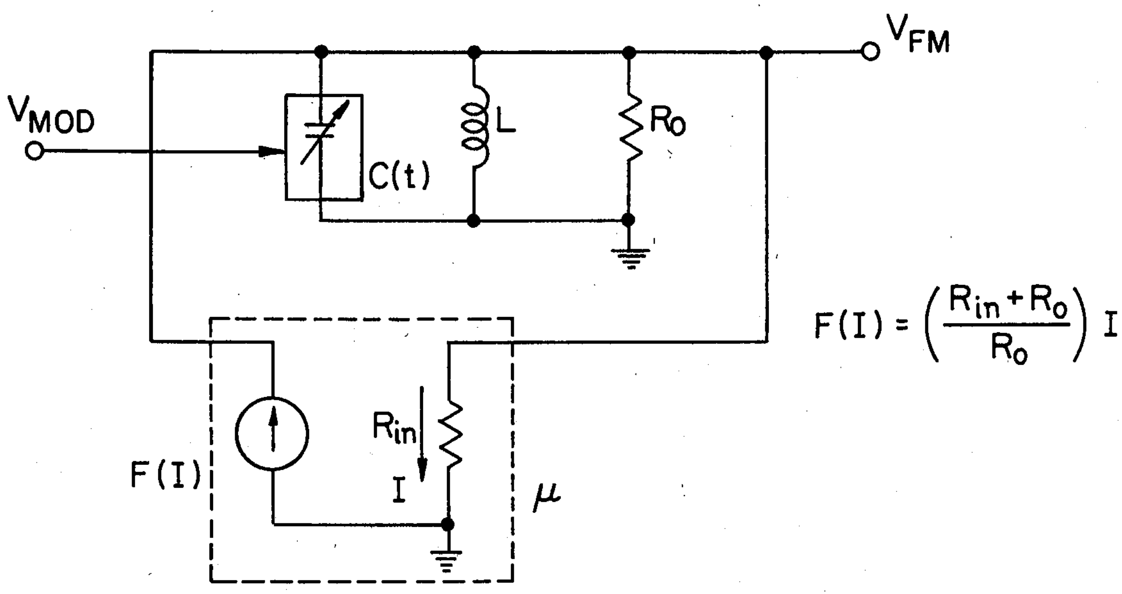


Figure 6. Parallel LC(t) Modulator

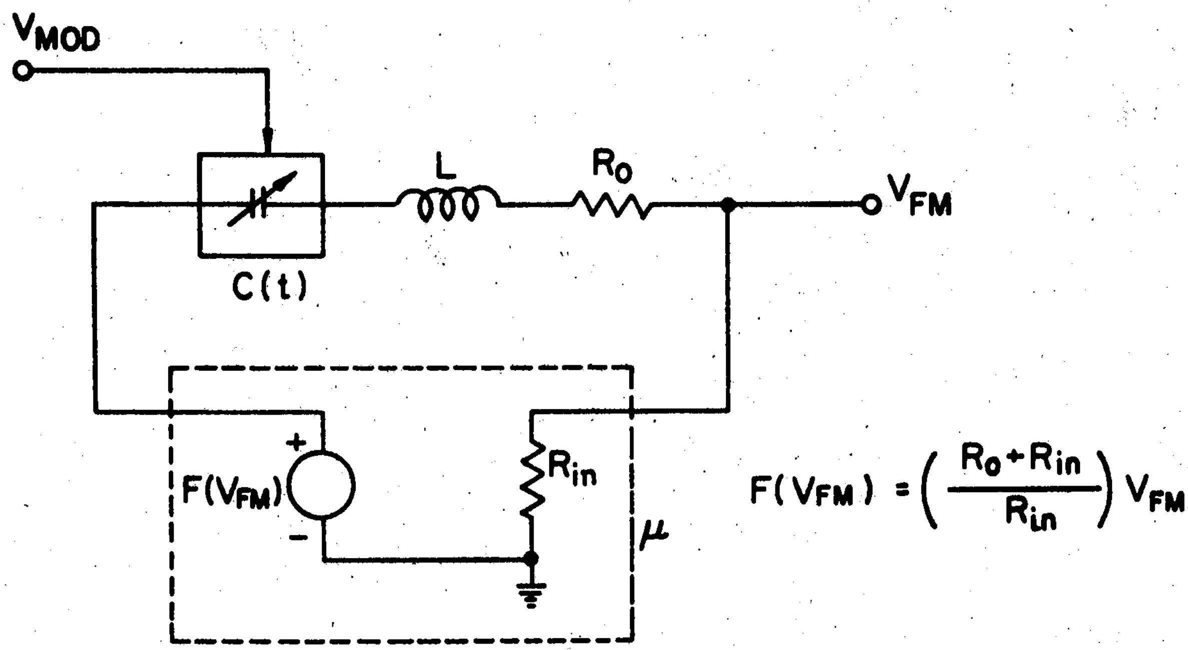


Figure 7. Series LC(t) Modulator

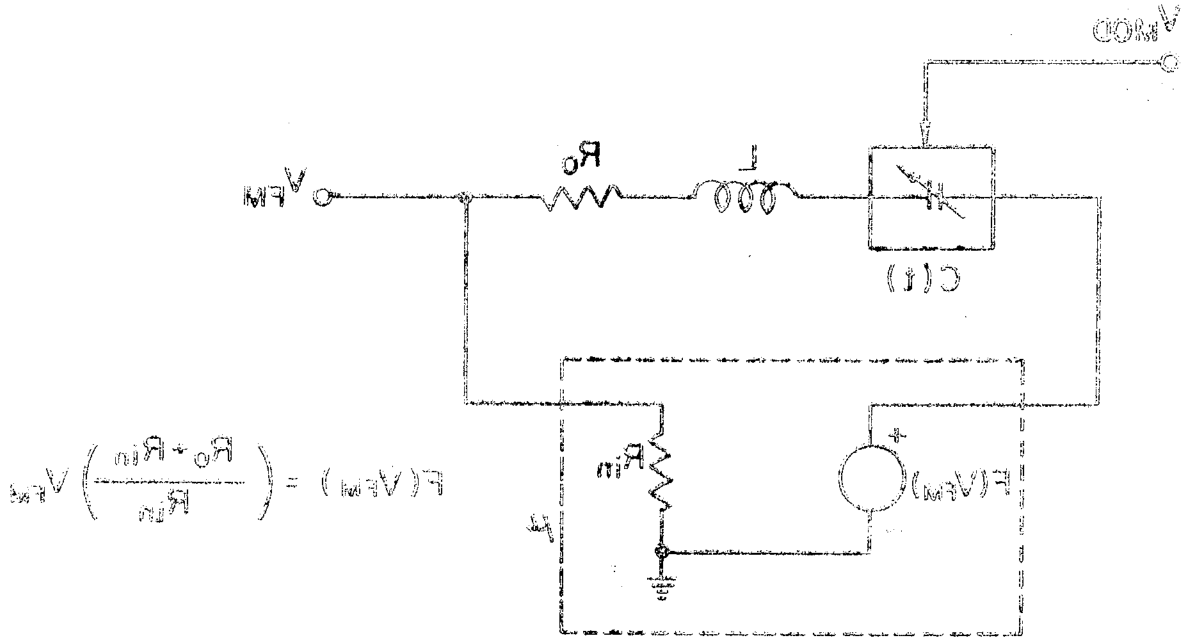


Figure 7. Series LC(t) Modulator

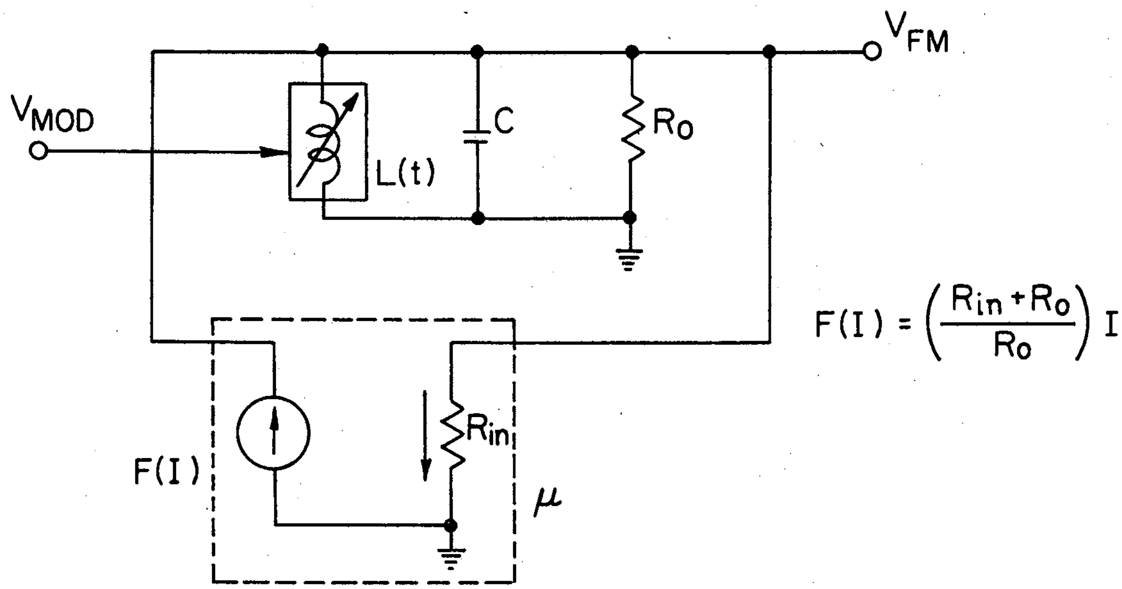


Figure 8. Parallel L(t)C Modulator

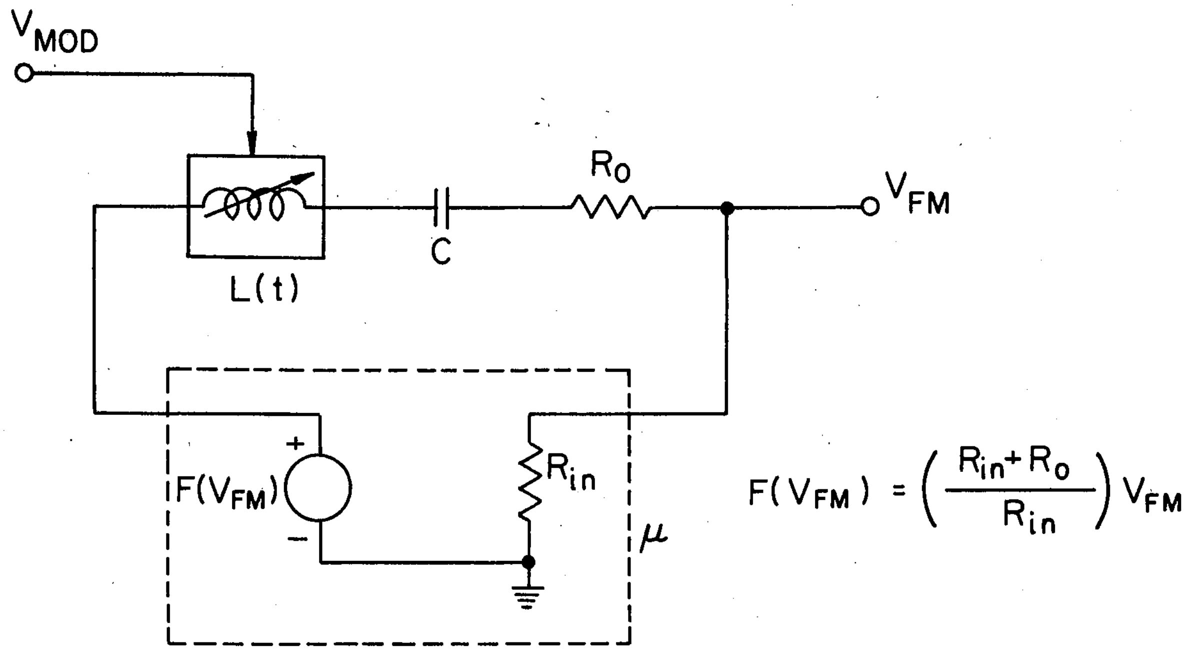


Figure 9. Series L(t)C Modulator

where the load $R = R_2/\varepsilon = R_1/2\varepsilon$. Note that $\varepsilon = 0$ corresponds to the no load condition, which in turn corresponds to the rate distortionless case previously discussed. In terms of the applied modulation,

$$1/R_1C_1 = \omega_0 + \Delta\omega\ \omega_m(t)\ , \tag{4.38}$$

and thus

$$\varphi(t) = \int_0^t (\dot{C}_1/C_1 + 1/R_1C_1)\ d\tau\ .$$

By carrying out the indicated integration

$$\varphi(t) = \log_e\ [C_1(t)/C_1(0)] + \Omega(t)\ .$$

Furthermore,

$$e^{-2\varphi(t)} = e^{-2\Omega(t)}\ C_1^2(0)/C_1^2(t)\ . \tag{4.39}$$

Taking the absolute value of (4.37) yields

$$\left|\ \dot{g}_1\ \right| \le \left|\ \varepsilon\ \frac{[\omega_0+\Delta\omega\ \omega_m(t)]^3}{[\omega_0+\Delta\omega\ \omega_m(0)]^2}\ e^{-2\Omega(t)}\right| \tag{4.40}$$

For the special case of single tone modulation with $\omega_m(t) = \sin pt$ and $\omega_m(0) = 0$. Hence,

$$e^{-2\Omega(t)} \le 1\ , \qquad \text{for all } t \ge 0\ .$$

Thus, equation (4.40) gives

$$\left| \dot{g}_1 \right| \leq \epsilon\, \omega_0\, (1+\Delta\omega/\omega_0)^3 \tag{4.41}$$

and the bound for the total rate distortion from equation (4.30) is given as

$$\left| \omega_d \right| \leq 3/2\, \epsilon\, \omega_0\, (1+\Delta\omega/\omega_0)^3 \, . \tag{4.42}$$

It should be noted from equation (4.37) that the rate distortive effect of loading diminishes exponentially as t becomes large. Also, $\omega_d = 0$ when $\epsilon = 0$, which corresponds to the distortionless case.

