ADAPTIVE ESTIMATION AND STOCHASTIC

CONTROL FOR UNCERTAIN MODELS

By

ANDREW YEOH-YIH LEE

Diploma
Taiwan Provincial Taipei Institute of Technology
Taipei, Taiwan
1960

Master of Science
Kansas State University
Manhattan, Kansas
1968

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
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Thesis Approved:

Craig S. Sims
Thesis Adviser

Henry R. Sebesta

Paul A. McCollum

Charles M. Bacon

D. Durham
Dean of the Graduate College

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CHAPTER I

INTRODUCTION

1.1 Uncertainty in Modeling

Significant advances have been made in the area of sequential state estimation due to the works of Kalman and Bucy[1-3]. Their approach has led to the well known Kalman-Bucy filter. Since the publication of this result, there have been numerous papers written on alternative ways of deriving the conditions for the optimal linear filter and on extensions of the original work[4-6]. The area of stochastic control theory [27,28] is an important application of sequential estimation techniques, and is itself a research topic of considerable interest. This investigation is concerned with extending results in estimation theory and stochastic control to cases where the model is uncertain.

It is ordinarily assumed that the parameters describing the system are completely known including the statistic of the associated noise vectors. In many engineering problems, however, the parameters may not be completely known. For example, an aerospace system such as a high performance aircraft may have aerodynamic parameters which change as a function of altitude, speed, and angle of attack. The estimation of the state of such a system is relatively simple if one can directly measure the uncertain parameters. In this case, when parameters are randomly changing, estimation schemes have been developed[29]. There are physical processes, however, where one cannot measure the parameters which describe the system, and has to rely on available statistical data for modeling the system. One way to handle the indicated type of problem is to develop an estimation technique which takes into account the uncertainty of the model. The formulation of this type of problem has been referred to as "Estimation-Under-Uncertainty"[14].

Just as state estimation for uncertain models may be treated by considering the uncertainty directly in the problem formulation, controls for uncertain systems may be developed in the same way. The philosophy which motivates this work is that one should include in the initial problem statement the fact that modeling of physical systems is seldom perfect. Although such an approach adds to the complexity of both the estimation and control problems, in many cases it may be preferable to ignoring the inherent uncertainty of modeling.

1.2 The Approach

This section indicates some of the significant aspects of the approach taken. Motivation and relationships to research articles which are of central importance to this work are also pointed out. Among the key features of the approach taken, the following are important points.

(1) Uncertainty of the model is treated by assuming that the true model is one of a set of a finite number of possible models.

(2) It is assumed that each candidate model has linear continuous dynamical behavior and linear observations which are discrete in time.

(3) A recursive estimation algorithm and system identification

technique are developed.

(4) A feedback control strategy is developed using the results of estimation-under-uncertainty.

1.2.1 Candidate Models

As mentioned previously, consideration of uncertainty in system parameters is important since systems are seldom known precisely. If some knowledge about a system is available, then it may be reasonable to assume that the system may be modeled by one of a finite set of possible candidate models. This set consists of the possible systems which one might in fact have. Parameters, as well as the order of the models, may differ. A set of prior probabilities associated with each model may be approximated or guessed. The approach indicated here is fundamental to this investigation. Much of this thesis is based on the work of Hilborn and Lainiotis[9], who also considered the true system to be one of a finite number of possible systems. A slightly different approach is adopted in this investigation, in which each of the dynamical models is assumed to be continuous in time while the observations are assumed to be discrete.

1.2.2 Continuous-Discrete Modeling

While most physical processes have dynamical behavior best described using differential equations, practical considerations often suggest that a discrete observation model is appropriate. The continuous-discrete modeling approach is motivated in part by problems arising in estimation of state variables, for example, finding Cartesian position and velocities of a target, using radar observations which are discrete in time. In such problems, the difficulty is one of estimating the state variables of a continuous-time dynamical system where some of describing parameters are uncertain. At discrete instants of time, certain output variables, for example, range, azimuth, elevation, etc., are observed in the presence of measurement noise. The solution of this type of continuous-discrete estimation problem without uncertainty of the system model has been developed by Jazwinski[22], and Athans, et al[25], where more general nonlinear dynamical systems were considered. Jazwinski's result is combined with Hilborn and Lainiotis' approach to obtain a recursive filter for the continuous-discrete model with uncertain parameters. The researchers mentioned above provide a convenient basis for the estimation portion of this thesis, and a logical starting point for the control section.

1.2.3 Recursive Filters and System Identification

The problem of estimating the state of each candidate model from noisy observations is a classical problem, where the recursive filter may be broken into two parts: in between observations the estimator is merely a predictor, and after a measurement is taken, information is available to the corrector which may update the estimates of each predictor[22]. The optimal state estimate of the system is the sum of the state estimates of the candidate models weighted by the appropriate posterior probabilities[9].

The posterior probability of each candidate model is updated whenever measurements are available[9]. The posterior probability also provides a measure indicating the closeness of the candidate model to the actual system. Hence, system identification may be achieved concurrently with state estimation. If the posterior probability of one of the candidate models reaches unity after many observations, then the true system is known, and the estimation problem reduces to that of estimation under certainty.

1.2.4 The Control Algorithm

A closed-loop control algorithm is developed, which can be implemented on-line. That is, the computation of the control law can be carried out at the same time as the estimation and system identification are taking place. This is in contrast to the situation in which one must first identify the system, and then design a controller. The control strategy is referred to as a closed-loop or feedback control law since it makes use of all information available up until the time of application. The closed-loop nature of the control is important, since in stochastic problems open-loop controls are usually unacceptable resulting in increased performance measures.

1.3 Objectives and Findings

There are two basic objectives that this study is intended to accomplish. The first is to develop a general formulation and solution of the state estimation and control problems for a large class of linear, stochastic continuous processes with discrete observations and uncertain models. The second objective is to exemplify the usefulness of the proposed techniques by developing the algorithms in detail for some typical models and illustrating the techniques by simulation.

The first objective is accomplished in the following manner:

(1) For estimation problems, the predictor and corrector equations are derived to generate conditional mean estimates. The

identification or posterior probability of each candidate model is obtained by employing Bayesian techniques.

(2) For control problems, a control strategy is obtained using dynamic programming, and the separation principle[20,26]. In a general sense, the separation principle implies that a stochastic control problem can be solved by considering estimation and control separately.

The second objective is achieved by providing numerous illustrative examples where the results are compared with the case in which the system is completely known. An application to the problem of stream quality control is also presented in detail to demonstrate the techniques.

In the estimation portion of the investigation, algorithms are obtained in which simultaneous estimation, and system identification are achieved using a recursive technique. The algorithms are optimal in the sense that a conditional mean estimate is developed. The development makes use of linear requirements on the dynamical and observation models, and Gaussian assumptions for the stochastic processes considered. If these assumptions are violated in practice, the resulting application, of course, will be suboptimal. In the case where it is possible for switching from one system to another to occur in the interval of interest, it is found that practical constraints force one to consider a suboptimal rather than an optimal solution.

In the control portion of the study, feedback controls which utilize the state estimates are obtained, where the control criterion is a quadratic performance measure. As in the estimation portion, suboptimal results are more practical if switching is allowed during the control interval. In the extensive example concerned with stream quality control, it is clear that the system identification aspect of the algorithm may be of considerable importance, and furthermore it is seen that the technique works reasonably well even when certain fundamental assumptions are violated.

1.4 Organization

The remainder of this study is arranged in the following way. Chapter II presents a brief review of classical estimation theory and stochastic control. The existing estimation and control schemes under uncertainty are also discussed and compared.

The estimation problem is formulated and solved in Chapter III, where optimal and suboptimal techniques of estimation are developed. In Chapter IV, the application of uncertain estimation techniques to stochastic control problems is developed. Examples are included to compare the results with alternative control strategies.

In Chapter V a stochastic model describing water quality is proposed and an application of the techniques developed in Chapter III and Chapter IV to the stream quality control problem is presented. Simulation results are also included in detail. Chapter VI contains a summary and conclusions of the results obtained in the dissertation. Suggestions for further research and extensions are also presented.

CHAPTER II

A SURVEY OF BACKGROUND MATERIAL

2.1 Introduction

The purpose of this chapter is to review various existing techniques in estimation and stochastic control problems. In Section 2.2 some of the classical estimation techniques are discussed for static as well as dynamic systems. Section 2.3 is concerned with the topic of estimation for systems with uncertain parameters. These subjects are presented since they form the basis of the estimation techniques in the next chapter. Section 2.4 is intended to provide background in the area of stochastic control, and in particular, the control of stochastic systems with uncertain models.

2.2 Estimation Problems

The problem of estimation in a system arises whenever an accurate measurement of the variable of interest cannot be made directly or indirectly due to the presence of noise in the observations. Two common types of estimation problems are:

(1) Parameter estimation, where the value of a parameter of a static system is of interest.

(2) State estimation, where the estimation of the state of a dynamic system is required.

In parameter estimation, a parameter may be deterministic or random. A deterministic but unknown parameter[29] is usually estimated by using statistical methods such as moving average[30] and least squares[16]. In the case of a random parameter with a known distribution, Bayesian estimation techniques[31] are often employed yielding "maximum a posteriori probability"(MAP) or "minimum variance"(MV)[17,32] estimates depending upon the chosen cost criterion. When the prior distribution on a parameter is not known, but the statistical knowledge about the noise is known, then the "maximum likelihood"(ML)[17] estimation scheme is often used. These schemes are used extensively in practice but certainly do not exhaust the available estimation techniques.

In the case of state estimation, the initial state vector may be random. In addition, the system may be excited by noise and the observation may also be corrupted by noise. If the statistical information concerning the noise and the state are not known, the "least square"(LS) method is commonly used. Whenever partial or complete statistical knowledge on the system dynamic and observation structure is available, extended versions of MAP, MV, and ML algorithms are applicable[17]. For a linear stochastic Gauss-Markov system with normally distributed additive noise and linear observations, all the above mentioned schemes yield the same optimal sequential filter, referred to as the <u>Kalman filter</u>[1-3]. Kalman derived this result using the principle of orthogonal projection. The same result has been derived by various methods such as dynamic programming[33], maximum principle[5], invariant imbedding[34], and, more recently, the "innovation" approach[6]. The Kalman filtering algorithm is of fundamental importance to this study, providing a convenient recursive procedure to achieve the desired estimates. The algorithm is therefore presented below for various cases of importance.

The Kalman filter for the cases of discrete systems, continuous systems[27], and continuous-discrete systems[22] is summarized in the following.

For the discrete version of the Kalman filter, it is assumed that the discrete system is described by the difference equation[27]

$$x(k+1) = \Phi(k+1,k)x(k) + \Gamma(k+1,k)w(k) \tag{2-1}$$

with observation

$$z(k+1) = H(k+1)x(k+1) + v(k+1) \tag{2-2}$$

where x is an nth order state vector, w is a qth order plant disturbance vector, z is an mth order measurement vector, v is a measurement noise vector, and k=1,2,..., is the discrete-time index. In addition, Φ is an nxn state transition matrix, Γ is an nxq disturbance transition matrix, and H is an mxn measurement matrix. The processes $\{w(k)\}$ and $\{v(k)\}$ are independent gaussian white sequences with zero means and covariance matrices

$$E\{w(k)w^T(j)\} = Q(k)\delta_{kj}$$

$$E\{v(k)v^T(j)\} = R(k)\delta_{kj} \tag{2-3}$$

The initial state x(0) is a gaussian random vector with mean

$$E\{x(0)\} = \bar{x}(0) \tag{2-4}$$

and variance

$$\mathrm{VAR}\,\{x(0)\} = E\,\{[x(0)-\bar{x}(0)][x(0)-\bar{x}(0)]^T\} = V_x(0) \tag{2-5}$$

It is assumed that $x(0)$ is independent of $\{w(k)\}$ and $\{v(k)\}$ for all k. Given a set of measurement data $Z_k = \{z(1), z(2), \ldots, z(k)\}$ and a new measurement $z(k+1)$, the optimal filtered estimate $\hat{x}(k+1/k+1)$ is given by the recursive relation

$$\hat{x}(k+1/k+1) = \Phi(k+1,k)x(k/k) + K(k+1)[z(k+1) - H(k+1)\,\Phi(k+1,k)\hat{x}(k/k)] \tag{2-6}$$

for $k = 0, 1, 2, \ldots$, where $\hat{x}(k/k)$ is defined as the conditional mean estimate,

$$\hat{x}(k/k) = E\,\{x(k)/Z_k\} \tag{2-7}$$

with $\hat{x}(0/0) = \bar{x}(0)$. The Kalman gain $K(k+1)$ is obtained from the expression

$$K(k+1) = V_x(k+1/k)H^T(k+1)[H(k+1)V_x(k+1/k)H^T(k+1) + R(k+1)]^{-1} \tag{2-8}$$

where the predicted conditional covariance matrix $V_x(k+1/k)$ is defined as

$$V_x(k+1/k) = \mathrm{VAR}\,\{x(k+1)/Z_k\} \tag{2-9}$$

which is given by the relationship

$$V_x(k+1/k) = \Phi(k+1,k)V_x(k/k)\,\Phi^T(k+1,k) + \Gamma(k+1,k)Q(k)\,\Gamma^T(k+1,k) \tag{2-10}$$

and the updated conditional covariance matrix in Eq. 2-10 satisfies the equation

$$V_x(k+1/k+1) = [I - K(k+1)H(k+1)]V_x(k+1/k) \tag{2-11}$$

with $V_x(0/0) = V_x(0)$.

Equations 2-6 through 2-11 provide a recursive means of calculating conditional mean estimates, for known discrete time systems.

For the continuous time version of the Kalman filter[27], the linear continuous system is assumed to described by the dynamic and observational equations

$$\dot{x}(t) = F(t)x(t) + G(t)w(t) \tag{2-12}$$

and

$$z(t) = H(t)x(t) + v(t) \tag{2-13}$$

The stochastic processes $\{w(t)\}$ and $\{v(t)\}$ are zero mean gaussian white noise with covariance matrices

$$\begin{aligned} E\{w(t)w^T(\tau)\} &= Q(t)\,\delta(t-\tau) \\ E\{v(t)v^T(\tau)\} &= R(t)\,\delta(t-\tau) \end{aligned} \tag{2-14}$$

The optimal continuous filtered estimate for the system described by Eqs. 2-12 and 2-13 is governed by the relation

$$\dot{\hat{x}}(t) = F(t)\hat{x}(t) + K(t)[z(t) - H(t)\hat{x}(t)] \tag{2-15}$$

for $t \geq t_o$, where the conditional mean $\hat{x}(t)$ is defined as

$$\hat{x}(t) = E\{x(t)/Z_t\} \tag{2-16}$$

with $\hat{x}(t_o) = \bar{x}(t_o)$ and $Z_t = \{z(\tau),\ t_o \leq \tau \leq t\}$. The filter gain K(t) is given by the expression

$$K(t) = V_x(t)H^T(t)R^{-1}(t) \tag{2-17}$$

where the conditional covariance matrix $V_x(t)$ is defined as

$$V_x(t) = VAR\{x(t)/Z_t\} \tag{2-18}$$

with $V_x(t_o) = V_x(o)$, and is obtained from the matrix Riccati equation

$$\dot{V}_x(t) = F(t)V_x(t) + V_x(t)F^T(t) - V_x(t)H^T(t)R^{-1}(t)H(t)V_x(t) + G(t)Q(t)G^T(t) \tag{2-19}$$

for $t \geqslant t_o$.

The continuous-discrete type of Kalman filter[22] is a combination of the above mentioned types which plays a role of central importance in this thesis. The linear continuous system with discrete observation is described by

$$\dot{x}(t) = F(t)x(t) + G(t)w(t) \tag{2-20}$$

and

$$z(k) = H(t_k)x(t_k) + v(k) \tag{2-21}$$

In between the observations, the estimator is a continuous filter, a <u>predictor</u>, without new data available to correct the estimate. The estimate is governed by the relation

$$\dot{\hat{x}}(t) = F(t)\hat{x}(t) \tag{2-22}$$

and the conditional covariance matrix is given by the solution to the matrix differential equation

$$\dot{V}_x(t) = F(t)V_x(t) + V_x(t)F(t) + G(t)Q(t)G^T(t) \tag{2-23}$$

for $t_k \leqslant t \leqslant t_{k+1}$ and $k = 0,1,2,\ldots$, where $\hat{x}(t_o) = \bar{x}(t_o)$ and $V_x(t_o) = V_x(0)$. At an observation instant, the estimator is a discrete filter, a corrector. The estimate is updated by the additional information obtained and is given by the difference equation

$$\hat{x}(t_k/Z_k) = \hat{x}(t_k/Z_{k-1}) + K(t_k)[z(k) - H(t_k)\hat{x}(t_k/Z_{k-1})] \tag{2-24}$$

where the Kalman gain is given as

$$K(t_k) = V_x(t_k/Z_{k-1})H^T(t_k)[H(t_k)V_x(t_k/Z_{k-1})H^T(t_k) + R(k)]^{-1} \tag{2-25}$$

and the conditional mean estimate $\hat{x}(t_k/Z_{k-1})$ and the conditional covariance matrix $V_x(t_k/Z_{k-1})$ are defined as

$$\hat{x}(t_k/Z_{k-1}) = E\{x(t_k)/Z_{k-1}\} \tag{2-26}$$

and

$$V_x(t_k/Z_{k-1}) = VAR\{x(t_k)/Z_{k-1}\} \tag{2-27}$$

which are the results obtained from the predictor equations, Eqs. 2-22 and 2-23. The covariance matrix is also updated according to the relationship

$$V_x(t_k/Z_k) = [I - K(t_k)H(t_k)]V_x(t_k/Z_{k-1}) \tag{2-28}$$

It is this last version, the continuous-discrete Kalman filter algorithm, which is used in the computational algorithms developed in the remaining chapters.

A basic assumption common to all the various approaches mentioned above is that the coefficients associated with the system and observation models as well as the statistical parameters of the noise terms are known. This assumption is not valid in many practical situations so that an entirely new class of estimation problems may be considered, i.e., problems of estimation with parameter uncertainty.

2.3 Estimation-Under-Uncertainty

There has been considerable effort in recent years directed at the problem of state estimation for systems with uncertain parameters. Although both continuous[35] and discrete problems have been considered, only discrete versions of the estimation-under-uncertainty problem are presented in this section.

The linear system of interest is given in Eqs. 2-1 and 2-2. The uncertainty may exist in the elements of the matrices, Φ , Γ , and H as well as in the statistical parameters of noise terms, w and v, and initial condition x(0). In general, the uncertainties may be classified in two categories:

(1) The nonswitching case, where it is assumed that all unknown parameters are members of a finite set with known probability distribution, and parameters are unchanging within the duration of interest.

(2) The switching case, where it is assumed that the unknown parameter may take on any particular value from their respective finite sets, and are randomly switching during the interval of interest.

In the following subsections, results of previous works, related to this study, from the categories indicated above are presented.

2.3.1 Nonswitching Case

Magill[8] made one of the first contributions in this problem area, in a paper in which he considered a scalar system subject to uncertainty. Hilborn and Lainiotis[9] extended the problem to the vector case. Middleton and Esposito[7] also worked on a closely related problem concerning the presence or absence of a signal. Dajani[10] has investigated a class of problem where a random coefficient is associated with either the plant noise vector, output vector, or observations. The work of Hilborn and Lainiotis is indicated below as typical of the approaches used in estimation-under-uncertainty problems.

The system under consideration is described by Eqs. 2-1 and 2-2, where only the matrix Φ is subject to uncertainty. It is assumed that N models are possible, each with a known prior probability of any candidate model, i.e., $P_r[\Phi = \Phi_i] = P_i$ is assumed known. It is desired to have a minimum variance estimate $\hat{x}(k)$ of the state vector $x(k)$ based on the set of data available up to stage k, $Z_k = \{z(1), z(2), \ldots, z(k)\}$. The performance index to be minimized is

$$J = E\ \{[x(k) - \hat{x}(k)]^T[x(k) - \hat{x}(k)]/Z_k\} \qquad (2\text{-}29)$$

The estimate which minimizes this performance measure is the conditional mean,

$$\hat{x}(k) = \int_{-\infty}^{\infty} x(k)p(x(k)/Z_k)dx(k) \qquad (2\text{-}30)$$

Since the uncertain system matrix Φ may have N possible values, Eq. 2-4 may be expressed as

$$\hat{x}(k) = \int_{-\infty}^{\infty} x(k) \sum_{i=1}^{N} p(x(k), \Phi_i/Z_k)dx(k) \qquad (2\text{-}31)$$

The joint density function $p(x(k), \Phi_i/Z_k)$ can be written as the product of the marginal density and the conditional density.

$$p(x(k), \Phi_i/Z_k) = p(\Phi_i/Z_k)p(x(k)/Z_k, \Phi_i)$$

By interchanging the order of summation and integration, Eq. 2-31 may be further simplified. The resulting equation is

$$\hat{x}(k) = \sum_{i=1}^{N} p(\Phi_i/Z_k)\int_{-\infty}^{\infty} x(k)p(x(k)/Z_k, \Phi_i)dx(k) \qquad (2\text{-}32)$$

The expression under integral sign is identical to the conditional mean of $\hat{x}(k)$ given $\Phi = \Phi_i$ and the data Z_k, for i = 1,2,..., N. Thus

$$\hat{x}(k) = \sum_{i=1}^{N} p(\Phi_i/Z_k)\hat{x}_i(k) \qquad (2\text{-}33)$$

where

$$\hat{x}_i(k) = E\{x(k)/Z_k, \Phi_i\}$$

The estimate is then obtained by summing all the estimates given the value of Φ_i, multiplied by corresponding conditional posterior probabilities. The posterior probabilities are obtained using Bayes' rule

$$p(\Phi_i/Z_k) = \frac{p(\Phi_i)p(Z_k/\Phi_i)}{\sum_{j=1}^{N} p(\Phi_j)p(Z_k/\Phi_j)} \qquad (2\text{-}34)$$

The estimation scheme is summarized pictorially in Fig. 1.

For a linear Gauss-Markov process with linear observation structure and a specified coefficient Φ_i, the optimal filter which gives the conditional estimate is the Kalman filter described in

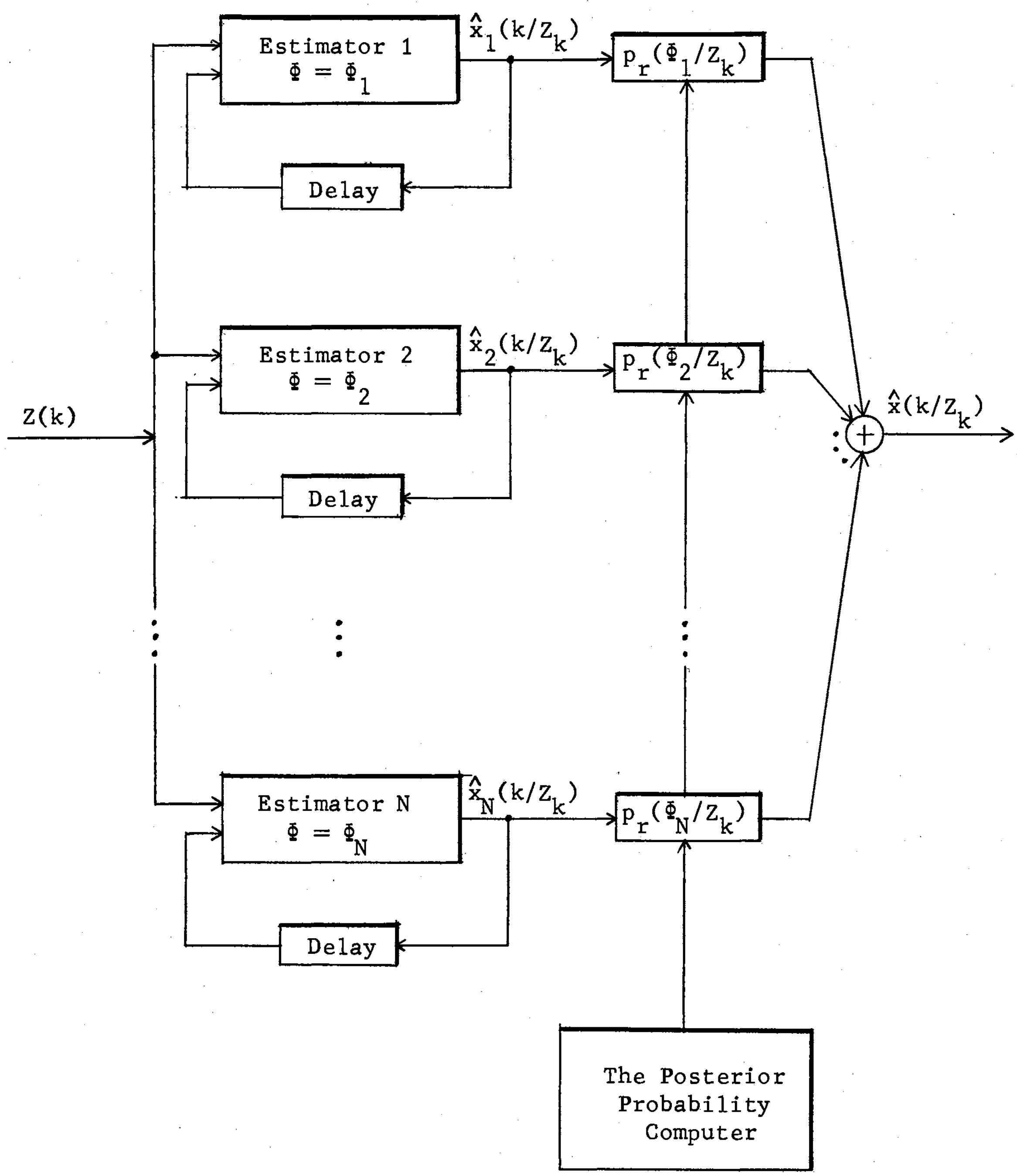


Figure 1. An Optimal Adaptive Estimation with Unknown Coefficients

Section 2.2. The estimation algorithm then includes a bank of Kalman filters, and the optimal estimate is the sum of all the filter outputs weighted by their associated posterior probability.

Hilborn and Lainiotis[9] also showed that if one of the posterior probabilities approaches 1 for k large, then the estimate is unbiased. It is also convergent in performance, in that the covariance of error approaches a steady state value. In another research result[35], where Lainiotis considered continuous time systems, the model was allowed to vary both parametrically and structurally, and the optimum estimate of the output, a linear combination of the state variables, was obtained. This type of structural uncertainty is also considered in this thesis.

2.3.2 Switching Case

Nahi[11] developed an optimal estimation algorithm for a restricted class of systems with an uncertain observation structure, having randomly switching coefficients, Ackerson and Fu[12], later obtained a suboptimal estimation technique for switching environments. They showed that optimal estimation is impossible due to evergrowing memories required with increasing observations. Parekh and Melsa[14] have extended Nahi's results to include uncertainty in the system matrix, Φ . In Dajani's[13] work, the case of random coefficients under "hypothesis" switching has also been considered. All of these works have resulted in suboptimal estimation strategies.

In the remainder of this section, the optimal estimation strategy is derived, and indications are given where difficulties arise. The problem formulation of Parekh and Melsa is used here, as it is typical of the approaches used on such problems.

The system dynamic and observation structures considered are the same as in Eqs. 2-1 and 2-2, except that matrices Φ and H are subject to uncertainty as well as randomly switching in time. It is assumed that the matrices Φ and H may be selected from finite sample spaces $\mathbf{\Phi}$ and $\mathbb{H}$, i.e.,

$$\mathbf{\Phi} = \{ \Phi_1 , \Phi_2 ,\ldots, \Phi_n \}$$

and

$$\mathbb{H} = \{H_1, H_2,\ldots, H_m \}$$

In addition, T and Δ are defined as the transition probability matrices whose ijth element represents the probability that event i will occur on the kth stage, given that event j occurred on the stage (k-1). At any stage k, the probability $P_r(\Phi = \Phi_i)$ and $P_r(H = H_i)$ are described by

$$P_r(\Phi(k) = \Phi_i) = \sum_{j=1}^{n} T_{ij} P_r(\Phi(k-1) = \Phi_j) \tag{2-35}$$

$$P_r(H(k) = H_i) = \sum_{j=1}^{m} \Delta_{ij} \; P_r(H(k-1) = H_j) \tag{2-36}$$

Given a sequence of data $Z_k = \{z(1), z(2),\ldots, z(k)\}$ and assuming that the initial condition on x(0) is $N[x(0), V_x(0)]$ and the prior probabilities $p_r(\Phi(0) = \Phi_i) = p_i$ and $p_r(H(0) = H_i) = q_i$, it is desired to obtain the minimum mean square estimate of $\hat{x}(k)$ such that

$$J_k = E \{ [x(k)-\hat{x}(k)]^T[x(k)-\hat{x}(k)]/Z_k \}$$

is minimized.

The solution to the problem is the conditional mean estimate of the form

$$\hat{x}(k) = \int_{-\infty}^{\infty} x(k) \sum_{\Omega_k} \sum_{\Lambda_k} p(\omega(k-1),\ \lambda(k)/Z_k) p(x(k)/\omega(k-1),\ \lambda(k), Z_k) dx(k) \tag{2-37}$$

where $\omega(k-1)$ and $\lambda(k)$ represent the specific sequence of events which have occurred and are defined as

$$\omega\,(k-1) = \{\Phi(0),\ \Phi(1), \ldots,\ \Phi(k-1)\}$$

and

$$\lambda\,(k) = \{H(1),\ H(2), \ldots,\ H(k)\}$$

and Ω_k and Λ_k are the sample spaces of the sequences $\omega(k-1)$ and $\lambda\,(k)$, and constitute N^k and M^k, respectively mutually exclusive and exhaustive sequences. The joint posterior probability $p[\omega(k-1), \lambda\,(k)/Z_k]$ can be expressed as

$$p[\omega(k-1), \lambda\,(k)/Z_k] = p[\omega(k-1)/Z_k] p[\lambda\,(k)/\omega(k-1),\ Z_k] \tag{2-38}$$

whereas each term in the right hand side of the equation can be expressed in terms of new measurement by using Bayes' rule,

$$p[\lambda(k)/\omega(k-1), Z_k] = \frac{p[z(k)/\lambda(k), \omega(k-1), Z_{k-1}] p[\lambda(k)/\omega(k-1), Z_{k-1}]}{\sum_{\Lambda_k} p[z(k)/\lambda(k), \omega(k-1), Z_{k-1}] p[\lambda(k)/\omega(k-1), Z_{k-1}]} \tag{2-39}$$

and

$$p[\ (k-1)/Z_k] = \frac{\sum_{\Lambda_k} p[z(k)/\lambda(k), \omega(k-1), Z_{k-1}] p[\lambda(k)/\omega(k-1), Z_{k-1}]}{\sum_{\Omega_k} \sum_{\Lambda_k} p[z(k)/\lambda(k), \omega(k-1), Z_{k-1}] p[\lambda(k)/\omega(k-1), Z_{k-1}]}$$

$$\frac{\cdot\ p[\ \omega(k-1)/Z_{k-1}]}{\cdot\ p[\ \omega(k-1)/Z_{k-1}]} \tag{2-40}$$

Since it is assumed that the sequence $\lambda(k)$ is independent of the sequence $\omega(k-1)$, the posterior probability can be written as

$$p[\lambda(k)/\omega(k-1), Z_{k-1}] = p[\lambda(k)/Z_{k-1}] \tag{2-41}$$

The probabilities $p[\lambda(k)/Z_{k-1}]$ and $p[\omega(k-1)/Z_{k-1}]$ can be obtained from the probability transition matrices, i.e.,

$$p[\omega(k-1) = \Phi_i/Z_{k-1}] = T_{ij} p[\omega(k-2) = \Phi_j/Z_{k-1}] \tag{2-42}$$

and

$$p[\lambda(k) = H_i/Z_{k-1}] = \Delta_{ij} p[\lambda(k-1) = H_j/Z_{k-1}] \tag{2-43}$$

Equation 2-37 may be rewritten in the form

$$x(k) = \sum_{\Omega_k} \sum_{\Lambda_k} p[\omega(k-1), \lambda(k)/Z_k] \hat{x}[k/\omega(k-1), \lambda(k), Z_k] \tag{2-44}$$

where $\hat{x}(k/\omega(k-1), \lambda(k), Z_k)$ represents the Kalman filter estimate under the assumption that the sequences $\omega(k-1)$ and $\lambda(k)$ took place. It is seen that the space Ω_k and Λ_k increase rapidly with k increasing. Figure 2 shows the tree type representations of the sample spaces Ω_k, Λ_k as well as the combined space $\Omega_k x \Lambda_k$. This leaves the optimal solution impossible in a practical sense, and the need for a suboptimal solution is evident.

Parekh and Melsa have investigated several suboptimal strategies. Their single stage strategy is also used by others [12], and is used also in this investigation. Basically the method consists of truncating the growing memory requirements of the filter one stage back.

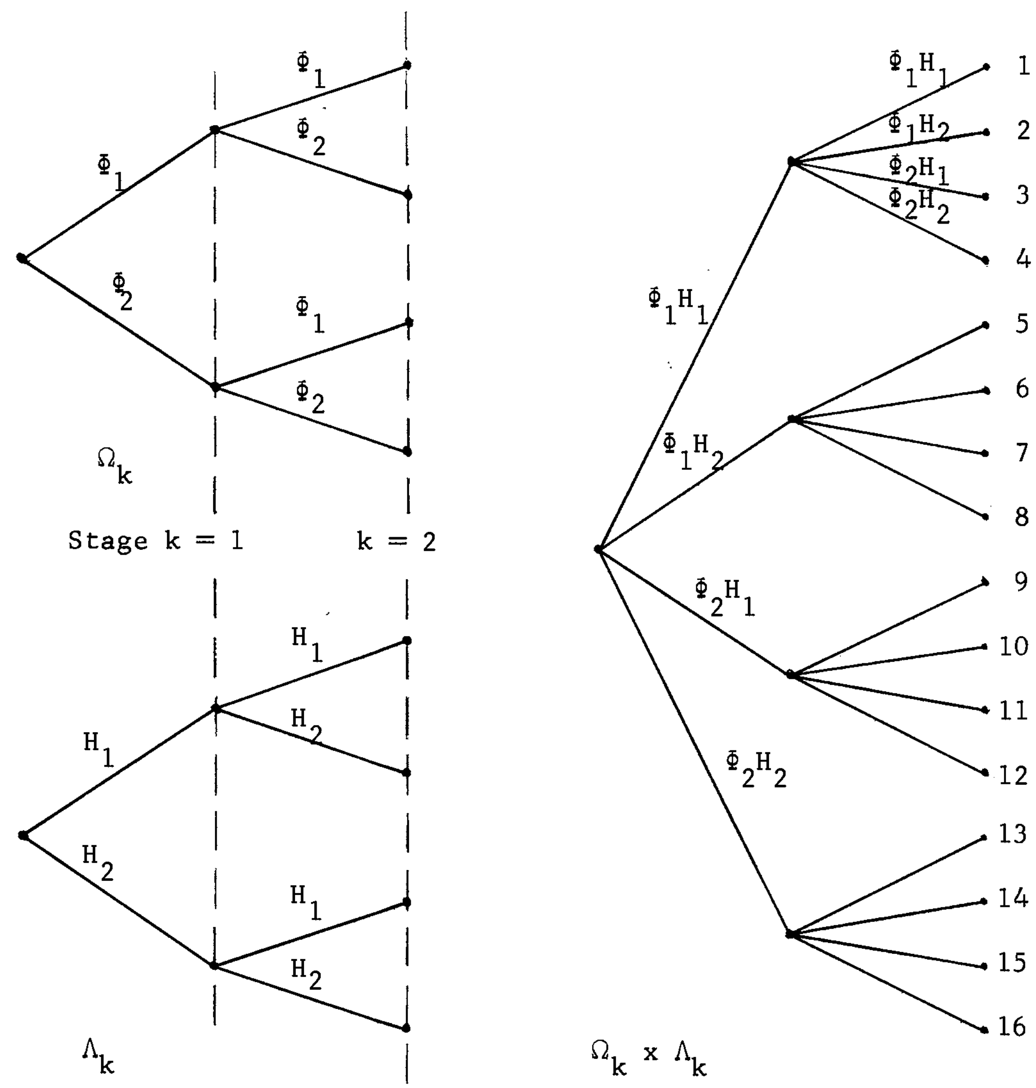


Figure 2. Sample Spaces Ω_k, Λ_k and $\Omega_k \times \Lambda_k$ for N=M=2

2.4 Stochastic Linear Regulator Problems

The problem of stochastic optimal control has received considerable attention. It has been shown that the general solution to the problem is extremely difficult to obtain[15,16,17]. There are cases, however, where relatively simple solutions are possible. Joseph and Tou[18] have shown that for linear discrete stochastic systems with quadratic cost functions and independent normally distributed observation noise, plant disturbance, and initial states, the optimal stochastic controller is separated into two parts: a Kalman filter to sequentially calculate the conditional mean estimate of the state, and a deterministic optimal linear controller. Gunkel and Franklin[20] described more general conditions under which the so-called separation theorem is valid. Recently Curry[19] has extended the result of the separation theorem to the problem of nonlinear measurements. Sims and Melsa[21] have shown that the optimal solution given by the separation theorem is not a unique result. The stochastic linear regulator problem for the discrete-time case and the continuous-time case[27] are summarized below.

For the discrete-time case, the system of interest is the similar to Eq. 2-1, except that a control vector is added, i.e.,

$$x(k+1) = \Phi(k+1,k)x(k) + \Gamma(k+1,k)w(k) + \psi(k+1,k)u(k) \qquad (2\text{-}45)$$

The observation is given as

$$Z(k+1) = H(k+1)x(k+1) + v(k+1) \qquad (2\text{-}46)$$

The problem is to find a control sequence $\{u(k),\ k = 0,1,\ldots,\ N-1\}$ over a fixed interval of time $[0,N]$ such that a given quadratic performance measure

$$J = E\{ \sum_{i=1}^{N} [x^T(i)A(i)x(i) + u^T(i-1)B(i-1)U(i-1)]\} \quad (2\text{-}47)$$

is minimized.

The optimal control strategy for the problem consists of the optimal linear filter cascaded with the optimal feedback gain matrix of the deterministic linear regulator. That is

$$u(k) = L(k)\hat{x}(k/k) \quad (2\text{-}48)$$

where

$$L(k) = -[\psi^T(k+1,k)S(k+1)\psi(k+1,k) + B(k)]^{-1} \cdot \psi^T(k+1,k)S(k+1)\Phi(k+1,k) \quad (2\text{-}49)$$

and

$$S(k) = \Phi^T(k+1,k)S(k+1)\Phi(k+1,k) + \Phi^T(k+1,k)S(k+1)\psi(k+1,k)L(k) + A(k) \quad (2\text{-}50)$$

for $k = N-1, N-2, \ldots, 0$, and $S(N) = A(N)$. The evaluation of $\hat{x}(k/k)$ is as indicated in Eqs. 2-6 through 2-11, except that in Eq. 2-6 the control term is added, i.e.,

$$\hat{x}(k+1/k+1) = \Phi(k+1,k)\hat{x}(k/k) + \psi(k+1,k)u(k) + K(k+1) \cdot \{z(k+1)-H(k+1)[\Phi(k+1,k)\hat{x}(k/k) + \Gamma(k+1,k)u(k)]\} \quad (2\text{-}51)$$

For the continuous-time case, the system of interest is described by the differential equation

$$\dot{x}(t) = F(t)x(t) + G(t)w(t) + C(t)u(t) \quad (2\text{-}52)$$

the performance measure to be minimized is

$$J = E\ \{x^T(t_f)A(t_f)x(t_f) + \int_{t_o}^{t_f} [x^T(t)A(t)x(t) + u^T(t)B(t)u(t)]dt\} \tag{2-53}$$

The optimal control law for the continuous stochastic linear regulator problem is characterized by the set of relations

$$u(t) = L(t)\hat{x}(t) \tag{2-54}$$

$$L(t) = -B^{-1}(t)C^T(t)S(t) \tag{2-55}$$

$$\dot{S}(t) = -F^T(t)S(t) - S(t)F(t) + S(t)C(t)B^{-1}(t)S(t) - A(t) \tag{2-56}$$

and the optimal linear filter is as presented in Eq. 2-15 through Eq. 2-19, except that the estimate is

$$\dot{\hat{x}}(t) = F(t)\hat{x}(t) + C(t)u(t) + K(t)[z(t) - H(t)\hat{x}(t)] \tag{2-57}$$

The solution to the control problem for the case of continuous-discrete model is derived in detail in Chapter IV. This control strategy is a new and significant step in obtaining the control algorithm for the class of uncertain models considered in this research.

The problems of stochastic control under uncertainty has not yet received great attention, except in the area of adaptive stochastic control with unknown gain. This topic has been investigated by several researchers[15,16,23,24]. Dajani[13] has obtained a controller for a class of problems subject to uncertainty, where the separation theorem is applicable. The result reported by Dajani is presented below, since it is closely related to the findings presented in Chapter IV of this study.

2.4.1 Stochastic Control for Linear Discrete System with Uncertain Constants.

In Dajani's work, the class of uncertain stochastic systems is described by

$$x(k+1) = \Phi(k+1),k)x(k) + \psi(k+1,k)u(k) + \gamma\Gamma(k+1,k)w(k) \qquad (2\text{-}58)$$

$$y(k+1) = \eta\, x(k+1) \qquad (2\text{-}59)$$

$$z(k+1) = \xi\, H(k+1) + v(k+1) \qquad (2\text{-}60)$$

where w(k) and v(k) are independent gaussian white noise terms with known statistics. Constants γ, η, and ξ are multivalued statistical parameters representing system uncertainty. Three types of problems are investigated

Problem I: $\eta = \xi = 1$

$\gamma = \gamma_i$ for $i = 1,2,\ldots, M$ with $P_r(\gamma = \gamma_i) = p_i$

Problem II: $\gamma = \xi = 1$

$\eta = \eta_i$ for $i = 1,2,\ldots, M$ with $P_r(\eta = \eta_i) = p_i$

Problem III: $\gamma = \eta = 1$

$\xi = \xi_i$ for $i = 1,2,\ldots, M$ with $P_r(\xi = \xi_i) = p_i$

The assumed prior statistics are

$$E\{x(0)\} = \bar{x}(0) \qquad VAR\{x(0)\} = V_x(0)$$

$$E\{w(k)\} = \bar{w}(k) \qquad VAR\{w(k)\} = Q(k)$$

$$E\{v(k)\} = 0 \qquad VAR\{v(k)\} = R(k)$$

$$COV\{w(k),v(k)\} = COV\{x(k),w(k)\} = COV\{x(k),v(k)\} = 0$$

Problem III is discussed below. It is desired to determine an optimal feedback control law that will minimize the cost function

$$J = E \left\{ \sum_{k=1}^{N} \left[x^T(k)A(k)x(k) + u^T(k-1)B(k-1)u(k-1) \right] \right\} \tag{2-61}$$

At any stage k, the solution to the problem based on the given data $Z_k = \{z(1), z(2), \ldots, z(k)\}$ is summarized as follows. The control strategy is

$$u(k) = \sum_{i=1}^{M} P_r(\xi_i/Z_k)u_i(k) \tag{2-62}$$

where

$$u_i(k) = L(k)\hat{x}_i(k) \tag{2-63}$$

$$L(k) = -[\psi^T(k+1,k)S(k+1)\psi(k+1,k) + B(k)]^{-1} \cdot \psi^T(k+1,k)S(k+1)\Phi(k+1,k) \tag{2-64}$$

$$S(k) = \Phi^T(k+1,k)S(k+1)\Phi(k+1,k) + \Phi^T(k+1,k)S(k+1)\psi(k+1,k)L(k) + A(k) \tag{2-65}$$

for $S(N) = A(N)$. The conditional mean estimate $x_i(k)$ is

$$\hat{x}_i(k) = E\{x(k)/\xi_i, Z_k\}$$

which is the estimate of a discrete Kalman filter given that ξ_i is known.

It is noted that the result presented here are not optimal when the parameters are subject to switching during the control interval.

2.5 Summary

The formulations of the problem of estimation-under-uncertainty as presented in this chapter are restricted in a certain sense. There are situations arising in practice where the mathematical models describing the system are more complicated, especially since the uncertainties might exist in all the parameters of the plant dynamics and the observation model, or even the order of the system. Only Lainiotis has considered this degree of generality as was the case in the treatment of continuous time problems presented in [36]. General solutions, as indicated in this thesis, are obtained by extending or combining various features of the methods which have been discussed.

The stochastic linear regulator problem which has been considered has importance when a feedback controller is needed[36]. A general solution to the stochastic control problem under uncertainty may be obtained, using the results presented here as a basis.

The main theme of this investigation is the derivation of general estimation-under-uncertainty algorithms for both nonswitching and switching cases as well as feedback controllers for the same class of systems. The material presented in this chapter should serve as an appropriate background for the remainder of this dissertation.

CHAPTER III

OPTIMAL AND SUBOPTIMAL SOLUTIONS OF ESTIMATION-UNDER-UNCERTAINTY

3.1 Introduction

Problems of estimation-under-uncertainty are considered in this chapter. Nonswitching and switching cases are treated separately. The general statement of the problems are given in Section 3.2. The optimal and suboptimal solutions to the problems are presented in Sections 3.3 and 3.4 for nonswitching and switching cases, respectively. Illustrative examples are included at the end of these sections.

3.2 Problem Statement

The purpose of this section is to define in mathematical terms the estimation-under-uncertainty problem for a stochastic linear continuous dynamical system with discrete observations. It is assumed that a set of candidate systems are modeled where each candidate represents one possible dynamical structure for the system. The actual system which is active in the time interval of interest is also one of candidates. If θ_i is used to index the ith candidate model, the structures of the models are described by the linear stochastic differential equations

$$\theta_i: \quad \dot{x}_i(t) = F_i(t)x_i(t) + C_i(t)u(t) + G_i(t)w_i(t) \qquad (3\text{-}1)$$

for $t \geq t_o$, and $i = 1,2,\ldots,N$, N is finite. Where $x_i(t)$ is an n_i vector which represents the state of the ith system, u(t) is an r-dimensioned control vector, and $w_i(t)$ is a disturbance vector whose q_i elements are zero mean white noise. The output of the ith model is a linear transformation of the state,

$$y_i(t) = H_i(t)x_i(t) \qquad (3\text{-}2)$$

where $y_i(t)$ is an m vector. The matrices $F_i(t)$, $C_i(t)$, $G_i(t)$, and $H_i(t)$ are, in general, function of time, and are subject to uncertainty. The order of the vectors u(t) and $y_i(t)$ is unsubscripted, since the dimension of the input and output of a system is usually fixed. The observations are taken at discrete instants of time

$$z(k) = y_i(t_k) + v_i(k) \qquad (3\text{-}3)$$

for $k = 1,2,\ldots,$ and depend on which model is active during the time interval of interest. The measurement noise term $v_i(k)$ is an m vector of zero mean discrete Gaussian processes. It is assumed that the plant disturbance and the measurement noise are independent. The covariance matrices of the noise terms are

$$E\{w_i(t)w_i^T(\tau)\} = Q_i(t)\delta(t-\tau)$$

and

$$E\{v_i(k)v_i^T(j)\} = R_i(k)\delta_{kj}$$

for $i = 1,2,\ldots, N$. The statistics concerning initial conditions of candidate model are assumed known,

$$E\{x_i(t_o)\} = \overline{x}_i(t_o)$$

and

$$\mathrm{VAR}\ \{x_i(t_o)\} = E\ \{[x_i(t_o) - \overline{x}_i(t_o)][x_i(t_o) - \overline{x}_i(t_o)]^T\} = V_{x_i}(t_o)$$

Here, again, the matrices $Q_i(t)$, $R_i(k)$ and $V_{x_i}(t_o)$ as well as initial vector $\overline{x}_i(t_o)$ may reflect the uncertainty of the system.

Given a set of prior probability $p_r(\theta_i)$, the problem is to find the best estimate of the output of the true system (in the sense of mean square errors) based on the observations available. Two kinds of situations are considered:

(1) The nonswitching case, where a set of dynamical parameters which describes the system are fixed but unknown.

(2) The switching case, where the operation of the system may switch from one model to the another during the time interval of interest.

The generalized problem formulation presented here may reduce to the cases discussed in the previous chapter with the exception of the continuous-discrete model formulation. In the nonswitching case,

(1) if the uncertainties are in the matrix G_i or H_i, Dajani's problem is obtained[10],

(2) if only the C_i's are uncertain, it is the problem of unknown gain[27,29].

In the switching case,

(1) if the uncertainties are in the matrices F_i and H_i, it is Parekh and Melsa's problem[14],

(2) if the statistics of w_i is uncertain, Ackerson and Fu's problem [12] is obtained.

The algorithms for the solution of the problems considered are presented in the following sections.

3.3 <u>Estimation-Under-Uncertainty</u>--Nonswitching Case

It is desired to estimate the output y(t) from the observations available over the time interval (t_o, t). The measurement record up to time t is denoted by $Z_t = \{z(j);\ t_j \leqslant t,\ j = 1,2,\ldots\}$. The conditional mean estimate is sought, since it is optimum with respect to a quadratic error criterion, and other performance measures under certain restrictions[17]. The best estimate of y(t) is then determined by the conditional probability density function $p(y(t)/Z_t)$. In view of Eq. 3-2 if $p(y_i(t)/Z_t)$ denotes the conditional probability density function given by the model θ_i, and $p_r(\theta_i/Z_t)$, the posterior probability of the model active at time t, the conditional density $p(y(t)/Z_t)$ can be described by [33]

$$P(y(t)/Z_t) = \sum_{i=1}^{N} p_r(\theta_i/Z_t)p(y_i(t)/Z_t) \tag{3-4}$$

Since $y_i(t)$ is related to $x_i(t)$ by Eq. 3-2, the density function $p(y_i(t)/Z_t)$ can be obtained from the conditional density function $p(x_i(t)/Z_t)$ provided that $H_i(t)$ is known.

