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BUILDING FENCES AROUND THE CHROMATIC COEFFICIENTS

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BUILDING FENCES AROUND THE CHROMATIC COEFFICIENTS

A Dissertation APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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CHAPTER 1

INTRODUCTION

For over a century, one of the most tantalizing problems in all of mathematics was the Four Color Conjecture which states that any map on a plane or a sphere can be colored using four colors so that no two adjacent countries have the same color. The notion of a formula for counting colorings was introduced in 1912 by Birkhoff [3] in his work on the four-color problem, and later addressed in his paper, *Chromatic Polynomials*, with Lewis [4]. Scores of mathematicians worked on the conjecture, which was finally settled in the affirmative by Appel, Haken, and Koch ([1] and [2]); the proof was first published in 1977. Meanwhile, the study of chromatic polynomials had assumed a life of its own and continues to be a very active research area.

In general it remains an unsolved problem to determine which polynomials are chromatic. The purpose of this paper is to establish controls over the allowable values and patterns in the coefficients of chromatic polynomials.

More specifically, associated to each graph G is its chromatic polynomial $f(G,t)$ and we associate to $f(G,t)$ the sequence $\alpha(G)$ of the norms of its coefficients. A stringent partial ordering is established for such sequences. First, we show that if H is a subgraph of G then $\alpha(H) \leq \alpha(G)$. The main result is that for any graph G with q edges we have $\alpha(R_q) \leq \alpha(G) \leq \alpha(S_q)$, where R_q and S_q are specified graphs with q edges. The particular partial ordering employed illuminates the manner in which the coefficients for G can deviate from the coefficients for R_q and S_q.

When the chromatic polynomial of a graph G is expressed in terms of the falling factorials basis (as opposed to the standard basis discussed above), then the polynomial itself reveals the chromatic number of G. For this reason and others, it is natural to examine the coefficient sequence $\beta(G)$ of a chromatic polynomial $f(G,t)$ which has been expressed in terms of falling factorials. If G has m missing edges we find that $\beta(T_{\overline{m}}) \leq \beta(G) \leq \beta(X_{\overline{m}})$ where $T_{\overline{m}}$ and $X_{\overline{m}}$ are specified graphs with m missing edges.

Although these bounding conditions do not allow us to completely predict all chromatic polynomials, they do serve to severely limit the form of polynomials considered to be candidates for the chromatic polynomial of some graph.

CHAPTER 2

A SHORT COURSE ON CHROMATIC POLYNOMIALS

In this chapter we establish terminology, review the procedures for producing a chromatic polynomial, and present some important results from the literature. In some cases, the proofs presented here differ from the proofs in the cited references.

Basic definitions

Graph theory lacks a universally accepted nomenclature. For the sake of completeness and clarity we include here all the basic terminology to be used in this paper (see Harary [7]).

A *graph* G is a finite nonempty set $V(G)$ of *points* (or *vertices*) together with a set $E(G)$ whose elements, called *edges* (or *lines*), are unordered pairs of distinct points of G. If $x = \{u,v\} = uv = vu$ is an edge in G we say that x is *incident* with u (and with v), x joins u and v, and, u and v are *adjacent points*. If x and y are distinct edges of G, and both are incident with some point v of G, then x and y are said to be *adjacent edges*. It is often convenient to represent a graph as a diagram with distinguished points symbolizing elements of $V(G)$ and connecting line segments denoting elements of $E(G)$. Two graphs G_1 and G_2 are *isomorphic*, written $G_1 \cong G_2$ or sometimes $G_1 = G_2$, if there is a bijection between their point sets which preserves adjacencies. A graph with p points and q edges is denoted as a *(p,q) graph*. The (1,0) graph is called the *trivial graph*.

The *degree* of a point v in G, *deg v*, is the number of lines incident with v. If $deg\ v = 0$ then v is called an *isolated point*; if $deg\ v = 1$ then v is called an *endpoint*.

A *subgraph* of G is a graph having all of its points and edges in G. If H is a subgraph of G, then G is a *supergraph* of H. If v is a point in G then $G - v$ is the maximal subgraph of G not containing v; that is, $G - v$ consists of all points of G except v and all edges of G not incident with v. Likewise, if x is an edge in G then $G - x$ is the maximal subgraph of G not containing x; thus $G - x$ contains all the points and edges of G except x. On the other hand, if u and v are nonadjacent points in G, and we set $y = uv$, then $G + y$ is the smallest supergraph of G containing the line y.

A *path* is an alternating sequence of points and edges $v_0x_1v_1x_2 \ldots x_nv_n$ comprised of distinct points and in which $x_i = v_{i-1}v_i$ for $1 \leq i \leq n$. The indicated path would be called a $v_0 - v_n$ path. A *cycle* is an alternating sequence of points and edges $v_0x_1v_1x_2 \ldots x_nv_n$ in which $n \geq 3$, v_0 through v_{n-1} are all distinct, $x_i = v_{i-1}v_i$ for $1 \leq i \leq n$, and $v_n = v_0$. We usually describe a path or cycle simply by listing its sequence of points, since the edges are evident from context. The length of a path or of a cycle is the number of edges it contains. P_n denotes the path of length n and C_n denotes the cycle of length n (also called an *n-cycle*). C_3 is called a triangle.

G is said to be *connected* if for every pair of points u and v there is a $u - v$ path in G. A *component* of G is a maximal connected subgraph. A point v is a *cutpoint* of G if $G - v$ has more components than G; an edge x is a *bridge* of G if $G - x$ has more components than G. A graph is *nonseparable* if it is connected, nontrivial, and has no cutpoints. The *blocks* of a graph are its maximal nonseparable subgraphs.

If a graph is connected and has no cycles it is called a *tree*. A specific tree which we will use later is the *star* with q edges, which has one central point incident with each of the q edges, and thus has a total of $q + 1$ points. Harary [7] presents several characterizations of trees; we will use the equivalences that G is a tree $\Leftrightarrow$ G is connected and $p = q + 1$ $\Leftrightarrow$ G is connected and every edge is a bridge.

Each graph G has an associated graph $\overline{G}$ called the *complement* of G. $V(\overline{G}) = V(G)$. Two points u and v are adjacent in $\overline{G}$ if and only if u and v are not adjacent in G. The graph with p points and all possible edges, $q = \binom{p}{2}$, is called the *complete graph* on p points, denoted K_p. Then $\overline{K}_p$ has p points and 0 edges and is said to be *totally disconnected*, except when $p = 1$ and then $\overline{K}_1 = K_1$ is the trivial graph.

We can take two existing graphs G_1 and G_2 and form their *union* $G_1 \cup G_2$ with point set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Their *join*, $G_1 \dotplus G_2$, has $V(G_1 \dotplus G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \dotplus G_2) = E(G_1) \cup E(G_2) \cup \{x \mid x = ab \text{ with } a \in V(G_1) \text{ and } b \in V(G_2)\}$.

If $x = ab$ is an edge in G then we can define, up to isomorphism, a new graph G/x by performing an *elementary contraction along edge* x. In the resulting graph we have point set $V(G/x) = V(G) - a$ and edge set $E(G/x) = E(G - a) \cup \{x = bc \mid ac \in E(G) \text{ and } c \neq b\}$. We visualize this process as releasing the edges at a, reattaching them at b, and removing point a. We could of course exchange the roles of a and b; the resulting graphs are naturally isomorphic under the bijection $a \rightleftarrows b$ with identity elsewhere. If G is a (p,q) graph then G/x has $p - 1$ points and at most $q - 1$ edges. Note that $G - a$ is a subgraph of G/x, but, in general, G/x is not a subgraph of G.

On the other hand, if u and v are nonadjacent points in G then we can produce an *elementary homomorphism*, εG, which identifies u and v. Effectively, $\varepsilon G = (G + y)/y$ where $y = uv$.

Graph coloring and the chromatic polynomial

A proper coloring of a graph G is an assignment of colors to its points such that no two adjacent points receive the same color. Henceforth we consider only proper colorings and so we simply say colorings. The minimum number of colors needed for a coloring of G is called the *chromatic number* of G, denoted $\chi(G)$. In a coloring of a complete graph K_p, a different color is needed for each point (because the points are all mutually adjacent) and so $\chi(K_p) = p$. In a graph with no edges only one color is needed so $\chi(\overline{K}_p) = 1$. In general, the determination of $\chi(G)$ is a lengthy task.

For a fixed graph G and a fixed set of n colors, $f(G,n)$ is the number of different colorings of G from the n colors (called *n-colorings*). Two given colorings are considered distinct if there is any point of G which is colored differently by them. If $n < \chi(G)$ then of course $f(G,n) = 0$. In a coloring of K_p from $t \geq \chi(K_p) = p$ colors we have t choices for the first point colored, $t - 1$ choices for the second point colored (since we cannot use any color twice), $t - 2$ choices for the third point, thus yielding the counting formula $f(K_p,t)=t(t-1)\cdots(t-p+1)$. We note that this formula produces valid results even when $0 \leq t < \chi(K_p)$, for then it gives 0. Then we have found a polynomial function in the variable t which, when evaluated at any nonnegative integer n, produces $f(K_p,n)$. The expression for $f(K_p,t)$ is called the *falling factorial of degree p*, and is denoted by t_p. In a coloring of $\overline{K}_p$ each point may be colored independently of the others; if $t \geq 1$ colors are available we have t choices for each point and thus $f(\overline{K}_p,t) = t^p$. Once again we have a polynomial formula in t that is valid for evaluation at any nonnegative integer n for calculating $f(\overline{K}_p,n)$. It is not immediately clear that such a polynomial function should exist for calculating $f(G,n)$ when G is an arbitrary graph, but the following theorem provides the necessary machinery.