It should be observed that the Wei-Norman global representation has been used to study the matrix differential equation (4.5) from the points of view of amplitude and phase distortion. In the next chapter, the fundamental solution matrix of equation (4.5) with the Wei-Norman representation, is employed to study the stability and asymptotic stability of the solutions of second-order linear time varying matrix differential equations.

CHAPTER V

STABILITY OF LINEAR TIME VARYING SYSTEMS

Stability Aspects

In the preceding chapter, the Wei-Norman representation was employed to study the solutions of second order matrix linear differential equations as a product of finite matrix exponentials. This particular representation makes it possible for the solutions to be characterized as sinusoids, with the exponents giving rise to amplitude and phase modulation terms, which are continuous functions of time. In this chapter, the fundamental solution matrix or the equivalent scalar amplitude term generated by the Wei-Norman basis, is studied closely in order to yield information about the stability of the system characterized by the matrix linear differential equation. A qualitative measure of the behaviour of the system can be obtained without actually solving the nonlinear differential equations which the scalar exponents in the Wei-Norman basis satisfy. Stability studies can be made by using Lyapunov's indirect method which dispenses with the necessity of solving the system of differential equations and which yields sufficient conditions for the stability or asymptotic stability of the equilibrium state. In fact, a necessary and sufficient conditions for the asymptotic stability of the equilibrium solution of

$$\dot{\underline{x}} = \underline{A}(t)\ \underline{x},\ \underline{X}(0) = \underline{I} \qquad (5.1)$$

is given by a theorem due to Gorbunov [23] which necessitates the existence of two quadratic forms $\underline{V} = \underline{x}^T \underline{P}\, \underline{x}$, $\underline{W} = \underline{x}^T \underline{Q}\, \underline{x}$, related by

$$\dot{\underline{P}} + \underline{A}^T \underline{P} + \underline{P}\,\underline{A} = -\underline{Q}\ , \tag{5.2}$$

which are positive definite for every finite fixed t, and of such a nature that the integral

$$\int_{t_0}^{t} \underline{V}(\tau)\, d\tau$$

increases beyond any bound as $t \to \infty$. Even if the actually solutions of (5.1) are not needed, the above theorem requires the solving of the matrix Riccati equation (5.2) which is in general difficult. In the special case of second order coefficient matrices, the Wei-Norman representation provides the sufficient conditions for the stability and asymptotic stability of the equilibrium state relatively easily, as the boundedness of the fundamental solution matrix is reduced to the study of a single scalar amplitude term. Indeed, boundedness of the fundamental solution matrix is necessary and sufficient for the equilibrium state stability [24].

Before proceeding further in the stability analysis of (5.1), the fundamental solution matrix $\underline{X}(t)$ of equation (5.1) using the Wei-Norman representation is given by

$$\underline{\phi}(t) = \underline{X}(t) = \prod_{i=1}^{4} \exp g_i(t)\, \underline{L}_i \tag{5.3}$$

with the exponents g_i's satisfying the following nonlinear differential equations:

$$\dot{g}_1 = \tfrac{1}{2}(a_{12}-a_{21}) - \tfrac{1}{2}(a_{11}-a_{22}) \sin 2g_1 - \tfrac{1}{2}(a_{12}+a_{21}) \cos 2g_1 ,$$

$$\dot{g}_2 = \tfrac{1}{2}(a_{11}-a_{22}) \cos 2g_1 - \tfrac{1}{2}(a_{12}+a_{21}) \sin 2g_1 ,$$

$$\dot{g}_3 = e^{-2g_2} [(a_{11}-a_{22}) \sin 2g_1 + (a_{12}+a_{21}) \cos 2g_1] ,$$

$$\dot{g}_4 = \tfrac{1}{2}(a_{11}+a_{22}) , \qquad (5.4)$$

where the entries a_{11}, a_{12}, a_{21}, a_{22} of $\underline{A}(t)$ are continuous functions of time, so that the continuity of $\underline{A}(t)$ assures the existence of the solutions of equation (5.4).

Upon carrying out the matrix multiplication in equation (5.3), the fundamental solution matrix becomes

$$\underline{\Phi}(t) = \begin{bmatrix} e^{g_2+g_4} \cos g_1 & (g_3\, e^{g_2+g_4} \cos g_1 + e^{-g_2+g_4} \sin g_1) \\ -e^{g_2+g_4} \sin g_1 & (-g_3\, e^{g_2+g_4} \sin g_1 + e^{-g_2+g_4} \cos g_1) \end{bmatrix}$$

(5.5)

The solutions of the vector-matrix differential equations associated with (5.1) with the initial conditions x_{10} and x_{20} is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underline{X}(t) \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \qquad (5.6)$$

which after some manipulation becomes

$$\begin{bmatrix} x_1(t) \\ \\ \dot{x}_2(t) \end{bmatrix} = (\xi^2+\zeta^2)^{\frac{1}{2}} e^{g_4} \begin{bmatrix} \sin (g_1+\theta) \\ \\ \cos (g_1+\theta) \end{bmatrix} \qquad (5.7)$$

where $\xi = (x_{10}+x_{20}\, g_3)\, e^{g_2}$, $\zeta = x_{20}\, e^{-g_2}$ and $\tan\theta = \xi/\zeta$. The amplitude of the sinusoids is

$$M(t) = \{(x_{10}+x_{20}\, g_3)^2\, e^{2(g_2+g_4)} + x_{20}^2\, e^{-2(g_2-g_4)}\}^{\frac{1}{2}} \qquad (5.8)$$

Since the nature of the amplitude of oscillation determines the stability of the system, the following two theorems, relating to stability and asymptotic stability are developed from the properties of $M(t)$.$M(t)$ being bounded is equivalent to the boundedness of the fundamental solution matrix $\underline{\Phi}$ that in turn guarantees the stability of (5.1).

THEOREM 1 : If (i) $(a_{11}+a_{22}) < 0$ and (ii) $\int_0^t |a_{11}+a_{22}|\, d\tau$ is bounded for all $t \geq 0$, then a sufficient condition for the equilibrium state of (5.1) to be stable is that $|a_{11}-a_{22}| + |a_{12}+a_{21}| \leq |a_{11}+a_{22}|$.

