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CATEGORIZING STUDENTS' DIFFICULTIES WITH MATHEMATICAL PROOFS: DEVELOPIONG A MODEL OF THE STRUCTURE OF PROOF CONSTRUCTION

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CATEGORIZING STUDENTS' DIFFICULTIES WITH MATHEMATICAL PROOFS: DEVELOPING A MODEL OF THE STRUCTURE OF PROOF CONSTRUCTION

A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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Abstract

Proof is an essential skill in mathematics and a key component in mathematics education. However, studies have shown that many students encounter various difficulties with proof at all levels. Studies have also shown that proof is challenging not only for students to learn but also for instructors to teach. Researchers in mathematics education have endeavored to provide an effective teaching method to help students with proof construction. However, there seems to be no effective and decisive method that is widely accepted by the mathematics community. The purposes of my dissertation were to reveal the sources of students' difficulties and provide effective methods to help them overcome their difficulties. In order to achieve these, I first created a model of the structure of proof construction. The model provided a comprehensive view of proof construction, which could encompass the aspects, factors, patterns, and features involved in cognitive process in proof construction. In light of the structure of proof construction, I examined students' proofs from undergraduate Algebra, Analysis, and Topology courses. The model of the structure of proof construction enabled me to identify, analyze, and explain their difficulties in an organized and systematic way. The findings from the analysis of students' proofs and the knowledge derived from the model of the structure of proof construction led me to produce an algorithm for proof construction that can be applicable to various proofs. The algorithm can serve as metacognitive knowledge for helping students, especially those who struggle with proof construction, to overcome their difficulties. It is the aspiration of this study to contribute to the development of a teaching method to help students learn proofs effectively.

Chapter 1: Introduction

1.1 Motivation of the Study

Proof has been a central topic for discussion among researchers in mathematics education for some decades. Based on the existing literature, it seems most researchers are in agreement that proof is one of the key components in mathematics education: "the essence of mathematics" (Baylis,1983); "the guts of mathematics" (Wu, 1996); "important at all grades and in all content domains" (Kilpatrick, Swafford & Findell, 2001); "the fundamental tool for extending the field of mathematics" (Driscoll, 1983); "the heart of mathematical practice" (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002); and "the basis of mathematical understanding" (Cirillo & Herbst, 2012). Wu (1996) suggested that anyone who wanted to learn mathematics had to learn proofs. Hanna (2000) proclaimed "students cannot be said to have learned mathematics, or even about mathematics, unless they have learned what proof is" (p. 24).

While proving is an essential skill in collegiate mathematics, studies have shown that students encounter various difficulties with proofs at all levels (Paola & Inglis, 2011; Pfeiffer, 2009; Stylianides, Stylianides, & Philippou, 2007; Harel & Sowder, 2007; Weber, 2006; Baker & Campbell, 2004; Epp, 2003; Weber, 2001; Dreyfus, 1999; Moore, 1994; Ruthven & Coe, 1994). CadwalladerOlsker and Miller (2013) made a representative statement, saying "it was notoriously difficult for students to develop the ability to write and read proofs" (p. 379). Mariotti (2006) reported proofs were so difficult that many teachers even gave up teaching proofs. Hanna and Villiers (2007) claimed that the challenges that educators faced in teaching proof had increased as proof had been more valued in mathematics learning.

1.2 Significance of the Study

Although students' difficulties with proof construction have been well researched, the issue still seems to be wide open for further discussion and investigation. Dreyfus (2012) believes various questions regarding students' cognitive difficulties with proof construction must be answered. For example, "what is involved in cognitive processes in proof construction?" Ayalon and Even (2008) claimed the need for establishing views and approaches to deductive reasoning. CadwalladerOlsker, Miller, and Hartmann (2013) stressed the significance of elucidating what constituted a proof. Knuth (2002) emphasized the importance of clarifying the nature and components of proof. It is also crucial to provide an effective method to help students with deductive reasoning. Harel and Sowder (1998) urged the necessity of fostering students' skills for logical deduction. Weber and Alcock (2004) indicated the importance of practicing both syntactic and semantic approaches for deductive reasoning. However, there seems to be little research that provided specific and practical knowledge to help students with proof construction based on logical deduction. Harel and Sowder (2007) also raised a question of critical importance: "What instructional interventions can bring students to see an intellectual need to refine and alter their current proof schemes into deductive proof schemes?" (p. 47).

Ball, Hoyles, Jahnke, and Movshoitz-Hadar (2002) exhorted the need of accumulating empirical studies on students' difficulties with proofs in order to develop effective teaching strategies to teach proofs. Brown, Bransford, Ferrara, and Campione (1983) suggested that highly developed metacognitive skills are one of the crucial factors for help students successfully solve problems. Metacognitive knowledge means the "knowledge of one's knowledge, processes, and cognitive and affective states; and the ability to consciously and deliberately monitor and regulate one's knowledge, processes and cognitive and affective states" (Papaleontiou-Louca, 2003, p.10-11). The significance of this study is to fill the gaps among the research literature by providing a model of the structure of proof construction. The aims of this study is to contribute to the body of innovative instructional methods to teach proofs by providing metacognitive knowledge that can help students with proof construction.