The conditional mean estimate

$$\hat{y}(t) \triangleq E\{y(t)/Z_t\} \tag{3-5}$$

can be obtained using the fundamental theorem of expectation as

$$\hat{y}(t) = \int_{-\infty}^{\infty} y(t)p(y(t)/Z_t)dy(t) \tag{3-6}$$

from the relationships in Eq. 3-2, which is linear between $y_i(t)$ and $x_i(t)$, and Eq. 3-4, the conditional mean estimate of Eq. 3-6 may be rewritten in the form

$$\hat{y}(t) = \sum_{i=1}^{N} p_r(\theta_i/z_t)\int_{-\infty}^{\infty} H_i(t)x_i(t)p(x_i(t)/Z_t)dx_i(t)$$

$$= \sum_{i=1}^{N} p_r(\theta_i/z_t)H_i(t)\hat{x}_i(t/Z_t) \tag{3-7}$$

where $\hat{x}_i(t/Z_t)$ is conditional mean estimate of the state given model θ_i,

$$\hat{x}_i(t/Z_t) = E\{x_i(t)/Z_t\}$$

$$= \int_{-\infty}^{\infty} x_i(t)p(x_i(t)/Z_t)dx_i(t) \tag{3-8}$$

The problem of finding the conditional mean estimate $\hat{y}(t)$ reduces to the problem of finding the conditional mean estimate of each candidate, $\hat{x}_i(t/Z_t)$, and the posterior probability $p(\theta_i/Z_t)$. In the remainder of this section the derivation of algorithms are presented. Convergence properties and deficiencies of the identification algorithm are discussed. Examples are given to illustrate the results obtained.

3.3.1 Estimation Algorithm

Since throughout this study, Gauss-Markov stochastic processes are considered, the random vector $x_i(t)$ is normally distributed. The conditional mean and covariance completely describe the density function of the state vector. The solution to the estimation problem is obtained in a recursive manner consisting of two basic parts:

(1) In between observations, the estimator is a predictor.

(2) At the observation instants, the measurements provide the information to the corrector portion which updates the estimate of the predictor.

The predictor:

At any time t in an observation interval $t_i \leqslant t < t_{k+1}$, the available measurements are $Z_k = \{z(1), z(2), \ldots, z(k)\}$. The conditional mean and covariance of the state of each candidate model are described by[22]

$$\dot{\hat{x}}_i(t/Z_k) = F_i(t)\hat{x}_i(t/Z_k) + C_i(t)u(t) \tag{3-9}$$

and

$$\dot{V}_{x_i}(t/Z_k) = F_i(t)V_{x_i}(t/Z_k) + V_{x_i}(t/Z_k)F_i^T(t) + G_i(t)Q_i(t)G_i^T(t) \tag{3-10}$$

for i = 1,2,...,N. Where $\hat{x}_i(t/Z_k)$ is defined in Eq. 3-8, and $V_{x_i}(t/Z_k)$ is defined as

$$V_x(t/Z_k) = VAR\{x_i(t)/Z_k\}$$

$$= E\{[x_i(t) - \hat{x}_i(t/Z_k)][x_i(t) - \hat{x}_i(t/Z_k)]^T\} \tag{3-11}$$

The initial conditions at time t_k are $\hat{x}_i(t_k/Z_k)$ and $V_{x_i}(t_k/Z_k)$, while at k = 0, they are $\bar{x}_i(t_o)$. The conditional density function $p(x_i(t)/Z_k)$ is gaussian with mean vector $\hat{x}_i(t/Z_k)$ and covariance matrix $V_{x_i}(t/Z_k)$. Thus, it is described by the expression

$$p(x_i(t)/Z_k) \sim N[\hat{x}_i(t/Z_k), V_{x_i}(t/Z_k)]$$

$$= [(2\ )^{\frac{-n_i}{2}} \mid V_{x_i} \mid^{-\frac{1}{2}}]EXP\{-\tfrac{1}{2}[(x_i - \hat{x}_i)^T V_{x_i}^{-1}(x_i - \hat{x}_i)]\} \tag{3-12}$$

where $\hat{x}_i$ and V_{x_i} are short for the conditional mean and the covariance matrix.

From Eq. 3-7, the best estimate of the output during the time interval $t_k \leqslant t < t_{k+1}$ is given by

$$\hat{y}(t/Z_k) = \sum_{i=1}^{N} p_r(\theta_i/Z_k)H_i(t)\hat{x}_i(t/Z_k) \tag{3-13}$$

The Corrector:

At time instant t_{k+1}, a measurement $z(k+1)$ is obtained. One must update the density function $p(x_i(t_{k+1})/Z_k)$ utilizing this extra information. That is, the relationships between the conditional means and covariance matrices, which describe the conditional density functions $p(x(t_{k+1})/Z_k)$ and $p(x(t_{k+1})/Z_{k+1})$, and the new measurement is sought.

Since the conditional density function $p(x_i(t_{k+1})/Z_{k+1})$ can be written as the density function $p(x_i(t_{k+1}), Z_k)$, using Bayes' rule, the relationship between $p(x_i(t_{k+1})/Z_k)$ and $p(x_i(t_{k+1})/Z_{k+1})$ can be obtained as

$$p(x_i(t_{k+1})/Z_{k+1}) = p(x_i(t_{k+1})/z(k+1), Z_k)$$

$$= \frac{p(z(k+1)/x_i(t_{k+1}), Z_k)p(x_i(t_{k+1})/Z_k)}{p(z(k+1)/Z_k, \theta_i)} \tag{3-14}$$

where

$$p(z(k+1)/Z_k, \theta_i) = \int_{-\infty}^{\infty} p(z(k+1)/x_i(t_{k+1}), Z_k)p(x_i(t_{k+1})/Z_k)dx_i(t_{k+1}) \tag{3-15}$$

Furthermore, since the measurement noise is assumed white gaussian with zero mean in Eq. 3-3, it follows that

$$p(z(k+1)/x_i(t_{k+1}), Z_k) = p(z(k+1)/x_i(t_{k+1})) \tag{3-16}$$

which is gaussian from Eq. 3-2 and 3-3, i.e.,

$$p(z(k+1)/x_i(t_{k+1})) \sim N[H_i(t_{k+1})x_i(t_{k+1}),\ R_i(k+1)] \tag{3-17}$$

Thus, Eq. 3-14 can be rewritten as

$$p(x_i(t_{k+1})/Z_{k+1}) = \frac{p(z(k+1)/x_i(t_{k+1}))p(x_i(t_{k+1})/Z_k)}{p(z(k+1)/Z_k,\theta_i)} \tag{3-18}$$

By substituting from Eqs. 3-17 and 3-12 in the numerator of Eq. 3-18, collecting terms involving $x_i(t_{k+1})$, and completing the quadratic forms, one may obtain an expression for $p(x_i(t_{k+1})/Z_{k+1})$ in a form similar to Eq. 3-12 [see Appendix A]. The normal density has updated conditional mean and covariance matrix indicated by the relations

$$\hat{x}(t_{k+1}) = \hat{x}_i(t_{k+1}/Z_k) + K_i(t_{k+1})[z(k+1) - H_i\hat{x}_i(t_{k+1}/Z_k)] \tag{3-19}$$

and

$$V_{x_i}(t_{k+1}/z_{k+1}) = V_{x_i}(t_{k+1}/Z_k) - K_i(t_{k+1})H_iV_{x_i}(t_{k+1}/Z_k)$$

$$= [I-K_iH_i]V_{x_i}(t_{k+1}/Z_k)[I-k_iH_i]^T + K_iR_iK_i^T \tag{3-20}$$

where

$$K_i(t_{k+1}) = V_{x_i}(t_{k+1}/Z_k)H_i^T[H_iV_{x_i}(t_{k+1}/Z_k)H_i^T + R_i(k+1)]^{-1} \tag{3-21}$$

which is referred to as the Kalman gain[27]. These equations 3-19 through 3-21 are corrector equations for every model. It is noted that the last form of Eq. 3-20 has a computational advantage which will assure one that the covariance matrix V_{x_i} will never take on a negative value due to inherent computer errors.

3.3.2 Identification Algorithm

The posterior probability $p_r(\theta_i/Z_k)$ in Eq. 3-13 provides a measure of the closeness of the candidate model to the actual system. Whenever a measurement is obtained, this information may be used to update the probability. At time t_{k+1}, the posterior probability is

$$p_r(\theta_i/Z_{k+1}) = p_r(\theta_i/z(k+1),\ Z_k)$$

Using Bayes' rule, it can be expressed as

$$p_r(\theta_i/z_{k+1}) = \frac{p(z(k+1)/Z_k,\ \theta_i)p_r(\theta_i/Z_k)}{\sum_{j=1}^{N} p(z(k+1)/Z_k,\ \theta_j)p_r(\theta_j/Z_k)}$$

$$= [1 + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{p(z(k+1)/Z_k,\ \theta_j)}{p(z(k+1)/Z_k,\ \theta_i)} \frac{p_r(\theta_j/Z_k)}{p_r(\theta_i/Z_k)}]^{-1} \tag{3-22}$$

Defining the likelihood ratio as

$$L_{ji} = \frac{p(z(k+1)/Z_k,\ \theta_j)}{p(z(k+1)/Z_k,\ \theta_i)}$$

which is the ratio of two conditional functions and is also a measure of likelihood of two distributions upon a measurement value, Eq. 3-22 may be simplified,

$$p_r(\theta_i/Z_{k+1}) = [1 + \sum_{\substack{j=1 \\ j\neq i}}^{N} L_{ji} \frac{p_r(\theta_j/Z_k)}{p_r(\theta_i/Z_k)}]^{-1} \tag{3-23}$$

The conditional density functions $p(z(k+1)/Z_k,\ \theta_i)$ are given in Eq. 3-15, which may be expressed as

$$p(z(k+1)/Z_k,\ \theta_i) \sim N[H_i(t_{k+1})\hat{x}_i(t_{k+1}/Z_k),\ H_i V_{x_i}(t_{k+1}/Z_k)H_i^T + R_i(k+1)] \tag{3-24}$$

[see Appendix A]. The likelihood ratio can then be evaluated by

$$L_{ji} = \left| \frac{H_i V_{x_i} H_i^T + R_i}{H_j V_{x_j} H_j^T + R_j} \right|^{\frac{1}{2}} \mathrm{EXP}\{-\tfrac{1}{2}[\cdot]\} \tag{3-25}$$

where

$$[\cdot] = [z(k+1) - H_j\hat{x}_j(t_{k+1}/Z_k)]^T[H_j V_{x_j} H_j^T + R_j]^{-1}[z(k+1) - H_j\hat{x}_j(t_{k+1}/Z_k)]$$

$$-[z(k+1) - H_i\hat{x}_i(t_{k+1}/Z_k)]^T[H_i V_{x_i} H_i^T + R_i]^{-1}[z(k+1) - H_i\hat{x}_i(t_{k+1}/Z_k)] \tag{3-26}$$

It is seen that Eq. 3-23, may be computed recursively and the memory requirements are finite. Most of the elements in Eq. 3-26 are already computed in Eqs. 3-19 and 3-21, and may be stored for use in evaluating likelihood ratios.

3.3.3 A Summary of the Results

Combining the results of sections 3.3.1 and 3.3.2, the algorithm for estimation-under-uncertainty for the nonswitching case is summarized as indicated below.

ALGORITHM 3-1

The solution to the nonswitching case of the estimation problem stated in Section 3.2 is presented in recursive form. Between observations the estimation is merely prediction, and at an observation time instant, the estimate as well as the posterior probability of each candidate model is updated.

At any time t, the best estimate of the output is

$$\hat{y}(t/Z_k) = \sum_{i=1}^{N} p_r(\theta_i/Z_k)H_i(t)\hat{x}_i(t/Z_k) \tag{3-13}$$

for $t_k \leqslant t < t_{k+1}$, $k = 1,2,\ldots N$, where N is finite and the estimate of each model is given as follow

(1) In between observations, $\hat{x}_i$ is described by

$$\dot{\hat{x}}_i(t/Z_k) = F_i(t)\hat{x}_i(t/Z_k) + C_i(t)u(t) \tag{3-9}$$

and its covariance matrix by

$$\dot{V}_{x_i}(t/Z_k) = F_i(t)V_{x_i}(t/Z_k) + V_{x_i}(t/Z_k)F_i^T(t) + G_i(t)Q_i(t)G_i^T(t) \tag{3-10}$$

for $i = 1,2,\ldots,N$, where the initial conditions at time t_k are $\hat{x}_i(t_k/Z_k)$ and $V_{x_i}(t_k/Z_k)$ and, for $k = 0$, $\bar{x}_i(t_o)$ and $V_{x_i}(t_o)$.

(2) At an observation instant t_{k+1}, $\hat{x}_i$ and V_{x_i} are updated by

$$\hat{x}_i(t_{k+1}/Z_{k+1}) = \hat{x}_i(t_{k+1}/Z_k) + K_i(t_{k+1})[z(k+1) - H_i(t_{k+1})\hat{x}_i(t_{k+1}/Z_k)] \tag{3-19}$$

and

$$V_{x_i}(t_{k+1}/Z_{k+1}) = V_{x_i}(t_{k+1}/Z_k) - K_i(t_{k+1})H_iV_{x_i}(t_{k+1}/Z_k) \tag{3-20}$$

where the Kalman gain $K_i(t_{k+1})$ is

$$K_i(t_{k+1}) = V_{x_i}(t_{k+1}/Z_k)H_i^T[H_i V_{x_i} H_i^T + R_i]^{-1} \qquad (3\text{-}21)$$

for i = 1,2,...,N. The posterior probability of each candidate model is updated according to the relationship

$$p_r(\theta_i/Z_{k+1}) = [1 + \sum_{\substack{j=1 \\ j\neq i}}^{N} L_{ji} \frac{p_r(\theta_j/z_k)}{p_r(\theta_i/z_k)}]^{-1} \qquad (3\text{-}24)$$

where

$$L_{ji} = \left| \frac{H_i V_{x_i} H_i^T + R_i}{H_j V_{x_j} H_j^T + R_j} \right|^{\frac{1}{2}} \text{EXP}\{-\tfrac{1}{2}[(z-H_j\hat{x}_j)^T(H_j V_{x_j} H_j^T + R_j)^{-1}(z-H_j\hat{x}_j)$$

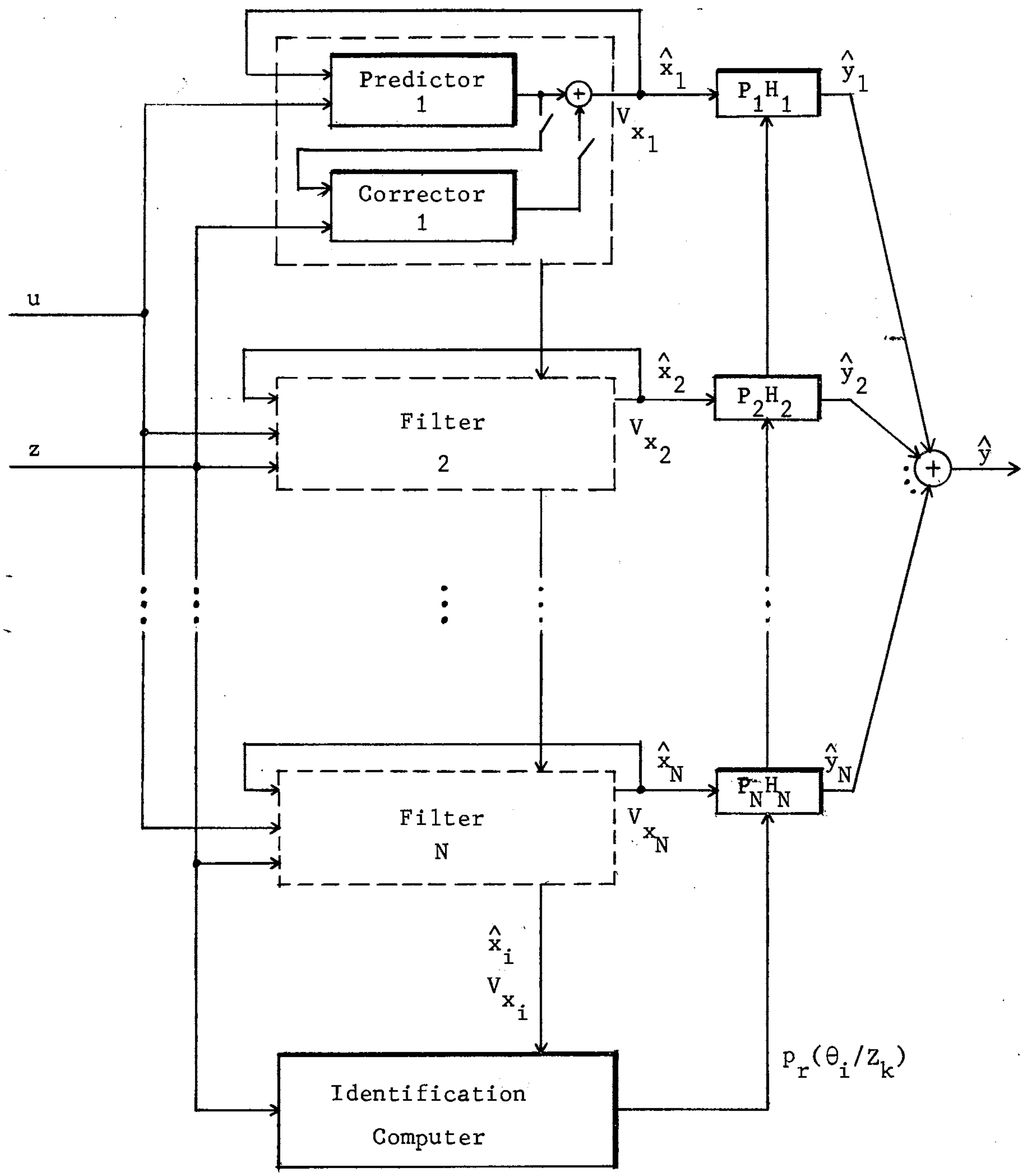
$$-(z-H_i\hat{x}_i)^T(H_i V_{x_i} H_i^T + R_i)^{-1} (z-H_i\hat{x}_i)]\} \qquad (3\text{-}25)$$

The algorithm is shown schematically in Figure 3.

3.3.4 Convergence Test

The convergence properties of the identification algorithm developed in the previous section are of importance, since one needs to know what to expect from the procedure. In this section, it is shown that a sequence of conditional probabilities $p_r(\theta_i/Z_i)$, $p_r(\theta_i/Z_2)$,..., converges to 1 or 0 according to whether or not $\theta_i = \theta_a$, when θ_a denotes the actual model.

In order to investigate this topic, rigorous mathematical theorems from the theory of random processes are needed. Some of the related definitions and theorems are presented here without proof. Especially useful is the concept of a bounded martingale and related theorems.



Predictor: Eqs. (3-9) and (3-10)

Corrector: Eqs. (3-29) and (3-30)

Identification: Eq. (3-35)

Figure 3. Block Diagram of ALGORITHM 3-1, Estimation-Under-Uncertainty-Nonswitching Case

<u>Definition</u>[40]: A sequence of random variables $\{g_k\}$ is a bounded "martingale," if

$$E\{\ |g_k|\ \} \qquad \text{and } E\{g_{k+1}/g_k\} = g_k \tag{3-26}$$

<u>Theorem 1</u>[40]: A bounded martingale $\{g_k\}$ converges to a limit g_∞ with probability 1 (WP1), i.e.,

$$\lim_{k\to\infty} g_k = g_\infty \qquad \text{WP1}$$

A random variable Y which is to be estimated from a set of data $Y_1, Y_2, \ldots$ is considered. If the sequence defined by

$$Z_n = E\{Y/Y_1, Y_2, \ldots, Y_n\} \tag{3-27}$$

is a martingale and such that $E\{\ |Y|\ \} < \infty$, it follows from Theorem 1 that

$$\lim_{n\to\infty} Z_n = Z = E\{Y/Y_1, Y_2, \ldots, Y_n, \ldots\} \tag{3-28}$$

with probability 1. If, moreover, the probability space of Y is defined on Y_n, then

$$\lim_{n\to\infty} Z_n = Y$$

The result is summarized in the following theorem:

<u>Theorem 2</u>[41]: If a sequence defined by Eq. 3-27 is such that $E\{\ |Y|\ \} < \infty$, then

$$\lim_{n\to\infty} Z_n = Z$$

When the probability space of Y is defined on Y_n, then Eq. 3-28 reduces to

$$\lim_{n\to\infty} Z_n = Y$$

Furthermore, if $Y = I_B$ where the event B is defined on Y_n, and B is a "property" of the sequence, then the conditional density function

$$\lim_{n\to\infty} p(B/Y_1,Y_2,\ldots,Y_n) = I_B \qquad \text{WP1}$$

The above is stated in the following corollary.

<u>Corollary 1</u>[41]: The sequence of conditional probability densities $p(B/Y_1,Y_2,\ldots,Y_n)$ of a property B of the sequence $Y_1,Y_2,\ldots,Y_n$, converges to 1 or 0 with probability 1 depending on whether the sequence has this property or not.

It is seen that the convergence test hinges on constructing a bounded martingale sequence. In Agrawala's[51] paper, theorems for constructing a martingale sequence are developed, and the convergence of the estimation of a parameter with a continuous probability function is proved. These results are stated in the following theorems:

<u>Theorem 3</u>[42]: Any sequence $\{g_1,g_2,\ldots,g_k\}$ such that

$$g_k = \int_\Phi f(\theta)p(\theta/z_{tk})d\theta \qquad (3\text{-}29)$$

is a bounded martingale, if

(1) $f(\theta)$ is any non-negative Lebesque measurable function,

(2) $\max f(\theta) = M < \infty$, bounded.

where p is a conditional probability density function and Φ is the space on which θ is defined.

Theorem 4[42]: If there exists a sequence of functions $\{g_m(Z_m)\}$ defined on Φ such that $\lim_{m\to\infty} g_m = \theta_o$, WP1, where θ_o is the true value of θ, then

$$\lim_{k\to\infty} p(\theta/Z_k) = \delta(\theta - \theta_o) \qquad \text{WP1} \tag{3-30}$$

The problem described in the previous section is a discrete type of problem where θ takes on discrete values θ_i, $i = 1,2,\ldots,N$, and N is finite. The theorems presented can be used to show that the sequence of conditional probabilities $p_r(\theta_i/Z_1)$, $p_r(\theta_i/Z_2),\ldots,$ converges to 1 or 0 according to whether or not $\theta_i = \theta_a$, for θ_a denoting the actual value of θ. That is, if the actual system is described by one of the candidate models selected, the probability of that candidate model will go to 1 after enough observations.

Defining I_a as a random variable indicating the event $\theta_i = \theta_a$, i.e.,

$$I_a = \begin{cases} 1 & \theta_i = \theta_a \\ 0 & \theta_i = \theta_a \end{cases} \tag{3-31}$$

It is desired to show that

$$\lim_{k\to\infty} p_r(\theta_i/Z_k) = I_a \tag{3-32}$$

for $i = 1,2,\ldots,N$, with probability 1.

Modifying the continuous formulation of a sequence in Eq. 3-29 for a discrete probability mass, one may construct a sequence $\{g_k\}$ as

$$g_k = \sum_{i=1}^{N} f(\theta_i)p_r(\theta_i/Z_k) \tag{3-33}$$

where $f(\theta_i)$ is bounded for all i. To show that g_k is a bounded martingale, conditions in Eq. 3-26 have to be satisfied. For the first condition, since $f(\theta_i) \leqslant M < \infty$ for all i, it follows that

$$g_k = \sum_i f(\theta_i) p_r(\theta_i/Z_k) \leqslant M \sum_i p_r(\theta_i/Z_k) = M < \infty$$

So,

$$E\{ |g_k| \} \leqslant M < \infty \tag{3-34}$$

and the sequence is bounded. For the second condition, one may consider

$$\begin{aligned} E\{g_{k+1}/g_k\} &= E\{\sum_i f(\theta_i) p_r(\theta_i/Z_{k+1})/Z_k\} \\ &= \sum_i f(\theta_i) E\{p_r(\theta_i/Z_{k+1})/Z_k\} \end{aligned} \tag{3-35}$$

From Eq. 3-22, it can be seen that

$$\begin{aligned} p_r(\theta_i/Z_{k+1}) &= \frac{p(z(k+1)/\theta_i,\ Z_k) p_r(\theta_i/Z_k)}{\sum_j p(z(k+1)/\theta_j,\ Z_k) p_r(\theta_j/Z_k)} \\ &= \frac{p(z(k+1)/\theta_i,\ Z_k) p_r(\theta_i/Z_k)}{p(z(k+1)/Z_k)} \end{aligned} \tag{3-36}$$

The expected value of $p(\theta_i/Z_{k+1})$ conditioned on Z_k is evaluated over all the possible values of $z(k+1)$ conditioned on Z_k.

$$E\{p_r(\theta_i/Z_{k+1})/Z_k\} = \int_{-\infty}^{\infty} \frac{p(z(k+1)/\theta_i,\ Z_k)p_r(\theta_i/Z_k)}{p(z(k+1)/Z_k)} p(z(k+1)/Z_k)dz(k+1)$$

$$= p_r(\theta_i/Z_k)\int_{-\infty}^{\infty} p(z(k+1)/\theta_i,\ Z_k)dz(k+1)$$

$$= p_r(\theta_i/Z_k) \qquad (3\text{-}37)$$

Thus,

$$E\{g_{k+1}/g_k\} = \sum_i f(\theta_i)p_r(\theta_i/Z_k) = g_k \qquad (3\text{-}38)$$

From Eqs. 3-34 through 3-38, one may conclude that the sequence given by Eq. 3-33 is a bounded martingale. It may also be seen from Theorem 1 that

$$\lim_{k\to\infty} g_k = g_\infty \qquad \text{WP1}$$

independent of Z_k. It remains to show that $p(\theta_i/Z_k)$ approaches I_a for $\theta_i = \theta_a$.

By chosing $f(\theta_i) = I_a$, it is seen that the conditions of Theorem 3 are satisfied. The constructed sequence takes the form,

$$g_k = \sum_i I_a p_r(\theta_i/Z_k) \qquad (3\text{-}39)$$

From corollary 2, the convergence follows

$$\lim_{k\to\infty} g_k = I_a \qquad \text{WP1}$$

and therefore, one may conclude from Eq. 3-29 that

$$\lim_{k\to\infty} p_r(\theta_i/Z_k) = I_a \qquad \text{WP1} \qquad (3\text{-}40)$$

This complete the proof. It is summarized in the following theorem.

Theorem 5 If there exists a sequence $\{g_k(Z_k)\}$ defined by Eq. 3-33 on a discrete probability space Φ_N such that

$$\lim_{k\to\infty} g_k = I_a \qquad \text{WP1}$$

where I_a is defined by Eq. 3-31, then

$$\lim_{k\to\infty} p_r(\theta_i/Z_k) = I_a \qquad \text{WP1}$$

For completeness, the results of Hilborn and Lainiotis[9] concerning the properties of convergence of the adaptive filters in the discrete case are stated below as theorems without proof. The theorems do not depend on the strictly discrete formulation used and hence may be extended to the continuous-discrete case considered here.

Theorem 6[9]: Suppose that, given a positive definite symmetric matrix Q, there exists a bound $M<\infty$ such that

$$E\{[\hat{x}^T(k)Q\hat{x}(k)]^r\} < \infty \tag{3-41}$$

for all k and for $r \geqslant 1$. Where the estimate is described by

$$\hat{x}(k) = \sum_i p_r(\theta_i/Z_k)\hat{x}_i(k)$$

Then if $p_r(\theta_i/Z_k) \rightarrow I_a$ WP1, and θ_a is the true model

$$\lim_{k\to\infty} E\{[\hat{x}(k) - \hat{x}_a(k)]^T Q[\hat{x}(k) - \hat{x}_a(k)]\} = 0 \tag{3-42}$$

This theorem implies the convergence of $[\hat{x}(k) - \hat{x}_a(k)]$ to zero in a quadratic mean sense. In addition, the optimal quadratic performance J_k^* of the adaptive filter converges to the optimal quadratic performance J_k^{**} of the filter with known actual model, where J_k^* and J_k^{**} are defined

$$J_k^* = E\{[x(k) - \hat{x}(k)]^T Q[x(k) - \hat{x}(k)]\}$$

and

$$J_k^{**} = E\{[x(k) - \hat{x}_a(k)]^T Q[x(k) - \hat{x}_a(k)]\}$$

The next theorem indicates this fact.

Theorem 7[9]: For the same condition given in Eq. 3-41, if $p_r(\theta_i/Z_k) \rightarrow I_a$ WP1, then

$$\lim_{k \rightarrow \infty} [J_k^* - J_k^{**}] = 0 \qquad (3\text{-}43)$$

The above guarantees convergence in "performance" so that after enough data has been obtained, the true system is known, and the estimation-under-uncertainty problem has become an estimation-under-certainty problem. This of course depends on having selected one of the candidate models as the true one, and in practice, one could rarely expect that this would be the case. The next section deals with the difficulty of having none of the candidate models describe the actual system.

3.3.5 Supplement to the Identification Problems

In the previous sections, the development of the algorithms is based on the assumption that the true model is one of the candidate models. This assumption may not be valid in practical applications where the candidate models are usually obtained by guessing based on past experiences. There are cases for which either the posterior probability may not converge or converges to the wrong solution due to the indicated difficulties. This section describes an approach which one may use when none of the candidate models is true.

At any observation time, the estimate of each candidate model is corrected according to Eq. 3-19, where the measurement z(k+1) is used for all models. The term

$$r_i(k+1/Z_k) = z(k+1) - H_i(t_{k+1})\hat{x}_i(t_{k+1}/Z_k) \qquad (3\text{-}44)$$

is defined as the <u>predicted measurement residual error</u> or the <u>predicted residual</u>. The correction to the state estimate is then proportional to the predicted residual. If the i-th model represents the true system then

$$\hat{r}_i(k+1/Z_k) = E\{r_i(k+1/Z_k)\} = 0 \qquad (3\text{-}45)$$

for all k, and the variance of r_i is expressed by

$$V_{r_i}(k+1/Z_k) = VAR\{r_i(k+1/Z_k)\}$$

$$= H_i(t_{k+1})V_{x_i}(t_{k+1}/Z_k)H_j^T(t_{k+1}) + R_i(k+1) \qquad (3\text{-}46)$$

The predicted residual provides a useful tool for judging the performance of the filter. By checking whether residuals indeed possesses their statistical properties, one is able to assess the performance of the filter.

The problem of filter divergence has been discussed in the literature[22,27,37], for cases in which the filter is constructed on the basis of an erroneous model. The problem is particularly acute when the plant noise is small and the measurement noise is large. Eventually, in this case, the filter gain becomes small, and subsequent observations have little effect on the estimate. When the filter tends to become decoupled from the system, as happens with a small gain, then the residual computed from Eq. 3-44 will be nonzero mean, if the filter is not matched to the true system. The filters described by the algorithms in previous sections which are different from the actual model should manifest the divergence property. The statistic provided by Eq. 3-44 may serve as a measurement criterion for testing whether a model is true or not. That is, one may check to see whether the residual error is zero mean, and in this way determine whether or not the candidate model of interest is true active, or whether it was only the best choice from a selection of poor candidates.

At a time instant t_k, there are k sets of observations available. The time average of the residuals is defined by

$$\bar{r}_i(k) = \frac{1}{k} \sum_{j=1}^{k} r_i(j/Z_{j-1}) \tag{3-47}$$

The time average can be recursively computed using the following relationship.

$$\bar{r}_i(k+1) = \frac{1}{k+1} \left[\sum_{j=1}^{k} r_i(j/Z_{j-1}) + r_i(k+1/Z_k) \right]$$

$$= \frac{1}{k+1} \left[k\bar{r}_i(k) + r_i(k+1/Z_k) \right] \tag{3-48}$$

The time averages presented here are not the same as the ensemble average residual defined according to Eq. 3-45, since the systems are not even assumed to be stationary. They may still be used as a measure of whether or not a candidate model is erroneous, however. In general, one should use a normalized form of the time average residual[22] rather than the residual itself, to help reduce the effects of the different levels of measurement noise.

The time average of the normalized residuals is defined as

$$\gamma_i(k) \triangleq \frac{1}{k} \sum_{j=1}^{k} \frac{r_i(j/Z_{j-1})}{|R_i(j)|^{\frac{1}{2}}} \tag{3-49}$$

and the recursive relationship is expressed by

$$\gamma_i(k+1) = \frac{1}{k+1} \left[k\gamma_i(k) + \frac{r_i(k+1/Z_k)}{|R_i(k+1)|^{\frac{1}{2}}}\right] \tag{3-50}$$

Since the residual at any stage is available in the estimation algorithm, it is easy to accomplish these computations. It should be noted that the arguments presented here depend on the noise level and the separation between models. That is, under large noise disturbances or if the distributions of the state of each candidate model are almost equal. Eq. 3-47 or Eq. 3-49 may not give satisfactory indications to judge which model do possess the statistical property. The use of time average residuals is shown in one of the examples of the following section.

3.3.6 Examples

Two examples are presented here. The first example is used to demonstrate the proposed estimation and identification algorithm.

The second example is given to examine the relationship of noise levels to identification.

(1) Example 3-1

A second order differential equation with the damping ratio subject to uncertainty is considered

$$\ddot{\eta}(t) + 2\xi\omega_n\dot{\eta}(t) + \omega_n^2\eta(t) = w(t)$$

where ξ is the damping ratio, ω_n is the natural frequency, and w is the plant noise. Rewriting the differential equation in state model form, one obtains

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \eta(t) \\ \dot{\eta}(t) \end{bmatrix}$$

The output and measurement are assumed to be

$$y(t) = [1 \quad 0]x(t)$$

and

$$z(k) = y(t_k) + v(k)$$

The covariance of plant and measurement noise are given by

$$E\{w(t)w^T(\tau)\} = Q(t)\delta(t-\tau)$$

and

$$E\{v(k)v^T(j)\} = R(k)\delta_{kj}$$

The problem is simulated on the digital computer. The gaussian white noise is generated using a standard algorithm[38]. White noise of a continuous type is implemented using a discrete white noise equivalent with $\frac{Q(t)}{\Delta t}$ as the approximated covariance [27], where Δt is the integration step size. The rectangular integration method is adopted here for simplicity. In the simulation results presented below, the observation is scheduled once every twenty integration steps.

Result 1: In this case, four values for the uncertain parameter are considered, i.e., ξ = 1.2, 1.0, 0.8, and 0.4. While the value of ω_n is fixed as 1.0. The candidate models under consideration are indicated as

$$\theta_i: \quad \dot{x}_1(t) = \begin{bmatrix} 0. & 1. \\ -1. & -2.4 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

$$\theta_2: \quad \dot{x}_2(t) = \begin{bmatrix} 0. & 1. \\ -1. & -2. \end{bmatrix} x_2(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

$$\theta_3: \quad \dot{x}_3(t) = \begin{bmatrix} 0. & 1. \\ -1. & -1.6 \end{bmatrix} x_3(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

$$\theta_4: \quad \dot{x}_4(t) = \begin{bmatrix} 0. & 1. \\ -1. & -0.8 \end{bmatrix} x_4(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

The model with ξ = 1.0, or θ_2, is the true (or active) system. The initial statistic of $x(t_o)$ and θ are given as

$$x_i(t_o) = \begin{bmatrix} 10. \\ 0. \end{bmatrix}, \qquad V_{x_i}(t_o) = \begin{bmatrix} 1.0 & 0. \\ 0. & .5 \end{bmatrix}$$

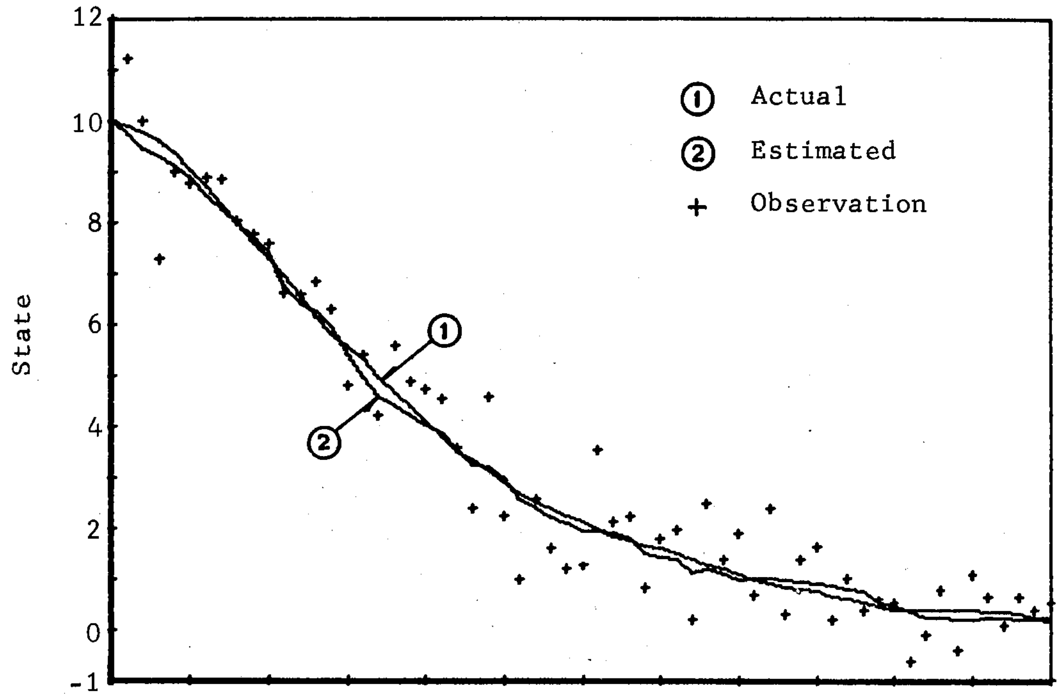
and

$$p_r(\theta_i) = .25$$

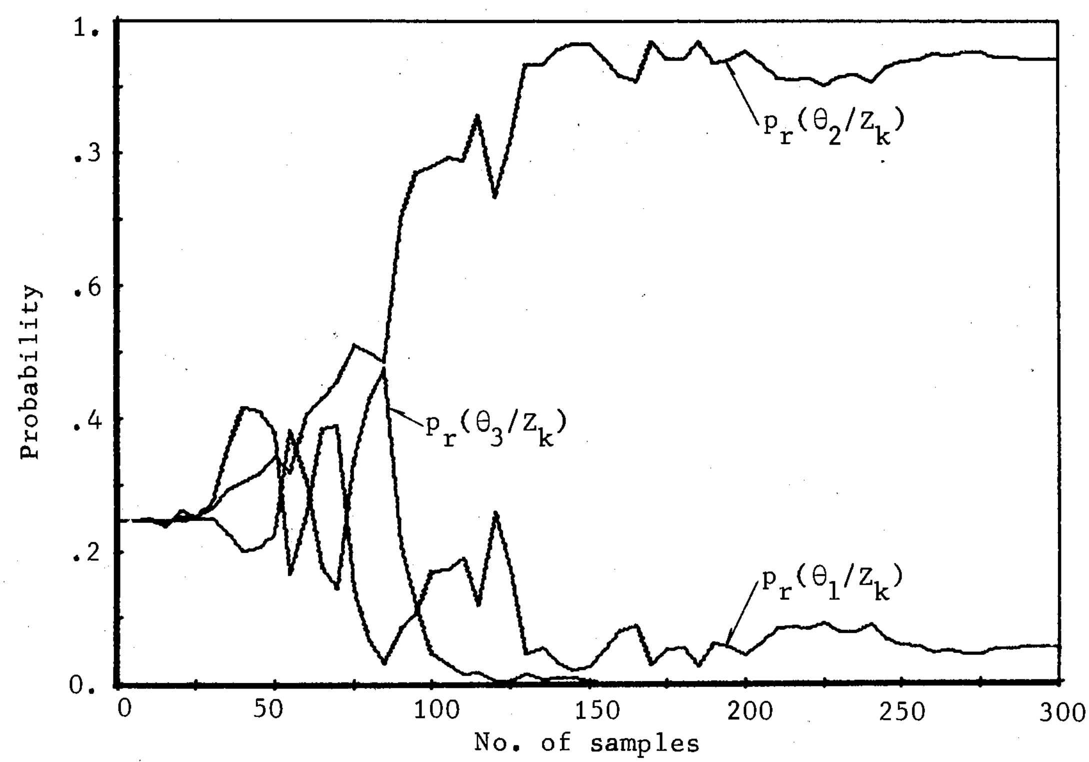
for all i.

A typical single run of the estimated output and the true output are shown in Fig. 4(a). The associated identification capability is also plotted in Fig. 4(b). It is seen that the estimated output is very closed to the true output. Identification of the true model θ_2 is indicated after 100 samples, by the greater posterior probability, $p(\theta_2/Z_k)$. In Fig. 4(b) the plot of the posterior probability of model θ_4 is not shown, because it goes to zero rapidly.

The average performance as well as the identification capability of the algorithm are examined by applying different noise sequences for 26 runs. The trajectories of the averaged estimated output and the true output are plotted in Fig. 5. The plots of the averaged posterior probabilities and their related variances are presented in Fig. 6. In Fig. 6(b), it is seen that the variation of the posterior probabilities of model θ_3 is as great as that of the true model, θ_2. In certain runs model θ_3 achieved favorable posterior probability. In fact, in 1 out of 26 runs the identification algorithm confirmed that model θ_3 is the correct model. This is due to an extraordinary noise sequence. A question arises from this result. What is the effect of plant noise on the identification algorithm? This question is examined and illustrated in a later example.

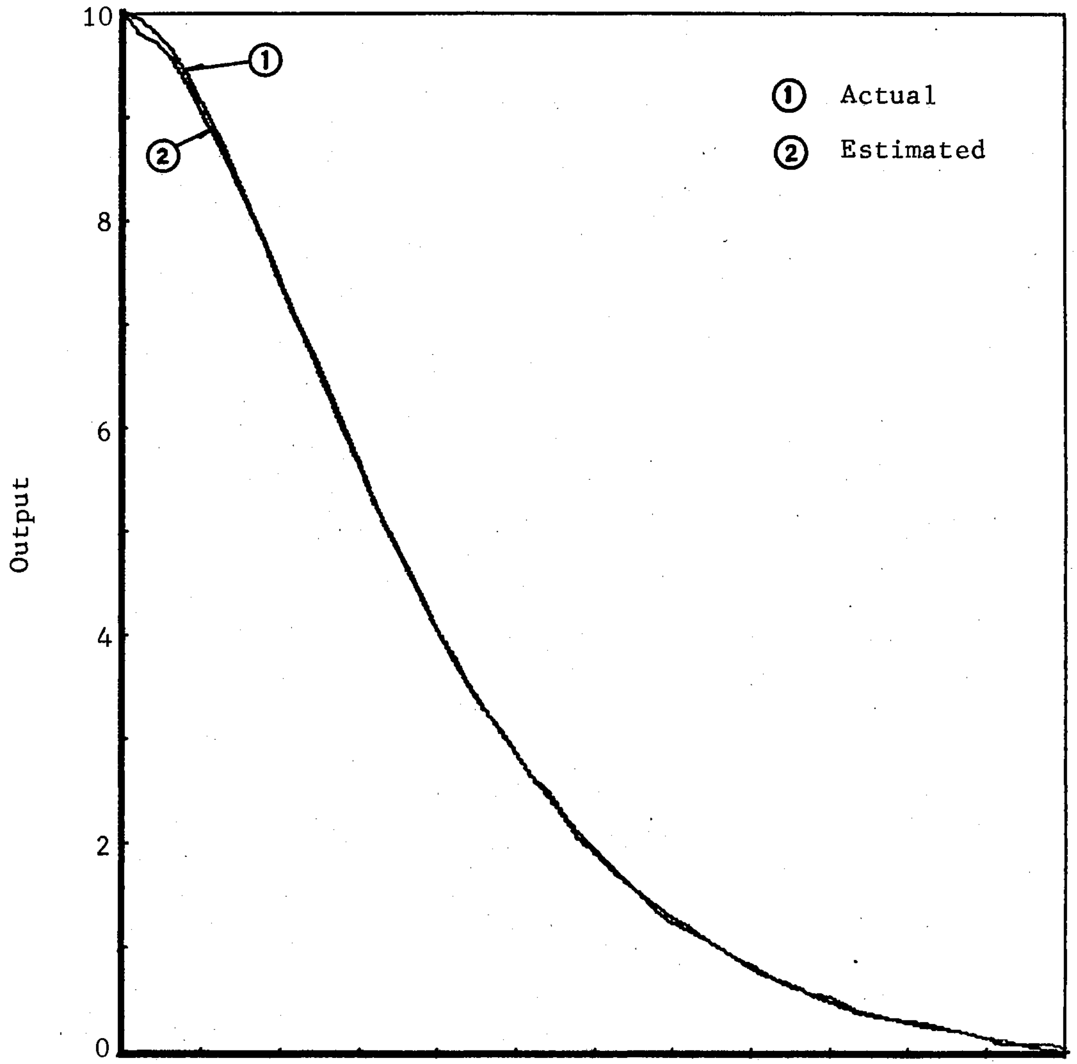


(a) Output Trajectories

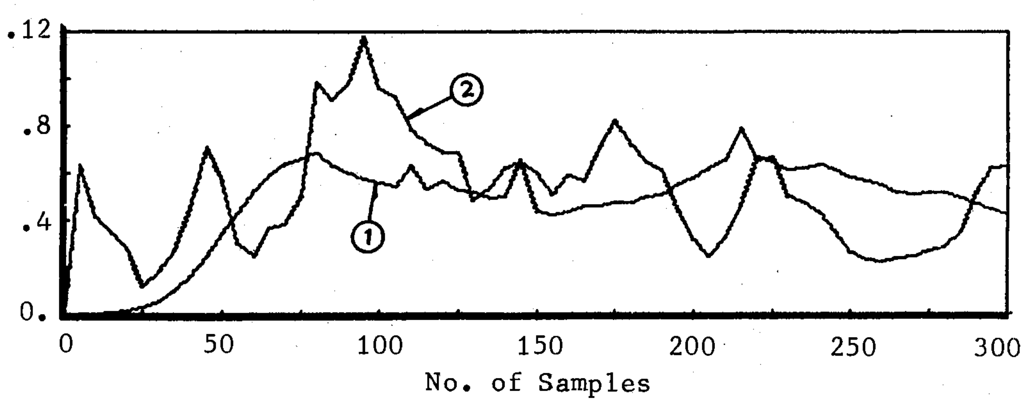


(b) Posterior Probabilities

Figure 4. Simulation Result 1 for Example 3-1 (Single Run)

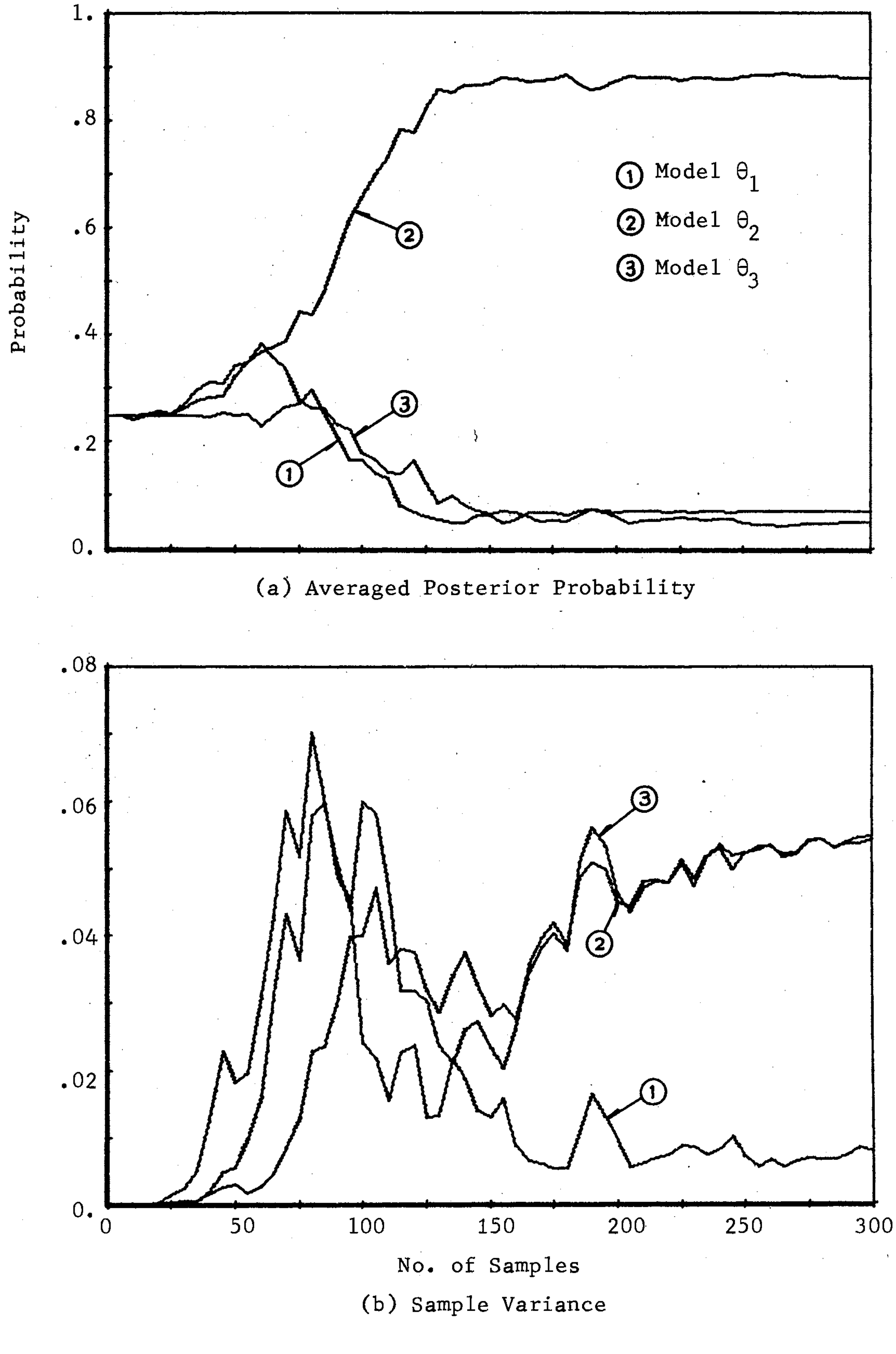


(a) Averaged Output Trajectories



(b) Sample Variance

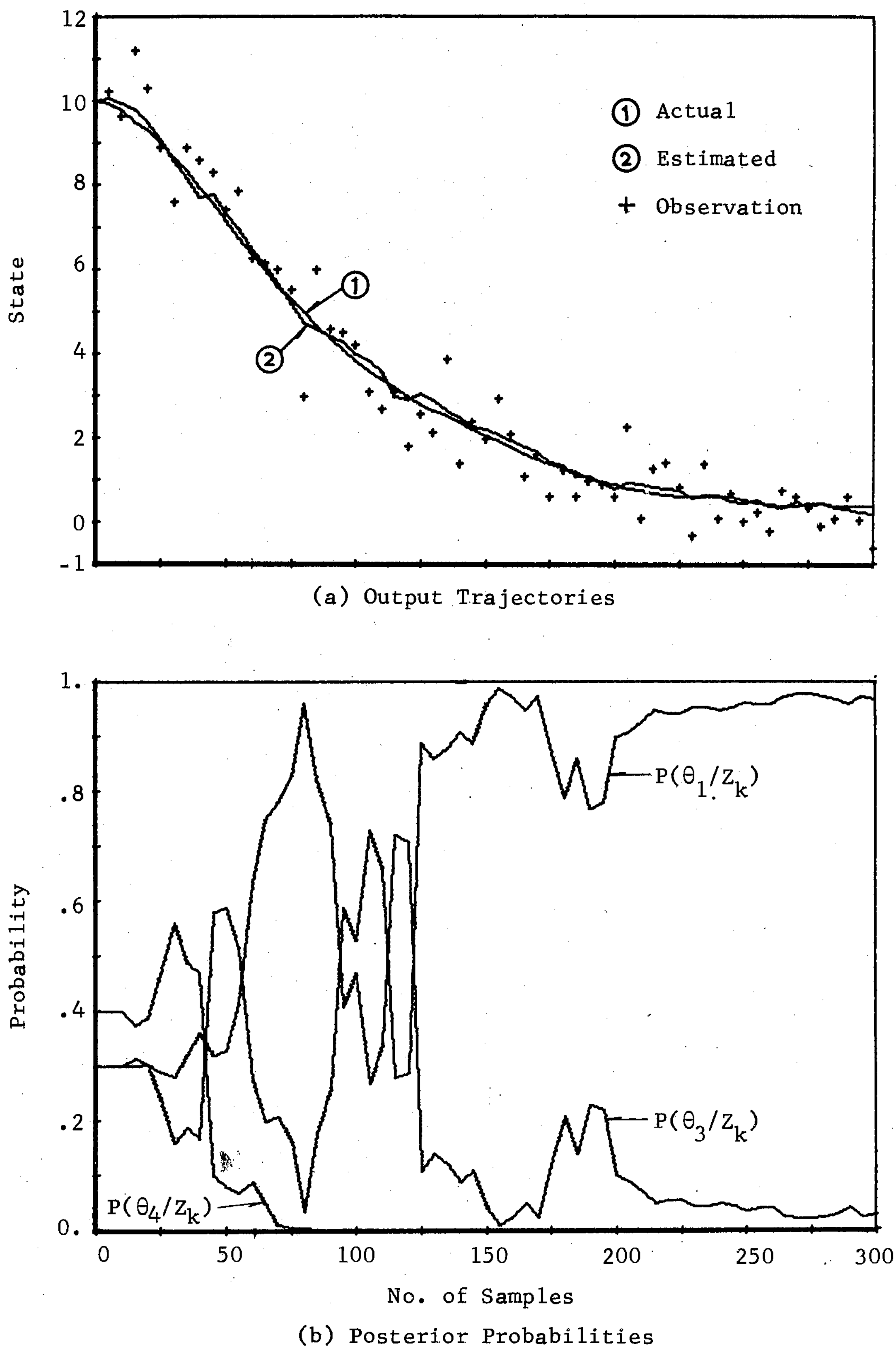
Figure 5. Average Performance for Example 3-1



