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GRADUATE COLLEGE

TOPICS IN FEEDBACK CONTROL STABILIZATION

A DISSERTATION
SUBMITTED TO THE GRADUATE FACULTY
in partial fulfillment of the requirements for the
degree of
DOCTOR OF PHILOSOPHY

by
Chi Chen
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TOPICS IN FEEDBACK CONTROL STABILIZATION

A DISSERTATION

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

By

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Abstract

In this dissertation, we first show that for a class of uncertain nonlinear systems, the robust output feedback stabilizability is equivalent to the existence of robust output Lyapunov functions with the small control property. This is a generalization of a previous result of Tsinias and Kalouptsidis[1][2]. Then we construct state feedback and output feedback controls for some specific uncertain systems using either variable structure controls or continuous feedback controls. The feedback controls are designed to compensate for uncertainties and disturbancees present in the systems. Some control designs are robust versions of those proposed by Gu[6].

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0. Introduction

Feedback stabilization of linear and nonlinear systems at a specified equilibrium is a central topic in control theory and has been studied by many authors ([1]-[37]). One approach is via Lyapunov's second method, where the feedback laws and Lyapunov functions are applied to stabilize the closed loop system (see, for example, [1]-[5], [6]-[8], [11], [12], [16]). Tsinias and Kalouptsidis ([1], [2]) studied the output feedback problem for the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x \in R^n, u \in R^l \\ y &= h(x) \in R^k. \end{aligned} \tag{0.1}$$

They showed that the stabilization at the origin by means of an output feedback law $u = \Phi(y)$, where $\Phi : R^k \to R^l$, is equivalent to the existence of an output feedback Lyapunov function with the small control property.

In section 1.1 of Chapter 1 of this dissertation we will show that the above statement is true for a broader class of systems which involve an uncertainty w that takes values in a known compact set:

$$\begin{aligned} \dot{x} &= F(x, u, w) \\ y &= h(x) \end{aligned} \tag{0.2}$$

This generalization is theoretically important. However, the proof is based on the partition of unity techniques, which are inherently nonconstructive and thus do not offer a practical means of constructing the feedback control laws when the Lyapunov functions are given. Specific applications require methods for explicit construction of the feedback control laws. Sontag and Wang ([11], [16]) gave explicit formulas for the output feedback laws for the system (0.1) under full state feedback and with no uncertainty. In section 1.2, we will generalize this result to allow an uncertainty in $f(x)$ through the consideration of a so-called marginal function associated to the system.

The more practical problem is how to find the Lyapunov functions if some information about the system is known. One approach to the control of uncertain systems was proposed by Leitmann and coworkers ([23]-[26]). This approach requires that the nominal system, that is, the system without uncertainty, be already uniformly asymptotically stable and that a Lyapunov function for this nominal system be known. Then the stabilization of the system with uncertainty proceeds from this point. Motivated by this idea, we construct several feedback control laws throughout the remainder of this dissertation. In section 1.2, we construct a simple partial feedback control law for the diagonally uncertain system

$$\begin{aligned} \dot{x} &= (A + wI_n)x + Bu \\ y &= x_2, \ x(0) = x_0 \end{aligned} \tag{0.3}$$

under the assumption that (A, B) is controllable. The essence of the proof is that with proper choice of the linear state feedback for the nominal system, we can make the resulting system matrix diagonal and negative. Then we can keep the fast modes untouched and design a feedback law according to the slow modes. Under reasonable assumptions, this will result in a closed loop system that is stable.

In Chapters 2–4, we will construct state feedback control laws for partially matched systems with uncertainty

$$\dot{x} = (A + \Delta A(w(t)))x + B(1 + \Delta b(w(t)))u + B\Delta D(w(t)) \tag{0.4}$$

using Variable Structure Control (VSC) or differential inclusions. The VSC approach results in a feedback control which is discontinuous on some switching surface in the state space ([27]-[36]). Although the feedback control law is not continuous along the so-called "sliding surface" $s(x) = 0$, when the equivalent control on the sliding surface exists (that is, when there exists a controller u_{ea} such that $\dot{s}(x) = 0$), then the sliding surface is invariant under the closed loop dynamics. Therefore we can apply the Lyapunov function to stabilize the closed loop system. Furthermore, if we have $s^{\top}(x)\dot{s}(x) < 0$ in a neighborhood of a region of the sliding surface $s(x) = 0$ (in which case we have achieved "sliding mode control"), then the system dynamics on the sliding surface are insensitive to the uncertainty ([30], [31]). Differential inclusions are introduced in section 3.1 to overcome technical difficulties when the equivalent control strategy is implemented in multiple-input/multiple-output (MIMO) systems. As a byproduct, in Chapter 3 we present some related results to another class of systems with delayed perturbation:

$$\dot{x} = (A + \Delta A(w(t)))x + E(t)x[t - h(t)] + B(1 + \Delta b(w(t)))u + B\Delta D(w(t)) \quad (0.5)$$

Also in Chapters 2–4, we show that for systems that satisfy rather stringent structure conditions one can actually obtain continuous output feedback control laws. The presence of these strong structure assumptions is hardly surprising since the possible loss of state information inherent in output feedback laws will invariably limit their applicability [6]. Our output feedback laws will drive the closed loop system flows to some fixed ball, whose radius, in some cases, can be made as small as we wish.

We conclude this introduction by giving some background of the structure of (0.4). In [6]–[8], the system

$$\dot{x} = (A + \Delta A(w(t)))x + B(1 + \Delta b(w(t)))u$$

was studied under the so-called "matching condition"

$$\Delta A = B\Delta E, \quad \Delta B = B\Delta b$$

To make the system more robust, we will drop the matching condition for ΔA and add the term $B\Delta D$ to the system. Thus some of the results in this dissertation can be viewed as extensions of [6]-[8].

As a further extension, in Chapter 4, we consider a modified version of the system (0.4) which has a partitioned structure into a family of subsystems of the same general form. Each subsystem has an associated input and output, distinct from those of the other subsystems, and all of the states enter into each subsystem linearly. What is surprising is that, in addition to the presence of uncertainties and disturbances (on which we must impose priori bounds), we can achieve either asymptotic or practical stabilizability even in the case where each subsystem is affected by outputs from the remaining subsystems in a rather arbitrary(e. g., unbounded and/or nonlinear) manner. These results are related to, and partially generalize, those in [21].

1. ROLC Vs ROFS

1.1. A Sufficient And Necessary Condition For Robust Control

We consider a nonlinear input-output system with uncertainty of the form

$$\begin{aligned} \dot{x} &= F(x,u,w) \\ y &= h(x) \end{aligned} \tag{1.1}$$

where $x \in R^n, y \in R^k (k \leq n), u \in R^l, w \in R^d$. Here x is the state variable, u is the control, w is the disturbance, and y is the output. We further assume that F is smooth and u-affine, (i. e., for any a, b and u_1, u_2

$$F(x, au_1 + bu_2, w) = aF(x, u_1, w) + bF(x, u_2, w) \ ),$$

$F(0,u,w) = 0$, h is an open map such that $h(0) = 0$, and $w \in W$, a compact subset in R^d containing 0.

Definition 1.1 (1.1) satisfies the robust output Lyapunov condition (ROLC) if there exists a real-valued function $V : R^n \to R^+$, which is smooth in a punctured neighborhood of 0 and satisfies:

(1) V is positive definite; i. e., $V(0) = 0$ and $V(x) > 0$ for every $x \neq 0$ in a compact neighborhood S of zero;

(2) there exist a neighborhood K of $0 \in R^k$ with $K \subseteq h(S)$ and a continuous, positive-definite function $c : R^+ \to R^+$ (i. e., $c(0) = 0$, and $c(t) > 0$ for $t \neq 0$) such that for any $y \in K$ there exists $u = u(y) \in R^l$ depending on y with $u(0) = 0$ and

$$\bar{F}(x, u(y), w) \leq -c(||x||) \tag{1.2}$$

for any $x \in (h^{-1}(y) \cap S)\backslash\{0\}$ and $w \in W$, where $\bar{F}(x,u,w) = \nabla V \cdot F(x,u,w)$, $h^{-1}(y) = \{x \in R^n : h(x) = y\}$, and $||\cdot||$ is the usual norm.

Note in particular that

$$\bar{F}(x,0,w) \leq -c(||x||)$$

for any $x \in (h^{-1}(0) \cap S)\backslash\{0\}$ and $w \in W$.

Definition 1.2 (1.1) satisfies the ROLC with small control property if it satisfies the ROLC and if there exists a positive continuous function $L(y)$ defined on a neighborhood K of $0 \in R^k$ such that $L(y) \to 0$ as $y \to 0$ and for every $y \in K$ there exists $u = u(y) \in R^l$ depending on y such that $||u(y)|| < L(y)$ for $y \neq 0$ and

$$\bar{F}(x,u(y),w) \leq -c(||x||) \tag{1.3}$$

for any $x \in (h^{-1}(y) \cap S)\backslash\{0\}$ and $w \in W$.

Definition 1.3 (1.1) said to be robust output feedback stabilized (ROFS) if there exist a neighborhood K of $0 \in R^k$ and a mapping $u : R^k \to R^l$ which is continuous on K, smooth on $K\backslash\{0\}$, and is such that the closed-loop analog of (1.1) after feedback, given by

$$\dot{x} = F(x, u(h(x)), w),$$

satisfies:

(1) robust stability: $\forall\epsilon > 0$ there exists a $\delta > 0$ such that

$$||x(t, x_0, w)|| \leq \epsilon$$

for all $w \in W$ whenever $|x_0| \leq \delta$ and $t \geq 0$;

(2) robust attraction: $\forall\epsilon > 0$ and $r > 0$ there exists $T > 0$ such that for every $w \in W$,

$$||x(t, x_0, w)|| \leq \epsilon$$

whenever $||x_0|| < r$ and $t \geq T$.

Lemma 1.1. If (1.1) satisfies ROLC, and the set $h^{-1}(0) \cap S$ is positively invariant for the system $\dot{x} = F(x, 0, w)$, then (1.1) is OFS (output feedback stabilized).

The corresponding output feedback law $\Phi : R^k \to R^l$ is smooth in a punctured neighborhood of 0.

Proof. Let

$$Q(x,u,w) = \bar{F}(x,u,w) + \frac{1}{2}c(||x||).$$

Since (1.1) satisfies ROLC, for every $y \in K\backslash\{0\}$ there exists $u = u(y) \in R^l$, such that

$$\begin{aligned} Q(x,u(y),w) &= \bar{F}(x,u(y),w) + \frac{1}{2}c(||x||) \\ &\leq -\frac{1}{2}c(||x||) \\ &< 0 \end{aligned}$$

$\forall x \in h^{-1}(y) \cap S$, $\forall w \in W$. Since Q is continuous and both $h^{-1}(y) \cap S$ and W are compact, there exists a $\alpha > 0$ such that $Q(x,u,w) \leq -\alpha$ for every $x \in h^{-1}(y) \cap S$ and $w \in W$. Thus there is an open neighborhood O of $h^{-1}(y) \cap S$ such that for every $x \in O$ and $w \in W$ we have

$$Q(x,u,w) \leq -\frac{\alpha}{2} < 0$$

Since h is open, we infer that $h(O)$ is open neighborhood of y. We claim that we can always find a closed ball $B(y)$ such that

$$h^{-1}(B(y)) \cap S \subseteq O.$$

Otherwise, there exists a sequence $\{B_i(y)\}_{i=1}^{\infty}$ of balls, where $B_i(y)$ has radius $r_i > 0$ and $\lim_{i\to\infty} r_i = 0$, and a sequence $\{x_i\}_{i=1}^{\infty}$ such that $x_i \in h^{-1}(B_i(y)) \cap S$ and $x_i \notin O$ for every i. Note that S is compact, so after passing to a subsequence we can assume $x_i \to x_0 \in S$. By continuity, $h(x_i) \to h(x_0)$, which implies $x_0 \in h^{-1}(y) \cap S$. However, this contradicts the fact that $x_0 \in R^n\backslash O$. Consequently, there exists $B(y)$ such that $y \in B(y)$, $K \supseteq B(y)$ and $h^{-1}(B(y)) \cap S \subseteq O$ and $\forall x \in O$, $\forall w \in W$, we have

$$Q(x,u,w) \leq -\frac{\alpha}{2} < 0$$

where $0 \notin h^{-1}(B(y))$ and $u = u(y)$.

Since we can obtain such a ball $B(y)$ for every $y \in K\backslash\{0\}$, there exists a partition of unity $\{B_i, p_i\}$, where $B_i = B(y_i)$, $\cup IntB_i \supseteq K\backslash\{0\}$, $p_i : R^k \to R^+$ is a smooth map supported on B_i, and $\sum p_i(y) = 1 \ \forall y \in K\backslash\{0\}$.

Let

$$\Phi(y) = \begin{cases} \sum_{i=1}^{\infty} u(y_i)p_i(y), & \text{for } y \neq 0 \\ 0, & \text{for } y = 0. \end{cases}$$

Observe that Φ is well defined on K and smooth on $K\backslash\{0\}$, since for $y \neq 0$ there are a only finite number of indices i such that $y \in B_i$.