1.3 Research Purposes and Questions

The aims of this study were to establish a model of the structure of proof construction, to identify students' difficulties and clarify the sources of their difficulties, and to provide pedagogical suggestions to help students with proof construction. This study attempted to meet the needs for providing a comprehensive view of proof, which revealed the components and nature of proof construction to students. This study also attempted to fill the gaps caused by a deficiency of an effective method to help students with proof construction based on logical deduction. In order to help students grasp a comprehensive view of proof construction and enhance their skills for logical deduction, the following research questions were considered.

Research Questions

1. What is a suitable model for characterizing the structure of proof construction?

2. What difficulties do students have with proof construction and what are the sources of their difficulties in light of the structure of proof construction?

3. How useful is the model of the structure of proof construction?

4. What pedagogical suggestions can be drawn to help students with proof construction?

1.4 Overview of the Study

This study examined students' difficulties with proof construction in light of the structure of proof construction. The structure of proof construction provides a comprehensive view of proof construction, which encompass the aspects, factors, patterns, and features that involve in cognitive process. Chapter 2 examines the relevant literature regarding the theoretical perspectives on proof construction, students' difficulties with proof construction, and pedagogical approaches to help students with proof construction. Chapter 3 presents how a model of the structure of proof construction was created, elaborates the model of the structure of proof construction, offers the framework for analyzing students' difficulties with proof construction, and discusses the reliability of the model while relating some other relevant theoretical frameworks for a support. Chapter 4 describes the methodology that justified the method this study adopted and details the ways to collect and analyze students' proofs. Chapter 5 presents various examples collected from students' proofs to describe possible sources of students' difficulties with proof construction and to analyze their proofs based on the analysis framework. Chapter 6 highlights the findings from the analysis of students' proofs while relating them to the literature and provides specific and practical suggestions to help students overcome their difficulties with proof construction.

Chapter 2: Literature Review

2.1 Introduction

This chapter gives a comprehensive review on literature regarding the issues of proof construction while identifying the problems to be examined and the gaps to be filled. This chapter consists of three parts: theoretical perspectives on proof construction; existing research on students' difficulties with proof construction; and pedagogical approaches for helping students with proof construction.

2.2 Theoretical Perspectives

This section discusses the following three theoretical frameworks for proof construction: (1) proof schemes (Harel & Sowder, 1998); (2) syntactic and the semantic approaches (Weber & Alcock, 2004); and (3) *the formal-rhetorical* and *problem-centered* parts (Selden & Selden, 2007).

Harel and Sowder (1998) classified student proof schemes into three levels: external, empirical, and analytical. In the external proof schemes, students convince themselves and others based on external sources such as (a) the ritual of the appearance of the argument (the ritual proof scheme); (b) the word of a textbook or a teacher (the authoritarian scheme); and (c) some symbolic manipulation without understanding the meaning of the symbol (the symbolic scheme). The empirical proof schemes include verifying the validity of their reasoning by using some specific examples (the inductive scheme) or through their rudimentary mental images (the perceptual scheme). The analytical proof schemes include validating through the use of logical deduction. Analytical proof schemes include considering the generality, setting a goal, and transforming images. Several researchers suggested that students should grow out of their external, empirical, and pictorial proof schemes, and acquire analytical proof scheme (Finlow-Bates, Lerman, & Morgan, 1993; Harel & Sowder, 1998; Recio & Godino, 2001; Zaslavsky & Shir, 2005; Stylianou, Chae, & Blanton, 2006). Those studies revealed that students had difficulties with practicing analytical proof scheme. Instead, they tended to depend on diagrams, pictures, and specific examples for reasoning.

Weber and Alcock (2004) presented a theoretical framework for the types of thinking process in proving. They introduced two types of proof production: syntactic and semantic. The former represents the proof production in which students derive valid inferences by manipulating definitions and symbols. The latter represents the proof production in which students draws formal inferences while using instantiations of mathematical concepts. More specifically, the semantic approach is the one in which students explore and figure out the way to reach the conclusion while understanding the situation, creating examples, applying relevant facts, and checking what they have done. Semantic approach may depend on mathematical contents while syntactic approach may not. Weber and Alcock (2009) suggested that both syntactic and semantic approaches must concur for proof production based on deductive reasoning.

Selden and Selden (2007) offered a model of the structure of a proof. They claimed that a proof consisted of two parts: the *formal-rhetorical* part and *problem-centered* part. The *formal-rhetorical* part stands for the part of a proof that can be obtained by unpacking a concept into the definition. The *problem-centered* part is the remaining part, which is the core of problem solving done through rigorous thoughts, a deep understanding, and intuition. It seems the *formal-rhetorical* part corresponds to the product obtained through a syntactic approach and the *problem-centered* part corresponds to the product obtained through a semantic approach. They offered a proof framework as an instructional method to help students with proof construction. According to their method, students should first write a hypothesis, leave a blank space, put the conclusion at the end. Next, they fill the earlier blank space by inserting the beginning and end of the unpacked conclusion. Students write a proof from both ends toward the middle. However, it is not completely clear how they should advance a reasoning process in the remaining part. In addition, their method of leaving a blank space at the beginning of proof construction may not be perfectly practical.

The above frameworks were used for the following two reasons. First, they helped to understand students' approaches for proof construction and logical deduction, and the structure of a proof. Second, they helped to identify the position of my study among various studies on proof construction. More specifically, they made it clear that my study was centered at an exploration of metacognitive knowledge for the skills of logical deduction in proof construction. The frameworks also provided the gaps to fill and the demands to meet. The frameworks led my study to explore the skills for practicing analytical proof scheme, clarify the mechanism of syntactic approach and semantic approaches, and provide an effective method to write a proof from the top down.