Theorem 2.1. If u and v are nonadjacent points in a graph G, and ε is the elementary homomorphism which identifies u with v, then we have $f(G,n) = f(G + uv,n) + f(\varepsilon G,n)$ for any nonnegative integer n.

Proof. (Theorem 12.32 in [7]) The n-colorings of G can be partitioned into two sets:

A = {n-colorings in which u and v receive different colors}

and

B = {n-colorings in which u and v receive the same color}.

The set A is precisely the set of n-colorings of $G + uv$, so $|A| = f(G + uv,n)$.

Now we claim that $|B| = |\{n\text{-colorings of } \varepsilon G\}|$. Suppose we have an n-coloring of G in which u and v receive the same color, say blue. Then no point adjacent to u is blue and no point adjacent to v is blue, and so the coloring carries directly over to a coloring of εG in which the point identifying u and v is colored blue. Since this is reversible we have $|B| = f(\varepsilon G,n)$.

Then $f(G,n) = |A| + |B| = f(G + uv,n) + f(\varepsilon G,n)$. ■

We recall that $Z[t]$ is the ring of polynomials in one variable with integer coefficients (see Lang [8]).

Corollary 2.2. For each graph G there is a polynomial $f(G,t) \in Z[t]$ such that evaluation of $f(G,t)$ at any nonnegative integer n produces $f(G,n)$.

Proof. (Corollary 12.32a in [7]) If G is a complete graph on p points then we saw previously that the polynomial $f(G,t) = (t)(t-1)\cdots(t-p+1) \in Z[t]$, evaluated at any nonnegative integer n, produces $f(G,n)$. If G is not complete we can find nonadjacent points u and v so that $f(G,n) = f(G + uv,n) + f(\varepsilon G,n)$ where ε is the elementary homomorphism identifying u with v, and n is any fixed nonnegative integer. We repeat this procedure if necessary until we arrive at $f(G,n) = \sum_{i=1}^{m} f(G_i,n)$ where $m \in \mathbb{N}$ and each G_i is a complete graph. For each i there is a polynomial $f(G_i,t) \in Z[t]$ such that $f(G_i,n) = f(G_i,t)(n)$. Then we define $f(G,t) = \sum_{i=1}^{m} f(G_i,t) \in Z[t]$, and we have for any nonnegative integer n:

$$f(G,n) = \sum_{i=1}^{m} f(G_i,n) = \sum_{i=1}^{m} f(G_i,t)(n) = \left(\sum_{i=1}^{m} f(G_i,t)\right)(n) = f(G,t)(n).$$ ■

$f(G,t)$ is called the *chromatic polynomial* of G.

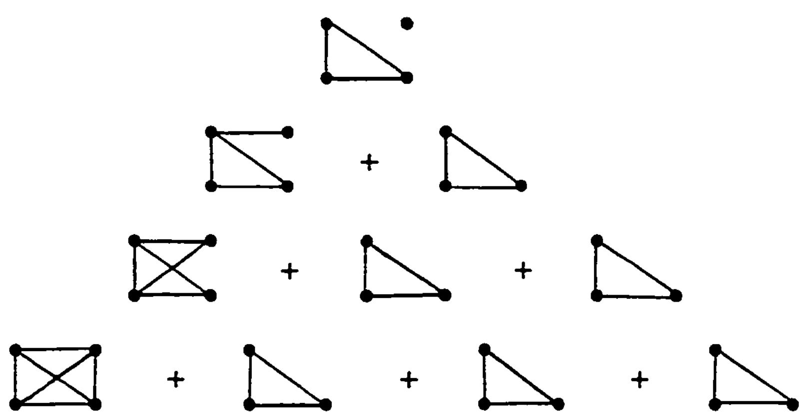
It is a well-known result in algebra that if two polynomials in $\mathbb{Z}[t]$ agree in their evaluation at all $n \in \mathbf{N}$ then in fact the two polynomials are identical (see Corollary 4.5 in [8]). Then we can restate Theorem 2.1 as follows: If u and v are nonadjacent points in G, and ε is the elementary homomorphism that identifies u with v then $f(G,t) = f(G + uv,t) + f(\varepsilon G,t)$. We will call this the *completion formula* because both $G + uv$ and εG are more nearly complete than G, in the sense of having fewer missing edges.

Corollary 2.3. *The Reduction Formula.* If $x = uv$ is an edge in G then $f(G,t) = f(G - x,t) - f(G/x,t)$.

Proof. Let $H = G - x$. Then by the completion formula we have ($\star$): $f(H,t) = f(H + x,t) + f(\varepsilon H,t)$ where u and v are identified through the homomorphism ε. Since $H + x = G$ and $\varepsilon H \cong G/x$, ($\star$) becomes $f(G - x,t) = f(G,t) + f(G/x,t)$, or $f(G,t) = f(G - x,t) - f(G/x,t)$. This is called the reduction formula because both $G - x$ and G/x have fewer edges than G. ∎

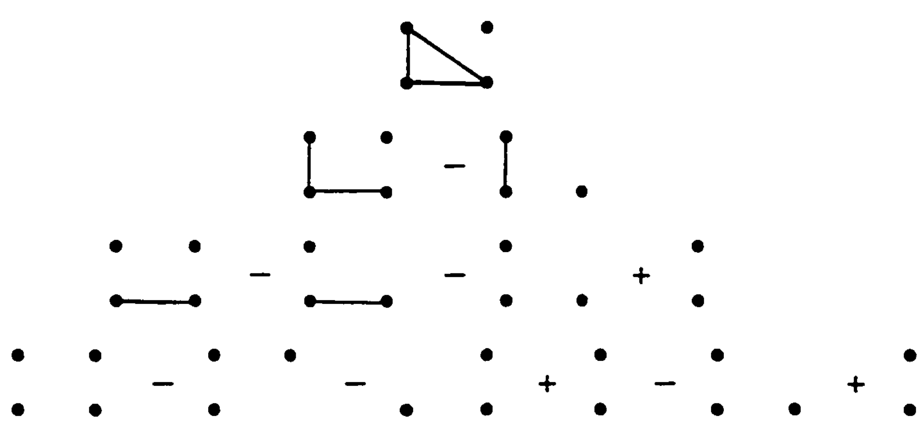
Just as we could use the completion formula to eventually write $f(G,t)$ in terms of complete graphs, we can use the reduction formula to eventually express $f(G,t)$ in terms of totally disconnected and/or trivial graphs. Figure 1 provides a visual representation of these processes (see Zykov [10]).

Using the completion formula with $G = C_3 \cup K_1$:



$$f(G,t) = t_4 + 3t_3 = t^4 - 3t^3 + 2t^2$$

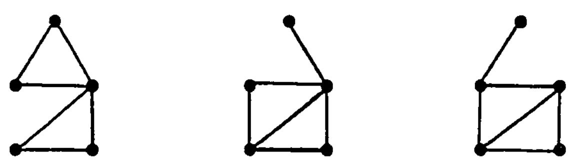
Using the reduction formula with $G = C_3 \cup K_1$:



$$f(G,t) = t^4 - 3t^3 + 2t^2$$

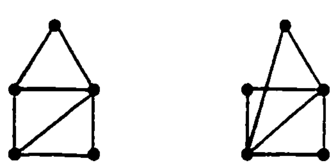
Figure 1 Two determinations of $f(G,t)$

It is of course true that isomorphic graphs have the same chromatic polynomial, however it is also true that nonisomorphic graphs, and even nonisomorphic blocks, can have the same chromatic polynomial.



$$f(G,t) = t^5 - 6t^4 + 13t^3 - 12t^2 + 4t$$

Figure 2
Nonisomorphic graphs with the same chromatic polynomial



$$f(G,t) = t^5 - 7t^4 + 18t^3 - 20t^2 + 8t$$

Figure 3
Nonisomorphic blocks with the same chromatic polynomial

General properties of chromatic polynomials

Chromatic polynomials enjoy many interesting properties. All of the known general properties (meaning that they apply to every chromatic polynomial $f(G,t)$ regardless of the construction of G) are contained in the following theorem. Statements (a) through (g) have all been known for at least 30 years. Statement (h) was contained in a 1986 paper by Chia [5] and is the only known recent addition to the list of general properties. The proof of (h) is not hard, but it would require an excursion into block-cutpoint trees, and so we have omitted it. We note that the results to be developed in Chapters 3 and 4 are new additions to the list of general properties.

Theorem 2.4. If G is a (p,q) graph then

(a) $f(G,t) = \prod_{i=1}^{k} f(G_i,t)$ where G has the k components $G_1,\ldots,G_k$.

(b) If $q \geq 1$ then the sum of the coefficients in $f(G,t)$ is 0.

(c) $f(G,t)$ has degree p.

(d) The coefficient of t^p in $f(G,t)$ is 1.

(e) The coefficient of t^{p-1} in $f(G,t)$ is $-q$.

(f) The coefficients of $f(G,t)$ alternate in sign.

(g) t^i has nonzero coefficient in $f(G,t) \Leftrightarrow k \leq i \leq p$.