Now, for every $y \in K\backslash\{0\}$ there exists a positive integer q such that $y \notin B_i$ for $i \geq q+1$. Thus for every $x \in h^{-1}(y) \cap S$ and $w \in W$ the fact that $\bar{F}$ is u-affine yields

$$\begin{aligned} \bar{F}(x, \Phi(y), w) &= \bar{F}(x, \sum_{i=1}^{q} p_i(y)u(y_i), w) \\ &= (\sum_{i=1}^{q} p_i(y))\bar{F}(x, u(y_i), w) \\ &\leq (\sum_{i=1}^{q} p_i(y))(-c(||x||)) \\ &= -c(||x||). \end{aligned}$$

Therefore

$$\bar{F}(x, \Phi(h(x)), w) \leq -c(||x||) \tag{1.4}$$

$\forall x \in h^{-1}(K\backslash\{0\}) \cap S$, $\forall w \in W$. We also have $\forall x \in (h^{-1}(0) \cap S)\backslash\{0\}$, $w \in W$,

$$\begin{aligned} \bar{F}(x, \Phi(0), w) &= \bar{F}(x, 0, w) \\ &\leq -c(||x||). \end{aligned} \tag{1.5}$$

From (1.4) and (1.5), one can see that

$$\dot{V}(x) \mid_{(1.1)} \leq -c(||x||) \tag{1.6}$$

Since $F(x, \Phi(h(x)), w)$ is smooth for $x \in h^{-1}(0)$ near 0 and $h^{-1}(0) \cap S$ is invariant,the inequality (1.6) implies that $0 \in R^n$ is a stable equilibrium of the closed loop system $\forall w \in W$, whereas every trajectory $x(t, x_0, w)$ of the closed loop system is defined for all $t > 0$ and x_0 in a neighborhood of zero.

We further claim that the equilibrium $0 \in R^n$ is attractive. Otherwise we would have

$$\lim_{t \to \infty} x(t, x_0, w) \neq 0$$

for some $w \in W$ and $x_0 \in S$. Because $V(x(t, x_0, w))$ is monotonically decreasing, we can assume that $\lim_{t \to \infty} V(x(t, x_0, w)) = v_0 > 0$ for some positive real number v_0. The positive definiteness of V implies the existence of a $\delta > 0$ such that $|x(t, x_0, w)| \geq \delta$ for every $t \geq 0$, and this in turn yields a real number $a > 0$ such that $c(|x(t, x_0, w)|) > c(a)$ for all $t \geq 0$. From (1.6) we obtain

$$V(x(t, x_0, w)) - V(x_0) \leq -tc(a) \quad \forall t > 0. \tag{1.7}$$

However, the right hand side of (1.7) tends to $v_0 - V(x_0)$ as $t \to \infty$, and this fact is clearly incompatible with (1.7). This proves that the equilibrium $0 \in R^n$ is indeed attractive. Therefore, $u = \Phi(y)$ is the desired output feedback stabilizing controller. QED.

Theorem 1.1. (1.1) is ROFS iff (1.1) is ROLC with the small control property.

Proof. ($\Leftarrow$)

If (1.1) is ROLC with the small control property, then, similar to the proof of Lemma 1.1, we can find a locally finite sequence of closed balls $\{B_i\}_{i=1}^{\infty}$ such that $\| u(y_i) \| < L(y)$ for $y \in B_i$ and $\cup_i Int B_i \supseteq K \backslash \{0\}$, $\dot{V} < -c(||x||)$ for all $x \in h^{-1}(B_i) \cap S$, $u(y_i)$ satisfies $\| u(y_i) \| < L(y)$, for $y \in B_i$, $i = 1, 2, \cdots$.

Now, take $u = \Phi(y)$ as in the proof of Lemma 1.1, then it suffices to show that $\Phi(y)$ is continuous at zero.

Indeed $\forall y \in K\backslash\{0\}$, without loss of generality, we can assume $y \notin B_i$, $i = q+1, q+2, \cdots$. Then

$\Rightarrow \| u(y_i) \| < L(y), i = 1, 2, \cdots$

$\Rightarrow \| \Phi(y) \| \leq \sum_{i=1}^{q} \| u(y_i) \| p_i(y) < L(y)$

$\Rightarrow \| \Phi(y) \| < L(y)$ as $y \to 0$

$\Rightarrow u = \Phi(y)$ is continuous at $y = 0$, where $\Phi(0) = 0$.

Note that with $\Phi(y)$ is smooth for $y \neq 0$, continuous at $y = 0$, it follows that every trajectory $x(t, x_0, w)$ of the closed loop system is forward complete, ie, defined for all $t \geq 0$.

($\Rightarrow$)

If (1.1) is ROFS, then there exists a feedback control law $u = u(y)$ that is continuous on a neighborhood K of $0 \in R^l$, smooth on $K\backslash\{0\}$, and such that the closed loop system

$$\dot{x} = F(x, u(h(x)), w) \tag{1.8}$$

is asymptotically stable on S uniformly with respect to $w \in W$. Therefore, by Theorem 2.9 of [4] we deduce that (1.1) is ROLC. The small control property follows from the continuity of $u(y)$ at $y = 0$. QED.

We conclude this section by presenting a result which illustrates the limited nature of the class of systems for which one can expect to achieve robust output feedback stabilization. Consider the following stand-alone disturbance system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + w \\ y &= h(x) \end{aligned} \tag{1.9}$$

where $x \in R^n, y \in R^k, u \in R^l, w \in R^n, k \leq n$. We assume f, g, h are smooth, $h^{-1}(0) \neq \{0\}$, and $f(0) = 0, g(0) = 0, h(0) = 0$. We further assume that the disturbance has an a priori bound $\| w \| \leq \beta$ for some positive constant β.

Corollary 1.1. (1.9) is never ROFS.

Proof. If (1.9) were ROFS, then, according to Theorem 1.1, (2.1) would be ROLC with the small control property, ie,there was a positive definite $V : R^n \to R$, with $V(0) = 0, V(x) > 0$ for $x \neq 0$, such that

$$\nabla V \cdot (f(x) + g(x)u + w) < -c(||x||) \tag{1.10}$$

for all $\| w \| \leq \beta$, where c is also positive definite.
Now, for $u = 0$, we have

$$\nabla V \cdot (f(x) + w) < -c(||x||) \tag{1.11}$$

for $\forall x \in (h^{-1}(0) \cap S)\backslash\{0\}$, $\| w \| \leq \beta$.
Observe that (1.3) can never hold, because when $x \to 0$, we can pick $w = -f(x)$ which will contradict (1.3). Note also that $h^{-1}(0) \neq \{0\}$. QED.

An illustrative example. Consider an example of a salt solution of two tanks[37] with some uncertainties on the flow rates:

$$\begin{aligned}
\dot{x}_1 &= -(\frac{\bar{Q}_1}{L_1})x_1 + (\frac{\bar{Q}_2}{L_2})x_2 + (\bar{Q}_1 - \bar{Q}_2)u \\
\dot{x}_2 &= (\frac{\bar{Q}_1}{L_1})x1 - (\frac{\bar{Q}_1}{L_2})x_2 \\
y &= (\frac{1}{L_2})x_2
\end{aligned}$$

where $\bar{Q}_1 = Q_1 + w_1$, $\bar{Q}_2 = Q_2 + w_2$ for small w_1, w_2 and $Q_1 > Q_2$.
Take $u = u(y) = 0$ as the control; then since the system matrix without uncertainties has two negative eigenvalues, for small enough w_1 and w_2 we still have asymptotical stability of the state variables x_1 and x_2. Therefore the system is ROFS.
According to Theorem 1.1, the system is ROLC with small control property. In fact, when take $u = u(y) = 0$, since the closed loop system without uncertainties is asymptotically stable, there is a positive $V(x) = x^\top Px$ such that $\dot{V} < -x^\top Qx$

for some positive Q. Certainly this V tolerates some small perturbations of Q_1 and Q_2.

1.2. Construction of Feedback Controls Using Lyapunov Functions

The proof of Theorem 1.1 is based on partitions of unity and is highly nonconstructive. In this section we consider some special situations where robust feedback controls can be explicitly constructed if the Lyapunov functions are known. The results are direct extensions of those in [11] and [16]. Then we will give another construction that applies to certain linear systems.

Consider the nonlinear system

$$\dot{x} = f(x, w) + G(x)u \tag{1.12}$$

where all entries of the vector f and $n \times m$ matrix G are smooth functions of their arguments and $f(0, w) = 0 \ \forall w$. As before w is the uncertainty and we assume $w \in B_l = \{w \in R^l, ||w|| \leq 1\}$.

For any positive definite function $V(x)$, we denote

$$\begin{aligned} a(x, w) &= \nabla V(x) \cdot f(x, w) \\ B(x) &= \nabla V(x) \cdot G(x) = (b_1(x), \cdots, b_m(x)) \end{aligned}$$

Lemma 1.2. [11] Fix $w = 0$. If there is a smooth control Lyapunov function V for the system (1.12) (with $w = 0$), then there is a feedback stabilizer $u = k(x)$ which is smooth for $k \neq 0$. Moreover, if V satisfies the small control property, then $k(x)$ can be chosen to be continuous at $x = 0$.

In fact, in [11] it is shown that k is given by the formula

$$u = k(x) = (u_1(x), \cdots, u_m(x))$$
$$u_i(x) = -b_i(x)\phi(a(x), \beta(x))$$

where $\beta(x) = \sum_{i=1}^{m} b_i^2(x) = ||B(x)||^2$, $\phi(a, 0) = 0$ for $a < 0$, and for $a \geq 0$

$$\phi(a, b) = \frac{a + \sqrt{a^2 + b^4}}{b}.$$

The result of Lemma 1.2 can be extended to the system (1.12) with the uncertainty as follows.

Theorem 1.2. If (1.12) has a RCLF $V(x)$, then the continuous feedback law $u = (u_1(x), \cdots, u_m(x))$, where

$$u_i(x) = -b_i(x)\phi(a(x), \beta(x)),$$

will be a stabilizer of (1.12), where ϕ is defined as above and

$$a(x) = \max_{w \in B_l} a(x, w)$$

Moreover, if V satisfies the small control property, then the stabilizer is continuous at $x = 0$.

Proof. Note that the continuity of $a(x)$ is guaranteed by Corollary 3.6 of [17]. Also note that $\phi(a, b)$ is well defined on the set in $\{(a, b) \in R^2 : b > 0 \text{ or } a < 0\}$ and for every x the pair $(a(x), \beta(x))$ will satisfy $a(x) < 0$ whenever $\beta(x) = 0$. We claim that the same V will serve as the Lyapunov function for the closed loop system. For if $\beta(x) = 0$, the RCLF property implies $a(x) < 0$, so that

$$\dot{V} = a(x, w) \leq a(x) < 0.$$

On the other hand if $\beta(x) > 0$, then

$$\begin{aligned}\dot{V} &= a(x,w) + \sum_{i=1}^{m} b_i(x)u_i(x) \\ &\leq a(x) + (-a(x) + \sqrt{a^2+b^4}) \\ &= -\sqrt{a^2+b^4} \\ &< 0.\end{aligned}$$

Therefore, u is a stabilizer of (1.12). The second statement about the continuity of the stabilizer follows from Lemma 1.2. QED.

Remark. A sufficient condition for $a(x)$ to be smooth, which would then imply that the stabilizer $u(x)$ is smooth, can be found in [17].

Corollary 1.2. If $f(x,w) = f(x) + Q(x)w$ and if $q(x) = \nabla V(x) \cdot Q(x)$ is never zero, then

$$a(x) = \nabla V(x) \cdot f(x) + ||q(x)||$$

is smooth, whence $u(x)$ is also smooth.

The feedback control law $u(x)$ given by Theorem 1.2 may be unbounded. If one modifies Definition 1.1 to require that $u \in B_m = \{||u|| \leq 1\}$, then one can argue in a manner similar to [16] to obtain a bounded feedback control.

Lemma 1.3. If (1.12) has a RCLF V with $u \in B_m$, then the feedback control law of Theorem 1.2, with the function ϕ modified as

$$\phi(a,b) = \frac{a+\sqrt{a^2+b^4}}{b(1+\sqrt{1+b^2})}, \quad b \neq 0,$$

will be a stabilizer of (1.12).

In the remainder of this section, we will construct a simple output feedback control for a class of linear input-output systems of the form

$$\begin{aligned} \dot{x} &= (A + wI_n)x + Bu \\ y &= x_2, \; x(0) = x_0 \end{aligned} \tag{1.13}$$

where $x = (x_1, x_2) \in R^{n-m} \times R^m = R^n (m < n)$ and the uncertainty w occurs diagonally and is bounded by $|w| \leq \alpha$. The decomposition of the state space as $R^n = R^{n-m} \times R^m$ allows us to write

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \; A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \; B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

If (A, B) is controllable, then for any given family of real numbers $\{\lambda_1, \cdots, \lambda_n\}$ such that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < -\alpha < 0,$$

we can find an $n \times m$ matrix $F = (F_1, F_2)$ such that $A^* = A + BF$ has eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. In particular, $u(x) = F_1 x_1 + F_2 x_2$ is a stabilizer of (1.13), or equivalently the closed loop system

$$\dot{x} = (A + BF)x = A^* x \tag{1.14}$$

is asymptotically stable. Let P be the nonsingular $n \times n$-matrix such that

$$P(A + BF)P^{-1} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$$

where $\Lambda_1 = \text{diag}(\lambda_1, \cdots, \lambda_{n-m})$, $\Lambda_2 = \text{diag}(\lambda_{n-m+1}, \cdots, \lambda_n)$. Write A^* in block form according to the decomposition of the state space

$$A^* = A + BF = \begin{pmatrix} A_{11} + B_1 F_1 & A_{12} + B_1 F_2 \\ A_{21} + B_2 F_1 & A_{22} + B_2 F_2 \end{pmatrix},$$

and denote

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}$$

where $P_{11},\ Q_{11} \in R^{(n-m)\times(n-m)}$, $P_{22},\ Q_{22} \in R^{m\times m}$, $P_1 \in R^{(n-m)\times n}$, and $Q_1 \in R^{n\times(n-m)}$. Then the solution of (1.13) can be written as

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= P^{-1} e^{(\Lambda + wI_n)t} P x_0 \\ &= \begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix} e^{(\Lambda_1 + wI_{n-m})t} P_1 x_0 + \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} e^{(\Lambda_2 + wI_m)t} P_2 x_0 \end{aligned} \tag{1.15}$$

Now, if we for the moment ignore the first term in (1.15), then

$$x_1(t) = Q_{12} e^{(\Lambda_2 + wI_m)t} P_2 x_0$$
$$x_2(t) = Q_{22} e^{(\Lambda_2 + wI_m)t} P_2 x_0$$

so that

$$x_1(t) = Q_{12} Q_{22}^{-1} x_2(t). \tag{1.16}$$

Next replace $x_1(t)$ in $u = F_1 x_1 + F_2 x_2$ by (1.16) to obtain

$$u(t) = (F_1 Q_{12} Q_{22}^{-1} + F_2) y(t) \tag{1.17}$$

With the output feedback (1.17), the closed loop system is

$$\dot{x} = (\bar{A} + wI_n)x \tag{1.18}$$

where

$$\bar{A} = \begin{pmatrix} A_{11} & A_{12} + B_1 F_2 + B_1 F_1 Q_{12} Q_{22}^{-1} \\ A_{21} & A_{22} + B_2 F_2 + B_2 F_1 Q_{12} Q_{22}^{-1} \end{pmatrix}.$$

Theorem 1.3. The matrix $\bar{A} + wI_n$ has $\lambda_{n-m+1} + w < 0, \cdots, \lambda_n + w < 0$ as m of its eigenvalues. The remaining $n-m$ eigenvalues are determined by the matrix $P_1 \bar{A} Q_1 + wI_{n-m}$.