(b) Sample Variance

Figure 6. Averaged Identification Capability for Example 3-1

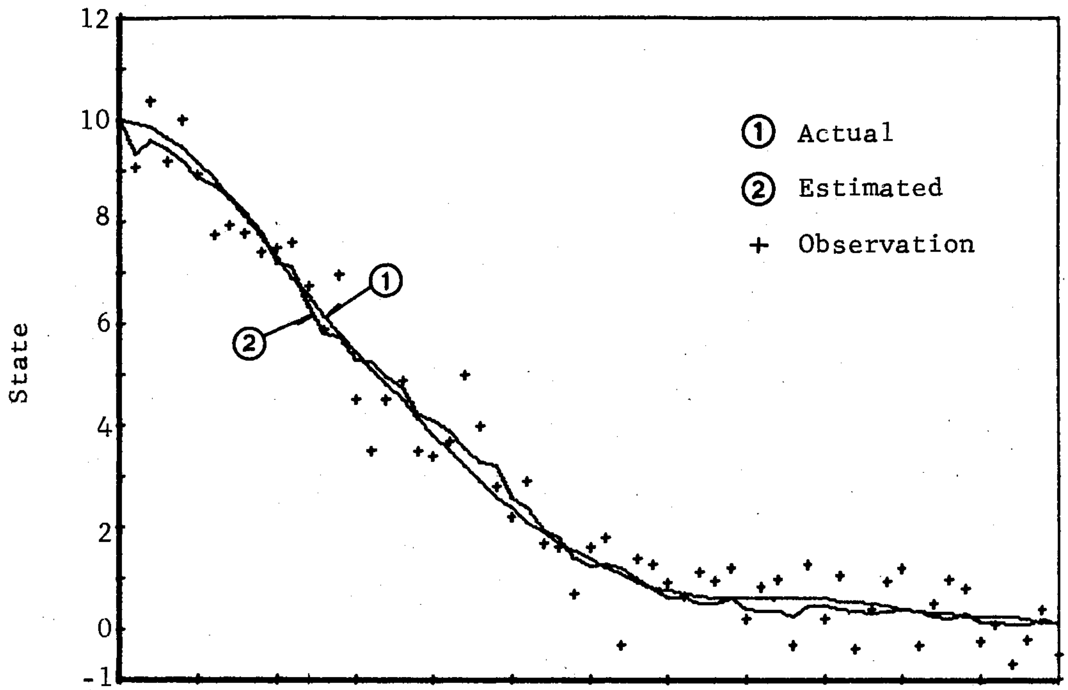
Result 2: It is interesting to consider the case where the true model is not included among the candidate models, i.e., the candidate models are θ_1, θ_3, and θ_4, and θ_2 is the active system. All other aspects of the problem correspond to those of result 1, except that the prior probabilities of the candidate models are $p_r(\theta_1) = .3$, $p_r(\theta_3) = .3$, and $p_r(\theta_4) = .4$. The estimation and identification capability for two typical single runs are shown in Fig. 7 and Fig. 8 where in Fig. 7 the model θ_1 is dominant and in Fig. 8 the model θ_3 is dominant. It is seen that even though the posterior probability indicates that the algorithm may converge to either θ_1 or θ_3, the estimated output is still fairly close to the true output. The average identification capability for 20 runs is shown in Fig. 9, where mean and variance of the posterior probabilities are indicated. From Fig.9, it is learned that the variations of model θ_1 and θ_3 are almost identical. One may guess that the posterior probability of model θ_1 and θ_3 may be close together if averaged over more runs. The identification then depends on the closeness of the model to the true model. Nevertheless, if the range of the modeled values includes the correct value, the algorithm gives fairly good estimates. Furthermore, the algorithm has the advantage that after more information is obtained, the range of the value of parameter subject to uncertainty may be reduced. A new set of candidate models may then be postulated. One may either apply the algorithm for the new models on-line or off-line (using the obtained data). The true value of the uncertain parameter may then be identified after enough iterations. From single run results, it is suggested that even though the posterior probability may approach 1, it does not guarantee that the true model is identified,



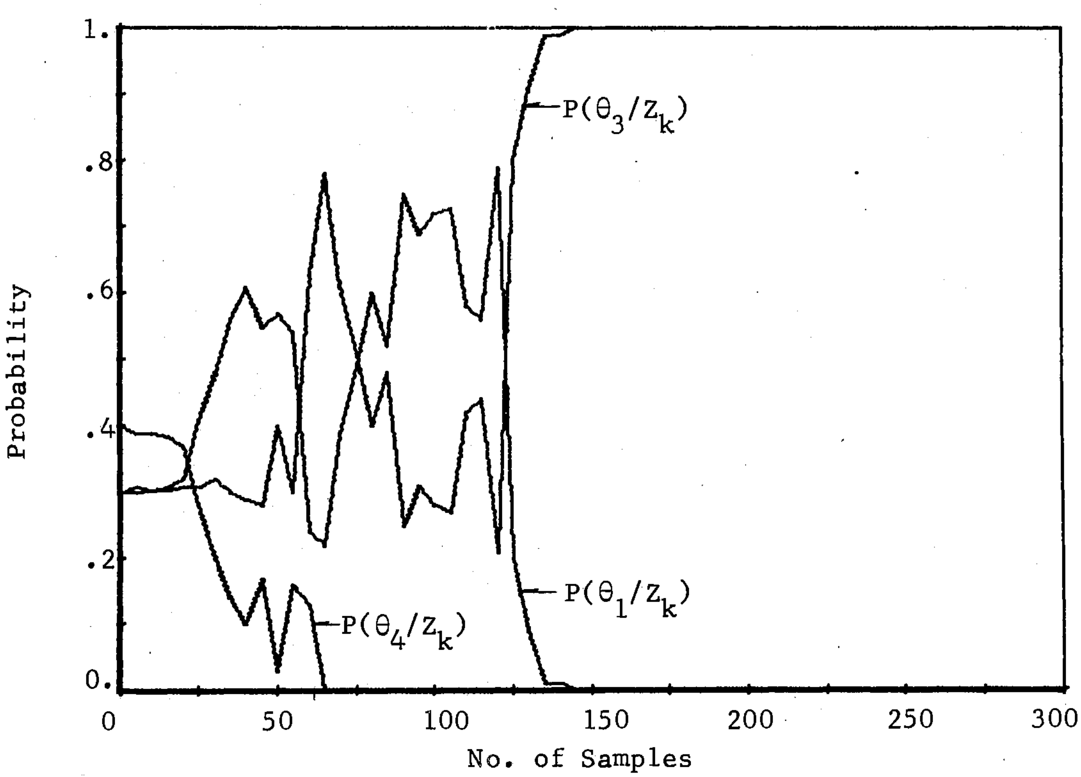
(a) Output Trajectories

(b) Posterior Probabilities

Figure 7. Simulation Result 2 for Example 3-1-single run (Converged to Model θ_1)

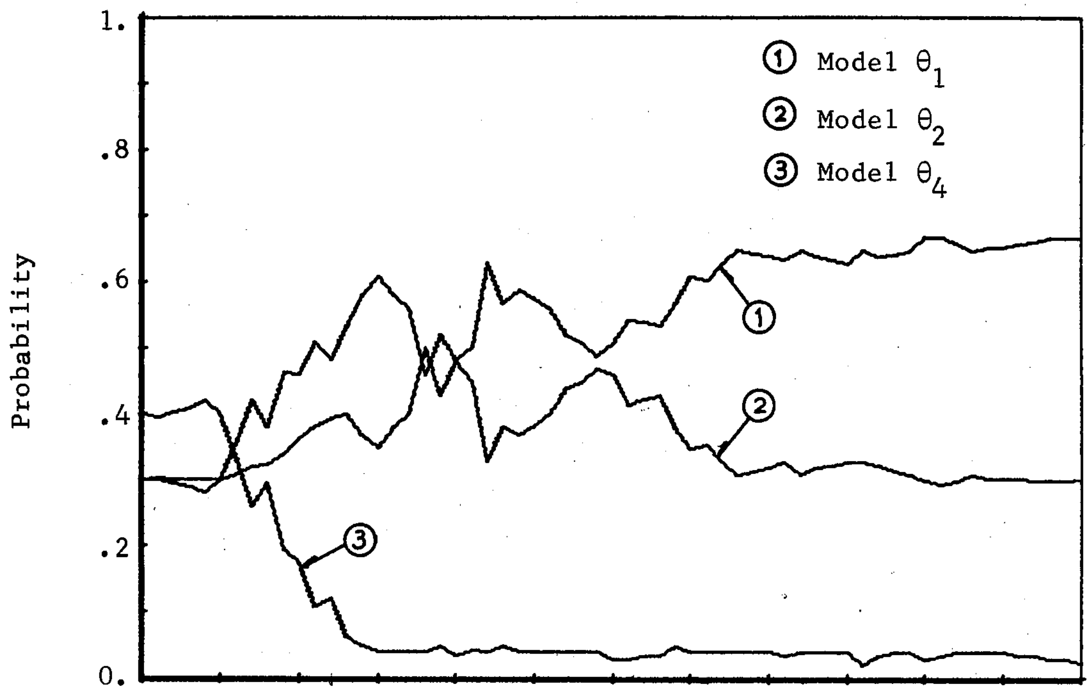


(a) Output Trajectories

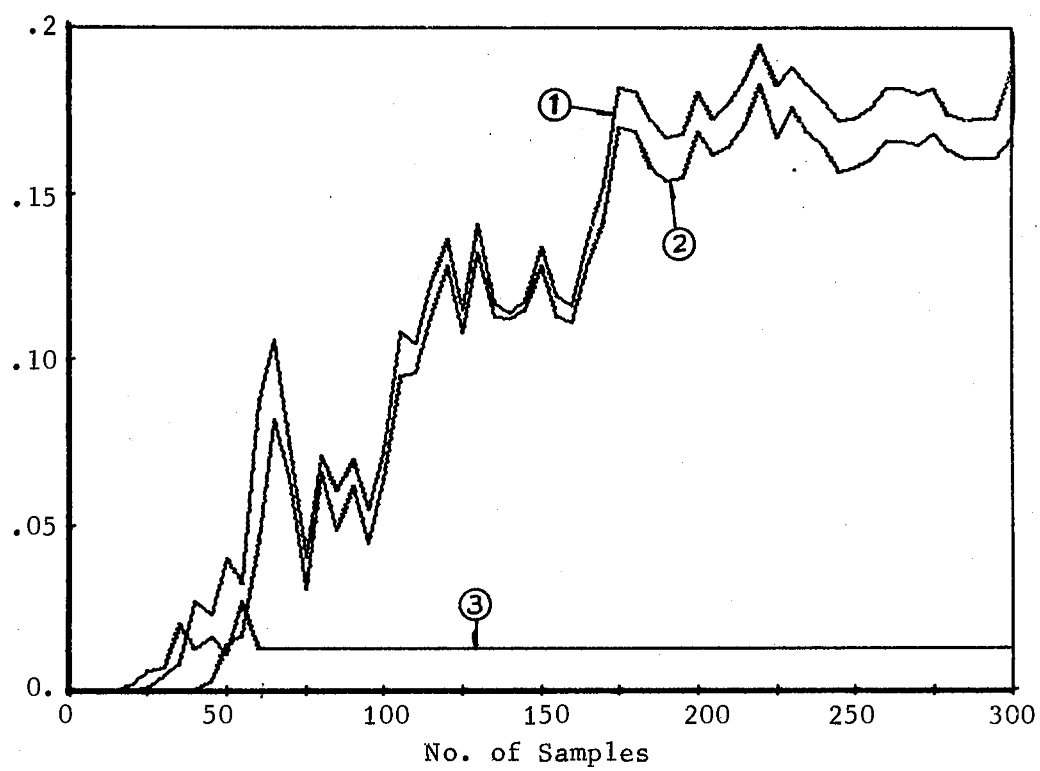


(b) Posterior Probabilities

Figure 8. Simulation Result 2 for Example 3-1-single run (Converged to Model θ_3)



(a) Averaged Posterior Probabilities



(b) Sample Variance

Figure 9. Simulation Result 2 for Example 3-1—20 runs

as indicated in Section 3.3.5. The time average of the normalized residuals associated with the cases of Figures 4, 7, and 8 are shown in Table I. It is seen that in the case of Fig. 7 and Fig. 8, the residuals do not show much different between models, but in the case of Fig. 4, the residual of the true model is less than the others, as one would expect.

TABLE I

THE RESIDUALS

	Model θ_1	Model θ_2	Model θ_3	Model θ_4 [1]
Fig. 4	-0.08641	0.02947	0.15250	--
Fig. 7	-0.12049	--	0.12186	--
Fig. 8	-0.13780	--	0.10319	--

[1] The residuals of model θ_4 are not obtained due to the fact that its posterior probability goes to zero rapidly, and the further computations eventually may be ignored.

(2) Example 3-2

This example is presented to elaborate on a question which occurred in result 1 of the previous example. What is the effect of the plant disturbance on the identification algorithm investigated? Intuitively, identification depends on the shape of the posterior probability density functions and the separation between them[48]. At an observation instant t_k, the density function of each prospective system is $p(x_i(t_k)/Z_k)$. The shape of the density function depends on the covariance matrix V_{x_i} and the separation depends on the mean. Moreover, the covariance matrix V_{x_i} is related to Q(t) and R(k) are described in Eqs. 3-10 and 3-12. In Fig. 10, two sets of density functions with the same separation are indicated. Case (a) seems "more identifiable" than (b) due to the small variance V_{x_i}. The likelihood ratio would also indicate that (a) is more favorable toward model θ_1 than (b).

In the previous example where the nonzero initial estimates $\hat{x}_i(t_o)$ were given, the dynamics of the filter provides good separation. It, therefore, follows from the above arguments that smaller plant noise results in better identification. It is of interest to consider the case where the initial estimates $\hat{x}_i(t_o)$ are zero, so that the density functions all have the same mean value initially. The response is governed by the noise input, rather than the homogeneous portion.

A scalar example with two alternatives is simulated with $\hat{x}_i(t_o)$ given as zero. The dynamics are described by

$$\theta_1: \quad \dot{x}_1(t) = -1\ x_i(t) + w(t)$$

$$\theta_2: \quad \dot{x}_2(t) = -5\ x_2(t) + w(t)$$

with the observation

$$z(k) = x_i(t_k) + v(k)$$

The noise terms are zero mean white gaussian with variances

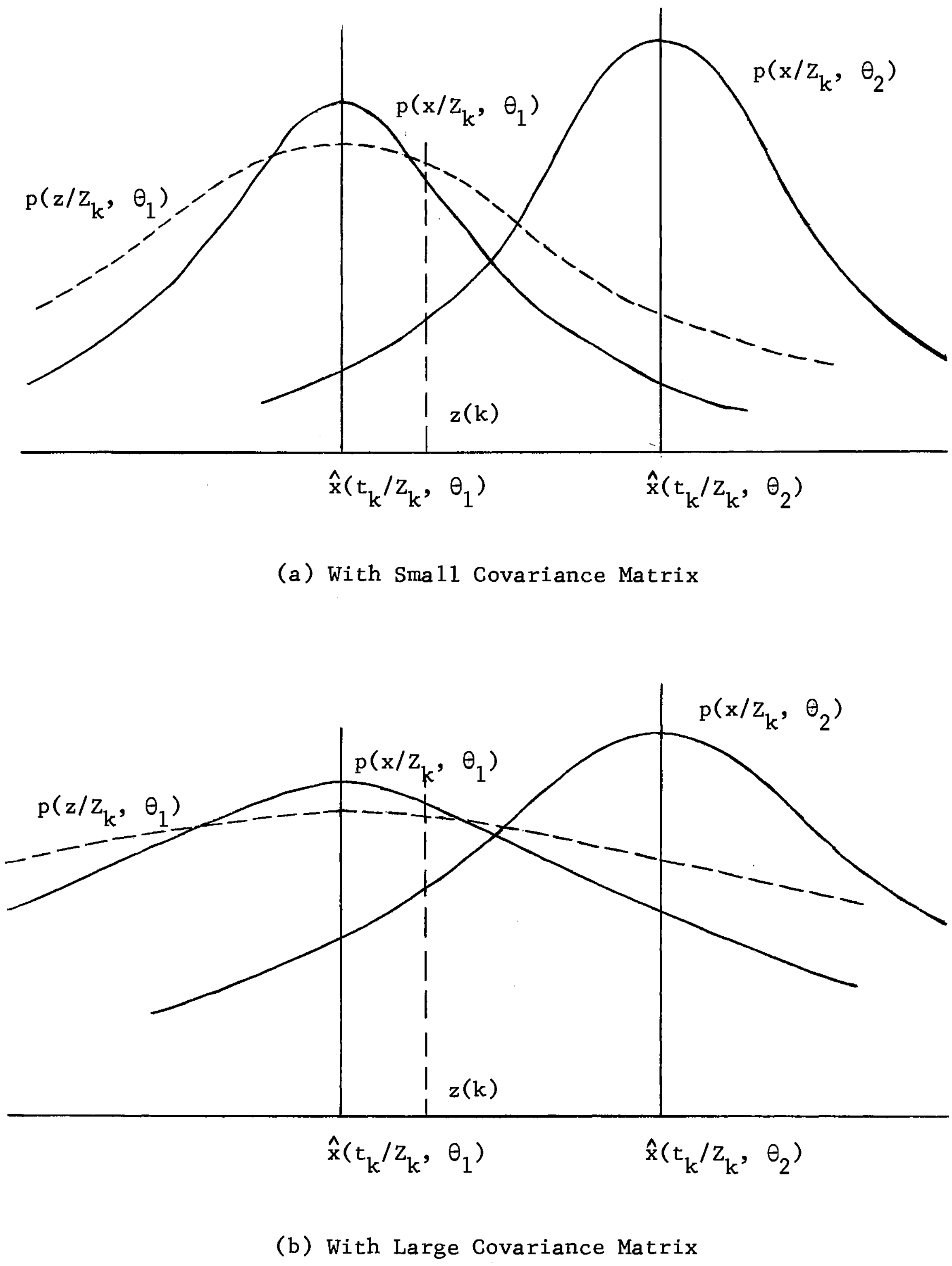


Figure 10. Illustration for Identification Capability

$$VAR\{w(t)\} = Q$$

and

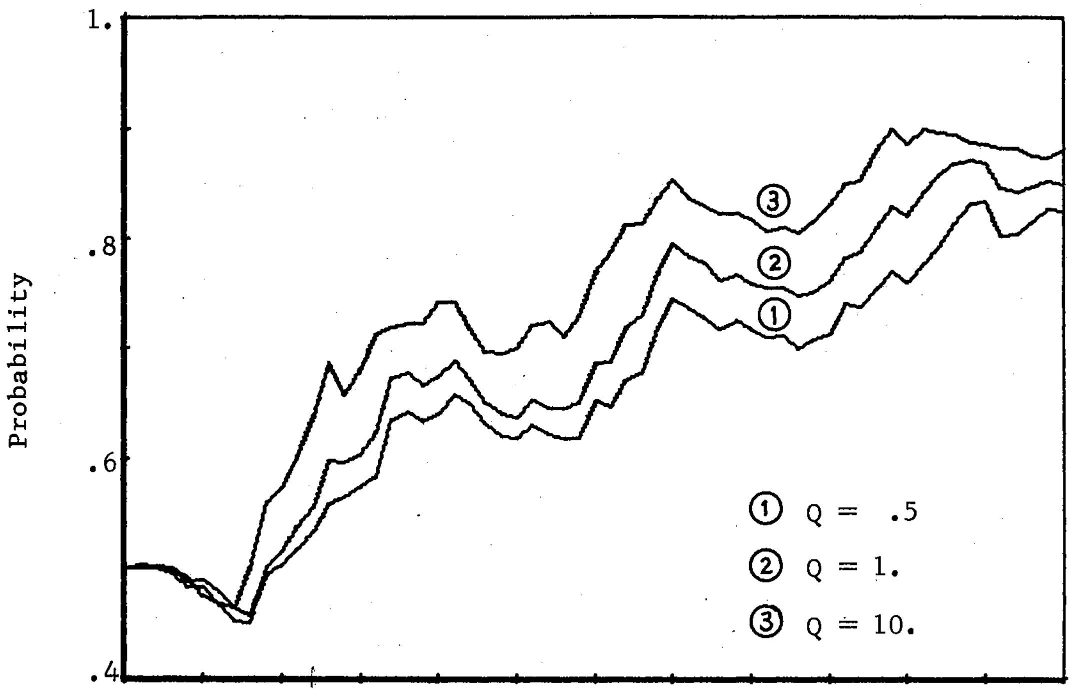
$$VAR\{v(k)\} = R = 1.$$

With the same prior probability $p_r(\theta_i) = .5$ and using the same noise sequence for cases with different values of Q, the identification capability of the algorithm is shown in Fig. 11 averaged over 15 runs. In Fig. 11(b), the variance for various value of Q appears almost the same. It is shown that with larger variance for the noise, better separation is eventually achieved, and hence better system identification.

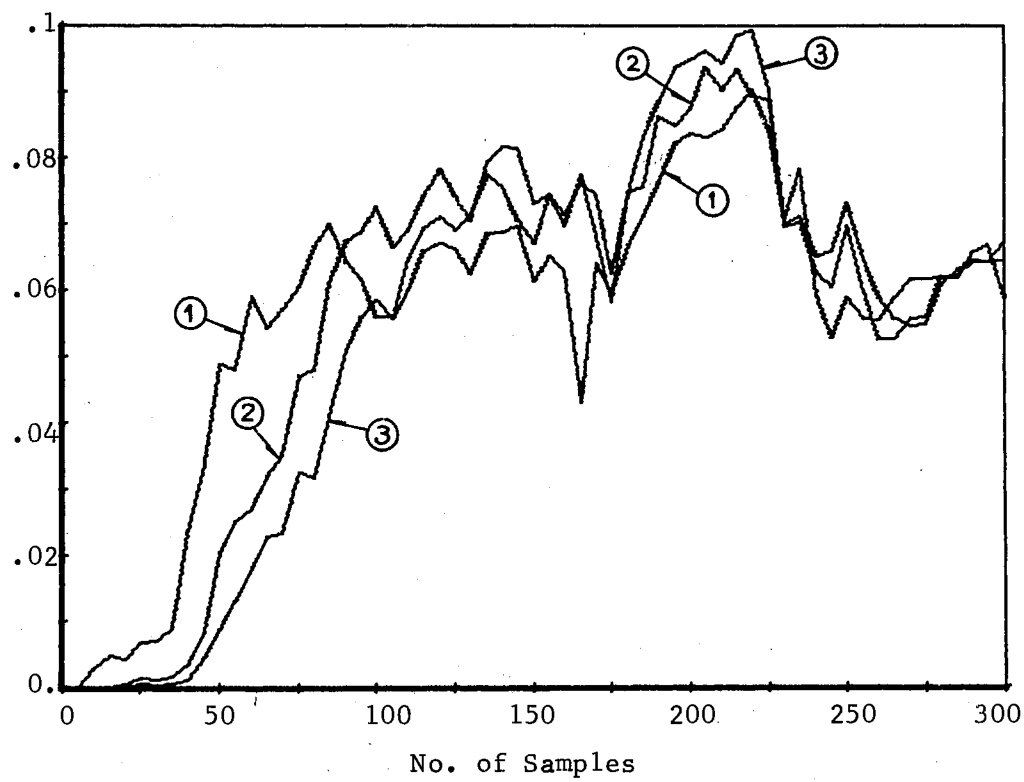
From this example, the following trends may be noted. In the initial phase, when the response is dominated by initial conditions, good identification requires small noise. Identification is enhanced by plant noise, however, after the response due to initial conditions dies out. In general, more observations seem to be required when the response is primary due to the input rather than the initial conditions.

3.4 Estimation-Under-Uncertainty--Switching Case

The system description remains the same as previously defined in Eq. 3-1 through Eq. 3-3, except that the active candidate model may switch from one to another. The switching between models is governed by a transition probability matrix Δ, where the i-j th element represents the probability that model θ_i is active on the k-th observation interval given that model θ_j was active on the k-1 st observation interval. The probability that θ_i is active on the k-th



(a) Averaged Posterior Probabilities



(b) Sample Variance

Figure 11. The Relation of Plant Noise and Identification

observation interval is described by

$$p_r(\theta(t) = \theta_i) = \sum_{j=1}^{N} \Delta_{ij} p_r(\theta(\tau) = \theta_j) \qquad (3\text{-}51)$$

where $t_k \leqslant t < t_{k+1}$ and $t_{k-1} \leqslant \tau < t_k$. Here, only one switching between observation is assumed.

In addition, the order of the state vector of each model is assumed to be the same. This is due to the fact that when the system switches from one model to another the information possessed by every state has to be transferred to the corresponding state in a different model. This means that the structure of each model is fixed which is different from the nonswitching case.

At the k-th observation stage, a sequence $\lambda(k)$ is defined as

$$\lambda(k) = \{\theta(1),\ \theta(2), \ldots,\ \theta(k)\} \qquad (3\text{-}52)$$

where $\theta(i)$ is the label of a model from which the measurement $z(i)$ is sampled, and the space Λ_k on which $\lambda(k)$ is defined has N^k elements. Furthermore, a specific sequence of length k having last element of label $\theta(k) = \theta_i$ is defined by

$$\lambda_i(k) = \{\lambda_j(k-1),\ \theta(k) = \theta_i\} \qquad (3\text{-}53)$$

It is seen that there are N^{k-1} such sequences with the terminal label $\theta(k) = \theta_i$.

Given a sequence of data $Z_k = \{z(1),\ z(2), \ldots,\ z(k)\}$, the optimal and suboptimal solutions for state estimation problems are presented in the following subsections.

3.4.1 The Optimal Solution

The times between observations and at observation instants are considered separately as in the nonswitching case. At time t_k, it is assumed that the probability of the sequences $\lambda(k-1)$ is known, or $p_r(\lambda(k-1)/Z_k)$ is given for $\lambda(k-1) \in \Lambda_{k-1}$. At stage $t_k \leqslant t < t_{k+1}$, each current sequence may take one of the N possible labels as the next element. The prior probability of the generated sequence is governed by

$$p_r(\lambda_i(k)/Z_k) = \Delta_{ij} p_r(\lambda_j(k-1)/Z_k) \tag{3-54}$$

for $i = 1,2,\ldots,N$, and the sequence $\lambda_i(k)$ as defined in Eq. 3-53. The estimate at time t_k of each sequence $\lambda(k-1)$ provides the initial conditions for the N additional filters, needed for the next time interval. In between observations, the best estimate of the system is then expressed by

$$\begin{aligned}\hat{x}(t/Z_k) &= \sum_{\Lambda_{k-1}} \sum_{i=1}^{N} p_r(\lambda_i(k)/Z_k)\hat{x}(t/\theta_i,\ \lambda(k-1),\ Z_k) \\ &= \sum_{\Lambda_k} p_r(\lambda(k)/Z_k)\hat{x}(t/\lambda(k),\ Z_k)\end{aligned} \tag{3-55}$$

where $\hat{x}(t/Z_k)$ is the best state estimate and $\hat{x}(t/\theta_i,\ \lambda(k-1),\ Z_k)$ is the state estimate of a sequence given terminal label $\theta(k) = \theta_i$. The filter structure for a specific sequence given the terminal label is the same as in Eqs. 3-9 and 3-10 of the nonswitching case. That is,

$$\dot{\hat{x}}(t/\theta_i,\ \lambda(k-1),\ Z_k) = F_i(t)\hat{x}(t/\theta_i,\ \lambda(k-1),\ Z_k) + C_i(t)u(t) \tag{3-56}$$

and

$$\dot{V}_x(t/\theta_i,\ \lambda(k-1),\ Z_k) = F_i(t)V_x(t/\theta_i,\ \lambda(k-1),\ Z_k)$$

$$+ V_x(t/\theta_i,\ \lambda(k-1),\ Z_k)F_i^T(t) + G_i(t)Q_i(t)G_i^T(t) \tag{3-57}$$

where the initial conditions at t_k are $\hat{x}(t_k/\lambda(k-1),\ Z_k)$ and $V_x(t_k/\lambda(k-1),\ Z_k)$, and at t_o, $\hat{x}(t_o)$ and $V_x(t_o)$.

At the observation instant t_{k+1}, the conditional mean estimate and the covariance matrix are corrected according to Eq. 3-19 through Eq. 3-21 as in the nonswitching case, conditioned on the terminal label of the specific sequence given, i.e.,

$$\hat{x}(t_{k+1}/\theta_i,\ \lambda(k-1),\ Z_{k+1}) = \hat{x}(t_{k+1}/\theta_i,\ \lambda(k-1),\ Z_k) + K(t_{k+1}/\theta_i,\lambda(k-1))$$

$$\times[z(k+1) - H_i(t_{k+1})\ \hat{x}(t_{k+1}/\theta_i,\lambda(k-1),Z_k)] \tag{3-58}$$

and

$$V_x(t_{k+1}/\theta_i,\lambda(k-1),Z_{k+1}) = V_x(t_{k+1}/\theta_i,\lambda(k-1),Z_k)$$

$$-K(t_{k+1}/\theta_i,\lambda(k-1))H_i(t_{k+1})V_x(t_{k+1}/\theta_i,\lambda(k-1),Z_k) \tag{3-59}$$

where

$$K(t_{k+1}/\theta_i,\ \lambda(k-1)) = V_x(t_{k+1}/\theta_i,\ \lambda(k-1),\ Z_k)H_i(t_{k+1})[H_iV_xH_i^T + R_i]^{-1} \tag{3-60}$$

The probability associated with each sequence then may be updated by a relationship similar to the nonswitching case,

$$p_r(\lambda(k)/Z_{k+1}) = \frac{p(z(k+1)/\lambda(k),\ Z_k)p_r(\lambda(k)/Z_k)}{\sum\limits_{\Lambda_k} p(z(k+1)/\lambda(k),\ Z_k)p_r(\lambda(k)/Z_k)} \tag{3-61}$$

Moreover, if a sequence is terminated by label θ_i,

$$p(z(k+1)/\lambda(k),\ Z_k) = p(z(k+1)/\theta_i,\ \lambda(k-1),\ Z_k)$$

which may be evaluated as in Eq. 3-24. The best estimate at this moment is then

$$\hat{x}(t_{k+1}/z_{k+1}) = \sum_{\Lambda_k} p_r(\lambda(k)/Z_{k+1})\hat{x}(t_{k+1}/\lambda(k),\ Z_{k+1}) \tag{3-62}$$

The summation is taken over all the possible sequence $\lambda(k)$.

The results of Eqs. 3-58 and 3-59 are then used as initial conditions for the next observation interval. The technique presented here is still recursive, except that the number of filters increases with measurements as mentioned in Chapter II. Consequently, a suboptimal technique which utilized a finite memory is sought.

3.4.2 Suboptimal Solution--Single Stage Estimation

The single stage estimation is obtained by truncating the growing memory requirements of the optimal algorithm one stage back. In fact, at the k-th observation stage, the optimal estimate may be obtained by grouping together the estimates of sequences ending with the same label. Interchanging the summation of the first expression in Eq. 3-55, and using the relationship

$$p_r(\lambda_i(k)/Z_k) = p_r(\theta(k) = \lambda_i,\ \lambda(k-1)/Z_k)$$

$$= p_r(\theta(k) = \theta_i/\lambda(k-1), Z_k)p_r(\lambda(k-1)/Z_k) \tag{3-63}$$

equation 3-55 may be rewritten as

$$\hat{x}(t/Z_k) = \sum_{i=1}^{N} \sum_{\Lambda_{k-1}} p_r(\theta(k) = \theta_i/\lambda(k-1),\ Z_k)p_r(\lambda(k-1)/Z_k)\hat{x}(t/\theta_i,\lambda(k-1),Z_k) \tag{3-64}$$

for $t_k \leqslant t < t_{k+1}$. If the condition $\lambda(k-1)$ on the estimate $\hat{x}(t/\theta_i,\ \lambda(k-1),\ Z_k)$ is dropped, Eq. 3-64 may be approximated by

$$\hat{x}(t/Z_k) = \sum_{i=1}^{N} p_r(\theta(k) = \theta_i/Z_k)\hat{x}(t/\theta_i,\ Z_k) \tag{3-65}$$

where

$$p_r(\theta(k) = \theta_i/Z_k) = \sum_{\Lambda_{k-1}} p_r(\theta_i/\lambda(k-1),\ Z_k)p_r(\lambda(k-1)/Z_k) \tag{3-66}$$

which is the marginal probability of $\theta(k) = \theta_i$ over the space Λ_{k-1}. In this fashion, a suboptimal algorithm involving only N filters is obtained. Since the memory is restricted one stage back, the best estimate at time t_k for each filter is the estimate $\hat{x}(t_k/Z_k)$ instead of the estimate of each filter $\hat{x}(t_k/\theta_i,\ Z_k)$. The probability expressed in Eq. 3-66 may be modified as

$$p_r(\theta(k) = \theta_i/Z_k) = \sum_{j=1}^{N} \Delta_{ij}p_r(\theta(k-1) = \theta_j/Z_k) \tag{3-67}$$

The estimates and covariance matrices associated with each filter are then the same as given by Eqs. 3-9 and 3-10.

At an observation time instant t_{k+1}, the estimate as well as the covariance matrix of each filter is corrected as indicated by Eq. 3-19 through Eq. 3-21. The probability of each filter should be updated at the new observation. The updated probability is given by

$$p_r(\theta(k) = \theta_i/Z_{k+1}) = \frac{p(z(k+1)/\theta_i,\ Z_k)p_r(\theta(k) = \theta_i/Z_k)}{\sum_{j=1}^{N} p(z(k+1)/\theta_j,\ Z_k)p_r(\theta(k) = \theta_j/Z_k)}$$

$$= [1 + \sum_{\substack{j=1 \\ j\neq i}}^{N} L_{ji} \frac{p_r(\theta(k) = \theta_j/Z_k)}{p_r(\theta(k) = \theta_i/Z_k)}]^{-1} \tag{3-68}$$

The expression is the same as Eq. 3-23 and where L_{ji} is given in Eq. 3-25.

The computational algorithm suggested here is essentially the same as for the nonswitching case, except for the consideration of the initial conditions at each observation and the fact that system identification is not considered. The one stage algorithm is summarized below.

<u>ALGORITHM 3-2</u> (Suboptimal)

A suboptimal solution to the switching case of the estimation problem stated in Section 3.2 with N filters is presented in recursive form. At any time $t_k \leqslant t < t_{k+1}$, the best estimate is expressed as

$$\hat{x}(t/Z_k) = \sum_{i=1}^{N} p_r(\theta(k) = \theta_i/Z_k)\hat{x}(t/\theta_i,\ Z_k) \tag{3-69}$$

where the probability of each filter is evaluated through the transition probability matrix.

$$p_r(\theta(k) = \theta_i/Z_k) = \sum_{j=1}^{N} \Delta_{ij} p_r(\theta(k-1) = \theta_j/Z_k) \tag{3-70}$$

with $p_r(\theta(0) = \theta_i/Z_o) = p_r(\theta_i)$.

The estimate and covariance matrix of a given filter are described as follows.

(1) In between observations

$$\dot{\hat{x}}(t/\theta_i,\ Z_k) = F_i(t)\hat{x}_i(t/\theta_i,\ Z_k) + C_i(t)u(t) \tag{3-71}$$

and

$$\dot{V}_x(t/\theta_i,Z_k) = F_i(t)V_x(t/\theta_i,Z_k) + V_x(t/\theta_i,Z_k)F_i^T(t) + G_i(t)Q_i(t)G_i^T(t) \tag{3-72}$$

for $i = 1,2,\ldots,N$, and with initial condition at time t_k,

$$\hat{x}(t_k/\theta_i,\ Z_k) = \hat{x}(t_k/Z_k) \tag{3-73}$$

and

$$\begin{aligned} V_x(t_k/\theta_i,Z_k) &= V_x(t_k/Z_k) \\ &= \sum_{i=1}^{N} p_r(\theta(k-1) = \theta_i/Z_k)V_x(t_k/\theta_i,Z_k) \end{aligned} \tag{3-74}$$

for all i, and $\hat{x}(t_o/\theta_i) = \hat{x}_i(t_o)$ and $V_x(t_o/\theta_i) = V_{x_i}(t_o)$.

(2) At observation time t_{k+1},

$$\begin{aligned} \hat{x}(t_{k+1}/\theta_i,Z_{k+1}) = \hat{x}(t_{k+1}/\theta_i,Z_k) &+ K(t_{k+1}/\theta_i)[z(k+1) \\ &- H_i(t_{k+1})\hat{x}(t_{k+1}/\theta_i,Z_k)] \end{aligned} \tag{3-75}$$

and

$$V_x(t_{k+1}/\theta_i,\ Z_{k+1}) = V_x(t_{k+1}/\theta_i,Z_k) - K(t_{k+1}/\theta_i)H_i(t_{k+1})V_x(t_{k+1}/\theta_i,\ Z_k)$$

(3-76)

where the filter gain is

$$K(t_{k+1}/\theta_i) = V_x(t_{k+1}/\theta_i,Z_k)H_i^T(t_{k+1})[H_iV_x(t_{k+1}/\theta_i,Z_k)H_i^T + R_i(k+1)]^{-1}$$

(3-77)

for $i = 1,2,\ldots,N$. The posterior probability of each filter is expressed as

$$p_r(\theta(k) = \theta_i/Z_{k+1}) = [1 + \sum_{\substack{j=1 \\ j\neq i}}^{N} L_{ji} \frac{p_r(\theta(k) = \theta_j/Z_k)}{p_r(\theta(k) = \theta_i/Z_k}]^{-1} \qquad (3\text{-}78)$$

where L_{ji} is the likelihood ratio given in Eq. 3-25.

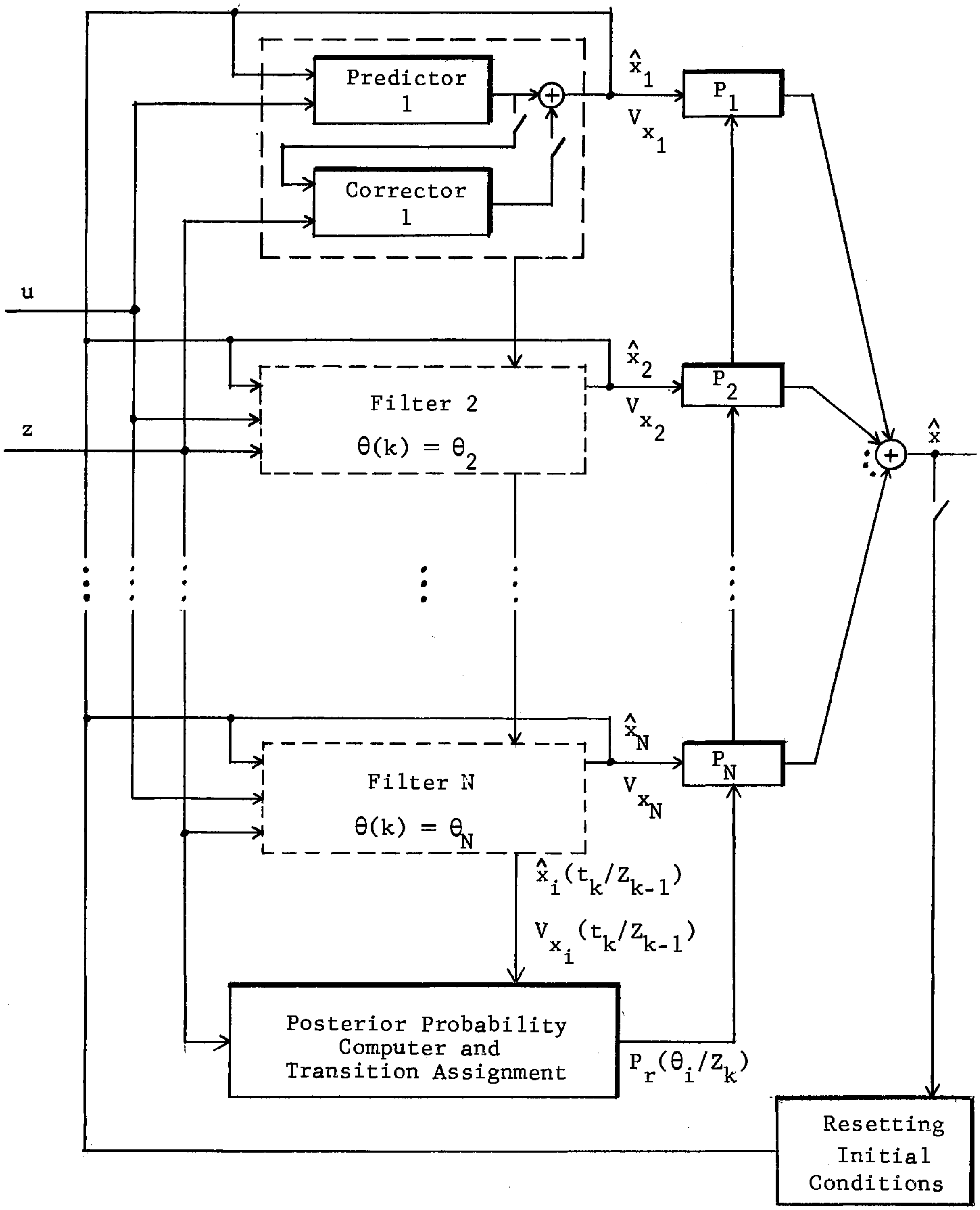
The block diagram of the algorithm is presented in Fig. 12.

In a practical situation, the transition probability matrix may be difficult to obtain, but one should be able to make an appropriate assignment of transition probability values based on past experiences.

3.4.3 Example 3-3

The system under consideration is the same as in example 3-1, where four candidate models are also adopted. The transition probability matrix is assumed to be

$$\Delta = \begin{bmatrix} .5 & .2 & .2 & .1 \\ .3 & .5 & .2 & .1 \\ .2 & .2 & .5 & .1 \\ .25 & .25 & .25 & .25 \end{bmatrix}$$



Predictor: Eqs. 3-71 and 3-72

Corrector: Eqs. 3-75 and 3-76

Identification: Eqs. 3-78

Figure 12. Block Diagram of ALGORITHM 3-2 Switching Case

The noise terms have zero mean and variances $Q(t) = 1$ and $R(k) = 1$. In simulation, the assignment of the active model is determined by a number from a random number generator.

Two results are obtained where both small and large noise disturbance are considered separately.

Result 1: With the initial mean values and covariance matrices specified as

$$\hat{x}_i(t_o) = \begin{bmatrix} 10. \\ 0. \end{bmatrix} \quad \text{and} \quad V_{x_i}(t_o) = \begin{bmatrix} 1. & 0. \\ 0. & .5 \end{bmatrix}$$

for all i, the results of suboptimal estimation and estimation under certainty (with known active model) are compared in Eqs. 3-11 and 3-12. A typical single run result is plotted in Fig. 13 with 50 sampled points, while Fig. 14 is the plot of averaging over 15 runs. It is seen that the suboptimal estimates are very closed to the optimal estimates with the known active model at each instant, for this case with small disturbances.

Result 2: For the situation in which there are large noise disturbances, good results are also obtained. The initial mean value $\hat{x}_i(t_o) = \begin{bmatrix} 1. \\ 0. \end{bmatrix}$ is assumed, and the covariance is as in the previous case. Results of typical single run and averaging over 15 runs are presented in Figures 15 and 16, respectively. It is shown that the suboptimal estimates are close to optimal estimate under certainty. Moreover, the results reveal that after gathering enough information, the estimates are close to the true state.