(h) $f(G,t) = \dfrac{\prod_{i=1}^{n} f(B_i,t)}{t^{n-1}}$ if G is connected and $B_1,\ldots,B_n$ are the blocks of G.

Proof. (Whitney [9] discusses (b) and (f). The remaining properties - except (h) - are stated in [7] without proof)

Property (a) follows from the observation that the components of G can be colored independently.

For property (b), we note that if $q \geq 1$ then G cannot be colored using only one color. Then $f(G,1) = 0$, and any polynomial evaluated at 1 yields the sum of its coefficients.

Properties (c) through (g) will be proved using induction on q and the reduction formula. If $q = 0$ then G is a totally disconnected graph with $k = p$ and $f(G,t) = t^p$, and we see that properties (c) through (g) hold. Now assume $q \geq 1$ and let x be an edge in G. By the reduction formula we have $f(G,t) = f(G - x,t) - f(G/x,t)$.

$G - x$ has p points, $q - 1$ edges and either k or $k + 1$ components depending on whether or not x is a bridge. The contraction graph G/x has $p - 1$ points, q' edges for some $q' \leq q - 1$, and k components. By induction, we may assume that properties (c) through (g) are true for $G - x$ and G/x. Now we verify properties (c) through (g) for G.

(c) and (d) Since $f(G - x,t)$ is monic of degree p (and $f(G/x,t)$ has degree $p - 1$) we know that $f(G,t)$ is monic of degree p.

(e) Since $f(G - x,t)$ has coefficient $-(q-1)$ for t^{p-1}, and $f(G/x,t)$ has coefficient 1 for t^{p-1}, then $f(G,t)$ has coefficient $-(q-1) - 1 = -q$ for t^{p-1}.

(f) Both $f(G - x,t)$ and $f(G/x,t)$ have coefficients which alternate in sign, both begin with a positive coefficient, and $f(G/x,t)$ has degree one less than $f(G - x,t)$, so $f(G - x,t) - f(G/x,t) = f(G,t)$ has coefficients which alternate in sign.

(g) When $i > p$, and when $i < k$, both $f(G - x,t)$ and $f(G/x,t)$ have coefficient 0 for t^i, so $f(G,t)$ has coefficient 0 when $i > p$ and when $i < k$. When $k \le i \le p$ either $f(G - x,t)$ or $f(G/x,t)$, or both, have a nonzero coefficient for t^i. Then it follows from the degree and alternation discussion for (f) that $f(G,t)$ has a nonzero coefficient for t^i when $k \le i \le p$. ■

We say that a polynomial is chromatic if it is the chromatic polynomial of some graph. In order for a polynomial to be chromatic it must of course satisfy all the appearance properties in Theorem 2.4 (parts (b) through (g)). Unfortunately this is not sufficient to guarantee that a given polynomial is chromatic. Consider, for example, $p(t) = t^6 - 6t^5 + 15t^4 - 18t^3 + 14t^2 - 6t$. Then $p(t)$ is monic, with alternating coefficients summing to zero, and we might suspect that it is the chromatic polynomial of some (6,6) graph. We will prove later that $p(t)$ is not chromatic, even if we increment all the exponents by any specified amount.

Much of the research on chromatic polynomials follows a particular pattern: a type of graph construction is described and then properties of the associated chromatic polynomial are determined (for a recent example see Chia [6]). However interesting and useful these results might be, we cannot expect that this approach will find all the chromatic polynomials because we have no classification theorem describing all possible graph constructions.

We have already calculated $f(G,t)$ in the two extreme (and easiest) cases: when G has all possible edges and when G has no edges at all. We need just one more construction-specific result before moving on to our main chapter.

Theorem 2.5. A graph with p points is a tree if and only if $f(G,t) = t(t-1)^{p-1}$.

Proof. (Theorem 12.35 in [7]) Suppose G is a tree. If

$p = 1$ then $G = K_1$ and $f(G,t) = t = t(t-1)^{p-1}$. Now assume $p \geq 1$. Let P be a longest path in G and suppose u and v are the endpoints of P. We note that v must have degree 1 since the existence of a second edge incident to v would lead to either a longer path in G or a cycle in G. In a coloring of G, v can have any color except the one assigned to the adjacent point, so $f(G,t) = (t-1)f(G-v,t)$. $G-v$ is a tree with $p-1$ points so we may assume by induction that $f(G-v,t) = t(t-1)^{p-2}$. We conclude that $f(G,t) = (t-1)t(t-1)^{p-2} = t(t-1)^{p-1}$.

Now suppose $f(G,t) = t(t-1)^{p-1}$. Then $f(G,t)$ has degree p, so G has p points; the coefficient of t^{p-1} is $-\binom{p-1}{1}$ so G has $p-1$ edges; and, t has nonzero coefficient, so G is connected. Thus G is a tree. ■

CHAPTER 3

CONTROLLING THE CHROMATIC COEFFICIENTS

Chromatic sequences

If G is a (p,q) graph with k components and chromatic polynomial $f(G,t) = a_0t^p - a_1t^{p-1} + \cdots + (-1)^{p-k}a_{p-k}t^k$ then we define the *chromatic sequence* of G to be $\alpha(G) = (a_0, a_1, \ldots, a_m)$ where $m = p - k$. It follows from the general properties of chromatic polynomials that $a_0 = 1$ and a_1 is the number of edges in G, if any exist. If G has no edges, its chromatic sequence is simply $\alpha(G) = (1)$. By design, the chromatic sequence does not record the number of points in G, and graphs with differing numbers of points can have the same chromatic sequence. In particular, if $H = G \cup K_1$ then $f(H,t) = tf(G,t)$ and thus $\alpha(H) = \alpha(G)$; that is, isolated points have no impact on the chromatic sequence. Figure 4 illustrates the chromatic sequences up to $q = 6$.

After some initial set-up with sequence operations and partial orderings, we will first compare $\alpha(H)$ with $\alpha(G)$ when H is a subgraph of G. Then, among all graphs with q edges we will identify graphs R_q (containing many cycles) and S_q (containing no cycles) whose chromatic coefficients form bounds for the chromatic coefficients of any graph

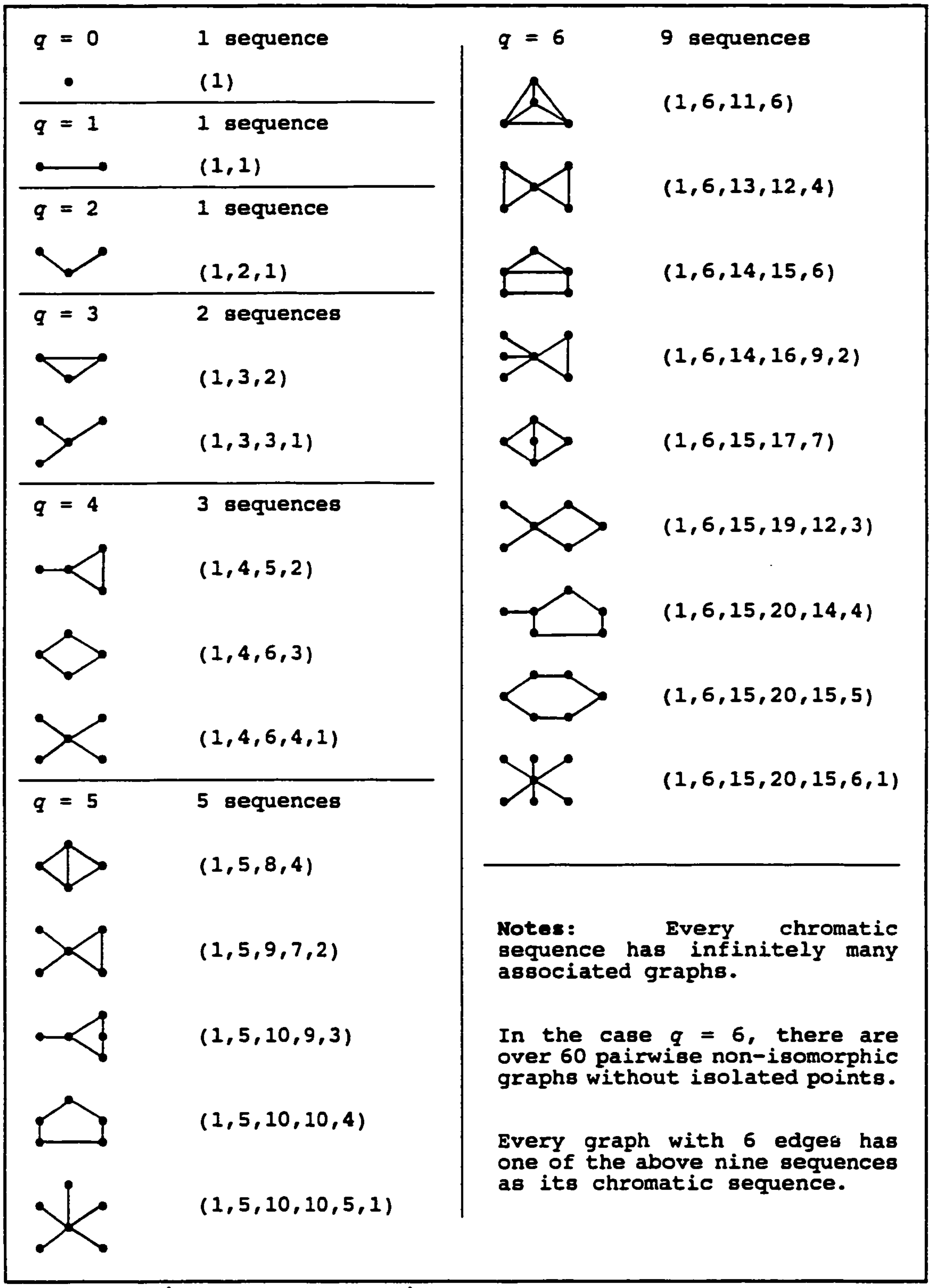


Figure 4 Chromatic α-sequences up to q=6

with q edges. The partial ordering which we employ imposes strict conditions on the manner in which the coefficients for G can differ from the coefficients for R_q and S_q.