Proof. Since

$$
\begin{aligned}
P(\bar{A}+wI_n)P^{-1} &= \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \bar{A} \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} + wI_n \\
&= \begin{pmatrix} P_1\bar{A}Q_1 & P_1\bar{A}Q_2 \\ P_2\bar{A}Q_1 & P_2\bar{A}Q_2 \end{pmatrix} + wI_n
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{A}Q_2 &= \begin{pmatrix} A_{11} & A_{12}+B_1F_2+B_1F_1Q_{12}Q_{22}^{-1} \\ A_{21} & A_{22}+B_2F_2+B_2F_1Q_{12}Q_{22}^{-1} \end{pmatrix} \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} \\
&= \begin{pmatrix} A_{11}+B_1F_1 & A_{12}+B_1F_2 \\ A_{21}+B_2F_1 & A_{22}+B_2F_2 \end{pmatrix} \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} \\
&= A^*Q_2
\end{aligned}
$$

we obtain

$$
\begin{aligned}
P(\bar{A}+wI_n)P^{-1} &= \begin{pmatrix} P_1\bar{A}Q_1 & P_1\bar{A}Q_2 \\ P_2\bar{A}Q_1 & P_2\bar{A}Q_2 \end{pmatrix} + wI_n \\
&= \begin{pmatrix} P_1\bar{A}Q_1 & P_1\bar{A}^*Q_2 \\ P_2\bar{A}Q_1 & P_2\bar{A}^*Q_2 \end{pmatrix} + wI_n \\
&= \begin{pmatrix} P_1\bar{A}Q_1 + wI_{n-m} & 0 \\ P_2\bar{A}Q_1 & \Lambda_2 + wI_m \end{pmatrix},
\end{aligned}
$$

where $P_1\bar{A}^*Q_2 = 0$, $P_2\bar{A}^*Q_2 = \Lambda_2$. Therefore $\bar{A}+wI_n$ has as m of its eigenvalues $\lambda_{n-m+1}+w < 0, \cdots, \lambda_n + w < 0$, and the remaining $n-m$ eigenvalues are determined by the matrix $P_1\bar{A}Q_1 + wI_{n-m}$.

Corollary 1.3. If the eigenvalues of $P_1\bar{A}Q_1$, denoted by $\lambda_i(P_1\bar{A}Q_1)$, satisfy

$$
Re(\lambda_i(P_1\bar{A}Q_1)) < -\alpha, \; i = 1, 2, \cdots, n-m,
$$

then the closed loop system (1.18) is asymptotically stable under the output feedback (1.17).

We examine more closely the eigenvalues $\lambda_i(P_1\bar{A}Q_1)$ of $P_1\bar{A}Q_1$. Since $P_1A^*Q_1 =$

Λ_1, we have

$$\begin{aligned} P_1\bar{A}Q_1 &= \Lambda_1 + P_1(\bar{A} - A^*)Q_1 \\ &= \Lambda_1 + P_1 \begin{pmatrix} -B_1F_1 & B_1F_1Q_{12}Q_{22}^{-1} \\ -B_2F_1 & B_2F_1Q_{12}Q_{22}^{-1} \end{pmatrix} \begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix} \\ &= \Lambda_1 + P_1BF(Q_{12}Q_{22}^{-1}Q_{21} - Q_{11}) \end{aligned}$$

It is easy to show that

$$P_{11}^{-1} = -Q_{12}Q_{22}^{-1}Q_{21} + Q_{11},$$

so we obtain

$$P_1\bar{A}Q_1 = \Lambda_1 - P_1BFP_{11}^{-1}.$$

Corollary 1.4. If

$$Re\lambda(\Lambda_1 - P_1BFP_{11}^{-1}) < -\alpha,$$

then the closed loop system (1.18) is asymptotically stable under the output feedback (1.17).

An illustrative example. Consider a 1+1 dimensional system

$$A = \begin{pmatrix} -5 & 2 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |w| \le \alpha = 1.$$

An elementary computation shows that (A, B) is controllable. If we take $F = (-1, -10)$, then

$$A + BF = \begin{pmatrix} -6 & -8 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues $\lambda_1 = -4$, $\lambda_2 = -2$. The matrix P that diagonalizes $A + BF$ is seen to be

$$P = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}, \; P^{-1} = \begin{pmatrix} 4 & -1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

so that $Q_{12}Q_{22}^{-1} = -2$. According to (1.17), we can take $u(t) = -(-2)x_2 - 10x_2 = -8x_2$, which results in the closed loop system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -5+w & -6 \\ 1 & w \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

with eigenvalues $-2+w \leq -1$ and $-3+w \leq -2$ (the inequalities follow from the assumed bound on w). Therefore the closed loop system is asymptotically stable.

2. Feedback Designs For A Class of SISO Systems

In this chapter we consider a class of so-called "partially matched" linear systems with uncertainty of the form

$$\dot{x} = (A + \Delta A(w(t)))x + b(1 + \Delta b(w(t)))u + b\Delta D(w(t)) \tag{2.1}$$

where $x \in R^n, u \in R, b \in R^n$ (a constant vector), and ΔA, Δb, ΔD are continuous functions of a scalar parameter. It is assumed that the uncertainty $w(t)$ is Lebesgue measurable function of t which takes values in a fixed compact set $\Omega \subseteq R$. The terms ΔA, Δb, ΔD represent the disturbances, which are either known or unknown. Similar systems have been studied in [6]-[9].

We make the following assumptions

(H2.1) There exist $\alpha > 0, 0 < \beta < 1, \delta > 0$ such that

$$||\Delta A(w(t))|| \leq \alpha, ||\Delta b(w(t))|| \leq \beta, ||\Delta D(w(t))|| \leq \delta,$$

where $||\cdot||$ denotes the usual norm.

(H2.2) There exist $p \in R^n$ and an $n \times n$ symmetric, positive definite matrix P such that the matrix $A_0 = A + b < p, \cdot >$ and P result in

$$Q = A_0^\top P + PA_0$$

being negative definite; i. e., $< Qx, x > \leq -2\lambda < x, x >$ for some $\lambda > 0$.

The following lemma establishes a sufficient condition for (H2.2) for a class of systems.

Lemma 2.1. If the nominal system of (2.1) (i. e., with $\Delta A, \Delta b$, and ΔD all zero) can be decomposed as

$$\dot{x}_1 = A_1 x_1 + A_{12} x_2 + b_1 u \tag{2.2}$$

$$\dot{x}_2 = A_2 x_2 + b_2 u \tag{2.3}$$

where A_1 is Hurwitz and (A_2, b_2) is controllable, then (H2.2) holds.

Proof. Since (A_2, b_2) is controllable, there is p_2 such that $M_2 = A_2 + b_2 < p_2, \cdot >$ is Hurwitz. Setting $p = (0, p_2)^\top$, we see that

$$\begin{aligned} A_0 &= A + b < p, \cdot > \\ &= \begin{pmatrix} A_1 & A_{12} + b_1 p_2^\top \\ 0 & A_2 + b_2 p_2^\top \end{pmatrix} \\ &= \begin{pmatrix} A_1 & A_{12} + b_1 p_2^\top \\ 0 & M_2 \end{pmatrix}. \end{aligned} \tag{2.4}$$

Since both A_1 and M_2 are Hurwitz, it follows easily that A_0 is Hurwitz. Therefore, there is a symmetric, positive definite matrix P such that

$$Q = A_0^\top P + P A_0$$

is negative definite. Hence (H2.2) is satisfied. QED.

Remark. Generally speaking, the existence of P in (H2.2) is still an open problem [9].

2.1 State Feedback Design

Under assumptions (H2.1) and (H2.2), we seek a state feedback of the form

$$u(x) =< p, x > +v(x) \tag{2.5}$$

where $v(x)$ is to be determined. The substitution of (2.5) into (2.1) results in the closed loop system

$$\dot{x} = (A_0 + \Delta A + b\Delta b < p, \cdot >)x + b(1 + \Delta b)v(x) + b\Delta D \tag{2.6}$$

We will determine conditions under which $V(x) = \frac{1}{2} < Px, x >$ is a Lyapunov function for (2.6), where P is as in (H2.2). A straightforward computation of $\dot{V}$ and the bounds in (H2.1) yields

$$
\begin{aligned}
\dot{V}\mid_{(2.6)} &=< Px(t), \dot{x}(t) > \\
&=< Px(t), A_0x(t) > + < Px(t), (\Delta A + b\Delta b < p, \cdot >)x(t) > \\
&+ < Px(t), b(1+\Delta b)v(x) + b\Delta D > \\
&= \frac{1}{2} < (PA_0 + A_0^\top P)x(t), x(t) > + < Px(t), \Delta Ax(t) > \\
&+ \Delta b < Px(t), b >< p, x(t) > + < Px(t), b(1+\Delta b)v(x) + b\Delta D > \\
&\leq -\lambda < x, x > + \alpha||P|| < x, x > \\
&+ \beta| < Px, b > || < p, x > | + < Px(t), b(1+\Delta b)v(x) + b\Delta D >
\end{aligned}
\tag{2.7}
$$

We define

$$
v(x) = \begin{cases} -\frac{2\delta + \beta|<p,x>|}{1-\beta}\mathrm{sgn}(< Px, b >), & \text{for } b^\top Px \neq 0 \\ \bar{u}(x) - < p, x >, & \text{for } b^\top Px = 0, \end{cases}
\tag{2.8}
$$

where $\bar{u}(x)$ is the equivalent control on the sliding surface $s(x) =< Px, b >$; i. e., it is the solution to

$$
\begin{aligned}
\dot{s}(x) &= b^\top P[(A_0 + \Delta A + b\Delta b < p, \cdot >)x \\
&+ b(1+\Delta b)\bar{u} + b\Delta D] \\
&= 0,
\end{aligned}
$$

which yields

$$
\bar{u}(x) = -\frac{1}{1+\Delta b}(b^\top Pb)^{-1}[b^\top P(A_0 + \Delta A + b\Delta b < p, \cdot >)x + b^\top Pb\Delta D]
$$

(note $b^{\top}Pb \neq 0$ and $1+\Delta b > 0$).

Substituting (2.8) in the last expression in (2.7), we obtain

$$\begin{aligned}
&\beta| < Px, b > | \, | < p, x > | + < Px(t), b(1+\Delta b)v(t) + b\Delta D > \\
&= \beta| < Px, b > | \, | < p, x > | + (1+\Delta b) < Px, b > v(t) + < Px, b > \Delta D \\
&= \beta| < Px, b > | \, | < p, x > | - (1-\beta)\frac{2\delta + \beta| < p, x > |}{1-\beta} \, | < Px, b > | \\
&+ \delta| < Px, b > | \\
&\leq 0.
\end{aligned}$$

Therefore, (2.7) yields

$$\dot{V} \,|_{(2.6)} \leq -(\lambda - \alpha||P||) < x(t), x(t) > . \tag{2.9}$$

This estimate leads immediately to the following result.

Theorem 2.1. If (H2.1) and (H2.2) hold and if $\lambda - \alpha||P|| > 0$, then the state feedback $u(x)$ defined by (2.5), where $v(x)$ is defined by (2.8), will stabilize the closed loop system (2.6).

Proof. Since the matrix P is positive definite and symmetric, there exist $k_1 > 0, k_2 > 0$ such that

$$k_1 < x, x > \leq V(x) = \frac{1}{2} < Px, x > \leq k_2 < x, x >$$

This and the inequality (2.9) imply that $\dot{V} \leq -\lambda_0 \frac{V}{k_1}$, where $\lambda_0 = \lambda - \alpha||P|| > 0$. Hence we obtain

$$\begin{aligned}
&V \leq \exp(-\tfrac{\lambda_0}{k_1}t)V_0 \\
&\Longrightarrow k_1 < x(t), x(t) > \leq \exp(-\tfrac{\lambda_0}{k_1}t)k_2||x_0||^2 \\
&\Longrightarrow ||x(t)||^2 \leq \tfrac{k_2}{k_1}\exp(-\tfrac{\lambda_0}{k_1}t)||x_0||^2 \to 0 \text{ as } t \to \infty. \text{ QED.}
\end{aligned}$$

Remark. The feedback design (2.8) is such that the sliding surface $s(x) = 0$ is invariant for the closed loop system, though there is some design uncertainty on the sliding surface. However, the sliding surface motions are insensitive to this uncertainty because $s^{\mathsf{T}}(x)\dot{s}(x) < 0$ in a neighborhood of a region of the sliding surface. To verify this we simply compute

$$\begin{aligned}
s^{\mathsf{T}}(x)\dot{s}(x) = &< Pb, x >< Pb, (A + \Delta A(w(t)))x \\
&+ b(1 + \Delta b(w(t)))u + b\Delta D(w(t)) > \\
&\leq (||APb|| + \alpha + ||b||\ ||p||)||x||| < Pb, x > | \\
&+ \delta < Pb, b > | < Pb, x > | - 2\delta < Pb, b > | < Pb, x > | \\
&= (||APb|| + \alpha + ||b||\ ||p||)||x||| < Pb, x > | \\
&- \delta < Pb, b > | < Pb, x > |.
\end{aligned}$$

It follows that if

$$||x|| < \frac{\delta < Pb, b >}{||APb|| + \alpha + ||b||\ ||p||},$$

then

$$s^{\mathsf{T}}(x)\dot{s}(x) < 0.$$

An illustrative example. Consider the system

$$\begin{aligned}
\dot{x}_1 &= -x_1 + ax_2 + \frac{1}{2}(1 + b)u + \frac{1}{2}D \\
\dot{x}_2 &= ax_1 + x_2 + (1 + b)u + D
\end{aligned}$$

where all the variables are scalars. Then we can take $\alpha = |a|$ and

$$\lambda = -\frac{1}{2}, p = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, P = I_2, Q = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}.$$

Therefore, if $|b| < 1$ and $\frac{1}{2} - |a| > 0$ (i. e, $-\frac{1}{2} < a < \frac{1}{2}$ and $-1 < b < 1$), then the state feedback $u(x)$ defined by (2.5) and (2.8) will drive the closed loop system to zero. Note that there is no restriction on δ in this example.