2.3 Students' Difficulties with Proof Construction

This section reviews the literature on students' difficulties with proof construction in terms of four aspects: *background knowledge*, *reasoning activity*, *mental attitudes*, and *affect and beliefs*. The followings are the definition of each term.

- *Background knowledge*: the knowledge necessary for solving a given proof problem. It includes concepts, definitions, notations, theorems, proposition, mathematical laws, and problem solving techniques.
- *Reasoning activity*: cognitive actions or operations for advancing a reasoning process.
- *Mental attitudes*: tenacity, persistence, flexibility, carefulness, and precision.
- *Affect and beliefs*: emotions, feelings, self-confidence, beliefs, and perceptions toward mathematics, proofs, and logics.

The above aspects seem to cover the categories that Schoenfeld (2010) included in his theoretical framework for problem-solving. The following are the categories Schoenfeld included: (1) knowledge base (what students know); (2) problem-solving strategies (the tools or the techniques for solving problems); (3) self-regulation or monitoring (monitoring and assessing progress); (4) beliefs (one's understanding, feelings, perceptions, decision). (1) *Background knowledge* directly corresponds to knowledge base. (2) *Reasoning activity* may correspond to problem-solving strategies because the r*easoning activity* can be all considered to be strategies and techniques for constructing a proving argument. (3) *Mental attitudes* may correspond to self-regulation or monitoring because the components of mental attitudes (tenacity, persistence, flexibility, carefulness, alertness, and precision) can be considered to be mind tools for self-regulation. (4) *Affect* and *beliefs* may directly correspond to beliefs because both share the same components such as feelings and beliefs.

2.3.1 Background Knowledge

In order to construct a successful proof, students need to be fully equipped with the knowledge around a given proof problem. In particular, their knowledge about the concepts involved in the proof is indispensable. This section discusses the significance of students' knowledge of concepts from the following three angles: mathematical language; definitions; and abstract thinking.

Mathematical language

As Tall (1991) pointed out, mathematical language is one of the difficulties that students face in starting to learn proofs. Moore (1994) also provided students' difficulties with mathematical language as one of the major sources of their difficulties with proof construction. Mathematical language consists of mathematical terms, notations, and logical words. For example, "continuous," "differentiable," and "homomorphism" are examples of mathematical terms. "$R / Ker\phi$," "$Z[x]$," and "$Z(G)$" are examples of mathematical notations. "If," "then," "for all," and "there exists" are examples of logical words. Students' ability of using mathematical language around a concept depends on their levels of understanding of the concept. Moore (1994) pointed out that students' lack of understanding of concepts can hinder them from correctly using the language and notation.

Definitions

Definitions of mathematical concepts are central and fundamental mathematical language. Students have two types of difficulties with definitions: not understanding the contents of definitions; being unable to use the definitions. There are four factors that cause their difficulties: (a) the gap between students' concept image and concept definition; (b) the nature of definitions as stipulated language; (c) the difference between mathematical definition and everyday language; and (d) the learning way to approach definitions.

Tall and Vinner (1981) introduced the notions of concept images and concept definition" Concept image is the total of the mental pictures of concepts and related properties. Concept definition is a formal definition of concept, which can be found in a textbook. Alcock (2007) observed that students made extra assumptions that concepts did not include. Their incomplete concept image led to their difficulties with using definitions to express their ideas. Students need to take effort to narrow the gap between their concept image and concept definition. It takes time for them to be "formally operable" so that they can use a definition or a theorem to create a formal argument (Bills & Tall, 1998).

Edwards and Ward (2008) attributed a chief role of definitions to the creation of concepts. They considered definitions as stipulated language, in which the meaning-relation was explicitly and self-consciously set up. When students deal with a definition, they face a difficulty of building up their thoughts on the concept. Zaslavsky & Shir (2005) suggested that students should know the role of definitions to overcome their difficulties with understanding definitions.

The difference between mathematical definitions and everyday language also causes student's difficulties with definitions (Edwards & Ward, 2004; Epp, 2003; Selden & Selden, 2007). The definitions in everyday language are extracted from examples and evidence while the definitions in mathematical language are defined by stipulation (Edwards & Ward, 2004). Some mathematical language are used in everyday language but they have different meanings. Frid (1994) and Cornu (1991) observed that students interpreted the language "limit" in terms of everyday meaning. Awareness of the distinction between mathematical language and everyday language helps students integrate new mathematical rules into their cognitive framework (Epp, 2003).

A mathematical definition has a logical structure. Selden and Selden (2007) pointed out that every part of the mathematical definition contributes to supporting the structure. For example, small parts of a definition such as "for any …" "for some …" play an important role in deciding the structure of the concept. Mathematical definitions often involve a conditional statement in it. For example, the definition of continuity of $f(x)$ at $x = a$ is that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Selden and Selden (2007) suggested that students should pay attention to all parts of a definition.

The way to approach definitions influences one's ability to understand definitions. Students and mathematicians took different cognitive processes in comprehending definitions (Vinner, 1991). Paramerswaran (2010) explored how mathematicians approached definitions. They found three factors that helped mathematicians enhance their understanding of definitions: (1) examples; (2) reformulations of definitions with a related theorem; (3) resolutions to the conflicts evoked in facing an example that contradicts their concept image of the definition. Paramerswaran (2010) provided necessary stages in understanding definitions: (1) familiarizing oneself with the notation; (2) building a concept image; (3) acquiring examples; and (4) learning how to use a definition.