Proof: Equation (5.4) gives

$$|g_2| \leq \int_0^t \tfrac{1}{2} \{ |a_{11}-a_{22}| + |a_{12}+a_{21}| \}\, d\tau \qquad (5.9)$$

and

$$\left| g_4 \right| = \int_0^t \tfrac{1}{2} \left| a_{11}+a_{22} \right| d\tau, \text{ for all } t \geq 0. \tag{5.10}$$

If $\left| a_{11}-a_{22} \right| + \left| a_{12}+a_{21} \right| \leq \left| a_{11}+a_{22} \right|$ for all $t \geq 0$, then

$$\left| g_2 \right| \leq \left| g_4 \right| = -g_4, \text{ for all } t \geq 0 \tag{5.11}$$

as $g_4 < 0$ from the hypothesis of the theorem.
Equation (5.11) yields

$$g_4 \leq g_2 \leq -g_4,$$

that is,

$$g_2 + g_4 \leq 0,$$

$$g_2 - g_4 \geq 0, \text{ for all } t \geq 0. \tag{5.12}$$

Also $\left| \dot{g}_3 \right|$ from equation (5.4) is bounded by

$$\left| \dot{g}_3 \right| \leq e^{-2g_4} \left\{ \left| a_{11}-a_{22} \right| + \left| a_{12}+a_{21} \right| \right\} .$$

If $\left| a_{11}-a_{22} \right| + \left| a_{12}+a_{21} \right| \leq \left| a_{11}+a_{22} \right|$, for all $t \geq 0$, then

$$\left| \dot{g}_3 \right| \leq e^{-2g_4} \left| a_{11}+a_{22} \right|$$

$$\leq e^{-2g_4} \left| 2\dot{g}_4 \right| = - 2\dot{g}_4 \, e^{-2g_4} .$$

Upon integration,

$$\left| g_3 \right| \le e^{-2g_4} < \infty, \text{ for all } t \geq 0 \tag{5.13}$$

as $g_4 < \infty$ from assumption (ii) of the hypothesis. So, equations (5.12) and (5.13) result in M(t) being bounded for all $t \geq 0$. This completes the proof.

THEOREM 2 : If $a_{11} < 0$ and $a_{22} < 0$, for all $t \geq 0$, then a sufficient condition for the equilibrium state of (5.1) to be asymptotically stable is that $\left| a_{11}-a_{22} \right| + \left| a_{12}+a_{21} \right| < \left| a_{11}+a_{22} \right|$.

Proof: Equation (5.4) gives

$$\left| g_2 \right| \le \int_0^t \tfrac{1}{2} \left\{ \left| a_{11}-a_{22} \right| + \left| a_{12}+a_{21} \right| \right\} d\tau \tag{5.14}$$

and

$$\left| g_4 \right| = \int_0^t \tfrac{1}{2} \left| a_{11}+a_{22} \right| d\tau \ . \tag{5.15}$$

If $\left| a_{11}-a_{22} \right| + \left| a_{12}+a_{21} \right| < \left| a_{11}+a_{22} \right|$, for all $t \geq 0$, then

$$\left| g_2 \right| < \left| g_4 \right| = - g_4 \tag{5.16}$$

as $g_4 < 0$ from the hypothesis of the theorem. Hence,

$$g_4 < g_2 < - g_4$$

and

$$g_2 + g_4 < 0,$$

$$g_2 - g_4 > 0, \text{ for all } t \geq 0. \tag{5.17}$$

Also,

$$g_2+g_4 = \int_0^t [\tfrac{1}{2}(a_{11}-a_{22})\cos 2g_1 - \tfrac{1}{2}(a_{12}+a_{21})\sin 2g_1 + \tfrac{1}{2}(a_{11}+a_{22})]\, d\tau \tag{5.18}$$

and

$$g_2-g_4 = \int_0^t [\tfrac{1}{2}(a_{11}-a_{22})\cos 2g_1 - \tfrac{1}{2}(a_{12}+a_{21})\sin 2g_1 - \tfrac{1}{2}(a_{11}+a_{22})]\, d\tau . \tag{5.19}$$

Again. if $\left| a_{11}-a_{22} \right| + \left| a_{12}+a_{21} \right| < \left| a_{11}+a_{22} \right|$, for all $t \geq 0$, then

$$g_2+g_4 < 0, \ g_2+g_4 \to -\infty \text{ as } t \to \infty \tag{5.20}$$

and

$$g_2-g_4 > 0, \ g_2-g_4 \to \infty \text{ as } t \to \infty . \tag{5.21}$$

Also,

$$\left| \dot{g}_3 \right| \leq e^{-2g_4} \{ \left| a_{11}-a_{22} \right| + \left| a_{12}+a_{21} \right| \}$$

and

$$\left| \dot{g}_3 \right| < \left| a_{11}+a_{22} \right| e^{-2g_4} \tag{5.22}$$

if $\left| a_{11}-a_{22} \right| + \left| a_{12}+a_{21} \right| < \left| a_{11}+a_{22} \right|$, for all $t \geq 0$.

Upon integration, equation (5.22) yields

$$\left| g_3 \right| < e^{-2g_4}, \text{ for all } t \geq 0 \text{ .} \tag{5.23}$$

Equation (5.23) implies that g_3 is bounded on every finite interval of t.

In order to prove that $M(t) \rightarrow 0$ as $t \rightarrow \infty$, it is required that

$$\lim_{t \rightarrow \infty} g_3 \, e^{g_2+g_4} = 0.$$

From equation (5.20), this is true for g_3 bounded for all $t \geq 0$, or for $g_3 \rightarrow 0$ as $t \rightarrow \infty$. However, for g_3 unbounded, the above limit is indeterminate. By L'Hospital's Rule,

$$\lim_{t \rightarrow \infty} g_3 \, e^{g_2+g_4} = \lim_{t \rightarrow \infty} \frac{\dot{g}_3}{-(\dot{g}_2+\dot{g}_4) \, e^{-(g_2+g_4)}} \tag{5.24}$$

Substitution from equation (5.4) reduces equation (5.24) to

$$\lim_{t \rightarrow \infty} g_3 \, e^{g_2+g_4}$$

as

$$\lim_{t \to \infty} g_3 e^{g_2+g_4} = \lim_{t \to \infty} e^{-(g_2-g_4)} *$$

$$\left[\frac{(a_{11}-a_{22})\sin 2g_1+(a_{12}+a_{21})\cos 2g_1}{-\frac{1}{2}(a_{11}-a_{22})\cos 2g_1+\frac{1}{2}(a_{12}+a_{21})\sin 2g_1-\frac{1}{2}(a_{11}+a_{22})} \right]. \qquad (5.25)$$

If $\left| a_{11}-a_{22} \right| + \left| a_{12}+a_{21} \right| < \left| a_{11}+a_{22} \right|$, for all $t \geq 0$, the denominator is always positive, while the expression in the numerator within the brackets is always bounded. Also,

$$e^{-(g_2-g_4)} \to 0 \text{ as } t \to \infty$$

from (5.21). Hence, from equation (5.25),

$$\lim_{t \to \infty} g_3 e^{g_2+g_4} = 0.$$

So, equations (5.20), (5.21), (5.23) and (5.25) result in M(t) being bounded for all finite t and in addition $M(t) \to 0$ as $t \to \infty$. Hence, M(t) is bounded, that is, stable for all $t \geq 0$ and $M(t) \to 0$ as $t \to \infty$. This completes the proof.