2.2 Output Feedback Design

Next we consider the system

$$\dot{x}_1 = A_1 x_1 + \Delta A_{11} x_1 + \Delta A_{12} x_2 + b_1(1+\Delta b)u + b_1 \Delta D \tag{2.10}$$

$$\dot{x}_2 = A_2 x_2 + \Delta A_{21} x_1 + \Delta A_{22} x_2 + b_2(1+\Delta b)u + b_2 \Delta D \tag{2.11}$$

$$y = < c, x > = < c_1, x_1 > + < c_2, x_2 >$$

where the notations are similar to those used in (2.1). The following assumptions are made.

(H2.3) $A_1 + A_1^\top$ is negative definite

(H2.4) (A_2, b_2, c_2) is minimum phase and (2.11) has nonsingular high-frequency gain; i. e., $\det(c_2 b_2) \neq 0$.

Lemma 2.2([6]). If (H2.4) holds, then there is a symmetric, positive definite matrix P_2 such that for some constant k the matrix

$$Q_2 = \frac{1}{2}[(A_2 + kb_2c_2)^\top P_2 + P_2(A_2 + kb_2c_2)] \tag{2.12}$$

is negative definite and $c_2 = b_2^\top P_2$.

Under assumptions (H2.3) and (H2.4) we seek a stabilizing output feedback of the form

$$u(y) = ky + v(y) \tag{2.13}$$

where $v(y)$ is yet to be determined. An application of the feedback (2.13) yields the closed loop system

$$\begin{aligned} \dot{x}_1 &= [A_1 + \Delta A_{11} + kb_1(1+\Delta b)C_1]x_1 \\ &\quad + [kb_1(1+\Delta b)c_2 + \Delta A_{12}]x_2 + b_1\Delta D(t) + b_1(1+\Delta b)v(t) \end{aligned} \tag{2.14}$$

$$\dot{x}_2 = [A_2 + \Delta A_{22} + kb_2(1+\Delta b)C_2]x_2 + [kb_2 + (1+\Delta b)c_1 + \Delta A_{21}]x_1$$
$$+ b_2 \Delta D(t) + b_2(1+\Delta b)v(t)$$

$$(2.15)$$

$y = c_1 x_1 + c_2 x_2.$

As the Lyapunov function, we take $V(x) = \frac{1}{2}(< x_1, x_1 > + < P_2 x_2, x_2 >)$. Then a direct computation using (2.14) and (2.15) gives

$$\begin{aligned}
\dot{V} &=< x_1, \dot{x}_1 > + < P_2 x_2, \dot{x}_2 > \\
&=< x_1, A_1 x_1 > + < x_1, < x_1, [\Delta A_{11} + kb_1(1+\Delta b)c_1]x_1 > \\
&+ < x_1, [kb_1(1+\Delta b)c_2 + \Delta A_{12}]x_2 > + < x_1, b_1 > \Delta D \\
&+ < x_1, b_1 > (1+\Delta b)v(t) + < P_2x_2, [kb_2(1+\Delta b)c_1 + \Delta A_{21}]x_1 > \\
&+ < P_2 x_2, (A_2 + kb_2 c_2)x_2 > + < P_2 x_2, (\Delta A_{22} + kb_2 \Delta b c_2)x_2 > \\
&+ < P_2 x_2, b_2 > \Delta D + < P_2 x_2, b_2 > (1+\Delta b)v(t) \\
&\leq -\lambda_1 < x_1, x_1 > + [\alpha + |k| \, ||b_1|| \, ||c_1||(1+\beta)] < x_1, x_1 > \\
&+ [\alpha + |k|| \, |b_1|| \, ||c_2||(1+\beta)]||x_1|| \, ||x_2|| + < x_1, b_1 > \Delta D \\
&+ < x_1, b_1 > (1+\Delta b)v(t) + [\alpha + |k| \, ||P_2|| \, ||b_2|| \, ||c_1||(1+\beta)]||x_1|| \, ||x_2|| \\
&+ < Q_2 x_2, x_2 > + ||P_2||(\alpha + \beta|k| \, ||b_2|| \, ||c_2||) < x_2, x_2 > \\
&+ < P_2 x_2, b_2 > \Delta D + < P_2 x_2, b_2 > (1+\Delta b)v(t)
\end{aligned}$$

$$(2.16)$$

where $-\lambda_1 < 0$ is the maximum eigenvalue of $\frac{1}{2}(A_1 + A_1^\top)$.

Since $< P_2 x_2, b_2 > =< c_2, x_2 > = y- < c_1, x_1 >$, we obtain

$$\dot{V} \le -(\lambda_1 - \alpha - |k|(1+\beta)||b_1||\,||c_1||) < x_1, x_1 >$$
$$+ [2\alpha + |k|(1+\beta)(||b_1||\,||c_2||| + ||P_2||\,||b_2||\,||c_1||)]||x_1||\,||x_2||$$
$$- (\lambda_2 - ||P_2||(\alpha + \beta|k|\,||b_2||\,||c_2||) < x_2, x_2 > +\delta|y(t)|$$
$$+ \delta(||c_1|| + ||b_1||)||x_1||$$
$$+ (1+\Delta b)y(t)v(t) + (1+\beta)(||b_1|| + ||c_1||)|v(t)|\,||x_1|| \tag{2.17}$$

We define

$$v(t) = \begin{cases} -\frac{\delta}{1-\beta}\mathrm{sgn}(y(t)), & \text{for } |y(t)| > \epsilon \\ -\frac{\delta}{1-\beta}\frac{y(t)}{\epsilon}, & \text{for } |y(t)| \le \epsilon. \end{cases} \tag{2.18}$$

Observe that for $|y(t)| > \epsilon$ we have

$$\delta|y(t)| + (1+\Delta b)y(t)v(t)$$
$$\le \delta|y(t) - (1-\beta)\frac{\delta}{1-\beta}|y(t)|$$
$$= 0,$$

while for $|y(t)| \le \epsilon$ we have

$$\delta|y(t)| + (1+\Delta b)y(t)v(t)$$
$$\le -\delta(\frac{1}{\epsilon}y^2 - |y|)$$
$$\le \frac{\delta\epsilon}{4};$$

also in either case we have

$$\delta(||c_1|| + ||b_1||)||x_1|| + (1+\beta)(||b_1|| + ||c_1||)|v(t)|\,||x_1||$$
$$\le \delta(1 + \frac{1+\beta}{1-\beta})(||c_1|| + ||b_1||)||x_1||$$
$$= \frac{2\delta}{1-\beta}(||c_1|| + ||b_1||)||x_1||.$$

Therefore, these estimates and (2.17) yield

$$\dot{V}(x) \leq -a_1||x_1||^2 - a_2||x_2||^2 + 2a_{12}||x_1||\ ||x_2|| + a_3||x_1|| + \frac{\delta\epsilon}{4}, \tag{2.19}$$

where

$$\begin{aligned}
a_1 &= \lambda_1 - \alpha - |k|(1+\beta)||b_1||\ ||c_1||, \\
a_2 &= \lambda_2 - ||P_2||(\alpha + \beta|k|\ ||b_2||\ ||c_2||), \\
a_{12} &= \frac{1}{2}[2\alpha + |k|(1+\beta)(||b_1||\ ||c_2|| + ||b_2||\ ||c_1||)], \\
a_3 &= \frac{2\delta}{1-\beta}(||b_1|| + ||c_1||),
\end{aligned}$$

and ϵ is any real number satisfying $0 < \epsilon < 1$.

The following theorem is an immediate consequence of (2.19).

Theorem 2.2. Suppose (H2.1), (H2.3) and (H2.4) hold. If $a_1 > 0$, $a_2 > 0$, and $a_1 a_2 > a_{12}^2$, then there exists $r > 0$ such that the output feedback $u(t)$ defined by (2.13) and (2.18) will drive the solutions of the closed loop system (2.14) and (2.15) to the ball $B_r = \{||x|| \leq r\}$.

Remarks.

1. It is easy to see that with $a_1 > 0$, $a_2 > 0$, $a_1 a_2 > a_{12}^2$, the surface

$$G(x_1, x_2) = -a_1||x_1||^2 - a_2||x_2||^2 + 2a_{12}||x_1||\ ||x_2|| + a_3||x_1|| + \frac{\delta\epsilon}{4}$$

is a parabaloid which opens downward in the ambient three dimensional space in which it is plotted. Therefore, there exists $r > 0$ such that $x_1^2 + x_2^2 > r^2$ implies $G(x_1, x_2) < 0$. Moreover, if we choose positive constants $k_1 > 0$, $k_2 > 0$ such that

$$k_1 < x, x > \leq V(x) \leq k_2 < x, x >,$$

then we see that $\dot{V} < 0$ whenever $x_1^2 + x_2^2 > r^2$, so the solutions of (2.14) and (2.15) converge to B_r. However, the explicit computation of r requires tedious algebraic computations and so is omitted here.

2. ϵ is design parameter which can be used to adjust the value of r. In principle, it should be small.

3. Note that δ only appears in the expression for a_3. This means that the hypotheses $a_1 > 0$, $a_2 > 0$, $a_1 a_2 > a_{12}^2$ are independent of the bound on ΔD.

4. If $\Delta D(t)$ is replaced by $\Delta D(t, x)$, with $||\Delta D|| \leq \delta + \xi ||x||$, then one can see that the same design works with proper adjustments of the parameters. Consequently, this uncertain system may tolerate some nonlinearities of the system.

An illustrative example. Consider the system

$$\begin{aligned}
\dot{x}_1 &= -10x_1 + a_{11} + a_{12}x_2 + \frac{1}{2}(1+b)u + \frac{1}{2}D \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + x_2 + (1+b)u + D,
\end{aligned}$$

where all variables are scalars and we assume that $|a_{ij}| \leq \frac{1}{20}$, $|b| \leq \frac{1}{2}$, and $|D| \leq \delta$. The aforementioned formulas for a_1, a_2, a_3, and a_{12} yield

$$a_1 = \frac{109}{20} > 0, \ a_2 = \frac{9}{20} > 0, \ a_{12} = \frac{46}{20}, \ a_3 = 6\delta,$$

whence we obtain $a_1 a_2 > a_{12}^2$ and

$$\begin{aligned}
G(x_1, x_2) &= -\frac{109}{20}x_1^2 - \frac{9}{20}x_2^2 + \frac{49}{2}|x_1||x_2| + 6\delta|x_1| + \frac{\delta\epsilon}{4} \\
&\leq -\frac{1}{5}x_1^2 - \epsilon x_2^2 + 6\delta|x_1| + \frac{\delta\epsilon}{4}
\end{aligned}$$

where ϵ is a small positive constant. One verifies that $x_1^2 + x_2^2 > (31\delta)^2$ implies $G(x_1, x_2) < 0$ when we take ϵ sufficiently small, so $r = 31\delta$ is the radius of the stable ball.

3. Feedback Designs For A Class of MIMO Systems

In this section we will consider partially matched systems with uncertainty of the form

$$\dot{x} = (A + \Delta A(w(t)))x + B(1 + \Delta b(w(t)))u + B\Delta D(w(t)) \tag{3.1}$$

where $x \in R^n$, $u \in R^k$, A, B are constant matrices of the appropriate dimensions, and $\Delta A : R^s \to R^{n\times n}$, $\Delta b : R^s \to R$, and $\Delta D : R^s \to R^k$ are continuous. The uncertainty $w : R \to R^s$ is assumed to be a Lebesgue measurable mapping taking values in a fixed compact set $\Omega \subset R^s$. ΔA, Δb, ΔD represent disturbances which are either known or unknown.

We make the following assumptions.

(H3.1) There exist $\alpha > 0$, $0 < \beta < 1, \delta > 0$ such that

$$||\Delta A(w(t))|| \le \alpha, ||\Delta b(w(t))|| \le \beta, ||\Delta D(w(t))|| \le \delta$$

where $||\cdot||$ is the usual norm.

(H3.2) There exist a $k \times n$ matrix F, and an $n \times n$ symmetric, positive definite matrix P such that for $A_0 = A + BF$ we have

$$Q = A_0^\top P + PA_0$$

is negative definite; i. e., there exists $\lambda > 0$ such that $< Qx, x >\le -2\lambda < x, x >$ for every $x \in R^n$.

The following lemma establishes a sufficient condition for (H3.2) to hold in a specific class of systems.

Lemma 3.1. If the nominal system of (2.1) can be decomposed as

$$\dot{x}_1 = A_1 x_1 + A_{12} x_2 + B_1 u \tag{3.2}$$

$$\dot{x}_2 = A_2 x_2 + B_2 u, \tag{3.3}$$

where A_1 is Hurwitz and (A_2, B_2) is controllable, then (H3.2) holds.