Sequence operations and partial ordering

To establish the necessary setting for comparisons of chromatic sequences we let $\mathcal{P}$ be the collection of all nonempty finite sequences of positive integers. Given $A = (a_0, a_1, \ldots, a_m) \in \mathcal{P}$ we write $\ell(A) = m$ and adopt the convention that $a_i = 0$ if $i > \ell(A)$. Since the indexing of the sequence elements begins with 0, $\ell(A)$ does not exactly coincide with our usual understanding of the length of a sequence. We will make use of standard sequence addition and scalar multiplication, and also define a new sequence operation, $\overset{\rightarrow}{+}$, in which the elements of the second summand are shifted to the right by one index position before being added term by term to the first summand. Given sequences $A = (a_0, a_1, \ldots, a_m) \in \mathcal{P}$ and $B = (b_0, b_1, \ldots, b_n) \in \mathcal{P}$ and $d \in \mathrm{N}$ we define

$A + B = (a_0 + b_0,\ a_1 + b_1, \ldots,\ a_t + b_t)$ where $t = \max(m, n)$,
$A \overset{\rightarrow}{+} B = (a_0,\ a_1 + b_0, \ldots, a_t + b_{t-1})$ where $t = \max(m,\ n + 1)$,
and $dA = (da_0,\ da_1,\ \ldots,\ da_m)$.

We also adopt the convention that $A + OB = A \overset{\rightarrow}{+} OB = A$ even though $OB \notin \mathcal{P}$. We note that $\overset{\rightarrow}{+}$ is not associative since

$$(A \overset{\rightarrow}{+} B) \overset{\rightarrow}{+} C = (a_0,\ a_1 + b_0 + c_0,\ a_2 + b_1 + c_1, \ldots) \text{ and}$$
$$A \overset{\rightarrow}{+} (B \overset{\rightarrow}{+} C) = (a_0,\ a_1 + b_0\ ,\ a_2 + b_1 + c_0, \ldots).$$

However, we have the following useful relationship.

Proposition 3.1. If A, B, C, and $D \in \mathcal{P}$ then
$$(A \overset{\rightarrow}{+} B) \overset{\rightarrow}{+} (C \overset{\rightarrow}{+} D) = A \overset{\rightarrow}{+} [(B + C) \overset{\rightarrow}{+} D].$$

Proof. $(A \overset{\rightarrow}{+} B) \overset{\rightarrow}{+} (C \overset{\rightarrow}{+} D)$

$= (a_0,\ a_1 + b_0,\ a_2 + b_1, \ldots) \overset{\rightarrow}{+} (c_0,\ c_1 + d_0,\ c_2 + d_1, \ldots)$

$= (a_0,\ a_1 + b_0 + c_0,\ a_2 + b_1 + c_1 + d_0,\ a_3 + b_2 + c_2 + d_1, \ldots)$

$= A \overset{\rightarrow}{+} (b_0 + c_0,\ b_1 + c_1 + d_0,\ b_2 + c_2 + d_1, \ldots)$

$= A \overset{\rightarrow}{+} [(B + C) \overset{\rightarrow}{+} D]$. ■

We may compare sequences A and B in $\mathcal{P}$ and write $A \le B$ if

(τ_1) $a_0 = b_0$,

(τ_2) $a_i \le b_i$ for all i, and

(τ_3) if $a_j = b_j$ with $1 \le j \le \ell(B)$ then $a_i = b_i$ for $1 \le i \le j$.

That is, $A \le B$ means that either $A = B$ or that A and B agree for a certain number of entries after which any remaining entries in A are strictly smaller than the corresponding entries in B. Note that $A \leq B$ necessitates $\ell(A) \leq \ell(B)$. To specify that $A \leq B$ but $A \neq B$ we write $A < B$; to specify that $A < B$ and $a_i = b_i$ only for $i = 0$ we write $A << B$.

To see that $\leq$ is transitive, suppose A, B, and C from $\mathcal{P}$ satisfy $A \leq B$ and $B \leq C$. Then $a_0 = b_0 = c_0$ and $a_i \leq b_i \leq c_i$ for all i. If $a_j = c_j$ for some $1 \leq j \leq \ell(C)$ then $c_j \neq 0$ implies $a_j \neq 0$ and so $j \leq \ell(A) \leq \ell(B)$. Then $a_j = b_j$ and $j \leq \ell(B)$ imply $a_i = b_i$ for $1 \leq i \leq j$. Likewise $b_j = c_j$ and $j \leq \ell(C)$ imply $b_i = c_i$ for $1 \leq i \leq j$, yielding $a_i = c_i$ for $1 \leq i \leq j$. Thus $A \leq C$, which shows that $\leq$ is transitive.

Now suppose $A < B$ and $B < C$. From the prior discussion we can immediately conclude that $A \leq C$. If $A = C$ we would have $a_i = b_i = c_i$ for $1 \leq i \leq \ell(C)$, which violates $A < B$ (and $B < C$). Thus $A < C$, and so $<$ is transitive. In fact it easily follows that ($A < B$ and $B \leq C$) implies ($A < C$), and ($A \leq B$ and $B < C$) implies ($A < C$).

Finally, assume $A << B$ and $B << C$. By the arguments in the prior paragraph, we may immediately conclude that $A < C$. Suppose for some $i > 0$ we have $a_i = c_i$; then $a_i = b_i$ violates $A << B$. Thus we know $A << C$, and so $<<$ is transitive also.

Caution is required when combining the sequence operations with the orderings. For example, ($A \le B$ and $C \le D$) does not imply ($A \overset{\rightarrow}{+} C \le B \overset{\rightarrow}{+} D$) as can be seen with $A = (1,2)$, $B = (1,2,1)$ and $C = D = (1,2,3)$. Then $A \overset{\rightarrow}{+} C = (1,3,2,3)$ and $B \overset{\rightarrow}{+} D = (1,3,3,3)$, two sequences which cannot be compared using $\le$. All the arithmetic properties which we will need are contained in the following proposition.

Proposition 3.2. Let $A, B, C \in \mathcal{P}$ and $d, e \in \mathbb{N} \cup \{0\}$.

(a) If $A \le C$ and $B < C$ then $A \overset{\rightarrow}{+} B < C \overset{\rightarrow}{+} C$.

(b) If $B \le C$ and $\ell(A) \le \ell(C) + 1$ then $A \overset{\rightarrow}{+} B \le A \overset{\rightarrow}{+} C$.

(c) If $A \le B$ then $A \overset{\rightarrow}{+} dA \le B \overset{\rightarrow}{+} dB$.

(d) If $A \le B$ and $A \le C$ then $A + A \le B + C$.

(e) If $d \le e$ then $A \overset{\rightarrow}{+} dA \le A \overset{\rightarrow}{+} eA$.

Proof. (a) Assume $A \le C$ and $B < C$. The leading term of $A \overset{\rightarrow}{+} B$ is a_0, and the leading term of $C \overset{\rightarrow}{+} C$ is c_0, and $a_0 = c_0$, which verifies τ_1. Since $a_i \le c_i$ for all i, and $b_i \le c_i$ for all i, then $a_i + b_{i-1} \le c_i + c_{i-1}$ for all $i \ge 1$ which verifies τ_2.

Now assume that we have $a_j + b_{j-1} = c_j + c_{j-1}$ for some $1 \le j \le \ell(C \overset{\rightarrow}{+} C) = \ell(C) + 1$. Then $a_j = c_j$ and $b_{j-1} = c_{j-1}$ where $j - 1 \le \ell(C)$. If it were true that $j - 1 = \ell(C)$ then we would have $b_i = c_i$ for all $i \le \ell(C)$, giving $B = C$

which violates $B < C$. So we know that $j - 1 \le \ell(C) - 1$. Then $a_j = c_j$ with $j \le \ell(C) \Rightarrow a_i = c_i$ for $1 \le i \le j$, and $b_{j-1} = c_{j-1}$ with $j - 1 < \ell(C) \Rightarrow b_i = c_i$ for $1 \le i \le j - 1$. We automatically get $b_0 = c_0$ from $B < C$, so altogether we have $a_i + b_{i-1} = c_i + c_{i-1}$ for $1 \le i \le j$. This verifies τ_3, so we know that $A \mathbin{+\!\!\to} B \le C \mathbin{+\!\!\to} C$.

Since $B < C$ we know there is some $k \le \ell(B)$ such that $b_k < c_k$. Then $a_{k+1} + b_k < c_{k+1} + c_k$, so $A \mathbin{+\!\!\to} B \ne C \mathbin{+\!\!\to} C$. Thus $A \mathbin{+\!\!\to} B < C \mathbin{+\!\!\to} C$.