The definition of a concept plays a significant role in proof construction. A proving argument is constructed based on the definition. However, definitions can be difficult for students to understand and use due to their complex nature, which everyday language does not have. Students need to know how to learn and apply definitions. The next section discusses abstract thinking, which plays an important role in connecting *background knowledge* and *reasoning activity* in proof construction.

Abstract thinking

Abstract thinking is the ability to generalize and synthesize objects into concepts through representations. Dreyfus (2002) viewed abstraction as the most important processes to be developed in advanced mathematical thinking. He claimed achieving the ability of abstraction may well be considered as the most important goal of advanced mathematics learning. Frasier and Panasuk (2013) agreed that abstract thinking was central to conceptual understanding and was an essential part of mathematics learning. In particular, they placed proofs as the typical instance of abstract thinking. Dreyfus (2002) included generalizing and abstracting, synthesizing and abstracting, and representing and abstracting as basic component processes of abstract thinking.

Generalizing and Abstracting. Both generalizing and abstracting are the processes to construct a cognitive structure of a concept. The ability to generalize and abstract is crucial for students' formal thinking in proof construction. Dreyfus (2002) defined generalizing to be a process of "deriving or inducing from particulars, to identify commonalities, to expand domains of validity (p. 35)" and defined abstracting to be a process of "building of mental structures from properties of and relationships between mathematical objects (p. 37)". There is a concise distinction between them. The former involves an expansion of the individual's knowledge structure while the latter involves a mental re-construction of the existing structure. Tall (1991, p. 12) gave a clear illustration of the difference between them, using the following example:

> For instance, we generalize the solution of linear equations in two and three dimensions to n-dimensions and we abstract from this context the notion of a vector space. In doing so, two very different mental objects are produced: The generalization R^n and the abstraction, a vector space V over a field F. The generalization R^n simply extends the chain of ideas from R^1 to R^2 and from R^2 to R^3, and so on, which is described by applying the usual arithmetic processes to each coordinate. The abstraction V is a very different mental object, which is defined by a list of axioms. While the former simply involves an extension of familiar processes, the latter requires a massive mental reorganization.

Dreyfus (2002) considered generalizing, synthesizing, and representing as a prerequisite basis to abstracting: generalizing and synthesizing (p. 34). Abstraction requires more cognitive load than generalizing and synthesizing because abstraction requires more attention to the structure of the properties and their relationships than the objects themselves. For example, when his students tried to understand the concept of field, they focused on the relationships that existed between numbers rather than on the numbers themselves (Dreyfus, 2002).

Synthesizing and Abstracting. According to Dreyfus (2002), synthesizing is a process of "combining or composing parts in such a way that they form a whole or an entity (p. 35)." It is a process of integrating separate facts into a complete picture, which is not just a sum of parts but a structure of interrelated facts. This can be applied to the following example. Suppose that a student has the following mental representation for continuity: $f(x)$ is continuous at $x = c$ if and only if for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Then, suppose that the student learns another representation for continuity of a function at $x = c$: a function $f(x): X \to Y$, where X and Y are topological spaces, is continuous at $x = c$ if and only if for every open neighborhood V of $f(c)$, there exists an open neighborhood U of c such that $f(U) \subset V$. At first, those two concepts may exist independently of each other in the students' mind, but later, the student may relate and connect each other to establish a stronger concept of continuity. Thus, integrating different mental presentations for the same concept into an interrelated structure can be considered as synthesizing.

Synthesizing is a crucial process to construct a more powerful mental representation for a concept, namely, to build a more solid understanding of the concept. The ability of synthesizing makes a difference in understanding a concept and solving a problem. For example, in a mathematician's mind, several mental representations for a concept may be strongly linked to form a broad and strong world for the concept. Then, the mathematician can grasp one representation from several angles. The representation is supported by other representations in the mind. The ability of synthesizing allows the mathematician to make seemingly independent mental representations complementary. Therefore, the ability of synthesizing increases the power of flexibility to switch representations, which makes it easier for the mathematician to deal with problems. In addition, the mathematician can see the mental world of the concept from a bird's eye view, which makes it easier to make use of those representations according to the given situations.

On one hand, students may have difficulties with abstract thinking because they may fail to synthesize mental representations for a concept. Even if students have the same mental representations as mathematicians do, if they fail to synthesize them, they would grasp one representation by itself. In addition, that would make it difficult for them to know where they stand in the world around the concept and to make good use of available representations in solving a given problem.

Moreover, mathematicians find themselves in a world around a concept that they have formed through synthesizing some mental representations. This may prove difficult when trying to communicate with students who has a narrower and unorganized world around the concept. This may cause conflicts between instructors' teaching and students' learning.

Representing and Abstracting. A representation is a realization of a mathematical notion. Representations take several forms: algebraic expressions, notations, graphs, figures, tables, matrices, arrow-diagrams, and words. In addition, representations can become tools to solve problems and construct meanings (Davis & Maher, 1997, Radford, 2000, Sfard, 2000). Dreyfus (2002) viewed representing and abstracting as complementary processes in opposite directions. "A concept is often abstracted from several of its representations, on the other hand, representations are always representations of some more abstract concept (p. 38)." Dreyfus (2002) further claimed that learning processes evolved in four stages: using a single representation, using more than one representation in parallel, making links between parallel representations, and integrating representation and flexible switching between them. Once this process has been completed, one has formed an abstract notion of a concept.