It should be noted that equations (5.20), (5.21) and (5.25) require that $\underline{\Phi}(t) \to 0$ as $t \to \infty$. Since $M(t) \to 0$ as $t \to \infty$ independently of the initial conditions, the system is asymptotically stable in the large (ASL).

Illustrative Examples

The following examples are presented to indicate the range of applications of the stability results proven.

Example 1 :

Consider $d^2x/dt^2 + p(t)\ x = 0.$

With $dx/dt = y$, the given differential equation is transformed to the following

$$d^2y/dt^2 - (\dot{p}/p)\ dy/dt + p(t)\ y = 0.$$

With $y_2 = y$ and $y_1 = y + \dot{y}$, the following state equation result:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \dot{p}/p + 1 & -(\dot{p}/p + p + 1) \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

For ASL,

(i) $\dot{p}/p < -1$

(ii) $\left| \dot{p}/p + 2 \right| + \left| \dot{p}/p + p \right| < \left| \dot{p}/p \right|$.

Condition (i) gives either $\dot{p} < 0$, $p(t) > 0$ or $\dot{p} > 0$, $p(t) < 0$. The case of $\dot{p} > 0$, $p(t) < 0$ violates condition (ii) as $\left| \dot{p}/p + p \right| > \left| \dot{p}/p \right|$ Hence, the conditions for ASL are from (i) and (ii):

$$p(t) > 0 \text{ and } \dot{p}(t) < 0, \text{ for all } t \geq 0.$$

With $x_1 = x$ and $\dot{x} = y$, the original differential equation yields the following state equations:

$$\dot{x} = y$$

$$\dot{y} = - p(t)\ x,\ p(t) > 0 \text{ for all } t \geq 0.$$

The Lyapunov function $V = (p\ x^2 + y^2)\ e^p$ yields

$$\dot{V} = \dot{p}\ [(p\ x^2 + y^2)\ e^p + e^p\ x^2].$$

So, $\dot{V} < 0$ if $\dot{p} < 0$. So, the sufficient conditions $p(t) > 0$, $\dot{p}(t) < 0$ for ASL are also provided by the theorem.

Example 2 :

Consider the Sturm-Liouville equation

$$d^2y/dt^2 + p(t)\ dy/dt + q(t)\ y = 0.$$

With $y_2 = y$ and $y_1 = y + \dot{y}/q^{\frac{1}{2}}$, the following state equations result to satisfy the requirements of the second theorem as to the main diagonal entries of $\underline{A}(t)$:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -(p+\dot{q}/2q - q^{\frac{1}{2}}) & (p+\dot{q}/2q - 2q^{\frac{1}{2}}) \\ q^{\frac{1}{2}} & -q^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

For ASL, the theorem requires

(i) $q^{\frac{1}{2}} > 0$

(ii) $p + \dot{q} / 2q > q^{\frac{1}{2}}$

(iii) $\left| p+\dot{q}/2q - 2q^{\frac{1}{2}} \right| + \left| p+\dot{q}/2q - q^{\frac{1}{2}} \right| < \left| p+\dot{q}/2q \right|$,

for all $t \geq 0$. Calling $p+\dot{q}/2q = m$ and observing that $m > 0$ from (ii) and (i) and $(m-q^{\frac{1}{2}}) > 0$ from (ii), condition (iii) yields either

$$+ (m-2q^{\frac{1}{2}}) + (m-q^{\frac{1}{2}}) < m$$

giving $m < 3q^{\frac{1}{2}}$, or

$$- (m-2q^{\frac{1}{2}}) + (m-q^{\frac{1}{2}}) < m$$

giving $m > q^{\frac{1}{2}}$. Hence, the conditions for ASL are

$$q(t) > 0 \;;$$

$$q^{\frac{1}{2}} < p+\dot{q}/2q < 3q^{\frac{1}{2}}, \text{ for all } t \geq 0 \,.$$

With $y = x_1$, $\dot{y} = x_2$, the Lyapunov function

$$V = x_1^2 + x_2^2 / q(t), \; q(t) > 0 \text{ yields}$$

$$\dot{V} = -(2p\,q + \dot{q})\, x_2^2 / q^2 \,.$$

In comparison, Lyapunov's method gives $q > 0$, $2p\,q + \dot{q} > 0$, for all $t \geq 0$ as sufficient conditions for ASL, which are also provided by the theorem.

Example 3 :

Consider the coefficient matrix given by

$$\underline{A}(t) = \underline{A} = \begin{bmatrix} -1 & 100 \\ 0 & -2 \end{bmatrix} .$$

The characteristic equation yields $\lambda_1 = -1$ and $\lambda_2 = -2$ as the eigenvalues which assure ASL. In contrast, $\underline{A}(t)$ matrix does not satisfy the conditions given by the theorem as

$$\left| -1 + 2 \right| + \left| 100 \right| > \left| -3 \right| .$$

So, this counter example proves that the conditions given by the theorem are not necessary, but only sufficient.

Example 4 :

Consider the coefficient matrix which apparently has poles fixed in the left half plane:

$$\underline{A}(t) = \begin{bmatrix} -1+a\cos^2 t & 1-a\sin t\cos t \\ -1-a\sin t\cos t & -1+a\sin^2 t \end{bmatrix} .$$

For ASL, the second theorem requires

(i) $-1+a\cos^2 t < 0$

(ii) $-1+a\sin^2 t < 0$

(iii) $\left| a\cos 2t \right| + \left| -a\sin 2t \right| < \left| -2+a \right|$, for all $t \geq 0$.

Conditions (i) and (ii) give $a < 1$ for ASL. The left hand side (LHS) of the inequality (iii) is the greatest when $\sin 2t = \cos 2t = 1/\sqrt{2}$. Hence,

$$\{ \left| a \right| + \left| a \right| \} 1/\sqrt{2} < \left| -2+a \right| .$$

That is,

$$\sqrt{2} \left| a \right| < \left| -2+a \right| .$$

Squaring,

$$2a^2 < a^2 - 4a + 4$$

or, on rearranging

$$a^2 + 4a - 4 < 0$$

giving either a < 0.828 or a < -4.828. For a < 0.828, the LHS of the quadratic is negative while for a < -4.828, the quadratic is positive. So, a < 0.828 yields a sufficient condition for ASL.

In comparison, the fundamental solution matrix computed bt Marcus and Yamabe [25] yields

$$\underline{\Phi}(t) = \begin{bmatrix} e^{(a-1)t} \cos t & e^{-t} \sin t \\ -e^{(a-1)t} \sin t & e^{-t} \cos t \end{bmatrix} .$$

Clarly, a < 1 is the necessary and sufficient condition for ASL.

Example 5 :

Consider

$$\underline{A}(t) = \underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $\underline{A}$ is time-invariant or the constant coefficient matrix.