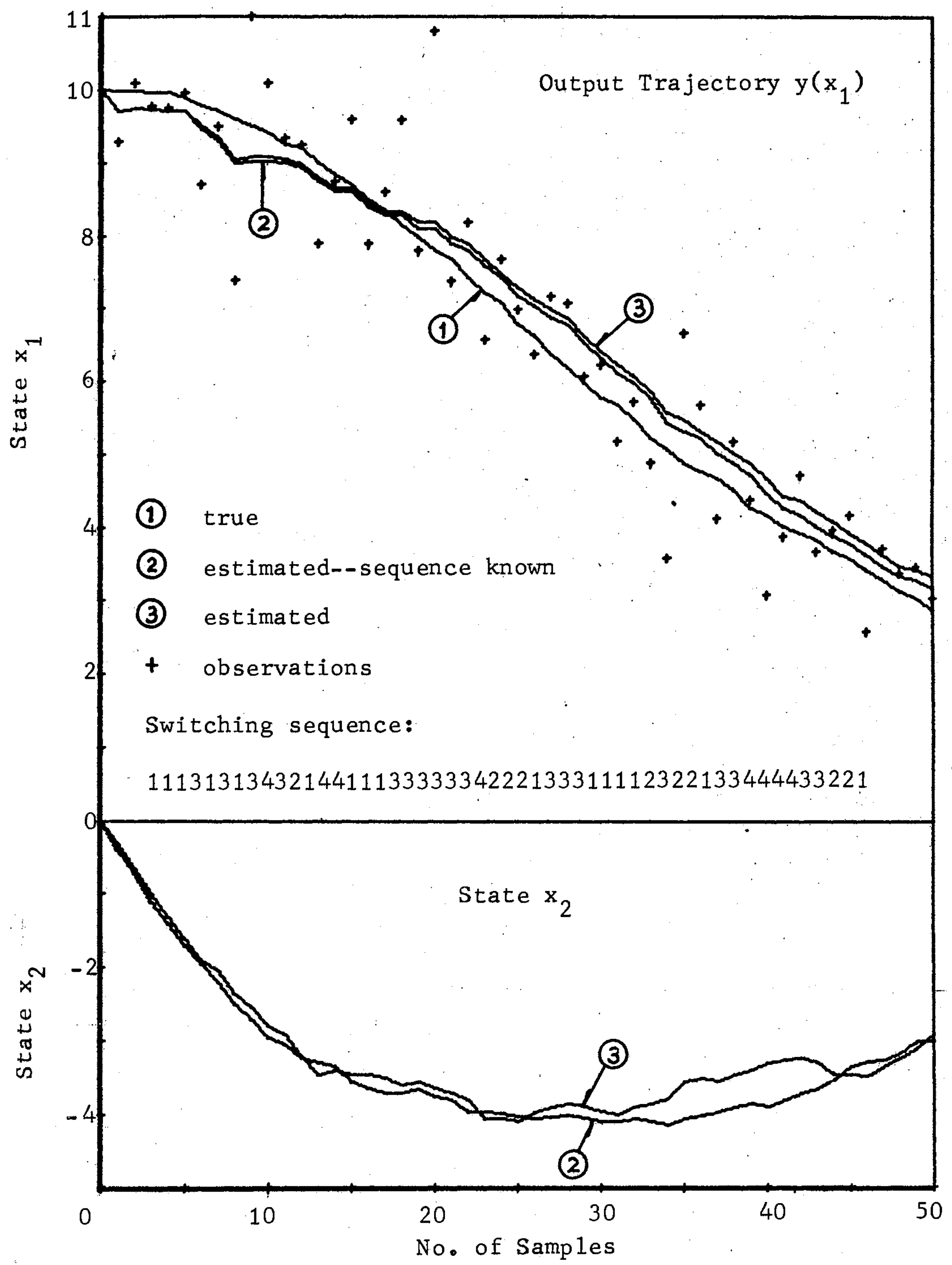


Figure 13. A Typical Single Run Result for Example 3-3 Switching Case--Result 1 (small noise)

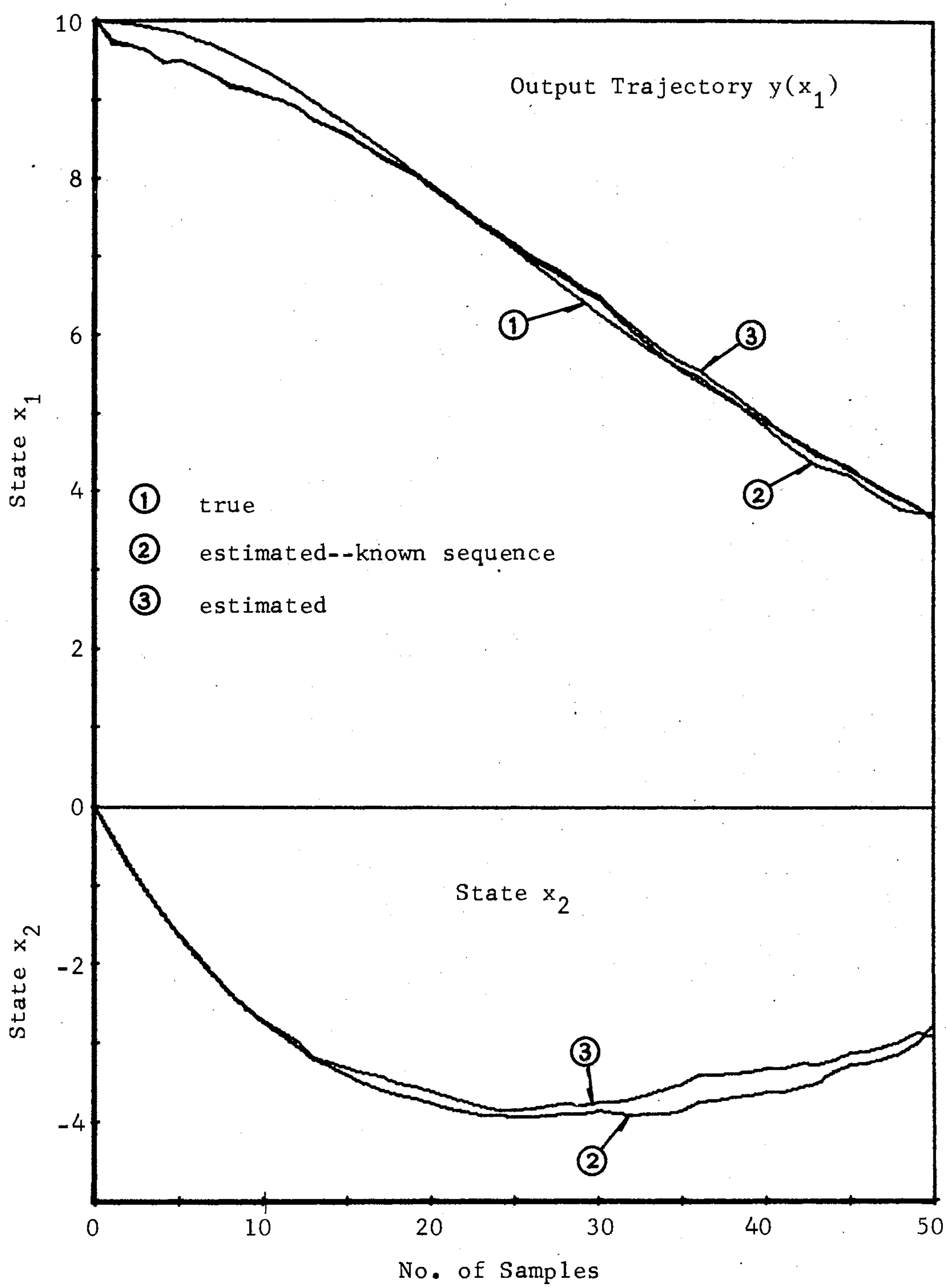
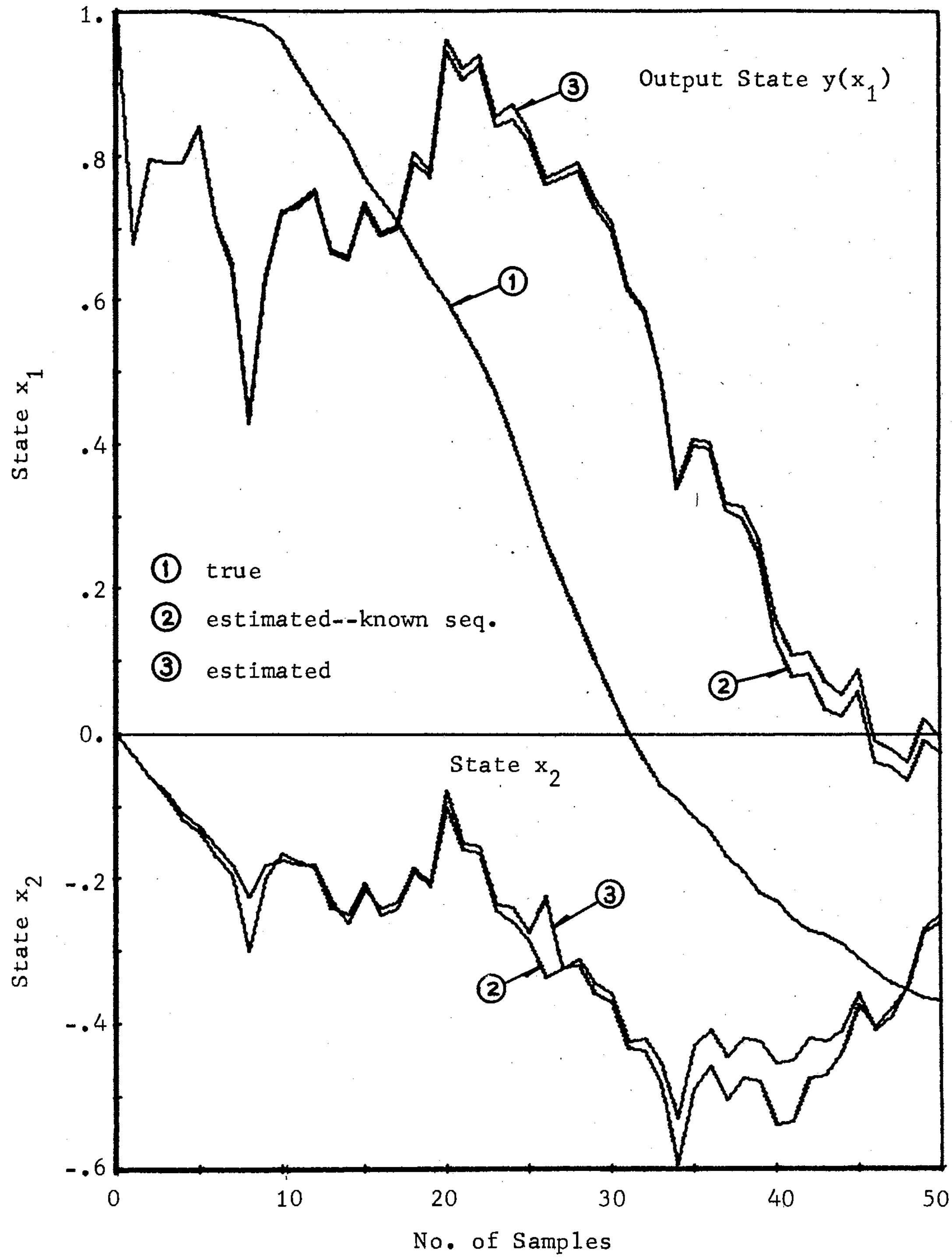


Figure 14. Averaged Performance for Example--Result 1



Switching sequence:

11131342121441113333334222133311112322333444313221

Figure 15. A Typical Single Run Result for Example 3-3 Switching Case--Result 2 (large noise)

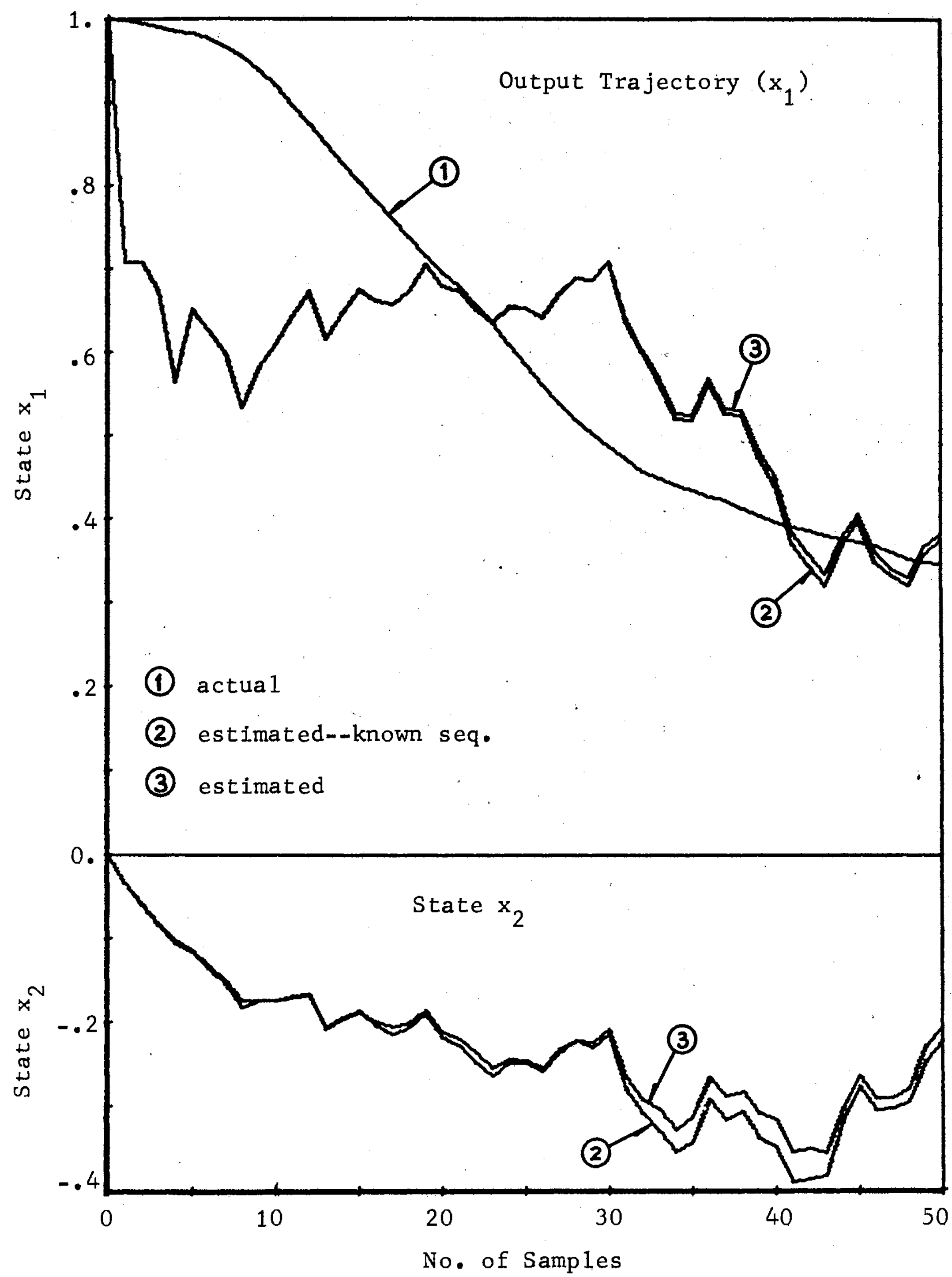


Figure 16. Averaged Performance for Example 3-3 (Result 2)

3.5 Summary

The formulation of the estimation-under-uncertainty problem as presented in this chapter encompasses many special cases discussed in the previous chapter. The solutions presented in Algorithms 3-1 and 3-2 are therefore quite general. The convergence properties investigated in Section 3.3.4 show that system identification can be achieved whenever one of the candidate models is the true system. In practice, however, one may not be assured that a set of candidate models includes the true model. In a case where the active model may not be included, computation of the time average residuals as suggested in Section 3.3.5 may be included as a part of the algorithm. The comparison of residuals of models, in turn, may indicate whether or not the selection of models is good.

In some cases where the noise disturbances are colored or correlated, the algorithms presented may be modified to suit the situation. In the case of colored plant noise, the algorithms are applied without modification, except that the order of the state vector of interest is increased. In the correlated noise cases, the predictor equations of each candidate models have to be modified. The expressions are available in references[17], [27], and [28].

System identification is an important application of Algorithm 3-1. In physical systems the problem of finding a mathematical model which describes the processes is often encountered. If the stochastic approach is adopted, it is seen that the estimation algorithm presented may serve as a system identification procedure. A most probable model or a set of probable models may be obtained from a set or many sets of data. This application feature is illustrated in Chapter V.

There is an aspect of digital computation which the author has noted. The finite word length of a computer imposes computational errors. Once one of the posterior probabilities approaches 1 within the accuracy of a fixed word length, the identification algorithm fails to function. That is, if the posterior probability of a wrong model approaches 1 due to an abnormal noise sequence and computational errors, the additional data will not effect the identification algorithm so that the true model is eventually identified, as indicated by the convergence theorem presented in section 3.3.4. The indicated difficulty can be avoided by perturbing the posterior probability from its unity value and testing to see whether it will return to a value of unity, given more data.

CHAPTER IV

STOCHASTIC LINEAR REGULATOR PROBLEMS

4.1 Introduction

The stochastic linear regulator problem with uncertain dynamics is examined in this chapter. Two classes of regulator problems are considered. In the nonswitching case, where the order of the candidate models may be different, the "output" regulator[43] problem is investigated. In the switching case, the order of each system is the same, and the "state" regulator[43] problem is considered.

In Section 4.2, the control strategy for the output regulator problem with certain parameters is obtained for the continuous-discrete model. In subsequent sections, the nonswitching and switching cases subject to uncertainty are examined.

4.2 The Stochastic Control Problem--Continuous-Discrete Systems

4.2.1 The Problem Statement

The dynamics of the system to be controlled are governed by the linear stochastic differential equation

$$\dot{x}(t) = F(t)x(t) + C(t)u(t) + G(t)w(t) \tag{4-1}$$

where x is an n vector which represents the state of the system, u is an r dimensioned control vector, and w is a disturbance vector whose q elements are represented by zero-mean white noise. The observations are taken at discrete instants of time

$$z(k) = y(t_k) + v(k) \tag{4-2}$$

where

$$y(t) = H(t)x(t) \tag{4-3}$$

is the m-dimensioned output, and v is a zero mean white noise. The noise terms are assumed uncorrelated with covariance matrices

$$E\{w(t)w^T(\tau)\} = Q(t)\delta(t-\tau)$$

$$E\{v(k)v^T(j)\} = R(k)\delta_{kj}$$

It is assumed that the mean and the covariance of the initial conditions of the state are known

$$E\{x(t_o)\} = \hat{x}(t_o)$$

$$VAR\{x(t_o)\} = V_x(t_o)$$

The problem is finding the control u(t) based on observations such that a quadratic performance measure

$$J = E\{y^T(t_f)\Lambda y(t_f) + \int_{t_o}^{t_f} [y^T(t)A(t)y(t) + u^T(t)B(t)u(t)]dt\} \tag{4-4}$$

is minimized. Where Λ, A(t), B(t) are weighting matrices. The expectation is taken over the joint probability distribution space of w(t) v(k), and $x(t_o)$. This is equivalent to taking the expectation over the joint distribution space of x(t) and Z_k.

In deriving the solution, a technique is used which is well suited to the class of problems being considered, and convenient to apply. First the system model is discretized so that dynamic programming can be employed. Then the continuous time solution is obtained by considering the limit of the discrete solution as the time increment approaches zero.

4.2.2 A Discretized Formulation

The dynamic programming scheme is used to find the control u(t) in a manner similar to that used in references [27,28]. The discretized form of Eq. 4-1 is

$$x(t + \Delta t) = \Phi(t + \Delta t,t)x(t) + \psi(t + \Delta t,t)u(t) + \Gamma(t + \Delta t,t)w(t) \tag{4-5}$$

where

$$\Phi(t + \Delta t,t) = I + F(t)\Delta t + O(\Delta t^2) \tag{4-6}$$

$$\psi(t + \Delta t,t) = C(t)\Delta t + O(\Delta t^2) \tag{4-7}$$

$$\Gamma(t + \Delta t,t) = G(t)\Delta t + O(\Delta t^2) \tag{4-8}$$

and t is the discrete time index $\{t = t_o + j\Delta t, j = 0,1,\ldots\}$ with $\Delta t > 0$, and $O(\Delta t^2)$ indicates the collection of terms involving Δt with power of two and higher. In Eqs. 4-6 through 4-8 only first order terms in Δt are retained. The plant noise $\{w(t), t = t_o + j\Delta t, j = 0,1, \ldots\}$ is zero mean white gaussian with covariance matrix

$$E\{w(t)w^T(\tau)\} = \frac{Q(t)}{\Delta t}\,\delta_{jk} \tag{4-9}$$

where τ is the discrete time index $\{\tau = t_o + k\Delta t, k = 0,1,\ldots\}$.

Utilizing the partitioning as above, the performance measure Eq. 4-4 may be rewritten as

$$J = E\{y^T(t_f)\Lambda y(t_f) + \sum_{i=0}^{\xi-1} [y^T(t_o + i\Delta t)A(t_o + i\Delta t)y(t_o + i\Delta t)$$

$$+ u^T(t_o + i\Delta t)B(t_o + i\Delta t)u(t_o + i\Delta t)]\Delta t\} \qquad (4\text{-}10)$$

where the values of y(t) and u(t) are defined at the left end points of the partitions, and ξ is the number of intervals of interest.

At any time instant t, the output signals $Z_k = \{z(1), z(2), \ldots, z(k)\}$ have been observed. Equation 4-10 may be split in two parts

$$J = E\{\sum_{i=0}^{\eta-1} [y^T(t_o + i\Delta t)A(t_o + i\Delta t)y(t_o + i\Delta t)$$

$$+ u^T(t_o + i\Delta t)B(t_o + i\Delta t)u(t_o + i\Delta t)]\Delta t\}$$

$$+ E\{\sum_{i=\eta}^{\xi-1} [y^T(t_o + i\Delta t)A(t_o + i\Delta t)y(t_o + i\Delta t)$$

$$+ u^T(t_o + i\Delta t)B(t_o + i\Delta t)u(t_o + i\Delta t)]\Delta t + y^T(t_f)\Lambda y(t_f)\}$$

$$(4\text{-}11)$$

where $\eta\Delta t = t - t_o$. Using the principle of optimality [27], the problem of minimizing J is reduced to minimize the second term of J at time t, for all $t = \eta\Delta t$ in the control interval, i.e.,

$$\min_u J = \min_u J_1$$

$$= \min_u E\{y^T(t_f)\Lambda y(t_f) + \sum_{i=\eta}^{\xi-1} [y^T(t_o + i\Delta t)A(t_o + i\Delta t)y(t_o + i\Delta t)$$

$$+ u^T(t_o + i\Delta t)B(t_o + i\Delta t)u(t_o + i\Delta t)]\Delta t\} \qquad (4\text{-}12)$$

It is noted that the expectation is taken over the joint probability space of y(t) and Z_k, Equation 4-12 may be expressed as [28]

$$\min_u J_1 = E\{\min_u E\{y^T \Lambda y + \sum [y^T A y + u^T B u]\Delta t / Z_k\}\} \qquad (4\text{-}13)$$

where the time argument for variables are dropped for simplicity. The inner expectation in Eq. 4-13 is the conditional expectation given Z_k, and the minima are taken with respect to all strategies which express u(t) as a function of Z_k. Examining the cases for single stage, double stage, and multiple stage backward from time t_f [27], or for $t = t_f - \Delta t$, $t_f - 2\Delta t, \ldots$, under the assumption that all minima exist and unique, it is seen that

$$\min_{u(t),\ldots,u(t_f-\Delta t)} E\{y^T \Lambda y + \sum [y^T A y + u^T B u]\Delta t\} = EV(Z_k, t)$$

where the minima are taken with respect to all admissible control strategies, and the function V is defined as

$$V(Z_k, t) = \min_u E\{y^T \Lambda y + \sum [y^T A y + u^T B u]\Delta t / Z_k\} \qquad (4\text{-}14)$$

and satisfies the following functional equation

$$V(Z_k, t) = \min_u E[y^T(t)A(t)y(t) + u^T(t)B(t)u(t)]\Delta t + V(Z_k', t+\Delta t)/Z_k\} \qquad (4\text{-}15)$$

which is the Bellman equation [44], and where Z_k' depends on whether or not an additional observation is obtained at time t, i.e.,

$Z_k' = Z_k$ if there is no new measurement

and (4-16)

$Z_k' = Z_{k+1}$ if there is a new measurement

It is a complicated equation for the reason that the dimension of Z_k increases with the number of observations obtained. Some simplifications should be done such that Eq. 4-15 is easy to solve. The case where t falls in between observations is considered first, and the case of t occurring at an observation instant is then developed.

(1) Between observations: In the time interval $t_k \leqslant t < t_{k+1}$, the observed data available is Z_k. From the estimation algorithm developed, it is seen that the conditional distribution of the output $y(t + \Delta t)$ given Z_k is uniquely determined by the conditional distribution of $x(t + \Delta t)$ given Z_k, i.e.,

$$E\{y(t + \Delta t)/Z_k\} = H(t + \Delta t)E\{x(t + \Delta t)/Z_k\}$$

or

$$\hat{y}(t + \Delta t/Z_k) = H(t + \Delta t)\hat{x}(t + \Delta t/Z_k);$$

and

$$\begin{aligned} V_y(t + \Delta t/Z_k) &= \text{VAR}\{y(t + \Delta t)/Z_k\} \\ &= H(t + \Delta t)\text{VAR}\{x(t + \Delta t)/Z_k\}H^T(t + \Delta t) \\ &= H(t + \Delta t)V_x(t + \Delta t/Z_k)H^T(t + \Delta t) \end{aligned}$$

In view of the fact that the statistics of $x(t)$ given Z_k are the sufficient statistics for the normal conditional distribution, of $y(t)$ given Z_k, it is convenient to introduce the notation

$$\begin{aligned} W(\hat{x}(t/Z_k),t) &= V(Z_k,t) \\ &= \min_u E\{y^T\Lambda y + \sum [y^T A y + u^T B u]\Delta t/\hat{x}(t/Z_k)\} \end{aligned} \quad (4\text{-}17)$$

for $t_k \leq t < t_{k+1}$. Here the conditional expectation is understood to be with respect to both $\hat{x}(t/Z_k)$ and $V_x(t/Z_k)$, but since V_x is non-random and can be precomputed, only $\hat{x}$ is indicated as a sufficient statistic. In this way, Eq. 4-15 may be rewritten as

$$W(\hat{x}(t/Z_k),t) = \min_u E\{[y^T(t)A(t)y(t) + u^T(t)B(t)u(t)]\Delta t + W(\hat{x}(t + \Delta t/Z_k),t)/\hat{x}(t/Z_k)\} \quad (4\text{-}18)$$

This is a considerable simplification because the argument $\hat{x}$ of W is of constant dimension. The terminal condition for W at t_f is

$$\begin{aligned} W(\hat{x}(t_f/Z_\xi),t_f) &= E\{y^T(t_f)\Lambda y(t_f)/\hat{x}(t_f/Z_\xi)\} \\ &= \hat{y}^T(t_f/Z_\xi)\Lambda\hat{y}(t_f/Z_\xi) + t_f\Lambda V_y(t_f/Z_\xi) \end{aligned} \quad (4\text{-}19)$$

where Z_ξ indicates the data available before the time t_f. The form of the solution of the Bellman equation, Eq. 4-18, is assumed to be [44,28]

$$W(\hat{x}(t/Z_k),t) = \hat{y}^T(t/Z_k)S(t)\hat{y}(t/Z_k) + \varphi(t) \quad (4\text{-}20)$$

with initial condition specified by Eq. 4-19. From optimal filtering result for discrete stochastic system [27], it follows that

$$\hat{x}(t + \Delta t/Z_k) = \Phi(t + \Delta t,t)\hat{x}(t/Z_k) + \psi(t + \Delta t,t)u(t) \quad (4\text{-}21)$$

and

$$V_x(t+\Delta t/Z_k) = \Phi(t+\Delta t,t)V_x(t/Z_k)\Phi^T(t+\Delta t,t) + \Gamma(t+\Delta t,t)Q(t)\Gamma^T(t+\Delta t,t) \quad (4\text{-}22)$$

Consequently, the expectation of $\hat{y}(t+\Delta t/Z_k)$ given $\hat{x}(t/Z_k)$ is

$$E\{\hat{y}(t+\Delta t/Z_k)/\hat{x}(t/Z_k)\} = \hat{y}(t+\Delta t/Z_k)$$

$$= H(t+\Delta t)[\Phi(t+\Delta t,t)\hat{x}(t/Z_k) + \psi(t+\Delta t,t)u(t)] \tag{4-23}$$

and variance is

$$VAR\{\hat{y}(t+\Delta t/Z_k)/\hat{x}(t/Z_k)\} = 0 \tag{4-24}$$

Equation 4-18 may be evaluated as

$$W(\hat{x},t) = \min_u \{[\hat{y}^T(t/Z_k)A(t)\hat{y}(t/Z_k) + t_r\, A(t)V_y(t/Z_k)$$

$$+ u^T(t)B(t)u(t)]\Delta t + [\hat{y}^T(t+\Delta t/Z_k)S(t+\Delta t)\hat{y}(t+\Delta t/Z_k) + \varphi(t+\Delta t)]\}$$

$$= \min_u \{[\hat{x}^T H^T A H\hat{x} + t_r A V_y + u^T B u]\Delta t$$

$$+ [(\Phi\hat{x} + u)^T H^T S(t+\Delta t)H(\Phi\hat{x} + u) + \varphi(t+\Delta t)]\} \tag{4-25}$$

By expanding the second term of Eq. 4-25 collecting terms of $\hat{x}$ and u together, and completing a quadratic form, the following expression is obtained

$$W(\hat{x},t) = \min_u \{\hat{x}^T[H^T A H\Delta t + \Phi^T H^T S(t+\Delta t)H\Phi]\hat{x}$$

$$- \hat{x}^T L^T(t)[B\Delta t + \psi^T H^T S(t+\Delta t)H\psi]L(t)\hat{x}$$

$$+ (u+L\hat{x})^T[B\Delta t + \psi^T H^T S(t+\Delta t)H\psi](u+L\hat{x}) + \varphi(t+\Delta t) + t_r A V_y \Delta t\} \tag{4-26}$$

where

$$L(t) = [B\Delta t + \psi^T H^T S(t+\Delta t)H\psi]^{-1}\psi H^T S(t+\Delta t)H\Phi \tag{4-27}$$

The control which achieves the minimum value is the given by

$$u(t) = -L(t)\hat{x}(t/Z_k) \tag{4-28}$$

Equating both sides of Eq. 4-26 and using the expression in Eq. 4-21 for $W(\hat{x},t)$, the equations for S and φ are obtained.

$$H^T S(t)H = H^T AH\Delta t + \Phi^T H^T S(t+\Delta t)H\Phi - L^T[B\Delta t + \psi^T H^T S(t+\Delta t)H\psi]L \tag{4-29}$$

and

$$\varphi(t) = \varphi(t+\Delta t) + t_r AV_y \Delta t \tag{4-30}$$

It is convenient to define

$$\overline{S}(t) = H^T S(t)H \tag{4-31}$$

then Eq. 4-30 may be rewritten as

$$\overline{S}(t) = H^T AH\Delta t + \Phi^T \overline{S}(t+\Delta t)\Phi - L^T[B\Delta t + \psi^T \overline{S}(t+\Delta t)\psi]L \tag{4-32}$$

The control strategy in between observations is given by Eq. 4-28, where the control gain L(t) depends on the solution of Eq. 4-29 or Eq. 4-32.

(2) At the observation instant t_{k+1}, the Bellman equation may be expressed as

$$\begin{aligned} W(\hat{x}(t_{k+1}/Z_k),t_{k+1}) = \min_u E\{&y^T(t_{k+1})A(t_{k+1})y(t_{k+1}) \\ &+ u^T(t_{k+1})B(t_{k+1})u(t_{k+1})]\Delta t \\ &+ W(\hat{x}(t_{k+1}+\Delta t/Z_{k+1}),t_{k+1}+\Delta t)/\hat{x}(t_{k+1}/Z_k)\} \end{aligned} \tag{4-33}$$

After an observation is taken, the state estimate is updated using the corrector equation

$$\hat{x}(t_{k+1}/Z_{k+1}) = \hat{x}(t_{k+1}/Z_k) + K(t_{k+1}) \cdot [z(k+1) - H(t_{k+1})\hat{x}(t_{k+1}/Z_k)] \quad (4\text{-}34)$$

and

$$V_x(t_{k+1}/Z_{k+1}) = V_x(t_{k+1}/Z_k) - K(t_{k+1})H(t_{k+1})V_x(t_{k+1}/Z_k) \quad (4\text{-}35)$$

where the distribution of the term $z - H\hat{x}$ is normal with zero mean and covariance matrix $[H^T V_x(t_{k+1}/Z_k)H + R(k+1)]$. The expection of $\hat{y}(t_{k+1} + \Delta t/Z_{k+1})$ given $\hat{x}(t_{k+1}/Z_k)$ is

$$E\{\hat{y}(t_{k+1} + \Delta t/Z_{k+1})/\hat{x}(t_{k+1}/Z_k)\}$$
$$= H(t_{k+1})[\Phi(t_{k+1} + \Delta t, t_{k+1})\hat{x}(t_{k+1}/Z_k) + \psi(t_{k+1} + \Delta t, t_{k+1})u(t_{k+1})] \quad (4\text{-}36)$$

which is similar to Eq. 4-24, and the covariance matrix is

$$VAR\{\hat{y}(t_{k+1} + \Delta t/Z_{k+1})/\hat{x}(t_{k+1}/Z_k)\}$$
$$= H(t_{k+1})K(t_{k+1})[HV_x(t_{k+1}/Z_k)H^T + R]K^T H^T \quad (4\text{-}37)$$

The expression, Eq. 4-33, may be rewritten as

$$W(\hat{x}, t_{k+1}) = \min_u \{[\hat{x}^T(t_{k+1}/Z_k)H^T AH\hat{x}(t_{k+1}/Z_k) + u^T(t_{k+1})B(t_{k+1}) \cdot u(t_{k+1})]\Delta t + [(\Phi\hat{x} + \psi u)^T H^T S(t_{k+1} + \Delta t)H(\Phi\hat{x} + \psi u) + t_r S(t+\Delta t)HK[HV_x H^T + R]K^T H^T + \varphi(t+\Delta t)] \quad (4\text{-}38)$$

It can be noted that Eq. 4-38 differs from Eq. 4-25 by an extra term $t_r S(t+\Delta t)HK[HV_x H^T + R]K^T H^T$, which only effects φ. The control strategy is then indicated by Eq. 4-29 and the equation for S or $\bar{S}$ remains the same as indicated in Eq. 4-29 or Eq. 4-32. The equation for φ is

$$\varphi(t) = \varphi(t+\Delta t) + t_r AV_y \Delta t + t_r S(t+\Delta t)HK[HV_x H^T + R]K^T H^T$$

$$= \varphi(t+\Delta t) + t_r AV_y \Delta t + t_r \bar{S}(t+\Delta t)K[HV_x H^T + R]K^T \qquad (4\text{-}39)$$

4.2.3 The Continuous Solution

If one applies a limiting procedure, letting $\Delta t \to 0$ in Eqs. 4-27, 4-30, and 4-32, and using the expressions for Φ and ψ from Eqs. 4-6 and 4-7, the continuous form for the control gain L(t) is obtained.

$$L(t) = \lim_{\Delta t \to 0} [B\Delta t + C^T\bar{S}(t+\Delta t)C(\Delta t)^2]^{-1} C^T\bar{S}(t+\Delta t)\Delta t[I + F\Delta t]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} [B + C^T\bar{S}(t+\Delta t)C\Delta t]^{-1} C^T\bar{S}(t+\Delta t)[I + F\Delta t]\Delta t$$

$$= B^{-1}(t)C^T(t)\bar{S}(t) \qquad (4\text{-}40)$$

The limiting form of the equation for $\bar{S}$ is

$$\bar{S}(t) = H^T AH\Delta t + (I + F\Delta t)^T\bar{S}(t+\Delta t)(I + F\Delta t)$$

$$- L^T[B\Delta t + C^T\bar{S}(t+\Delta t)C(\Delta t)^2]L$$

or

$$\frac{\bar{S}(t) - \bar{S}(t+\Delta t)}{\Delta t} = F^T\bar{S}(t+\Delta t) + \bar{S}(t+\Delta t)F + F^T\bar{S}(t+\Delta t)F\Delta t$$

$$- L^T[B + C^T S(t+\Delta t)C\Delta t]L + H^T AH$$

letting $\Delta t \to 0$,

$$\dot{\bar{S}}(t) = -F^T\bar{S}(t) - \bar{S}(t)F + L^T BL - H^T AH$$

$$= -F^T\bar{S} - \bar{S}F + \bar{S}CB^{-1}C^T\bar{S} - H^T AH \qquad (4\text{-}41)$$

The same process may be applied to the equation for φ, resulting in the expression

$$\dot{\varphi}(t) = -t_r AV_y \qquad (4\text{-}42)$$

At an observation instant t_k, φ is updated according to the relationship

$$\varphi(t_k^+) = \varphi(t_k^-) + t_k\bar{S}(t_k)K(t_k)[HV_x(t_k/Z_{k-1})H^T + R(k)]K(t_k) \qquad (4\text{-}43)$$

where "+" and "-" indicate the quantity of φ after and before a measurement is obtained. The boundary conditions for Eqs. 4-41 and 4-42 are

$$\bar{S}(t_f) = H^T\Lambda H \qquad \text{and } \varphi(t_f) = t_r\Lambda V_y(t_f/Z_\xi)$$

The expected performance measure is the expected value of W at time t_o. It may be evaluated from the equation

$$\min J = E\{W(\hat{x}(t_o),t_o)\} = E\{\hat{y}^T(t_o)S(t_o)\hat{y}(t_o) + \varphi(t_o)\}$$

$$= \hat{x}^T(t_o)\bar{S}(t_o)\hat{x}(t_o) + \sum_{k=1}^{\xi} [t_r\bar{S}(t_k)K[HV_xH^T + R]K^T$$

$$+ \int_{t_o}^{t_f} t_r AV_y dt + t_r\Lambda V_y\} \qquad (4\text{-}44)$$

The foregoing derivation is summarized below.

ALGORITHM 4-1:

The solution of the optimal control problem for the continuous time system with discrete time observations is governed by the following equations which describe the control strategy.

$$u(t) = -L(t)\hat{x}(t/Z_k) \tag{4-39}$$

where

$$L(t) = B^{-1}(t)C^T(t)\bar{S}(t) \tag{4-40}$$

$$\dot{\bar{S}}(t) = -F^T\bar{S} - \bar{S}F + \bar{S}CB^{-1}C^T\bar{S} - H^TAH \qquad \bar{S}(t_f) = H^T\Lambda H \tag{4-41}$$

Between observations, the conditional mean estimate $\hat{x}(t/Z_k)$ is the solution to

$$\dot{\hat{x}}(t/Z_k) = F(t)\hat{x}(t/Z_k) + C(t)u(t) \tag{4-45}$$

The associated covariance matrix is obtained by solving the equation

$$\dot{V}_x(t/Z_k) = F(t)V_x(t/Z_k) + V_x(t/Z_k)F(t) + G(t)Q(t)G^T(t) \tag{4-46}$$

At observation instant t_{k+1}, the mean and covariance are updated

$$\hat{x}(t_{k+1}/Z_{k+1}) = \hat{x}(t_{k+1}/Z_k) + K(t_{k+1})[z(k+1) - H\hat{x}(t_{k+1}/Z_k)] \tag{4-47}$$

and

$$V_x(t_{k+1}/Z_{k+1}) = V_x(t_{k+1}/Z_k) - K(t_{k+1})HV_x(t_{k+1}/Z_k) \tag{4-48}$$

where

$$K(t_{k+1}) = V_x(t_{k+1}/Z_k)H^T[HV_xH^T + R]^{-1} \tag{4-49}$$

Equations 4-45 through 4-49 comprise the estimation algorithm for continuous-discrete system presented in Chapter II. The minimal expected performance measure is indicated in Eq. 4-44. The algorithm is schematically indicated in Fig. 17.

It is noted that the separation theorem is applicable in the continuous-discrete formulation of the stochastic linear regulator problems. The control strategy for the case of model uncertainty is derived in the following sections, utilizing results developed in this section. The cases of nonswitching and switching are treated separately as in the previous chapter.

4.3 The Control Problem Under Uncertainty--Nonswitching Case

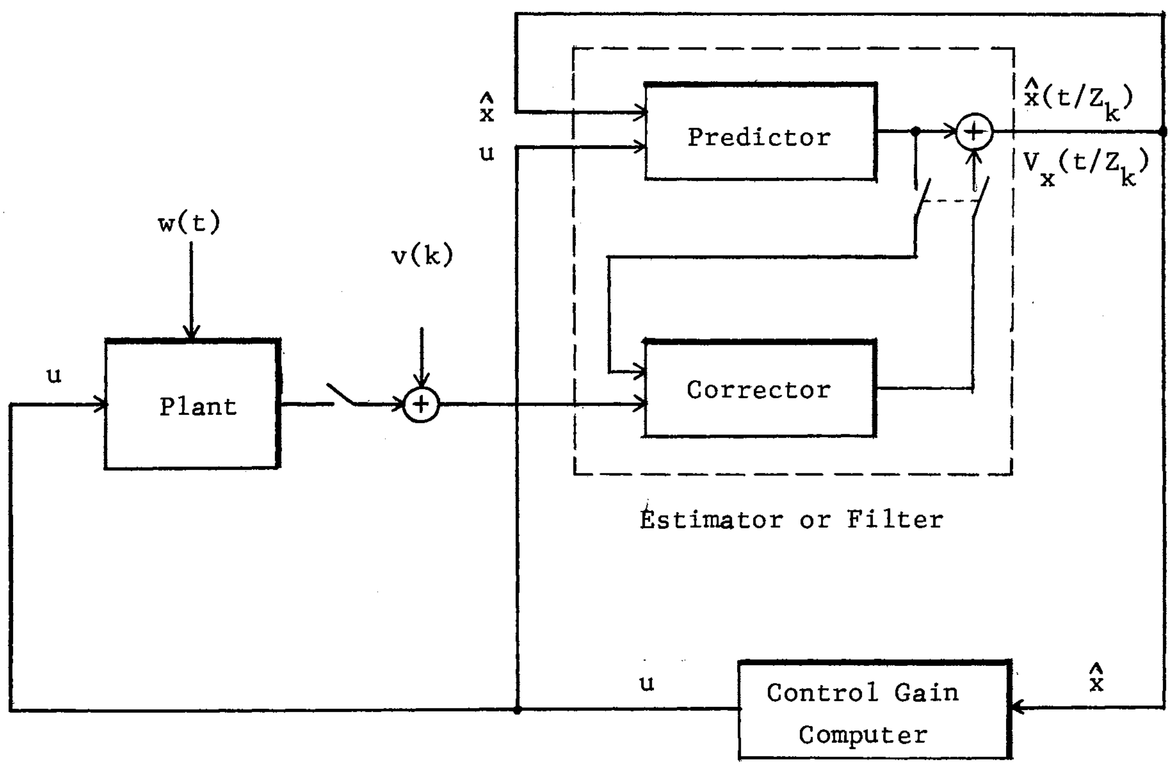
The system dynamics and observation model are as indicated in Eqs. 4-1 through 4-3, except that the model is subject to uncertainty. The control u(t) is to be found such that a performance measure J is minimized, where J is given by Eq. 4-4.

As in the previous chapter the uncertainties are treated by assuming that there are N possible candidate models. Each candidate model has the form

$$\theta_i: \quad \dot{x}_i(t) = F_i(t)x_i(t) + C_i(t)u(t) + G_i(t)w_i(t) \qquad (4\text{-}50)$$

for $i = 1,2,\ldots,N$. The control u(t) is the same for all models.

The expectation in the performance measure in Eq. 4-4 is now evaluated over the joint probability space of x(t), Z_k, and θ_i. Dynamic programming is used to obtain the control strategy, as in the previous section.



Predictor: Eq. (3-25) and Eq. (3-26)

Corrector: Eq. (3-37) and Eq. (3-38)

Control Gain: Eq. (4-37) and Eq. (4-35)

Figure 17. Block Diagram for Optimal Linear Combined Estimation and Control System

4.3.1 Discretized Formulation

It is convenient to consider the discretized form of the problem first. The performance measure stated in Eq. 4-12 may be rewritten as

$$\min J_1 = \min_u E\{y^T(t_f)\Lambda y(t_f) + \sum_{j=\eta}^{\xi-1} [y^T(t_o + j\Delta t)A(t_o + j\Delta t)y(t_o + j\Delta t)$$

$$+ u^T(t_o + j\Delta t)B(t_o + j\Delta t)u(t_o + j\Delta t)]\Delta t\} \qquad (4\text{-}51)$$

where the minima is taken with respect to all admissible control strategies which give u(t) as a function of Z_k. In a manner similar to that of the previous section, one may define $V(Z_k,t)$ as in Eq. 4-14, which satisfies the functional equation 4-15. Equation 4-15 may be reduced further to take into account the probability space of the model index, θ_i.

$$V(Z_k,t) = \min_u \sum_{i=1}^{N} p_r(\theta_i/Z_k)E\{[y^T(t)A(t)y(t) + u^T(t)B(t)u(t)]\Delta t$$

$$+ V(Z_k', t+\Delta t)/Z_k,\theta_i\} \qquad (4\text{-}52)$$

where Z_k' is defined by Eq. 4-16. Here the problem of the growing dimensionality of the measurement record Z_k still exists. The simplifications for time between observations and at an observation instant are considered separately.

4.3.2 In Between Observations

In the time interval $t_k \leqslant t < t_{k+1}$, the measurement record Z_k' in Eq. 4-52 is simply Z_k. Defining auxiliary variables

$$V_i(Z_k,t) = E\{y^T(t_f)\Lambda y(t_f) + \sum_{j=\eta}^{\xi-1} [y^T(t_o+j\Delta t)A(t_o+j\Delta t)y(t_o+j\Delta t)$$

$$+ u^T(t_o+j\Delta t)B(t_o+j\Delta t)u(t_o+j\Delta t)]\Delta t/Z_k,\theta_i\}$$

$$= E\{y^T(t_f)\Lambda y(t_f) + \sum[y_i^T A y_i + u^T B u]\Delta t/Z_k\} \qquad (4\text{-}53)$$

where u(t) is not dependent on i, one obtains

$$V(Z_k,t) = \min_u \sum_{i=1}^{N} p_r(\theta_i/Z_k)V_i(Z_k,t) \qquad (4\text{-}54)$$

Given the model θ_i, $\hat{x}_i(t/Z_k)$ and $V_{x_i}(t/Z_k)$ are sufficient statistics for the conditional distribution $p(y_i(t)/Z_k)$. Thus, one may define an auxiliary variable

$$W_i(\hat{x}_i,t) = V_i(Z_k,t)$$

$$= E\{y_i^T(t_f)\Lambda y_i(t_f) + \sum[y_i^T A y_i + u^T B u]\Delta t/\hat{x}_i\} \qquad (4\text{-}55)$$

It is seen that if $p_r(\theta_i/Z_k) = I_a$, where I_a is defined by Eq. 3-29, the problem reduces to that of the previous section under certain conditions. For a given model θ_i, $W_i(\hat{x}_i,t)$ satisfies the following Bellman equation

$$W_i(\hat{x}_i,t) = \min E\{[y_i^T(t)A(t)y_i(t) + u^T(t)B(t)u(t)]\Delta t$$

$$+ W_i(\hat{x}_i,t+\Delta t)/\hat{x}_i\} \qquad (4\text{-}56)$$

The solution to Eq. 4-56 is assumed, as in the previous section, to be of the form

$$W_i(\hat{x}_i,t) = \hat{y}_i^T(t)S(t)\hat{y}_i(t) + \varphi_i(t) \qquad (4\text{-}57)$$

Using the relationship indicated in Eq. 4-56, Eq. 4-52 may be rewritten as

$$\sum_{i=1}^{N} P_r(\theta_i/Z_k)W_i(\hat{x}_i,t) = \min_u \sum_{i=1}^{N} P_r(\theta_i/Z_k)E\{[y_i^T(t)A(t)y_i(t) + u^T(t)B(t)u(t)]\Delta t + W_i(\hat{x}_i,t+\Delta t)/\hat{x}_i\} \tag{4-58}$$

by substituting the solution of $W_i(\hat{x}_i,t+\Delta t)$ from Eq. 4-57 into Eq. 4-58, one obtains,

$$\begin{aligned}\sum_{i=1}^{N} P_r(\theta_i/Z_k)W_i(\hat{x}_i,t) &= \min_u \sum_{i=1}^{N} P_r(\theta_i/Z_k)\{[\hat{y}_i^T A\hat{y}_i + t_r AV_{yi} + u^T Bu]\Delta t \\ &\quad + \hat{y}_i^T(t+\Delta t)S(t+\Delta t)\hat{y}_i(t+\Delta t) + \varphi_i(t+\Delta t)\} \\ &= \min_u \sum_{i=1}^{N} P_r(\theta_i/Z_k)\{[\hat{x}_i^T H_i^T AH_i\hat{x}_i + t_r AV_{y_i} + u^T Bu]\Delta t \\ &\quad + \hat{x}_i^T(t+\Delta t)H_i^T S(t+\Delta t)H_i\hat{x}_i(t+\Delta t) + \varphi_i(t+\Delta t)\}\end{aligned} \tag{4-59}$$

From the optimal filtering results indicated in Eqs. 4-21 and 4-22, one can see that Eq. 4-59 reduces to

$$\begin{aligned}\sum_{i=1}^{N} P_r(\theta_i/Z_k)W_i(\hat{x}_i,t) &= \min_u \sum_{i=1}^{N} P_r(\theta_i/Z_k)[(\hat{x}_i^T H_i^T AH_i\hat{x}_i + t_r AV_{yi} + u^T Bu)\Delta t \\ &\quad + (\Phi_i\hat{x}_i + \psi_i u)^T H_i^T S(t+\Delta t)H_i(\Phi_i\hat{x}_i + \psi_i u) \\ &\quad + \varphi_i(t+\Delta t)]\end{aligned} \tag{4-60}$$

By collecting terms, the following quadratic form results,

$$\sum_{i=1}^{N} P_r(\theta_i/Z_k) W_i(\hat{x}_i,t) = \min_u \sum_{i=1}^{N} P_r(\theta_i/Z_k)[\hat{x}_i^T(H_i^T A H_i \Delta t + \Phi_i^T H_i^T S(t+\Delta t) H_i \Phi_i)\hat{x}_i] - [\sum_{j=1}^{N} P_r(\theta_j/Z_k) L_j \hat{x}_j]^T$$

$$\cdot [B\Delta t + \sum_{j=1}^{N} P_r(\theta_j/Z_k)\psi_j^T H_j^T S(t+\Delta t) H_j \psi_j]$$

$$\cdot [\sum_{j=1}^{N} P_r(\theta_j/Z_k) L_j \hat{x}_j] + (u + \sum_{j=1}^{N} P_r(\theta_j/Z_k) L_j \hat{x}_j)^T$$

$$\cdot [B\Delta t + \sum_{j=1}^{N} P_r(\theta_j/Z_k)\psi_j^T H_j^T S(t+\Delta t) H_j \psi_j]$$

$$\cdot (u + \sum_{j=1}^{N} P_r(\theta_j/Z_k) L_j \hat{x}_j) + \sum_{i=1}^{N} P_r(\theta_i/Z_k)$$

$$\cdot [t_r A V_{yi} \Delta t + \varphi_i(t+\Delta t)] \qquad (4\text{-}61)$$

where

$$L_i(t) = [B\Delta t + \sum_{j=1}^{N} P_r(\theta_j/Z_k)\psi_j^T H_j^T S(t+\Delta t) H_j \psi_j]^{-1} \psi_i^T H_j^T S(t+\Delta t) H_i \Phi_i \qquad (4\text{-}62)$$

The control strategy is then obtained by finding the value of u which minimizes the right hand side of Eq. 4-61.

$$u(t) = - \sum_{i=1}^{N} P_r(\theta_i/Z_k) L_i(t) \hat{x}_i(t/Z_k) \qquad (4\text{-}63)$$

Equation 4-61 then may be rewritten as

$$\sum_{i=1}^{N} P_r(\theta_i/Z_k)(\hat{x}_i^T H_i S H_i \hat{x}_i + \varphi_i)$$

$$= \sum_{i=1}^{N} P_r(\theta_i/Z_k)\hat{x}_i^T[H_i^T A H_i \Delta t + \Phi_i^T H_i^T S(t+\Delta t) H_i \Phi_i]\hat{x}_i$$

$$- (\sum_{j=1}^{N} P_r(\theta_j/Z_k) L_j \hat{x}_j)^T [B\Delta t + \sum_{j=1}^{N} P_r(\theta_j/Z_k)\psi_j^T H_j^T S(t+\Delta t) H_j \psi_j]$$

$$\cdot (\sum_{j=1}^{N} P_r(\theta_j/Z_k) L_j \hat{x}_j) + \sum_{i=1}^{N} P_r(\theta_i/Z_k)[t_r A V_{yi} \Delta t + \varphi_i(t+\Delta t)] \qquad (4\text{-}64)$$

It is seen that the expression above does not simplify, indicating that the form assumed as the solution to the Bellman equation 4-58 is not correct in general, and consequently, the control strategy obtained may not be optimum in all cases. By substracting and adding terms

$$\sum_{i=1}^{N} P_r(\theta_i/Z_k)\hat{x}_i^T L_i^T[B\Delta t + \sum_{j=1}^{N} P_r(\theta_j/Z_k)\psi_i^T H_i^T S(t+\Delta t)H_i\psi_i]L_i\hat{x}_i$$

in Eq. 4-59, one obtains the expression

$$\begin{aligned}
\sum_{i=1}^{N} & P_r(\theta_i/Z_k)(\hat{x}_i^T H_i^T S H_i \hat{x}_i + \varphi_i) \\
&= \sum_{i=1}^{N} P_r(\theta_i/Z_k)\hat{x}_i[H_i^T A H_i \Delta t + \Phi_i^T H_i^T S(t+\Delta t)H_i\Phi_i \\
&\quad - L_i^T[B\Delta t + \sum_{j=1}^{N} P_r(\theta_j/Z_k)\psi_j^T H_j^T S(t+\Delta t)H_j\psi_j]L_i]\hat{x}_i + t_r A V_{yi}\Delta t \\
&\quad + \varphi_i(t+\Delta t) + \sum_{i=1}^{N} P_r(\theta_i/Z_k)[(L_i\hat{x}_i)^T \\
&\quad \cdot [B\Delta t + \sum_{j=1}^{N} P_r(\theta_j/Z_k)\psi_j^T H_j^T S(t+\Delta t)H_j\psi_j](L_i\hat{x}_i - \sum_{j=1}^{N} P_r(\theta_j/Z_k)L_j\hat{x}_j)]
\end{aligned} \tag{4-65}$$

The expression in the bracket of the first term of Eq. 4-65 is the same as in Eq. 4-26 given model θ_i with certainty. If

$$u_i(t) = -L_i(t)\hat{x}_i(t/Z_k) \tag{4-66}$$

is the optimal control strategy given that model θ_i is true, the expression in the second term of Eq. 4-65 is then the cost of applying the control u(t) to the model θ_i which differs from $u_i(t)$. It is seen that if $P_r(\theta_i/Z_k) = I_a$, or a true system is identified, the cost due to the second term of Eq. 4-65 is zero. Therefore, the control strategy becomes optimal when a true system is identified. Denoting

$$\tilde{u}_i(t) = u(t) - u_i(t)$$
$$= L_i\hat{x}_i - \sum_{j=1}^{N} P_r(\theta_j/Z_k)L_j\hat{x}_j \qquad (4\text{-}67)$$

as the control error between model θ_i and the actual control, then

$$\lambda(t) = \sum_{i=1}^{N} P_r(\theta_i/Z_k)t_r[B\Delta t + \sum_{j=1}^{N} P_r(\theta_j/Z_k)\psi_j^T H_j^T S(t+\Delta t)H_j\psi_j]$$
$$\cdot [\tilde{u}_i u_i^T] + \lambda(t+\Delta t) \qquad (4\text{-}68)$$

can be evaluated as the extra cost due to uncertainty. Furthermore, having defined

$$S_i(t) = H_i^T S(t)H_i$$

one may obtain expressions for S_i and φ_i

$$S_i(t) = H_i A H_i \Delta t + \Phi_i^T S_i(t+\Delta t)\Phi_i - L_i^T[B\Delta t + \sum_{j=1}^{N} P_r(\theta_j/Z_k)\psi_j^T S_j(t+\Delta t)\psi_j]L_i \qquad (4\text{-}69)$$

$$\varphi_i(t) = t_r A V_{yi}\Delta t + \varphi_i(t+\Delta t) \qquad (4\text{-}70)$$

Letting Δt approaches zero, one may obtain the continuous version of Eqs. 4-62, 4-68, 4-69, and 4-70. The results are indicated below for each candidate model.

$$L_i(t) = B^{-1}(t)C_i^T(t)S_i(t) \qquad (4\text{-}71)$$

$$\dot{\lambda}(t) = -\sum_{i=1}^{N} P_r(\theta_i/Z_k)t_r B[\tilde{u}_i u_i^T] \qquad (4\text{-}72)$$

$$\dot{S}_i(t) = -F_i^T(t)S_i(t) - S_i(t)F_i(t) + S_i C_i B^{-1} C_i S_i - H_i^T A H_i \qquad (4\text{-}73)$$

$$\dot{\varphi}_i(t) = -t_r A V_{y_i} \tag{4-74}$$

for i = 1,2,...,N. The boundary conditions for Eqs. 4-72, 4-73, and 4-74 are

$$S_i(t_f) = H_i \Lambda H_i, \qquad \varphi_i(t_f) = t_r \Lambda V_{y_i}(t_f/Z_\xi)$$

and

$$\lambda(t_f) = 0$$

4.3.3 At Observation Instants

At an observation instant t_{k+1}, the measurement record Z_k' in Eq. 4-52 is Z_{k+1}, and the control $u(t_{k+1})$ is applied before the new measurement is obtained. Using the sufficient statistic $\hat{x}_i(t_{k+1}/Z_k)$ and $V_{x_i}(t_{k+1}/Z_k)$, Eq. 4-58 is rewritten as

$$\begin{aligned}\sum_{i=1}^{N} & P_r(\theta_i/Z_k)W_i(\hat{x}_i(t_{k+1}/Z_k),t_{k+1}) + \lambda(t_{k+1}) \\ &= \min_u \sum_{i=1}^{N} P_r(\theta_i/Z_k)E\{[y_i^T(t_{k+1})A(t_{k+1})y_i(t_{k+1}) \\ &+ u^T(t_{k+1})B(t_{k+1})u(t_{k+1})]\Delta t + W_i(\hat{x}_i(t_{k+1}/Z_{k+1}),t_{k+1}+\Delta t) \\ &/\hat{x}_i(t_{k+1}/Z_k)\} + \lambda(t_{k+1}+\Delta t)\end{aligned} \tag{4-75}$$

Since the filtered estimate of each model is updated according to Eqs. 3-19 and 3-20, Eq. 4-75 may be put in the following form

$$\begin{aligned}\sum_{i=1}^{N} & P_r(\theta_i/Z_k)W_i(\hat{x}_i,t_{k+1}) + \lambda(t_{k+1}) \\ &= \min_u \sum_{i=1}^{N} P_r(\theta_i/Z_k)\{[\hat{y}_i^T A\hat{y}_i + t_r A V_{y_i} + u^T B u]\Delta t \\ &+ \hat{y}_i^T(t_{k+1}+\Delta t)S(t_{k+1}+\Delta t)\hat{y}_i(t_{k+1}+\Delta t) + \varphi_i(t_{k+1}+\Delta t) \\ &+ t_r S(t_{k+1}+\Delta t)H_i K_i[H_i V_{x_i} H_i^T + R]K_i^T H_i^T\} + \lambda(t_{k+1}+\Delta t)\end{aligned} \tag{4-76}$$

Comparing Eqs. 4-76 and 4-59, a result similar to that of Eq. 4-39 is obtained, i.e.,

$$\varphi_i(t_{k+1}) = \varphi_i(t_{k+1}+\Delta t) + t_r AV_{y_i} + t_r S_i(t+\Delta t)K_i \cdot [H_i V_{x_i} H_i^T + R_i]K_i^T \quad (4\text{-}77)$$

and the equation for S_i, Eq. 4-69, remain unchanged.

