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Oklahoma City, Oklahoma

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SCHEDULING PHYSICIANS IN AN OUTPATIENT CLINIC MODELED AS A TRANSIENT QUEUE

A DISSERTATION

BY

DISSERTATION COMMITTEE

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With a certain element of regret, I see this dissertation come to completion for it signals the end of frequent contact with Dr. Carl Nau, a man I have come to love. As Chairman of the University of Oklahoma Department of Environmental Health and as my advisor, he was always present with encouragement and the right advice.

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CHAPTER I

INTRODUCTION

Health services in the United States are being increasingly delivered through outpatient clinics as opposed to inpatient treatment. According to Somers and Somers (36)

> "...there can be no doubt as to the current inadequacy-quantitative and qualitative-of outpatient facilities. Although the statistical evidence is notoriously inadequate and the phenomenon has never received the professional or administrative attention it deserves, there has been a 50 per cent greater expansion in outpatient visits than inpatient admissions during the last decade (1957-1967)."

In 1966 an estimated 120 million visits were made to outpatient clinics and outpatient departments. Approximately 60 per cent of these visits were non-emergency and theoretically could be scheduled (39).

Among the reasons for the increased use of outpatient clinics is the entrance of the federal government as a third party in medical payments through programs such as Medicare (P. L. 89-97 Title XVIII) and Medicaid (P. L. 89-

97 Title XIX) and through the creation of neighborhood clinics under the sponsorship of the Office of Economic Opportunity (OEA).

Prior to Medicare and Medicaid third party payment plans such as Blue Cross did not generally pay for outpatient services. According to Snyder (35) as of 1967 all states and territories which had participated in Medicaid provided for some outpatient services. Medicaid is expected to surpass Medicare in size and spending due to its wider coverage. In New York 44 per cent of the population is eligible to participate in Medicaid.

Since 1966 the Office of Economic Opportunity offered grants to community action agencies, hospitals, medical schools, health departments, medical societies and other nonprofit agencies interested in setting up and operating neighborhood health centers for the development of comprehensive health services for the poor (31). President Johnson recommended that $60 million of the OEA budget be spent on Neighborhood Health Centers. The goal was to have 50 centers in operation by the end of 1968. In order to meet the needs of the poor alone, it is estimated that 850 health centers throughout the country would be needed. Individuals coming to these health centers would be administered on an outpatient basis.

Most everyone who has come to an outpatient clinic for health services is impressed (or more properly depressed) by the length of time spent waiting to see a physician. Dr. John Knowles (21), General Director of the Massachusetts General Hospital, is a severe critic of the present-day clinic with "the long hard bench, the four-hour wait, multiple referals, incredible discontinuity of care and various other indignities suffered in an anti-social and decadent environment." Field (10) points out that the clinic patient is, to begin with, the economically deprived patient who can least financially bear the burden of time lost from work. Worry about loss of earnings often leads to an intensification of symptoms followed by more clinic visits. Excessively long waits may discourage the patient from coming to the clinic until his disease has progressed to a more critical stage.

Most outpatient clinics are staffed with physicians who attend the clinic for only a portion of the time the clinic is open. This is especially true of the OEA Neighborhood Clinics which are open during the evening hours and partly staffed by physicians who have a private practice during the day. Thus, physicians attending outpatient clinics are being scheduled by someone (perhaps themselves). The question arises as to whether the schedule physicians follow at a clinic is particularly satisfactory from the patients' view point.

Queueing Systems

Fundamentally a queueing system consists of an in-coming flow of descrete units demanding service at some service facility. A queueing system requires an arrival process, a queue discipline, and a service mechanism. The arrival process generates customers who are served according to a queue discipline by the service mechanism. After service for a given customer is completed he departs from the system. The queueing system is completely described once the arrival process, queue discipline and service mechanism is specified.

The state of the queueing system, $n(t)$, at any time t, $t \leq 0$, is the total of the number of customers being served plus the number of customers waiting to be served. When service is completed for a customer and there are other customers waiting to be served, service will immediately commence for one of the waiting customers.

In all but the most trivial queueing models the state of the system at time t, $t > 0$, cannot be predicted with certainty but rather $n(t)$ must be treated as a random variable.

Let $P_{in}(t)$ be the probability the system is in state n at time t for $n = 0, 1, \ldots$ and $t > 0$, given the system was in state i at $t = 0$ and let

$$P_{in} = \lim_{t \to \infty} P_{in}(t)$$

If P_{in} exists for $n = 0,1,2, \ldots$ the system is said to be capable of achieving statistical equilibrium.

The set (P_{in}, n = 0,1,2, . . .) is said to be the steady state solution of the queueing system. If $\frac{dP_{in}(t)}{dt} = 0$ for all n and all $t > t_o$ the system is said to be transient. A queueing system is either transient or has achieved statistical equilibrium.

If $P_{in} = P_{jn}$ for all pairs of i, j = 0,1,2 . . . the steady state solution is said to be independent of the initial state at t = 0.

The arrival process refers to the way in which customers may become part of the queueing system. Customers may join the system individually or in bulk. To describe the arrival process the probability distribution of the times between customers joining the system must be specified. This distribution may or may not be independent of the state of the system and the length of time the arrival process has been in operation.

Queue discipline is the manner in which customers form a queue, how they behave while waiting to be served and the method by which customers are chosen to be served. Customers may be selected for service on a first-come-first serve basis or they may be randomly selected. When a customer arrives at the queue, he may balk or elect not to join the queue. After a customer has entered the system, he may renege or decide to leave the system before he is served. Some customers may be served before others regardless of their order of arrival or they are served according to their order of priority. If there is more than one server, customers may change waiting lines or jockey from one waiting line to another.

The service mechanism is specified by the number of servers and the probability distribution of the customer service time. The distribution of service times may or may not be the same for all customers and all servers.

Kendall (20) proposed the following convention for classifying queueing systems:
"arrival process/service mechanism/number of servers." He used the following set of symbols for arrival and service processes.

M--Poisson arrival process or exponentially distributed service times.

G--No assumption made about arrival process or service mechanism.

GI--The only assumption made concerning the time between arrivals or the service times is that they are independently distributed.

E_k--The time between arrivals or the service times are distributed according to an Erlang distribution with parameter k.

D--The time between arrivals or the service time is constant for each customer.

The number of servers is some positive interger.

Using this system, $D/E_3/2$ would represent a queueing system with constant arrival times, service times distributed according to an Erlang distribution with parameter k = 3 and a service mechanism with two servers.

This notation has come into general acceptance in queueing literature and will be employed here.

CHAPTER II

HISTORICAL BACKGROUND

One of the earliest reported applications of queueing theory to outpatient clinics was by Bailey (1) in 1952. Bailey, interested in reducing the time patients waited in an outpatient clinic before being seen by a physician, considered setting up an appointment system. The patients had previously been arriving simultaneously at the beginning of the clinic. Bailey used a $D/E_k/1$ model with k = 2, 3, and 4. The method of analysis was simulation and essentially a transient model was used. Bailey felt that an analytic solution was impractical.

Bailey (2) also constructed a queueing model to describe the demand for beds in a hospital and the expected wait before being admitted to a hospital. The model he used was an M/M/s model where s represented the number of beds in the hospital. In the same research he considered an outpatient clinic with a bulk service mechanism. He assumed the queueing system achieved statistical equilibrium and was interested in determining the average length of the waiting line. White and Pike (40) investigated the effect of patient punctuality on waiting time and devised an appointment system using bulk arrivals of two or three patients. The effectiveness of this type of appointment system was determined via computer simulation. Fetter and Thompson (9) constructed simulation models of a maternity suite, an outpatient clinic and a surgical pavilion. The purpose of the models was to make possible "the testing of hypothesis concerning hospital operation, design and organization". The work they reported was oriented primarily toward examining the effect that changes in patient behavior (arrival rates, lateness, elective services, etc.) have on the utilization of hospital facilities and staff.

Jackson (16), using a simulation model, for various ratios of service rates and arrival rates, investigated the relationship between patient's waiting time and the idle time of the doctor in an outpatient clinic. For the data Jackson used, it was found that the ratio between the arrival rate and the service rate could get close to 1.0 and not cause an excessively long wait for patients. His model was essentially an $D/E_4/1$ model. Jackson's results agree quite well with Bailey (2).

Sorians (37) examined appointment systems using a D/M/1 model with single arrivals and a D/M/1 model with batch arrivals of size two. Using various ratios of arrival rates and service rates, he compared the expected wait per patient and expected physician idle time for the two models. It was assumed in the analysis that statistical equilibrium had been reached.

Katz (19) described a simulation program written in FORTRAN which will simulate an outpatient clinic. The user has the option of an individual or bulk arrival process. The simulator allows more than one physician to attend the clinic but they all must attend the clinic from the time it begins until it ends, that is, the program allows no scheduling of physicians. The simulator does allow the addition of multiple service mechanisms such as laboratories and X-ray.

In all of the above models of outpatient clinics, either it was assumed statistical equilibrium was reached (2,37) or the behavior of the model was simulated rather than determined analytically (1,2,9,16,40). It is important to note, however, that outpatient clinics probably do not approach statistical equilibrium for a significant portion of the time that the clinic is operational. Clinics open each session without holdovers from the previous session. Also, if there is no limit in the size of the queue and the arrival rate exceeds the service rate, $\lim_{t \to \infty} P_{in}(t)$ does not exist. Thus, analytic models assuming equilibrium has been reached may not accurately describe the behavior of an outpatient clinic.

Simulating the outpatient clinic has the advantage of greater flexibility in the type of arrival process and service mechanism used. An additional advantage of a simulation model is that it does not assume that the clinic has achieved statistical equilibrium. However, statistical error is present in simulation models, and the reduction in the statistical error is proportion to $\sqrt{N}$, where N is the number of simulation runs made. Thus, for example, if the statistical error is to be reduced by a factor of 10 the number of simulation runs must be increased by a factor of 100.

Using simulation as an optimization tool can involve extra ordinary amounts of computer time if the set of decision variables is large. Consider a simulation run where there are only three decision variables and each decision variable is allowed to take on ten different values. The number of points to be evaluated is 1000 and each point may involve many simulations to reduce statistical error to a tolerable level. In addition, simulation may not shed a great deal of light on how the different parameters of the model interact.

The emphasis in all of the above models is on scheduling patients. A careful search of the literature indicates that there has been no work in the area of scheduling the physicians who attend outpatient clinics.

Transient Behavior of Queues

The simplest queueing model is the M/M/1 model. According to Satty (32), it's behavior for statistical equilibrium was first analyzed by A. K. Erlang in 1909. Clarke (6) in 1953 published the first transient solution of the M/M/1 model. Ledermann and Reuter (22) used spectral theory of birth-death processes and Bailey (3) used a standard generating function technique to derive the M/M/1 transient solution. The transient solution to the M/M/1 model is:

$$P_n(t) = e^{-(1+\lambda)t}[\lambda^{1/2(n-a)} I_{n-a}(2t\sqrt{\lambda}) + \lambda^{(n-a+1)/2}$$

$$I_{n+a+1}(2t\sqrt{\lambda})] + (1-\lambda)\lambda^n \sum_{r=a+n+2}^{\infty} \lambda^{-1/2r} I_r(2t\sqrt{\lambda})$$

where a) the service rate $\mu = 1.0$

b) the system is in state a at $t = 0$

c) $I_r(x)$ is a modified Bessel function of the first kind.

Karlin and McGregor (18) found the transient solution to the M/M/s model using spectral theory and Satty (32) found the transient solution using generating functions. In both cases explicit expressions for $P_n(t)$ were not found. Karlin and McGregor showed:

$$P_n(t) = \pi_n \int_0^{\infty} e^{-xt} Q_k(x) Q_n(x)\, d\psi(x)$$

where: $Q_0(x) = 1$

$$-xQ_0(x) = -\lambda Q_0(x) + \lambda Q_1(x)$$

$$-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda+\mu_n)Q_n(x) + \lambda Q_{n+1}(x)$$

and ψ is a regular measure on $0 \leq x \leq \infty$ for which the orthogonality relations

$$\int_0^\infty Q_i(x)Q_j(x)d\psi(x) = \frac{\delta_{ij}}{\pi_j} \qquad i, j = 0,1, \ldots$$

hold.

$$\pi_0 = 1$$

$$\pi_n = \frac{\lambda^n}{\mu_1 \mu_2 \cdots \mu_n}$$

and k is the state of the queue at $t = 0$.

$$\mu_n = \begin{cases} n\mu \text{ if } n \leq s \\ s\mu \text{ if } n > s \end{cases}$$

Saaty showed

$$P_n(t) = \frac{1}{2\pi i} \int_{a-i\pi}^{a+i\pi} e^{st} P_n^*(s)ds$$

where $P_n^*(s)$ is the Laplace transform of $P_n(t)$ and is obtained through the use of the generating function of $P_n(t)$. The general expression for $P_n^*(s)$ is extremely complex and involves a series of gamma functions in both the numerator and denominator.

The work of Karlin and McGregor (18) and Saaty (32) indicate that it is not feasible to obtain explicitly transient solutions of even the simple M/M/s queueing model.

Bhat (4) considered the GI/M/s transient queue but again the solution to the model is in terms of transformations whose inverse must be found through numerical methods.

Optimization of Queueing Systems

The bulk of the research in optimization of queueing systems has been directed towards optimization of systems which have achieved statistical equilibrium.

Mangelsdorf (23) investigated the optimal assignment of machines to an operator. Using an M/M/1 model and an M/D/1 model with a finite population he minimized a linear function of the cost of an idle machine and the cost of an operator. Using an M/M/s model with an infinite population he minimized a linear function of the expected queue length and the cost of an idle server. In all three models he assumed statistical equilibrium had been achieved.

Hillier (15) constructed three optimization models. The first model was essentially the same as Mangelsdorf's third model. His second model was an M/M/1 model but the population was divided into k sub-populations of equal density. The decision variable was the value of k which would minimize a linear function of the cost of service per customer, the cost of the expected customer waiting time, and the cost of customers traveling to the server. His third model was an M/M/s model and was designed to minimize a linear function of the cost of operating the service mechanism and the cost of customers waiting for service. His design parameters were the service rate and the number of servers. In all three models Hillier assumed statistical equilibrium had been achieved and he suggested finding the optimal design parameters by trial and error methods.

Brigham (5), DeCani (8), and Morse (25) set up queueing models for optimization similar to Mangelsdorf and Hillier and assumed the queueing systems had reached statistical equilibrium. The optimum solutions were found graphically or by methods of calculus.

Moder and Philips (24) considered an M/M/s model where s = f(n). There was an upper bound placed on s and the model assumed that statistical equilibrium had been achieved. The behaviour of the model was investigated but no objective function was minimized.

Jannson (17) considered both the transient and steady state solution of a D/M/1 queueing model and found an expression for the expected waiting time of an arrival for both the transient case and for the case of statistical equilibrium. The decision variables were the initial state of the system and the arrival rate. He minimized a linear function of the cost of idle servers and the expected waiting time of customers.

Yadin and Naor (42) considered an M/G/1 model. When the queue was empty service was discontinued until the queue built up to R customers. There was a cost associated with setting up and closing down the service mechanism. There was also a cost associated with the expected queue length and a negative cost associated with the savings brought about by elimination of the service facility during the time it would have been idle due to lack of customers. A linear objective function was used and the optimum value of R was found using calculus.

Optimization of many types of queueing models by classical methods is often difficult or impractical due to the complexity of the equations associated with the model. Alternative approaches to the optimal design of queueing models have been taken by Kumin (22) and Heyman (14).

Kumin proposed a design theory which made use of the transition matrix of the Markov chain associated with the queueing system. He constructed an algorithm which appeared to converge numerically to the optimal design parameters of the models he considered under conditions of statistical equilibrium.

Heyman considered the same model as that of Yadin and Naor (42) but used a dynamic programing formulation. He also considered a more general problem in that the costs of operating the queueing system were discounted over time.

It is to be noted that a careful search of the literature in queueing theory shows only one article (17) on the optimization of transient queueing models.

CHAPTER III

DEVELOPMENT OF THE MATHEMATICAL MODEL

Mathematical Background

Characteristic Values and Vectors

Consider a real n x n matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n1} & a_{n2} & & & & a_{nn} \end{bmatrix} \qquad (3.1)$$

Definition 1. A real or complex number λ is a characteristic value of an n x n matrix A if there exist some non-zero vector x such that

$$XA = \lambda X. \qquad (3.2)$$

A non-zero vector satisfying (3.2) is a characteristic vector of A.

From (3.2)

$$XA - \lambda X = 0$$

$$X(A - \lambda I) = 0$$

or $(A - \lambda I)$ is a matrix which maps a non-zero vector X into the zero vector. From the theory of homogenous linear equations this occurs if and only if

(3.3)

$$\text{Det}(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

Solving for the value of λ such that (3.3) is satisfied is equivalent to finding the roots of the nth degree polynomial

(3.4) $\quad \phi(\lambda) = -1^n\lambda^n+b_1\lambda^{n-1}+\ldots\ldots+.b_{n-1}\lambda+b_n = 0$

where the coefficients, $b_1, b_2, \ldots, b_{n-1}$, are sums of products of $\{a_{ij}\}$. Therefore, there are n numbers (real or complex and not necessarily distinct) which satisfy (3.4). Equation (3.4) is defined as the <u>characteristic polynomial</u> of A.

Once the characteristic values of A are found an independent set of characteristic vectors $\{X_i$, i=1, 2,

. . .,n} can be found (if they exist) using the definition

$$X_i A = \lambda X_i .$$

Listed below are several properties of characteristic values and characteristic vectors.

__Property 1.__ If the vector $X_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ is a characteristic vector of A associated with λ_i, the ith characteristic value of A, and if the set of vectors $(X_i, i=1,2, \ldots, n)$ exist and are independent, then the matrix $P = (x_{ij})$ is such that

$$\text{(3.5)} \qquad PAP^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_n \end{bmatrix}$$

__Property 2.__ If the matrix A is symmetric the characteristic values of A are real.

__Definition__ Two matrices A and B are similar if there exists a nonsingular matrix P such that $PAP^{-1} = B$.

__Property 3.__ If two matrices A and B are similar they have identical characteristic values.

__Property 4.__ If A is a symmetric matrix there exists a unitary matrix P such that

a) $PP^T = I$

b) $PAP^T = D$

where D is a diagonal matrix.

Solution of a System of Differential Equations

Consider a system of first order linear differential equations

(3.6) $$d\dot{X}(t) = \dot{X}(t)A$$

$$X(0) = C$$

where C is a 1 x n vector and A is an n x n matrix.