(b) Assume $B \le C$ and $\ell(A) \le \ell(C) + 1$. We want to show that $A \mathbin{+\!\!\to} B \le A \mathbin{+\!\!\to} C$. τ_1 is easy to confirm since both $A \mathbin{+\!\!\to} B$ and $A \mathbin{+\!\!\to} C$ have leading term a_0. Since $b_i \le c_i$ for all i we have $a_i + b_{i-1} \le a_i + c_{i-1}$ for all $i \ge 1$, and so τ_2 is verified.

Suppose that $a_j + b_{j-1} = a_j + c_{j-1}$ for some j satisfying $1 \le j \le \ell(A \mathbin{+\!\!\to} C)$. Then $b_{j-1} = c_{j-1}$. Also consider that $\ell(A \mathbin{+\!\!\to} C) = \max\{\ell(A), \ell(C) + 1\} = \ell(C) + 1$ and therefore $j - 1 \le \ell(C)$. Then $b_i = c_i$ for $1 \le i \le j - 1$, and we automatically get $b_0 = c_0$ from $B \le C$, which results in $a_i + b_{i-1} = a_i + c_{i-1}$ for $1 \le i \le j - 1$. This confirms τ_3 and we conclude that $A \mathbin{+\!\!\to} B \le A \mathbin{+\!\!\to} C$.

(c) Assume $A \le B$ to show $A \stackrel{\rightarrow}{+} dA \le B \stackrel{\rightarrow}{+} dB$. If $d = 0$ there is nothing to show, so assume $d \ge 1$. Since $a_0 = b_0$ we see that τ_1 is satisfied. τ_2 is also easy to verify : $a_i \le b_i$ for all $i \ge 0 \Rightarrow a_i + da_{i-1} \le b_i + db_{i-1}$ for all $i \ge 1$.

Assume $a_j + da_{j-1} = b_j + db_{j-1}$ for some j which satisfies $1 \le j \le \ell(B \stackrel{\rightarrow}{+} dB) = \ell(B) + 1$. Then $a_j = b_j$ and $a_{j-1} = b_{j-1}$ where $0 \le j - 1 \le \ell(B) \Rightarrow a_i = b_i$ for $1 \le i \le j - 1$, and of course $a_0 = b_0$. Then we have $a_i + da_{i-1} = b_i + db_{i-1}$ for $1 \le i \le j$, confirming τ_3.

(d) Suppose $A \le B$ and $A \le C$. Since $a_0 = b_0 = c_0$ we see that $A + A$ has the same leading term as $B + C$, so τ_1 is verified. For all i we have $a_i \le b_i$ and $a_i \le c_i$ and thus $a_i + a_i \le b_i + c_i$ for all i, so τ_2 is satisfied.

Suppose $a_j + a_j = b_j + c_j$ for some $1 \le j \le \ell(B + C)$. Then $b_j + c_j \ne 0 \Rightarrow a_j + a_j \ne 0$ yielding $j \le \ell(A) \le \ell(B)$ and $j \le \ell(A) \le \ell(C)$. Now $a_j = b_j$ and $1 \le j \le \ell(B)$ implies $a_i = b_i$ for $1 \le i \le j$. Likewise $a_j = c_j$ and $1 \le j \le \ell(C)$ implies $a_i = c_i$ for $1 \le i \le j$. Then $a_i + a_i = b_i + c_i$ for $1 \le i \le j$, which verifies τ_3, and we conclude that $A + A \le B + C$.

(e) It is surely clear in this case that $d = e \Rightarrow A \overset{\rightarrow}{+} dA = A \overset{\rightarrow}{+} eA$ and $d < e \Rightarrow A \overset{\rightarrow}{+} dA < A \overset{\rightarrow}{+} eA$. ■

The chromatic sequence of a subgraph

Our first task is to translate the reduction formula, $f(G,t) = f(G - x,t) - f(G/x,t)$, into a related statement about chromatic sequences. Consider for example $G = K_4$ and x an edge in G. Then

$$f(K_4,t) = f(K_4 - x,t) - f(K_4/x,t) = \begin{array}{r} t^4 - 5t^3 + 8t^2 - 4t \\ -\quad\underline{(t^3 - 3t^2 + 2t)} \\ t^4 - 6t^3 + 11t^2 - 6t. \end{array}$$

Examining the associated chromatic sequences, we see that $(1,6,11,6) = (1,5,8,4) \overset{\rightarrow}{+} (1,3,2)$; $\alpha(K_4) = \alpha(K_4 - x) \overset{\rightarrow}{+} \alpha(K_4/x)$.

It is true in general that $\alpha(G) = \alpha(G - x) \overset{\rightarrow}{+} \alpha(G/x)$ because both $f(G - x,t)$ and $f(G/x,t)$ have coefficients which alternate in sign, and $f(G/x,t)$ has degree one less than $f(G - x,t)$.

Lemma 3.3. If H is a subgraph of G then $\alpha(H) \ll \alpha(G)$ unless H and G have the same edges, and then $\alpha(H) = \alpha(G)$.

Proof. If H and G have the same edges then they differ only in their isolated points. Since isolated points have no impact on the chromatic sequence we have $\alpha(H) = \alpha(G)$.

Otherwise H is obtained, up to isolated points, by removing edges from G. Assume that H has n fewer edges than G for some $n \geq 1$, and let $x_1, \ldots, x_n$ be the edges of G which do not belong to H. Since G/x_1 has the same number of components as G, and one less point, then the length of $\alpha(G/x_1)$ is exactly one less than the length of $\alpha(G)$. The reduction formula says $\alpha(G) = \alpha(G - x_1) \overset{\rightarrow}{+} \alpha(G/x_1)$ and we can conclude that $\alpha(G)$ and $\alpha(G - x_1)$ have the same leading term while any subsequent terms in $\alpha(G - x_1)$ are strictly smaller than the corresponding terms in $\alpha(G)$. That is, $\alpha(G - x_1) \ll \alpha(G)$. With n applications of this procedure we eventually arrive at $\alpha(H) = \alpha(G - x_1 - \cdots - x_n) \ll \cdots \ll \alpha(G - x_1) \ll \alpha(G)$. ■

By the technique of the proof we know the following: if $H \subset G$ with $\alpha(H) = (1, h_1, \ldots, h_k)$ and $\alpha(G) = (1, g_1, \ldots, g_m)$, and H has n fewer edges than G, then $h_i \leq g_i - n$ for $1 \leq i \leq k$. Consider the special case when $g_m = 1$ and $H = G - x$, where x is any edge in G; then necessarily $k < m$ so we conclude that H has more components than G and thus x is a bridge. Therefore, whenever $\alpha(G)$ has terminal value 1 we know that G has no cycles. The converse follows from the upcoming proof of Theorem 3.4.

The upper bound for chromatic sequences

For $q \geq 0$ we could choose S_q to be any acyclic graph with q edges; for the sake of being specific, we choose S_q to be the star. If x is an edge of S_q then the reduction formula gives $\alpha(S_q) = \alpha(S_q - x) \dotplus\!\!\rightarrow \alpha(S_q/x) = \alpha(S_{q-1}) \dotplus\!\!\rightarrow \alpha(S_{q-1})$. If $m < n$ we have $S_m \subset S_n$ and Lemma 3.3 implies $\alpha(S_m) \ll \alpha(S_n)$.

Theorem 3.4. If G has q edges then $\alpha(G) \leq \alpha(S_q)$. Equality occurs if and only if G has no cycles.

Proof. From Theorem 12.35 in [7] we have: A graph G with p points is a tree if and only if $f(G,t) = t(t-1)^{p-1}$. In particular, $f(S_q,t) = t(t-1)^q$.

If $\alpha(G) = \alpha(S_q)$ then $\alpha(G)$ has a terminal value 1, and thus G has no cycles. If G has no cycles, we apply Theorem 12.35 to the k components of G and since the chromatic polynomial is multiplicative over components, we have $f(G,t) = t^k(t-1)^q$ yielding $\alpha(G) = \alpha(S_q)$.

What remains is to show that if G has q edges and at least one cycle then $\alpha(G) < \alpha(S_q)$. There is nothing to show when

$q = 0, 1$, or 2. Assume $q \geq 3$. Let C be a cycle in G and let $x \in E(C)$. Since $G - x$ has $q - 1$ edges and G/x has q' edges for some $q' \leq q - 1$ we may assume, by induction, that $\alpha(G - x) \leq \alpha(S_{q-1})$ and $\alpha(G/x) \leq \alpha(S_{q'})$. If G/x has a cycle then, again by induction, $\alpha(G/x) < \alpha(S_{q'}) \leq \alpha(S_{q-1})$. If not, then collapsing edge x caused the cycle C in G to dissipate, and so C is a triangle. Therefore $q' \leq q - 2$ giving $\alpha(G/x) = \alpha(S_{q'}) \leq \alpha(S_{q-2}) < \alpha(S_{q-1})$. Then either way $\alpha(G/x) < \alpha(S_{q-1})$, and the reduction formula and Proposition 3.2 (a) provide the desired result:

$$\alpha(G) = \alpha(G - x) \overset{\rightarrow}{+} \alpha(G/x) < \alpha(S_{q-1}) \overset{\rightarrow}{+} \alpha(S_{q-1}) = \alpha(S_q). \quad \blacksquare$$

This result would allow us to say, for example, that the sequence $A = (1,6,15,18,14,6)$ is not the chromatic sequence of any graph. If A were the chromatic sequence for some graph G, then G would necessarily have 6 edges and by Theorem 3.4 we would have $\alpha(G) \leq \alpha(S_6) = (1,6,15,20,16,6,1)$. Since A does not compare properly with $\alpha(S_6)$ we know that no such graph exists. It is this result which allows us to say that the polynomial $p(t) = t^6 - 6t^5 + 15t^4 - 18t^3 + 14t^2 - 6t$ cannot be the chromatic polynomial of any graph, even if every exponent is raised by the same increment.