Proof. Since (A_2, B_2) is controllable, there is a matrix F_2 such that $M_2 = A_2 + B_2 F_2$ is Hurwitz. Setting $F = (0, F_2)$, we have

$$\begin{aligned} A_0 &= A + BF \\ &= \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} (0, F_2) \\ &= \begin{pmatrix} A_1 & A_{12} + B_1 F_2 \\ 0 & M_2 \end{pmatrix}. \end{aligned}$$

Since both A_1 and M_2 are Hurwitz, it follows that A_0 is Hurwitz, which proves the Lemma. QED.

3.1 State Feedback Design

Under assumptions (H3.1) and (H3.2) we will seek to stabilize (3.1) by a full state feedback control law the form

$$u(x) = Fx + v(x) \tag{3.4}$$

where $v(x) \in R^k$ is yet to be determined.

The substitution of (3.4) into (3.1) yields the closed loop system

$$\dot{x} = (A + BF + \Delta A + \Delta b BF)x + B(1 + \Delta b)v(x) + B\Delta D \tag{3.5}$$

We take $V(x) = \frac{1}{2} < Px, x >$ as the candidate for the Lyapunov function of (3.5) and compute $\dot{V}$ along trajectories of (3.5) to obtain

$$
\begin{aligned}
\dot{V} &=< Px(t), \dot{x}(t) > \\
&=< Px, A_0 x > + < Px, (\Delta A + \Delta b BF)x > \\
&+ < Px, B(1+\Delta b)v(x) + B\Delta D > \\
&= \frac{1}{2} < (PA_0 + A_0^\top P)x, x > + < Px, (\Delta A + \Delta b BF)x > \\
&+ < Px, B(1+\Delta b)v(x) + B\Delta D > \\
&\leq -(\lambda - \alpha||P||) < x, x > \\
&+ < Px, \Delta b BFx > + < Px, B(1+\Delta b)v(x) + B\Delta D >
\end{aligned}
\tag{3.6}
$$

We next define

$$
v(x) = \begin{cases} -\frac{\mu\delta + \nu\beta||x^\top F^\top||}{1-\beta} \frac{B^\top Px}{||B^\top Px||}, & \text{for } x \notin \ker(B^\top P) \\ \bar{v}(x), & \text{for } x \in \ker(B^\top P). \end{cases}
\tag{3.7}
$$

where $\mu \geq 1$ and $\nu \geq 1$ are constants to be determined later and $\bar{v}(x)$ is defined as follows.

(1) If $B^\top PB$ is invertible, i. e., B has full column rank, then $\bar{v}(x) = \bar{u}(x) - Fx$, where $\bar{u}(x)$ is the equivalent control on the sliding surface $s(x) = B^\top Px$. That is, $\bar{u}(x)$ is the solution of

$$
\begin{aligned}
\dot{s}(x) &= B^\top P[(A + \Delta A(w(t)))x \\
&+ B(1 + \Delta b(w(t)))\bar{u} + B\Delta D(w(t))] \\
&= 0
\end{aligned}
$$

which yields

$$
\bar{u}(x) = -\frac{1}{1+\Delta b}(B^\top PB)^{-1}[B^\top P(A+\Delta A)x + B^\top PB\Delta D]
$$

where $1 + \Delta b > 0$.

(2) If $B^\top PB$ is not invertible, we will define $\bar{v}(x)$ by

$$\bar{v}(x) = \bigcap_{\epsilon > 0} \overline{co}\, f(x + \epsilon B)$$

where B is the unit ball in R^n, $\overline{co}(K)$ denotes the closed convex hull of K, and

$$f(x) = -\frac{\delta + \beta ||x^\top F^\top||}{1 - \beta} \frac{B^\top Px}{||B^\top Px||}.$$

Because $\bar{v}(x)$ is set valued in case (2) above, it follows that the closed loop system will become a differential inclusion

$$\dot{x}(t) \in Q(x)$$

where $Q(x)$ is a set valued map. Observe that $Q(x)$ consists of a single point if $x \notin \ker(B^\top P)$; on the other hand, if $x \in \ker(B^\top P)$, then $Q(x)$ may consist of more than one point. However, in any case one can verify that $Q(x)$ is upper semicontinuous with compact, convex values. Therefore, according to [18], the existence of absolutely integrable solutions to $\dot{x}(t) \in Q(x)$ is guaranteed (of course, when $x \notin \ker\{B^\top P\}$, then $\dot{x}(t) =$ the single value of $Q(x)$).

Thus for $x \notin \ker(B^\top P)$ we see that

$$\begin{aligned}
&< Px, \Delta bBFx > + < Px, B(1 + \Delta b)v(t) + B\Delta D > \\
&\leq -(1 - \beta)\frac{\mu\delta + \nu\beta||x^\top F^\top||}{1 - \beta}\, ||B^\top Px|| \\
&+ \delta||B^\top Px|| + \beta||x^\top F^\top||\, ||B^\top Px|| \\
&\leq 0,
\end{aligned}$$

while for $x \in \ker(B^\top P)$ we see that

$$< Px, \Delta bBFx > + < Px, B(1 + \Delta b)v(t) + B\Delta D >= 0$$

Therefore, from (3.6), we have

$$\dot{V}(x) \leq -(\lambda - \alpha||P||) < x(t), x(t) > \tag{3.8}$$

Hence, in a manner similar to Theorem 2.1, we have established the following result.

Theorem 3.1. Suppose that (H3.1) and (H3.2) hold. If $\lambda > \alpha||P||$, then the feedback control law $u(x)$ defined by (3.4) and (3.8) will render the closed loop system (3.5) to be asymptotically stable.

Remark. The fact that $\lambda > \alpha||P||$ is independent of β and δ.

Remark. In case (1), we can design μ, ν such that (3.7) is a sliding mode control. The sliding surface is $s(x) = B^\top Px = 0$. Since B is of full rank, one can see that $B^\top PB$ is positive definite. Take $q = \min\{\lambda(B^\top PB)\} > 0$. Then

$$(B^\top Px)^\top B^\top PB(B^\top Px) \geq q||B^\top Px||^2$$

Therefore, we have

$$\begin{aligned}
s^\top(x)\dot{s}(x) &= (B^\top Px)^\top B^\top P[(A + \Delta A)x \\
&+ B(1 + \Delta b)u + B\Delta D] \\
&= (B^\top Px)^\top B^\top P[(A_0 + \Delta A + B\Delta bF)x \\
&+ B(1 + \Delta b)v + B\Delta D] \\
&\leq (||B^\top PA_0|| + \alpha||B^\top P||)||x|| + \beta||B^\top PB||\ ||Fx|| \\
&- (\mu\delta + \nu\beta||FX||)\frac{(B^\top Px)^\top B^\top PB(B^\top Px)}{||B^\top Px||} \\
&+ \delta||B^\top PB||\ ||B^\top Px|| \\
&\leq (||B^\top PA_0|| + \alpha||B^\top P||)||x||\ ||B^\top Px|| + \beta||B^\top PB||\ ||Fx||\ ||B^\top Px|| \\
&- q||B^\top Px||(\mu\delta + \nu\beta||Fx||) \\
&+ \delta||B^\top PB||\ ||B^\top Px||
\end{aligned}$$

Now set

$$\mu = \max\{1, \frac{2||B^\top PB||}{q}\},\ \nu = \max\{1, \frac{1}{q}\}$$

to obtain

$$s^{\top}(x)\dot{s}(x) \leq (||B^{\top}PA_0|| + \alpha||B^{\top}P||)||x||\ ||B^{\top}Px|| \\ - \delta||B^{\top}PB||\ ||B^{\top}Px||.$$

Hence, if

$$||x|| < \frac{\delta||B^{\top}PB||}{||B^{\top}PA_0|| + \alpha||B^{\top}P||}$$

then

$$s^{\top}(x)\dot{s}(x) < 0.$$

Remark. In case (2), one can see that if $x \in \ker\{B^{\top}P\}$, then

$$Q(x) = \frac{\mu\delta + \nu\beta||x^{\top}F^{\top}||}{1-\beta} \bigcap_{B^{\top}Px \neq 0} \bar{co}\,(\frac{B^{\top}Px}{||B^{\top}Px||})$$

is a disk determined by the surface $B^{\top}Bx = 0$, whose radius is

$$r = \frac{\mu\delta + \nu\beta||x^{\top}F^{\top}||}{1-\beta}$$

Therefore, we can simply take $\mu = \nu = 1$.

An illustrative example. Consider the example in 1.1 where we allow some uncertainty on the solution source: that is,

$$\begin{aligned} \dot{x}_1 &= -(\frac{Q_1}{L_1})x_1 + (\frac{Q_2}{L_2})x_2 + (Q_1 - Q_2)u + (Q_1 - Q_2)\Delta D \\ \dot{x}_2 &= (\frac{Q_1}{L_1})x1 - (\frac{Q_1}{L_2})x_2 \\ y &= (\frac{1}{L_2})x_2 \end{aligned}$$

where $(Q_1 - Q_2)\Delta D$ is some unknown source, and $|\Delta D| \leq \delta$. In terms of the notations in Theorem 3.1, $\alpha = \beta = 0$, $F = 0$, $B = (Q_1 - Q_2, 0)^{\top}$, $A_0 = A$. For simplicity we assume $L_1 = L_2 = L$. Take $P = I$ to obtain

$$A_0 + A_0^{\top} = \begin{pmatrix} -\frac{2Q_1}{L} & \frac{Q_1+Q_2}{L} \\ \frac{Q_1+Q_2}{L} & -\frac{2Q_2}{L} \end{pmatrix},$$

which is negative definite. Therefore, when take

$$\mu = \max\{1, \frac{2(Q_1 - Q_2)^2}{(Q_1 - Q_2)^2}\} = 2$$

according to Theorem 3.1, the control for $x_1 \neq 0$

$$u(x) = -2\delta \operatorname{sgn}(x_1)$$

will asymptotically stabilize the closed loop system.

Note. Physically, the above example means that if we have some unknown solution source added to the first tank, we should pump out some solution from the first tank to guarantee the solution will be eventually drained.

Next, we will extend Theorem 3.1 to a case involving delayed perturbations. The following lemma is a modification of a result in [15].

Lemma 3.2. If P is a positive definite matrix, then

$$\begin{aligned} G(t) &= x^\top(t)PE(t)x[t-h(t)] - \xi[1-\dot{h}(t)]x^\top[t-h(t)]x[t-h(t)] \\ &\leq \frac{\zeta^2}{4\xi(1-\dot{h}(t))}||P||^2\ ||x||^2 \end{aligned}$$

where $0 \leq \dot{h}(t) < 1$, $\xi > 0$, $||E(t)|| \leq \zeta$.

Proof. Simply set $\gamma = \frac{1}{\xi(1-\dot{h}(t))} > 0$ and compute to obtain

$$\begin{aligned} G(t) &= -\frac{1}{\gamma}[x(t-h(t)] - \frac{1}{2}\gamma E^\top Px]^\top[x(t-h(t)] - \frac{1}{2}\gamma E^\top Px] \\ &\quad + \frac{1}{4}\gamma x^\top PEE^\top Px \\ &\leq \frac{1}{4}\gamma x^\top PEE^\top Px \\ &\leq \frac{\gamma}{4}\zeta^2||P||^2\ ||x||^2 \\ &\leq \frac{\zeta^2}{4\xi(1-\dot{h}(t))}||P||^2\ ||x||^2 \end{aligned}$$

which completes the proof. QED.

Now we consider a generalization of (3.1), which contains a delay term, of the form

$$\dot{x} = (A + \Delta A(w(t)))x + E(t)x[t - h(t)] + B(1 + \Delta b(w(t)))u + B\Delta D(w(t)) \quad (3.9)$$

where $||E(t)|| \leq \zeta$, $0 \leq \dot{h}(t) < 1$, $0 \leq h(t) \leq r$ for some $r > 0$ and the remaining terms are as before.

Corollary 3.1. Suppose that (H3.1) and (H3.2) hold. If there exists $\xi > 0$ such that

$$\lambda > \alpha||P|| + \xi + \frac{\zeta^2}{4\xi(1 - \dot{h}(t))}||P||^2 > 0,$$

then the feedback control law defined by (3.4) and (3.8) will render the closed loop system to be asymptotically stable.

Proof. Define $V : R \times C([-r, 0], R^n) \to R^+$

$$V(t, \phi) = \frac{1}{2} < P\phi(0), \phi(0) > +\xi \int_{-h(t)}^{0} \phi^\top(t + \theta)\phi(t + \theta)d\theta,$$

and observe that V is a positive definite, since

$$\frac{1}{2}| < P\phi(0), \phi(0) > | \leq V(t, \phi) \leq (\frac{1}{2} + r\xi)|\phi|^2.$$

Using (3.4), (3.8) and Lemma 3.2, we have

$$\begin{aligned}\dot{V}\,|_{(3.9)} &\leq -(\lambda - \alpha||P||) < x(t), x(t) > +\xi < x(t), x(t) > \\ &\quad + x^\top(t)PE(t)x[t - h(t)] - \xi[1 - \dot{h}(t)]x^\top[t - h(t)]x[t - h(t)] \\ &\leq -(\lambda - \alpha||P|| - \xi - \frac{\zeta^2}{4\xi(1 - \dot{h}(t))}||P||^2) < x(t), x(t) >\end{aligned}$$

Therefore if

$$\lambda > \alpha||P|| + \xi + \frac{\zeta^2}{4\xi(1 - \dot{h}(t))}||P||^2 > 0$$

then the closed loop system will be asymptotically stable(see [38]). QED.

3.2 Output Feedback Design

Consider the following system

$$\dot{x}_1 = A_1x_1 + \Delta A_{11}x_1 + \Delta A_{12}x_2 + B_1(1+\Delta b)u + B_1\Delta D \tag{3.10}$$

$$\dot{x}_2 = A_2x_2 + \Delta A_{21}x_1 + \Delta A_{22}x_2 + B_2(1+\Delta b)u + B_2\Delta D \tag{3.11}$$

$$y = C_1x_1 + C_2x_2$$

where the notations have similar meanings as in (3.1) and C_1, C_2 are constant matrices. The following assumptions are made.