If n = 1 then

$$x(t) = e^{At}$$

(3.7) $$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

The right hand side of (3.7) also has meaning if n>1 where

$$A^k \equiv A \cdot A \cdots A \quad \text{(k factors)}$$

Then

(3.8) $$e^{At} = \sum_{k=o}^{\infty} \frac{1}{k!} A^k t^k$$

It can be shown that the infinite series of matrices converges uniformly for t in any bounded interval.

Differentiation of (3.8) term by term yields

$$\frac{de^{At}}{dt} = \sum_{k=1}^{\infty} \frac{1}{k!} A^k k t^{k-1}$$

$$= A\sum_{k=1}^{\infty} \frac{1}{(k-1)!} A^{k-1} t^{k-1}$$

$$= A\sum_{m=0}^{\infty} \frac{A^m t^m}{k!}$$

Thus using the above definitions

$$X(t) = e^{At}$$

is seen to be a solution to (3.6) for n>1. If n independent characteristic vectors of A exist, then

$$e^{At} = e^{P^{-1}DPt}$$

where P is the matrix of characteristic vectors of A and D is the diagonal matrix whose elements are the characteristic values corresponding to the characteristic vectors of P. Thus,

$$e^{At} = \sum_{k=0}^{\infty} \frac{(P^{-1}DP)^k t^k}{k!}$$

But

$$(P^{-1}DP)\cdot(P^{-1}DP) = P^{-1}D^2P$$

or in general

$$e^{At} = \sum_{k=0}^{\infty} \frac{(P^{-1}D^kP)\,t^k}{k!}$$

$$= P^{-1} \sum_{k=0}^{\infty} \frac{D^k t^k}{k!}\, P$$

where

$$D^k = \begin{bmatrix} \lambda_1^k & & & & & \\ & \lambda_2^k & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \lambda_n^k \end{bmatrix}$$

Therefore

$$X(t) = X(0)P^{-1}\text{Diag}\ (e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})P.$$

Development of the Model

Consider a multichannel queueing system with s servers. Let h_i, i=1,2, . . . be the time of the arrival of the ith customer. The ith customer will be accepted into the queue only if

$$0 \leq h_i \leq L .$$

Let t_i, i=1,2, . . .,s be the length of time the ith server is available for accepting customers.

$$0 \leq U_i \leq L$$

and

$$0 \leq t_i + U_i \leq L$$

for i=1,2, . . .,s.

Let s(t) be the number of servers accepting customers at time t where

$$s(t) = 0,1,2, \ldots,s$$

If $U_1 = L$ then $s(t) \geq 1$ for $0 \leq t \leq L$.

Let N(t) be defined as the state of the queueing system at time t and let N be defined as the maximum number of customers allowed in the system. If $n(h_i) = N$ the ith customer is not accepted into the queue.

The above queueing system is shown schematically in Figure 1.

If it is assumed that the customers arrive according to a Poisson distribution and that the time required to service a customer follows a negative exponential distribution, the following equations can be written.

$$P_{i0}(t+\Delta t) = P_{i0}(t)[1-\lambda\Delta t] + P_{i1}(t)\mu\Delta t$$

$$P_{in}(t+\Delta t) = P_{in-1}(t)[\lambda\Delta t] + P_{in}(t)[1-\lambda\Delta t][1-n\mu\Delta t] + P_{in+1}(t)[(n+1)\mu\Delta t] \qquad 0 < n < s(t)$$

$$P_{in}(t+\Delta t) = P_{in-1}(t)[\lambda\Delta t]+P_{in}(t)[1-\lambda\Delta t][1-s(t)\mu\Delta t] + P_{in+1}(t)[s(t)\mu\Delta t] \qquad s(t) \leq n < N$$

$$P_{iN}(t+\Delta t) = P_{iN-1}(t)[\lambda\Delta t] + P_{iN}(t)[1-s(t)\mu\Delta t]$$

Multiplying and rearranging terms yields:

$$P_{i0}(t+\Delta t) - P_{i0}(t) = -\lambda P_{i0}(t)\Delta t + \mu P_{i1}(t)\Delta t$$

$$P_{in}(t+\Delta t) - P_{in}(t) = \lambda P_{in-1}(t)\Delta t - [\lambda+n\mu]P_{in}(t) + (n+1)\mu P_{in+1}(t) \qquad 0 < n < s(t)$$

$$P_{in}(t+\Delta t) - P_{in}(t) = \lambda P_{in-1}(t)\Delta t-[\lambda+s(t)\mu]P_{in}(t)$$

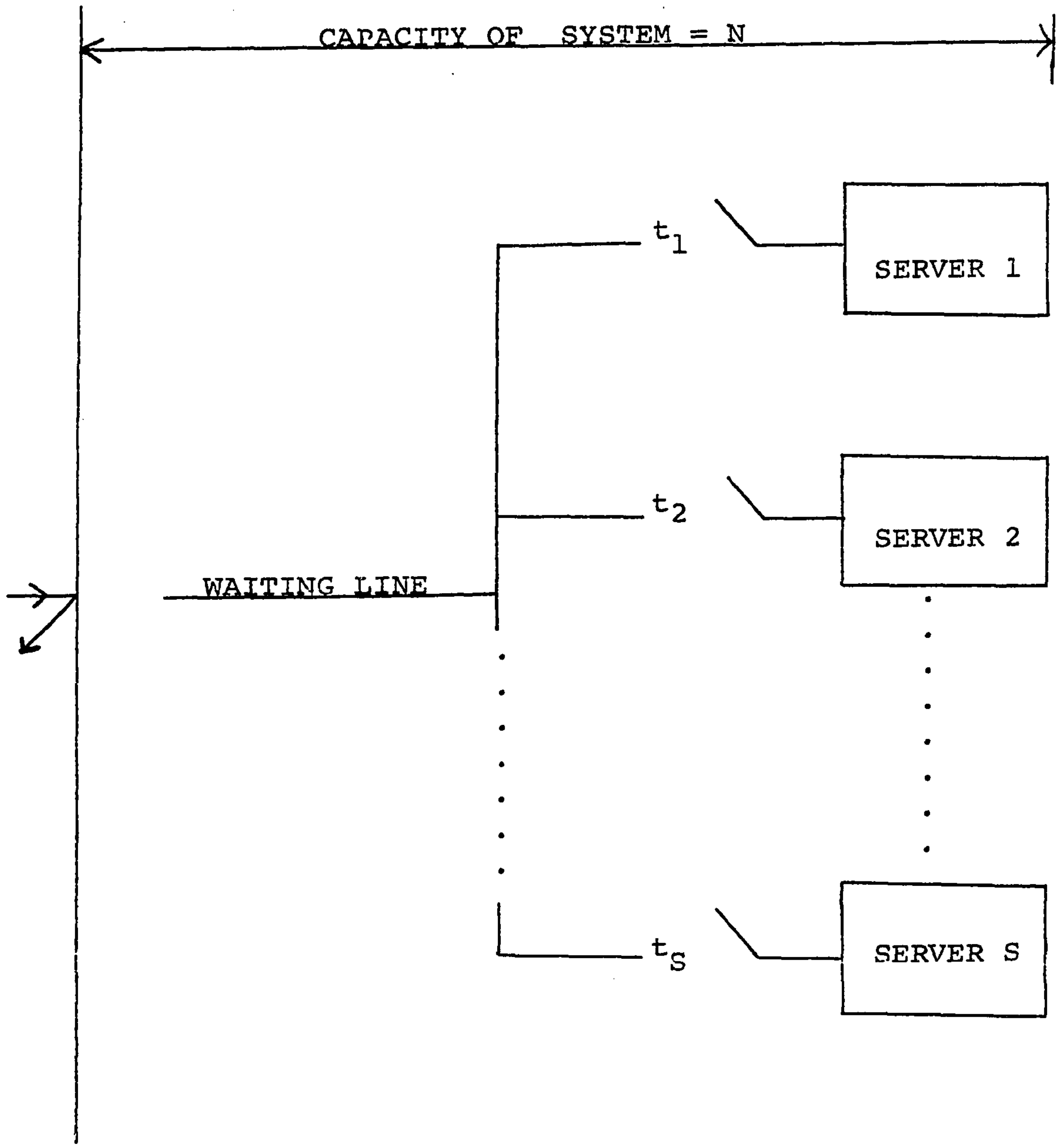

Fig. 1 Schematic of a multichannel queueing system with finite capacity and schedulable service.

$$+ s(t)\mu P_{in+1}(t) \qquad s \leq n < N$$

$$P_{iN}(t+\Delta t) - P_{iN}(t) = \lambda P_{iN-1}(t) - s(t)\mu P_{iN}(t)$$

Dividing the above equations by Δt and taking the limit as $\Delta t \to 0$ yields

$$\frac{dP_{i0}(t)}{dt} = -\lambda P_{i0}(t) + \mu P(t)$$

$$\frac{dP_{in}(t)}{dt} = \lambda P_{in-1}(t) - (\lambda+n\mu)P_{in}(t) + (n+1)\mu P_{in+1}(t)$$

$$0 < n < s(t)$$

$$\frac{dP_{in}(t)}{dt} = \lambda P_{in-1}(t) - (\lambda+s(t)\mu)P_{in}(t) + s(t)\mu P_{in}(t)$$

$$s(t) \leq n < N$$

$$\frac{dP_{iN}(t)}{dt} = \lambda P_{iN-1}(t) - s(t)\mu P_{iN}(t)$$

or by matrix notation

$$\frac{dP_{in}(t)}{dt} = P_{in}(t)A_{s(t)}$$

subject to: $P_{in}(0) = P_0$

where P_0 is a probability vector describing the state of the system at $t = 0$

and

$$A_{s(t)} = \begin{bmatrix} -\lambda & \lambda & 0 & & & & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & & & 0 \\ 0 & -2\mu & -(\lambda+2\mu) & \lambda & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & & & & & & \\ & & & s(t)\mu & -\lambda+s(t)\mu & \lambda & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & s(t)\mu & -s(t)\mu \end{bmatrix}$$

The above model describes an outpatient clinic if the physicians staffing the clinic are viewed as the servers and the patients attending the clinic are viewed as the customers. The model assumes the clinic operates as follows:

a. The ith physician is continuously assigned to the clinic from t_i to t_i+U_i, $i = 1,2, \ldots, s$.

b. The length of time a patient is seen by a physician is distributed according to a negative exponential distribution and is independent of which physician sees the patient.

c. The patients have no preference as to which physician they see.

d. The arrival pattern of patients is Poisson.

e. If a patient arrives when there are currently N patients in the clinic, he is not accepted in the clinic and does not attempt to enter the clinic at a later time.

f. The clinic does not carry patients over from the previous day.

g. There is at least one physician who is attending the clinic from the time it opens until it closes.

Criteria for Evaluation of the Effectiveness of a Given Schedule

The behavior of the model described above depends upon the following parameters:

λ--The average rate at which patients arrive at the clinic.

$\frac{1}{\mu}$--The average time a patient is seen by a physician.

N--The maximum number of patients allowed in the clinic at one time.

M--The number of doctors attending the clinic.

t_i--The time the ith physician begins seeing patients, $i = 2,3, \ldots, M$.

U_i--The length of time the ith physician sees patients, $i = 2,3, \ldots, M$.

L--The length of time the clinic is open for accepting patients.

The "schedule" of physicians is the M-1 tuple $(t_2, t_3, \ldots, t_M)$ and is represented schematically in Figure 2.

If there are M physicians who can be scheduled in the clinic then s(t) can change values K times during the

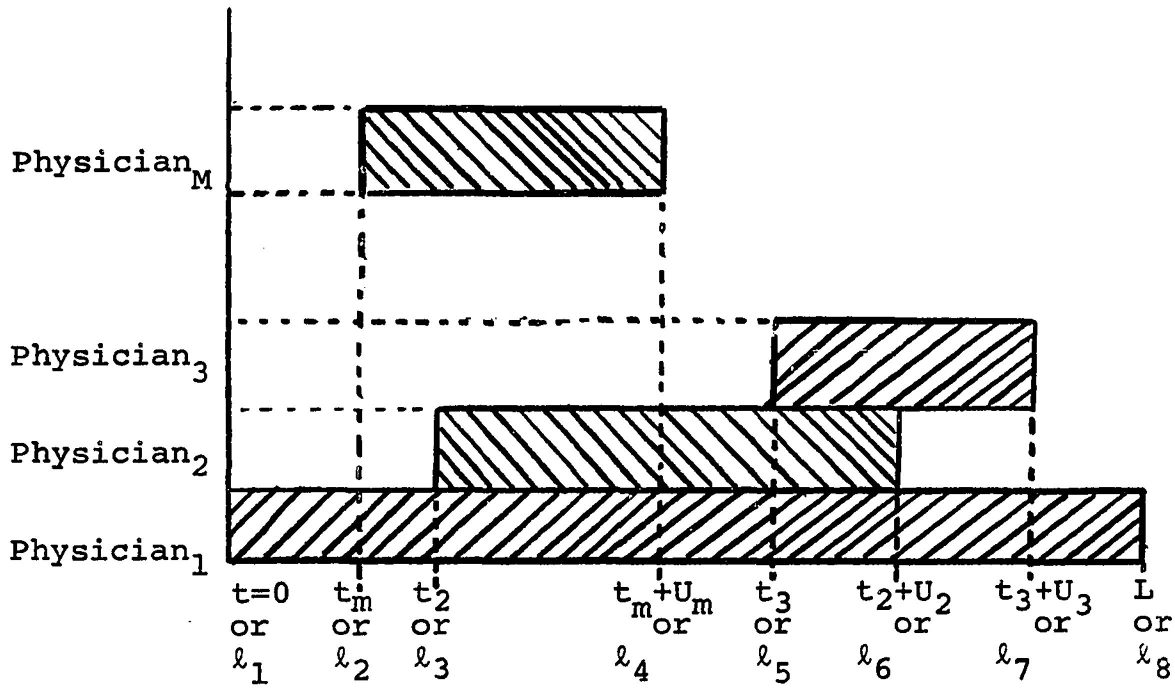

Fig. 2. Schedule of M physicians in a clinic which accepts patients between t = 0 and t = L.

interval [0,L], where K = 2M.

Let the sequence

$$L = (\ell_1, \ell_2, \ldots, \ell_K), \qquad \ell_1 \leq \ell_2 \leq \ldots \leq \ell_K$$

correspond one-to-one with the sequence

$$(0, t_2, t_3, \ldots, t_M, L, t_2+U_2, t_3+U_3, \ldots, t_M+U_M).$$

Consider the counting sequence

$$E = (e_j, j = 1,2, \ldots, k)$$

where

$$e = 1$$

$$e_j = 1 \text{ if } \ell_j \text{ corresponds to some } t_i, i = 2,3, \ldots, M$$

$e_j = -1$ if ℓ_j corresponds to some t_2+U_i

$$i = -2,3, \ldots, M$$

$e_K = -1$

and let

$$I_K = \sum_{j=1}^{K} e_j .$$

Using the essence of Kendall's (18) notation, the above model could be classified as a sequence of transient M/M/s(t) models, with a finite queue capacity where

$$s(t) = I_k, \quad k = 1,2, \ldots, K$$

for t within the interval $[\ell_k, \ell_{k+1})$

For each of the M/M/[s(t) = I_i] models there is a matrix A_{I_i}, $i = 1,2, \ldots, K$ such that the differential equations of the model may be written for $\tau \geq t-\ell_i$, as in (3.6),

$$\frac{\overline{dP_{in}(\tau)}}{dt} = P_{in}(\tau) A_{I_k}$$

subject to $\overline{P_{in}(0)} = P_k$

From this point, all quantities which are barred shall indicate a row vector.

The row vector $\overline{P_k}$ is the probability of the system being in state $0,1, \ldots, N$ at the instant the system changes from $M/M/I_{k-1}$ to $M/M/I_k$. Let the vector $\overline{P_1}$ be the probability of the system being in state $0,1, \ldots, N$ at

$t = 0$.

Let R_{I_j} be the matrix of characteristic vectors of A_{I_j} and let $\mathrm{Diag}_{I_j}[e^{Zs}]$ be a diagonal matrix with z_i corresponding with the ith characteristic vector in R.