The lower bound for chromatic sequences

Having shown that graphs with no cycles produce upper bounds for the chromatic sequences, it is natural to expect that graphs with many cycles produce lower bounds.

We have seen that for the complete graphs we have $f(K_1, t) = t$ and $f(K_p, t) = t(t-1)\cdots(t-p+1) = (t-(p-1))f(K_{p-1}, t)$ if $p \geq 2$; that is $\alpha(K_1) = (1)$ and whenever $p \geq 2$ we have $\alpha(K_p) = \alpha(K_{p-1}) \stackrel{\rightarrow}{+} (p-1)\alpha(K_{p-1})$.

For a fixed q we let $\pi(q) = \max\{i \mid \binom{i}{2} \leq q\}$ so that $q = \binom{m}{2} + r$ with $m = \pi(q)$ and $0 \leq r \leq m-1$. Then define R_q to be the unique connected graph (up to isomorphism) with q edges such that $K_m \subseteq R_q \subset K_{m+1}$. If $r = 0$ then $R_q = K_m$, and if $r > 0$ then $R_q = K_m \cup \{\text{a point}\} \cup \{r \text{ edges}\}$. Hence $f(R_q, t) = f(K_m, t)$ if $r = 0$, and $f(R_q, t) = (t-r)f(K_m, t)$ if $r > 0$, yielding $\alpha(R_q) = \alpha(K_m) \stackrel{\rightarrow}{+} r\alpha(K_m)$. If $j < k$ we have $R_j \subset R_k$ and conclude from Lemma 3.3 that $\alpha(R_j) << \alpha(R_k)$.

The following rather tedious numerical result provides a crucial step in the upcoming Theorem 3.7.

Proposition 3.5. When $q \geq 1$ and $0 \leq d \leq \pi(q) - 1$, then $\alpha(R_q) \leq \alpha(R_{q-d}) \stackrel{\rightarrow}{+} d\alpha(R_{q-d})$.

Proof. We have $q = \binom{m}{2} + r$ where $m = \pi(q)$ and $0 \leq r \leq m - 1$. If $d \leq r$ then $q - d = \binom{m}{2} + r - d$ where $0 \leq r - d \leq m - 1$. Thus, writing $A = \alpha(K_m)$ and using Propositions 3.1, 3.2(b) and 3.2(e) we have

$$\begin{aligned}
\alpha(R_{q-d}) \stackrel{\rightarrow}{+} d\alpha(R_{q-d}) &= [A \stackrel{\rightarrow}{+} (r - d)A] \stackrel{\rightarrow}{+} d[A \stackrel{\rightarrow}{+} (r - d)A] \\
&= A \stackrel{\rightarrow}{+} [rA \stackrel{\rightarrow}{+} d(r - d)A] \\
&\geq A \stackrel{\rightarrow}{+} rA \\
&= \alpha(R_q).
\end{aligned}$$

Thus we may suppose that $d > r$. Then $q - d = \binom{m}{2} + r - d = \binom{m-1}{2} + \delta$

where $0 \leq \delta = m - 1 + r - d \leq m - 2$. If $z = d - r > 0$ then

$$\begin{aligned}
d\delta &= d(m - 1 + r - d) \\
&= (r + z)(m - 1 - z) \\
&= r(m - 1) + z(m - 1 - r - z) \\
&\geq r(m - 1) \qquad\qquad (\blacklozenge)
\end{aligned}$$

since $r + z = d \leq m - 1$. Writing $A = \alpha(K_m)$ and $B = \alpha(K_{m-1})$ and using Propositions 3.1, 3.2(b) and 3.2(e) we have

$$\begin{aligned}
\alpha(R_q) &= A \stackrel{\rightarrow}{+} rA \\
&= [B \stackrel{\rightarrow}{+} (m - 1)B] \stackrel{\rightarrow}{+} r[B \stackrel{\rightarrow}{+} (m - 1)B] \\
&= B \stackrel{\rightarrow}{+} [(m - 1 + r)B \stackrel{\rightarrow}{+} r(m - 1)\ B]
\end{aligned}$$

and

$$\begin{aligned}\alpha(R_{q-d}) \overset{\to}{+} d\alpha(R_{q-d}) &= [B \overset{\to}{+} \delta B] \overset{\to}{+} d[B \overset{\to}{+} \delta B] \\ &= B \overset{\to}{+} [(\delta + d)B \overset{\to}{+} d\delta B] \\ &\geq \alpha(R_q)\end{aligned}$$

since $\delta + d = m - 1 + r$ and $d\delta \geq r(m - 1)$ by (♦). ■

Proposition 3.5 fails if $d \geq \pi(q)$. Consider $q = 5$ (so $\pi(q) = 3$) and $d = 3$. Then $\alpha(R_5) = (1,5,8,4) \not\leq (1,5,7,3) = (1,2,1) \overset{\to}{+} 3(1,2,1) = \alpha(R_2) \overset{\to}{+} 3\alpha(R_2)$.

Proposition 3.6. If G has $q \geq 1$ edges then there is some point in G with degree d satisfying $0 < d \leq \pi(q) - 1$.

Proof. Let H be the subgraph of G consisting of all of G's edges and all of G's nonisolated points. H has q edges. Among all graphs with q edges, R_q has the fewest points, so the average degree in H cannot exceed the average degree in R_q (see Theorem 2.1 in [7]). R_q is a proper subgraph of the complete graph on $\pi(q) + 1$ points and thus the average degree in R_q is strictly less than $\pi(q)$. Then H has average degree strictly less than $\pi(q)$, which guarantees the existence of a point w with $0 < deg\ w \leq \pi(q) - 1$. ■

Now we can prove the main result.

Theorem 3.7. If G has q edges then $\alpha(R_q) \leq \alpha(G)$.

Proof. We proceed by induction on q. If $q=0$ we have $\alpha(R_q) = (1) = \alpha(G)$. Now assume that $q \geq 1$ and write $q=\binom{m}{2}+r$ where $m = \pi(q)$ and $0 \leq r \leq m-1$. By Proposition 3.6 there is a point w in G with degree d satisfying $0 < d \leq m-1$. Let $L = G - w$ and let $x_1,\ldots,x_d$ be the edges incident with w. Repeated application of the reduction formula and Proposition 3.1 yields $\alpha(G) = \alpha(G-x_1) \overset{\rightarrow}{+} \alpha(G/x_1)$ when $d=1$, and when $d \geq 2$:

$$
\begin{aligned}
\alpha(G) &= \alpha(G-x_1) \overset{\rightarrow}{+} \alpha(G/x_1) \\
&= (\alpha(G-x_1-x_2) \overset{\rightarrow}{+} \alpha((G-x_1)/x_2)) \overset{\rightarrow}{+} \alpha(G/x_1) \\
&= \alpha(G-x_1-x_2) \overset{\rightarrow}{+} (\alpha((G-x_1)/x_2) + \alpha(G/x_1)) \\
&\vdots \\
&= \alpha(G-x_1-\cdots-x_d) \overset{\rightarrow}{+} (G/x_1 + \sum_{i=2}^{d} \alpha((G-x_1-\cdots-x_{i-1})/x_i))\,.
\end{aligned}
$$

Note that $\alpha(L) = \alpha(G-x_1-\cdots-x_d)$. Furthermore, L is a subgraph of each of the contraction graphs in the summation, so by Lemma 3.3 and Proposition 3.2(d) we have

$$d\alpha(L) \le \alpha(G/x_1) + \sum_{i=2}^{d} \alpha((G - x_1 - \cdots - x_{i-1})/x_i)\,.$$

By Proposition 3.2(b) we have $\alpha(G) \ge \alpha(L) \overset{\rightarrow}{+} d\alpha(L)$. Since $|E(L)| = q - d \le q - 1$ we may assume, by induction, that $\alpha(R_{q-d}) \le \alpha(L)$ and thus Propositions 3.5 and 3.2(c) yield: $\alpha(R_q) \le \alpha(R_{q-d}) \overset{\rightarrow}{+} d\alpha(R_{q-d}) \le \alpha(L) \overset{\rightarrow}{+} d\alpha(L) \le \alpha(G)$. ■

Summary

Theorems 3.4 and 3.7 combine to produce the main result: if G is a graph with q edges then $\alpha(R_q) \le \alpha(G) \le \alpha(S_q)$. Continuing our prior discussion we know, for example, that if G is a graph with 6 edges then its chromatic sequence must satisfy $(1,6,11,6) \le \alpha(G) \le (1,6,15,20,15,6,1)$. We should bear in mind that there are sequences which satisfy the required inequalities but are not chromatic. For example, $\alpha(R_6) \le (1,6,12,7) \le \alpha(S_6)$, but $(1,6,12,7)$ is not the chromatic sequence for any graph. However the chromatic sequences for the R_q and S_q graphs are particularly easy to calculate, and the bounding conditions provide us with a rapid means for dispensing with many polynomials which might otherwise appear to be potentially chromatic.