For ASL, the second theorem requires

(i) $a < 0$

(ii) $d < 0$

(iii) $|a-d| + |b+c| < |a+d|$, for all $t \geq 0$.

The first two imply $(a+d) < 0$.

Upon squaring the third inequality,

$$(a-d)^2 + (b+c)^2 + 2\left| a-d \right| \left| b+c \right| < (a+d)^2$$

or

$$(b+c)^2 + 2\left| a-d \right| \left| b+c \right| < 4ad \text{ results. As } 4bc \leq (b+c)^2,$$

$$4bc+2\left| a-d \right| \left| b+c \right| \leq (b+c)^2 + 2\left| a-d \right| \left| b+c \right| < 4ad$$

giving $2\left| a-d \right| \left| b+c \right| < 4(ad-bc)$ or

$$(ad-bc) > \tfrac{1}{2}\left| a-d \right| \left| b+c \right| .$$

The conditions for ASL are

$$a < 0,\ d < 0 \text{ and}$$

$$(ad-bc) > \tfrac{1}{2}\left| a-d \right| \left| b+c \right| .$$

In contrast, the characteristic equation

$$\lambda^2 - (a+d)\ \lambda + (ad-bc) = 0$$

yields $(a+d) < 0$ and $(ad-bc) > 0$ as necessary and sufficient conditions for ASL. Again the second theorem gives only the sufficient conditions for the stability of the equilibrium state.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

Based upon the mathematical methods of Lie algebras, this thesis has considered the theory and applications of exponential representations for the solutions of linear state equations. This research has led to the following results and investigations:

1. With special reference to (2x2) continuous real matrices, three new algebraic bases were discovered for the global exponential representation of the solutions of linear state equations as a finite product of matrix exponentials. A preliminary investigation into the applicability of these new bases has been conducted.

2. The state-variable analysis of a class of rate distortionless frequency modulators was discussed. A two-parameter design technique was developed to reduce the state coefficient matrix to a fundamental formulation which yields an orthogonal pure FM solution matrix and a scalar AM distortion factor. This technique gives only sufficient conditions for the generation of rate distortionless FM, and necessary and sufficient conditions if the pure FM component is further constrained to be represented by an orthogonal solution matrix. An application to the generation of FM with gyrators under ideal conditions was discussed

along with two classes of RC network modulators.

3. The state variable analysis of the FM differential equation in relation to modulation rate distortion as a degradation of pure FM signals was discussed by Lie algebraic methods. Use was made of the Wei-Norman basis development for the exponential representation of the modulation and distortion in terms of a finite matrix product with the associated exponents identified as the sources of distortion. The distinguishing feature of this analysis is that the exponent g_1, associated with the principal phase distortion, satisfies a nonlinear differential equation which is decoupled from the other nonlinear equations satisfied by the other exponents of the Wei-Norman basis. Furthermore, the principal rate distortion $\dot{g}_1$ was shown to be represented as an implicit function of the relative phase distortion. Given the solution of g_1, the other distortion terms were determined by simple quadratures. The bounds for the total rate distortion and the amplitude distortion terms were found.

4. Several applications of the theory of modulation rate distortion were considered. The rate distortive effects of loading with resistance on the Wein-bridge RC(t) modulator were investigated and it was shown that the distortion has an exponential decay rate. The series resonant LC modulator was discussed with respect to single tone modulation and a useful approximation for the rate distortion frequency was derived for narrowband FM.

5. The Wei-Norman representation was extended to study the stability aspects of the second order linear time varying matrix state equa-

tions. Two theorems, one for stability and the other for asymptotic stability of the system, were given, both results provide sufficient conditions for the stability of the equilibrium state. The results were applied to several classical examples of second order linear differential equations.

Recommendations

As with all research, the inquiries made in this thesis have led to new and unanswered questions. While these questions are of prime interest and importance, they lie for the most part outside the scope of this thesis. The following areas are suggested by the author as fruitful topics for future investigation:

Linear Second Order Equations

The analytical study made in this thesis reveals certain relationships existing between the physical structure of a linear system and the mathematical structure of its underlying Lie algebraic basis. A number of physical systems obeying the classical second order linear differential equations like Bessel's, Legendre's, Hermite's, Mathieu's, Laguerre's and the hypergeometric equations can be studied using the Lie algebraic methods developed in this thesis. The physical structure represented by these classical differential equations may be then analyzed in a new light.

Periodic Linear Systems

Few explicitly defined representations are known for the general linear system described by

$$\dot{\underline{x}} = \underline{A}(t)\, \underline{x} \, .$$

One important special case, first determined by Floquet [26] is for periodic systems. The matrix $\underline{A}(t)$ is called T-periodic if there exists a scalar $T > 0$ such that

$$\underline{A}(t+T) = \underline{A}(t)$$

for all t. Linear systems described by such periodic matrices can be represented by

$$\underline{\Phi}(t) = \underline{Q}(t)\, e^{\underline{R}t}$$

where $\underline{Q}(t)$ is T-periodic and $\underline{R}$ is a constant matrix. In general $\underline{Q}$ and $\underline{R}$ are not explicitly determined. The relationship between this Floquet representation and the product exponential representation of this thesis is presently unknown. It is believed that a tie between the two representation theories would be of use in the determination of periodic solutions and the investigation of the stability of linear periodic systems. It is well known that such systems have many important engineering applications.

Higher Order Global Bases

In general, the exponential representations studied in this thesis are local, existing only in some open neighborhood of $t = 0$. Few global results are presently known and it is strongly suspected that the representation theory is basis dependent. The only global result known for a reasonably large class of problems is the Wei-Norman basis for

(2x2) real systems. An investigation of higher order basis developments is badly needed. Several interesting engineering applications await global representations. For example, nonuniform transmission linear theory is defined over (2x2) complex coefficient matrices. A global representation theory would permit an analysis of the line nonuniformity as a distortion similar to the modulation techniques employed in this thesis.

The vector space of all (3x3) skew-symmetric matrices defines a 4-dimensional Lie algebra under the usual commutator product. This system describes a body rotating in free space. A global representation theory would enable the control of such systems. Consider the very natural selection of a basis consisting of the four matrices

$$\underline{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \underline{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\underline{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \underline{A}_4 = \underline{I}$$

which describe the rotations about the three standard Cartesian axes. Wei and Norman [13] have shown that this basis is not global. This example points out very well the difficulties encountered by the global theory of exponential representations.

Stability

In general, the fundamental solution matrix represented by the finite product of matrix exponentials is valid locally, existing only in some open neighborhood of $t = 0$. With global exponential representations for higher order systems stability properties can be inferred from the fundamental solution matrix without actually solving the differential equations which the scalar exponents in the representations satisfy.

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APPENDIX

This appendix deals with the calculations necessary to establish the global bases discussed in chapter II.