Dreyfus (2002) classified representations into two kinds. One is a symbolic representation and the other is a mental representation. He made a distinction between them in the following way. "A symbolic representation is externally written or spoken, usually with the aim of making communication about the concept easier. A mental representation, on the other hand, refers to internal schemata or frames of reference which a person uses to interact with the external world" (p. 31).

Symbolic representations play an important role in mathematical thinking and learning. They make it easier and more convenient to express, convey, and understand mathematical ideas and arguments. As Olson and Campbell (1994) described, their role is to make an individual's implicit knowledge explicit in terms of symbols. On the other hand, a mental representation is a mental picture that occurs in the mind when an individual thinks of a mathematical notion. It involves his or her understanding of the notion. Therefore, it happens that different individuals have different mental representations for the same notion in accordance with their understanding levels.

Their concepts can be applied to the following example. A symbolic representation of a limit concept of a function as *x* approaches *a* is $\lim_{x \to a} f(x)$. However, students may have different mental representations for it. Some may mistakenly assume that it is the same as $f(a)$; some may correctly relate it to the fact that it exists if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$; some may even recall that $\lim_{x \to a} f(x) = L$ if and only if for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all real number x with $0 < x - a < \delta$ implies $|f(x) - L| < \varepsilon$. Similarly, for the notion of continuity, some students may vaguely picture a mental image of some kind of smooth curve that contains neither a jump discontinuity nor a removable discontinuity; some may recall that $f(x)$ is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$; some may conceive that $f(x)$ is continuous if and only if for every sequence x_n of points in the domain which converges to a, $\lim_{x \to a} f(x) = f(a)$; and some may even think of the topological idea that a function $f(x): X \to Y$, where X and Y are topological spaces, is continuous if and only if for every open set $V \subset Y$, the inverse image $f^{-1}(V)$ is an open subset of X.

Some form rich mental representations and others have limited mental representations for the same concept. Such a difference can become a source of students' difficulties with learning. For example, students encounter difficulties with understanding their instructors because the students and their instructor have different mental representations for the same concept. Dreyfus (2002, p.31) described this situation in the following way:

> a student's notion of a function may be limited to processes (of computation or mapping), whereas the teacher teaching indefinite integrals may think of the function in the integral as an object to be transformed. Such discrepancies easily lead to situations where students are unable to understand teachers.

Students' mental representations evoked by a notion can be limited or even incorrect. Then, they are unable to understand what their instructor means by it. This issue occurs not only when students learn from their instructor or a textbook but also when they solve problems. With richer representations of a concept, students can have more strategies to tackle problems. Kaput (1989) and Clements (1983) showed that the use of several representations was successful in helping students improve their understanding of a concept. With poor mental representations, the types of problems that students can get access to are more limited. Selden, Mason, and Selden (1989, pp. 48-49) gave several non-routine problems to average Calculus students to see how well they performed. The following was one of the problems: Decide if $x^{21} + x^{19} - \frac{1}{x} + 2 = 0$ has any roots between -1 and 0. They observed that nine out of seventeen students were unable to solve the problem. They were unable to think about the function $f(x) = x^{21} + x^{19} - \frac{1}{x} + 2$ in a graphical sense, take the derivative of the function to check if it increased or decreased on the given interval, and use the limit concept to figure out the end behavior of the function as x approached 0. Instead, they tried to manage the problem in a primitive way, plugging some numbers into x, using trial and error, or making a guess. This was an example showing that students were unable to solve a problem successfully because their mental representations were limited. More specifically, the given expression evoked those students just a symbolic representation, namely, a relatively complicated algebraic equation. It did not occur to them that the given expression might be a graphical representation of a relation of the two functions, which were $f(x) = x^{21} + x^{19} - \frac{1}{x} + 2$ and $g(x) = 0$. Their mental representations for the given expression were not broad enough to cover a graphical representation for it. However, it might not be enough for an individual to be equipped with rich representations in solving problems or in constructing proofs successfully. The ability to flexibly switch representations according to a situation would be also required. Representing can be considered to be one of the fundamental processes of abstract thinking taken not only in the construction of the knowledge of concepts but also in the reasoning activity of proof construction.

Thus, generalizing, synthesizing, and representing are possible factors that can influence students' use of their background knowledge in proof construction. The next sections reviews the studies that can directly affect thinking actions in advancing a reasoning process in proof construction.

2.3.2 Reasoning Activity

The reasoning activity for proof construction is the scene in which students advance their reasoning processes for reaching the goal while using the premises and transforming representations. Students transform objects by switching representations and construct an argument based on logical and deductive reasoning.

Switching representations

Rich mental representations help students tackle a problem by providing more information for dealing with it. At the same time, having rich mental representations may cause a difficulty to students: they may get confused in choosing the best representation from among several options. It can happen that students choose a less effective representation and fail to solve a problem efficiently or successfully. For example, suppose that students are given a function $f(x) = x^2 - x + 1$ and asked to obtain the derivative of the function at $x = 1$ by using the definition of derivative. Some students may apply $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to evaluate $f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

and others may apply $f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ to evaluate $f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$.

Both methods work, but the latter turns out to be easier and faster. This is just a small example, but in a more advanced mathematics, the situation can be more serious. Failing to adopting a more effective representation can lead the individual to a less effective problem solving strategy and may end up with confusing the individual. Thus, students' difficulties with problem solving can be partly due to the difficulty with deciding when to use which representation in accordance with a situation. Dreyfus (2002) asserted "an individual needs to be able to flexibly switch from one representation to a more efficient one in problem solving" (p. 32).