The overall expected performance measure may be evaluated as

$$J = E\{ \sum_{i=1}^{N} P_r(\theta_i) W_i(\hat{x}_i(t_o), t_o) + \lambda(t_o)$$

$$= \sum_{i=1}^{N} P_r(\theta_i)[\hat{x}_i^T(t_o) S_i(t_o)\hat{x}_i(t_o) + \varphi_i(t_o)] + \lambda(t_o) \quad (4\text{-}78)$$

where

$$\varphi_i(t_o) = \sum_{k=1}^{\xi} \{[t_r S_i(t_k) K_i [H_i V_{x_i} H_i^T + R]K_i^T + \int_{t_{k-1}}^{t_k} t_r AV_{y_i} dt\} \quad (4\text{-}79)$$

and

$$\lambda(t_o) = \sum_{k=0}^{\xi-1} \{ \sum_{i=1}^{N} P_r(\theta_i/Z_k) \int_{t_k}^{t_{k+1}} t_r B\tilde{u}_i u_i^T dt \} \quad (4\text{-}80)$$

Since the posterior probabilities depend on measurements, the cost due to $\lambda(t_o)$ is unpredictable. That is one may not precalculate the cost function associated with the control strategy derived. But as indicated previously, when the posterior probability $P_r(\theta_i/Z_k) = I_a$, the strategy becomes optimal, and one can evaluate the cost before hand. The algorithm is summarized as follows.

<u>ALGORITHM 4-2</u>:

A control strategy (which may be suboptimal) for the problem of continuous-discrete system with model uncertainty is given by

$$u(t) = \sum_{i=1}^{N} P_r(\theta_i/Z_k)u_i(t) \tag{4-63}$$

where

$$u_i(t) = -L_i(t)\hat{x}_i(t/Z_k) \tag{4-66}$$

$$L_i(t) = B^{-1}(t)C_i^T(t)S_i(t) \tag{4-71}$$

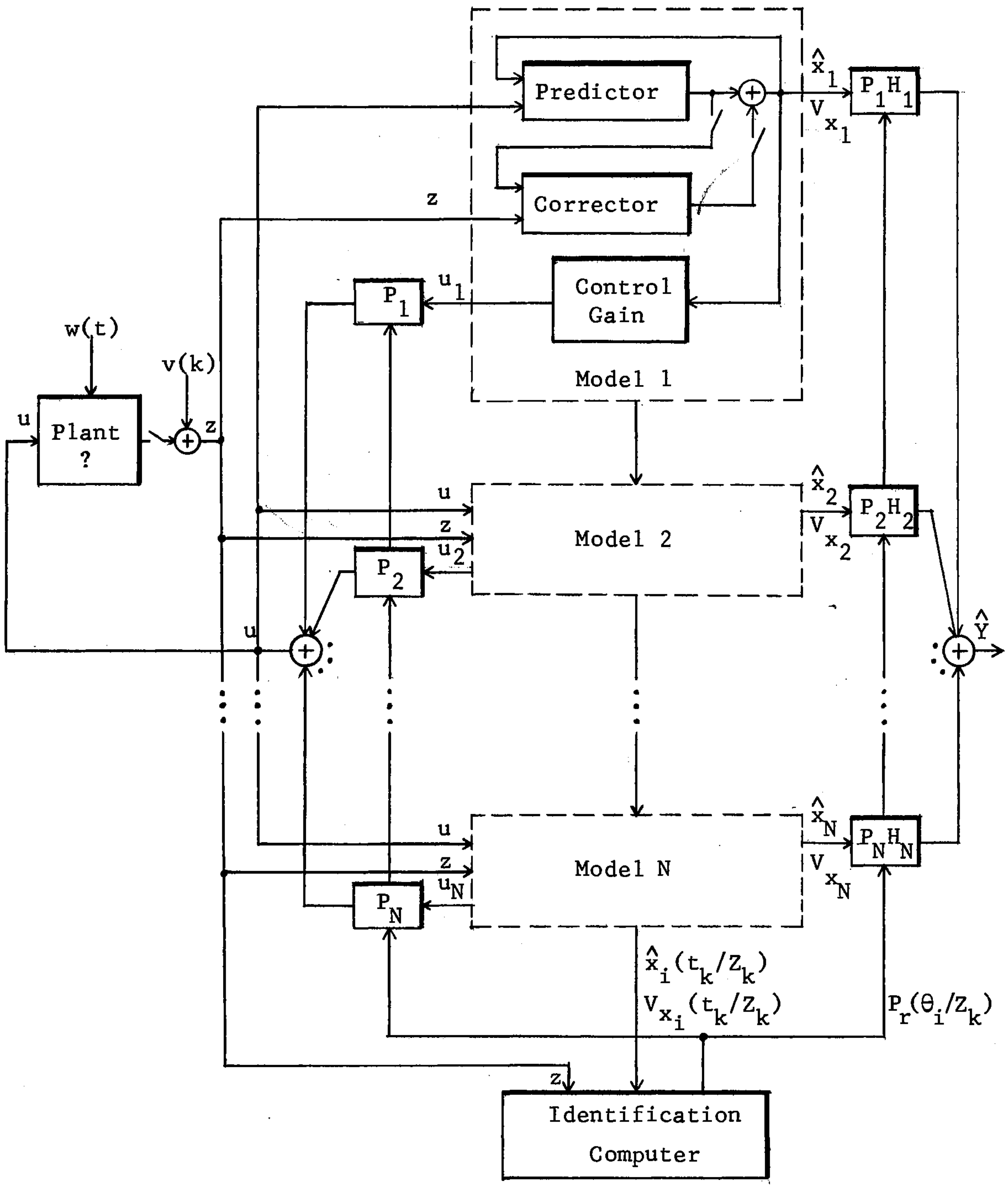
$$\dot{S}_i(t) = -F_i^T S_i - S_i F_i + S_i C_i B + C_i^T S_i - H_i^T \Lambda H_i \qquad S_i(t_f) = H_i^T \Lambda H_i \tag{4-73}$$

for $i = 1,2,\ldots,N$. These last three equations are obtained in Algorithm 4-1, and the conditional mean estimates used in the control algorithm are given in Algorithm 3-1. The block diagram representing Algorithm 4-2 is shown in Fig. 18.

The control algorithm developed in this chapter has been derived in such a way as to make use of the estimation algorithm obtained in Chapter III. Hence a separation principle (separation between control and estimation) has been imposed on the solution. A discrete version of the control strategy suggested here has recently been presented by Lainiotis et al.[46].

4.3.4 Examples

Three examples are presented in this section to demonstrate the control algorithm which has been derived. In order to help in evaluating performance, the results of control under uncertainty are compared with the results under certainty. A typical regulator problem is presented in the first example, where nonzero initial conditions and small disturbance are considered. The case where the system is dominantly driven by noise is examined in the second example. The



Predictor: Eqs. 3-25 and 3-26
Corrector: Eqs. 3-37 and 3-38
Identification: Eq. 3-39
Control Gain : Eqs. 4-63 and 4-65

Figure 18. Block Diagram for Algorithm 4-2

third example illustrates the situation where the true system is not included among the candidate models.

(1) Example 4-1:

A scalar system is described by the formal stochastic differential equation

$$\dot{x}(t) = Fx(t) + u(t) + w(t)$$

where F is a constant which is subject to uncertainty. The observations are taken at discrete intervals of time through a noisy channel

$$z(k) = x(t_k) + v(k)$$

The noise terms, w(t) and v(k), are independent, white, and gaussian with zero mean and covariance given by

$$E\{w(t)w^T(\tau)\} = 1.\delta(t-\tau)$$

and

$$E\{v(k)v^T(j)\} = .5\delta_{kj}$$

The mean and variance of the initial condition of the state are

$$\hat{x}(t_o) = E\{x(t_o)\} = 2.$$

and

$$V_x(t_o) = VAR\{x(t_o)\} = 1.$$

The performance measure to be minimized is

$$J = E\{\tfrac{1}{2}\int_o^1 [x^2(t) + .5u^2(t)]dt\}$$

Two candidate models are proposed:

$$\theta_i\text{:}\quad \dot{x}_1(t) = 4.x_1(t) + u(t) + w(t)$$

$$\theta_i\text{:}\quad \dot{x}_2(t) = x_2(t) + u(t) + w(t)$$

where the model θ_2 represents the true system. The optimal control strategy, given a model to be true, is given by Algorithm 4-1. The deterministic procedure for evaluating the control gains requires solving the Riccati equations

$$\theta_1: \quad \dot{S}_1(t) = -8S_1(t) + 2S_1^2(t) - 1 \qquad S_1(t_f) = 0.$$

$$\theta_2: \quad \dot{S}_2(t) = -2S_2(t) + 2S_2^2(t) - 1 \qquad S_2(t_f) = 0.$$

The solution for $S_1(t)$ and $S_2(t)$ may be obtained analytically [52]

$$S_1(t) = .5 \, \frac{\xi_1 + 4 - (4+\xi_1)\mathrm{EXP}[2\xi_1(t-t_f)]}{1 + (4+\xi_1)\mathrm{EXP}[2\xi_1(t-t_f)]/(\xi_1-4)}$$

and

$$S_2(t) = .5 \, \frac{\xi_2 + 1 - (1+\xi_2)\mathrm{EXP}[2\xi_2(t-t_f)]}{1 + (1+\xi_2)\mathrm{EXP}[2\xi_2(t-t_f)]/(\xi_2-1)}$$

where

$$\xi_1 = \sqrt{\frac{1}{.5} + 4^2} = 3\sqrt{2} \quad \text{and} \quad \xi_2 = \sqrt{\frac{1}{.5} + 1^2} = \sqrt{3}$$

The trajectories of the state and control are plotted in Figs. 19 and 20, while the identification capability is shown in Fig. 21. In part (a) of the figures, the sample mean trajectory for the case of model θ_2 having .5 prior probability is compared with the trajectory having model θ_2 known with certainty. In part (b) of the figures, the sample variance for the 15 simulated runs using different noise sequences is plotted. There are 100 observations for each run.

It is noted that since the given models have positive eigenvalues, the uncontrolled system is unstable and the magnitude of the state will

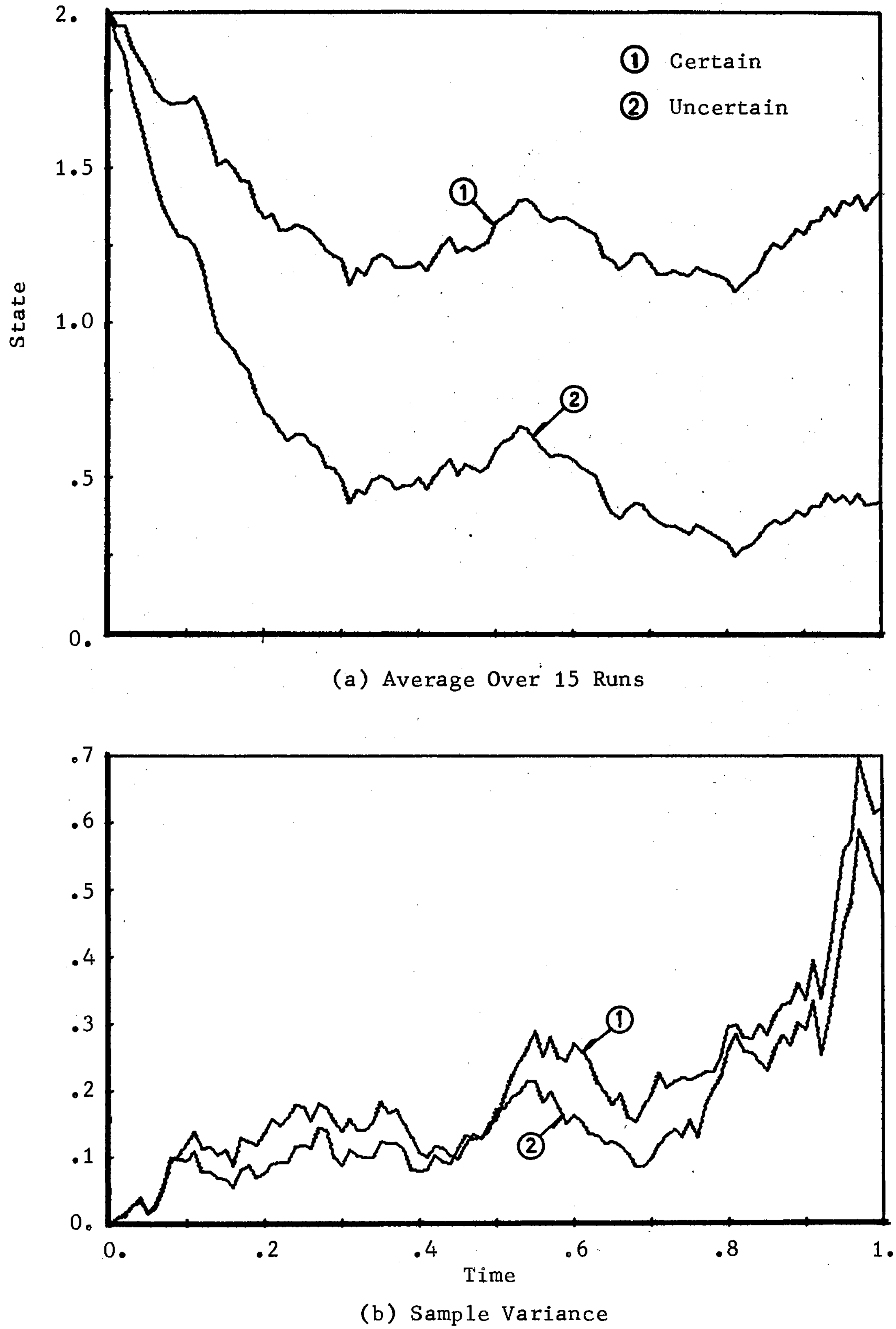
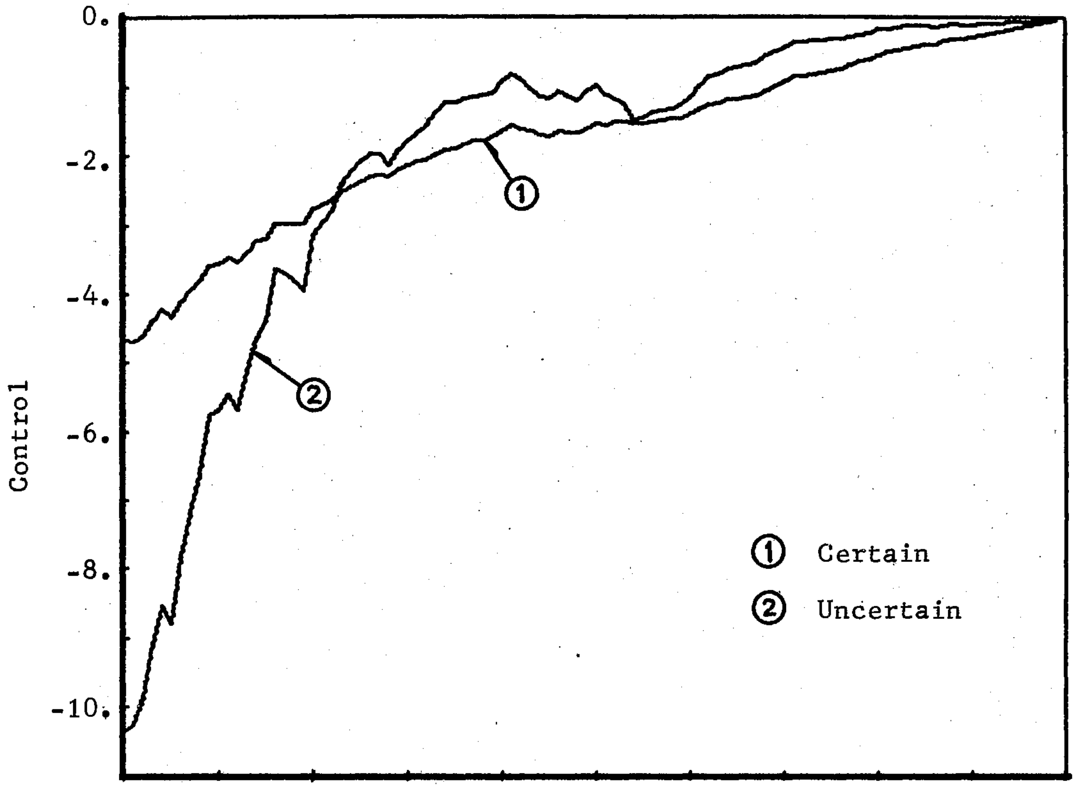
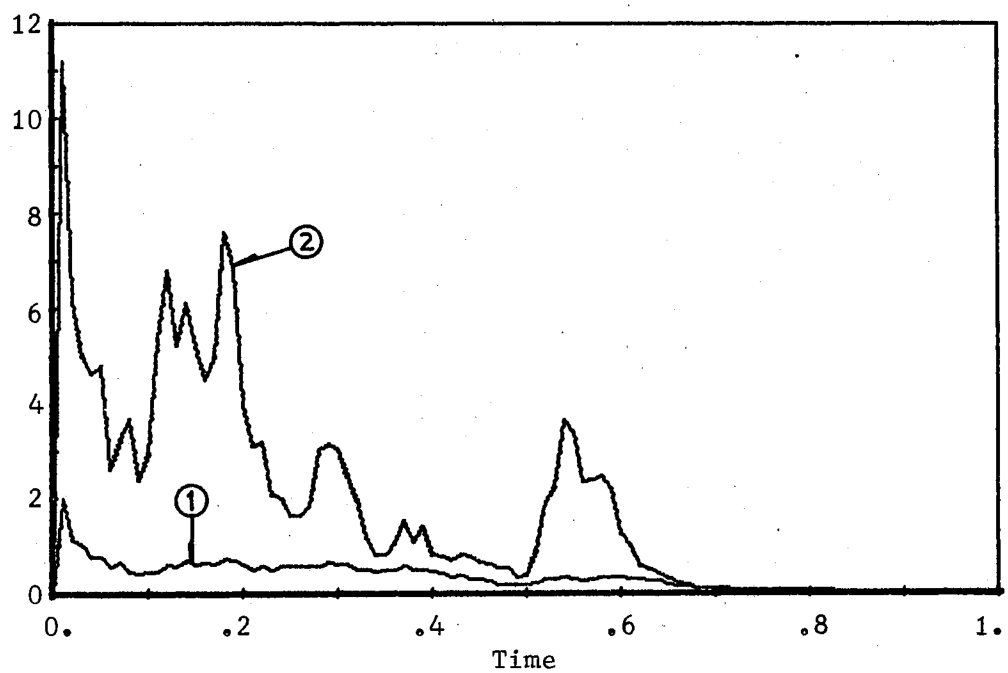


Figure 19. State Trajectories for Example 4-1

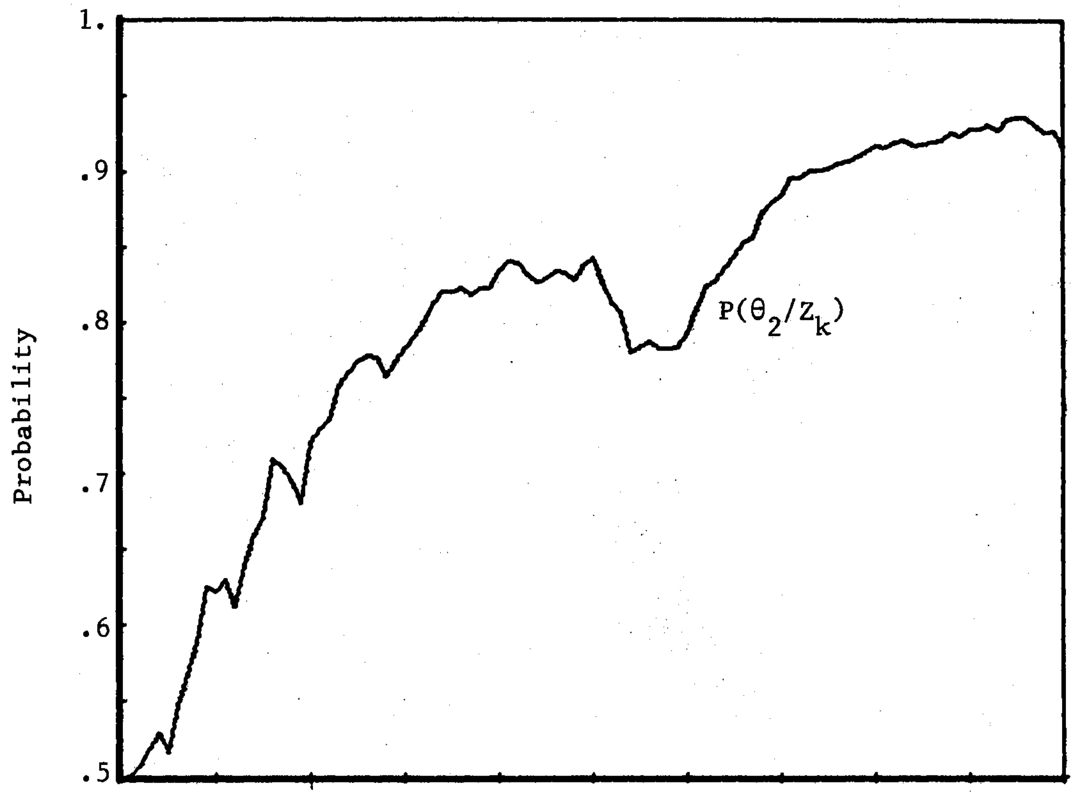


(a) Average Over 15 Runs

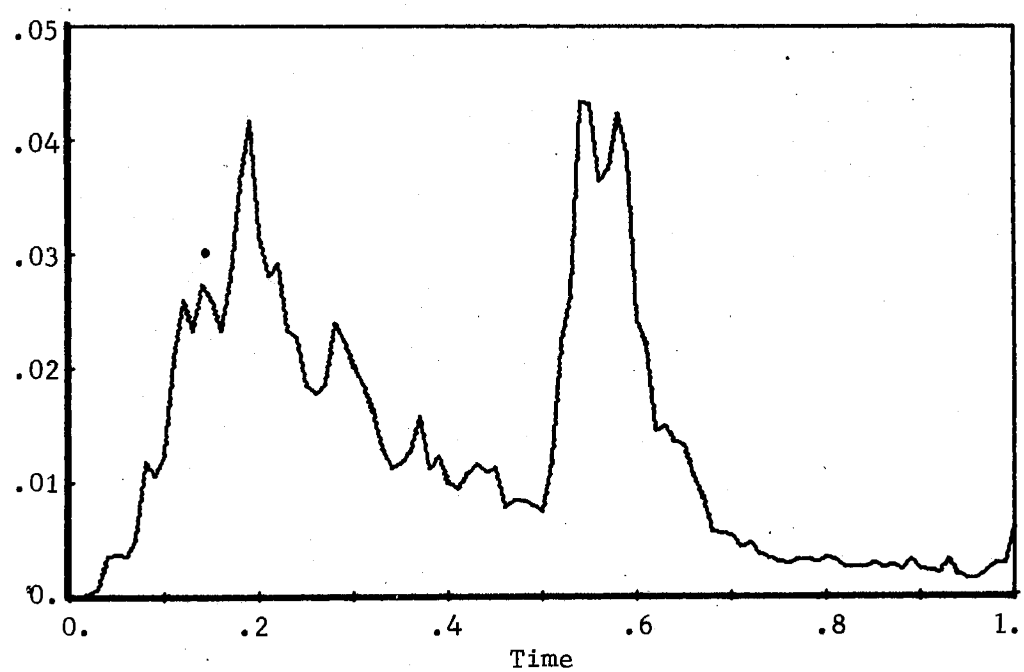


(b) Sample Variance

Figure 20. Control Trajectories for Example 4-1



(a) Average Over 15 Runs



(b) Sample Variance

Figure 21. Identification for Example 4-1

increase with time. From Fig. 19(a), one may argue that the trajectory for the uncertain case is better than that under certainty, but from Fig. 20(a), one may observe that the control effort for the uncertain case is greater than that under certain conditions. Therefore, the cost is higher as expected. In Table II the averaged performance value computed from Eq. 4-78 is compared to that the optimum result computed from Eq. 4-44 from Algorithm 4-1, under certainty. The expected cost without control is computed analytically, and compared with the other costs.

From Fig. 20(a), it is seen that the cost of control in the earlier stages of the control interval is relatively high. When the algorithm gathers enough informations to begin to identify the true system, as shown in Fig. 21, the cost of control is reduced.

TABLE II

PERFORMANCE VALUES FOR EXAMPLE 4-1

	Certain		Uncertain	Uncontrolled
	Expected	Actual	Actual	Expected
Average	3.348	2.266	3.498	8.535
Variance	--	0.615	1.087	--

(2) Example 4-2:

In this example, the situation is examined where the system is dominantly driven by the plant noise. The same dynamical structure and candidate models as in the previous example are considered. In this case the mean and variance of the initial state are given as

$$\hat{x}(t_o) = 0. \qquad \text{and} \qquad V_x(t_o) = 0.$$

and the covariance of the plant noise is larger.

$$E\{w(t)w^T(\tau)\} = 10.\delta(t-\tau)$$

The results of applying the control algorithm under these circumstances are shown in Figs. 22 through 24. The trajectories for the state, the control, and the posterior probability are plotted with the prior probability of the true model θ_2 having values .5 and .8. These trajectories are compared with the trajectories for the system optimally controlled under certainty and with no control applied. It is seen that the controlled trajectories compare favorably with the uncontrolled trajectories. It can also be seen that when the prior probability of the correct model increases, the trajectory under uncertainty approaches the trajectory under certainty as one would expect. The performance measures associated with various trajectories are indicated in Table III. It is seen that for the higher prior probability of the correct model, the cost is close to the cost under certainty.

(3) Example 4-3:

The purpose of this example is to illustrate the use of the scheme proposed in Algorithm 4-2, when the true system is not a candidate

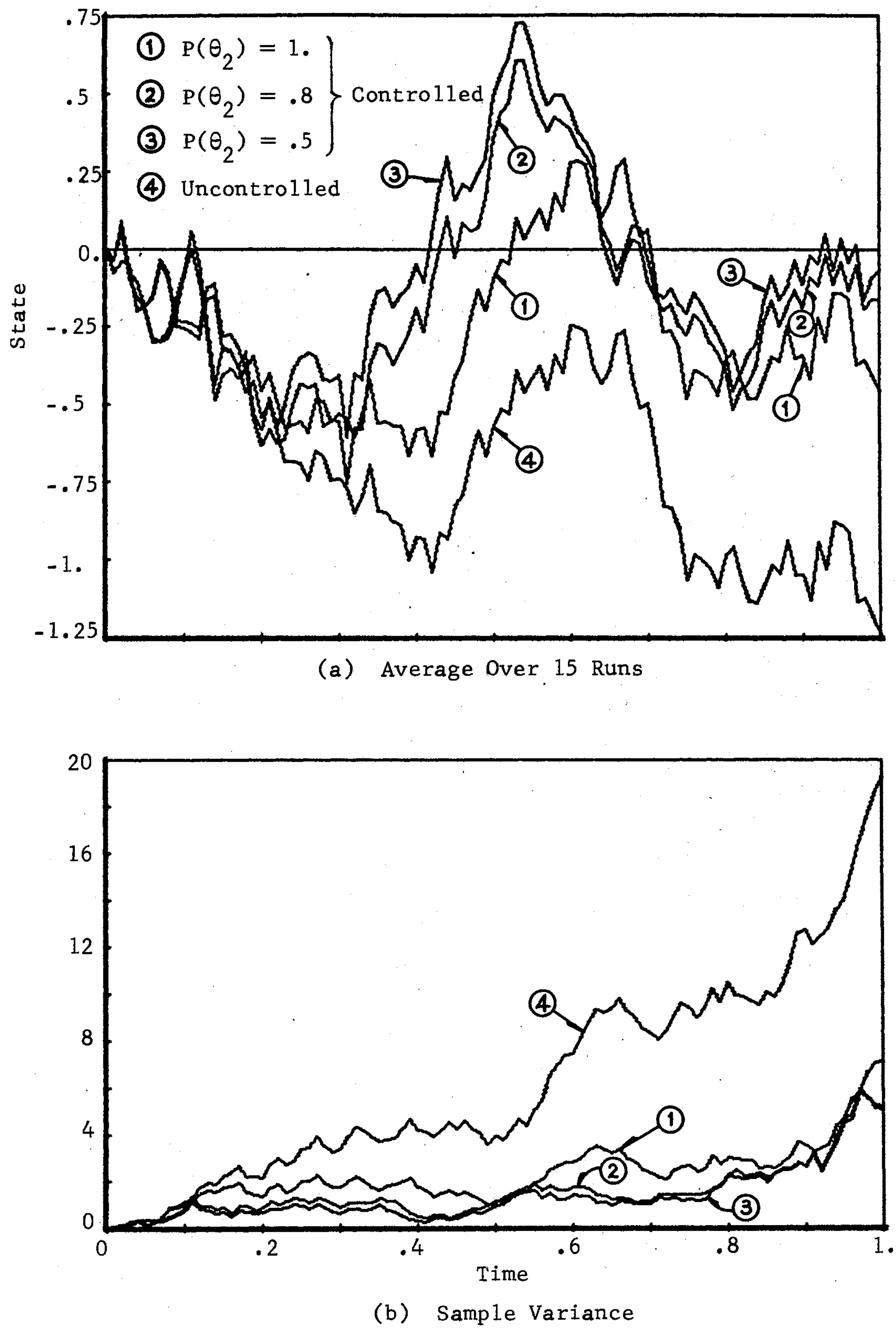
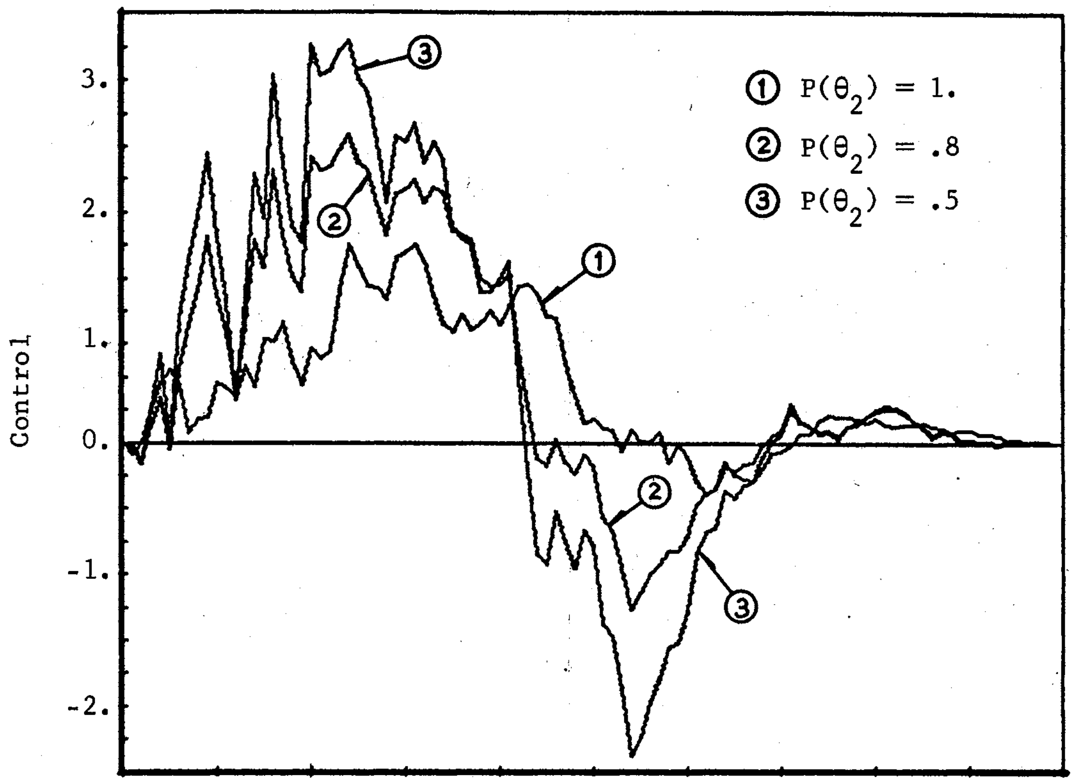
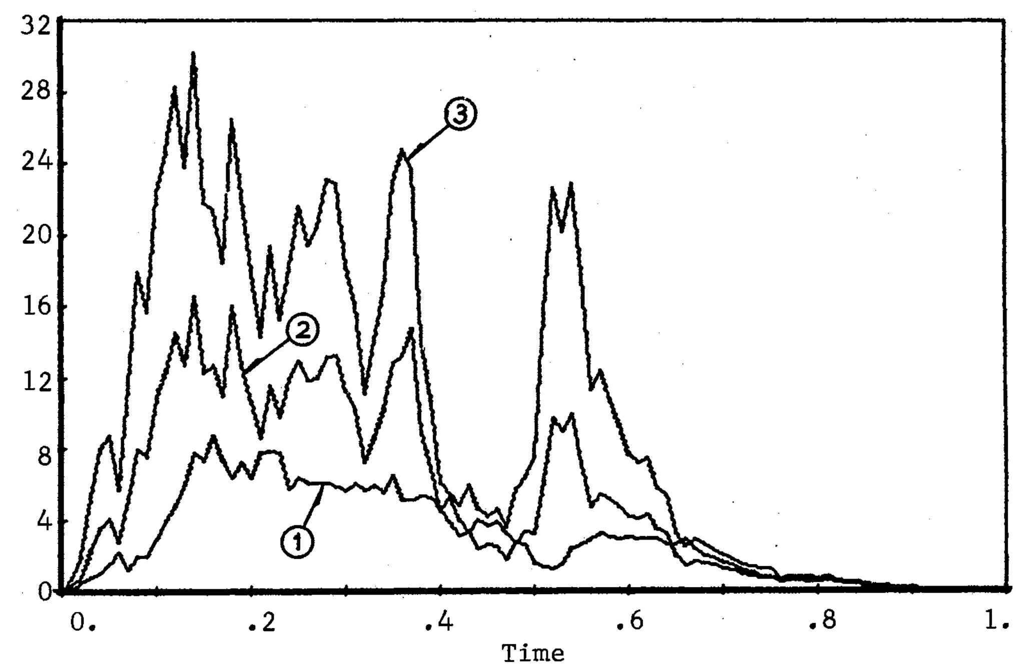


Figure 22. State Trajectories for Example 4-2

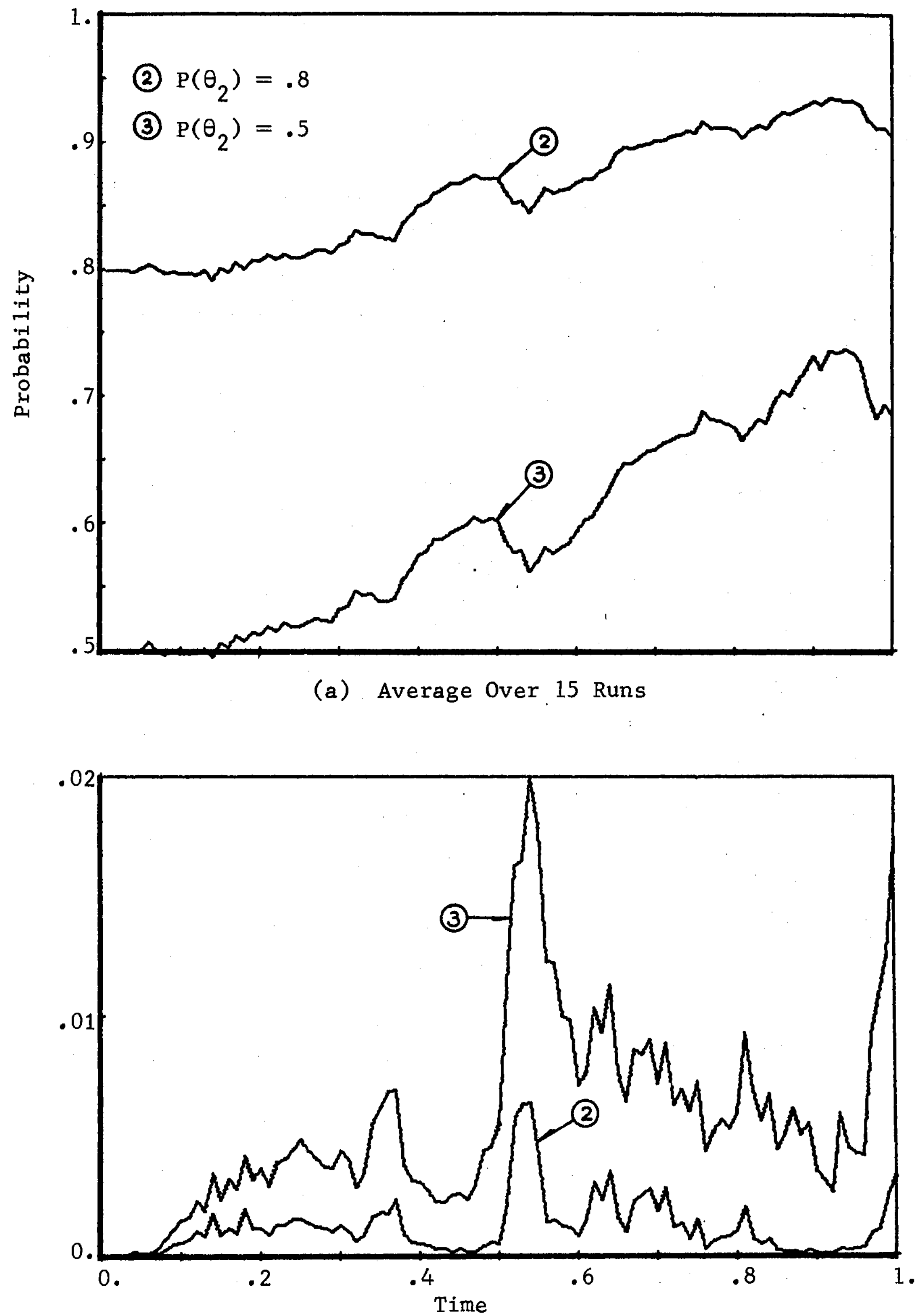


(a) Average Over 15 Runs



(b) Sample Variance

Figure 23. Control Trajectories for Example 4-2



(a) Average Over 15 Runs

(b) Sample Variance

Figure 24. Identification for Example 4-2

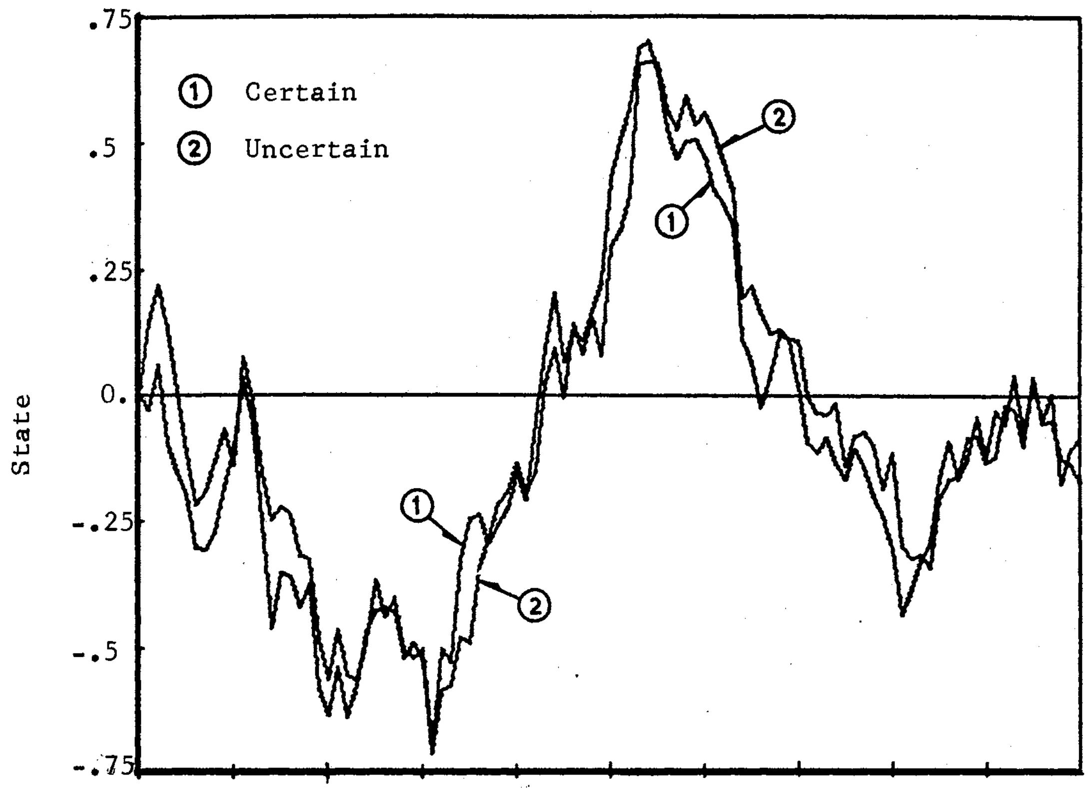
TABLE III

PERFORMANCE VALUES FOR EXAMPLE 4-2

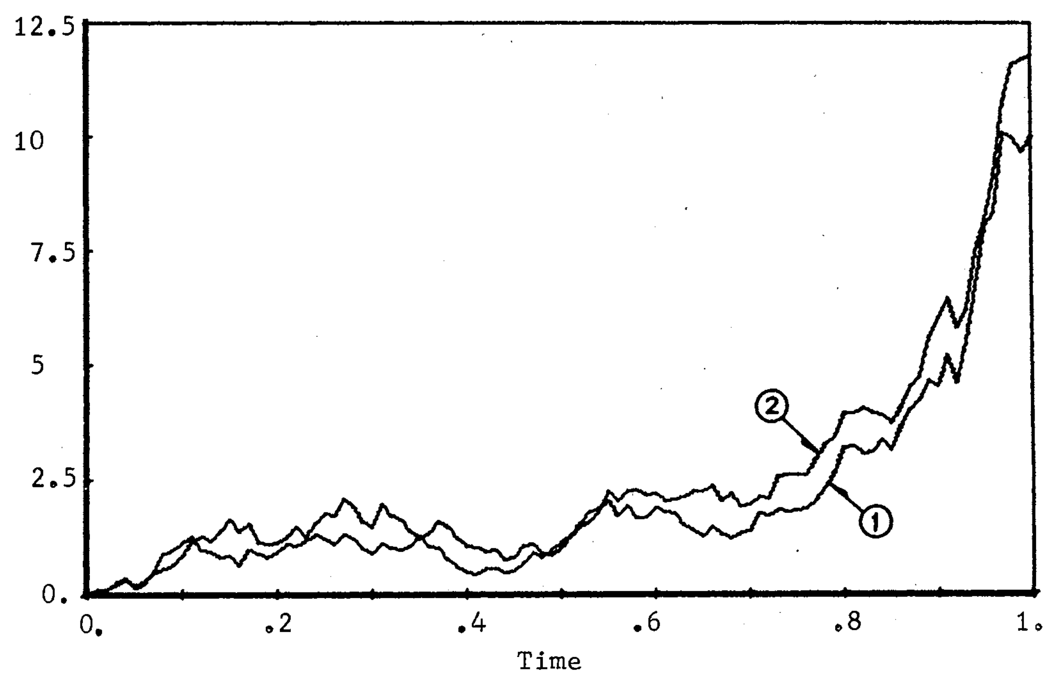
	Certain		Uncertain		Uncontrolled	
	Expected	Actual	$p(\theta_2)$=.5	$p(\theta_2)$=.8	Expected	Actual
Average	3.350	1.843	3.663	2.512	5.475	3.568
Variance	--	1.629	5.627	2.682	--	23.055

model, as might often occur in practice. The possible models are as in Example 4-2. Here, however, the true system has a parameter value F = 2.5. The initial conditions as well as the noise terms remain the same. With the prior probability $P_r(\theta_1) = P_r(\theta_2) = .5$, the control algorithm was applied. The resultant trajectories averaged over 15 runs are shown in Figs. 25 through 27. The averaged performance values are shown in Table IV. The trajectories are compared to the trajectories under certainty. Surprisingly, the two trajectories are close together. Since the true value of F falls in between of the model values, one may observe a "balance effect." That is, apparently if the eigenvalue of the true system is between the eigenvalues of the proposed candidate models, good results may be obtained even though the true model is not considered as a candidate.