(H3.3) $A_1 + A_1^\top$ is negative definite.

(H3.4) (A_2, B_2, C_2) is minimum phase and (3.11) has nonsingular high-frequency gain;i. e., $\det(C_2B_2) \neq 0$.

According to [6], if (H3.4) holds, then there exist a symmetric, positive definite matrix P_2, a nonsingular matrix K, and a real number ρ such that

$$Q_2 = \frac{1}{2}[(A_2 + \rho B_2KC_2)^\top P_2 + P_2(A_2 + \rho B_2C_2)]$$

is negative definite and $KC_2 = B_2^\top P_2$.

With assumptions (H3.1), H(3.3) and (H3.4), we will look for a stabilizing feedback control law for the system (3.10), (3.11) of the form

$$u(y) = \rho Ky + v(y) \tag{3.12}$$

The substitution of (3.12) into (3.10), (3.11) and the use of the output relation $y = C_1x_1 + C_2x_2$ results in the closed loop system

$$\dot{x}_1 = [A_1 + \Delta A_{11} + \rho B_1K(1+\Delta b)C_1]x_1 + [\rho B_1K(1+\Delta b)C_2 + \Delta A_{12}]x_2$$
$$+ B_1\Delta D(t) + B_1(1+\Delta b)v(t)$$

(3.13)

$$\dot{x}_2 = [A_2 + \Delta A_{22} + \rho B_2 K(1+\Delta b)C_2]x_2 + [\rho B_2 K(1+\Delta b)C_1 + \Delta A_{21}]x_1$$
$$+ B_2 \Delta D(t) + B_2(1+\Delta b)v(t)$$

(3.14)

$$y = C_1 x_1 + C_2 x_2$$

We take $V(x) = \frac{1}{2}(< x_1, x_1 > + < P_2 x_2, x_2 >)$ as our candidate for the Lyapunov function, and compute $\dot{V}$ along solution of (3.14) and (3.15) to obtain

$$\dot{V} =< x_1, \dot{x}_1 > + < P_2 x_2, \dot{x}_2 >$$
$$=< x_1, A_1 x_1 > + < x_1, < x_1, [\Delta A_{11} + \rho B_1 K(1+\Delta b)C_1]x_1 >$$
$$+ < x_1, [\rho B_1 K(1+\Delta b)C_2 + \Delta A_{12}]x_2 >$$
$$+ < x_1, B_1 \Delta D(t) + B_1(1+\Delta b)v(t) >$$
$$+ < P_2 x_2, [\rho B_2 K(1+\Delta b)C_1 + \Delta A_{21}]x_1 >$$
$$+ < P_2 x_2, (A_2 + \rho B_2 K C_2)x_2 >$$
$$+ < P_2 x_2, (\Delta A_{22} + \rho B_2 K \Delta b C_2)x_2 >$$
$$+ < P_2 x_2, B_2 \Delta D(t) + B_2(1+\Delta b)v(t) >$$
$$\leq \frac{1}{2} < (A_1 + A_1^{\top})x_1, x_1 > +[\alpha + |\rho|\ ||B_1||\ ||KC_1||(1+\beta)] < x_1, x_1 >$$
$$+ [\alpha + |\rho|\ ||B_1||\ ||KC_2||(1+\beta)]||x_1||\ ||x_2||$$
$$+ < x_1, B_1 \Delta D > + < x_1, (1+\Delta b)B_1 > v(t)$$
$$+ [\alpha + |\rho|\ ||P_2||\ ||B_2||\ ||KC_1||(1+\beta)]||x_1||\ ||x_2|| + < Q_2 x_2, x_2 >$$
$$+ ||P_2||(\alpha + \beta|\rho|\ ||B_2||\ ||KC_2||) < x_2, x_2 >$$
$$+ < P_2 x_2, B_2 \Delta D > + < P_2 x_2, B_2 > (1+\Delta b)v(t)$$

(3.15)

Since $B_2^\top P_2 x_2 = KC_2 x_2 = K(y - C_1 x_1) = Ky - KC_1 x_1$, we have

$$\begin{aligned}
\dot{V} \le & -a_1 < x_1, x_1 > +2a_{12}||x_1||\ ||x_2|| - a_2 < x_2, x_2 > +\delta||K||\ |y(t)| \\
& + \delta(||KC_1|| + ||B_1||)||x_1|| + (1 + \Delta b)v(t)^\top Ky(t) \\
& + (1 + \beta)(||B_1|| + ||KC_1||)|v(t)|\ ||x_1||
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
a_1 &= \lambda_1 - \alpha - |\rho|(1+\beta)||B_1||\ ||KC_1|| \\
a_2 &= \lambda_2 - ||P2||(\alpha + \beta|\rho|\ ||Bb_2||\ ||KC_2||) \\
a_{12} &= \frac{1}{2}[2\alpha + |\rho|(1+\beta)(||B_1||\ ||KCc_2|| + ||P_2||\ ||B_2||\ ||KC_1||)] \\
\lambda_1 &= \text{ minimum eigenvalue of } -\frac{1}{2}(A_1 + A_1^\top) > 0 \\
\lambda_2 &= \text{ minimum eigenvalue of } Q_2 > 0
\end{aligned}$$

We define the function $v(y)$ in the desired feedback control law by

$$v(t) = \begin{cases} -\frac{||K||\delta}{1-\beta} \frac{K^{-1^\top} y(t)}{||y(t)||}, & \text{for } ||y(t)|| > \epsilon \\ -\frac{||K||\delta}{1-\beta} \frac{K^{-1^\top} y(t)}{\epsilon}, & \text{for } ||y(t)|| \le \epsilon. \end{cases} \tag{3.17}$$

Observe that if $||y(t)|| > \epsilon$, then

$$\begin{aligned}
\delta||K||\ ||y(t)|| + (1+\Delta b)v^\top Ky(t) &\le \delta||K||\ ||y(t)|| - (1-\beta)\frac{||K||\delta}{1-\beta}\ ||y(t)|| \\
&= 0,
\end{aligned}$$

while if $||y(t)|| \le \epsilon$, then

$$\begin{aligned}
&\delta||K||\ ||y(t)|| + (1+\Delta b)v^\top Ky(t) \\
&\le \delta||K||\ ||y(t)|| - ||K||\delta y(t)^2 \\
&\le \frac{||K||\delta\epsilon}{4}
\end{aligned}$$

which yields

$$\delta||K||\ ||y(t)|| + (1+\Delta b)v^\top K y(t) \leq \frac{||K||\delta\epsilon}{4}$$

and

$$\begin{aligned}
&\delta(||C_1|| + ||KB_1||)||x_1|| + (1+\beta)(||B_1|| + ||KC_1||)||v(t)||\ ||x_1|| \\
&\leq \delta(1+\frac{1+\beta}{1-\beta})(||KC_1|| + ||B_1||)||x_1|| \\
&= \frac{2\delta}{1-\beta}(||KC_1|| + ||B_1||)||x_1||
\end{aligned}$$

Therefore we obtain

$$\dot{V}(x) \leq -a_1||x_1||^2 - a_2||x_2||^2 + 2a_{12}||x_1||\ ||x_2|| + a_3||x_1|| + \frac{||K||\delta\epsilon}{4} \tag{3.18}$$

where

$$a_3 = \frac{2\delta}{1-\beta}(||B_1|| + ||KC_1||)$$

and ϵ is any real number satisfying $0 < \epsilon < 1$.

In a manner similar to Theorem 2.2, these computations prove the following result.

Theorem 3.2. Suppose that (H3.1), (H3.3) and (H3.4) hold. If $a_1 > 0$, $a_2 > 0$, $a_1 a_2 > a_{12}^2$, then there exists $r > 0$ such that the output feedback $u(y)$ defined by (3.12) and (3.17) will drive the solutions of the closed loop system (3.13) and (3.14) to the ball $B_r = \{||x|| \leq r\}$.

Remark. Note that δ only appears in the formula for a_3. This means that the truth of the inequalities $a_1 > 0$, $a_2 > 0$, and $a_1 a_2 > a_{12}^2$ is independent of the term ΔD.

An illustrative example. We modify the example in section 1.1 to allow the input solution in the second tank to have some uncertainty on the solution source:

$$
\begin{aligned}
\dot{x}_1 &= -(\frac{Q_1}{L_1})x_1 + (\frac{Q_2}{L_2})x_2 \\
\dot{x}_2 &= (\frac{Q_1}{L_1})x1 - (\frac{Q_1}{L_2})x_2 + (Q_1 - Q_2)u + (Q_1 - Q_2)\Delta D \\
y &= (\frac{1}{L_2})x_2
\end{aligned}
$$

where $(Q_1 - Q_2)\Delta D$ is some unknown source, and $|\Delta D| \leq \delta$. In line with the notations in Theorem 3.2, $\alpha = \beta = 0$, $A_1 = -\frac{Q_1}{L_1}$, $A_2 = -\frac{Q_1}{L_2}$, $C_2 = \frac{1}{L_2}$, $B_1 = 0$, $B_2 = Q_1 - Q_2$.

Take $K = L_2(Q_1 - Q_2)$, $P_2 = 1$, $\rho = -1$, then according to Theorem 3.2, when $u = u(y) = -Ky + v(y)$,

$$
v(y) = \begin{cases} -\delta \mathrm{sgn}(x_2), & \text{for } |x_2| > L_2\epsilon \\ -\delta \frac{x_2}{L_2\epsilon}, & \text{for } |x_2| \leq L_2\epsilon. \end{cases}
$$

then

$$
\dot{V} \leq -\frac{Q_1}{L_1}x_1^2 - [\frac{Q_1}{L_2} + (Q_1 - Q_2)^2]x_2^2 + \frac{L_2(Q_1 - Q_2)\delta\epsilon}{4}
$$

Therefore, by taking ϵ small enough, we can make the ball of attraction as small as we wish, which means that for all practical purposes the solution will be drained eventually.

In the remainder of this section, we extend Theorem 3.2 to a class of systems involving delayed perturbations by the use of Lemma 3.2.

Consider the partitioned linear system with disturbances and delay

$$
\dot{x}_1 = A_1x_1 + \Delta A_{11}x_1 + \Delta A_{12}x_2 + E_1(t)x_1[t - h(t)] + B_1(1 + \Delta b)u + B_1\Delta D \quad (3.19)
$$

$$
\dot{x}_2 = A_2x_2 + \Delta A_{21}x_1 + \Delta A_{22}x_2 + E_2(t)x_2[t - h(t)] + B_2(1 + \Delta b)u + B_2\Delta D \quad (3.20)
$$

$$
y = C_1x_1 + C_2x_2
$$

where $||E_i(t)|| \le \zeta$,$i = 1,2$, $0 \le \dot{h}(t) < 1$, $0 \le h(t) \le r$ for some $r > 0$ and the remaining terms have the same meanings as before.

Corollary 3.2. Suppose that (H3.1), (H3.3) and (H3.4) hold. If $\bar{a}_1 > 0$, $\bar{a}_2 > 0$, and $\bar{a}_1\bar{a}_2 > a_{12}^2$, then there exists $r > 0$ such that the output feedback law $u(y)$ defined by (3.12) and (3.17) will drive the solutions of the closed loop system of (3.19) and (3.20) to a ball $B_r = \{||x|| \le r\}$, where $\bar{a}_1$ and $\bar{a}_2$ are certain constants (to be defined in the proof).

Proof. Similar to the proof of Corollary 3.1, if we take

$$V(x) = \frac{1}{2}(< x_1, x_1 > + < P_2x_2, x_2 >) + \xi \int_{t-h(t)}^{t} x^\top(\theta)x(\theta)d\theta$$

then V is positive definite.

Using (3.18) and Lemma 3.2, we have

$$\dot{V}\ |_{(3.19)+(3.20)} \le -\bar{a}_1||x_1||^2 - \bar{a}_2||x_2||^2 + 2a_{12}||x_1||||x_2|| + a_3||x_1|| + \frac{||K||\delta\epsilon}{4}$$

where

$$\bar{a}_1 = \lambda_1 - \alpha - \xi - |\rho|(1+\beta)||B_1||\ ||KC_1|| - \frac{\zeta^2}{4\xi(1-\dot{h}(t))}$$
$$\bar{a}_2 = \lambda_2 - \xi - ||P2||(\alpha + \beta|\rho|\ ||Bb_2||\ ||KC_2||) - \frac{\zeta^2}{4\xi(1-\dot{h}(t))}||P||^2$$

and a_{12}, and a_3 are defined as before. Therefore, if $\bar{a}_1 > 0$, $\bar{a}_2 > 0$, and $\bar{a}_1\bar{a}_2 > a_{12}^2$, then the output feedback $u(y)$ defined by (3.12) and (3.17) will drive the solutions of the closed loop system of (3.19) and (3.20) to a ball $B_r = \{||x|| \le r\}$. QED.

4. Additional Feedback Designs For A Class of MIMO Systems

4.1. State Feedback Design

In this chapter we consider a class of so-called "partially matched" input-output systems with uncertainty and having the partitioned structure

$$\dot{x}_i = (A_i + \Delta A_i(w(t)))x + b_i u_i + b_i \Delta D_i(w(t)) + b_i \sum_{j \neq i}^{l} f_{ij}(y_j) \tag{4.1}$$

$$y_i = c_i x_i \tag{4.2}$$

where $x = (x_1, \dots, x_l)^\top \in R^{n_1} \times \cdots \times R^{n_l} = R^n, u_i \in R, b \in R^{n_i}$ (a constant vector), $y_i \in R$ and ΔA_i, ΔD_i, and f_{ij} are continuous functions of their arguments. It is assumed that the uncertainty $w(t)$ is Lebesgue measurable function of t which takes values in a fixed compact set $\Omega \subseteq R^s$. The terms ΔA_i and ΔD_i represent the disturbances, which are either known or unknown.