Logical and deductive reasoning

Logical and deductive reasoning plays a central part of proof construction. Epp (2003) stressed the need of providing a method to help students develop their formal reasoning skills. Knapp (2005) pointed out that lack of awareness of the laws of logic and deductive reasoning was one of the causes of students' difficulties with proof construction. The laws of logic, including the laws of syllogism, form the system with which an argument can be validated. Deductive reasoning is the process of reaching a conclusion by using the given premises, assumptions, previous statements, and relevant theorems and propositions. If students fail to follow the laws of logic, their proofs will collapse. For example, Weber (2002) observed that students were unable to make their proving arguments complete because they abused the laws of logic. His students tried to prove a statement by finding a theorem that might support the statement and by proving the hypothesis of the theorem.

Stylianides and Stylianides (2007) conceptualized proof, focusing on deductive reasoning. They considered as the components of proof the following three factors: (1) the set of accepted statements such as definitions, axioms, and theorems; (2) the modes of argumentation such as application of logical rules of inference; (3) and the modes of argument representation such as verbal, pictorial and algebraic representations. In particular, they associated deductive reasoning with the modes of argumentation as logically necessary inferences while introducing the theory of Johnson-Laird's and Bera's (1984). According to their theory, deductive reasoning contains the three stages in proof construction: (1) constructing a mental model to represent the structure of the premises of a given statement, (2) scanning the model to acquire informative inference, and (3) searching for alternative mental models such as counterexamples. They considered as the main abilities for deductive reasoning the ability to build a model of the premise of a given statement as well as the linguistic competence to comprehend logical terms such as "and", "or", "not", "if", "none", "some", "all" in the premises. In particular, working memory capacity was the key to successful deductive reasoning. They claimed that practice helped students enhance their working memory, which might lead to the improvement of the ability of deductive reasoning and proving. They further suggested that practice might help students internalize the general logical structure of proof method such as proof by contradiction. Practice and the knowledge of the structure of proof method can help students enhance their proving ability. This study further aims to explore the knowledge of the general logical structure of proving itself beyond a proof method, investigating specifically what is involved in proof construction which may help students enhance their deductive reasoning.

Alcock and Weber (2005) investigated how students validated an argument, using a proof from real analysis. They found out that students were unable to validate a proof because they failed to infer and check warrants. In this case, warrants means asserted preceding statements, accepted knowledge, or accepted assertions. They suggested that the importance of inferring and evaluating implicit warrants used in proofs should be emphasized in a classroom in order to help students develop their proving skills and understand new proofs.

The start of a proof can be one of the key factors in the reasoning activity of proof construction. It is a scene in which students make sure what they are asked to prove and what proving strategy they use before writing out a proof. Baker and Campbell (2004) observed students failed to prove because they did not pay full attention to the given statement and did not think about the meaning of the statement. Selden and Selden (2003b) observed that students lacked their attention to global picture of an argument and focused on local issues in validating their arguments.

The role of mathematical language is closely related to logical and deductive reasoning. Selden and Selden (1995) discussed students' difficulties with understanding of the logic for validation. In particular, they noted students' difficulties with "unpacking" informal statements into formal statements. They used the term "unpacking" to mean changing an informal statement into a formal one so that it could involve those terms that played an important role in logic such as "if", "for all", and "then." They gave the following example of "unpacking." "A function is continuous whenever it is differentiable" is an informal statement. Students need to "unpack" the statement into "For all functions f, if f is differentiable, then f is continuous." In this example, the word “whenever” was considered to be an informal expression and the phrase “for all” was considered to be a formal expression.

Ball, Hoyles, Jahnke, and Movshovitz-Hadar (2002) defined mathematical reasoning as a set of practices and norms that served as an instrument of inquiry and justification. They found the reasoning of justification on (1) public knowledge evolving in mathematics community, including mathematical ideas, procedures, methods, concepts, axioms, and publicly accepted knowledge and (2) mathematical language, including symbols, notations, definitions, and representations, and rules of logic and syntax. Those factors can be categorized in a different way. For example, a distinction can be made between procedures and terms and between mathematical language and rules of logic. Moreover, notations and definitions can be included in mathematical knowledge. In this study, mathematical ideas, procedures, methods, and logic and syntax are considered to be “actions” or “operations” while concepts, axioms, symbols, definitions, and representations are considered to be “objects.”

Students’ knowledge and their reasoning activities are not the only factors that are involved in their cognitive activities in proof construction. The next section reviews the studies related to another aspect of students’ cognitive activities involved in proof construction.