For $0 = \ell_1 \le t \le \ell_2$

$$\overline{P_n(t)} = \overline{P_1} R_{I_1}^{-1} \mathrm{Diag}_{I_1}[e^{Zt}] R_{I_1}$$

For $\ell_2 \le t \le \ell_3$

$$\overline{P_2} = \overline{P_n(\ell_2)}$$

$$= P_1 R_{I_1}^{-1} \mathrm{Diag}_{I_i}[e^{Z\ell_1}] R_{I_1}$$

$$\overline{P_n(t)} = \overline{P_2} R_{I_2}^{-1} \mathrm{Diag}_{I_2}[e^{Z(\ell_2 - t)}] R_{I_2}$$

$$= \overline{P_1} R_{I_1}^{-1} \mathrm{Diag}_{I_1}[e^{Z\ell_1}] R_{I_1} R_{I_2}^{-1} \mathrm{Diag}[e^{Z(\ell_2 - t)}] R_{I_2}$$

And, by induction, it is easily shown for $\ell_j \le t \le \ell_{j+1}$, $j \ge 2$

$$\overline{P_n(t)} = P_1 \prod_{i=1}^{j-1} R_{I_i}^{-1} \mathrm{Diag}_{I_i}[e^{Z(\ell_{i+1} - \ell_i)}] R_{I_i}$$

$$\{R_{I_j}^{-1} \mathrm{Diag}_{I_j}[e^{Z(t - \ell_j)}] R_{I_j}\}$$

To describe the behavior of the outpatient clinic for various sets of parameters and various schedules (t_2, t_3, . . ., t_M) the following measures will be used:

$P_w(\tau_1, \tau_2)$--E (total time all patients have waited between the interval $[\tau_1, \tau_2]$)

$D_w(\tau_1, \tau_2)$--E (total time all physicians are idle between the interval $[\tau_1, \tau_2]$)

N_L --E (number of patients in the clinic at $t = L$)

λ_{eff} --E (number of patients accepted by the clinic between the interval 0, L)

For the above model if $\ell_k \leq \tau_1 < \tau_2 \leq \ell_{k+1}$

$$P_w(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} [\sum_{n=I_k}^{N} (n-I_k)P_n(t)]dt$$

Consider now the set of column vectors

$$B_j = (b_0, b_1, \ldots, b_N), \quad j = 0,1, \ldots, M$$

where

$$b_i = 0 \quad \text{for } i = 0,1, \ldots, j$$

$$b_i = i-j \quad \text{for } i = j+1, j+2, \ldots, N$$

For example, if N = 6

$$B_2 = (0,0,0,1,2,3,4)$$

By employing the set of vectors $[B_j]$, $P_w(\tau_1, \tau_2)$ can be written in matrix form rather than integral equations.

$$P_w(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \overline{P_n(t)} B_{I_k} dt$$

and from (3.8) for $\ell_k \leq \tau_1 < \tau_2 \leq \ell_{k+1}$

$$\int_{\tau_1}^{\tau_2} \overline{P_n(t)dt} = \overline{P_1} \prod_{i=1}^{k-1} [R_{I_i}^{-1} \text{Diag}_{I_i} [e^{Z(\ell_{i+1}-\ell_i)}] R_{I_i}].$$

$$R_{I_k}^{-1} \int_{\tau_1}^{\tau_2} [\text{Diag}_{I_k} [e^{Z(t-\ell_k)}] dt R_{I_K}$$

where $\prod_{i=1}^{k-1} (\cdot) = 1$ if $k = 1$

Let z_1 be the characteristic value which equals zero and let

$$V_{I_i}(S_1, S_2) = \int_{S_1}^{S_2} \text{Diag}_{I_i} [e^{z(t-\ell_j)}] dt$$

$$= \text{Diag}_{I_i} [t, \frac{e^{Z_z(t-\ell_j)}}{Z_z} \ . \ . \ . \ e^{Z_{n+1}(t-\ell_j)}] \Big|_{S_1}^{S_2}$$

$$= \text{Diag}_{I_i} [S_2 - S_1, \frac{e^{Z_2(S_2-\ell_j)} - e^{Z_2(S_1-\ell_j)}}{Z_2}, \ldots$$

$$., \frac{e^{Z_{N+1}(S_2-\ell_j)} - e^{Z_{N+1}(S_1-\ell_j)}}{Z_{N+1}}]$$

and let

$$D_{I_k}(U_2, \mu_1) = \text{Diag}_{I_k} [e^{Z_1(U_2-U_1)}, e^{Z_2(U_2-U_1)}, \ldots,$$

$$e^{Z_{N+1}(U_2-U_1)}]$$

then

$$P_w(\tau_1, \tau_2) = P_1 \prod_{i=1}^{k-1} [R_{I_i}^{-1} D_{I_k}(\ell_i, \ell_{i-1}) R_{I_i}].$$

$$R_{I_k}^{-1} D(\ell_k, \ell_{k+1}) R_{I_k} R_{I_k}^{-1} V(\tau_1, \tau_2) R_{I_k} B_{I_k}$$

If $\ell_i \leq \tau_i < \ell_{i+1} \leq \ell_n < \tau_2 \leq \ell_{m+1}$

$$P_w(\tau_1, \tau_2) = \int_{\tau_1}^{\ell_{i+1}} \sum_{n=I_i}^{N} [n-I_i] P_n(t) dt$$

$$+ \sum_{j=i+1}^{m-1} \int_{\ell_j}^{\ell_{j+1}} \sum_{n=I_j}^{N} [n-I_j] P_n(t) dt$$

$$+ \int_{\ell_m}^{\tau_2} \sum_{n=I_m}^{N} [n-I_m] P_n(t) dt$$

$$= \int_{\tau_i}^{\ell_i} \overline{P_n(t)} B_{I_i} dt$$

$$+ \sum_{j=i+1}^{m-1} \int_{\ell_j}^{\ell_{j+1}} \overline{P_n(t)} B_{I_i} dt$$

$$+ \int_{\ell_m}^{\tau_2} \overline{P_n(t)} B_{I_m} dt$$

$$= \overline{P_1} \prod_{k=1}^{i-1} [R_{I_k}^{-1} D_{I_k}(\ell_k, \ell_{k+1}) R_{I_k}].$$

$$R_{I_i}^{-1} D_{I_i}(\ell_i, \tau_i) R_{I_i} R_{I_i}^{-1} V_{I_i}(\tau_i, \ell_{L+1}) R_{I_i} B_{I_i}$$

$$+ \sum_{j=i+1}^{m-1} \overline{P_1} \prod_{k=1}^{j-1} [R_{I_k}^{-1} D_{I_k}(\ell_k, \ell_{k-1}) R_{I_k}] R_{I_j}^{-1} \cdot$$

$$V_{I_j}(\ell_j, \ell_{j+1}) R_{I_j} B_j$$

$$+ \overline{P_1} \prod_{k=1}^{m-1} [R_{I_K}^{-1} D_{I_k}(\ell_k, \ell_{k+1}) R_{I_k}] \cdot R_{I_m}^{-1} V_{I_m}(\ell_j, \ell_{j+1}) R_{I_m} B_{I_m}$$

and

$$P_w(0,L) = \overline{P_1} [R_{I_1}^{-1} V_{I_1}(0, \ell_2) R_{I_1} B_{I_1}$$

$$+ \sum_{i=2}^{k-1} [\prod_{j=1}^{L-1} [R_{I_j}^{-1} D_{I_j}(\ell_j, \ell_{j+1}) R_{I_i} V_{I_i}(\ell_i, \ell_{i+1}) R_{I_i} B_{I_i}]$$

$P_w(\tau_1, \tau_2)$ is thus expressed in terms of the state of the system at $t = 0$, the characteristic values and vectors of the different M/M/s queueing models and the "schedule" $(t_2, t_3, \ldots, t_M)$.

In a similar manner, an expression for $D_w(\tau_1, \tau_2)$ can be derived.

Let $\ell_k \leq \tau_1 < \tau_2 < \ell_{k+1}$

Then

$$D_w(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \sum (I_k - n) P_n(t) dt$$

Consider now a set of column vectors

$$C_j = (c_0, c_1, \ldots, c_N), \quad j = 1,2, \ldots, M$$

where

$$c_i = j-i \quad \text{for } i = 0,1, \ldots, j$$

$$c_i = 0 \quad \text{for } i = j+1, j+2, \ldots, N$$

For example, if N = 6

$$C_2 = (2,1,0,0,0,0)$$

By employing the set of vectors $\{C_j\}$, $D_w(\tau_1,\tau_2)$ can be written in matrix form rather than integral form.

For $\ell_k \leq \tau_1 < \tau_2 \leq \ell_{k+1}$

$$D_w(\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \overline{P_n(t)} C_{I_k} dt$$

$$= \overline{P_1} \prod_{j=1}^{k-1} [R_{I_j}^{-1} D_{I_j}(\ell_j, \ell_{j+1}) R_{I_j}] R_{I_k}^{-1} D(\ell_k \tau_1) R_{I_k} \cdot$$

$$\int_{\tau_1}^{\tau_2} R_{I_k} D_{I_k}(\tau_1,\tau_2) R_{I_k} C_{I_k} dt$$

where $\prod_{j=1}^{k-1}(.) = 1$ if $k = 1$

Letting $V_{I_i}(S_1 S_2)$ be defined as before

$$D_w(\tau_1,\tau_2) = \overline{P_1} \prod_{j=1}^{k-1} [R_{I_j}^{-1} D_{I_j}(\ell_j, \ell_{j+1}) R_{I_j}] R_{I_k}^{-1} D_{I_k}(\ell_k, \tau_1) R_{I_k} \cdot$$

$$R_{I_k}^{-1} V_{I_k}(\tau_1, \tau_2) R_{I_k}$$

Similarly for $\ell_i \leq \tau < \ell_{i+1} \leq \ell_m \leq \tau_2 \leq \ell_{m+1}$

$$D_w(\tau_1 \tau_2) = \int_{\ell i}^{\ell_{i+1}} \sum_{n=0}^{I_i} (I_i - n) P_n(t)\,dt$$

$$+ \sum_{j=i+1}^{m-1} \int_{\ell_j}^{\ell_{j+1}} \sum_{n=0}^{I_j} (I_j - n) P_n(t)\,dt$$

$$+ \int_{\ell_m}^{\tau_2} \sum_{n=0}^{I_m} (I_m - n) P_n(t)\,dt$$

$$= \int_{\tau_1}^{\ell_{i+1}} \overline{P_n(t)} C_{I_i}\,dt$$

$$+ \sum_{j=i+1}^{m-1} \int_{\ell_j}^{\ell_{j+1}} \overline{P_n(t)} C_{I_j}\,dt$$

$$+ \int_{\ell_m}^{\tau_2} \overline{P_n(t)} C_{I_m}\,dt$$

$$= \overline{P_1} \prod_{k=1}^{i-1} [R_{I_k}^{-1} D_{I_k}(\ell_k, \ell_{k+1}) R_{I_k}].$$

$$R_{I_i}^{-1} D_{I_i}(\ell_i \tau_i) R_{I_i} R_{I_i}^{-1} V(\tau_i, \ell_{i+1}) R_{I_i} C_{I_i}$$

$$+ \sum_{j=i+1}^{m-1} [\overline{P_1} \prod_{k=1}^{j=1} [R_{I_k}^{-1} D_{I_k}(\ell_k, \ell_{k+1}) R_{I_k} \cdot R_{I_j}^{-1} V_{I_j}(\ell_m, \tau_2) R_{I_j} C_{I_j}$$

$$+ \overline{P_1} \prod_{k=1}^{m-1} R_{I_k}^{-1} D_{I_k}(\ell_k, \ell_{k+1}) R_{I_m}^{-1} V_{I_m}(\ell_m, \tau_2) R_{I_m} C_{I_m}$$

$$D_W(0,L) = P_1 [R_{I_1} V_{I_1}(\ell_1, \ell_2) R_{I_1} C_{I_1}$$

$$= \sum_{i=2}^{k-1} \prod_{j=1}^{i-1} (R_{I_j}^{-1} D(\ell_j, \ell_{j+1}) R_{I_j}) R_i^{-1} V(\ell_i, \ell_{i+1}) R_i C_{I_i}$$

$D_w(\tau_1, \tau_2)$ is thus explicitly expressed as a function of the state of system at t = 0, the characteristic values and vectors of the different M/M/s queueing models and the schedule of physicians $(t_2, t_3, \ldots, t_M)$.

Similarly,

$$N_L = \sum_{n=0}^{N} n \, P_n(L)$$

$$= \overline{P_n(L)} B_0$$

$$= \overline{P_1} \prod_{i}^{k-1} [R_{I_i}^{-1} D(\ell_i, \ell_{i+1}) R_{I_i}] B_0$$

In the above model of the outpatient clinic, patients are accepted in the clinic provided there are less than N patients in the clinic. If λ is the mean rate of arrival, the expected number of arrivals during the interval [0,L] is $\int_0^L \lambda dt$. The probability that a patient arriving at time t will be accepted in the clinic is $P[n<N|t]$ and

$$P[n<N|t] = 1-P_N(t)$$

or the effective arrival rate is $\lambda[1-P_N(t)]$.

Thus

$$\lambda_{eff} = \int_0^L [1-P_N(t)]$$

$$= \sum_{i=1}^{k-1} \int_{\ell_i}^{\ell_{i+1}} \lambda(1-P_N(t)dt$$

Consider the N + 1 column vector

F = (0,0, . . .,1)

then

$$\lambda_{eff} = \lambda L-\overline{P_1}[R_{I_i}^{-1}V(0,\ell_2)R_{I_i}$$

$$+ \sum_{L=2}^{k-1} \prod_{j=1}^{L=1} [R_{I_j}^{-1}D(\ell_j,\ell_{j+1})R_{I_j}].$$

$$R_{I_i}^{-1}V(\ell_i,\ell_{i+1})R_{I_i}]F$$

"Optimally" scheduling physicians in an outpatient clinic as described above is now equivalent to:

$$\text{minimize } f(P_w, D_w, N_L, \lambda_{eff})$$

$$= f(\phi(t_2, t_3, \ldots, t_M))$$

where

$$P_w = g_1(t_2, t_3, \ldots, t_M)$$

$$D_w = g_2(t_2, t_3, \ldots, t_M)$$

$$N_L = g_3(t_2, t_3, \ldots, t_M)$$

$$\lambda_{eff} = g_4(t_2, t_3, \ldots, t_M)$$

subject to:

$$0 \leq t_i \leq L-U_i, \quad i = 2, 3, \ldots, M.$$

CHAPTER IV

NUMERICAL ANALYSIS OF THE MODEL

In order to measure the effectiveness of a schedule of physicians $(t_2, t_3, \ldots, t_M)$ in an outpatient clinic as modeled in Chapter III the characteristic values and characteristic values and characteristic vectors of the matrices

$$A_{I_k} = \begin{bmatrix} -\lambda & \lambda & 0 & \cdot & 0 \\ \mu_1 & -(\mu_1+\lambda) & \lambda & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mu_{n-2} & -(\mu_{n-2}+\lambda) & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \mu_{n-1} & -\mu_{n-1} \end{bmatrix} \quad (4.1)$$

$$\begin{aligned} \mu_i &= i\mu \quad \text{for } i = 1,2, \ldots, I_k \\ &= I_k\mu \quad \text{for } i > I_k \\ &\lambda, \mu > 0 \end{aligned}$$

must be found.

In order to locate

minimum $f(\phi(t_2, t_3, \ldots, t_M))$

subject to: $0 \leq t_i \leq L-U_i$

an analytic minimization procedure must be developed or a search for the optimum values of $(t_2, t_3, \ldots, t_M)$ must be made.

Before going on with the numerical analysis of the model, however, three theorems concerning the matrices A_{I_k} will be established.

<u>Theorem 1.</u> A tridiagonal matrix with positive off diagonal elements has real characteristic values.

Proof: Consider the tridiagonal matrix

$$B = \begin{bmatrix} a_1 & c_1 & 0 & \cdot & \cdot & \cdot & 0 \\ b_1 & a_2 & c_2 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & b_{n-2} & a_{n-1} & c_{n-1} \\ 0 & \cdot & \cdot & \cdot & \cdot & b_{n-1} & a_n \end{bmatrix}$$

From Chapter III by Property 2 and Property 3 of characteristic values and vectors if B is similar to a symmetric matrix A then B has the same characteristic values are real. Let D be the following diagonal matrix:

$$D = \begin{bmatrix} d_1 & 0 & \cdot & 0 \\ 0 & d_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & d_3 \end{bmatrix}$$

and then

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & . & 0 \\ 0 & \frac{1}{d_2} & . & 0 \\ . & . & . & . \\ . & . & . & 0 \\ 0 & 0 & 0 & \frac{1}{d_n} \end{bmatrix}$$

If B is premultiplied by D, every element in the ith row of B is multiplied by d_i. If DB is then post-multiplied by D^{-1}, every element in the ith column of DB is multiplied by $\frac{1}{d_j}$.

Let $d_i = \left[\frac{c_1 c_2 \ . \ . \ . \ c_{i-1}}{b_1 b_2 \ . \ . \ . \ b_{i-1}}\right]^{1/2}$

Then

$$DBD^{-1} = \begin{bmatrix} a_1 & (b_1c_1)^{1/2} & (b_{n-1}c_{n-1})^{1/2} & 0 \\ (b_1c_1)^{1/2} & a_2 & . & . \\ . & . & . & . \\ . & . & . & (b_{n-1}c_{n-1})^{1/2} \\ 0 & . & (b_{n-1}c_{n-1})^{1/2} & a_n \end{bmatrix} \quad (4.3)$$

Therefore B is similar to a symmetric matrix and thus has real characteristic values and vectors.

<u>Theorem 2.</u> For the matrix A_{I_k} defined in (4.1)

a) A_{I_k} has real characteristic values

b) The maximum characteristic value of A_{I_k} is 0 and is distinct.

c) The characteristic values of A_{I_k} are within the interval [2m, 0] where $m = -\inf(a_{ii},\ i = 1,2, \ldots, n)$.

Proof: A_{I_k} is a tridiagonal matrix whose off diagonal elements are positive, thus by Theorem 1, the characteristic values of A_{I_k} are real.

To prove the second and third statements of the theorem consider the matrix

$$Q = A_{I_k} + mI$$

where

$$m = -\inf(a_{ii},\quad i = 1,2, \ldots, n)$$

Q is then irreducible, tridiagonal, and non-negative.

To determine the characteristic values

$$\rho_i,\quad i = 1,2, \ldots, n$$

of Q the roots of the characteristic polynomial

$$\text{Det}\ (A_{I_k} + mI - I)$$

must be found.

Let X_i be the characteristic vector of Q associated with the characteristic value ρ_i. Then

$$X_i(A_{I_k} + mI) = \rho_i X_i$$

or

(4.4) $$X_i A_{I_k} = (\rho_i - m) X_i$$

Thus if ρ_i is a characteristic value of Q, then

(4.5) $$\zeta_i = \rho_i - m$$

is a characteristic value of A_{I_k} and if X_i is a character-istic vector of Q associated with ρ_i then X_i is also a characteristic of A_{I_k} and is associated with ζ_i.

From (4.1) it is seen that the matrix A_{I_k} is of rank n-1 or less. Therefore

$$\zeta_i = 0 \text{ for some } i = 1, 2, \ldots, n$$

For convenience, let $\zeta_1 = 0$.

Then by (4.4)

$$X_1 A_{I_k} = 0$$

$$= (\rho_1 - m) X_1$$

or

$$\rho_1 = m .$$

By employing the Perron-Frobenius Theorem (7), Q has a real distinct positive characteristic value α such that

(4.6) i) $|\rho_i| \leq \alpha$

ii) $$\alpha \leq \max_i \left[\sum_{j=1}^{n} q_{ij}\right]$$

$$\alpha \leq \max_j \left[\sum_{i=1}^{n} q_{ij}\right]$$

But

$$\sum_{j=1}^{n} q_{ij} = m \quad \text{for } i = 1,2, \ldots, n$$

Therefore $\rho_1 = m$ is the maximum characteristic value of Q and ρ_1 is distinct.

Therefore $\rho_i < m$ for $i = 2,3, \ldots, n$.

and by (4.5)

$$\zeta_i = \rho_i - m < 0$$

By (4.6)

$$-\rho_1 \leq \rho_i \leq \rho_1$$

therefore

$$-\rho_1 - m \leq \rho_i - m \leq \rho_1 - m$$

or

$$-2m \leq \zeta_i \leq 0$$

which completes the proof.