We will impose the following assumptions:

(H4.1) There exist nonnegative constants $\alpha_i \geq 0$, $\delta_i \geq 0$, $M_{ij} \geq 0$ such that

$$||\Delta A_i(w(t))|| \leq \alpha_i, ||\Delta D_i(w(t))|| \leq \delta_i, |f_{ij}(y_j)| \leq M_{ij}|y_j|,$$

and the functions f_{ij} satisfy a Lipschitz condition with constants $L_{ij} \geq 0$; that is, for an arbitrary pair of real numbers y_1, y_2 we have

$$|f_{ij}(y_1) - f_{ij}(y_2)| \leq L_{ij}|y_1 - y_2|.$$

(H4.2) The triple (A_i, b_i, c_i) is minimum phase and has nonsingular high-frequency gain in the sense that $c_i b_i \neq 0$.

According to [6], if (H2.4) holds, then there exist for each $i = 1, \ldots, l$ a symmetric, positive definite matrix P_i (of dimension $n_i \times n_i$) and $p_i \in R^{n_i}$ such that $c_i = b_i^\top P_i$ and for $A_{i0} = A_i + b_i < p_i, \cdot >$ the matrix

$$Q_i = A_{i0}^\top P_i + P_i A_{i0}$$

is negative definite; i. e., there exists $\lambda_i > 0$ such that $< Q_i x_i, x_i > \leq -\lambda_i < x_i, x_i >$ for some $x \in R^{n_i}$.

Under assumptions (H4.1) and (H4.2), we will seek a feedback control law of the form

$$u(x) = (u_1(x_1), \ldots u_l(x_l))$$

where for each $i = 1, \ldots l$ we have

$$u_i(x_i) = < p_i, x_i > -N_i c_i x_i - v_i(x_i) \tag{4.3}$$

where the scalars N_i and the functions $v_i(x_i)$ are to be determined later. It is essential to note that the ith component of the feedback law u_i will be designed so that it only depends on the ith component of the state variable x_i, even though the remaining outputs y_j, $j \neq i$, enter into the dynamics that determine x_i. Substituting (4.3) into (4.1) and (4.2), we obtain the closed loop system

$$\begin{aligned} \dot{x}_i &= (A_i + \Delta A_i)x_i + b_i(< p_i, x_i > -N_i y_i - v_i) \\ &\quad + b_i(\Delta D_i + \sum_{j \neq i}^{l} f_{ij}(y_j)) \\ y_i &= c_i x_i \\ (i &= 1, 2, \cdots, l) \end{aligned} \tag{4.4}$$

We next determine conditions under which $V(x) = \sum_{i=1}^{l} < P_i x_i, x_i >$ is a Lyapunov function for the closed loop system (4.4). The computation of $\dot{V}$ along the trajectories of (4.4) yields

$$
\begin{aligned}
\dot V \mid_{(4.4)} &= \sum_{i=1}^{l} 2 < P_i x_i(t), \dot x_i(t) > \\
&= \sum_{i=1}^{l} [< (A_{i0}\top P_2 + P_2 A_{i0}) x_i(t), x_i(t) > +2 < P_i x_i(t), \Delta A_i x_i(t) > \\
&- 2N_i < P_i b_i, x_i(t) > y_I + 2 < P_i b_i, x_i > \sum_{j \neq I}^{l} f_{ij}(y_j) \\
&+ 2 < P_i b_i, x_i > (\Delta D_i - v_i) \\
&\leq \sum_{i=1}^{l} -(\lambda_i - 2\alpha_i ||P_i||) < x_i, x_i > -2 \sum_{i=1}^{l} N_i y_i^2 \\
&+ 2 \sum_{i=1}^{l} (|y_i| \sum_{j \neq i}^{l} M_{ij} |y_j|) + 2 \sum_{i=1}^{l} (\Delta D_i - v_i) y_i \\
&\leq \sum_{i=1}^{l} -(\lambda_i - 2\alpha_i ||P_i||) ||x_i||^2 - 2\bar y^\top M \bar y \\
&+ 2 \sum_{i=1}^{l} (\Delta D_i - v_i) y_i
\end{aligned}
\tag{4.5}
$$

where $\bar y = (|y_1|, \cdots, |y_l|)^\top$ and

$$
M = \begin{pmatrix} N_1 & -M_{12} & \dots & -M_{1l} \\ -M_{21} & N_2 & \dots & -M_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ -M_{l1} & -M_{l2} & \dots & N_l \end{pmatrix}
$$

and the N_i's are yet to be determined. Take

$$
v_i(t) = \begin{cases} T_i \mathrm{sgn}(y_i), & \text{for } y_i(t) \neq 0 \\ \bar v_i(x), & \text{for } y_i(t) = 0. \end{cases}
$$

where $T_i > \delta_i$ is yet to be determined and $\bar v(x)$ is given by

$$
\bar v_i(x_i) =< p_i, x_i > -N_i y_i(t) - \bar u_i(x_i)
$$

where $\bar{u}_i(x_i)$ is the equivalent control of the sliding surface $s_i(x_i) = c_i x_i$; i. e., it is the solution of

$$\begin{aligned} \dot{s}_i(x_i) &= c_i[(A_i + \Delta A_i(w(t)))x_i \\ &\quad + b_i\bar{u}_i + b_i\Delta D_i(w(t)) + b_i \sum_{j\neq i}^{l} f_{ij}(y_j)] \\ &= 0 \end{aligned}$$

which yields

$$\bar{u}_i(x_i) = -\frac{1}{c_i b_i}[c_i(A_i + \Delta A_i(w(t)))x_i + c_i b_i \Delta D_i(w(t)) + c_i b_i \sum_{j\neq i}^{l} f_{ij}(y_j)]$$

For $c_i x_i \neq 0$ we see that

$$(\Delta D_i - v_i)y_i \leq \delta_i|y_i| - T_i|y_i| \leq 0,$$

while if $c_i x_i = 0$ we have

$$(\Delta D_i - v_i)y_i = 0.$$

Therefore, from (4.5) we have

$$\dot{V}\,|_{(4.4)} \leq \sum_{i=1}^{l} -(\lambda_i - 2\alpha_i||P_i||)\,||x_i||^2 - < (M + M^\top)\bar{y}, \bar{y} > \qquad (4.6)$$

Theorem 4.1. Suppose that (H4.1) and (H4.2) hold. If $\lambda_i > 2\alpha_i||P_i||$, then we can choose the constants $T_i \geq \delta_i$ and N_i sufficiently large so that the matrix $M + M^\top$ is positive definite and the feedback $u(x)$ defined by (4.3) will render the closed loop system (4.4) asymptotically stable.

Proof. Let $\bar{\lambda}_i = \lambda_i - 2\alpha_i||P_i|| > 0$ and observe that if $T_i \geq \delta_i$ and N_i are sufficiently large the matrix $M + M^\top$ is positive definite, whence we obtain

$$\dot{V}\,|_{(4.4)} \leq -\sum_{i=1}^{l} \bar{\lambda}_i||x_i||^2.$$

Since P_i is positive definite, there exist constants $k_{i1} > 0$, $k_{i2} > 0$ such that

$$k_{i1}||x_i||^2 \leq < P_i x_i, x_i > \leq k_{i2}||x_i||^2.$$

It follows that

$$\begin{aligned} \dot{V}\mid_{(4.4)} &\leq -\sum_{i=1}^{l} \frac{\bar{\lambda}_i}{k_{i1}} < P_i x_i, x_i > \\ &\leq -\lambda_0 V(x), \end{aligned}$$

where $\lambda_0 = \min\{\frac{\bar{\lambda}_i}{k_{i1}}\} > 0$. This results in

$V(x) \leq \exp(-\lambda_0 t)V(0)$

$\Longrightarrow \sum_{i=1}^{l} k_{i1}||x_i||^2 \leq \exp(-\lambda_0 t)\sum_{i=1}^{l} k_{i2}||x_{i0}||^2 \to 0$ as $t \to 0$

$\Longrightarrow x(t) = (x_1(t), \cdots, x_l(t))^\top \to 0$ as $t \to 0$. QED.

Remark. One can show that with the proper choice of the N_i's, (4.3) is a sliding mode control. Indeed let $s(x) = (s_i(x))^\top = (c_i x_i)^\top$. Then since $c_i b_i = b_i^\top P_i b_i > 0$, we have

$$\begin{aligned} s^\top(x)\dot{s}(x) &= (y_i)(c_i[(A_i + \Delta A_i)x + b_i u_i \\ &+ b_i \Delta D_i + b_i \sum_{j \neq i}^{l} f_{ij}(y_j)])^\top \\ &= \sum_{i=1}^{l} y_i\{< A_{i0}^\top c_i, x_i > + < c_i, \Delta A_i x_i >\} \\ &+ \sum_{i=1}^{l}\{c_i b_i[-(N_i y_i^2 + T_i|y_i|) \\ &+ y_i(\Delta D_i + \sum_{j \neq i}^{l} f_{ij}(y_j))]\} \\ &\leq \sum_{i=1}^{l}(||A_{i0}^\top c_i|| + \alpha||c_i||)||x_i||\ |y_i| \\ &- \sum_{i=1}^{l} c_i b_i(T_i - \delta_i)|y_i| \\ &- q_1 \sum_{i=1}^{l} N_i y_i^2 + q_2 \sum_{i=1}^{l}\sum_{j \neq i}^{l} M_{ij}|y_i|\ |y_j| \end{aligned}$$

where $q_1 = \min\{c_i b_i\} > 0$, $q_2 = \max\{c_i b_i\} > 0$. One can see that

$$-q_1 \sum_{i=1}^{l} N_i y_i^2 + q_2 \sum_{i=1}^{l} \sum_{j \neq i}^{l} M_{ij} |y_i|\ |y_j| = -\bar{y}^\top M^* \bar{y}$$

where

$$M^* = \begin{pmatrix} q_1 N_1 & -q_2 M_{12} & \dots & -q_2 M_{1l} \\ -q_2 M_{21} & q_1 N_2 & \dots & -q_2 M_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ -q_2 M_{l1} & -q_2 M_{l2} & \dots & q_1 N_l \end{pmatrix}$$

It is not hard to see that one can choose N_i's such that $M^* + M^{*\top}$ is positive definite. Hence

$$\begin{aligned} s^\top(x)\dot{s}(x) \leq & \sum_{i=1}^{l} (||A_{i0}^\top c_i|| + \alpha ||c_i||) ||x_i||\ |y_i| \\ & - \sum_{i=1}^{l} c_i b_i (T_i - \delta_i) |y_i| \end{aligned}$$

Therefore, if

$$||x_i|| < \frac{c_i b_i (T_i - \delta_i)}{||A_{i0}^\top c_i|| + \alpha ||c_i||}$$

then

$$s^\top(x)\dot{s}(x) < 0.$$

as required.

An illustrative example. Consider the following example:

$$\begin{aligned} \dot{x}_1 &= (A_1 + \Delta A_1) x_1 + b_1 u_1 + b_1 \Delta D_1 + b_1 f_{12}(y_2) \\ \dot{x}_2 &= (1 + \Delta A_2) x_2 + u_2 + \Delta D_2 + x_{11} + x_{12} \\ y_1 &= x_{11} + x_{12} \\ y_2 &= x_2 \end{aligned}$$

where, in light of the above notation, $x_1 = (x_{11}, x_{12})^\top$, $|\Delta A_1| \leq \alpha_1$, $||\Delta A_2|| \leq \alpha_2$, $|\Delta D_1| \leq \delta_1$, $|\Delta D_2| \leq \delta_2$, $f_{12}(y_2) = y_2$, $f_{21}(y_1) = \sin(y_1)$, $A_2 = 1$, $b_2 = 1$, $c_2 = 1$,

$c_1 = (1, 1)$, and

$$A_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},\ b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

One can see that $M_{12} = M_{21} = 1$. Take $p_2 = -2$, $p_1 = (-2, -2)^\top$, $P_1 = 1$, $P_2 = I$, then $\lambda_1 = \lambda_2 = 2$. According to (4.5), we can take the sliding mode control as

$$u_1 = < p_1, x_1 > -N_1 y_1 - v_1$$
$$u_2 = -2x_2 - N_2 y_2 - v_2$$

where

$$v_1 = T_1 \mathrm{sgn}(y_1)$$
$$v_2 = T_2 \mathrm{sgn}(y_2)$$

with $T_i > \delta_i$, and $N_1,\ N_2$ such that

$$\begin{pmatrix} N_1 & -1 \\ -1 & N_2 \end{pmatrix},\ and\ \begin{pmatrix} N_1 & -2 \\ -2 & N_2 \end{pmatrix}$$

are negative, eg,$N_1 = 3$, $N_2 = 2$. Therefore, according to Theorem 4.1, if $0 \leq \alpha_1 < 1$, $0 \leq \alpha_2 < 1$, the control will render the closed loop system asymptotically stable.

4.2. Output Feedback Design

We further refine the structure of the systems consider in the previous section by considering systems of the form

$$\dot{x}_{i1} = (A_{i1} + \Delta A_{i1}(w(t)))x_{i1} + b_{i1}(1 + \Delta b_i)u_i + b_{i1}\Delta D_i(w(t)) + b_{i1} \sum_{j \neq i}^{l} f_{ij}(y_j) \quad (4.7)$$

$$\dot{x}_{i2} = (A_{i2} + \Delta A_{i2}(w(t)))x_{i2} + b_{i2}(1 + \Delta b_i)u_i + b_{i2}\Delta D_i(w(t)) + b_{i2} \sum_{j \neq i}^{l} f_{ij}(y_j) \quad (4.8)$$

$$y_i = c_{i1}x_{i1} + c_{i2}x_{i2}$$

where $x = (x_{11}, x_{12}, \cdots, x_{l1}, x_{l2})^\top \in R^n$, $u_i \in R$, $y_i \in R$, $\Delta D_i \in R$, ΔA_{ij}, Δb_{ij}, ΔD_i are continuous, $w(t) \in \Omega$ is the Lebesgue measurable uncertainty taking values in a fixed compact set $\Omega \subset R^s$. As before, ΔA_{ij}, Δb_{ij}, ΔD_i represent the disturbances, which are either known or unknown.

We make the following assumptions.