2.3.3 Mental Attitudes

A proving activity involves not only a logical and reasoning mental-activity but also some psychological aspects. Furinghetti and Morselli (2009) asserted “purely cognitive behavior is extremely rare in performing mathematical activity.” Lai (2011) included making inferences, using deductive reasoning, judging or evaluating, making decisions, and solving problems in the component skills of critical thinking. She included inquisitiveness, flexibility, a propensity to seek reason, and a desire to be well-informed, in the cognitive dispositions required by critical thinking. These cognitive dispositions can be applied to students' proving activities. In proof construction, inquisitiveness and a propensity to seek reason may motivate students to tackle a given problem persistently. A desire to be well-informed may urge students to look for information widely to advance a reasoning process. Beyer (1985) noted precision and flexibility as factors of helping students with their reasoning processes. He considered a frame of mind and a number of specific mental operations to be two major dimensions of critical thinking. He included in the frame of mind "an alertness to the need to evaluate information and a willingness to test opinions, and a desire to consider all view-points" (p. 131). Baker and Campbell (2004) also pointed out the importance of students' precision in dealing with objects. They observed students had difficulties with correctly using mathematical language. The mental dispositions such as persistency, tenacity, flexibility, carefulness, alertness, and precision can play an important role as a helm of using knowledge and practicing the operations to advance a reasoning process in proof construction. While those mental attitudes influence students' performances as necessary psychological traits for proof construction, there are psychological traits that can also affect their performances, but are not necessary for them to be equipped with. The next section goes over the studies related to those traits.

2.3.4 Affect and Beliefs

Affect and beliefs such as emotions, feelings, moods, and beliefs toward logic, proofs, and mathematics are psychological traits that can influence students' performances on proof construction. However, unlike the mental attitudes such as tenacity, persistency, flexibility, carefulness, alertness, and precision, affect and beliefs are not the psychological traits that students are required to hold for proof construction. Different individuals have different affect and beliefs. Students' states of affect and beliefs influence their performances on proof construction from inside individually. McLeod (1994) included emotions, attitudes, and beliefs in the affective domain. Furinghetti and Morselli (2009) asserted a psychological trait such as affect should be a crucial factors influencing mathematical activities. It is easily conceivable that emotions that stem from fear, confidence, impatience, patience, anxiety, safeness, frustration, restlessness, and composure can influence a student's proving performance. Individual beliefs can also influence students' problem solving performances.

Skemp (1979) positioned emotions to be an important and essential part in human cognitive activities. Skemp introduced the concepts of *goals* and *anti-goals*. He defined *goals* to be the ones that students want to accomplish while he referred to *anti-goals* as the ones students may want to avoid. He claimed that students had different emotions according to which type of goal they went through. He associated *goals* with pleasure, confidence, frustration, un-pleasure, and *antigoals* with fear, security, anxiety, and relief. He made a distinction between the emotions. He indicated those emotions could influence students' learning positively or negatively. The same thing can be said to students' proving activities.

Goldin, Rosken, and Torner (2009) stressed that beliefs are important factors in teaching and learning of mathematics. For example, a belief that a definition is a static description of a term might mislead students to ignore the logical structure of definitions to help them advance a reasoning process in proof construction. Moore (1994) indicated that students who believed that an explanation was enough for proving might lack a rigor based on rules of logic in constructing a proof. In addition, beliefs and affect are not independent of each other but influence each other. Goldin, Rosken, and Torner (2009) proclaimed beliefs and perceptions might influence students' success in both learning and problem solving. Their claims can be applied to proof construction. Affects and beliefs cannot be ignorable factors in proof construction.

Students' proof schemes can be interpreted as representative examples of their beliefs toward proofs. As shown at the beginning of this chapter, Harel and Sowder (1998) introduced three types of proof schemes: external conviction; empirical, and analytical. Those proof schemes were further explored, developed, and refined by several researchers. Finlow-Bates, Lerman, and Morgan (1993) observed students with different types of proof schemes. The students with empirical proof schemes valued proofs that contained evidence. The students with pictorial proof schemes valued proofs explained with diagrams or figures. The students with analytical proof schemes valued deductive reasoning. The students with empirical and pictorial proof schemes must know the limitations of proofs constructed with their proof schemes in validation and persuasion.

Students' proof schemes can influence their performances in constructing proofs. Zaslavsky and Shir (2005) found that external schemes and empirical schemes can be the sources of students' difficulties with proving. They found out some students assumed that definitions accounted for the conclusion of a given statement, which represented their external proof schemes, and that others depended on examples to validate their arguments, which represented their empirical proof schemes. Students' external and empirical schemes hindered them from developing and practicing analytic schemes. Recio and Godino (2001) ascribed a possible source of their empirical schemes to their prior knowledge of the way to draw a conclusion that they acquired through social sciences.

Some researchers investigated the relationship between students' proof schemes and their problem solving strategies. For example, Stylianou, Chae, and Blanton (2006) explored the patterns of problem solving in proof construction in light of their proof schemes. Using some problems for proof construction in elementary number theory, they observed the followings. First, the students with external schemes used a definition of concept based on their incomplete concept images formed through what they saw in a textbook or on what they heard in a past classroom. Concept image is a mental picture of a concept built in the process of their learning experiences (Tall & Vinner, 1981). The students expected that a proof should follow from the definition without further exploration and discussion of it. Second, the students with empirical schemes resorted to a convincing pattern through numerical examples for supporting their proof. However, they rarely introduced definitions or symbolic representations of the problem. Third, the students with analytical schemes showed the abilities to set a goal for their subsequent activities, symbolize the definition, explore the definition, gain further information, link the new information to the initial problem, and keep goals while monitoring their actions.

The observations that Stylianou, Chae, and Blanton (2006) made indicated that students' perspectives toward proof construction might be a crucial factor for successful proof construction. The external schemes may lead students to depend on their memorization for constructing and validating their arguments. The empirical schemes may blind them from exploring a logical and deductive reasoning argument. It seems imperative to provide practical knowledge for helping students grow out of their external and empirical proof schemes and develop and enforce analytical schemes.