This "balance" effect may be important in practical application, especially, in the case where the mathematical model for a physical process may not be found exactly. When a set of candidate models expressing the possible mathematical models of the process is

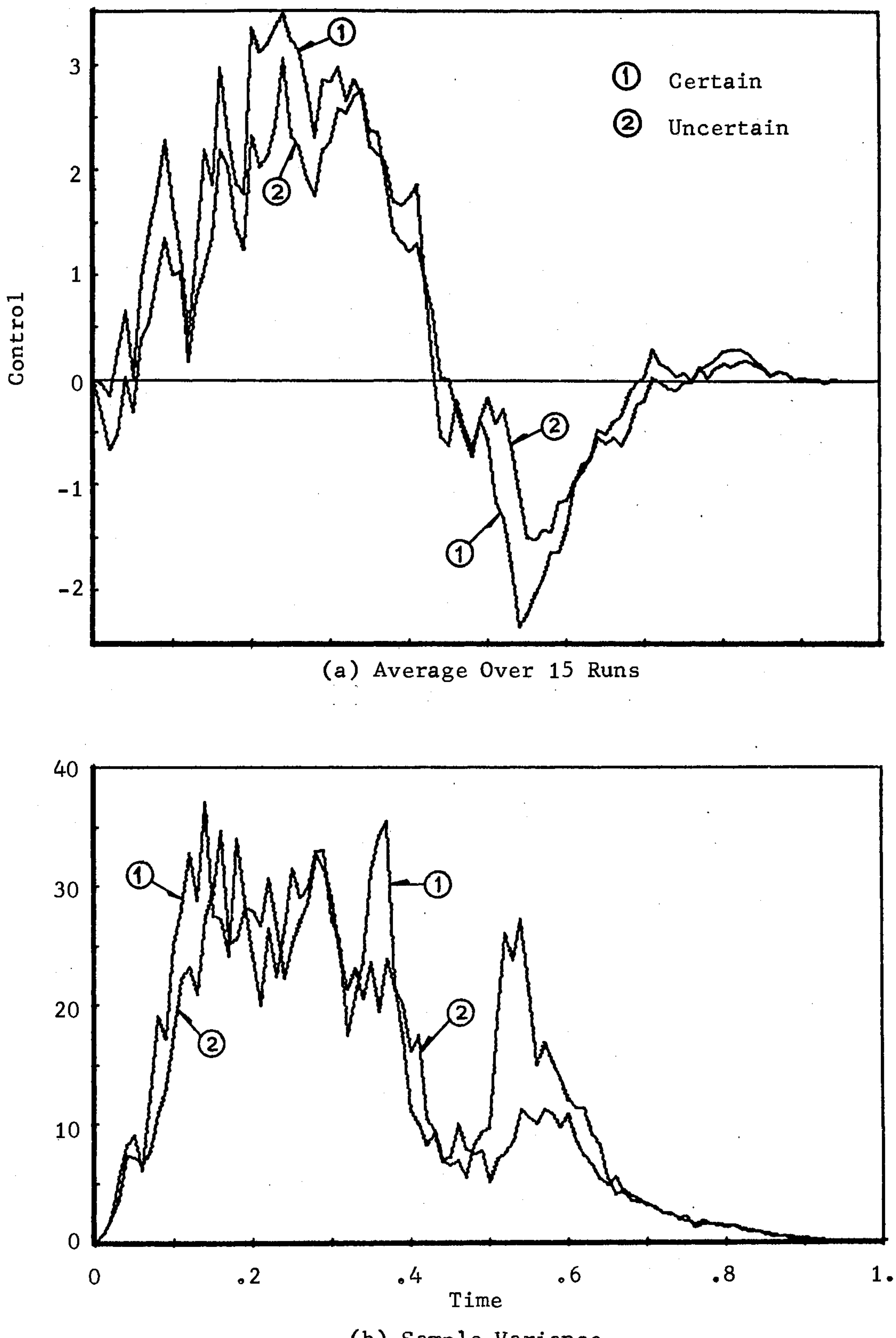


(a) Average Over 15 Runs



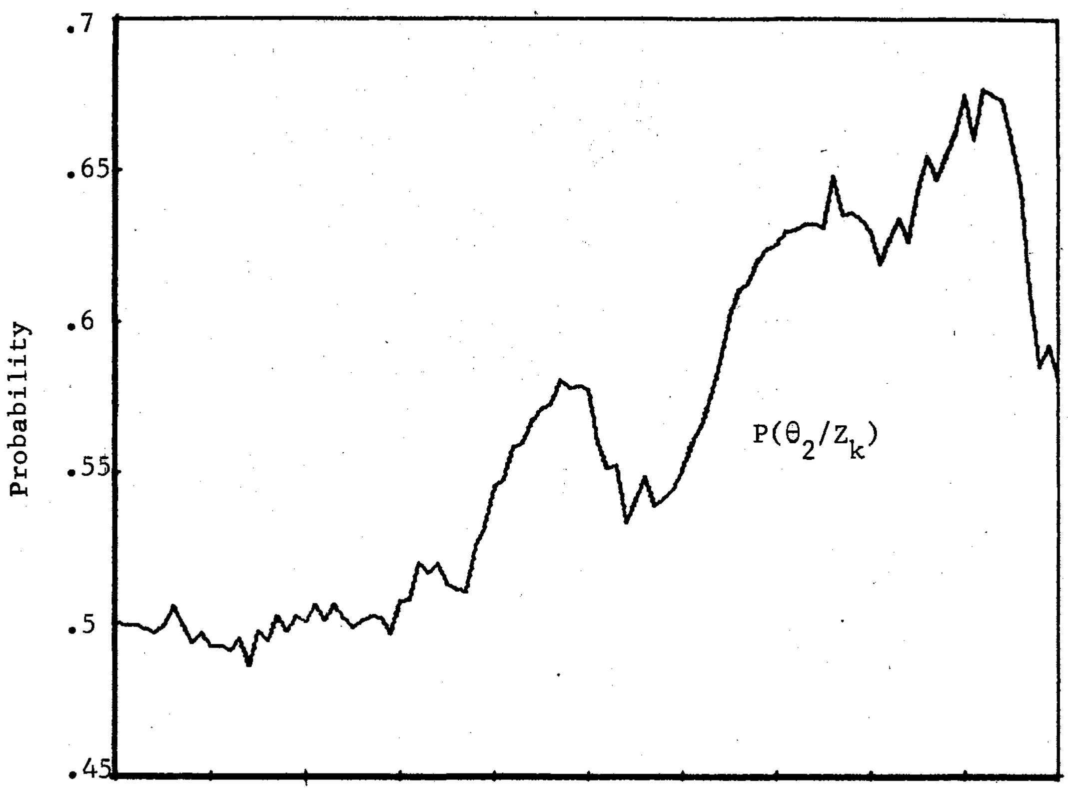
(b) Sample Variance

Figure 25. State Trajectories for Example 4-3

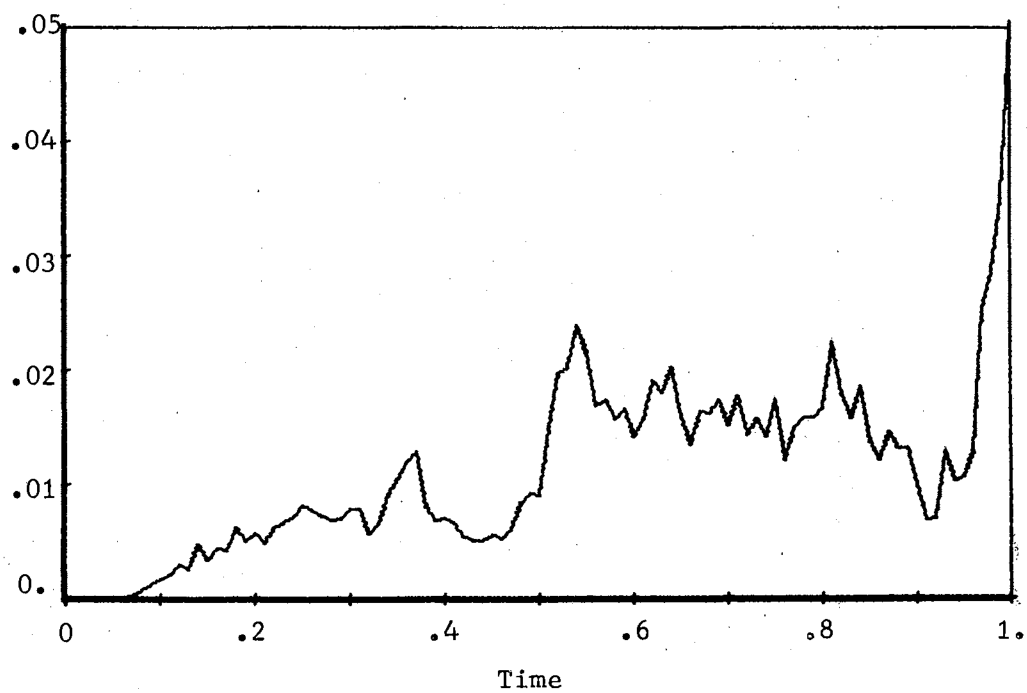


(b) Sample Variance

Figure 26. Control Trajectories for Example 4-3



(a) Average Over 15 Runs



(b) Sample Variance

Figure 27. Identification for Example 4-3

TABLE IV

PERFORMANCE VALUES FOR EXAMPLE 4-3

	Certain		Uncertain	Uncontrolled
	Expected	Actual	Actual	Expected
Average	7.299	3.710	5.197	57.012
Variance	--	4.856	11.940	--

available, the control strategies developed here may be implemented on-line. One may apply the control at the same time as the system identification is taking place. This is opposed to the procedure of first identifying the system, and then designing a control algorithm. This feature of the approach is important for application where the model may change before one could design an off-line control. This type of application is exemplified in Chapter V.

4.4 The Control Problem Under Uncertainty--Switching Case

In the last chapter, it is seen that the optimal estimation structure for the switching case requires a growing memory as the number of observations increase. This is due to the fact that a switching sequence may not be determined before hand. The same reasoning is applied to the control problem. Suboptimal control strategies which are reasonable to implement are desirable for this type of control problem.

The state estimate given the filter θ_i from Algorithm 3-2 is used as the sufficient statistic for the Bellman equation stated in Eq. 4-47. The suboptimal control strategy proposed in Algorithm 4-2 is applicable for this case when the output regulation is desired and one uses the estimate provided by Algorithm 3-2. In the case of the state regulator problem, the performance measure is described by

$$J = E\{x^T(t_f)\Lambda x(t_f) + \int_{t_o}^{t_f} [x^T(t)A(t)x(t) + u^T(t)B(t)u(t)]dt\} \tag{4-81}$$

and the Bellman equation of interest is

$$V(Z_k,t) = \min_u \sum_{i=1}^{N} P_r(\theta_i/Z_k)E\{[x^T(t)A(t)x(t) + u^T(t)B(t)u(t)]\Delta t + V(Z_k,t+\Delta t)/Z_k,\theta_i\} \tag{4-82}$$

A suboptimal control strategy similar to Algorithm 4-2 may be obtained with only slight modification due to the fact that the entire state vector may be of interest, as opposed to only the output. The algorithm is summarized as below.

<u>ALGORITHM 4-3</u>:

A suboptimal control strategy for the state regulator problem under uncertainty, and subject to switching operation, is given by

$$u(t) = \sum_{i=1}^{N} P_r(\theta(k) = \theta_i/Z_k)u_i(t) \tag{4-83}$$

where $u_i(t)$ is the control strategy of a given model θ_i, and is expressed by the following relationships

$$u_i(t) = -L_i(t)\hat{x}_i(t/Z_k) \tag{4-84}$$

$$L_i(t) = B^{-1}(t)C_i^T(t)S_i(t) \tag{4-85}$$

and

$$\dot{S}_i(t) = -F_i^T S_i - S_i F_i + S_i C_i B^{-1} C_i^T S_i - A \qquad S_i(t_f) = \Lambda \quad (4\text{-}86)$$

The only difference is in the last term of Eq. 4-86. The conditional estimate $\hat{x}_i(t/Z_k)$ is indicated in Algorithm 3-2. A schematic block diagram of the algorithm is provided in Fig. 28.

It is noted that if the switching sequence is known, the control problem reduces to the problem of "discontinuities in the system equations at interior points"[45]. The optimal control solution of this type of problem is obtained and compared to the results using Algorithm 4-3 in the following example.

Example 4-4:

The scalar system described in Example 4-3 is considered where F has a nominal value of 2.5 and may switch to some other value during the interval of interest. With the same noise terms as indicated in Example 4-3 and with the prior statistic of the state assumed to be

$$\hat{x}(t_o) = 5. \qquad \text{and} \qquad V_x(t_o) = 1.$$

the control strategy is to be determined such that a performance measure

$$J = E\{\tfrac{1}{2}\int_o^1 [x^2(t) + .5u^2(t)]dt\}$$

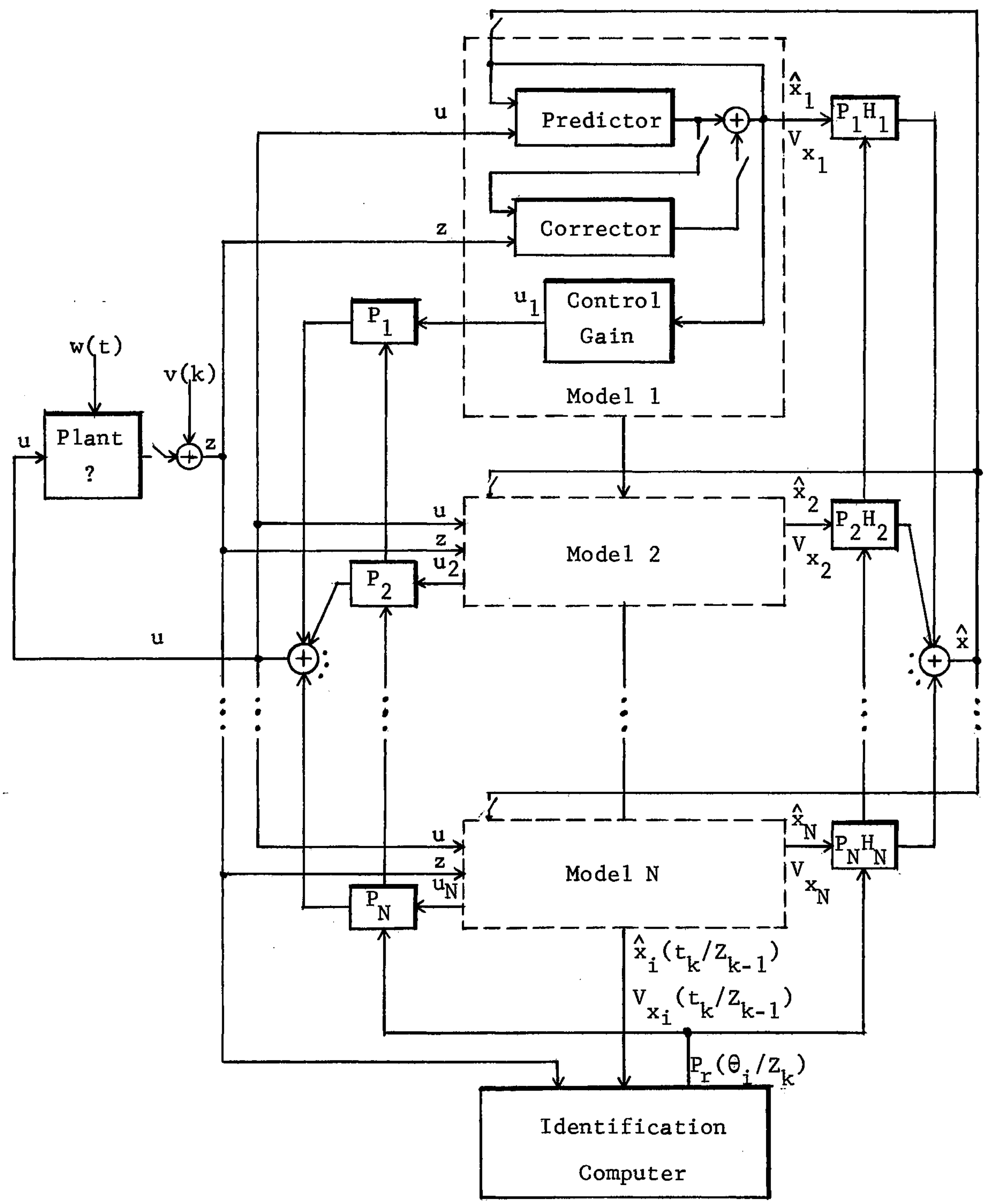
is minimized.

It is assumed that the value of F may jump to two other values, 4. and 1. That is, there are three structures under consideration.

$$\theta_1: \quad \dot{x}_1(t) = 4\ x_1(t) + u(t) + w(t)$$

$$\theta_2: \quad \dot{x}_2(t) = 2.5x_2(t) + u(t) + w(t)$$

$$\theta_3: \quad \dot{x}_3(t) = x_3(t) + u(t) + w(t)$$



Predictor: Eqs. 3-71 & 3-72

Corrector: Eqs. 3-75 & 3-76

Identification: Eq. 3-78

Control Gain: Eqs. 4-85 & 4-86

Figure 28. Block Diagram for Algorithm 4-3

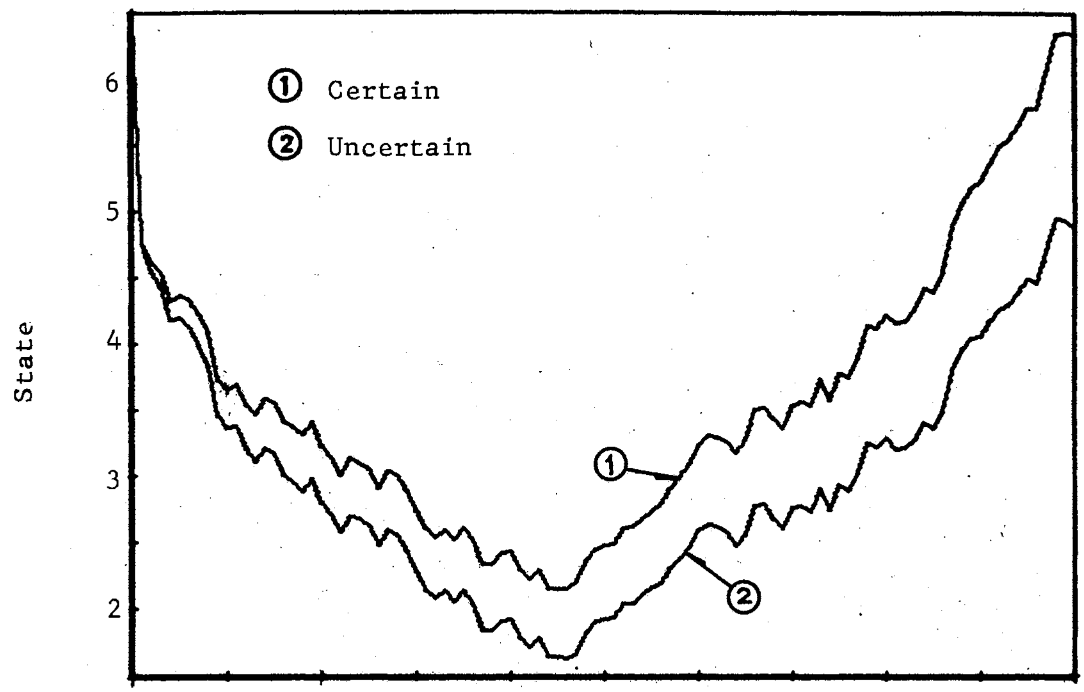
The prior probabilities of the models are

$$P(\theta_1) = P(\theta_3) = .25 \qquad \text{and} \qquad P(\theta_2) = .5$$

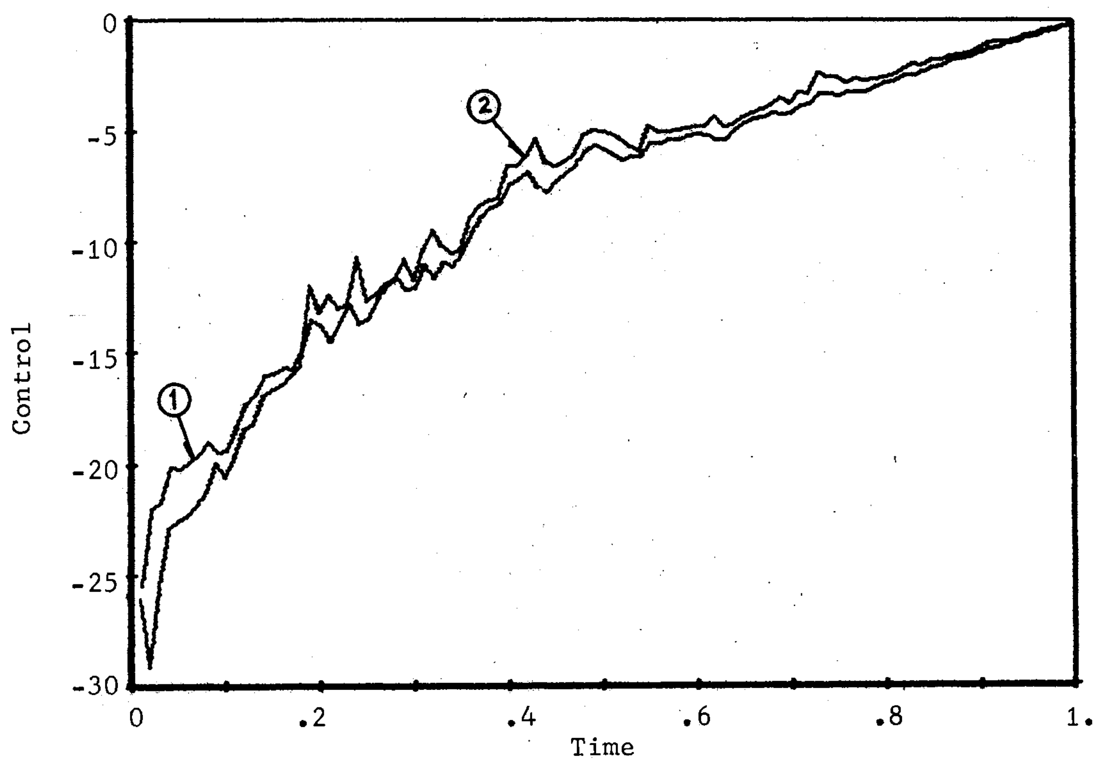
and the transition probability matrix assignment is given as

$$\Delta = \begin{bmatrix} .25 & .5 & .25 \\ .1 & .8 & .1 \\ .25 & .5 & .25 \end{bmatrix}$$

The results of simulation using Algorithm 4-3 with 100 observations during the time interval of interest are obtained and compared to the optimal control results when the switching sequence is known. In Fig. 29, typical single run trajectories for the state and control are presented. The switching sequence is also indicated below the figures. The results of averaging over 15 runs is shown in Figs. 30 and 31. In Fig. 30, the averaged state trajectories and their associated sample variance are plotted. The averaged control trajectories are shown in Fig. 31. It is interesting to note that the variance of the state trajectories is higher in the latter stages of the time interval. This is probably due to the fact that no penalty is imposed for final time errors. The variance of the control trajectories, on the contrary, is shown to be higher in the earlier stages. This is due to greater initial uncertainty of which system is active. It is also seen that the cost of control under uncertainty is very high at the earlier stages, but after more data is obtained it is reduced as in the nonswitching case. The averaged performance measures for 15 runs are indicated in Table V. It is seen that the averaged costs are very close. The proposed suboptimal control strategy compares

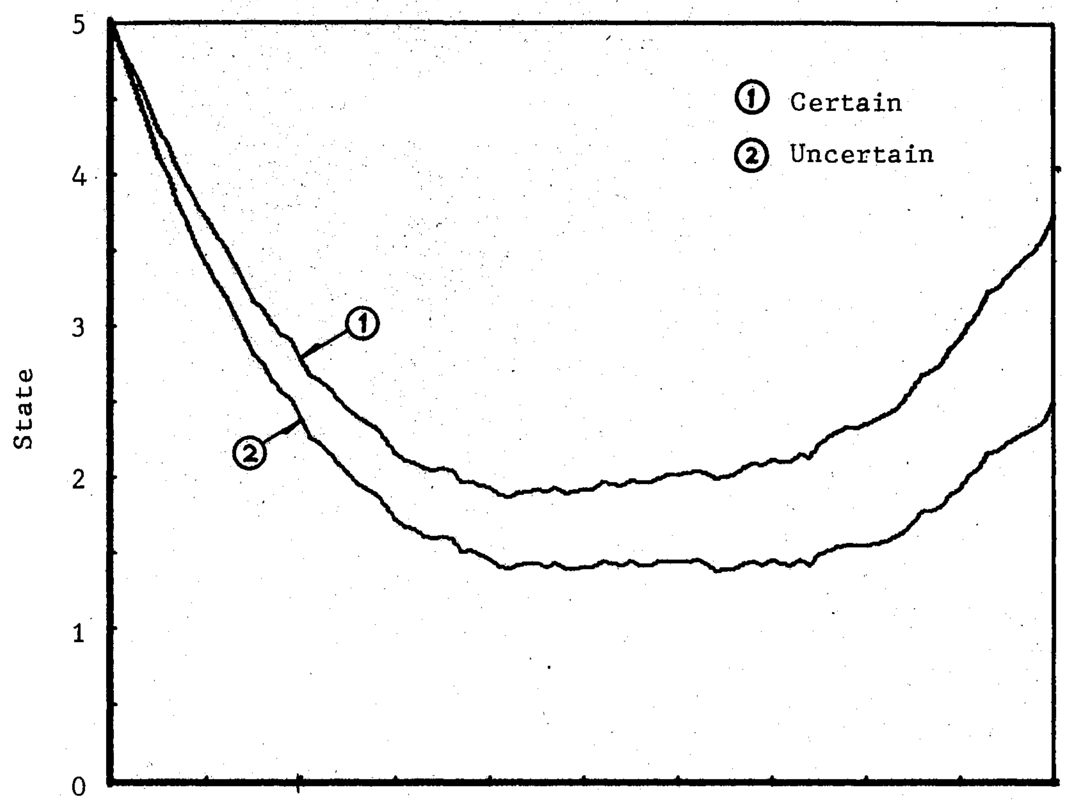


(a) Average Over 15 Runs

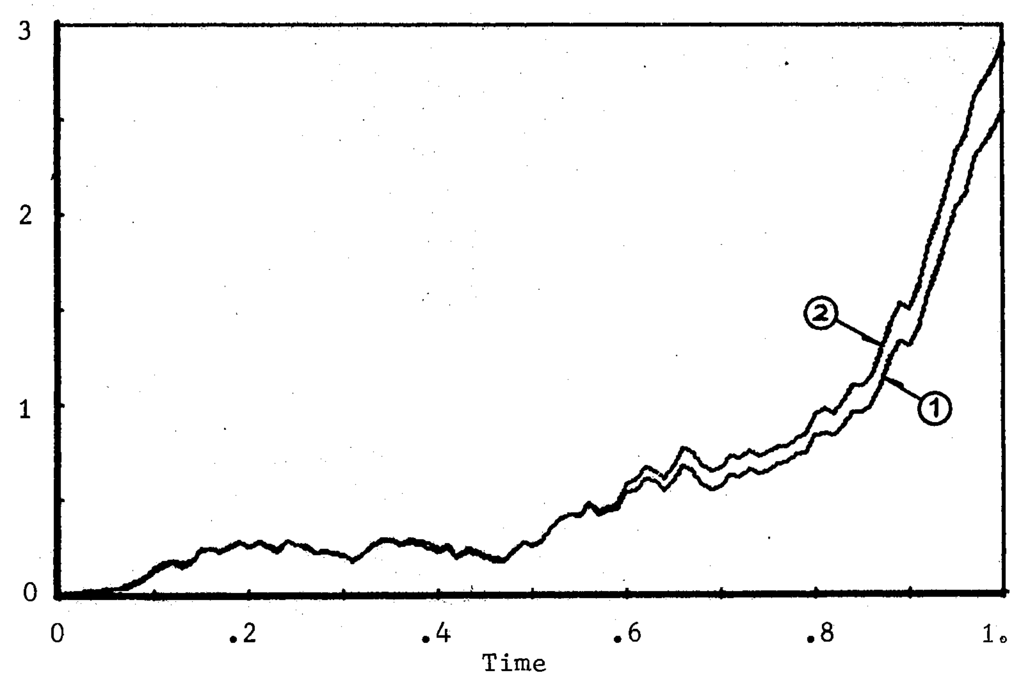


(b) Sample Variance

Figure 29. Typical Single Run for Example 4-4

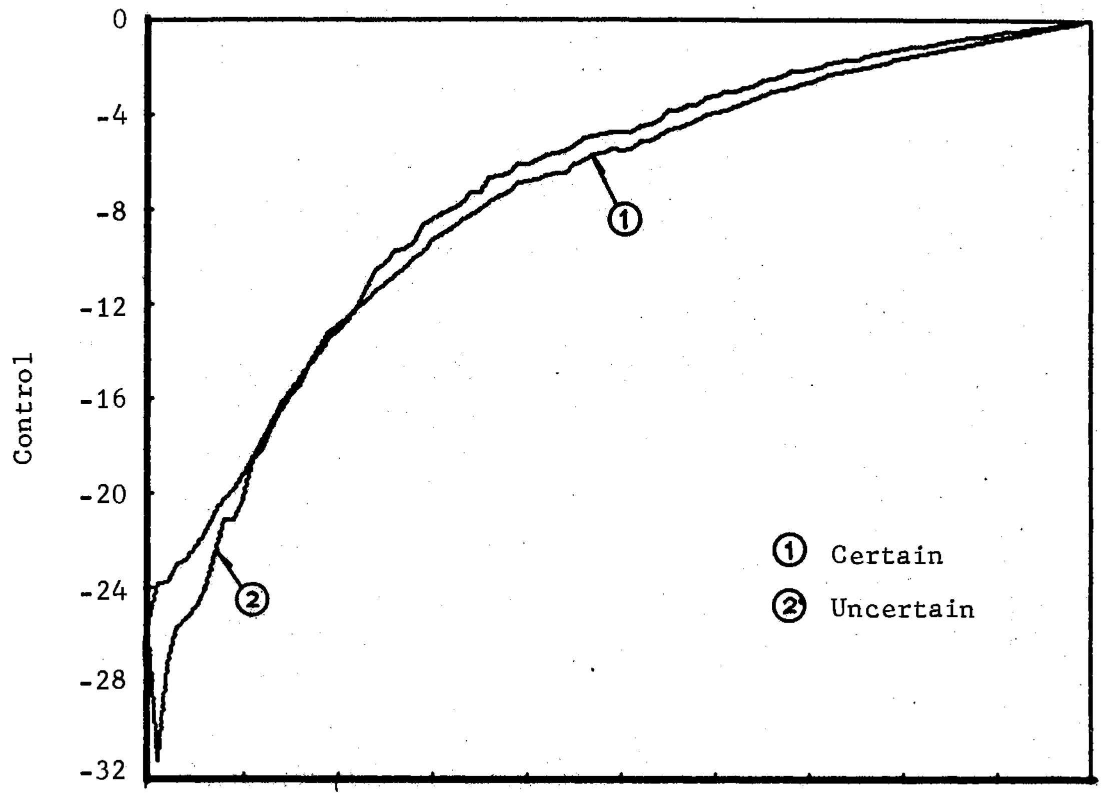


(a) Average Over 15 Runs

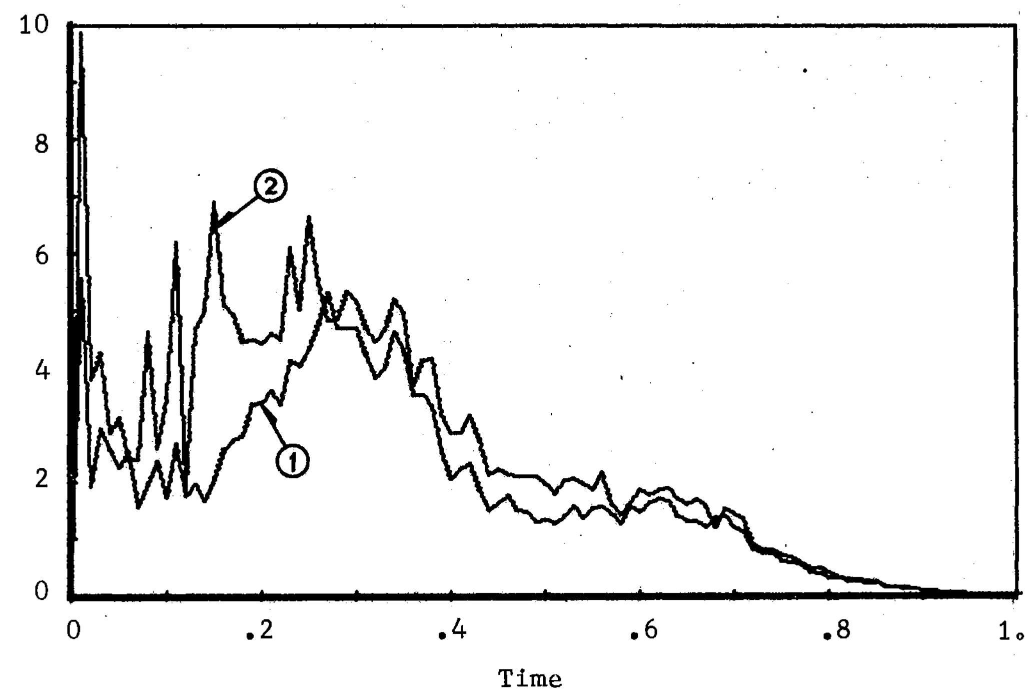


(b) Sample Variance

Figure 30. State Trajectories for Example 4-4



(a) Average Over 15 Runs



(b) Sample Variance

Figure 31. Control Trajectories for Example 4-4

favorably with the optimum control based on knowledge of the switching sequence.

TABLE V

PERFORMANCE VALUES FOR EXAMPLE 4-4

	Certain	Uncertain
Average	29.697	30.647
Variance	24.862	19.319

4.5 Summary

The control strategy for uncertain systems which has been developed here is in general suboptimal. It makes use of the estimation algorithms presented in previous chapter and is not of unreasonable complexity. In the nonswitching case, since system identification may be carried out on-line, the control strategy becomes optimal once a posterior probability approaches one. Even if the set of candidate models does not include the correct system, the "balance" effect may still provide good estimation as well as near optimal control.

The examples which have been presented in this chapter, although of an academic type, serve to demonstrate the ability of the suboptimal control algorithm to control unknown systems. In the following chapter, the control techniques which have been developed are applied to a stream quality control problem.

CHAPTER V

APPLICATION TO WATER POLLUTION PROBLEMS

5.1 Introduction

This chapter is included to demonstrate a possible application of the findings of Chapters III and IV. The problem under consideration is classified as a water quality control problem. A brief introduction to water quality problems is presented in Section 5.2, and the related mathematical models describing variables of interest are indicated. A stochastic model is proposed in Section 5.3 to account for random effects. The model then serves as a basis for applying algorithms developed in the previous chapters to the water quality control problems. Applications to system identification and control problems are presented in Section 5.4 and 5.5, where the results of simulation of the proposed system are indicated.

5.2 Water Quality Problems

A complete description of the physical, chemical, and biological aspects of water quality, as a function of natural and man-made factors, and the technological measures available for changing water quality can be found in the literature [47], [48], and [49]. Here, a partial background indicating some of the considerations of water quality control is presented.

The wastes having an impact on the water quality of a stream as a result of domestic, industrial, agricultural, and recreational activities can be classified in two categories, nondegradable and degradable wastes. Nondegradable wastes are usually diluted and may change form, but they are not appreciably reduced in weight in the receiving water. Degradable wastes are reduced in weight by the biological, physical, and chemical processes.

The effect of degradable waste, especially the biological effect of organic waste, in receiving water is of great interest since it threatens aquatic life and presents a health problem to those using the water.

5.2.1 Effects of Organic Waste Discharges on the Receiving Stream

The degradation process of organic waste in receiving waters is produced by the action of bacteria utilizing free oxygen. The imbalance between available oxygen and oxygen demand may proceed to the point where septic or anaerobic conditions result. Water quality, if this situation occurs, is undesirable. The problems in water quality control are to predict the time and spatial pattern of concentration of wastes in a stream, and to control the waste discharge so that in any portion of a stream anaerobic conditions will not result.

A measure of organic waste load is biochemical oxygen demand (BOD), which indicates the amount of oxygen drawn upon in the process of decomposition of the waste. The amount of oxygen demanded and the rate at which it is drawn upon are functions of the type and quantity of the waste and of other factors, among which the most important are the chemical characteristics and the temperature of the receiving water. The rate at which BOD is exerted combined with the rate at which oxygen is restored determines the level of dissolved oxygen (DO).

The rate of reaeration depends largely on the stream characteristics as they affect turbulance and the area of the air-water interface, the velocity of streamflow, and the net photosynthetic oxygen production.

In a stream, the combined effect of an organic waste discharged at a specific location and reaeration in the stream results first in a decrease and then an increase in DO as the waste is carried or moved downstream. This phenomenon is illustrated by a characteristic curve known as the "oxygen sag" [50,51]. This gives a crude way of predicting the time or spatial pattern of the concentration of wastes. But it is complicated by hydrological uncertainty [52,53], and the variation in quantity of waste discharged. Mathematical representations for the sag curve are discussed in the following sections.

5.2.2 Equations of the "Oxygen Sag"

The oxygen sag and the equations relating to the sag curve were first formulated by Streeter and Phelps [50]. Subsequently, various modifications have been made utilizing different assumptions and conditions [53-58]. Basically, the two processes involved in the oxygen sag are biochemical oxidation and reaeration. Under the assumptions that the stream has

(1) constant velocity,

(2) unidirectional flow,

(3) one-dimensional space dependence,

the oxygen sag can be characterized by a set of first-order differential equations

$$\text{BOD:} \qquad \frac{dL(t)}{dt} = -k_1 L(t) \qquad\qquad L(t_o) = L_o \qquad\qquad (5\text{-}1)$$

$$\text{DO:}\quad \frac{dC(t)}{dt} = k_2(C_s - C(t)) - k_1 L(t) + C_a \qquad C(t_o) = C_o \tag{5-2}$$

where

L = BOD concentration, mg/lit or ppm,

C = DO concentration, mg/lit or ppm,

C_s = DO concentration at saturation, mg/lit or ppm,

C_a = Average Photosynthesis-respiration rate, mg/lit-time or ppm/time,

k_1 = BOD removal coefficient, 1/time,

k_2 = Reaeration coefficient, 1/time,

t = time of travel or distance divided by velocity,

and L_o and C_o represent the initial concentrations.

The coefficient k_1 and k_2 represent the natural ability of a stream to handle the degradable waste load. They are functions of a number of variables. Various experiments have been conducted by researchers to determine the value of these coefficients under different conditions [59,60]. The experiments show that k_1 is sensitive to change of temperature and the type of waste, while k_2 is more difficult to determine. Primarily, k_2 depends on the slope, depth and velocity of the stream, temperature as well as the type of waste. C_s is also a parameter to be determined. In general, it depends on the same conditions as k_2, but temperature plays the dominant role.

Typical "oxygen sag" curves are shown in Fig. 32.

The mathematical model defined by Eqs. 5-1 and 5-2 is based on the assumption that there are only two major processes taking place, biochemical oxidation and reaeration. In addition to these processes, some or all of the following processes may be taking place in a stream.

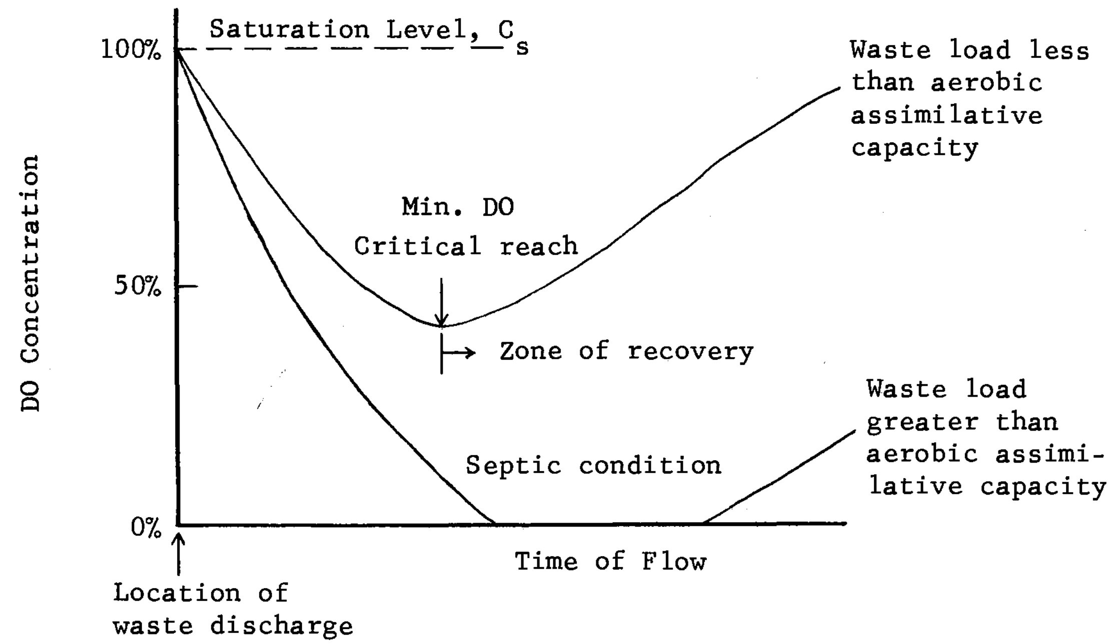


Figure 32. The Oxygen Sag

(1) The removal of BOD by sedimentation or absorption.

(2) The addition of BOD along the stream by the scour of bottom deposits or by the diffusion of partly decomposed organic products from the benthal layer into the water above.

(3) The addition of BOD along the stream by local run off.

(4) The removal of oxygen from the water by diffusion into the benthal layer to satisfy the oxygen demand in the aerobic zone of this layer.

(5) The removal of oxygen from the water by purging action of gases rising from the benthal layer.

(6) The continuous redistribution of both the BOD and DO by the effect of longitudinal dispersion.

These processes may effect either the BOD or DO equation or both.

The processes (1) through (3) only effect Eq. 5-1, producing the modified mathematical model [47,54]

$$\frac{dL(t)}{dt} = -(k_1 + k_3)L(t) + L_a(t) \tag{5-3}$$

where k_3 is a BOD removal coefficient of sludge sedimentation (process (1)), and L_a is the rate of addition of BOD due to processes (2) and (3). The processes (4) and (5) effect Eq. 5-2. The definition of C_a may be redefined, however, to include these processes, so that Eq. 5-2 remains unchanged.

In order to take into account the effects of process (6), Dobbins[60] has proposed two second-order models in the form of the partial differential equations

$$\frac{\partial L}{\partial t} = D_L \frac{\partial^2 L}{\partial \xi^2} - S\frac{\partial L}{\partial \xi} - (k_1+k_3)L + L_a \tag{5-4}$$

$$\frac{\partial C}{\partial t} = D_L \frac{\partial^2 C}{\partial \xi^2} - S\frac{\partial C}{\partial \xi} + k_2(C_s-C) - k_1L + C_a \tag{5-5}$$

where

D_L = the coefficient of longitudinal dispersion, ft^2/time,

S = the average stream velocity, ft/time

ξ = the distance along the stream, positive in the downstream direction, ft or mile

It should be noted that Eqs. 5-4 and 5-5 include both spatial and temporal distribution consideration. The values of L and C are dependent upon the variables ξ and t. More general representations can be found in [58]. The mathematical model indicated above has been widely used for the dissolved oxygen profile in tidal estuaries, sluggish streams, or ponds where longitudinal mixing occurs as a result of tides or wind-induced currents.

The mathematical models described by Eqs. 5-2 and 5-3 as well as Eqs. 5-4 and 5-5 are the types of models available water purification processes.

The process of purification of a stream, which usually is termed "stream self-purification," is a dynamic phenomenon that reflects hydrologic and biologic variations, the interrelations of which are not yet fully understood in precise terms. However, sufficient knowledge is available to permit quantitative definition of resultant stream condition under expected ranges of variation and to serve as a practical guide in decisions dealing with water resource use, development, and management.

The value of the minimum DO of the "oxygen sag" which is termed "critical reach" is a governing factor in planning the size or capacity of waste treatment plants. Sanitary engineering practice has typically been to calculate the oxygen sag at a given low level of streamflow and to determine the capacity and design of a waste treatment plant which will result in a specified level of dissolved oxygen in the critical reach of the oxygen sag. The stochastic nature of stream flow and its effect on the variation in the assimilative capacity of the stream has motivated consideration of the stream quality problem as an application of the theoretical results derived in previous chapters.

5.3 A Stochastic Model

As indicated in the previous section, the hydrologic uncertainty and the variation in waste run off make the prediction of time and spatial patterns of concentration difficult. Changes in hydrologic characteristics are such that the parameters in Eqs. 5-2 and 5-3 are subject to doubt. It appears that consideration of a stochastic system may be appropriate. In view of the fact that biological oxidation and reaeration are the processes of primary importance, other effects may be considered as disturbances to these major processes.

The mathematical models presented in Eqs. 5-2 and 5-3 may be modified so as to specifically indicate the stochastic nature of the models.

In order to be within the frame work of the class of problems presented in previous chapters, a Gaussian-Markov process must be assumed. The residual of BOD and DO concentration which provide the initial conditions for the system are assumed to be normally distributed random variables. Since the contribution of the term L_a is small and results in oxygen depletion, one may combine two parameters, L_a and C_a, into a disturbance variable, w(t) or noise term, to the reaeration equation 5-2. The noise term is assumed to be normally distributed. If the disturbances are rapid enough relative to the main processes, the noise may be assumed to be white. Otherwise, a colored noise model may be appropriate. For the sake of simplicity, white noise, w(t), with nonzero mean $\bar{w}(t)$ and correlation function

$$E\{w(t)w^T(\tau)\} = Q(t)\delta(t-\tau)$$

is assumed in the following developments.

By adding the control variables $u_1(t)$ and $u_2(t)$ in Eqs. 5-2 and 5-3, where $u_1(t)$ is the source of BOD, and $u_2(t)$ is the man-made reaeration DO source, and adding the random input w(t), a stochastic model is obtained.

$$\frac{dL(t)}{dt} = -(k_1(t) + k_3(t))L(t) + u_1(t) \qquad (5\text{-}6)$$

$$\frac{dC(t)}{dt} = -k_1(t)L(t) - k_2(t)C(t) + k_2(t)C_s(t) + u_2(t) + w(t) \qquad (5\text{-}7)$$

where the initial conditions $L(t_o)$ and $C(t_o)$ have mean $\bar{L}(t_o)$ and $\bar{C}(t_o)$, and variance $V_L(t_o)$ and $V_c(t_o)$, respectively.

Letting $x_1(t) = L(t)$ and $x_2(t) = C(t)$, Eqs. 5-6 and 5-7 may be rewritten in terms of the familiar notation used in previous chapters,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -k_1(t) - k_3(t) & 0 \\ -k_1(t) & -k_2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ k_2(t) \end{bmatrix} C_s(t)$$

$$+ \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) \qquad (5\text{-}8)$$

or in compact form

$$\dot{x}(t) = -F(t)x(t) + D(t)C_s(t) + C(t)u(t) + G(t)w(t) \qquad (5\text{-}9)$$

where

$$F(t) = \begin{bmatrix} k_1(t) + k_3(t) & 0 \\ k_1(t) & k_2(t) \end{bmatrix}, \quad D(t) = \begin{bmatrix} 0 \\ k_2(t) \end{bmatrix}$$

$$C(t) = I \quad \text{and} \quad G(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Equation 5-9 differs from the equations presented in previous chapters only by an extra term $D(t)C_s(t)$ and nonzero mean noise. Here, $C_s(t)$ may be unknown but not random. The coefficients $k_1(t)$, $k_2(t)$ and $k_3(t)$, as well as the statistical parameters of $w(t)$ and $x(t_o)$, are subject to uncertainty.

The consideration of time varying coefficients $k_1(t)$, $k_2(t)$, and $k_3(t)$ reflects the fact that there may be branches which merge with the stream of interest, and in such cases the hydrological characteristics of the stream definitely are changed as a function of time (or distance). The time varying coefficients may also apply if, in a section of the stream, the velocity varies significantly.

Although time is the independent variable considered, time can be converted to distance whenever the velocity of the stream is known as a function of distance. At discrete intervals, observation stations may be set up along the stream to measure conditions, and the data related to time. The observation models is

$$z(k) = H(t_k)x(t_k) + v(k)$$

for $k = 1,2,\ldots$, where $v(k)$ is the measurement error which is assumed to be gaussian with zero mean and covariance

$$E\{v(k)v^T(j)\} = R(k)\delta_{kj}$$

The procedure for measuring BOD is tedious [61] often requiring five days or more. The measurement of DO may be obtained rather easily. At each station only the DO measurement is assumed available, corresponding to a measurement matrix,

$$H(t_k) = [0 \quad 1]$$

The second order model indicated in Eqs. 5-4 and 5-5 is not considered due to the difficulties of considering distributed parameter stochastic processes. In the next section, the estimation algorithm developed in Chapter III is used with emphasis on system identification of the process described here.

5.4 The Problem of System Identification

It is of interest to sanitary engineers that the coefficients and parameters of various proposed mathematical models of degradable wastes in a receiving water can be determined logically. The assimilative capacity of the receiving water then can be determined and may serve as the basis of designing a waste treatment plant. It is also of interest to people in water resources management that the "critical reach" can be imposed in every tributary such that the use of the water of the entire stream system is optimized. Among the coefficients and the parameters mentioned k_1 and k_2 which are associated with BOD removal and DO recovery are the most important ones.

The value of the coefficient k_1 is usually determined by laboratory investigation. The procedures of analysis are well documented in Standard Methods [61]. In fact, the BOD concentration is measured by the same procedure. It requires five days or more to complete an investigation. Since non-turbulance is required for the procedure, the obtained value of k_1 is only true for a quiescent system.

The primary source of dissolved oxygen in a stream is the atmosphere. Reaeration is governed by two fundamental laws: the law of solution and the law of diffusion. The formulas for deriving the coefficient k_2 are available [62-65], and are based on these two laws together with stream conditions.

It is difficult to simulate the dynamics of a river environment, including the biological chains. The experimentally obtained value k_1 and k_2 are unsatisfactory to researchers. The conditions which are controlled in simulated streams are actually subject to uncertainty in the field. Hence, there is ample reason for adopting a systematic approach for identifying the parameters of interest, which takes into account the stochastic nature of the problem.

A range in a stream of interest may be divided into sections such that in each section the value of k_1, k_2 and some other parameters may be considered steady. A known amount of waste is discharged at a point upstream. There is no additional waste dumping or man-made aeration along the stream. The situation is equivalent to having no control term u(t) in Eq. 5-9, so that the model is of the form

$$\dot{x}(t) = -Fx(t) + D(t)C_s + Gw(t) \tag{5-11}$$

the corresponding predictor structure is

$$\dot{\hat{x}}(t) = -F\hat{x}(t) + D(t)C_s + G\bar{w}(t) \tag{5-12}$$

A set of measurement data may be obtained from a finite number of observation points along the stream, and the estimation algorithm of Chapter III may be applied. Candidate models may be selected on the basis of past experiments. The procedure for applying the algorithm is described in the following section.