Theorem 3. If the characteristic polynomial $\phi_n(z)$ of the matrix A_{I_k} is factored quadratically as

$$\phi_n(z) = (z^2+pz+q)(z^{n-2}+b_1 z^{n-3}+ \ldots +b_{n-2})$$

then $0 \leq p \leq 4m$

$0 \leq q \leq 4m^2$

where $m = -\inf\{(a_{ii}, i = 1,2, \ldots n), -1/4\}$

Proof: By Theorem 2, z is real and

$$-2m \leq z \leq 0$$

or

$$-2m \leq z = \frac{-p \pm \sqrt{p^2-4q}}{2} \leq 0$$

and letting $D = \sqrt{p^2-4q}$

$$-2m \leq \frac{-p+D}{2} \leq 0$$

and

$$-2m \leq \frac{-p-D}{2} \leq 0$$

or

$$0 \leq p \leq 4m.$$

Since z is real

$$p^2-4q \geq 0$$

or

$$q \leq \frac{p^2}{4} \leq 4m^2.$$

Also since $z \leq 0$

$$-p + \sqrt{p^2-4q} \leq 0$$

or

$$4q \geq 0$$

thus

$$0 \leq q \leq 4m^2$$

which completes the proof.

The numerical analysis literature abounds with methods and techniques for finding the characteristic values and characteristic vectors of a real matrix. Many of these techniques, however, are for special types of matrices.

In general methods for finding the characteristic values and vectors of a matrix can be classified into two groups.

a) Methods requiring the calculation of the coefficients of the characteristic polynomial of the matrix and the subsequent location of the roots of the polynomial.

b) Transformation methods whereby the matrix is transformed into a similar triangular or diagonal matrix. Two common methods of calculating the characteristic polynomial $\phi(z)$ of a general matrix A are: Krylov's method and reduction to a Hessenberg matrix.

Consider a general matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}$$

and the polynomial

$$\phi(z) = \text{Det} \begin{vmatrix} a_{11}-z & a_{12} & \cdot & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22}-z & \cdot & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & \cdot & a_{nn}-z \end{vmatrix}$$

The method of Krylov transforms $\phi(z)$ into an equivalent polynomial of the form

$$D(z) = \text{Det} \begin{vmatrix} b_{11}-z & b_{12} & \cdot & \cdot & \cdot & \cdot & b_{1n} \\ b_{21}-z^2 & b_{22} & \cdot & \cdot & \cdot & \cdot & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1}-z^n & b_{b2} & \cdot & \cdot & \cdot & \cdot & b_{nn} \end{vmatrix}$$

The expansion of D(z) in powers of z is obviously more easily accomplished than the expansion of $\phi(z)$. The characteristic vectors of A can also be determined by the method of Krylov using the matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1n} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{nn} \end{bmatrix}$$

the coefficients of the characteristic polynomial and the characteristic values of the matrix A.

If the coefficients of the characteristic equations can be determined accurately and the characteristic polynomial factored accurately, then Krylov's method gives accurate characteristic values and vectors. Unfortunately, to determine the coefficients of the characteristic polynomial of the matrix A, Krylov's method involves solving a system of n linear simultaneous equations whose coefficients are of highly different orders of magnitude.

An alternative method for determining the coefficients of the characteristic polynomial of the matrix A is by reducing A to an upper Hessenberg matrix.

Definition: An n x n matrix $C = \{c_{ij}\}$ is upper Hessenberg if $c_{ij} = 0$ for all $j > i + 1$, $i = 1,2, \ldots, n-2$. Consider the upper Hessenberg matrix H

$$H = \begin{bmatrix} h_{11} & a_1 & 0 & \cdot & 0 \\ h_{21} & h_{22} & a_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ h_{n1} & h_{n2} & h_{n3} & \cdot & h_{nn} \end{bmatrix}.$$

Let

$$\phi_i(z) = (h_{ii}-z)\phi_{i-1}(z) + \sum_{r=1}^{i-1} (-1)h_{ir}a_r \ldots a_{i-1}\phi_{r-1}(z)$$

$$\phi_0(z) = 1$$

$$\phi_1(z) = h_{11}-z$$

It can be shown that $\phi_n(z)$ is the characteristic polynomial of the matrix H (13).

A general matrix A can be reduced to a Hessenberg matrix by applying Householder's method (13). If A is symmetric, application of the Householder's method will reduce A to a tridiagonal matrix and when H is tridiagonal

$$(4.7) \qquad \phi_i(z) = (h_{ii}-z)\phi_{n-1}(z)-h_{i,i-1}a_{i-1}\phi_{i-2}(z)$$

$$\phi_0(z) = 1$$

$$\phi_1(z) = h_{11}-z.$$

The roots of $\phi_n(z)$ theoretically can be located using any of several methods. Two of the more common and frequently used methods are the NewtonRaphson method and Bairstow's method.

The NewtonRaphson method requires an initial approximation to the root of $\phi_n(z)$. The initial root is assumed to be complex and all arithematic is complex. When the criteria for convergence is met, there will generally be a real and imaginary part to the root even if the true root is not complex. If the imaginary component of the root is extremely small, relative to the real component, the modulus of the root is used and the root is considered real.

The Bairstow method for finding the roots of a polynomial $\phi_n(z)$ is based on quadratic factoring. In this method $\phi_n(z)$ is factored by synthetic division such that

$$\phi_n(z) = (z^2+pz+q)(z^{n-2}+b_1z^{n-3}+ - +b_{n-2}) + Rz + S.$$

Values of p and q are converged upon so that

$$R = 0$$

$$S = 0.$$

Initial approximations p_0 and q_0 for p and q are necessary. The Bairstow method avoids complex arithematic and often yields results when the NewtonRaphson method fails (33).

For example, for the polynomial

$$\phi(z) = z^{10} + 63z^9 + 1668z^8 + 24108z^7 + 206760z^6 + 1071000z^5 + 3262300z^4 + 5411100z^3 + 4149500z^2 + 989520z$$

the NewtonRaphson method was used and convergence had not occured after 10 minutes on an IBM 1800 computer. The same polynomial was factored and all roots located using the Bairstow method in less than 20 seconds. For these reasons the Bairstow method was used.

The difficulty with both of the above methods of factoring a polynomial is, that they may not converge due to a poor initial approximation of the roots (for the NewtonRaphson method) or the coefficients of the quadratic factors (Bairstow method). However, by Theorem 3, the coefficients of the quadratic factor of $\phi_n(z)$ can be bound. By setting

$$p_1 = 0$$

$$p_i = p_{i-1} + \frac{4m}{n/2} \qquad \text{for } i = 2,3, \ldots, \frac{n}{z}$$

and $$q_i = \frac{p_i^2}{4}$$

the incidence of convergence and rate of convergence should be improved.

Since the matrix A_{I_k} is tridiagonal to begin with Householder's method does not need to be used to reduce the matrix to Hessenberg form. Equation (4.7) was used to find the coefficients of the characteristic polynomial of A_{I_k} and Bairstow's method was used to factor the polynomial. Krylov's method was used for determining the characteristic vectors of A_{I_k}.

As an alternative to the above method for determining the characteristic values and vectors the Jacobi method was tried. The Jacobi method is a transformation method and is valid only for symmetric matrices. However, by Theorem 1 the matrix A_{I_k} is similar to a symmetric matrix. That is,

(4.8) $$GA_{I_k}G^{-1} = B$$

where B is symmetric and is in the form of (4.3).

Thus by Property 3 of characteristic values and vectors A_{I_k} and B have the same characteristic values.

By Property 4 of characteristic values and vectors the matrix B can be diagonalized by an orthogonal unitary matrix P

or

$$PBP^T = D$$

where D is a diagonal matrix.

Thus by (4.8)

$$PGA_{I_k}G^{-1}P^T = D$$

and

$$PGG^{-1}P^T = I.$$

Therefore, if the Jacobi method is used to find the characteristic values and vectors of B, it is a trivial matter to find the characteristic vectors of A_{I_k}.

The Jacobi method systematically annihilates selected off-diagonal elements of the symmetric matrix B. For a selected off diagonal element b_{pq} consider the matrix R such that

$$r_{pp} = \cos\theta$$

$$r_{qp} = -\sin\theta$$

$$r_{qp} = \sin\theta$$

$$r_{ii} = 1 \qquad i = p,q$$

$$r_{pq} = r_{iq} = r_{ik} = 0 \qquad i \neq p,q$$
$$k \neq p,q$$

and $$\theta = 1/2\tan^{-1} \frac{A_{pq}}{1/2(A_{pp}-A_{qq})}.$$

It is seen that R is an orthogonal unitary matrix and b_{pq} is annihilated by R^TBR.

Krylov's method (modified as described above) and the Jacobi method were tried on several matrices. It was found that for small matrices Krylov's method was superior; however, for larger matrices (N > 5) the Jacobi method was superior based on the mean ratio of the error norm to the vector norm. The results for the matrix

$$A = \begin{bmatrix} -4 & 4 & 0 & . & . & 0 \\ 4 & -8 & 4 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 4 & -8 & 4 \\ 0 & 0 & . & . & 4 & -4 \end{bmatrix}$$

are shown in Table 1.

Krylov's method and Jacobi's method were compared on an IBM 1800 computer with respect to computation time and the following times were noted for an 8 x 8 tridiagonal matrix.

Krylov's method (modified)	.55 minutes
Jacobi's method	.84 minutes.

The disadvantage of Krylov's method lies in the difficulty of obtaining the roots of the characteristic equation of A_{I_k}. Bairstow's method worked well in most cases but did at times fail to converge. The Jacobi method always converges, although it may take longer than Krylov's method.. The Jacobi method is more accurate, although the difference in the accuracy of the two methods did not cause the values of D_W, P_W, N_L, and λ_{eff} to differ by as much as 0.1 per cent when the model of the outpatient clinic was run on the IBM 1800 computer.

TABLE 1

ACCURACY OF CHARACTERISTIC VECTORS USING
KRYLOV'S METHOD AND JACOBI'S METHOD

	N = 3		N = 5		N = 10	
	Method		Method		Method	
	I	II	I	II	I	II
largest norm of P	3.93×10^{1}	1.0	6.9×10^{2}	1.00	3.74×10^{6}	1.00
smallest norm of P	2.26×10^{1}	1.0	4.13×10^{2}	1.00	5.98×10^{5}	1.00
largest norm of error	0.00	1.58×10^{-8}	2.03×10^{-4}	2.88×10^{-7}	3.58×10^{3}	1.18×10^{-6}
smallest norm of error	0.00	3.93×10^{-9}	1.87×10^{-5}	3.66×10^{-8}	0.00	1.03×10^{-7}
mean norm of P	0.00	1.00	5.20×10^{2}	1.00	1.23×10^{6}	1.00
mean norm of error	0.00	1.13×10^{-4}	1.17×10^{-4}	1.45×10^{-8}	5.6×10^{2}	8.20×10^{-7}
mean $\frac{\text{(norm of error)}}{\text{norm of P}}$	0.00	1.13×10^{-8}	2.33×10^{-7}	1.45×10^{-8}	2.56×10^{-4}	8.20×10^{-7}

Comparison of Krylov's Method (I) and Jacobi's Method (II) for calculating characteristic vectors of the matrix for M/M/1 model. $\lambda = 4$, $\mu = 4$

Primarily to avoid the problem of convergence for the Bairstow method of factoring the characteristic polynomial the Jacobi method was used in the computer runs of the model.

Once the characteristic values and vectors of set matrices A_{I_k} are found the behavior of the clinic for a given schedule $(t_2, t_3, \ldots, t_M)$ can be measured in terms of D_W, P_W, N_L, and λ_{eff}. Given an objective function $f(D_W, P_W, N_L, \lambda_{eff})$ one theoretically can find a schedule $(t_1^*, t_2^*, \ldots, t_M^*)$ which will minimize $f(.)$.

Due to the nature of the expressions for D_W, P_W, N_L, and λ_{eff} no technique such as linear programing, quadratic programing, or geometric programing is available which will minimize $f(.)$ and therefore, some search technique must be employed. In general, search techniques are guaranteed to converge to the functional minimum (in either a finite or infinite number of iterations) only if the function is convex or at least strictly quasiconvex (27). A function $f(x)$ is strictly quasiconvex if

$$f(x_2) \leq f(x_1) \quad \text{implies} \quad (x_2 - x_1) \cdot \nabla f(x_1) < 0$$

where

$$x_1 \neq x_2.$$

If the function is not strictly quasiconvex, a search may lead to a local minimum and not detect the existence of the global minimum.

A commonly used technique for searching out the minimum of a function that is not convex or strictly quasi-convex is to make several searches starting at different points. If the different searches lead to the same minimum, it is good evidence that the global minimum has been located.

Search techniques can be broadly divided into two classes: those making use of the derivative and those which do not. Directly calculating $\frac{\partial f(.)}{\partial t_i}$ $i = 2,3,...,M$ is not practical considering the expressions for D_W, P_W, N_L, and λ_{eff}. Accurately estimating $\frac{\partial f(.)}{\partial t_i}$ by examining the neighborhood of f(.) could be difficult due to the flatness of D_W, P_W, N_L, λ_{eff}. This point will be shown in Chapter V.

Function minimization without evaluating derivatives includes enumeration, random searches, patterned searches and conjugate direction methods. The latter method is generally considered to be more efficient than patterned and random searches or enumeration especially if the function is "well behaved". Fletcher (12) compared three gradient methods due to Powell (28), Smith (34), and Swann (39). He found that the method of Powell was clearly superior to the method of Smith. When the number of variables was small, Powell's method was superior to Swann's but became less favorable as the number of variables increased. Based on Fletcher's experiments with all three methods, Powell's method was chosen as the technique for minimizing f(.).

Powell's Method of Minimization Without Derivatives

Consider the function $f(P)$, $P = (p_1, p_2, \ldots, p_n)$. Let $\xi_1, \xi_2, \ldots, \xi_n$ be n linearly independent directions and let P_0 be some starting point.

Each iteration of Powell's method involves a search along each of the n linearly independent directions. Initially $\xi_1, \xi_2, \ldots, \xi_n$ are chosen as the coordinate directions e_1, $e_2, \ldots, e_n$. Thus for the first iteration, each search is a minimization in which only one parameter changes at a time. After a search has been made down each of the n directions a new linearly independent direction ξ^* is found and a new set of directions $\{\xi'\}$ is defined as

$$\xi_1' = \xi_2$$

$$\xi_2' = \xi_3$$

$$\vdots \qquad \vdots$$

$$\xi_n' = \xi^* .$$

In general the procedure is:

a) For $i = 1, 2, \ldots, n$

   i) Find γ_i such that $f(P_{i-1} + \gamma_i \xi_i)$ is minimized

   ii) Set $P_i = P_{i-1} + \gamma_i \xi_i$

b) For $i = 1, 2, \ldots, n-1$ set $\xi_i = \xi_{i+1}$

c) Set $\xi_n = P_n - P_0$

d) Find γ_0 such that $f(P_n+\gamma_0\xi_n)$ is minimized

e) Set $P_0 = P_0+\gamma_0\xi_n$.

Repeat steps a-e until the criteria for convergence is met.

Powell claimed this search technique would converge to the minimum of a quadratic in a finite number of iterations. Zangwill (43) pointed out, however, that the method may never converge to the minimum if the directions $\{\xi_i\}$ did not remain linearly independent. He then produced a counterexample for which the directions $\{\xi_i\}$ could become dependent and presented a modification of Powell's method which would avoid this problem. However, in order for the directions $\{\xi_i\}$ to become dependent the exact value of γ_i in min $f(P_n+\gamma_i\xi_i)$ must be found for some $i = 0,1,...,n$. If a finite point search were made, it is doubtful that the exact value of γ_i which would cause dependency would be found. For this reason, Powell's method, as originally put forth, is used here.

Application of Powell's Method of Minimization

Powell's method assumes the variables over which the search is made are unconstrained. However, the set of variables $(t_2,t_3,...,t_M)$ are constrained by

$$0 \leq t_i \leq L-U_i \quad i = 2,3,...,M.$$

Therefore, the transformation

$$t_i = (L-U_i)\sin^2\alpha_i \qquad i = 2,3, \ldots, M$$

is made.

Then

$$(\alpha_2, \alpha_3, \ldots, \alpha_M)$$

become the set of variables over which the search is made.

In step 1 and 4 of Powell's method, the location of

$$\min(F(P_{i-1}+\gamma_i\xi_i) \qquad i = 1,2, \ldots, M$$

involves a linear search over γ_i. The linear search technique used was a Fibonacci search (41). This search technique is used for searching only over a finite interval. However, the direction ξ_i need be searched only over a finite interval. To see this, consider an arbitrary direction

$$\xi_i = (\xi_i^2, \xi_i^3, \ldots, \xi_i^M)$$

and let
$$\xi^* = \min_j(|\xi_i^j| : \xi_i^j \neq 0)$$

and
$$a_i = \frac{1-\frac{\pi}{2}}{\xi^*}$$

$$a_2 = \frac{1+\frac{\pi}{2}}{\xi^*}.$$

If a search is made over the interval $[a_1, a_2]$ every nonzero element in ξ_i will be searched over at least the interval $\xi_i^j \pm \frac{\pi}{2}$ or equivalently each t_j corresponding to α_j varies over the interval $[0, L-U_i]$.

Since a Fibonacci search is a discrete point search, after n points on the interval $[a_1, a_2]$ have been examined, there remains an interval of uncertainty within which the optimum value of α_i may lie. If the starting point of the Fibonacci search is chosen optimally, this interval of uncertainty can be reduced to $\frac{a_2-a_1}{F_n}$

where

$$F_n = F_{n-1}+F_{n-2} \quad \text{for } n \geq 2$$

and

$$F_0 = F_1 = 1$$

CHAPTER V

RESULTS OF NUMERICAL ANALYSIS AND DISCUSSION

The model of the outpatient clinic was run 27 times. During these runs, the parameters of the model were varied over a wide range. The results of these runs are shown in Table 5, Appendix A. The runs were of two types: enumeration runs and search runs. In the enumeration runs, each t_i in the set $(t_2, t_3, \ldots, t_M)$ was varied over the interval $[0, L-U_i]$ and the intervals between t_i^k and t_i^{k+1} varied from 0.25 hours to 1.0 hours. In the search runs, Powell's method of searching as described in Chapter IV was used and the number of Finonacci points in each linear search varied from 7 to 10.

The objective function

(5.1) $$F = C_1 D_W(0,L) + C_2 P_W(0,L) + C_3 N_L + C_4 \lambda_{eff}$$

was used on all runs where C_1, C_2, C_3, and C_4 were constants and may represent costs, utilities, or some meaningful weighting factors. For certain of the enumeration runs, the objective function and measures of effectiveness were plotted as functions of $(t_2, t_3, \ldots, t_M)$ (Figures 3 through

15, Appendix B).