(H4.3) $A_{i1} + A_{i1}^\top$is negative definite and $c_{i1} = b_{i1}^\top$.

(H4.4) (A_{i2}, b_{i2}, c_{i2}) is minimum phase and (4.8) has nonsingular high-frequency gain; i. e., $\det(c_{i2}b_{i2}) \neq 0$.

According to [6], if (H4.4) holds, then there exist a symmetric, positive definite matrix P_i and a constant k_i such that $c_{i2} = b_{i2}^\top P_i$ and

$$Q_i = \frac{1}{2}[(A_{i2} + k_i b_{i2} c_{i2})\top P_2 + P_2(A_{i2} + k_i b_{i2} c_{i2})]$$

is negative definite; i. e., there exists $\lambda_{i2} > 0$ such that $< Q_i x_{i2}, x_{i2} > \leq -\lambda_{i2} < x_{i2}, x_{i2} >$ for all $x_{i2} \in R^{n_{i2}}$.

Under assumptions (H4.1), (H4.3), and (H4.4), we will seek a stabilizing feedback control law of the form

$$u(y) = (u_1(y_1)), \ldots, u_l(y_l))^\top,$$

where

$$u_i(y_i) = k_i y_i - N_i y_i + v_i(y_i) \tag{4.9}$$

and the scalars N_i and the functions $v_i(y_i)$ are yet to be determined.

Substituting (4.9) into (4.7) and (4.8), we obtain the closed loop system

$$
\begin{aligned}
\dot{x}_{i1} &= (A_{i1} + \Delta A_{i1})x_{i1} + b_{i1}(1 + \Delta b_i)(k_i y_i - N_i y_i)) \\
&\quad + b_{i1}(1 + \Delta b_i)(\Delta D_i + \sum_{j \neq i}^{l} f_{ij}(y_j)) + b_{i1}(1 + \Delta b_i)v_i
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
\dot{x}_{i2} &= (A_{i2} + \Delta A_{i2})x_{i2} + b_{i2}(1 + \Delta b_i)(k_i y_i - N_i y_i)) \\
&\quad + b_{i2}(1 + \Delta b_i)(\Delta D_i + \sum_{j \neq i}^{l} f_{ij}(y_j)) + b_{i2}(1 + \Delta b_i)v_i
\end{aligned} \tag{4.11}
$$

and

$$
\begin{aligned}
y_i &= c_i x_i \\
i &= 1, 2, \cdots, l
\end{aligned}
$$

We will determine conditions under which

$$
\begin{aligned}
V(x) &= \frac{1}{2}[< P_1 x_{12}, x_{12} > + < x_{11}, x_{11} >] \\
&\quad + \frac{1}{2}[< P_2 x_{22}, x_{22} > + < x_{21}, x_{21} >] \\
&\quad + \cdots \\
&\quad + \frac{1}{2}[< P_l x_{l2}, x_{l2} > + < x_{l1}, x_{l1} >] \\
&= \sum_{i=1}^{l} \frac{1}{2}[< P_i x_{i2}, x_{i2} > + < x_{i1}, x_{i1} >]
\end{aligned}
$$

is a Lyapunov function for the closed loop system (4.10), (4.11).

Indeed the computation of $\dot{V}$ along the trajectories of (4.10), (4.11) yields

$$
\begin{aligned}
\dot{V} &= \sum_{i=1}^{l}(< x_{i1}, \dot{x}_{i1} > + < P_i x_{i2}, \dot{x}_{i2} > \\
&= \sum_{i=1}^{l}\{\frac{1}{2} < [(A_{i2} + k_i b_{i2} c_{i2})^{\top} P_2 + P_2(A_{i2} + k_i b_{i2} c_{i2})] x_{i2}(t), x_{i2}(t) > \\
&+ < P_i x_{i2}(t), (\Delta A_{i2} + k_i b_{i2} \Delta b_i c_{i2}) x_{i2}(t) > \\
&+ < P_2 x_{i2}, k_i b_{i2}(1 + \Delta b_i) c_{i1} x_{i1} > -N_i(1 + \Delta b_i) < P_i b_{i2}, x_{i2}(t) > y_i \\
&+ < P_i b_{i2}, x_{i2}(t) > \Delta D_i + (1 + \Delta b_i) < P_i b_{i2}, x_{i2}(t) > v_i \\
&+ < P_i b_{i2}, x_{i2} > \sum_{j \neq i}^{l} f_{ij}(y_j)\} \\
&+ \sum_{i=1}^{l}\{\frac{1}{2} < (A_{i1} + A_{i1}^{\top}) x_{i1}(t), x_{i1}(t) > \\
&+ < x_{i1}(t), (\Delta A_{i1} + k_i b_{i1}(1 + \Delta b_i) c_{i1}) x_{i1}(t) > \\
&+ < x_{i1}, k_i b_{i1}(1 + \Delta b_i) c_{i2} x_{i2} > -N_i(1 + \Delta b_i) < b_{i1}, x_{i1}(t) > y_i \\
&+ < b_{i1}, x_{i1}(t) > \Delta D_i \\
&+ (1 + \Delta b_i) < b_{i1}, x_{i1}(t) > v_i + < b_{i1}, x_{i1} > \sum_{j \neq i}^{l} f_{ij}(y_j)\} \\
&\leq \sum_{i=1}^{l}\{-(\lambda_{i1} - \alpha_i - |k_i|(1 + \beta_i)||c_{i1}||^2)||x_{i1}||^2 \\
&+ 2|k_i|(1 + \beta_i)||b_{i1}||\,||c_{i2}||\,||x_{i1}||\,||x_{i2}|| \\
&- (\lambda_{i2} - ||P_i||\alpha_i - |k_i|\,|\beta_i|\,||c_{i2}||^2)||x_{i2}||^2 - (1 + \Delta b_i) N_i y_i^2 \\
&+ \Delta D_i y_i + (1 + \Delta b_i) v_i y_i + y_i \sum_{j \neq i}^{l} f_{ij}(y_j)\} \\
&\leq \sum_{i=1}^{l}\{-a_{i1}||x_{i1}||^2 + a_{i12}||x_{i1}||\,||x_{i2}|| - a_{i2}||x_{i2}||^2 + \delta_i|y_i| \\
&- (1 - \beta_i) N_i y_i^2 + (1 + \Delta b_i) v_i y_i + |y_i| \sum_{j \neq i}^{l} M_{ij}|y_j|\}
\end{aligned}
\tag{4.10}
$$

where

$$a_{i1} = \lambda_{i1} - \alpha_i - |k_i|(1+\beta_i)||c_{i1}||^2$$
$$a_{i1} = \lambda_{i2} - ||P_i||\alpha_i - |k_i|\ |\beta_i|\ ||c_{i2}||^2$$
$$a_{i12} = |k_i|(1+\beta_i)||b_{i1}||\ ||c_{i2}||.$$

If we define

$$v_i(y_i(t)) = \begin{cases} -\frac{\delta_i}{1-\beta_i}\mathrm{sgn}(y_i(t)), & \text{for } |y_i(t)| > \epsilon \\ -\frac{\delta_i}{1-\beta_i}\frac{y_i(t)}{\epsilon}, & \text{for } |y_i(t)| \le \epsilon, \end{cases} \tag{4.11}$$

then we see that

$$\delta_i|y_i| + (1+\Delta b_i)v_i(t)y_i \le \frac{\beta_i\epsilon}{4} \tag{4.12}$$

If we denote $\bar{y} = (|y_1|, \cdots, |y_l|)^\top$ and

$$M = \begin{pmatrix} (1-\beta_1)N_1 & -M_{12} & \dots & -M_{1l} \\ -M_{21} & (1-\beta_2)N_2 & \dots & -M_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ M_{l1} & -M_{l2} & \dots & (1-\beta_l)N_l \end{pmatrix}$$

then from (4.12) and (4.14) we obtain

$$\begin{aligned} \dot{V} \le & \sum_{i=1}^{l}\{-a_{i1}||x_{i1}||^2 + a_{i12}||x_{i1}||\ ||x_{i2}|| - a_{i2}||x_{i2}||^2 + \frac{\beta_i\epsilon}{4}\} \\ & -\frac{1}{2} < (M+M^\top)\bar{y}, \bar{y} > \end{aligned} \tag{4.13}$$

Thus, in a manner similar to Theorem 2.2, we have proved the following result.

Theorem 4.2. Suppose that (H4.1), (H4.3), and (H4.4) hold. If $a_{i1} > 0$, $a_{i2} > 0$, and $a_{i1}a_{i2} > a_{i12}^2$ for each $i = 1, \dots, l$, and if the constants N_i are sufficiently large that the matrix $M + M^\top$ is positive definite, then there exists $r > 0$ such that the feedback $u(y)$ defined by (4.9) and (4.13) will drive every solution of the closed loop system (4.10 and (4.11) to the ball $B_r = \{||x|| \leq r\}$. Furthermore, r can be made arbitarily small.

Proof. One can see that when the N_i are sufficiently large the matrix $M + M^\top$ is positive definite. Therefore

$$-\frac{1}{2} < (M + M^\top)\bar{y}, \bar{y} > \leq 0$$

Since $a_{i1} > 0$, $a_{i2} > 0$, and $a_{i1}a_{i2} > a_{i12}^2$, we can choose $0 < d_{i1} < a_{i1}$, $0 < d_{i2} < a_{i2}$ such that $d_{i1}d_{i2} = a_{i12}^2$. Denote $t_{i1} = a_{i1} - d_{i1} > 0$, $t_{i2} = a_{i2} - d_{i2} > 0$, and $\beta = \sum_{i=1}^{l} \beta_i$. Then from (4.15) we have

$$\begin{aligned}
\dot{V} &\leq -\sum_{i=1}^{l} \{-(d_{i1}^{\frac{1}{2}}||x_{i1}|| - d_{i2}^{\frac{1}{2}}||x_{i2}|)^\top + \frac{\epsilon}{4}\sum_{i=1}^{l} \beta_i \\
&- \sum_{i=1}^{l} (t_{i1}||x_{i1}||^2 + t_{i2}||x_{i2}||^2) \\
&\leq \frac{\beta\epsilon}{4} - \sum_{i=1}^{l} t_i||x_i||^2 \\
&\leq \frac{\beta\epsilon}{4} - t_0 \sum_{i=1}^{l} ||x_i||^2
\end{aligned}$$

where $t_i = \min\{t_{i1}, t_{i2}\} > 0$, $t_0 = \min\{t_i\} > 0$. It follows that if $||x|| > (\frac{\beta\epsilon}{4t_0})^{\frac{1}{2}}$, then $\dot{V} < 0$. Hence, with $r = (\frac{\beta\epsilon}{4t_0})^{\frac{1}{2}}$, the ball B_r is an attractive ball for the closed loop system (4.10) and (4.11). QED.

Remark. If we do not require $u(t)$ to be continuous, then one can see that with $v_i(t) = -\frac{\delta_i}{1-\beta_i}\mathrm{sgn}(y_i(t))$, the resulting $u(t)$ will render the closed loop system to be asymptotically stable.

An illustrative example. Consider an example without uncertainties on the input:

$$\dot{x}_{11} = (-3 + \Delta A_{11})x_{11} + u_1 + \Delta D_1 + sin(x_{21} + x_{22})$$
$$\dot{x}_{12} = (1 + \Delta A_{12})x_{12} + 2u_1 + 2\Delta D_1 + 2sin(x_{21} + x_{22})$$
$$\dot{x}_{21} = (-7 + \Delta A_{21})x_{21} + u_2 + \Delta D_2 + 2(x_{11} + 2x_{12})$$
$$\dot{x}_{22} = (1 + \Delta A_{22})x_{22} + u_1 + \Delta D_2 + 2(x_{11} + 2x_{12})$$
$$y_1 = x_{11} + 2x_{12}$$
$$y_2 = x_{21} + x_{22}$$

where, in light of the above notation, $|\Delta A_{1i}| \leq \alpha_1$, $|\Delta A_{2i}| \leq \alpha_2$, $|\Delta D_1| \leq \delta_1$, $|\Delta D_2| \leq \delta_2$, $f_{12}(y_2) = \sin(y_2)$, $f_{21}(y_1) = 2y_1$, $\beta_1 = \beta_2 = 0$.
Now, take $k_1 = -\frac{1}{2}$, $k_2 = -2$, then we can take $P_1 = P_2 = 1$, $\lambda_{11} = 3$, $\lambda_{12} = 1$, $\lambda_{12} = 7$, $\lambda_{22} = 1$, and the output feedback control as

$$u_1 = -\frac{1}{2}y_1 - N_1 y_1 + v_1$$
$$u_2 = -2y_2 - N_2 y_2 + v_2$$

where

$$v_1(t) = \begin{cases} -\delta_1 \mathrm{sgn}(y_1), & \text{for } |y_1| > 1 \\ -\delta_1 y_1, & \text{for } |y_i| \leq 1. \end{cases}$$
$$v_2(t) = \begin{cases} -\delta_2 \mathrm{sgn}(y_2), & \text{for } |y_2| > 1 \\ -\delta_2 y_2, & \text{for } |y_2| \leq 1. \end{cases}$$

and N_1 and N_2 are such that

$$M + M^\top = \begin{pmatrix} 2N_1 & -3 \\ -3 & 2N_2 \end{pmatrix}$$

is negative definite(e. g., $N_1 = N_2 = 2$). Since $\beta_i = 0$, according to Theorem 4.2, if

$$a_{11} = \frac{5}{2} - \alpha_1 > 0, \quad a_{12} = 1 - \alpha_1 > 0$$
$$a_{21} = 5 - \alpha_2 > 0, \quad a_{22} = 1 - \alpha_2 > 0$$

and

$$a_{11}a_{12} > 1, \quad a_{21}a_{22} > 4$$

where $a_{112} = 1$, $a_{212} = 2$, that is

$$0 \leq \alpha_1 < \frac{1}{2}, \quad 0 \leq \alpha_2 < 3 - 2\sqrt{2}$$

then the output feedback control will render the system to be asymptotically stable.

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