5.4.1 An Application of the Estimation Algorithm

The nonswitching algorithm for estimation-under-uncertainty presented in Chapter III has two important aspects, the optimal estimate of the output, and system identification. System identification is of primary importance here. Although the algorithm was not developed to be used repeatedly with the same data, one may use it in that way. Repeated application of the algorithm seems to be appropriate for the stream quality problem. The procedure used is summarized below.

(1) A set of candidate models may be chosen such that each model represents a possible description of the system.

(2) A set of prior probabilities is assigned to the models, and the algorithm for estimation-under-uncertainty is used to process the obtained data.

(3) After the first iteration when the data has been exhausted, if the posterior probabilities of the models do not show any significant difference, one may repeat the same procedure adopting the final posterior probabilities as the new prior probabilities.

(4) If the posterior probabilities persist without any significant difference after several iterations, a different set of models should be considered. If the posterior probabilities show some significant difference, or the posterior probabilities of some models are relatively higher than others, then models with lower posterior probabilities may be dropped from consideration. One may repeat the procedure by reassigning prior probabilities to the remaining models.

(5) If a model represents the true system or "close" to the true system, then the posterior probability should tend to one in a few iterations. In some cases, the posterior probability of two or three models may be significant. This is a sign that the true system may be in the range of the parameters of these models. A set of candidate models may be rechosen from the range indicated, and the procedure

reinitiated.

(6) There is an exceptional case when the true system is outside the range of the candidate models. The posterior probability of the "closest" model will reach unity rapidly. In this case also, a new set of candidates should be selected.

The procedures presented are illustrated in a flow chart in Fig.33. It is noted that this type of application repeated on the same data set violates the premises of the estimation algorithm developed. It is evident when one considers that the same noise set is encountered over and over, as the process is repeated, and this clearly violates the white noise assumption. From the following results, it is seen that the procedure may work well even when certain fundamental assumptions are violated. This is a desirable feature of any procedure, since applications seldom match theoretical developments exactly.

5.4.2 Example 5-1:

This example demonstrates the use of the procedure, just outlined, on the stream quality problem. It is assumed that the saturation DO level is known and constant, k_3 is zero, and the variance of the disturbance term w(t) is also known. The coefficients k_1 and k_2 as well as the mean of w(t), $\bar{w}$, are subject to uncertainty.

A set of data is generated by simulating the stream and measurements on the digital computer. The initial conditions of BOD and DO are assumed known. There are 35 data points obtained along the stream. The time between two data point is preprogrammed and is not constant.

The dynamical structure is described by Eq. 5-11, where $k_3 = 0$, $C_s = 8.5$, and the disturbance w(t) has mean $\bar{w}$ and variance Q = .01. Coefficient k_1 and k_2 as well as the mean of w(t), $\bar{w}$, are uncertain,

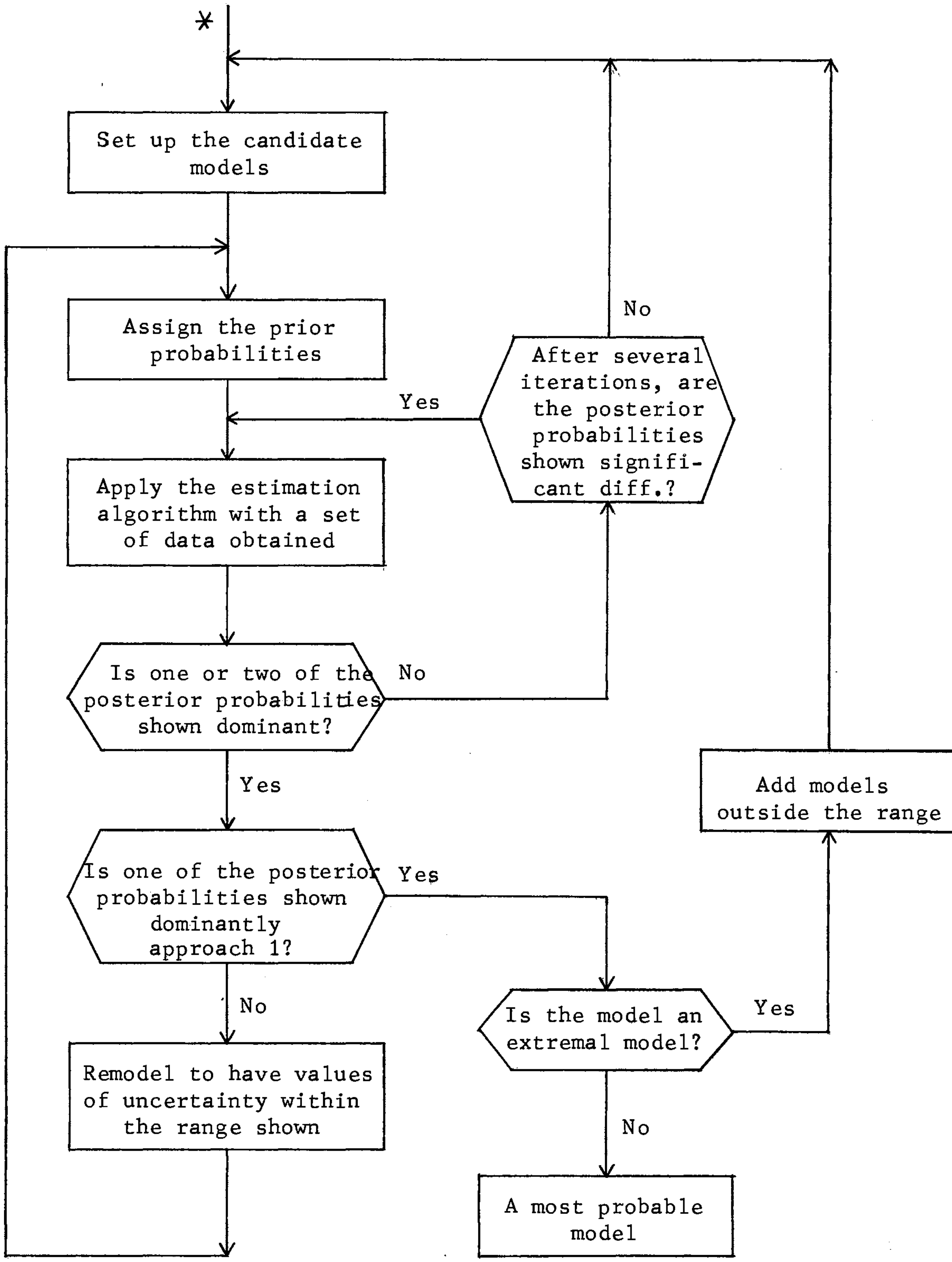


Figure 33. Flow Chart of the System Identification Procedures

but the range of each coefficient is known. The range of possible values is

k_1: .01 --- .02 hr^{-1}

k_2: .02 --- .03 hr^{-1}

$\bar{w}$: .01 --- .05 $mg(lit.-hr)^{-1}$

The observation model is given by Eq. 5-11, there the measurement noise v(k) is zero mean and has variance R = .05. The initial concentration of BOD and DO are 20. mg/lit and 7.0 mg/lit, respectively. The actual values of uncertain coefficients used in the simulation are $k_1 = .016$, $k_2 = .023$, and $\bar{w} = .02$.

As a first attempt, four models are chosen by selecting the possible parameter values

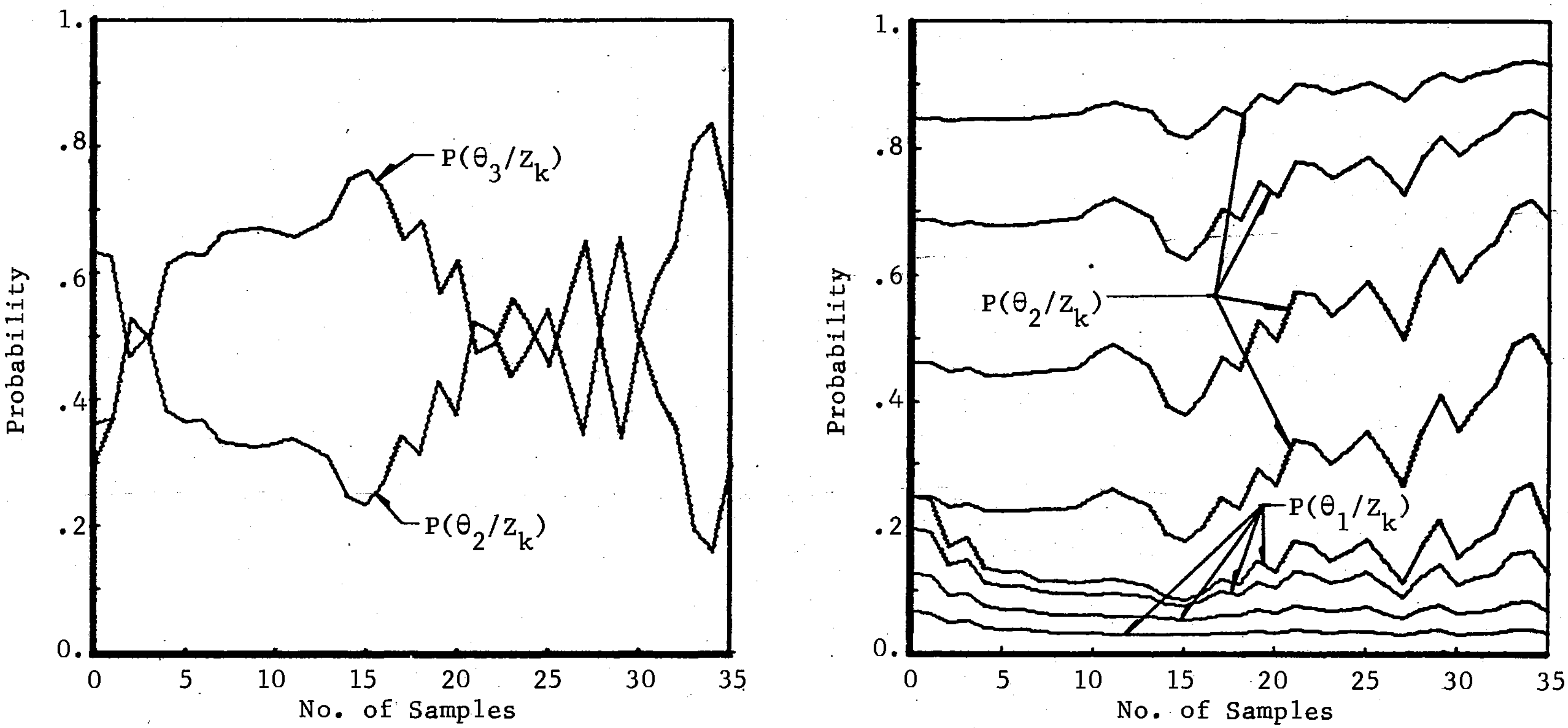
θ_1: $k_1 = .01$, $k_2 = .02$, $\bar{w} = .01$

θ_2: $k_1 = .013$, $k_2 = .023$, $\bar{w} = .02$

θ_3: $k_1 = .016$, $k_2 = .026$, $\bar{w} = .03$

θ_4: $k_1 = .019$, $k_2 = .029$, $\bar{w} = .04$

Using Algorithm 3-1 with equal prior probability of .25 for each model, after the first iteration the posterior probabilities of the models are $P(\theta_1) = .00485$, $P(\theta_2) = .56656$, $P(\theta_3) = .42820$, and $P(\theta_4) = .00039$. It is seen that models θ_2 and θ_3 are dominant. After three iterations, however, the posterior probabilities of models θ_2 and θ_3 do not show much different. The probability plots are shown in Fig. 34(a). One may postulate that the following alternatives may be true.



θ_1: k_1=.01, k_2=.02, $\bar{w}$=.01

θ_2: k_1=.013, k_2=.023, $\bar{w}$=.02

θ_3: k_1=.016, k_2=.026, $\bar{w}$=.03

θ_4: k_1=.019, k_2=.029, $\bar{w}$=.04

(a)

θ_1: k_1=.013, k_2=.023, $\bar{w}$=.02

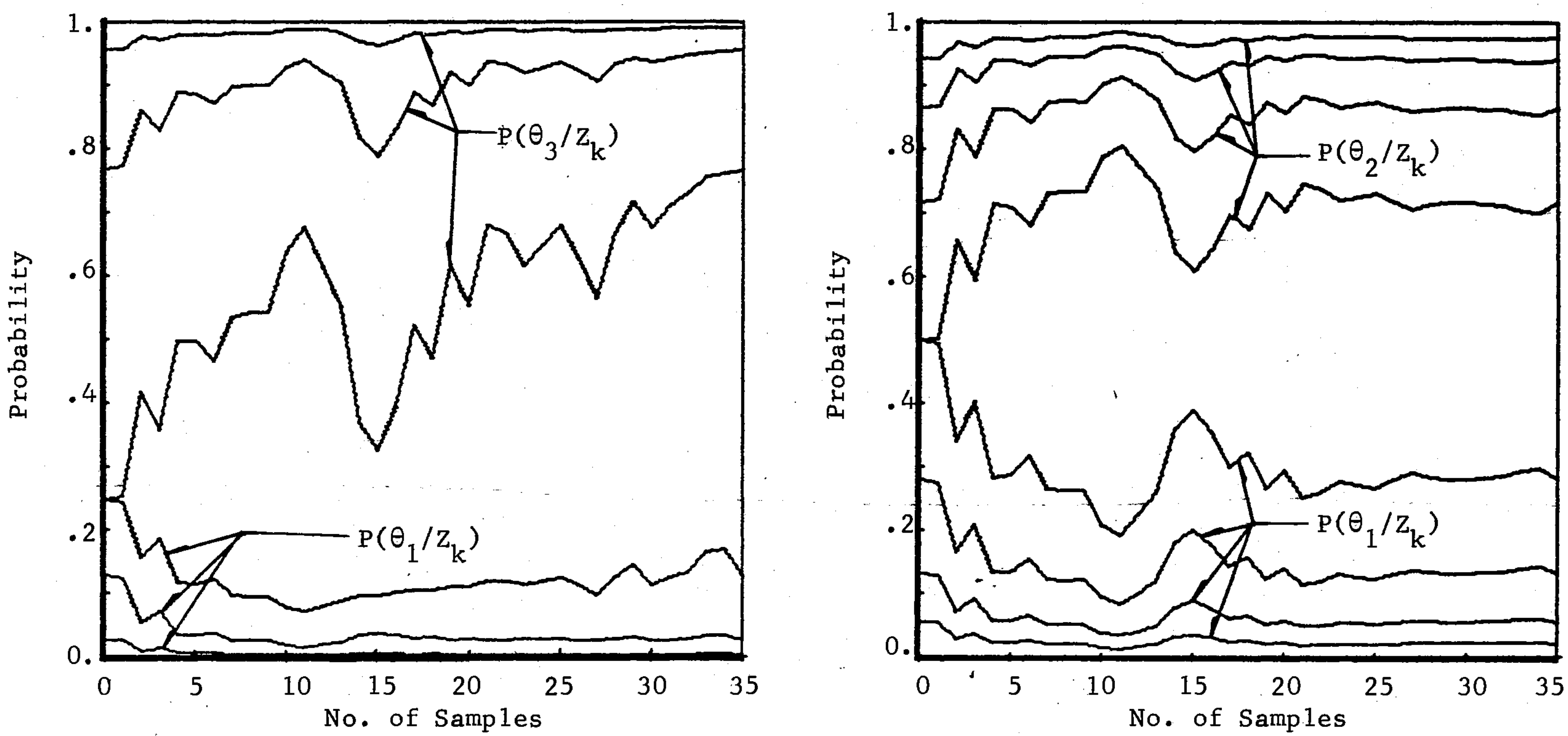
θ_2: k_1=.014, k_2=.024, $\bar{w}$=.02

θ_3: k_1=.015, k_2=.025, $\bar{w}$=.03

θ_4: k_1=.016, k_2=.026, $\bar{w}$=.03

(b)

Figure 34. Identification Plot for Example 5-1



(c)

θ_1: k_1=.013, k_2=.023, $\overline{w}$=.02

θ_2: k_1=.013, k_2=.026, $\overline{w}$=.03

θ_3: k_1=.016, k_2=.023, $\overline{w}$=.02

θ_4: k_1=.016, k_2=.026, $\overline{w}$=.03

(d)

θ_1: k_1=.014, k_2=.024, $\overline{w}$=.02

θ_2: k_1=.016, k_2=.023, $\overline{w}$=.02

Figure 34 (cont'd). Identification Plot for Example 5-1

(1) The true system has value of parameters inside the range of those of models θ_2 and θ_3.

(2) The true system has parameters which are associated with models θ_2 and θ_3, but the correct parameter set may not be obtained from just one of these.

For case (1), two additional models are chosen. The new set of models of interest is

θ_1: $k_1 = .013$, $k_2 = .023$, $\bar{w} = .02$

θ_2: $k_1 = .014$, $k_2 = .024$, $\bar{w} = .02$

θ_3: $k_1 = .015$ $k_2 = .025$, $\bar{w} = .03$

θ_4: $k_1 = .016$, $k_2 = .026$, $\bar{w} = .03$

The above candidates are postulated in the hope that by narrowing down the range of the values of uncertain parameters, the true system may be identified. By applying Algorithm 3-1 with equal prior probability again, posterior probabilities can be generated. After one iteration these are $P(\theta_1) = .1986$, $P(\theta_2) = .46255$, $P(\theta_3) = .18876$, and $P(\theta_4) = .15010$. It is seen that model θ_2 is dominant. Posterior probabilities plots for models θ_1 and θ_2 are shown in Fig. 34(b). Eventually, after several more iterations, the posterior probability of model θ_2 may approach 1. Therefore, model θ_2 is a likely model.

For case (2), a set of candidate models of interest is postulated.

θ_1: $k_1 = .013$, $k_2 = .023$, $\bar{w} = .02$

θ_2: $k_1 = .013$, $k_2 = .026$, $\bar{w} = .03$

θ_3: $k_1 = .016$, $k_2 = .023$, $\bar{w} = .02$

θ_4: $k_1 = .016$, $k_2 = .026$, $\bar{w} = .03$

With the same procedure as before, posterior probabilities after the first iteration are $P(\theta_1) = .12923$, $P(\theta_2) = .00262$, $P(\theta_3) = .77047$, and $P(\theta_4) = .09767$. The probability plots of models θ_1 and θ_3 are shown in Fig. 34(c). It is seen that after three iterations, the posterior probability of model θ_3 is very close to unity.

Model θ_2 of case (1) and model θ_3 of case (2) are favorable candidate models. The algorithm is applied to these models, and the posterior probability plots are indicated in Fig. 34(d). It may be concluded that model θ_3 of case (2) is the most probable system. This is in fact the model which was simulated . It is noteworthy that the algorithm was able to identify the correct system, even though the application did not strictly fit the theoretical development.

Remarks:

(1) In practice, due to different sets of data, different favorable systems may result. These models may be considered candidate models for future use.

(2) The estimated sag curve from the first set of trial models is presented in Fig. 35. It is seen that even when the correct system is unknown the estimate of the sag curve is still reasonably close to the actual trajectories.

5.5 The Control Problem

In this section, an open loop optimal control problem for the water pollution system is posed in a deterministic sense. The result is used in a stochastic model of the system. A regulator problem is formulated such that a feedback correction term to the open loop control is obtained to keep the response near the optimal trajectories. The

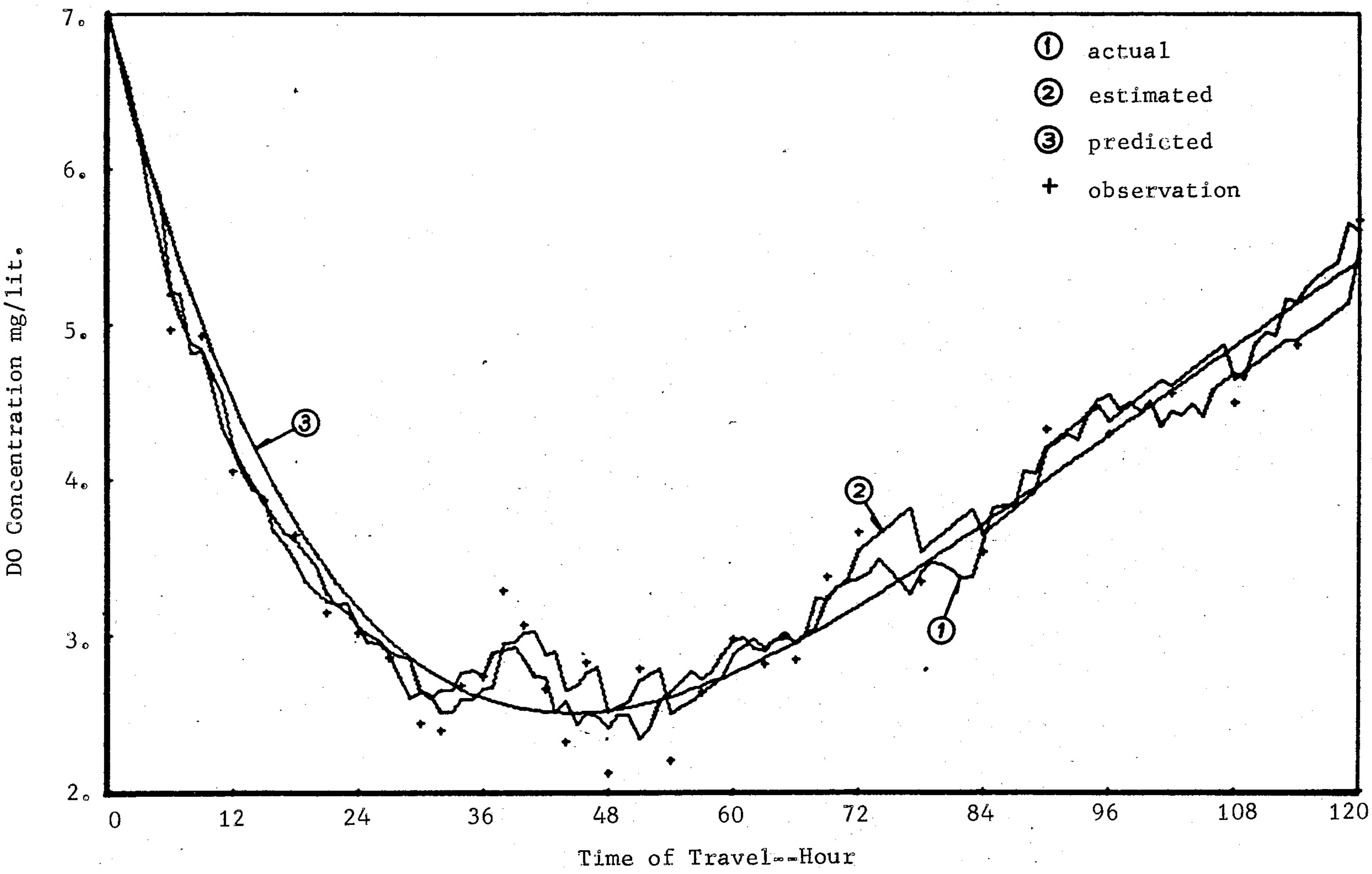


Figure 35. The Simulated Oxygen Sag Curve

situation of control under uncertainty is included to demonstrate a possible application of the algorithm developed in Chapter IV.

5.5.1 An Optimal Control Criterion

In current practice, two types of standards have been proposed for water resource management.

(1) Stream standard, which requires that the DO concentration in any part of a stream may not be lower than a specified level.

(2) Effluent standard, which defines the maximum BOD concentration at any outfall.

The stream standard limits maximal use of the stream assimilative capacity while the effluent standard sometimes may require increasing efficiency of a waste treatment to prevent violation of the standard. The criterion here is to optimally use the assimilative capacity of a stream such that stream standard is maintained, while a fixed effluent standard is imposed on the individual industrial pollutors as well as the domestic sewage plants.

The water pollution problem is usually serious in a section of stream close to a municipal area, where, in addition to the domestic sewage discharges, the industrial pollutors cause the stream to be seriously polluted. It is assumed that the wastes in the area are under certain supervision, or the wastes are collected together and then discharged along the stream under some programmed supervision. A certain length of a stream is said to be under control, where the concentrations of BOD and DO at the end of the section are specified such that further down stream the critical DO concentration will meet the standard requirements.

An optimal control criterion is proposed as follows. In a section of a stream, the waste should be discharged into the stream maximally, but the terminal conditions for BOD and DO must be met without introducing man-made reaeration. If stream velocity is known, time and distance may be related accordingly. For simplicity, the time domain formulation is presented here. The control criterion is equivalent to maximizing the following performance index.

$$J_1 = \int_0^{t_f} u_1(t)dt \qquad (5-13)$$

where $u_1(t)$ is the waste discharged in mg/lit-time, which is assumed to be discharged continuously along the stream, and t_f is the time of travel in between points of interest.

This type of optimal control problem is known as "bang-bang" or switching control problem [34]. The derivation of the solution is presented in Appendix B. In the case where only one switching is considered for easy implementation, the control strategy is described by

$$u_1(t) = \begin{cases} E & \text{for} \quad 0 \leqslant t \leqslant t_s \\ \varphi E & \text{for} \quad t_s \leqslant t \leqslant t_f \end{cases} \qquad (5-14)$$

where both E and φ are constants and positive real numbers, and t_s is the switching time. Given the boundary conditions and a specified value of E, one can find corresponding values of φ and t_s.

Since the initial concentrations of BOD and DO depend on the upstream residuals, they are random in nature. Due to this fact and other uncertain factors in the processes, the terminal conditions may differ from those desired levels. Hence, a closed loop control strategy is needed allowing minor variations from the open loop control so that the actual trajectory is close to the optimal trajectory.

5.5.2 Stochastic Regulator Control Problem

In this section, the solution to the stochastic regulator problem presented in Algorithm 4-1 is applied to the water quality control problem. An open loop optimal control strategy is to be modified using feedback so that variations about the nominal optimal trajectory are small. From the indicated control strategy u(t) in Eq. 5-14, the nominal predicted state trajectory may be obtained by solving the dynamic equation

$$\dot{\bar{x}}(t) = -F\bar{x}(t) + DC_s + C\bar{u}(t) + G\bar{w}(t) \qquad (5\text{-}15)$$

where $\bar{w}(t)$ is the mean of the disturbance, w(t), and the initial state $\bar{x}(0)$ is the given mean of initial concentrations of BOD and DO. A stochastic regulator problem is formulated to find a control u(t) to insure that the actual trajectory is near that determined from Eq. 5-15. To accomplish this, a secondary performance measure is defined. The quadratic performance measure

$$J_2 = E\{\tilde{x}^T(t_f)\Lambda\tilde{x}(t_f) + \int_0^{t_f}(\tilde{x}^T(t)A(t)\tilde{x}(t) + \tilde{u}^T(t)B(t)\tilde{u}(t))dt\} \qquad (5\text{-}16)$$

is to be minimized, where $\tilde{x}(t) = x(t) - \bar{x}(t)$ is the difference between the actual state and the nominal state, and $\tilde{u}(t) = u(t) - \bar{u}(t)$ is the difference between the applied control and the nominal control. The control criterion is somewhat arbitrary but should have the effect of minimizing the error deviations from the nominal trajectory.

Using Eqs. 5-9 and 5-15, the equation for the variations is obtained.

$$\dot{\tilde{x}}(t) = -F\tilde{x}(t) + C\tilde{u}(t) + G\tilde{w}(t) \tag{5-17}$$

where $\tilde{w}(t)$ is gaussian white noise with zero mean. If the measurements are obtained from observation stations measuring DO along the stream, the observation model is of the form

$$z(k) = H(t_k)x(t_k) + v(k)$$

$$= x_2(t_k) + v(k) \tag{5-18}$$

where $H = [0 \quad 1]$. The problem formulation is then similar to that which resulted in Algorithm 4-1. The solution to the problem may be stated accordingly.

(1) The variational control strategy is governed by

$$\tilde{u}(t) = -B^{-1}CS(t)\hat{\tilde{x}}(t) = -B^{-1}CS(t)[\hat{x}(t/Z_k) - \bar{x}(t)] \tag{5-19}$$

and

$$u(t) = \bar{u}(t) + \tilde{u}(t) \tag{5-20}$$

where

$$\dot{S}(t) = F^TS(t) + S(t)F + SCB^{-1}C^TS - A \qquad S(t_f) = \Lambda \tag{5-21}$$

(2) The estimated state state $x(t/Z_k)$ is given by the estimation algorithm.

(i) In between observations, $t_k \leqslant t < t_{k+1}$

$$\dot{\hat{x}}(t/Z_k) = -F\hat{x}(t/Z_k) + DC_s + Cu(t) + G\bar{w}(t) \tag{5-22}$$

while the variance satisfies the matrix differential equation

$$\dot{V}_x(t/Z_k) = -FV_x - V_xF^T + GQG^T \tag{5-23}$$

where Q is the covariance matrix of the plant disturbance. The initial conditions at t_k are $\hat{x}(t_k/Z_k)$ and $V_x(t_k/Z_k)$.

(ii) At an observation instant t_{k+1},

$$\hat{x}(t_{k+1}/Z_{k+1}) = \hat{x}(t_{k+1}/Z_k) + K(t_{k+1})[z(k+1) - H\hat{x}(t_{k+1}/Z_k)] \tag{5-24}$$

and

$$V_x(t_{k+1}/Z_{k+1}) = V_x(t_{k+1}/Z_k) - K(t_{k+1})HV_x(t_{k+1}/Z_k) \tag{5-25}$$

where

$$K(t_{k+1}) = V_x(t_{k+1}/Z_k)H^T[HV_xH^T + R]^{-1} \tag{5-26}$$

It is seen that the estimation portion is the same as indicated in Chapter II, except for the nonzero mean of the disturbance considered. The control strategy for u is obtained from Algorithm 4-1, and then added to the nominal control.

5.5.3 Stochastic Control Under Uncertainty

The stochastic regulator problem presented in the previous section for stream pollution control is extended to cover the case where the model is uncertain in this section. As has been indicated, coefficient k_1, k_2 and k_3 as well as statistical parameters of the initial state and disturbance term are subject to uncertainty due to hydrological and temperature effects. Since the section of a stream under control is assumed to be rather short, the parameters may be considered unchanging in a given range, and the nonswitching control algorithm may be applied.

A set of N candidate models is chosen where each model has a set of possible value of uncertain parameters. The ith candidate model has dynamics described as

$$\theta_i: \quad \dot{x}_i(t) = -Fx_i(t) + DC_{si} + Cu(t) + Gw_i(t) \qquad (5\text{-}27)$$

Given that the model θ_i is true, an open loop control $\bar{u}_i(t)$ is obtained such that the performance index Eq. 5-13 is maximized. For this control the optimal trajectory of the state for model θ_i is obtained by solving the dynamical equation

$$\bar{\theta}_i: \quad \dot{\bar{x}}_i(t) = -F_i\bar{x}_i(t) + DC_{si} + C\bar{u}_i(t) + G\bar{w}_i(t) \qquad (5\text{-}28)$$

The variational model is then written as

$$\theta_i: \quad \dot{\tilde{x}}_i(t) = -F_i\tilde{x}_i(t) + C\tilde{u}_i(t) + G\tilde{w}_i(t) \qquad (5\text{-}29)$$

for $i = 1,2,\ldots,N$ and where $\tilde{x}_i = x_i - \bar{x}_i$, $\tilde{u}_i = u - \bar{u}_i$, and $\tilde{w}_i = w_i - \bar{w}_i$.

With the performance measure Eq. 5-16 modified as

$$J_2 = \sum_{i=1}^{N} p_r(\theta_i/Z_k)E\{\tilde{x}_i^T(t_f)\Lambda\tilde{x}_i(t_f) + \int_o^{t_f}(\tilde{x}_i^T A\tilde{x}_i + \tilde{u}_i^T B\tilde{u}_i)dt/Z_k\} \qquad (5\text{-}30)$$

it is seen that the variational controls $\tilde{u}_i$ are different for every i, which complicates the finding of a solution. In order to correspond to the development of the previous chapter, one may assume that

$$\tilde{u} = u - \sum_{i=1}^{N} p_r(\theta_i/Z_k)\bar{u}_i \qquad (5\text{-}31)$$

is the normalized variational control, and the performance measure Eq. 5-30 is replaced by

$$J_2 = \sum_{i=1}^{N} p_r(\theta_i/Z_k)E\{\tilde{x}_i^T(t_f)\Lambda\tilde{x}_i(t_f) + \int_o^{t_f}(\tilde{x}_i^T A\tilde{x}_i + \tilde{u}^T B\tilde{u})dt/Z_k\} \qquad (5\text{-}32)$$

The problem is then cast in a framework similar to that which resulted in Algorithm 4-2. Hence, the solution of the control problem is summarized accordingly.

The control strategy is governed by

$$u(t) = \sum_{i=1}^{N} p_r(\theta_i/Z_k)u_i(t) \tag{5-33}$$

where

$$u_i(t) = \overline{u}_i(t) + \tilde{u}_i(t) \tag{5-34}$$

$$\tilde{u}_i(t) = -B^{-1}C^T S_i(t)[\hat{x}_i(t/Z_k) - \overline{x}_i(t)] \tag{5-35}$$

and

$$\dot{S}_i(t) = F_i^T S_i + S_i F_i + S_i C B^{-1} C^T S_i - A \qquad S_i(t_f) = \Lambda \tag{5-36}$$

The conditional estimate $\hat{x}_i(t/Z_k)$ is given by Algorithm 3-1 with minor modifications

(1) In between observations $t_k \leqslant t < t_{k+1}$,

$$\dot{\hat{x}}_i(t/Z_k) = -F_i\hat{x}_i(t/Z_k) + DC_{si} + Cu(t) + G\overline{w}_i(t) \tag{5-37}$$

and

$$\dot{V}_{x_i}(t/Z_k) = -F_i V_{x_i}(t/Z_k) - V_{x_i}(t/Z_k)F_i^T + GQ_iG^T \tag{5-38}$$

for $i = 1,2,\ldots,N$, where the initial conditions at time t_k are $\hat{x}_i(t_k/Z_k)$ and $V_{x_i}(t_k/Z_k)$.

(2) At an observation time t_{k+1},

$$\hat{x}_i(t_{k+1}/Z_{k+1}) = \hat{x}_i(t_{k+1}/Z_k) + k_i(t_{k+1})[Z(k+1) - H\hat{x}_i(t_{k+1}/Z_k)] \tag{5-39}$$

and

$$V_{x_i}(t_{k+1}/Z_{k+1}) = V_{x_i}(t_{k+1}/Z_k) - K_i(t_{k+1})HV_{x_i}(t_{k+1}/Z_k) \tag{5-40}$$

where

$$K_i(t_{k+1}) = V_{x_i}(t_{k+1}/Z_k)H^T[HV_{x_i}H^T + R]^{-1} \tag{5-41}$$

for i = 1,2,...,N. The posterior probability $p_r(\theta_i/Z_k)$ is updated at time t_{k+1} according to

$$p_r(\theta_i/Z_k) = [1 + \sum_{\substack{j=1 \\ j\neq i}}^{N} L_{ji} \frac{p_r(\theta_j/Z_k)}{p_r(\theta_i/Z_k)}]^{-1} \tag{5-42}$$

where L_{ji} is the likelihood ratio given by

$$L_{ji} = \left| \frac{HV_{x_i}H^T + R}{HV_{x_j}H^T + R} \right|^{\frac{1}{2}} \text{EXP}\{-\tfrac{1}{2}[(z-H\hat{x}_j)^T(HV_{x_j}H^T + R)^{-1}(z - H\hat{x}_j)]$$

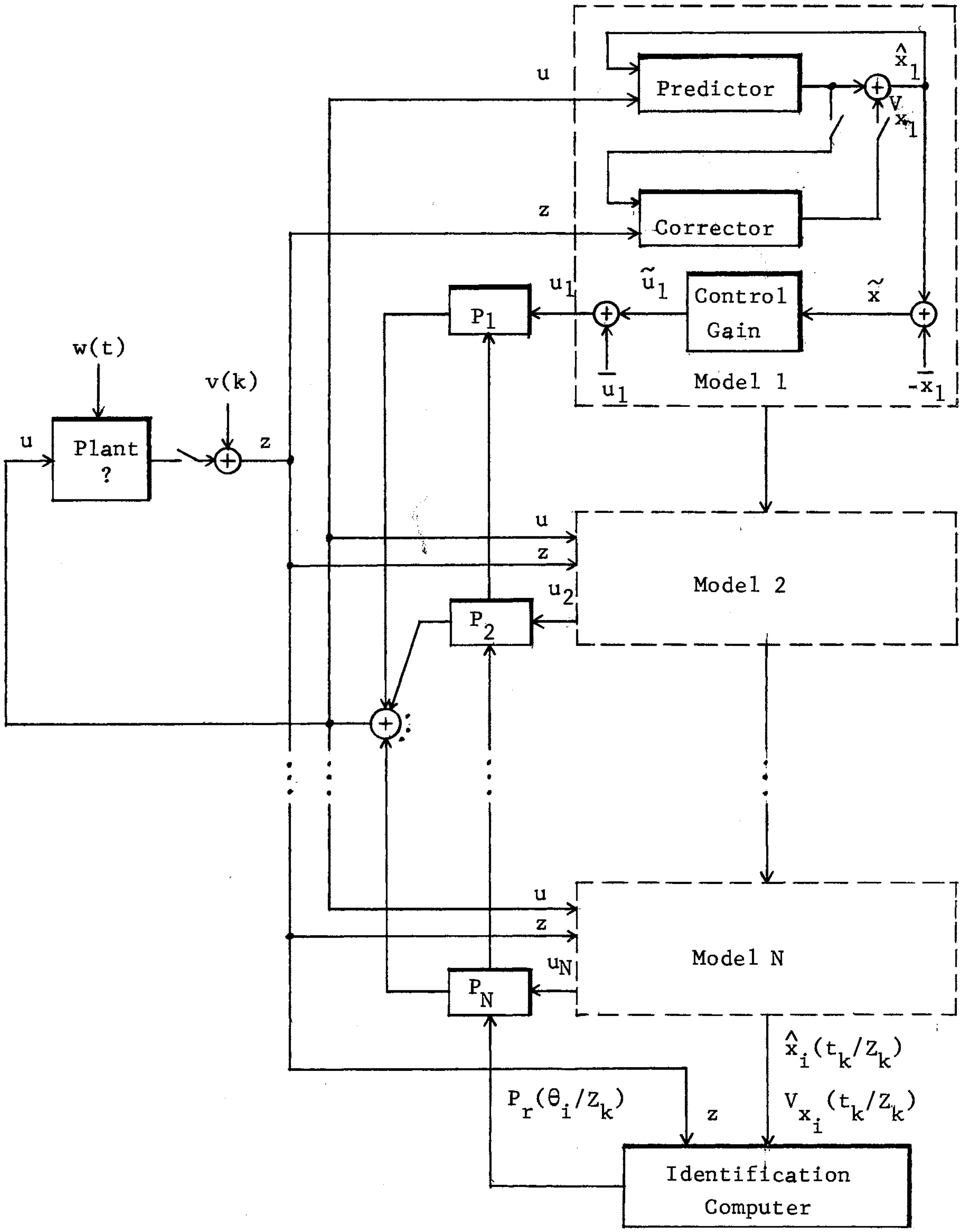
$$-(z - H\hat{x}_i)^T(HV_{x_i}H^T + R)^{-1}(z - H\hat{x}_i)]\} \tag{5-43}$$

The schematic block diagram is presented in Fig. 36. It is noted that the control strategy presented here is a typical application of linear regulator theory to keep errors about a nominal trajectory small [36].

5.5.4 Example 5-2

In this example the simulation is based on the following assumptions.

(1) The velocity in various sections of the range of a stream is



Predictor: Eqs. 5-26 and 5-29 Identification: Eqs. 5-31 and 5-32

Corrector: Eqs. 5-27 and 5-30 Control Gain : Eqs. 5-25 and 5-24

Figure 36. Block Diagram for the Control Algorithm with Reference Vectors

relatively fixed. The conversion between distance and time of travel may then be easily computed. The range of a stream under control is 15 hours of travel time, or $t_f = 15$, e.g., for the average stream velocity of 3 fps, the range under consideration is 30 miles.

(2) The coefficient k_1 and k_2 as well as the parameter w are subject to uncertainty. Within the time of travel, the environmental effects, such as the effect due to the changes of temperature or flow conditions, are insignificantly small, and the uncertainty is assumed to be primarily due to upstream conditions. The range of the values of the parameters are known to be

k_1:	.01	---	.02	hr^{-1}
k_2:	.02	---	.03	hr^{-1}
$\bar{w}$:	.01	---	.05	mg/lit-hr

(3) The initial concentration for BOD and DO are random variables with statistics

$$\text{BOD:} \quad x_1(0) \sim N[3.2, \quad .5]$$

$$\text{DO:} \quad x_2(0) \sim N[7., \quad .5]$$

The desired terminal concentrations for BOD and DO are assumed to be 11.88 mg/lit and 6.0 mg/lit, respectively, such that further down stream the critical reach will not be lower than the specified level (4.0 mg/lit). The other parameters are assumed known, i.e., $k_3 = 0$, $C_s = 8.5$ mg/lit, Q = .01, and R = .05.

(4) For simulation, the actual values for the uncertain parameter are assumed to be

$$k_1 = .016, \quad k_2 = .023, \quad \text{and} \quad \bar{w} = .02$$

Two sets of candidate models are under consideration. The values of uncertain parameters used for each candidate model are listed as follows.

Set A	θ_{A1}:	$k_1 = .01$,	$k_2 = .02$,	$\bar{w} = .01$
	θ_{A2}:	$k_1 = .013$,	$k_2 = .023$,	$\bar{w} = .02$
	θ_{A3}:	$k_1 = .016$,	$k_2 = .026$,	$\bar{w} = .03$
	θ_{A4}:	$k_1 = .019$,	$k_2 = .029$,	$\bar{w} = .04$
Set B	θ_{B1}:	$k_1 = .013$,	$k_2 = .023$,	$\bar{w} = .02$
	θ_{B2}:	$k_1 = .013$,	$k_2 = .026$,	$\bar{w} = .03$
	θ_{B3}:	$k_1 = .016$,	$k_2 = .023$,	$\bar{w} = .02$
	θ_{B4}:	$k_1 = .016$,	$k_2 = .026$,	$\bar{w} = .03$

where set B includes the actual model, θ_{B3}. For the given boundary conditions, the nominal dumping control for each candidate model is obtained as follows [see Appendix B].

θ_{A1}:	$E = 3.386$,	$\varphi E = .0228$	mg/lit-hr
θ_{A2} and θ_{B1}:	$E = 2.351$,	$\varphi E = .3$	mg/lit-hr
θ_{A3} and θ_{B4}:	$E = 1.686$,	$\varphi E = .485$	mg/lit-hr
θ_{A4}:	$E = 1.215$,	$\varphi E = .621$	mg/lit-hr
θ_{B2}:	$E = 3.238$,	$\varphi E = .099$	mg/lit-hr
θ_{B3}:	$E = .969$,	$\varphi E = .644$	mg/lit-hr

where the switching time is fixed at 3 for all models. The desired nominal trajectories of BOD and DO for each candidate model may be computed and stored. In actual operation, due to the randomness in initial concentrations as well as the noise disturbances the response will not follow any of the nominal trajectories exactly. The terminal condition, which is important in guaranteeing that the critical reach will not be lower than 4.0 mg/lit, may not be met. A closed loop regulator control is used to minimize these errors. The performance measure used is

$$J_2 = E\{5x_1^2(t_f) + 5x_2^2(t_f) + \int_0^{t_f} (x_1^2 + x_2^2 + u^2)dt\}$$

It is assumed that there are 15 observation stations along the stream.

The simulation results are presented in Fig. 37 through 42, where the following cases are considered and compared.

Case 1: The closed loop control under uncertainty is used with set A as the candidate models.

Case 2: Same as case 1 but set B is adopted.

Case 3: The closed loop control is applied with the correct model known with certainty.

Case 4: Only the open loop control is applied with certainty of the model.

Case 5: The open loop control is applied but the erroneous model θ_{A4} is used.

Case 6: The predicted nominal trajectories under certainty.