Symmetric Basis: The symmetric basis generates the coefficients given in terms of the entries of the coefficient matrix as follows:

$$a_1(t) = \tfrac{1}{2}\,[a_{12}(t)-a_{21}(t)] \;,$$

$$a_2(t) = \tfrac{1}{2}\,[a_{12}(t)+a_{21}(t)] \;,$$

$$a_3(t) = \tfrac{1}{2}\,[a_{11}(t)-a_{22}(t)] \;,$$

$$a_4(t) = \tfrac{1}{2}\,[a_{11}(t)+a_{22}(t)] \;,$$

and the basis vectors yield the following product relations:

$$[\underline{L}_1\ \underline{L}_2] = 2\underline{L}_3,\ [\underline{L}_1\ \underline{L}_3] = -2\underline{L}_2,\ [\underline{L}_2\ \underline{L}_3] = 2\underline{L}_1,\ [\underline{L}_1\ \underline{L}_4] = [\underline{L}_2\ \underline{L}_4] = [\underline{L}_3\ \underline{L}_4] = \underline{0}\ .$$

By the Baker-Hausdorff formula, the equations following result:

$$e^{\text{ad } g_1\underline{L}_1}\ \underline{L}_2 = \underline{L}_2 \cos 2g_1 + \underline{L}_3 \sin 2g_1 \;,$$

$$e^{\text{ad } g_2\underline{L}_2}\,\underline{L}_3 = \underline{L}_3 \cosh 2g_2 - \underline{L}_1 \sinh 2g_2 \ ,$$

$$e^{\text{ad } g_1\underline{L}_1}\, e^{\text{ad } g_2\underline{L}_2}\,\underline{L}_3 = -\,\underline{L}_1 \sinh 2g_2 - \underline{L}_2 \sin 2g_1 \cosh 2g_2$$

$$+\,\underline{L}_3 \cos 2g_1 \cosh 2g_2 \ ,$$

$$e^{\text{ad } g_3\underline{L}_3}\,\underline{L}_4 = \underline{L}_4 \ ,$$

and

$$e^{\text{ad } g_1\underline{L}_1}\, e^{\text{ad } g_2\underline{L}_2}\, e^{\text{ad } g_3\underline{L}_3}\,\underline{L}_4 = \underline{L}_4 \ .$$

Upon susbstituting the above results into equation (2.4) of the second chapter and equating like coefficients for the unique basis vector expansion for $\underline{A}(t)$, the following equations result:

$$a_1(t) = \dot{g}_1 - \dot{g}_3 \sinh 2g_2 \ ,$$

$$a_2(t) = \dot{g}_2 \cos 2g_1 - \dot{g}_3 \sin 2g_1 \cosh 2g_2 \ ,$$

$$a_3(t) = \dot{g}_2 \sin 2g_1 + \dot{g}_3 \cos 2g_1 \cosh 2g_2 \ ,$$

$$a_4(t) = \dot{g}_4 \ .$$

Thus, from the above, the nonlinear differential equations satisfied by the g_i exponents are

$$\dot{g}_1 = a_1(t)-a_2(t)\ \sin 2g_1\ \tanh 2g_2+a_3(t)\ \cos 2g_1\ \tanh 2g_2\ ,$$

$$\dot{g}_2 = a_2(t)\ \cos 2g_1+a_3(t)\ \sin 2g_1\ ,$$

$$\dot{g}_3 = -a_2(t)\ (\sin 2g_1/\cosh 2g_2)+a_3(t)\ (\cos 2g_1/\cosh 2g_2)\ ,$$

$$\dot{g}_4 = a_4(t)\ .$$

Wei-Norman 2 Basis: The coefficient matrix generates

$$a_1(t) = -\ a_{21}(t)\ ,$$

$$a_2(t) = a_{22}(t)\ ,$$

$$a_3(t) = a_{12}(t)+a_{21}(t)\ ,$$

$$a_4(t) = a_{11}(t)\ ,$$

and the basis vectors yield

$$[\underline{L}_1\ \underline{L}_2] = 2\underline{L}_3 - \underline{L}_1,\ [\underline{L}_1\ \underline{L}_3] = \underline{L}_4 - \underline{L}_2,\ [\underline{L}_1\ \underline{L}_4] = \underline{L}_1 - 2\underline{L}_3,\ [\underline{L}_2\ \underline{L}_3]$$

$$= -\ \underline{L}_3,\ [\underline{L}_2\ \underline{L}_4] = \underline{0},\ [\underline{L}_3\ \underline{L}_4] = -\ \underline{L}_3\ .$$

The Baker-Hausdorff formula gives

$$e^{\mathrm{ad}\ g_1\underline{L}_1}\ \underline{L}_2 = -\tfrac{1}{2}\ \underline{L}_1\ \sin 2g_1 + \tfrac{1}{2}\ \underline{L}_2\ (1+\cos 2g_1) + \underline{L}_3\ \sin 2g_1$$

$$+ \tfrac{1}{2} \underline{L}_4 (1-\cos 2g_1) ,$$

$$e^{\text{ad } g_2\underline{L}_2} \underline{L}_3 = \underline{L}_3 e^{-g_2} ,$$

$$e^{\text{ad } g_1\underline{L}_1} \underline{L}_3 = \tfrac{1}{2} \underline{L}_1 (1-\cos 2g_1) - \tfrac{1}{2} \underline{L}_2 \sin 2g_1 + \underline{L}_3 \cos 2g_1$$

$$+ \tfrac{1}{2} \underline{L}_4 \sin 2g_1 ,$$

$$e^{\text{ad } g_1\underline{L}_1} e^{\text{ad } g_2\underline{L}_2} \underline{L}_3 = e^{-g_2} [\tfrac{1}{2} \underline{L}_1 (1-\cos 2g_1) - \tfrac{1}{2} \underline{L}_2 \sin 2g_1$$

$$+ \underline{L}_3 \cos 2g_1 + \tfrac{1}{2} \underline{L}_4 \sin 2g_1] ,$$

$$e^{\text{ad } g_3\underline{L}_3} \underline{L}_4 = \underline{L}_4 - g_3 \underline{L}_3 ,$$

$$e^{\text{ad } g_2\underline{L}_2} e^{\text{ad } g_3\underline{L}_3} \underline{L}_4 = \underline{L}_4 - \underline{L}_3 g_3 e^{-g_2} ,$$

$$e^{\text{ad } g_1\underline{L}_1} e^{\text{ad } g_2\underline{L}_2} e^{\text{ad } g_3\underline{L}_3} \underline{L}_4 = [\tfrac{1}{2} \underline{L}_1 \sin 2g_1 - \underline{L}_3 \sin 2g_1$$

$$+ \tfrac{1}{2} \underline{L}_2 (1-\cos 2g_1) + \tfrac{1}{2} \underline{L}_4 (1+\cos 2g_1)] -g_3 e^{-g_2} *$$

$$[\tfrac{1}{2} \underline{L}_1 (1-\cos 2g_1) -\tfrac{1}{2} \underline{L}_2 \sin 2g_1 + \underline{L}_3 \cos 2g_1 +\tfrac{1}{2} \underline{L}_4 \sin 2g_1] .$$