As examples of two runs, consider run 15 and run 16. Run 15 is an enumeration run and run 16 is the corresponding search run. In both runs there were three physicians attending the clinic. Only two of the physicians can be scheduled since, according to the model developed, the first physician attends the clinic from the time it begins accepting patients to the time the clinic ceases accepting patients. Each of the second and third physicians can be assigned to the clinic for 4.0 hours. The patient arrival rate (λ) is 8.0 per hour and for each physician the average number of patients seen (μ) is 3.0 per hour. The clinic accepts patients for an 8.0 hour interval and the clinic can have as many as 7 patients present at any one time. The variables for both runs are the times at which the second and the third physicians begin accepting patients, that is, t_2 and t_3. In run 15, t_2 was varied from 0.00 to 4.00 hours at intervals of 2.00 hours; t_3 was varied from 0.00 to 4.00 hours at intervals of 0.50 hours. The objective function used in both runs was

$$F(t_2,t_3) = 0.00D_w + 1.00P_w + 0.00N_L + 0.00\lambda_{eff}.$$

These particular coefficients were chosen so that in run 16 a search was made on an objective function which was not convex, quasiconvex, concave, or quasiconcave (see Figure 9). The measures of effectiveness, $D_w(t_2,t_3)$, $P_w(t_2,t_3)$,

$N_L(t_2,t_3)$, and $\lambda_{eff}(t_2,t_3)$ were evaluated at 27 different points (t_2,t_3) and are plotted in Figure 8, 9, 10, and 11 respectively. The ranges of the measures of effectiveness for run 15 are listed in Table 5. For the 27 points (t_2^*, t_3^*) = (0,50,4.00) yielded the minimum value of $F(t_2,t_3)$. In Table 2, D_W^*, P_W^*, N_L^*, and λ_{eff} are the values of the measures of effectiveness at (t_2^*,t_3^*). The minimum value of $F(t_2,t_3)$ was 20.90.

In run 16, a numerical search using Powell's method was made for the values of (t_2,t_3) which minimized $F(t_2,t_3)$. Ten Fibonacci search points were evaluated along each conjugate direction searched. The starting point of the search was (t_2,t_3) = (0.00,0,00). The search indicated that (t_2^*, t_3^*) = (0.36,3.18) and $F(t_2^*,t_3^*)$ = 20.57 which agrees quite well with run 15. Also listed in Table 5, under run 16, are the values of D_W^*, P_W^*, N_L^*, and λ_{eff}^* for (t_2^*,t_3^*) = (0,36,3.18).

The bulk of the computations involved in the model are for:

a) calculation of the characteristic values and vectors of M matrices.

b) calculation of the measures of effectiveness.

Table 2 shows the execution time on an IBM 1800 computer for calculating the characteristic values and vectors for one matrix, for different values of N, using the Jacobi method.

TABLE 2

COMPUTATION TIME (MINUTES) FOR CHARACTERISTIC VALUES AND VECTORS

	N=4	N = 7	N = 11	N = 14
Time in Minutes	0.24	0.84	2.76	5.75

Table 3 shows the execution time on an IBM 1800 computer for calculating the measures of effectiveness for different values of N and M at a given point $(t_2, t_3, \ldots, t_M)$.

TABLE 3

COMPUTATION TIME (MINUTES) FOR THE MEASURES OF EFFECTIVENESS

M	N = 4	N = 7	N = 11	N = 14
2	0.120	0.280	0.360	0.500
3	0.180	0.320	0.580	0.840
4	0.219	0.380	0.660	0.955

From Tables 2 and 3, it appears that for large values of M and N the computation times can become quite long. For example, for run 26 with M = 4 and N = 12, 94.0 minutes were required and for run 27, 90.0 minutes were required. The IBM 1800 computer, however, is a relatively slow machine. Runs 26 and 27 were repeated on the IBM 360/40 computer and the computation times were 17.0 minutes and 16.5 minutes respectively. It should also be noted that if the number of Fibonacci search points were reduced from 10 to 7, the computation times for search runs could be reduced by nearly 30 per cent and yet the interval of uncertainty for $(t_2^*, t_3^*, \ldots, t_M)$ would still be less than 1/2 hour.

Each search was preceded by an enumeration run. When search runs were compared to enumeration runs, it appeared that in all but one case (run 20), the search procedure located the minimum of F(.) even though some of the component variables in (5.1) are not convex, quasiconvex, concave or quasiconcave functions of $(t_2, t_3, \ldots, t_M)$. See Figures 3-15. Run 21 is identical to run 20 except for the starting points of the search. Comparing run 21 to enumeration run 19, it appears that the minimum F(.) has been located.

Runs 16, 17, and 18 are identical except that different starting points for the searches were used. It should be noted that in all three runs the search terminated at or near the same point (t_2^*, t_3^*) even though the objective function was not convex, quasiconvex, concave or quasiconcave.

When the clinic was attended by only two physicians, the runs of the model seemed to indicate that the measures of effectiveness were either convex or concave (Figures 3, 4, 5, and 6).

The implication of this investigation must be discussed from two points of view: a) queueing models and b) scheduling physicians in an outpatient clinic.

Implications of the Investigation with Respect to Queueing Models

This investigation suggests it is practical to work with transient M/M/s queueing systems with finite storage although as the capacity of the queue increases the computation time increases also.

This investigation also suggests that a model of a finite capacity queueing system may serve as a good approximation to an infinite capacity queueing system. The finite capacity queue differs from the infinite capacity queue only in that arrivals will not be accepted in the queue when the capacity of the queue is reached. For example, in run 3 the expected number of arrivals generated was 16 (2 arrivals per hour x 8 hours) while the expected number of arrivals accepted into the system varied from 15.97 to 15.99 or the probability of an arrival not being accepted into the system was less than .00185. It would appear, then, that the model in run 3 was a good approximation to a model with infinite capacity. Again, in run 11, 96.5 to 98.5 per cent of all arrivals were accepted into the system, thus indicating that this model serves as a good approximation to its counterpart with an unlimited queue length.

In some of the runs, it was noted that the schedule which minimizes the objective function does not change significantly as the capacity of the queue increases. See runs 6, 8, 10, and 12. This would indicate that the models developed may serve adequately as a decision-making tool for the case of infinite capacity.

As a measure of the accuracy of the model developed, $\sum_{n=0}^{N} P_n(t=L)$ was calculated after each evaluation of $F(.)$. In no case did $\sum_{n=0}^{N} P_n(t=L)$ differ from 1.0 by more than 0.0001.

Implication of the Results with Respect to Scheduling Outpatient Clinics

From Table 4, it can be seen that the patient waiting time is significantly affected by the schedule physicians follow.

It was noted in the enumeration runs that the minimum value for P_w and the maximum value for λ_{eff} occur at or near the same point $(t_2, t_3, \ldots, t_M)$. It would appear, then, that the objective of seeing as many patients as possible and the objective of minimizing patient waiting time are complementary.

It was also found that by limiting the capacity of the clinic the average waiting time of a patient can be reduced without causing a proportional decrease in the number of patients accepted into the clinic. See Table 5, runs 1, 2, and 3; runs 5, 7, 9, and 11; and runs 13, 15, and 19.

TABLE 4

EXPECTED WAITING TIME PER PATIENT AS FOUND IN ENUMERATION RUNS

Run No.	N	Minimum E(Wait) in Minutes	Maximum E(Wait) in Minutes
1	7	13.8	20.2
2	9	19.0	25.9
3	11	19.2	26.1
5	5	20.3	23.5
7	7	26.9	32.9
9	11	34.0	44.3
11	14	35.9	50.7
13	5	15.1	21.4
15	7	25.4	31.8
19	11	38.8	50.9
22	14	44.8	64.2
24	7	13.4	20.3

CHAPTER VI

SUMMARY AND FURTHER RESEARCH

The purpose of this research was the development of a transient M/M/s queueing model which could be used for scheduling servers. An outpatient clinic was used as a specific application of the model; the physicians attending the clinic were considered as servers.

Previous queueing models of outpatient clinics have emphasized scheduling patients, rather than scheduling the physicians who attend the clinic. Also, with the exception of simulation models previous models of outpatient clinics assumed that statistical equilibrium had been reached. Such an assumption is not realistic since:

a) the number of physicians attending the clinic may be changing throughout the clinic session

b) the patient arrival rate may exceed the rate physicians at the clinic can attend the patients

c) most clinics do not begin each session with a queue held over from the previous session.

In the analysis it was assumed that there was an upper limit on the number of patients who could be present in the clinic at any one time, that is, a queueing model with finite capacity was used.

The schedule of physicians attending the clinic was considered as a decision variable and the measures of effectiveness of a given schedule were:

a) the total time physicians were idle

b) the total waiting time of the patients admitted to the clinic

c) the number of patients in the clinic when the clinic stopped accepting patients

d) the total number of patients accepted into the clinic.

Analytic closed-form expressions for the measures of effectiveness were derived in terms of the characteristic values and characteristic vectors of the system of differential equations associated with the different M/M/s queueing models of the clinic. Several numerical analysis techniques for determining the characteristic values and vectors were tried and evaluated.

The behavior of the measures of effectiveness as a function of the schedule physicians followed was then examined. It appeared that certain of these measures of effectiveness were neither convex, quasiconvex, concave, or quasiconcave.

The model was tested over a wide range of conditions. Using a linear objective function of these nonlinear measures of effectiveness, a search was made for the physician schedule which minimized the objective function. Numerical analysis of the model indicated:

a) the schedule physicians follow in the clinic can have a significant effect on the expected waiting time of patients

b) the objective of minimizing patient waiting time is complementary to the objective of maximizing the total number of patients accepted into the clinic

c) the average time patients waited in the clinic could be significantly reduced by restricting the number of patients accepted into the clinic

d) the transient behavior of M/M/s queueing models with infinite capacity can be approximated quite well by M/M/s queueing models with finite capacity.

One weakness of the model developed is the amount of computer time required to make a search for the schedule of physicians which minimizes the objective function. To overcome this difficulty the feasibility of using only the larger characteristic values and associated characteristic vectors should be investigated.

Extension of the model to a more general queueing model such as an $M/E_k/s$ model or an $E_k/M/s$ model should also be investigated. In addition, it would be of value to simulate an outpatient clinic with an arrival process other than Poisson and a service mechanism other than exponential and compare the results of such a simulation with the results of the model developed in this research.

APPENDIX A

RESULTS FROM NUMERICAL ANALYSIS OF MODEL

TABLE 5

RESULTS FROM NUMERICAL ANALYSIS OF MODEL

	Run 1	Run 2	Run 3
Type of Run	Enumeration	Enumeration	Enumeration
M	2	2	2
λ	2.00	2.00	2.00
μ	2.00	2.00	2.00
$U_2, U_3 - - - U_M$	4.00	4.00	4.00
L	8.00	8.00	8.00
N	4	9	11
Range of t_2	0, 4	0, 4	0, 4
$t_2{}^{k+1} - t_2{}^k$	0.25	0.25	0.25
Range of t_3	..	..	..
$t_3{}^{k+1} - t_3{}^k$	..	..	..
Range of t_4	..	..	..
$t_4{}^{k+1} - t_4{}^k$	..	..	..
C_1	1.00	1.00	1.00
C_2	2.00	2.00	2.00
C_3	2.00	2.00	2.00
C_4	1.00	1.00	1.00
Range of D_W	5.23, 5.69	4.75, 5.58	4.75, 5.58
Range of P_W	3.42, 4.42	5.06, 6.86	5.12, 6.96
Range of N_L^W	1.13, 1.95	1.43, 3.05	1.46, 3.13
Range of λ_{eff}	14.57,14.90	15.89,15.96	15.97,15.99
Range of F^{eff}	29.99,33.01	35.81,41.31	36.06,41.74
Search start pt.	..	..	..
$t_2^*, t_3^*, - - - t_M^*$	4.00	2.75	2.50
D_W^*	5.24	5.00	5.12
P_W^*	3.91	5.31	5.25
N_L^*	1.13	2.09	2.22
λ_{eff}^*	14.67	15.95	15.99
F*	29.99	35.81	36,06
No. of Fib. Exp.	..	..	..

TABLE 5 Continued

	Run 4	Run 5	Run 6
Type of Run	Search	Enumerator	Search
M	2	2	2
λ	2.0	5.0	5.0
μ	2.0	4.0	4.0
$U_2, U_3 - - - U_M$	4.0	2.0	2.0
L	8.0	8.0	8.0
N	9	5	5
Range of t_2	..	0.00,6.00	..
$t_2^{k+1} - t_2^k$	..	0.25	..
Range of t_3	..	..	..
$t_3^{k+1} - t_3^k$	..	..	..
Range of t_4	..	..	..
$t_4^{k+1} - t_4^k$	..	..	..
C_1	1.00	0.50	0.50
C_2	2.00	2.00	2.00
C_3	2.00	0.50	0.50
C_4	1.00	-1.00	-1.00
Range of D_W	..	2.22, 2.63	..
Range of P_W	..	11.43,12.78	..
Range of N_L	..	1.62,63.13	..
Range of λ_{eff}	..	32.58,33.66	..
Range of F	..	-8.09,-4.14	..
Search start pt.	0.00	..	0.00
$t_2^*, t_3^*, - - - t_M^*$	2.56	3.00	3.12
D_W^*	5.08	2.36	2.36
P_W^*	5.26	11.42	11.42
N_L^*	5.11	3.10	3.09
λ_{eff}^*	15.94	33.66	33.66
F*	35.80	-8.09	-8.09
No. of Fib. Exp.	10	..	10

TABLE 5 Continued

	Run 7	Run 8	Run 9
Type of Run	Enumeration	Search	Enumeration
M	2	2	2
λ	5.0	5.0	5.0
μ	4.0	4.0	4.0
$U_2, U_3 - - - U_M$	2.0	2.0	2.0
L	8.0	8.0	8.0
N	7	7	11
Range of t_2	0.00, 6.00	..	0.00, 6.00
$t_2^{k+1} - t_2^k$	0.25	..	0.25
Range of t_3	..	..	..
$t_3^{k+1} - t_3^k$	..	..	..
Range of t_4	..	..	..
$t_4^{k+1} - t_4^k$	..	..	..
C_1	0.50	0.50	0.50
C_2	2.00	2.00	2.00
C_3	0.50	0.50	0.50
C_4	-1.00	-1.00	-1.00
Range of D_W	1.81, 2.49	..	1.46, 2.44
Range of P_W	16.21,18.97	..	21.89,27.60
Range of N_L	2.01, 4.77	..	3.51, 7.13
Range of λ_{eff}	34.62,36.11	..	37.36,38.65
Range of F	-0.52, 6.85	..	9.27,21.27
Search start pt.	..	0.00	..
$t_2^*, t_3^*, - - - t_M^*$	3.00	3.12	2.75
D_W^*	2.05	2.04	1.88
P_W^*	16.21	16.21	21.94
N_L^*	4.31	4.28	6.14
λ_{eff}^*	36.11	36.12	38.63
F*	-0.52	-0.53	9.27
No. of Fib. Exp.	..	10	..

TABLE 5 Continued

	Run 10	Run 11	Run 12
Type of Run	Search	Enumeration	Search
M	2	2	2
λ	5.0	5.0	5.0
μ	4.0	4.0	4.0
$U_2, U_3 - - - U_M$	2.0	2.0	2.0
L	8.0	8.0	8.0
N	11	14	14
Range of t_2	..	0.00, 6.00	..
$t_2^{k+1} - t_2^k$	..	0.25	..
Range of t_3	..	..	..
$t_3^{k+1} - t_3^k$	..	..	..
Range of t_4	..	..	..
$t_4^{k+1} - t_4^k$	..	..	..
C_1	0.50	0.50	0.50
C_2	2.00	2.00	2.00
C_3	0.50	0.50	0.50
C_4	-1.00	-1.00	-1.00
Range of D_W	..	1.33, 2.44	..
Range of P_W	..	23.57,32.62	..
Range of N_L	..	4.05, 8.33	..
Range of λ_{eff}	..	38.62,39.43	..
Range of F	..	12.28,29.25	..
Search start pt.	0.00	..	0.00
$t_2^*, t_3^*, - - - t_M^*$	2.70	2.50	2.49
D_W^*	1.89	1.90	1.90
P_W^*	21.93	23.61	23.61
N_L^*	6.17	7.01	7.01
λ_{eff}^*	38.63	39.40	39.40
F*	9.26	12.28	12.28
No. of Fib. Exp.	10	..	10

TABLE 5 Continued

	Run 13	Run 14	Run 15
Type of Run	Enumeration	Search	Enumeration
M	3	3	3
λ	8	8	8
μ	3	3	3
$U_2, U_3 - - - U_M$	4.00, 4.00	4.00, 4.0	4.00, 4.0
L	8.0	8.0	8.0
N	5	5	7
Range of t_2	0.00, 4.00	..	0.00, 4.00
$t_2^{k+1} - t_2^k$	2.00	..	2.00
Range of t_3	0.00, 4.00	..	0.00, 4.00
$t_3^{k+1} - t_3^k$	0.50	..	0.50
Range of t_4	..	..	..
$t_4^{k+1} - t_4^k$	..	..	..
C_1	0.00	0.00	0.00
C_2	1.00	1.00	1.00
C_3	0.00	0.00	0.00
C_4	0.00	0.00	0.00
Range of D_w	1.91, 3.72	..	1.20, 3.30
Range of P_w	11.50,14.70	..	20.90,23.40
Range of N_L	2.73, 4.41	..	3.52, 6.40
Range of λ_{eff}	41.20,45.80	..	44.40,49.40
Range of F	11.50,14.70	..	20.90,23.40
Search start pt.	..	0.00, 0.00	..
$t_2^*, t_3^*, - - - t_M^*$	4.00, 0.50	3.86, 0.57	0.50, 4.00
D_w^*	1.79	1,89	1.17
P_w^*	11.50	11.50	20.90
N_L^*	3.42	3.42	4.97
λ_{eff}^*	46.00	46.01	49.40
F*	11.50	11.50	20.90
No. of Fib. Exp.	..	10	10

TABLE 5 Continued

	Run 16	Run 17	Run 18
Type of Run	Search	Search	Search
M	3	3	3
λ	8.0	8.0	8.0
μ	3.0	3.0	3.0
U_2, U_3- - -U_M	4.00, 4.00	4.00, 4.00	4.00, 4.00
L	8.0	8.0	8.0
N	7	7	7
Range of t_2	..	..	..
$t_2^{k+1} - t_2^k$	..	..	..
Range of t_3	..	..	..
$t_3^{k+1} - t_3^k$	..	..	..
Range of t_4	..	..	..
$t_4^{k+1} - t_4^k$	..	..	..
C_1	0.00	0.00	0.00
C_2	1.00	1.00	1.00
C_3	0.00	0.00	0.00
C_4	0.00	0.00	0.00
Range of D_W	..	..	..
Range of P_W	..	..	..
Range of N_L	..	..	..
Range of λ_{eff}	..	..	..
Range of F	..	..	..
Search start pt.	0.00, 0.00	2.00, 2.00	4.00, 4.00
t_2^*, t_3^*, - - - t_M^*	0.36, 3.18	0.38, 3.22	0.50, 3.24
D_W^*	1.50	1.48	1.46
P_W^*	20.57	20.57	20.58
N_L^*	6.10	6.07	6.06
λ_{eff}^*	49.61	49.64	49.69
F*	20.57	20.57	20.57
No. of Fib. Exp.	10	10	10