In the figures, the numbers assigned to the various results correspond to the cases stated above. In cases 1 and 2, the algorithm presented in Eqs. 5-33 through 5-43 is used with equal prior probabilities. While in case 3, the algorithm presented in Eqs. 5-19 through 5-26 is adopted. In Figs. 37 through 39, the results for BOD, DO, and

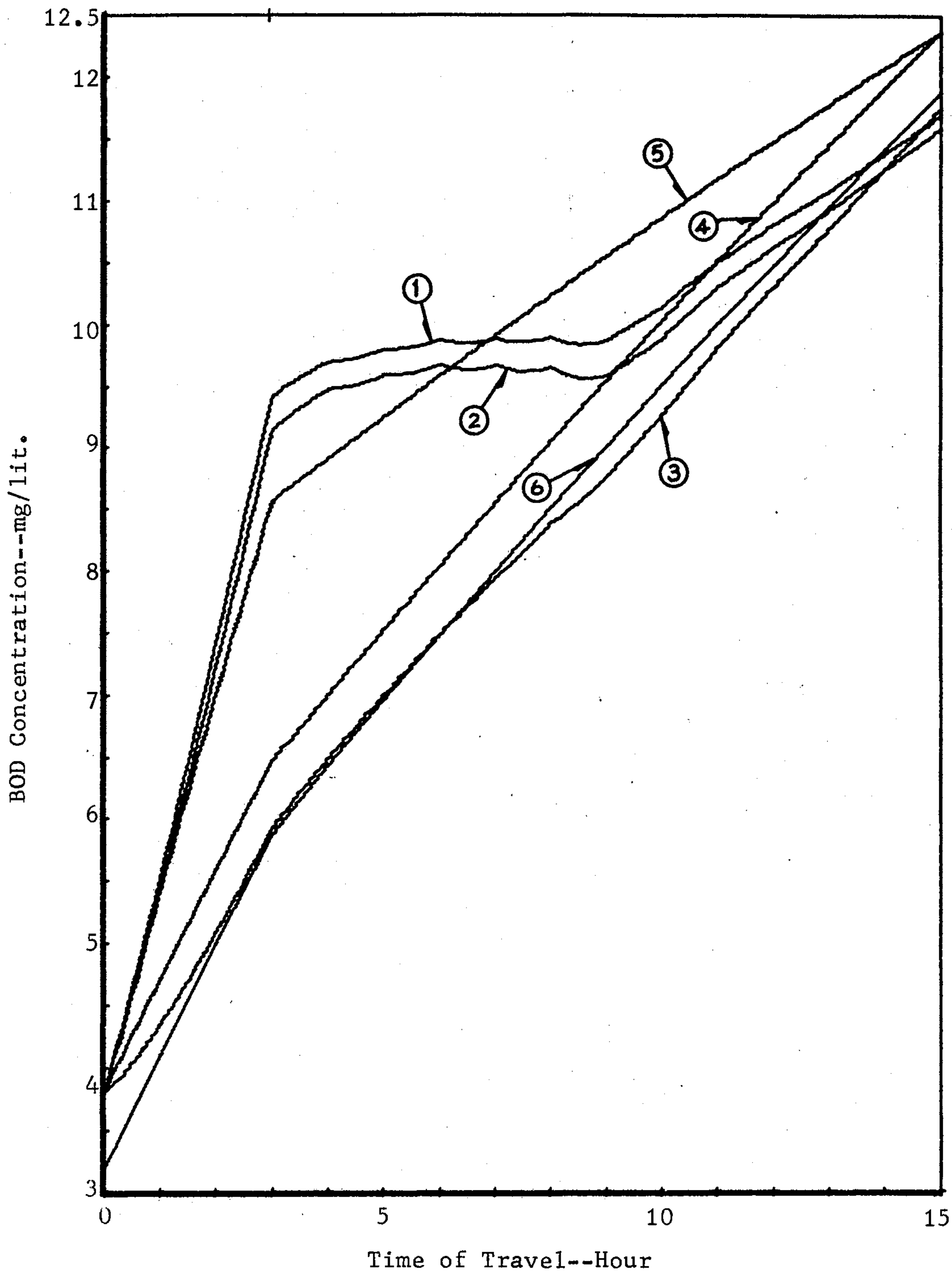


Figure 37. BOD Trajectories—Single Run

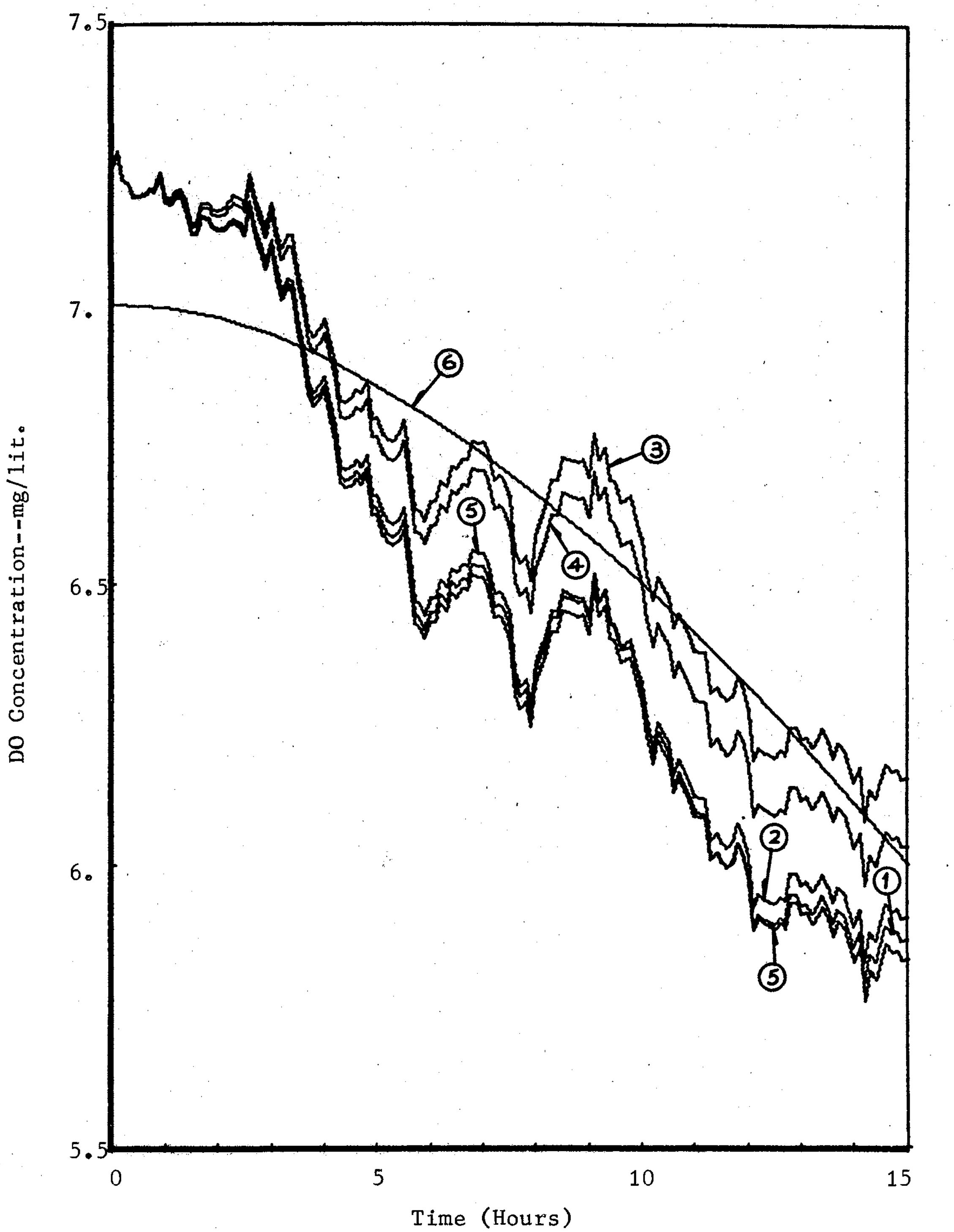


Figure 38. DO Trajectories—Single Run

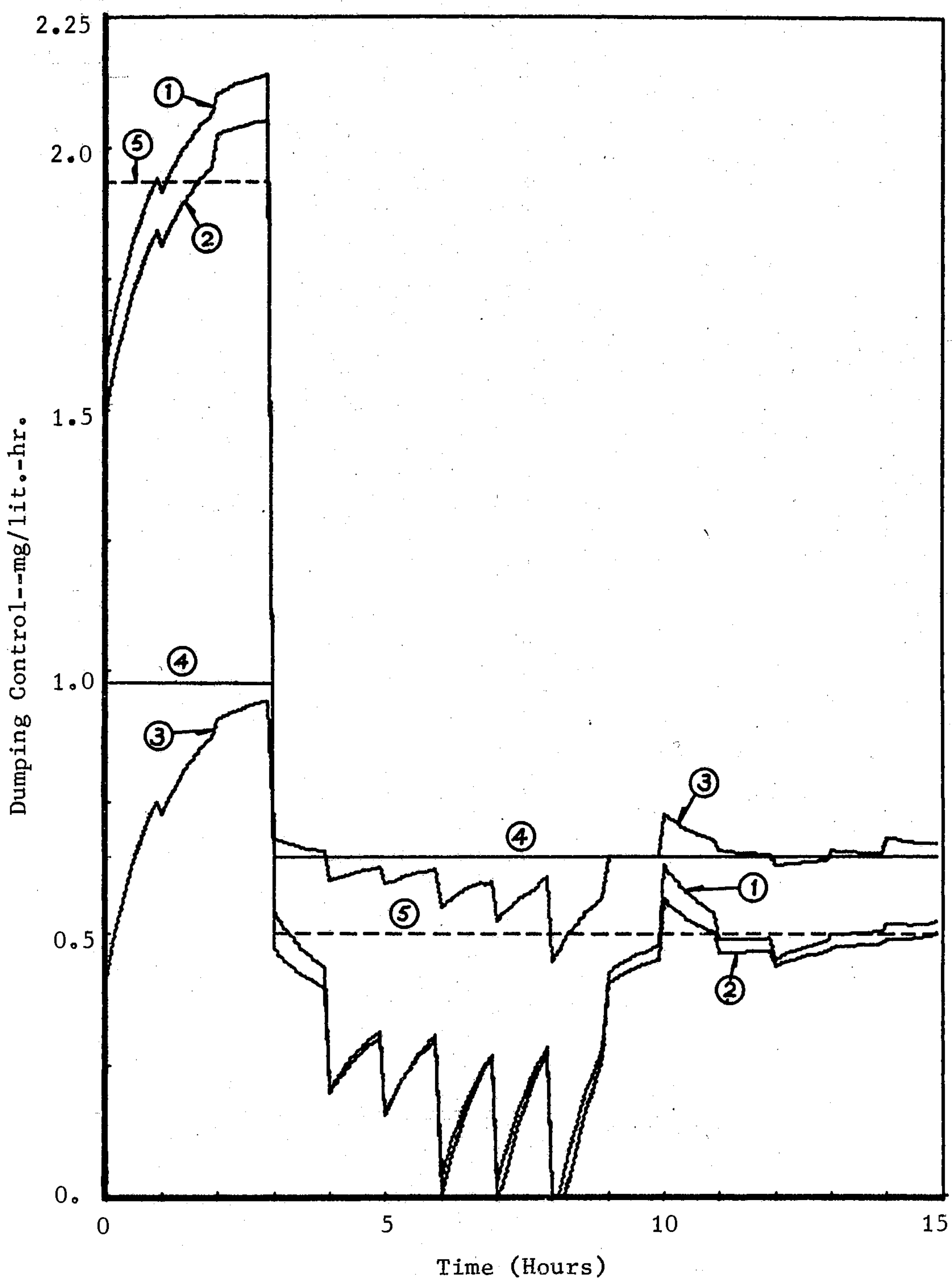


Figure 39. Control Trajectories—Single Run

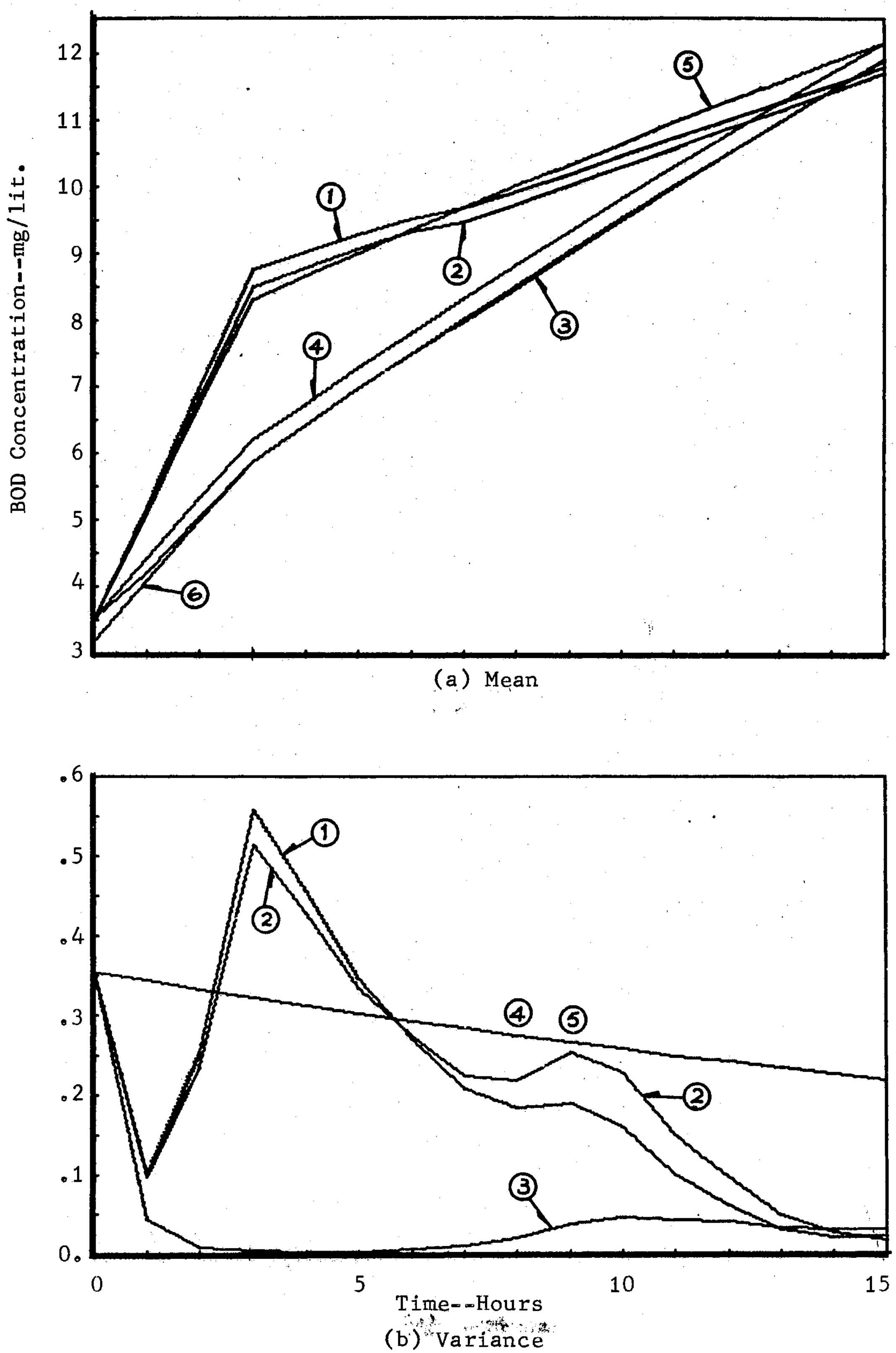


Figure 40. Average BOD Trajectories--20 Runs

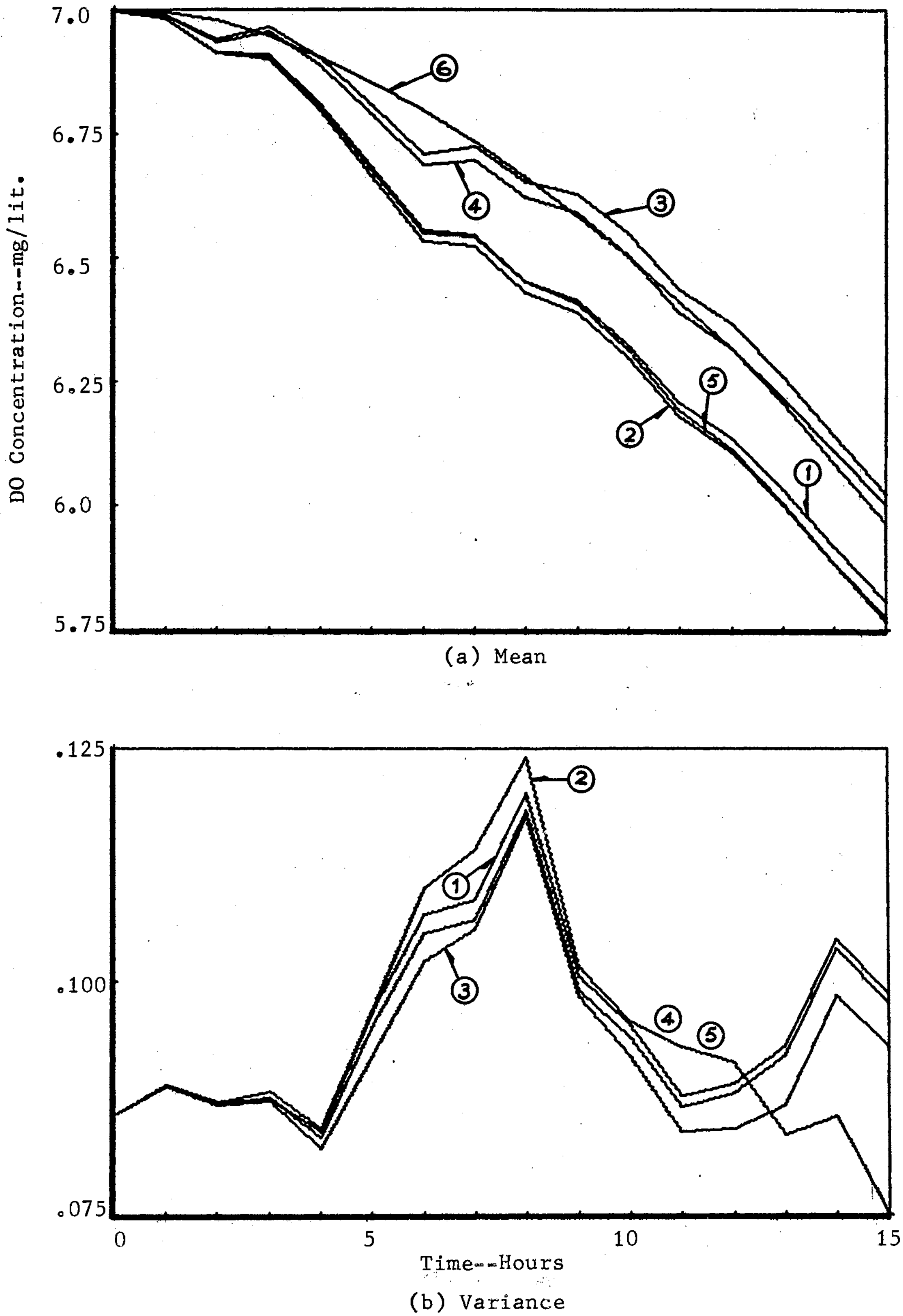
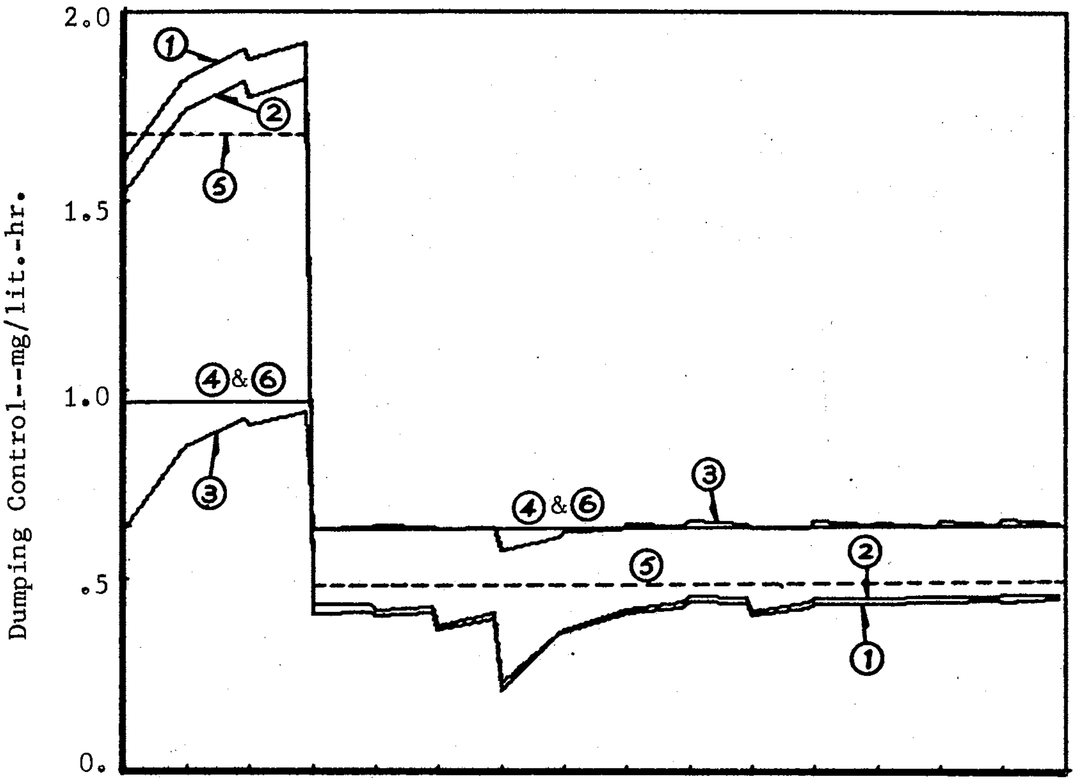
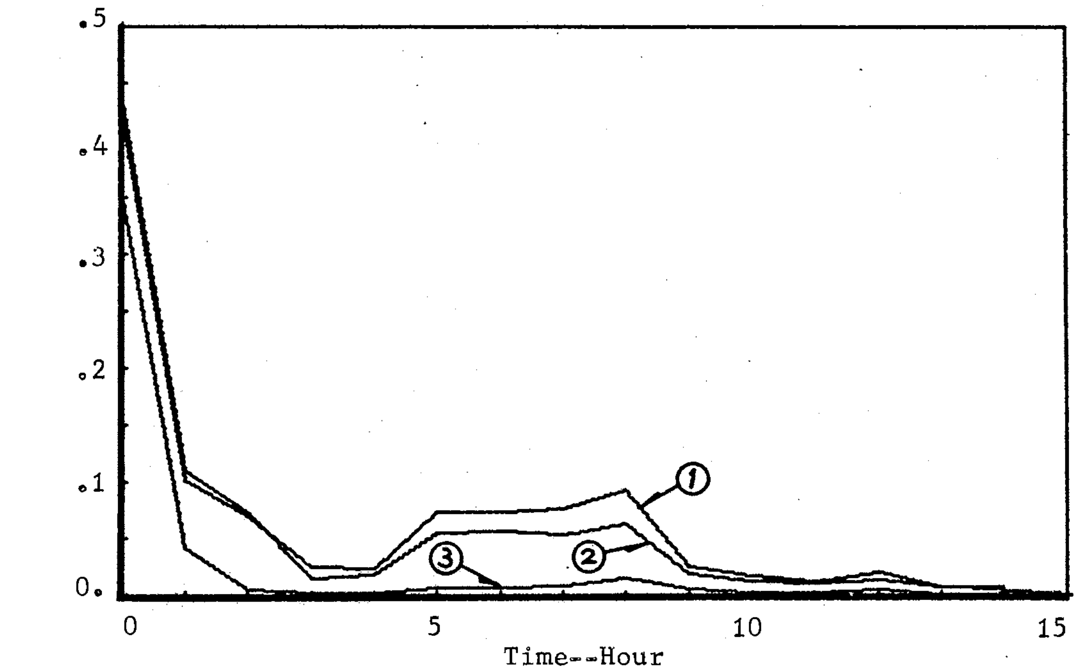


Figure 41. Average DO Trajectories--20 Runs



(a) Mean



(b) Variance

Figure 42. Average Dumping Control--20 Runs

the control u(t) for typical single run are presented. The average performances of 20 runs are shown in Figs. 40 through 42. The sample variance is also plotted as shown in part (b) of the figures. The total waste dumped into the stream for the various cases is compared in Table VI.

TABLE VI

WASTE DUMPED (mg/lit)

	Case 1	Case 2	Case 3	Case 4 & 6	Case 5
Single run	10.178	10.026	9.878	10.632	10.879
Average of 20 runs	10.463	10.394	10.303	10.632	10.879
Variance	0.365	0.371	0.407	0.	0.

The BOD trajectories shown in Figs 37 and 40 are rather smooth, this is partly due to the small variation in dumping control shown in Figs. 39 and 42. The terminal concentrations of BOD and DO associated with closed loop control under uncertainty are shown to be better than those achieved by open loop control. The total waste dumped for each is nearly the same as shown in Table VI. Slightly more waste is deposited with an open loop control for a known system, but the terminal concentrations are in error due to erroneous initial conditions.

The DO trajectories presented in Fig. 38 and 41 are not as smooth as the BOD trajectories. This is due chiefly to the effect of additive disturbances. The terminal concentrations are dependent on the initial concentrations. That is, with the higher initial DO concentration, the terminal concentration will also be higher. The average terminal DO concentration, when the feedback control under uncertainty is used, differs considerably from the nominal value, as shown in Fig. 41. If the penalty for terminal error in the cost function were increased, it is expected that the terminal concentrations would be close to the nominal values. Greater cost of control would probably result, meaning either too much waste deposited in the river or large variations about nominal controls which would cause difficulty in implementation.

The predicted critical reach for various cases are shown in Table VII where model θ_{B3} is known to be the actual system. The values inside the parentheses indicate the time of travel in hours, when the critical reach occurred. It is seen that cases 1 and 2 are better than cases 4 and 5, since the critical reach is not exceeded by as great a margin and the variance is smaller.

TABLE VII

THE CRITICAL DO CONCENTRATION[1]

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Single Run	4.04658 (37.17)	3.99943 (37.07)	4.10283 (38.83)	3.87372 (38.82)	3.79050 (37.72)	4.00117 (38.09)
Averaged 20 Runs	3.93421	3.97175	4.00479	3.90922	3.82528	4.00117
Variance	.05918	.05698	.04149	.17698	.18273	--

[1]Value in parenthesis indicates the time where critical reach occurs.

The control algorithm seems to work reasonably well for case 1, where the true system is not even a candidate model. When results are compared with that of case 2, where the true system is a candidate model, little difference is noted.

5.5.5 Implementation

A procedure for implementing the proposed control strategy for a given distance along the stream is described below.

It is assumed that the region under control and the observation stations are fixed. Therefore, whenever the stream velocity is changed, the time of travel from station to station as well as between the region of interest is changed. The observation time scheduled has to be redefined and the open loop optimal control strategy must be recomputed accordingly. Major changes in stream flow are usually seasonal, so that in daily operation one may consider the velocity as specified.

The implementation of a continuous dumping u(t) could be difficult, so discretized dumping is considered. That is, the discharge pipeline is arranged along the stream with outlets separated by a distance corresponding to a time increment Δt. It should be noted that Δt is assumed to vary with stream velocity, since the spacing increment, $\Delta \xi$ are fixed. At an outlet, the waste dumped is $u(t)\Delta t$. The concentration of waste to be dumped at the outlet is defined as

$$\lambda_u = u(t)\Delta t \qquad \text{mg/lit} \tag{5-43}$$

The relationship between BOD concentration before and after receiving waste in the stream is described by the following equation

$$\lambda_w \varphi_w + \lambda_s \varphi_s = \lambda_R \varphi_R = \lambda_R(\varphi_w + \varphi_s) \tag{5-44}$$

where λ_w and λ_s are the BOD concentrations of the waste and the stream, respectively, λ_R is the resulting concentration after mixing. φ_w and φ_s are the flow rates, in cubic feet per second (cfs) or million gallon per day (MGD), of the waste discharge and the stream, respectively. The various concentrations are also related by the expression

$$\lambda_R = \lambda_u + \lambda_s \tag{5-45}$$

In the case where $\varphi_s \gg \varphi_w$, the relation 5-45 may be approximated by

$$\lambda_w \varphi_w + \lambda_s \varphi_s = (\lambda_u + \lambda_s)\ \varphi_s$$

or

$$\lambda_w \varphi_w = \lambda_u \varphi_s \tag{4-46}$$

It is seen that one can control either the dumping concentration or the dumping flow rate. In practice, the dumping concentration is fixed from the output of a treatment plant. Therefore, the applied control is the dumping flow rate. This is achieved by adjusting the valve openings of the outlets for the waste discharge, and should not be difficult to implement. The waste flow rate should be adjusted according to the relationship

$$\varphi_w = \frac{\lambda_u}{\lambda_w}\ \varphi_s = \left(\frac{\varphi_s}{\lambda_w}\right) u(t)\Delta t \tag{5-47}$$

In many cases, the dumping waste has some DO residuals. This may be taken into account by inducing its effect in the mean value of the disturbance term, $\bar{w}$.

5.6 Summary

In this chapter, a potential application of the algorithms developed in previous chapters to a stream pollution problem has been described. There are certainly some factors which have not been taken into account individually, such as the effects due to temperature changes and small variations in flow conditions. These effects have been treated by assuming uncertainty and random disturbances in the problem formulation. The following remarks are made concerning this application.

(1) Since BOD and DO concentrations are nonnegative quantities, the assumption that the state variables are normally distributed is not true in a strict sense. The results of simulation indicates that violation of this assumption does not seriously detract from the effectiveness of the algorithm.

(2) The consideration of the disturbance term as white noise may not be appropriate. This is true especially when the presence of algae in the water is evident. In such a case a colored noise model might best describe the process. The order of the state is then increased by one and the associated coefficients must be identified.

(3) The seasonal effects are not considered in this application. Whenever the stream characteristics show great changes, the candidate models and the control strategy should be redefined. In daily operation, however, the effect of the photosynthesis due to the presence or absence of day light may be taken into account by considering a different set of candidate models.

(4) Abnormal situations, such as heavy rain run off, sudden weather changes, and so forth, may cause such severe changes that application

of the suggested algorithms would give poor results.

(5) The critical reach downstream from the controlled region may be monitored, and the information then could be used to update the desired terminal conditions such that the assimilative capacity of a stream may be maximally used.

Overall, the application described in this chapter is encouraging based on simulated results. The proposed techniques are relatively easy to implement, and can be extended to other engineering problems.

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 Summary

In this study a generalized treatment of estimation and control for systems with uncertain models has been presented. The techniques developed should be applicable to a large class of engineering problems. Systems with continuous dynamical structures and discrete observations have been studied since physical applications often fall within this framework. It has been demonstrated that a Bayesian approach could be successfully used to obtain the minimum mean square error estimate for a system which remains fixed during the interval of interest. In the switching case, however, the optimal solution requires an evergrowing memory. A suboptimal technique is achieved by truncating the memory requirements of the filter one stage back.

In the control portion of this study, strategies have been developed in which the control is a linear combination of the state estimates. This corresponds to the separation principle of stochastic control. The control gains are obtained using deterministic methods, not involving the estimation portion of the problem.

Illustrative examples were included to demonstrate the algorithms developed. A more extensive application to a water quality control problem was presented in Chapter V. It was demonstrated that even though some of the assumptions of the basic theoretical developments were violated, reasonably good results could be obtained.

6.2 Conclusions

It has been shown that, in the nonswitching case of estimation-under-uncertainty, the estimate given by Algorithm 3-1 is optimal. Moreover, if the true model is one of the candidate models, it has been seen that after a sufficient number of observations, the posterior probability of the correct model approaches unity with probability one. This system identification property has been demonstrated in Chapter V, where it was seen that the estimation algorithm may be used in identifying an unknown system, even though the same data set is used repeatedly. Since in this example, the algorithm was not used as designed, obviously one cannot conclude that good results will always be obtained when the algorithm is used in this way. However, the results obtained are encouraging. In the switching case, it was shown that an optimal solution is not practical. The simulation of the suboptimal solution has indicated that good result may be obtained under the assumption of a small signal to noise ratio. The estimation algorithms developed may be implemented either on-line or off-line, and this is an important aspect of the solutions.

The solution to the control problem has a cost of control related to the capability of identification. Once the true system is identified, the control strategy reduces to the optimal solution under certainty. Applying the developed control strategy to the water pollution problem has demonstrated the usefulness of the algorithms. Thus, the objectives stated in the introductory chapter are met. Although only one application area was considered in this study, it is thought that many types of applications are amenable to solutions using the estimation and control techniques developed in this research.

6.3 Suggestion for Further Research

There are many obvious extensions of this work, and a few of the more important topics are suggested below.

(1) It appears that the likelihood ratio test of decision theory, employing a general threshold criterion, could be put to use in simplifying the computational aspects of the estimation algorithm. That is, if at a given stage, a candidate model proved to be unlikely, using such a test, then the candidate could be removed from consideration. The remaining stages would then require less computation and storage.

(2) In cases where estimation of state trajectories is of importance, the development of optimal smoothing algorithms under uncertainty is needed. Extensions of this study to the smoothing case appear to be possible.

(3) In the switching case, it has been assumed that only one switching is possible between observations. A natural extension of the work presented here involves the development of solutions to estimation and control problems in which more than one switching may occur.

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APPENDIX A

DERIVATION OF THE UPDATED CONDITIONAL MEAN AND VARIANCE

It is the purpose of this appendix to derive the updated conditional mean and variance presented in Eqs. 3-19 and 3-20 from Eq. 3-18 whenever a new measurement z(k+1) is available at time t_{k+1}, i.e., the following relationships are to be derived.

$$p(x_i(t_{k+1})/Z_{k+1}) = \frac{p(z(k+1)/x_i(t_{k+1}))p(x_i(t_{k+1})/Z_k)}{p(z(k+1)/Z_k,\theta_i)} \qquad \text{(A-1)}$$

and

$$p(x_i(t_{k+1})/Z_{k+1}) \sim N[\hat{x}_i(t_{k+1}/Z_{k+1}),\ V_{x_i}(t_{k+1}/Z_{k+1})]$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n_i}{2}} \left| V_{x_i} \right|^{-\frac{1}{2}} \mathrm{EXP}[-\tfrac{1}{2}\,(x_i-\hat{x}_i)^T V_{x_i}^{-1}(x_i-\hat{x}_i)] \qquad \text{(A-2)}$$

where $\hat{x}(t_{k+1}/Z_{k+1})$ and $V_{x_i}(t_{k+1}/Z_{k+1})$ are to be obtained in terms of new measurement z(k+1) and the current estimate $\hat{x}(t_{k+1}/Z_k)$ and $V_{x_i}(t_{k+1}/Z_k)$.

Since the state variables, x_i, and the observation, z, are normally distributed variables, the corresponding density functions in Eq. A-1 may be evaluated as indicated below.

From the relationship that given a model θ_i, the observation is described by

$$z(k+1) = H_i(t_{k+1})x_i(t_{k+1}) + v_i(k+1) \tag{A-3}$$

The density function

$$p(z(k+1)/x_i(t_{k+1})) \sim N[\, H_i(t_{k+1})x_i(t_{k+1}),\ R_i(k+1)]$$

$$= (\frac{1}{2\pi})^{\frac{m}{2}} \, |R_i|^{-\frac{1}{2}} \ \text{EXP}[-\tfrac{1}{2}\,(z-H_i x_i)^T R_i^{-1}(z-H_i x_i)] \tag{A-4}$$

is obtained. Since $\hat{x}(t_{k+1}/Z_k)$ is the sufficient statistics for Z_k,

$$E\,\{z(k+1)/Z_k,\ \theta_i,\ t_{k+1}\} = H_i(t_{k+1})\hat{x}(t_{k+1}/Z_k)$$

$$\text{VAR}\,\{z(k+1)/Z_k,\ \theta_i,\ t_{k+1}\} = H_i(t_{k+1})V_{x_i}(t_{k+1}/Z_k)H_i^T(t_{k+1}) + R_i(k+1)$$

or

$$p(z(k+1)/Z_k,\ \theta_i,\ t_{k+1}) \sim N[H_i x_i(t_{k+1}/Z_k),\ H_i V_{x_i}(t_{k+1}/Z_k)H_i^T + R_i(k+1)]$$

$$= (\frac{1}{2\pi})^{\frac{m}{2}} \ |H_i V_{x_i} H_i^T + R_i|^{-\frac{1}{2}} \ \text{EXP}\,\{-\tfrac{1}{2}\,[z-H_i\hat{x}_i]^T\,[H_i V_{x_i} H_i + R_i]^{-1}$$

$$\cdot\,[z-H_i\hat{x}_i]\,\} \tag{A-5}$$

Furthermore, by definition

$$p(x_i,\ t_{k+1}/Z_k) \sim N[\hat{x}_i(t_{k+1}/Z_k),\ V_{x_i}(t_{k+1}/Z_k)]$$

$$= (\frac{1}{2\pi})^{\frac{n_i}{2}} \ |V_{x_i}|^{-\frac{1}{2}} \ \text{EXP}\ \ -\tfrac{1}{2}\,[x_i-\hat{x}_i]^T V_{x_i}^{-1}[x_i-\hat{x}_i] \tag{A-6}$$

Substituting Eqs. A-4, A-5, and A-6 into right hand side of Eq. A-1, one may obtain

$$p(x_i(t_{k+1})/Z_{k+1}) = \frac{\left| H_i V_{x_i} H_i^T + R_i \right|^{\frac{1}{2}}}{(2\pi)^{\frac{n_i}{2}} \left| R_i \right|^{\frac{1}{2}} \left| V_{x_i} \right|^{\frac{1}{2}}} \quad \text{EXP} \; -\tfrac{1}{2}[\;\cdot\;] \qquad \text{(A-7)}$$

where

$$[\;\cdot\;] = (z-H_i x_i)^T R_i^{-1}(z-H_i x_i) + (x_i-\hat{x}_i)^T V_{x_i}^{-1}(x_i-\hat{x}_i)$$

$$- (z-H_i\hat{x}_i)^T(H_i V_{x_i} H_i^T + R_i)^{-1}(z-H_i\hat{x}_i) \qquad \text{(A-8)}$$

By collecting terms and noting the fact that R_i and V_{x_i} are symmetric matrices, Eq. A-8 may be rewritten as

$$[\;\cdot\;] = x_i^T(H_i^T R_i^{-1} H_i + V_{x_i}^{-1})x_i - 2(z^T R_i^{-1} H_i + \hat{x}_i^T V_{x_i}^{-1})x_i + z^T R_i^{-1} z$$

$$+ \hat{x}_i^T V_{x_i}^{-1}\hat{x}_i - (z-H_i\hat{x}_i)^T(H_i V_{x_i} H_i^T + R_i)^{-1}(z-H_i\hat{x}_i)$$

$$\text{(A-9)}$$

Next, completing the squares by adding and subtracting

$$(z^T R_i^{-1} H_i + \hat{x}_i^T V_{x_i}^{-1})(H_i R_i^{-1} H_i + V_{x_i}^{-1})^{-1}(H_i R_i^{-1} z + V_{x_i}^{-1}\hat{x}_i)$$

Eq. A-9 may be further reduced

$$[\ \cdot\ ] = [x_i - (H_i R_i^{-1} H_i + V_{x_i}^{-1})^{-1}(H_i^T R_i^{-1} z + V_{x_i}^{-1}\hat{x}_i)]^T [H_i^T R_i^{-1} H_i + V_{x_i}^{-1}]$$

$$\cdot\ [x_i - (H_i^T R_i^{-1} H_i + V_{x_i}^{-1})^{-1}(H_i^T R_i^{-1} z + V_{x_i}^{-1}\hat{x}_i)] + r \qquad \text{(A-10)}$$

where

$$r = \hat{x}_i V_{x_i}^{-1}\hat{x}_i + z^T R_i^{-1} z - (z - H_i\hat{x}_i)^T (H_i V_{x_i} H_i^T + R_i)^{-1}(z - H_i\hat{x}_i)$$

$$-(H_i^T R_i^{-1} z + V_{x_i}^{-1}\hat{x}_i)^T (H_i R_i^{-1} H_i + V_{x_i}^{-1})^{-1}(H_i R_i^{-1} z + V_{x_i}^{-1}\hat{x}_i)$$

$$= \hat{x}_i^T [V_{x_i}^{-1} - H_i(H_i V_{x_i} H_i^T + R_i)^{-1} H_i - V_{x_i}^{-1}(H_i R_i^{-1} H_i + V_{x_i}^{-1})^{-1} V_{x_i}^{-1}]\hat{x}_i$$

$$+ 2z^T [(H_i V_{x_i} H_i^T + R_i)^{-1} H_i - R_i^{-1} H_i (H_i^T R_i^{-1} H_i + V_{x_i}^{-1})^{-1} V_{x_i}^{-1}]\hat{x}_i$$

$$+ z^T [R_i^{-1} - (H_i V_{x_i} H_i^T + R_i)^{-1} - R_i^{-1} H_i (H_i^T R_i^{-1} H_i + V_{x_i}^{-1})^{-1} H_i^T R_i^{-1}] z$$

$$= x_i^T \triangle_{11} x_i - 2z^T \triangle_{12} x_1 + z^T \triangle_{22} z \qquad \text{(A-11)}$$

Using the matrix inversion lemma[27,34]

$$(H_i^T R_i^{-1} H_i + V_{x_i}^{-1})^{-1} = V_{x_i} - V_{x_i} H_i^T (H_i V_{x_i} H_i^T + R_i)^{-1} H_i V_{x_i} \qquad \text{(A-12)}$$

The expression inside the bracket in Eq. A-11 can be evaluated as follow.

(1) First term,

$$\triangle_{11} = V_{x_i}^{-1} - H_i(H_i V_{x_i} H_i^T + R_i)^{-1} H_i - V_{x_i}^{-1} + H_i V_{x_i} H_i^T + R_i)^{-1} H_i = 0$$

(2) Second term,

$$\Delta_{12} = R_i^{-1} R_i (H_i V_{x_i} H_i^T + R_i)^{-1} H_i - R_i^{-1} H_i [I - V_{x_i} H_i^T (H_i V_{x_i} H_i^T + R_i)^{-1} H_i]$$

$$= R_i^{-1} [R_i + H_i V_{x_i} H_i^T][H_i V_{x_i} H_i^T + R_i]^{-1} H_i - R_i^{-1} H_i = 0$$

(3) Third term,

$$\Delta_{22} = R_i^{-1} - (H_i V_{x_i} H_i^T + R_i)^{-1} R_i R_i^{-1} - R_i^{-1} [I - H_i V_{x_i} H_i^T (H_i V_{x_i} H_i^T + R_i)^{-1}]$$

$$\cdot H_i V_{x_i} H_i^T R_i^{-1} + R_i^{-1} R_i (H_i V_{x_i} H_i^T + R_i)^{-1} H_i V_{x_i} H_i^T R_i^{-1}$$

$$- (H_i V_{x_i} H_i^T + R_i)^{-1} H_i V_{x_i} H_i^T R_i^{-1}$$

$$= R_i^{-1} - R_i H_i V_{x_i} H_i^T R_i^{-1} + R_i^{-1} (R_i + H_i V_{x_i} H_i^T)(H_i V_{x_i} H_i^T + R_i)$$

$$\cdot H_i V_{x_i} H_i^T R_i^{-1} - (H_i V_{x_i} H_i^T + R_i)^{-1} (R_i + H_i V_{x_i} H_i^T) R_i^{-1} = 0$$

Hence $r \equiv 0$. Comparing Eq. A-10 with the bracketed part inside the argument of the exponential function in Eq. A-2, one may conclude that

$$\hat{x}_i(t_{k+1}/Z_{k+1}) = [H_i^T R_i^{-1} H_i + V_{x_i}^{-1}(t_{k+1}/Z_k)]^{-1}$$

$$\cdot [H_i^T R_i^{-1} z(k+1) + V_{x_i}^{-1}(t_{k+1}/Z_k) \hat{x}_i(t_{k+1}/Z_k)] \qquad \text{(A-13)}$$

and

$$V_{x_i}(t_{k+1}/Z_{k+1}) = [H_i^T R_i^{-1} H_i + V_{x_i}^{-1}(t_{k+1}/Z_k)]^{-1} \qquad \text{(A-14)}$$

These relationships may be reduced further, using Eq. A-12 and

$$[H_i^T R_i^{-1} H_i + V_{x_i}^{-1}]^{-1} H_i R_i^{-1} = V_{x_i} H_i^T (H_i V_{x_i} H_i^T + R_i)^{-1} \tag{A-15}$$

The updated mean and variance are expressed as follow.

$$\hat{x}_i(t_{k+1}/Z_{k+1}) = \hat{x}_i(t_{k+1}/Z_k) + K_i(t_{k+1})[z(k+1) - H_i \hat{x}_i(t_{k+1}/Z_k)] \tag{A-16}$$

and

$$V_{x_i}(t_{k+1}/Z_{k+1}) = V_{x_i}(t_{k+1}/Z_k) - K_i(t_{k+1}) H_i V_{x_i}(t_{k+1}/Z_k)$$

$$= [I-K_i H_i]\, V_{x_i}(t_{k+1}/Z_k)[I-K_i H_i]^T + K_i R_i K_i^T \tag{A-17}$$

where

$$K_i(t_{k+1}) = V_{x_i}(t_{k+1}/Z_k) H_i^T (H_i V_{x_i} H_i^T + R_i)^{-1} \tag{A-18}$$

which is referred to as Kalman gain[27,34].

APPENDIX B

A CONTROL CRITERION FOR WATER QUALITY PROBLEMS

B.1 Open Loop Control Strategy

The solution to the open loop control problems stated in Section 5.5.1 is examined here in detail. The dynamical structure of the system is described by Eq. 5-9, or

$$\dot{x}(t) = f(x,u,t)$$
$$= -F(t)x(t) + D(t)C_s(t) + C(t)u(t) + G(t)w(t) \qquad \text{(B-1)}$$

where

$$D(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad G(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The problem is formulated in a deterministic sense, and the disturbance $w(t)$ is replaced by its mean $\bar{w}(t)$. Given the nominal initial conditions $\bar{x}(t_o)$ and the desired terminal conditions $\bar{x}(t_f)$, a control is to be selected to minimize a performance index

$$J = \int_{t_o}^{t_f} -u(t)\ dt \qquad \text{(B-2)}$$

The constraint Eq. B-1 is included in the performance measure by using Lagrange multipliers.

$$J = \int_{t_o}^{t_f} [-u + \lambda^T(f(x,u,t) - \dot{x})]\ dt \qquad \text{(B-3)}$$

where λ is a vector to be determined. The Hamiltonian in this case is

$$H = -u - \lambda_1(k_1+k_3)x_1 - \lambda_2(k_1x_1+k_2x_2) + \lambda_2k_2C_s + \lambda_2\bar{w} + \lambda_1u \qquad \text{(B-4)}$$

The necessary conditions for optimality associated with this problem are

$$\frac{\partial H}{\partial x_1} = -\dot{\lambda}_1 = -\lambda_1(k_1+k_3) - \lambda_2k_1 \qquad \lambda_1(t_f) = \lambda_{1f} \qquad \text{(B-5)}$$

$$\frac{\partial H}{\partial x_2} = -\dot{\lambda}_2 = -\lambda_2k_2 \qquad \lambda_2(t_f) = \lambda_{2f} \qquad \text{(B-6)}$$

$$\frac{\partial H}{\partial \lambda_1} = \dot{x}_1 = -(k_1+k_3)x_1 + u \qquad x_1(t_o) = x_{10} \qquad \text{(B-7)}$$

$$\frac{\partial H}{\partial \lambda_2} = \dot{x}_2 = -k_1x_1 - k_2x_2 + k_2C_s + \bar{w} \qquad x_2(t_o) = x_{20} \qquad \text{(B-8)}$$

and H is to be minimized with respect to the set of admissable controls. Where Eqs. B-7 and B-8 are as given in Eq. B-1, and λ_{1f} and λ_{2f} are chosen such that the terminal conditions of x are satisfied. This is a two point boundary value problem. The problem is also known as a "Bang-Bang" or switching problem. The switching takes place when the condition

$$\frac{\partial H}{\partial u} = -1 + \lambda_1 = 0 \qquad \text{(B-9)}$$

is satisfied. By combining Eqs. B-5 and B-6 in vector form

$$\dot{\lambda}(t) = \begin{bmatrix} k_1+k_3 & k_1 \\ 0 & k_2 \end{bmatrix} \lambda(t) = F^T\lambda(t) \qquad \lambda(t_f) = \lambda_f \qquad \text{(B-10)}$$

and by letting $\tau = t_f - t$

$$\dot{\lambda}(\tau) = -F^T\lambda(\tau) \qquad \lambda(0) = \lambda_f \tag{B-11}$$

The equation can be solved analytically when coefficients are constant. The solution is

$$\lambda_1(\tau) = \lambda_{1f}\ \text{EXP}[-(k_1+k_3)\tau] - \frac{k_1\lambda_{2f}}{k_2-k_1-k_3}\{\text{EXP}[-(k_1+k_3)\tau] - \text{EXP}[-k_2\tau]\} \tag{B-12}$$

and

$$\lambda_2(\tau) = \lambda_{2f}\ \text{EXP}[-k_2\tau] \tag{B-13}$$

It is assumed that only one switching takes place. This is indicated from Eq. B-12 since k_2 is usually greater than k_1 and k_3. It is also convenient from the view point of implementation. One may obtain a solution for Eq. B-1 by assuming a switching time at t_s and an allowable control strategy

$$u(t) = \begin{cases} E & \text{for } t_o \leqslant t \leqslant t_s \\ \varphi E & \text{for } t_s \leqslant t \leqslant t_f \end{cases} \tag{B-14}$$

where E and φ are non-negative real numbers.

In the time interval $t_o \leqslant t \leqslant t_s$, the analytical solution for the state of the system is

$$x_1(t) = x_{10}\text{EXP}[-(k_1+k_3)t] + \frac{E}{k_1+k_3}\{1-\text{EXP}[-(k_1+k_3)t]\} \tag{B-15}$$

and

$$x_2(t) = x_{20}EXP[-k_2t] - \frac{k_1x_{10}-E}{k_2-k_1-k_3}\{EXP[-(k_1+k_3)t] - EXP[-k_2t]\}$$

$$+ (C_s - \frac{E}{k_2} + \frac{\bar{w}}{k_2})\{1 - EXP[-k_2t]\} \qquad (B-16)$$

In the time interval $t_s \leqslant t \leqslant t_f$, the solution can be obtained backward in time. By letting $\tau = t_f - t$, the solution is

$$x_1(\tau) = x_{1f}EXP[(k_1+k_3)\tau] + \frac{\varphi E}{k_1+k_3}\{1- EXP[(k_1+k_3)\tau]\} \qquad (B-17)$$

and

$$x_2(\tau) = x_{2f}EXP[k_2\tau] - \frac{k_1x_{1f}-\varphi E}{k_2-k_1-k_3}\{EXP[(k_1+k_3)\tau] - EXP[k_2\tau]\}$$

$$+ (C_s - \frac{\varphi E}{k_2} + \frac{\bar{w}}{k_2})\{1-EXP[k_2\tau]\} \qquad (B-18)$$

At the switching time, the value of the state given by Eqs. B-15 and B-16 should be equal to that of Eqs. B-17 and B-18. Hence the following relationship must hold.

$$x_{10}EXP[-(k_1+k_3)t_s] + \frac{E}{k_1+k_3}\{1 - EXP[-(k_1+k_3)t_s]\}$$

$$= \frac{E}{k_1+k_3} + \{x_{1f}EXP[(k_1+k_3)t_f] - \frac{\varphi E}{k_1+k_3} EXP[(k_1+k_3)t_f]\}$$

$$\cdot EXP[-(k_1+k_3)t_s] \qquad (B-19)$$

and

$$x_{20}EXP[-k_2t_s] - \frac{k_1x_{10} - E}{k_2-k_1-k_3}\{EXP[-(k_1+k_3)t_s] - EXP[-k_2t_s]\}$$

$$+ (C_s - \frac{E}{k_2} + \frac{\bar{w}}{k_2})\{1 - EXP[-k_2t_s]\}$$

$$= x_{2f}EXP[k_2(t_f-t_s)] - \frac{k_1x_{1f}\varphi E}{k_2-k_1-k_3}\{EXP[(k_1+k_3)(t_f-t_s)] - EXP[k_2(t_f-t_s)]\}$$

$$+ (C_s - \frac{E}{k_2} + \frac{\bar{w}}{k_2})\{1 - EXP[k_2(t_f-t_s)]\} \qquad \text{(B-20)}$$

There are three unknowns in Eqs. B-19 and B-20, E, φ, and t_s. One of the unknowns may be determined arbitrarily. If a switching time is arbitrarily chosen, the determination of the other two unknowns is relatively easy. After algebraic manipulation of Eqs. B-19 and B-20, solutions for E and φ are obtained.

$$E = \{\frac{k_1+k_3}{(\frac{1}{A_s} - A)}(x_{10}-x_{1f}A) - \frac{k_2(k_2-k_1-k_3)}{(k_1+k_3)(\frac{1}{B_s} - B}[x_{20}-x_{2f}B - \frac{1}{k_2-k_1-k_3}(x_{10}-x_{1f}B)$$

$$-(C_s + \frac{\bar{w}}{k_2})(1-B) + \frac{k_3}{k_2-k_1-k_3}(x_{10}-x_{1f}A)\frac{A_s}{B_s}]\} \Big/ (\frac{A_s - 1}{1-AA_s} - \frac{B_s - 1}{1-BB_s})$$

(B-21)

and

$$\varphi = \frac{(k_1+k_3)(x_{10}-x_{1f}A)}{(\frac{1}{A_s} - A)E} - \frac{A_s - 1}{1 - AA_s} \qquad \text{(B-22)}$$

where

$$A = EXP[(k_1+k_3)t_f], \qquad B = EXP[k_2 t_f]$$

$$A_s = EXP[-(k_1+k_3)t_s], \quad \text{and} \quad B_s = EXP[-k_2 t_s]$$

Since the values of E and φ are non-negative, any solution of the above equations which violate this assumption is not acceptable.

B.2 Determination of Terminal Conditions

The criterion for determining the terminal BOD and DO concentrations is that the wastes discharged in the region under control will not cause the downstream critical reach to be lower than a specified level. Given a specified level of DO concentration, x_{2c}, the corresponding BOD concentration, x_{1c}, at that particular reach is determined by $\dot{x}_2(t_c) = 0$, or from Eq. B-1

$$\dot{x}_2(t_c) = -k_1 x_1(t_c) - k_2 x_2(t_c) + k_2 C_s + \bar{w} = 0 \qquad \text{(B-23)}$$

Thus,

$$x_{1c} = x_1(t_c) = \frac{k_2}{k_1}(C_s - x_{2c}) + \frac{\bar{w}}{k_1} \qquad \text{(B-24)}$$

where t_c is the time when the critical reach occurred. The problem of finding the concentrations of BOD and DO at terminal time t_f of the control region may be solved by computing the state trajectory backward in time from t_c. The result is similar to Eqs. B-17 and B-18.

$$x_{1f} = x_1(t_f) = x_{1c} EXP[(k_1+k_3)\tau] \qquad \text{(B-25)}$$

and

$$x_{2f} = x_2(t_f) = x_{2c}\text{EXP}[k_2\tau] - \frac{k_1 x_{1c}}{k_2-k_1-k_3}\{\text{EXP}[(k_1+k_3)\tau] - \text{EXP}[k_2\tau]\}$$

$$+ (C_s + \frac{\bar{w}}{k_2})\{1 - \text{EXP}[k_2\tau]\} \qquad \text{(B-26)}$$

where $\tau = t_c - t_f$. It is noted that τ should be chosen such that x_{2f} is less than x_{20}.

VITA

Andrew Yeoh-yih Lee

Candidate for the Degree of

Doctor of Philosophy

Thesis: ADAPTIVE ESTIMATION AND STOCHASTIC CONTROL FOR UNCERTAIN MODELS

Major Field: Electrical Engineering

Biographical:

Personal Data: Born in Taipei, Taiwan, July 17, 1938, the son of Mr. and Mrs. Shue-sue Lee.

Education: Graduated from Provincial Chien-kuo Middle School, Taipei, Taiwan in June, 1957; received the diploma in Electrical Engineering from Taiwan Provincial Taipei Institute of Technology in June, 1960; received the Master of Science degree from Kansas State University, Manhattan, Kansas in January, 1968; completed requirements for the Doctor of Philosophy degree at Oklahoma State University, Stillwater, Oklahoma in May, 1972.

Professional Experience: Engineer, Taiwan Public Works Bureau, Taipei, Taiwan from October, 1961 to March, 1965; Research Assistant, Electrical Engineering Department, Kansas State University from September, 1966 to August, 1967; Engineer, Tennessee Valley Authority from September, 1967 to August, 1968; Teaching and Research Assistant, School of Electrical Engineering and Center for Systems Science from September, 1968 to May, 1972.

Professional Organizations: Member of Eta Kappa Nu, Sigma Xi, and the Institute of Electrical and Electronics Engineers.