Again the a_i's in terms of $\dot{g}_i$'s are given below:

$$a_1(t) = \dot{g}_1 - \tfrac{1}{2}\,\dot{g}_2 \sin 2g_1 + \tfrac{1}{2}\,\dot{g}_3\, e^{-g_2} (1-\cos 2g_1)$$

$$+ \tfrac{1}{2}\,\dot{g}_4 \left[\sin 2g_1 - g_3\, e^{-g_2} (1-\cos 2g_1)\right] ,$$

$$a_2(t) = \tfrac{1}{2}\,\dot{g}_2 (1+\cos 2g_1) - \tfrac{1}{2}\,\dot{g}_3\, e^{-g_2} \sin 2g_1$$

$$+ \tfrac{1}{2}\,\dot{g}_4 \left[(1-\cos 2g_1) + g_3\, e^{-g_2} \sin 2g_1\right] ,$$

$$a_3(t) = \dot{g}_2 \sin 2g_1 + \dot{g}_3\, e^{-g_2} \cos 2g_1$$

$$- \dot{g}_4 (\sin 2g_1 + g_3\, e^{-g_2} \cos 2g_1) ,$$

$$a_4(t) = \tfrac{1}{2}\,\dot{g}_2 (1-\cos 2g_1) + \tfrac{1}{2}\,\dot{g}_3\, e^{-g_2} \sin 2g_1$$

$$+ \tfrac{1}{2}\,\dot{g}_4 \left[(1+\cos 2g_1) - g_3\, e^{-g_2} \sin 2g_1\right] .$$

The nonlinear differential equations satisfied by the g_i exponents are given below:

$$\dot{g}_1 = a_1(t) + \tfrac{1}{2}\, a_2(t) \sin 2g_1 + \tfrac{1}{2}\, a_3(t) (1-\cos 2g_1) - \tfrac{1}{2}\, a_4(t) \sin 2g_1,$$

$$\dot{g}_2 = \tfrac{1}{2}a_2(t) (1+\cos 2g_1) + \tfrac{1}{2}a_3(t) \sin 2g_1 + \tfrac{1}{2}a_4(t) (1-\cos 2g_1) ,$$

$$\dot{g}_3 = a_2(t)\ [\tfrac{1}{2}\ g_3\ (1-\cos\ 2g_1) - e^{g_2}\ \sin\ 2g_1]$$

$$+ a_3(t)\ [\ e^{g_2}\ \cos\ 2g_1 - \tfrac{1}{2}\ g_3\ \sin\ 2g_1]$$

$$+ a_4(t)\ [\ e^{g_2}\ \sin\ 2g_1 + \tfrac{1}{2}\ g_3\ (1+\cos\ 2g_1)]\ ,$$

$$\dot{g}_4 = \tfrac{1}{2}a_2(t)\ (1-\cos\ 2g_1) - \tfrac{1}{2}a_3(t)\ \sin\ 2g_1 + \tfrac{1}{2}a_4(t)\ (1+\cos\ 2g_1)\ .$$

Wei-Norman 3 Basis: The coefficients generated by the coefficient matrix are given as

$$a_1(t) = -\ a_{21}(t)\ ,$$

$$a_2(t) = a_{22}(t)-a_{11}(t)\ ,$$

$$a_3(t) = a_{12}(t)+a_{21}(t)\ ,$$

$$a_4(t) = a_{11}(t)\ ,$$

and the basis vectors yield

$$[\underline{L}_1\ \underline{L}_2] = 2\underline{L}_3 - \underline{L}_1,\ [\underline{L}_1\ \underline{L}_3] = \underline{L}_4 - 2\underline{L}_2,\ [\underline{L}_2\ \underline{L}_3] = -\ \underline{L}_3,\ [\underline{L}_1\ \underline{L}_4] =$$

$$[\underline{L}_2\ \underline{L}_4] = [\underline{L}_3\ \underline{L}_4] = \underline{0}\ .$$

By the Baker-Hausdorff formula the following result:

$$e^{\text{ad } g_1\underline{L}_1}\, \underline{L}_2 = -\tfrac{1}{2}\underline{L}_1 \sin 2g_1 + \underline{L}_2 \cos 2g_1 + \underline{L}_3 \sin 2g_1$$

$$+ \tfrac{1}{2}\underline{L}_4 (1-\cos 2g_1) ,$$

$$e^{\text{ad } g_2\underline{L}_2}\, \underline{L}_3 = \underline{L}_3\, e^{-g_2} ,$$

$$e^{\text{ad } g_1\underline{L}_1}\, e^{\text{ad } g_2\underline{L}_2}\, \underline{L}_3 = e^{-g_2} [\tfrac{1}{2}\underline{L}_1 (1-\cos 2g_1) - \underline{L}_2 \sin 2g_1$$

$$+ \underline{L}_3 \cos 2g_1 + \tfrac{1}{2}\underline{L}_4 \sin 2g_1] ,$$

$$e^{\text{ad } g_3\underline{L}_3}\, \underline{L}_4 = \underline{L}_4 ,$$

$$e^{\text{ad } g_1\underline{L}_1}\, e^{\text{ad } g_2\underline{L}_2}\, e^{\text{ad } g_3\underline{L}_3}\, \underline{L}_4 = \underline{L}_4 .$$

Substituting the above results in equation (2.4) and equating like coefficients for the unique basis vector expansion for $\underline{A}(t)$, the following result:

$$a_1(t) = \dot{g}_1 - \tfrac{1}{2}\dot{g}_2 \sin 2g_1 + \tfrac{1}{2}\dot{g}_3\, e^{-g_2} (1-\cos 2g_1) ,$$

$$a_2(t) = \dot{g}_2 \cos 2g_1 - \dot{g}_3\, e^{-g_2} \sin 2g_1 ,$$

$$a_3(t) = \dot{g}_2 \sin 2g_1 + \dot{g}_3\, e^{-g_2} \cos 2g_1 ,$$

$$a_4(t) = \tfrac{1}{2}\,\dot{g}_2\,(1-\cos 2g_1) + \tfrac{1}{2}\,\dot{g}_3\,e^{-g_2}\sin 2g_1 + \dot{g}_4 \,.$$

On solving the above, the nonlinear equations satisfied by the g_i's result as follows:

$$\dot{g}_1 = a_1(t) + \tfrac{1}{2}\,a_2(t)\sin 2g_1 + \tfrac{1}{2}\,a_3(t)\,(1-\cos 2g_1)\,,$$

$$\dot{g}_2 = a_2(t)\cos 2g_1 + a_3(t)\sin 2g_1\,,$$

$$\dot{g}_3 = e^{g_2}\,[-\,a_2(t)\sin 2g_1 + a_3(t)\cos 2g_1]\,,$$

$$\dot{g}_4 = \tfrac{1}{2}\,a_2(t)\,(1-\cos 2g_1) - \tfrac{1}{2}\,a_3(t)\sin 2g_1 + a_4(t)\,.$$

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