TABLE 5 Continued

	Run 19	Run 20	Run 21
Type of Run	Enumeration	Search	Search
M	3	3	3
λ	8.00	8.00	8.00
μ	3.00	3.00	3.00
$U_2, U_3 - - - U_M$	4.00, 4.00	4.00, 4.00	4.00, 4.00
L	8.00	8.00	8.00
N	11	11	11
Range of t_2	0.00, 4.00	..	..
$t_2^{k+1} - t_2^k$	2.00	..	..
Range of t_3	0.00, 4.00	..	..
$t_3^{k+1} - t_3^k$	0.50	..	..
Range of t_4	..	..	..
$t_4^{k+1} - t_4^k$	..	..	..
C_1	0.00	0.00	0.00
C_2	1.00	1.00	1.00
C_3	0.00	0.00	0.00
C_4	0.00	0.00	0.00
Range of D_w	1.10, 3.13	..	..
Range of P_w	34.90,41.50	..	..
Range of N_L	5.15,10.30	..	..
Range of λ_{eff}	48.90,54.00	..	..
Range of F	34.90,41.50	..	..
Search start pt.	..	0.00, 0.00	4.00, 4.00
$t_2^*, t_3^*, - - - t_M^*$	2.00, 0.50	1.38, 0.00	1.43, 0.51
D_w^*	1.52	1.84	1.83
P_w^*	34.90	35.62	34.89
N_L^*	10.10	10.10	10.26
λ_{eff}^*	53.90	52.70	52.76
F*	20.58	34.90	35.62
No. of Fib. Exp.	..	10	10

TABLE 5 Continued

	Run 22	Run 23	Run 24
Type of Run	Enumeration	Search	Search
M	3	3	4
λ	8.00	8.00	12.00
μ	3.00	3.00	4.00
$U_2, U_3 - - - U_M$	4.00, 4.00	4.00, 4.00	4.00,4.00,4.00
L	8.0	8.0	8.0
N	14	14	7
Range of t_2	0.00, 4.00	..	0.00, 4.00
$t_2^{k+1} - t_2^k$	2.00	..	1.00
Range of t_3	0.00, 4.00	..	0.00, 4.00
$t_3^{k+1} - t_3^k$	0.50	..	1.00
Range of t_4	..	..	0.00, 4.00
$t_4^{k+1} - t_4^k$	..	..	1.00
C_1	0.00	0.00	1.00
C_2	1.00	1.00	2.00
C_3	0.00	0.00	1.00
C_4	0.00	0.00	-1.00
Range of D_w	0.57, 3.12	..	2.23, 5.48
Range of P_w	42.61,55.66	..	17.11,22.50
Range of N_L	6.88,13.29	..	3.28, 6.50
Range of λ_{eff}	51.92,57.05	..	66.64,76.16
Range of F	42.61,55.66	..	-34.28,-7.57
Search start pt.	..	0.00, 0.00	0.00,0.00,0.00
$t_2^*, t_3^*, - - - t_M^*$	2.00, 0.50	0.45, 1.25	0.00,2.00,4.00
D_w^*	1.46	1.91	2.30
P_w^*	42.61	41.92	17.11
N_L^*	12.76	13.04	5.36
λ_{eff}^*	56.38	55.30	76.16
F*	42.62	41.92	-34.28
No. of Fib. Exp.	..		..

TABLE 5 Continued

	Run 25	Run 26	Run 27
Type of Run	Search	Enumeration	Search
M	4	4	4
λ	12.00	12.00	12.00
μ	4.00	4.00	4.00
U_2, U_3- - -U_M	4.00,4.00,4.00	4.00,4.00,4.00	4.00, 4.00, 4.00
L	8.0	8.0	8.0
N	7	11	11
Range of t_2	..	0.00, 4.00	..
$t_2^{k+1} - t_2^k$	..	1.00	..
Range of t_3	..	0.00, 4.00	..
$t_3^{k+1} - t_3^k$	..	1.00	..
Range of t_4	..	0.00, 4.00	..
$t_4^{k+1} - t_4^k$	..	1.00	..
C_1	1.00	1.00	1.00
C_2	2.00	2.00	2.00
C_3	1.00	1.00	1.00
C_4	-1.00	-1.00	-1.00
Range of D_w	..	1.36, 5.15	..
Range of P_w	..	32.32,38 18	..
Range of N_L	..	3.98,10.50	..
Range of λ_{eff}	..	69.90,82.73	..
Range of F	..	-5.42,50.45	..
Search start pt.	0.00,0.00,0.00	..	0.00, 0.00, 0.00
t_2^*, t_3^*, - - - t_M^*	0.00,2.23,4.00	0.00,2.00,4.00	
D_w^*	2.30	1.51	1.69
P_w^*	17.12	33.51	32.53
N_L^*	5.31	10.36	9.58
λ_{eff}^*	75.82	82.73	82.80
F*	-34.27	-5.42	-6.47
No. of Fib. Exp.	7	..	10