CHAPTER 4

CHROMATIC COEFFICIENTS IN THE FALLING FACTORIALS BASIS

$f(G,t)$ with respect to falling factorials

Recall that $f(K_p,t) = t(t-1)\cdots(t-(p-1)) = t_p$, the falling factorial of degree p. These falling factorials form a basis of $\mathbb{Z}[t]$ relative to which every chromatic polynomial is expressed entirely with nonnegative coefficients, as evidenced in the proof of Corollary 2.2. What follows are well-known properties.

Theorem 4.1. If G is a (p,q) graph and $f(G,t)$ is expressed with falling factorials then

(a) t_j has a positive coefficient if $\chi(G) \le j \le p$, and otherwise the coefficient is zero.

(b) t_p has coefficient 1.

(c) t_{p-1} has coefficient $\binom{p}{2}-q$.

Proof. We proceed by induction on $m = \binom{p}{2} - q$, the number of edges missing from G. If $m = 0$ then G is a complete graph, so $f(G,t) = t_p$, and we see that (a), (b), and (c) are true.

Now suppose $m \ge 1$ and let $x = uv$ be an edge missing from G.

By the completion formula, $f(G,t) = f(G+x,t) + f((G+x)/x,t)$ where both $G + x$ and $(G+x)/x$ have fewer missing edges than G. $G + x$ is a $(p,q+1)$ graph and $(G+x)/x$ is a $(p-1,q')$ for some $q' \leq q$.

By induction we may assume that $f(G+x,t)$ expressed in falling factorials has coefficient 1 for t_p, coefficient $\binom{p}{2}-(q+1)$ for t_{p-1}, and positive coefficients for t_j whenever $\chi(G+x) \leq j \leq p$, with all other coefficients zero. Likewise $f((G+x)/x,t)$ expressed in falling factorials has coefficient 1 for t_{p-1}, positive coefficients for t_j whenever $\chi((G+x)/x) \leq j \leq p - 1$ and coefficients equal to zero otherwise.

From the completion formula we determine that $\chi(G) = \min\{ n \mid f(G,n) > 0\} = \min\{ n \mid f(G+x,n) > 0$ or $f((G+x)/x, n) > 0\} = \min\{ \chi(G+x), \chi((G+x)/x)\}$. Also, the coefficient of t_j in $f(G,t)$ is the sum of the coefficients of t_j in $f(G+x, t)$ and in $f((G+x)x, t)$. We conclude that $f(G,t)$, when expressed with falling factorials:

(a) has a positive coefficient for t_j when $\chi(G) \leq j \leq p$, and otherwise the coefficient is zero,

(b) has coefficient 1 for t_p,

(c) has coefficient $\binom{p}{2} - q$ for t_{p-1}. ■

The β-sequences and duality

For a (p,q) graph G we earlier defined the sequence $\alpha(G)$ of the norms of the coefficients from the chromatic polynomial $f(G,t) = a_0t^p - a_1t^{p-1} + \cdots + (-1)^{p-k}a_{p-k}t^k$. Now we define $\beta(G)$ to be the sequence $(b_0, b_1, \ldots, b_{p-\chi(G)})$ of positive

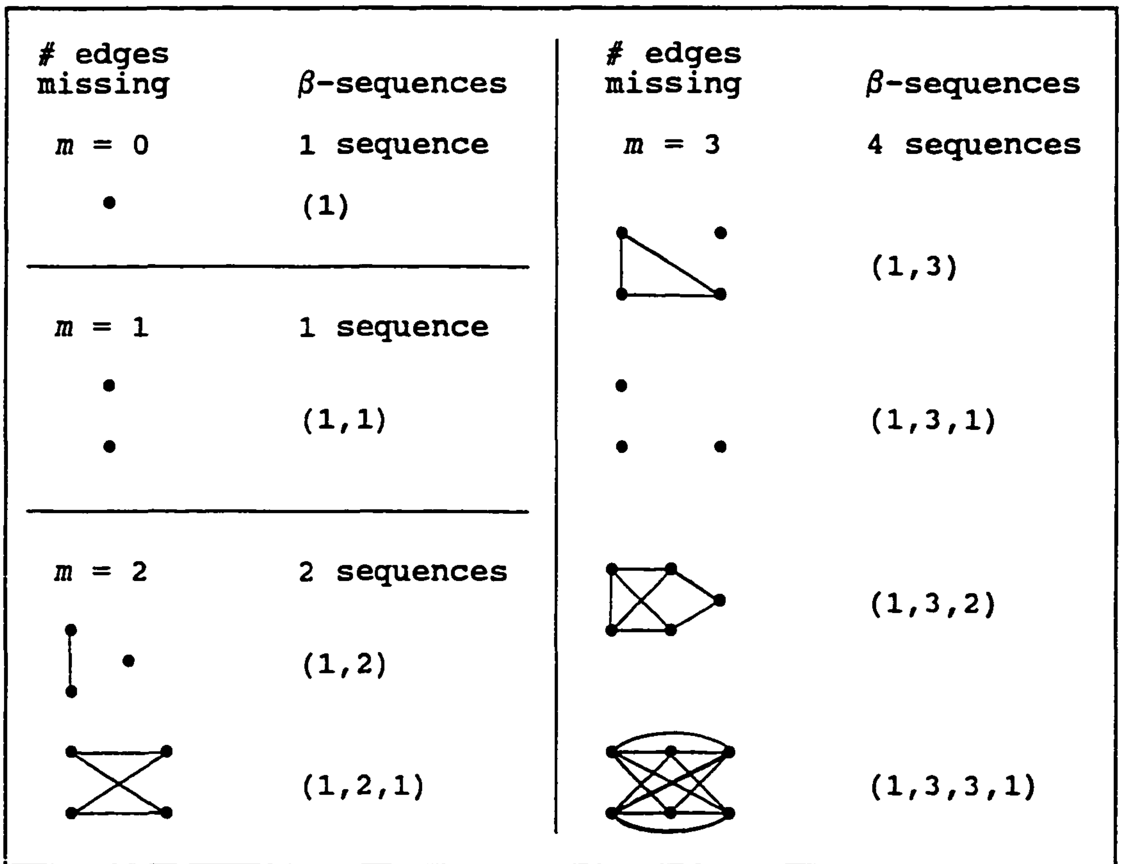


Figure 5 β-sequences up to $m = 3$

coefficients from $f(G,t) = b_0t_p + b_1t_{p-1} + \ldots + b_{p-\chi(G)}t_{\chi(G)}$, where $b_0 = 1$ and b_1 is the number of edges missing from G. Figure 5 illustrates the β-sequences for graphs with up to 3 missing edges.

The β-sequences belong to $\mathcal{P}$ and so they are subject to the arithmetic properties and partial ordering developed in Chapter 3.

In studying the α-sequences we worked with a translation of the reduction formula: $\alpha(G) = \alpha(G-x) \stackrel{\rightarrow}{+} \alpha(G/x)$ where x is an edge in G. Now we will work primarily with a translation of the completion formula for β-sequences : $\beta(G) = \beta(G + x) \stackrel{\rightarrow}{+} \beta((G + x)/x)$ where x is an edge missing from G. We found that α-sequences are unaffected by isolated points; the counterpart with respect to falling factorials is that β-sequences are unaffected by points of full degree.

Lemma 4.2. $\beta(G) = \beta(K_1 \dotplus G)$ for any graph G.

Proof. We proceed by induction on the number of edges missing from G. If G has 0 missing edges then both G and $K_1 \dotplus G$ are complete graphs and thus $\beta(G) = (1) = \beta(G \dotplus K_1)$. Otherwise, G has m missing edges for some $m \geq 1$. Let x be one of the missing edges. $\beta(G) = \beta(G + x) \stackrel{\rightarrow}{+} \beta((G + x)/x)$ where both $G + x$ and $(G + x)/x$ have fewer than m missing edges. Thus we may assume $\beta(G+x) = \beta(K_1 \dotplus (G+x))$ and $\beta((G+x)/x) = \beta(K_1 \dotplus ((G+x)/x))$. Then

$$
\begin{aligned}
\beta(G) &= \beta(K_1 \dotplus (G + x)) \mapsto \beta(K_1 \dotplus ((G + x)/x\ )) \\
&= \beta((K_1 \dotplus G) + x) \looparrowright \beta(((K_1 \dotplus G) + x)/x) \\
&= \beta(K_1 \dotplus G). \qquad \blacksquare
\end{aligned}
$$

The general result, $\beta(G) = \beta(K_j \dotplus G)$ for $j \geq 1$, follows from the observation that $K_j \dotplus G = (K_1 \dotplus (\cdots(K_1 \dotplus (K_1 \dotplus G))\cdots))$ where there are a total of j copies of K_1 in the iterated join.