APPENDIX B

GRAPHS OF MEASURES OF EFFECTIVENESS AND OBJECTIVE FUNCTION FOR SELECTED RUNS

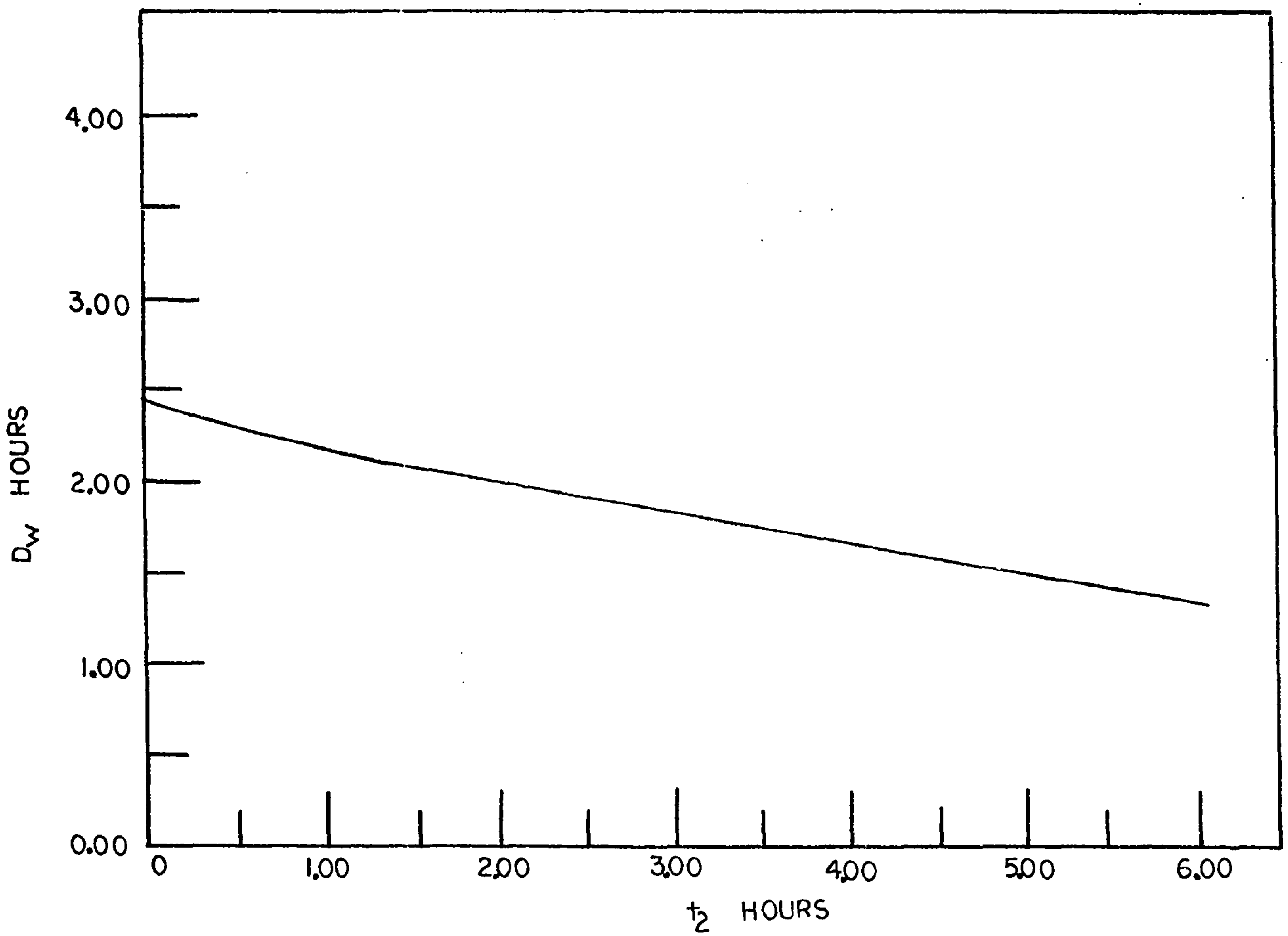

Fig. 3-Physician Idle Time for Run 11

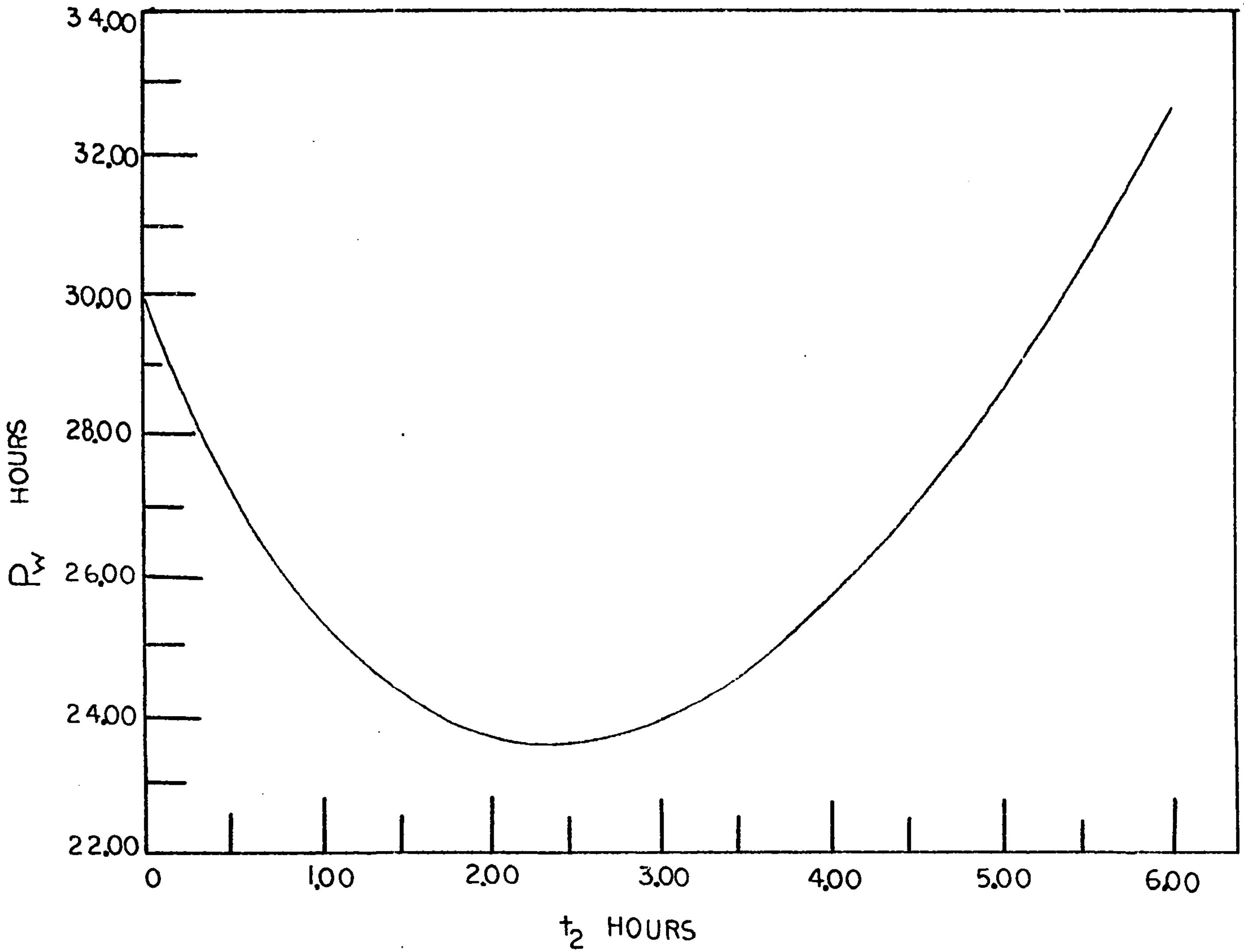

Fig. 4-Patient Waiting Time for Run 11

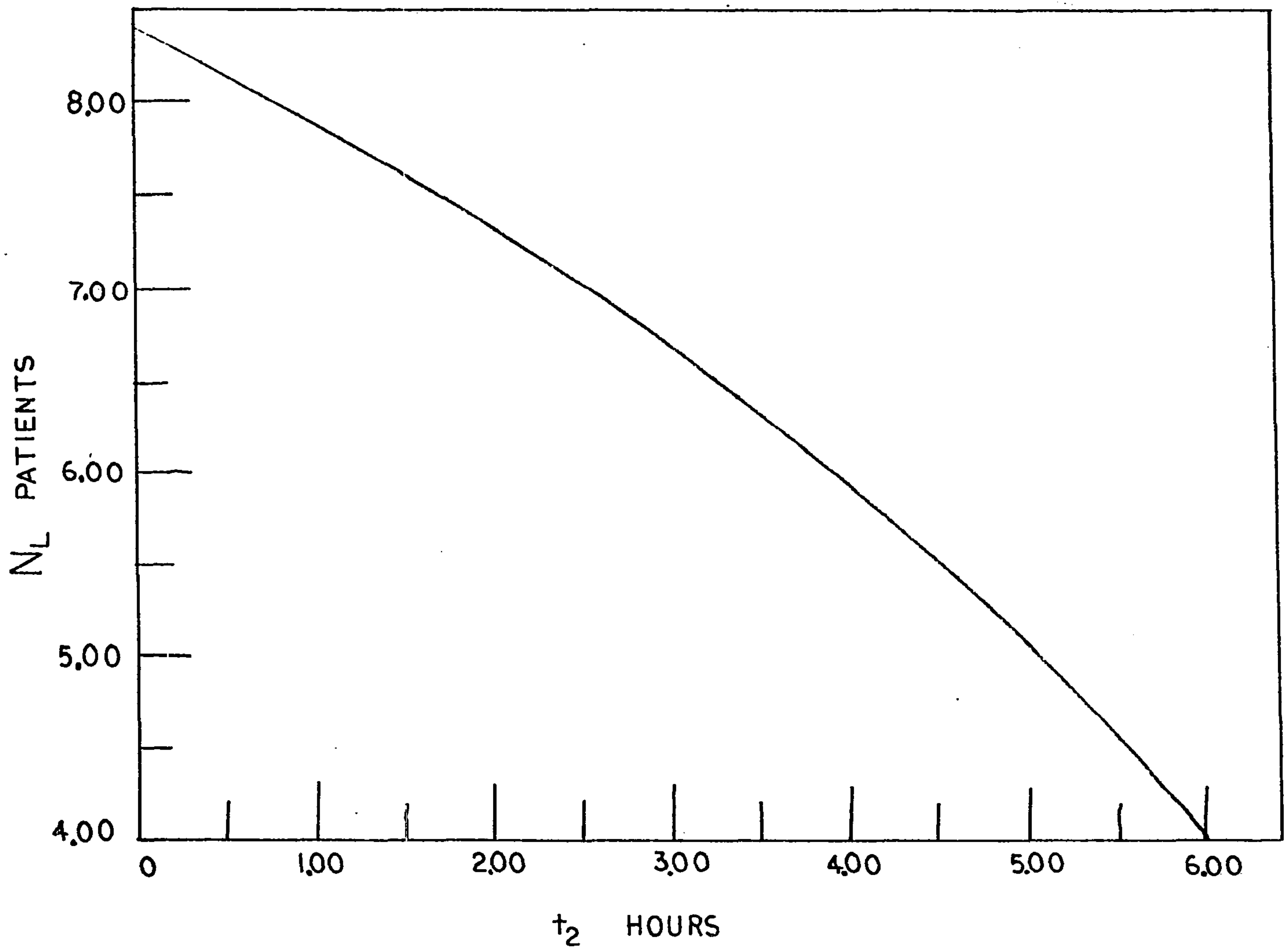

Fig. 5-Number of Patients in Clinic when Clinic Ends for Run 11

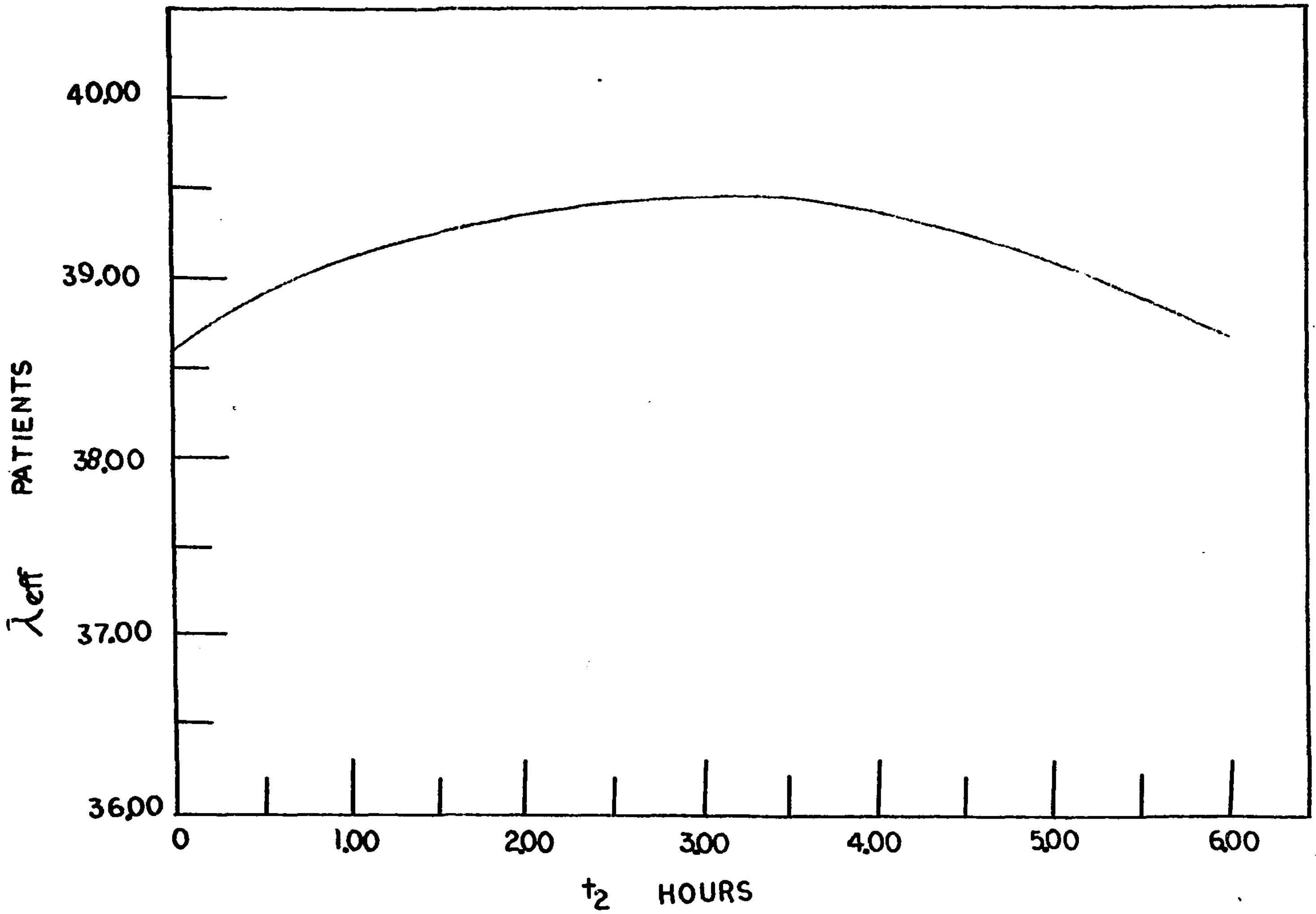

Fig. 6-Number of Patients Admitted to Clinic in Run 11

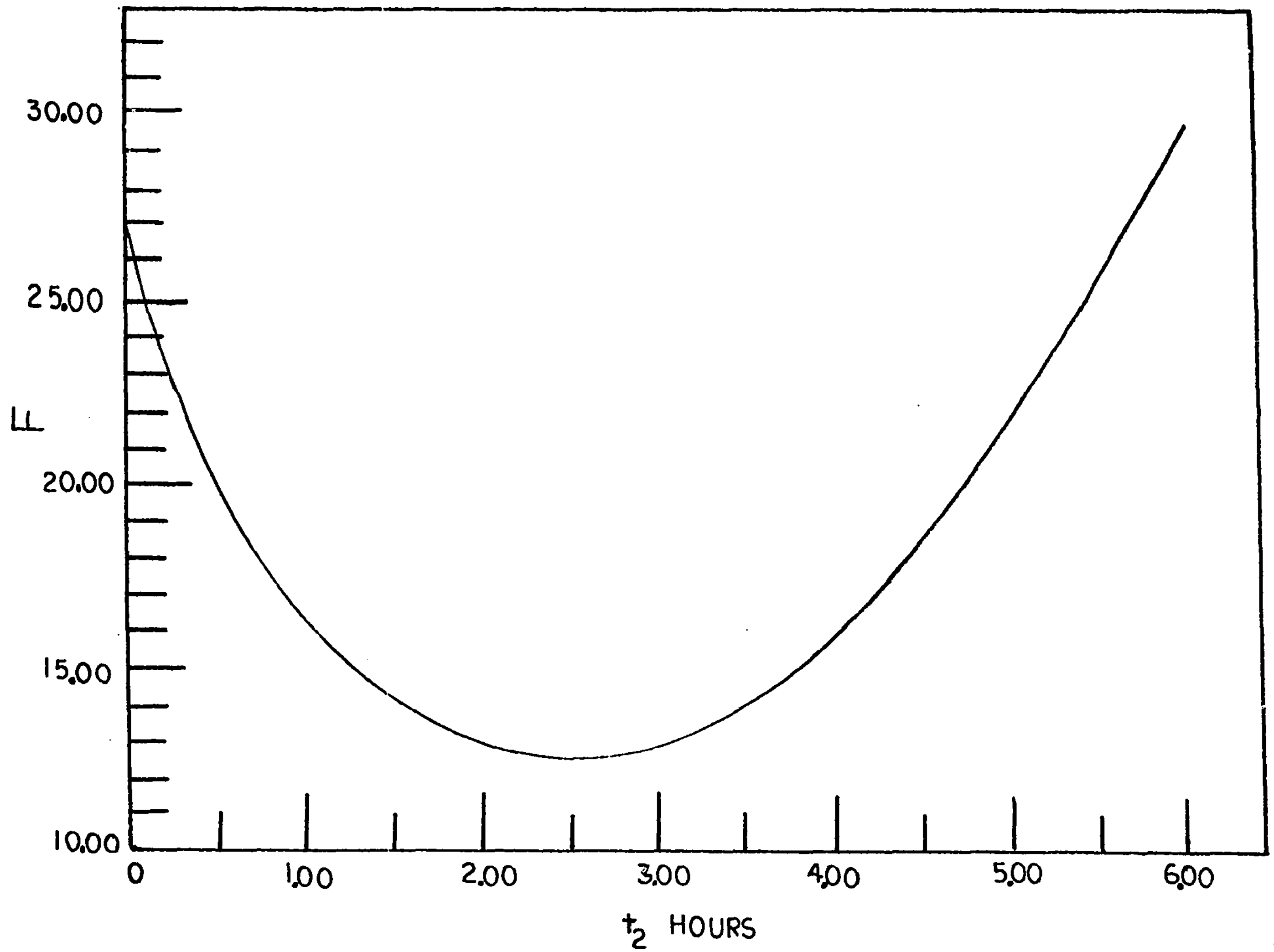

Fig. 7-Objective Function in Run 11

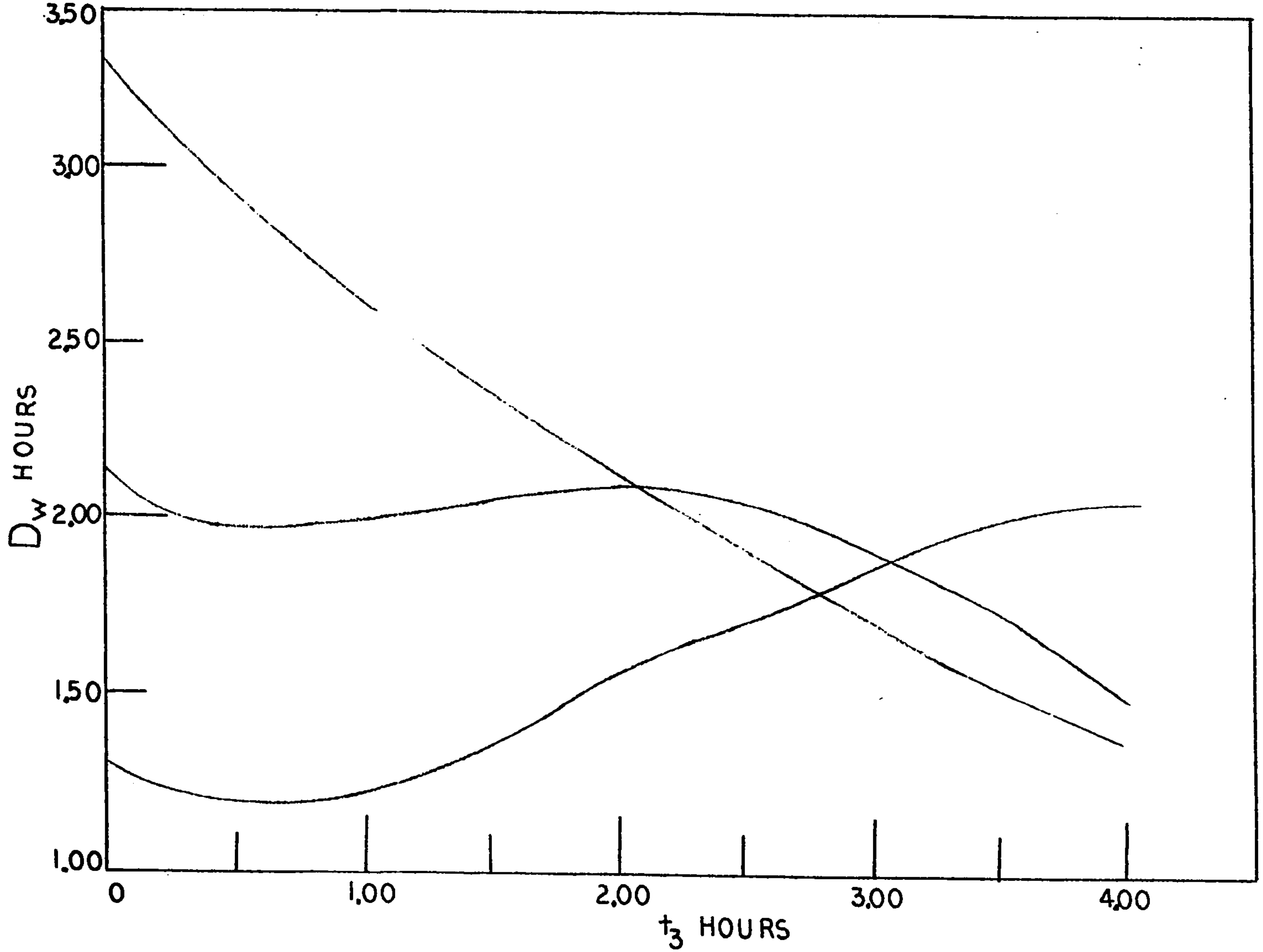

Fig. 8-Physician Idle Time for Run 15

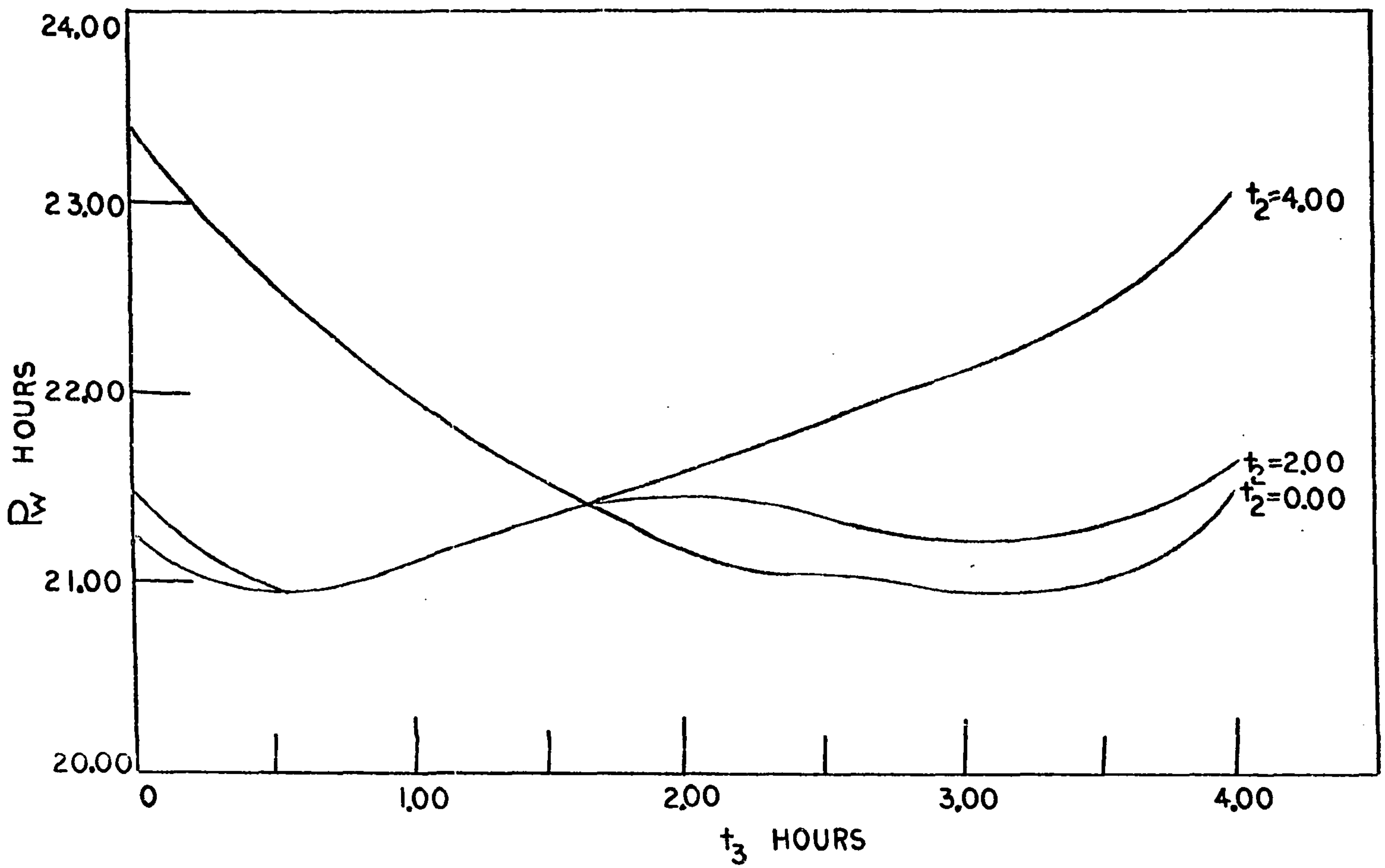

Fig. 9-Patient Waiting Time for Run 15

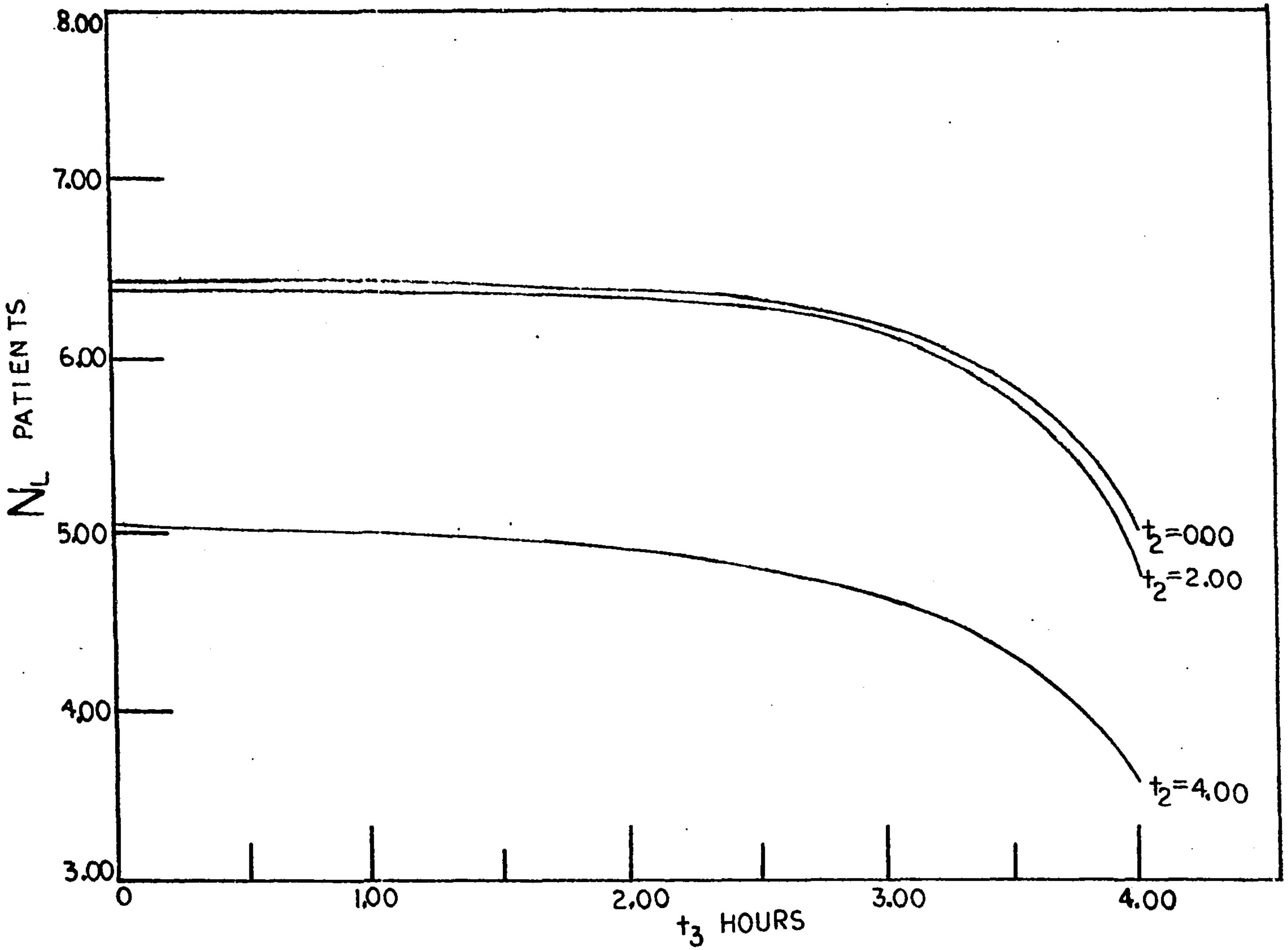

Fig. 10-Number of Patients in Clinic when Clinic Ends for Run 15

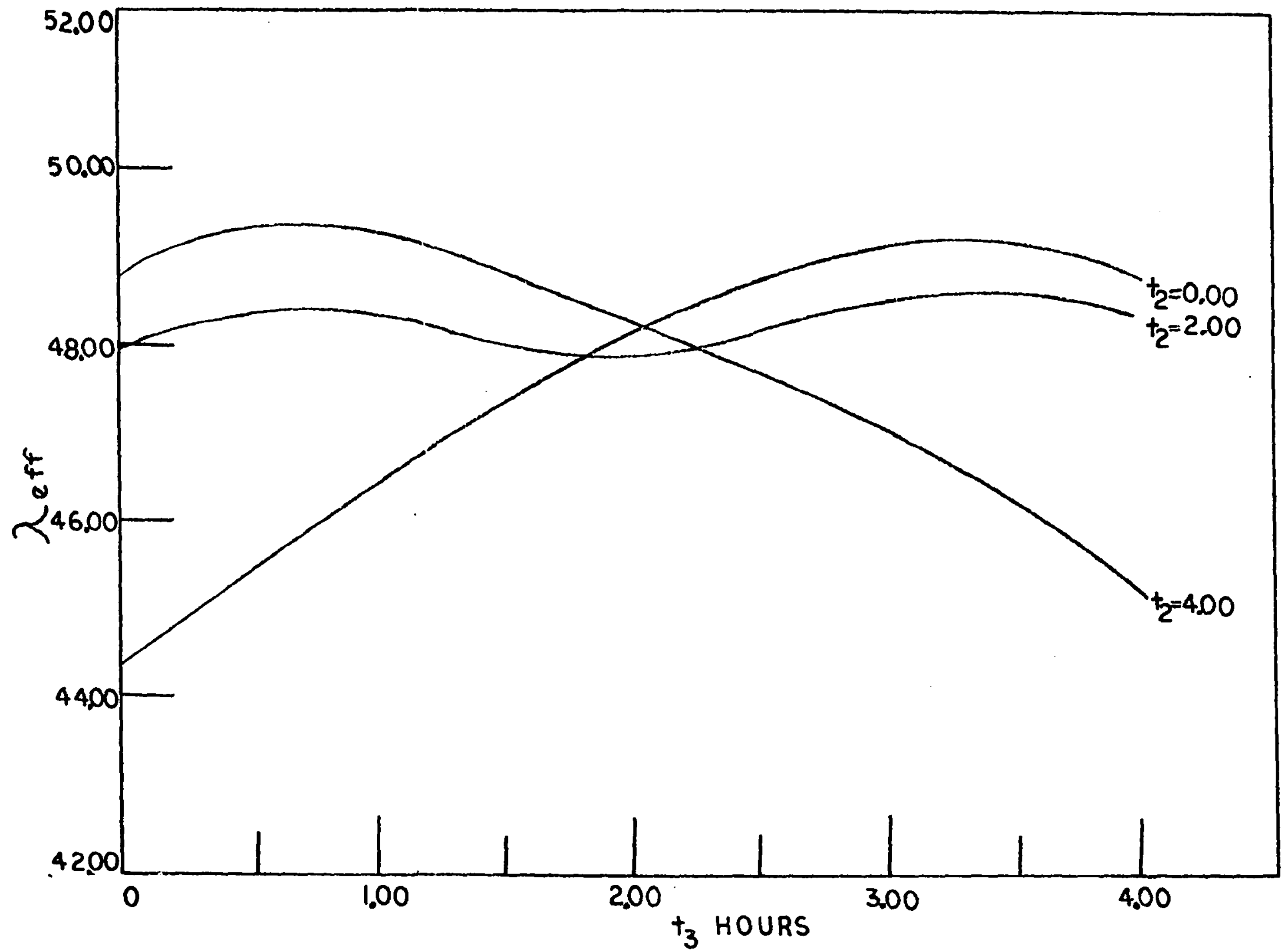

Fig. 11-Number of Patients Admitted to the Clinic in Run 15

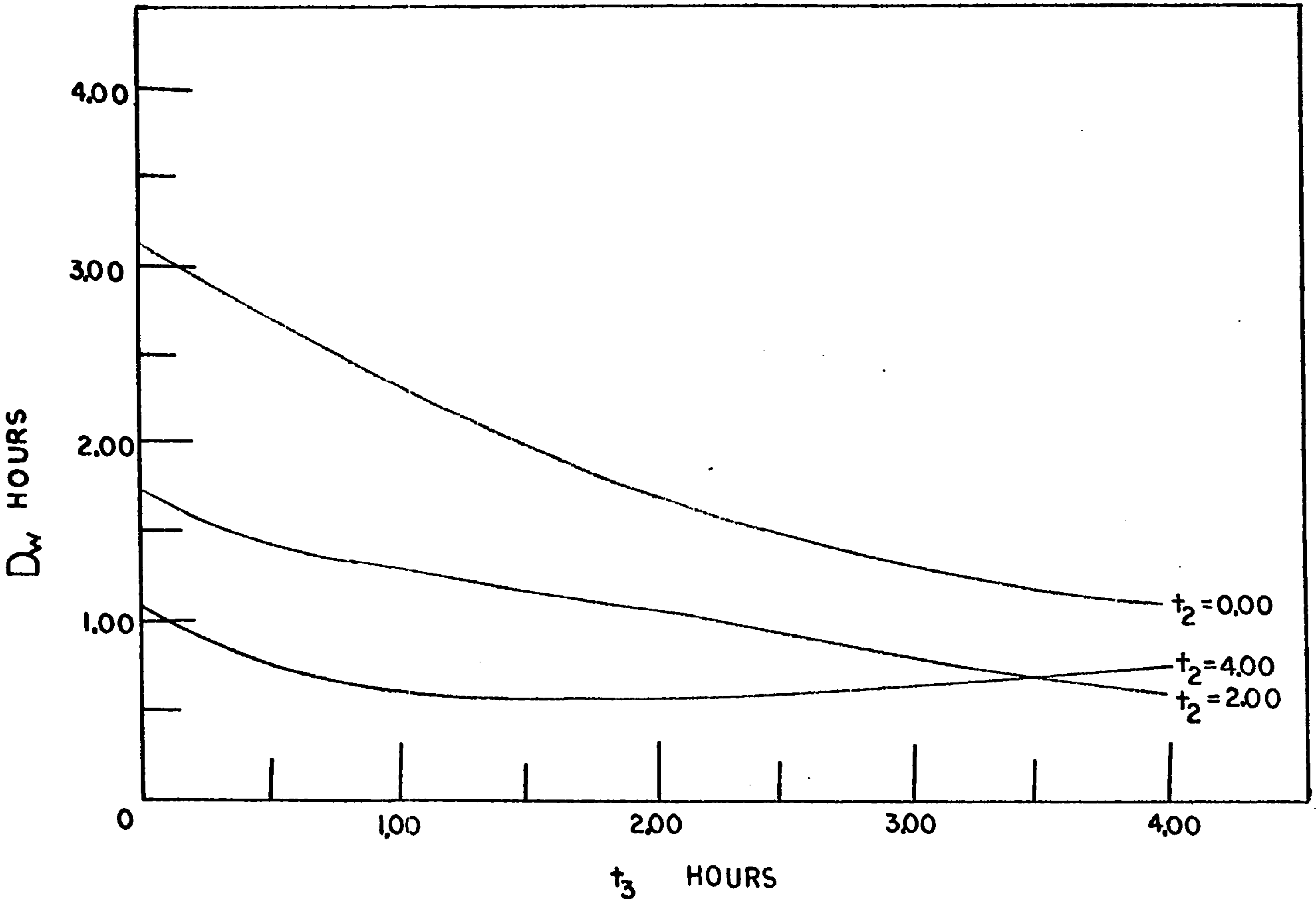

Fig. 12-Physician Idle Time for Run 22

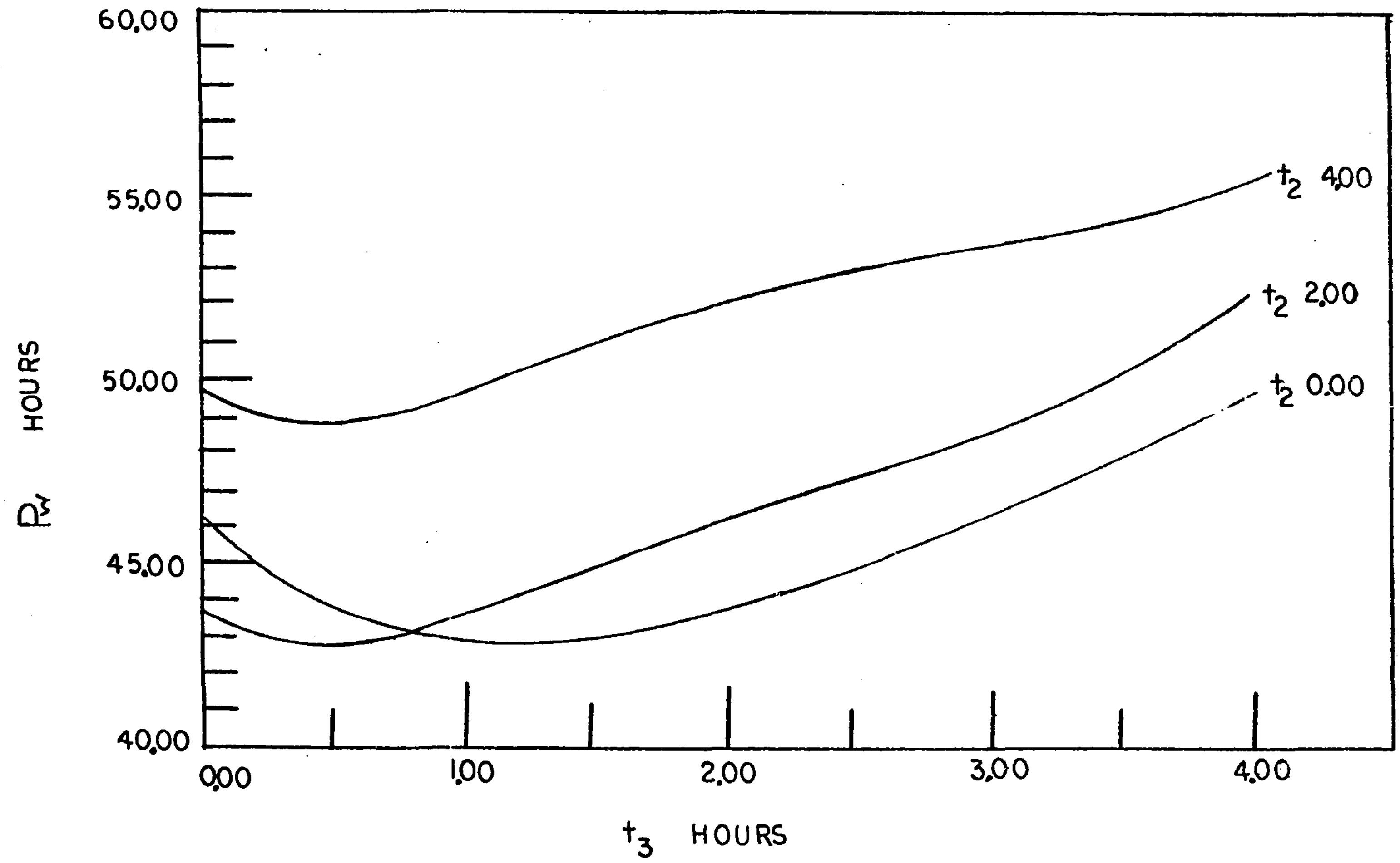

Fig. 13-Patient Idle Time for Run 22

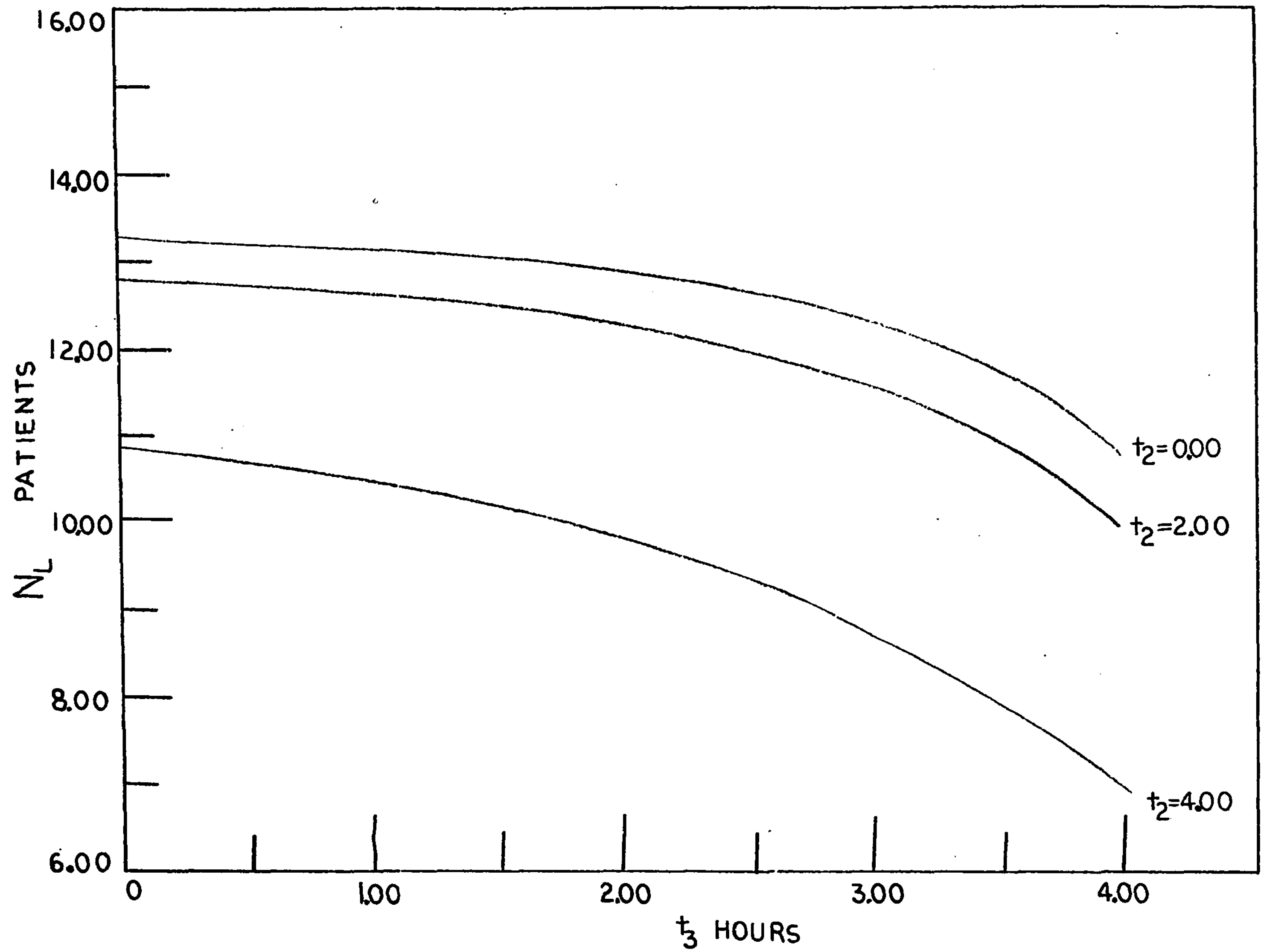

Fig. 14-Number of Patients in Clinic when Clinic Ends for Run 22

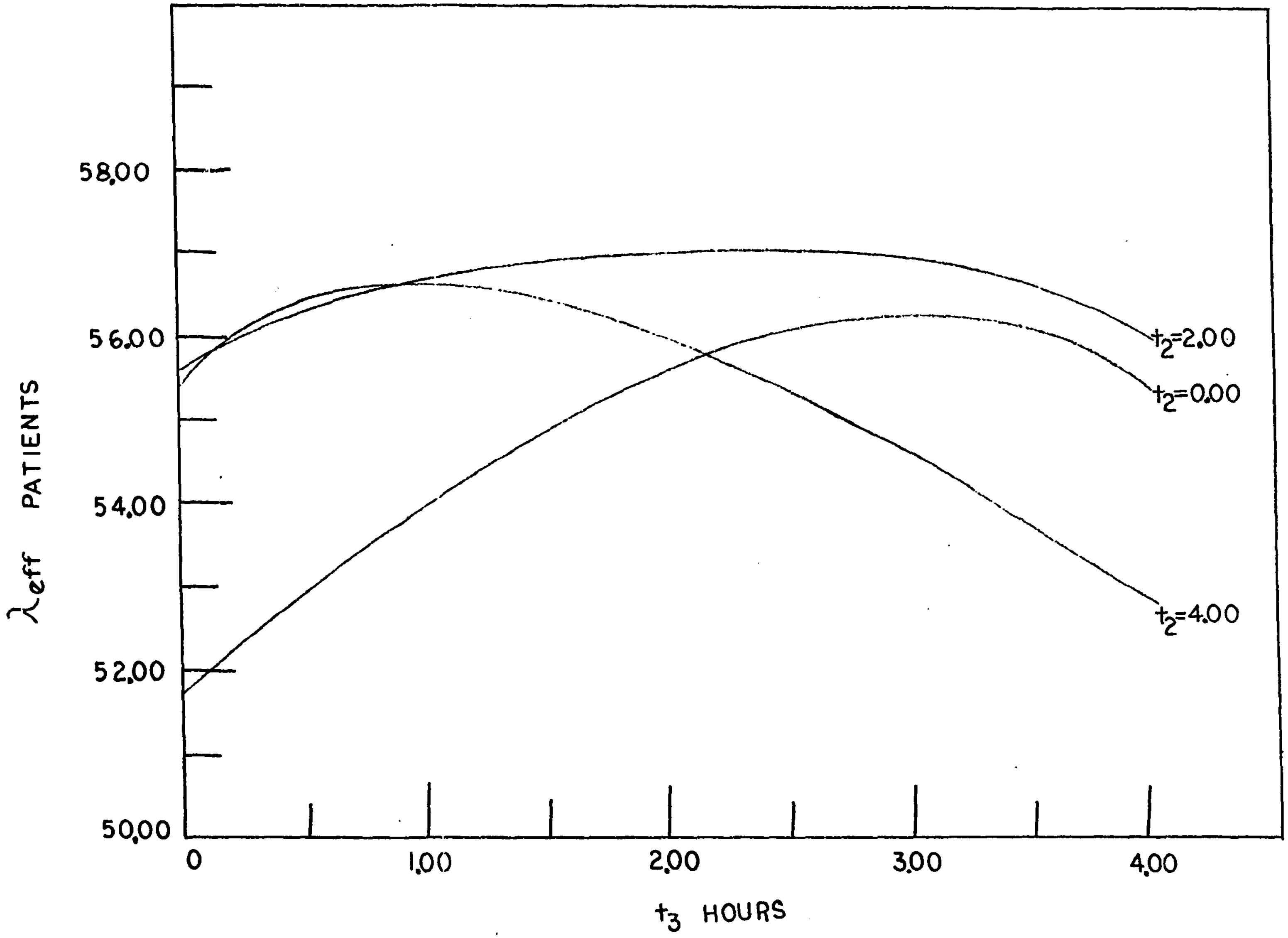

Fig. 15-Number of Patients Admitted to the Clinic in Run 22

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