We do not have a counterpart in the β-sequences for every result we found for the α-sequences. Recall that $H \subseteq G \Rightarrow \alpha(H) \leq \alpha(G)$. If the duality we have seen between the α and β sequences were complete, we would expect to find that $H \subseteq G \Rightarrow \beta(G) \leq \beta(H)$, but unfortunately this is not true; consider the graphs $P_3 \subseteq C_4$ where we have $\beta(P_3) = (1,3,1)$ and $\beta(C_4) = (1,2,1)$. It is likewise tempting to suspect that $\alpha(G) = \beta(\overline{G})$ since adding edges to $\overline{G}$ seems complimentary to subtracting edges from G. This is true when $G = \overline{K}_p$, and for another class of graphs described below, but typically there is great disparity between $\alpha(G)$ and $\beta(\overline{G})$ (see Figure 6).

Now we define a family of graphs which we will later show provide upper bounds for the β-sequences. Let $X_{\bar{0}} = K_1$, $X_{\bar{1}} = \overline{K}_2$, $X_{\bar{2}} = \overline{K}_2 \dotplus \overline{K}_2$, and for $m \geq 3$, $X_{\bar{m}}$ is the iterated join

$\alpha(G) = (1,4,5,2)$ $\alpha(\overline{G}) = (1,6,13,12,4)$

$\beta(G) = (1,6,6)$ $\beta(\overline{G}) = (1,4,2)$

Figure 6 Comparison of $\alpha(G)$, $\beta(G)$, $\alpha(\overline{G})$, $\beta(\overline{G})$

of m copies of $\overline{K}_2$. Note that $X_{\bar{m}}$ has exactly m missing edges. Consider the complimentary graphs: $\overline{X_{\bar{0}}} = K_1$, $\overline{X_{\bar{1}}} = K_2$, $\overline{X_{\bar{2}}} = \overline{(\overline{K_2} + \overline{K_2})} = K_2 \bigcup K_2$ and in general for $m \geq 3$, $\overline{X_{\bar{m}}}$ is the disjoint union of m copies of K_2. Then $\overline{X_{\bar{m}}}$ is a graph with m edges and no cycles, and by Theorem 3.4 we have $\alpha(\overline{X_{\bar{m}}}) = \alpha(S_m)$.

Lemma 4.3. $\beta(X_{\bar{m}}) = \alpha(\overline{X_{\bar{m}}})$ for all $m \geq 0$.

Proof. By previous comments, it will suffice to show that $\beta(X_{\bar{m}}) = \alpha(S_m)$ for all $m \geq 0$. We proceed by induction on m.

When $m = 0$ we have $\beta(X_{\bar{0}}) = \beta(K_1) = (1) = \alpha(S_0)$.
When $m = 1$ we have $\beta(X_{\bar{1}}) = \beta(\overline{K}_2) = (1,1) = \alpha(S_1)$.
Now assume $m \geq 2$ and let y be an edge missing from $X_{\bar{m}}$. Then $X_{\bar{m}} + y = K_2 + X_{\overline{m-1}}$ and $(X_{\bar{m}} + y)/y = K_1 + X_{\overline{m-1}}$. Applying the completion formula, Lemma 4.2, and induction we have

$$
\begin{aligned}
\beta(X_{\bar{m}}) &= \beta(K_2 + X_{\overline{m-1}}) \overset{\rightarrow}{+} \beta(K_1 + X_{\overline{m-1}}) \\
&= \beta(X_{\overline{m-1}}) \overset{\rightarrow}{+} \beta(X_{\overline{m-1}}) \\
&= \alpha(S_{m-1}) \overset{\rightarrow}{+} \alpha(S_{m-1}) = \alpha(S_m). \qquad \blacksquare
\end{aligned}
$$

We will need two arithmetic properties of the $X_{\bar{m}}$ graphs which are analogous to results for star graphs: for $m \geq 1$, $\beta(X_{\bar{m}}) = \alpha(S_m) = \alpha(S_{m-1}) \overset{\rightarrow}{+} \alpha(S_{m-1}) = \beta(X_{\overline{m-1}}) \overset{\rightarrow}{+} \beta(X_{\overline{m-1}})$, and if $m \leq n$ then $\beta(X_{\bar{m}}) = \alpha(S_m) \leq \alpha(S_n) = \beta(X_{\bar{n}})$.

Upper and lower bounds for the β-sequences

The lower bounds for the β-sequences are so easy to locate that it seems like cheating. For $m \geq 0$ define $T_{\bar{m}} = \overline{S}_m$. Then $T_{\bar{0}} = K_1$ and $T_{\bar{m}} = K_m \cup K_1$ when $m \geq 1$, so $T_{\bar{m}}$ has m missing edges.

Proposition 4.4. If G is a graph with m missing edges then $\beta(T_{\bar{m}}) \leq \beta(G)$.

Proof. If $m = 0$ then G is a complete graph, so $\beta(G) = 1$ and $\beta(T_{\bar{0}}) = \beta(K_1) = (1)$. Suppose $m \geq 1$. With m applications of the completion formula we arrive at $\beta(T_{\bar{m}}) = (1, m)$. From the general properties of the β-sequences we know that $\beta(G)$ has the form $(1, m, \ldots)$. Thus $\beta(T_{\bar{m}}) \leq \beta(G)$. $\blacksquare$

Now we need to demonstrate one relationship involving lengths of sequences in order to facilitate the final theorem, which will establish the upper bounds for the β-sequences.

Proposition 4.5. $\ell(\beta(G)) \geq \ell(\beta(G/x))$ for any edge x of G.

Proof. Suppose $x = uv$ and assume the points of G/x have been labelled so that $u \in G/x$ has all the adjacencies of both u and v in $G - x$. Fix a coloring of G/x using the smallest possible number of colors. Transfer the coloring back to G by assigning v a new color not used in the coloring of G/x, hence providing a coloring of G from $\chi(G/x) + 1$ colors. Thus we know $\chi(G) \leq \chi(G/x) + 1$. Then $\ell(\beta(G)) = p - \chi(G) \geq (p - 1) - \chi(G/x) = \ell(\beta(G/x))$, where p is the number of points in G. ■

Theorem 4.6. If G has m missing edges then $\beta(G) \leq \beta(X_{\bar{m}})$.

Proof. We proceed by induction on m. If $m = 0$ then both G and $X_{\bar{0}}$ are complete graphs; $\beta(G) = (1) = \beta(X_{\bar{0}})$. Now assume $m \geq 1$ and let y be one of the edges missing from G. By the completion formula, $\beta(G) = \beta(G + y) \overset{\rightarrow}{+} \beta((G + y)/y)$ where $G + y$ has $m - 1$ missing edges and $(G + y)/y$ has $m - j$ missing edges for some $j \geq 1$. By induction we may assume

$\beta(G + y) \le \beta(X_{\overline{m-1}})$ and $\beta((G+y)/y) \le \beta(X_{\overline{m-j}}) \le \beta(X_{\overline{m-1}})$. Either $\beta((G+y)/y) = \beta(X_{\overline{m-1}})$ or $\beta((G+y)/y)) < \beta(X_{\overline{m-1}})$.

First assume $\beta((G+y)/y) = \beta(X_{\overline{m-1}})$.

Then $\ell(\beta((G + y)/y)) = m - 1$. From Proposition 4.5 we have $\ell(\beta(G + y)) \ge \ell(\beta((G + y)/y)) = m - 1$, and $\beta(G+y) \le \beta(X_{\overline{m-1}})$ implies $\ell(\beta(G + y)) \le m - 1$. Then $\ell(\beta(G + y)) = m - 1$ and so $\beta(G+y) = \beta(X_{\overline{m-1}})$ because $\beta(X_{\overline{m-1}})$ has terminal value 1. Then $\beta(G) = \beta(G + y) \mathrel{+\!\!\rightarrow} \beta((G + y)/y) = \beta(X_{\overline{m-1}}) \mathrel{+\!\!\rightarrow} \beta(X_{\overline{m-1}}) = \beta(X_{\overline{m}})$.

Now suppose $\beta((G + y)/y) < \beta(X_{\overline{m-1}})$. Then by Proposition 3.2(a), $\beta(G) = \beta(G + y) \mathrel{+\!\!\rightarrow} \beta((G + y)/y) < \beta(X_{\overline{m-1}}) \mathrel{+\!\!\rightarrow} \beta(X_{\overline{m-1}}) = \beta(X_{\overline{m}})$.
■

By combining Proposition 4.4 and Theorem 4.6 we know that $\beta(T_{\overline{m}}) \le \beta(G) \le \beta(X_{\overline{m}})$ if G is any graph with m missing edges. Since $T_{\overline{m}}$ and $X_{\overline{m}}$ are specified graphs with m missing edges, we know that we have found the best possible bounds. It is fortunate that the sequences $\beta(T_{\overline{m}})$ and $\beta(X_{\overline{m}})$ are so easily determined: $\beta(T_{\overline{0}}) = (1) = \beta(X_{\overline{0}})$ and when $m \ge 1$ then we have $\beta(T_{\overline{m}}) = (1, m)$ and $\beta(X_{\overline{m}}) = (\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m})$.

It is important to note that a sequence can satisfy the bounding conditions and still not be chromatic; for example, $\beta(T_3) \le (1,3,3) \le \beta(X_{\bar{3}})$ but there is no graph with 3 missing edges whose β-sequence is (1,3,3). Thus the bounding conditions form a necessary, but not sufficient, condition for a sequence to be chromatic. Nevertheless, just as with the α-sequences, these bounding conditions with β-sequences serve to greatly narrow the field of polynomials considered to be candidates for the chromatic polynomial of some graph.

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