2.4 Comprehensive Views of Students' Difficulties

Selden and Selden (2003a) speculated that students' difficulties with reasoning errors resulted partly from underlying misconceptions. They described several factors as possible underlying misconceptions while showing examples mainly from introductory abstract algebra (2003a, p. 6-10).

M1. They can use the conclusion for an argument that should be proved.

M2. Anything that has a name always exists.

M3. Different symbols always represent different things.

M4. The converse is true.

M5. The rules used for real numbers are always applicable.

M6. Inequalities are conserved if the same operation is practiced to both sides.

M7. A set can be interchangeable with an element.

This study considers proof construction from four aspects: background knowledge, reasoning aspect, mental attitudes, and affect and beliefs. M1, M4, and M7 can be categorized in the knowledge of proving techniques. M6 can be categorized in the knowledge. M2, M3, and M5 can be categorized in the mental attitude of carefulness in dealing with an object. Their findings and the insights above might be useful in helping students with the same problems or similar problems in the same subjects. On the other hand, their findings and insights might be limited to certain types of problems in the same subject (introductory algebra). They might not be considered to be generalized sources of students' difficulties that can applicable to any problem from any subject. In addition, more types of possible misconceptions may be detected with a further investigation into other problems from any subject. It would be meaningful to investigate a method that can make it possible to categorize various sources of students difficulties with proof construction in an organized and systematic way which can be applicable to any problem from any mathematical subject.

Weber (2006) classified the causes of students' difficulties with proofs in three categories: (1) Students lack knowledge of mathematical proof; (2) Students misunderstand and misapply a concept or a theorem; and (3) Students do not know how to develop proving strategies. Weber set the framework for modeling proof construction from mainly two angles: knowledge base and how to use the knowledge to advance a reasoning process. (1) and (2) belong to the knowledge base and (3) belong to the use of knowledge. Students struggle with how to apply their knowledge to their proof construction. However, it seems little has been discussed about specifically what proving strategies are available for students. It would be meaningful to provide a specific proving strategy. Gibson (1998) set a framework for examining students' difficulties with proofs in terms of the following four factors: (1) the rules and nature of proof; (2) conceptual understanding; (3) proof techniques and strategies; and (4) cognitive load. What seems to be scarce are concrete suggestions to help students overcome their difficulties in each aspect. For example, (1) and (4) might be more specified in order to help students in a practical way. It would be meaningful to investigate exactly what the rules of proof exist, what nature of proof students should know, and what factors cause students cognitive heavy load.

Moore (1994, p. 251- 252) provided the sources of students' cognitive difficulties in a more specific way that can be applied to problems across mathematical subjects. He gave the following seven major sources.

> D1. The students did not know the definitions, that is, they were unable to state the definitions.
>
> D2. The students had little intuitive understanding of the concepts.
>
> D3. The students' concept images were inadequate for doing the proofs.
>
> D4. The students were unable, or unwilling, to generate and use their own examples.
>
> D5. The students did not know how to use definitions to obtain the overall structure of proofs.
>
> D6. The students were unable to understand and use mathematical language and notation.
>
> D7. The students did not know how to begin proofs.

Moore further categorized the above into three types in terms of the following factors: (a) concept understanding; (b) mathematical language and notation; and (c) getting started on a proof. He related D1, D2, D3, D4, and D5 to (a) concept understanding, D6 to (b) concept understanding, and D7 to (c) getting started on a proof. In addition, he pointed out that students' perceptions of mathematics and proof might affect their proof-writing performances negatively: (i) view mathematics as computations and symbol manipulations; (ii) view proof as procedures; and (iii) view proof as explanation without rigor.

The following are the correspondences between the factors in the above sources and the aspects from which this study views proof construction.

Affect and Beliefs: This aspect corresponds to students' views of mathematics and poof that Moore mentioned.

Background knowledge: This covers D1, D2, D3, and D6. D1 is about students' lack of knowledge of the definitions of concepts. D2 is about students' problems with understanding concepts and theorems, which results in their lack of knowledge of those concepts and theorems. D3 is about students' inabilities to express their concept images, which can be considered as their lack of knowledge. D6 is about students' lack of knowledge in notation and language.

Reasoning activity: This covers D4, D5 and D7. D4 is about students' ability to generate an example, which involves exploring, which is an operation practiced in reasoning process. D5 is about students' inability to use the definition of a concept to advance their reasoning process. D7 is about students' inability to advance their reasoning process at the beginning stage of proof construction.

Different researchers have examined the sources of students' difficulties from different angles. It seems most of the methods adopted in their studies were deriving their findings about students' difficulties directly from students' proofs. There seems to be little research that first set a framework for modeling proof construction and then examined students' proofs in light of the framework.

2.5 Pedagogical Approaches

Proof construction is difficult not only for students to learn but also for instructors to teach. Several researchers attributed students' inability to prove to the prior instruction they received. Epp (2003) indicated the instruction with too much emphasis on general principles or concrete problem-solving strategies might impede students from developing their logical, deductive, and formal reasoning-skill. It is conceivable that if students are accustomed to mathematics learning which focuses on problem solving techniques, they might get disturbed when they face proofs for understanding and constructing. According to Davis (1998, cited in Dreyfus, 2002, p.28):

> Most mathematics instruction, from elementary school through college courses, teaches what might be called rituals: 'do this, then do this, then do this …' and Teachers … will typically accept the correctly performed ritual as enough success for the time being.

This approach might mislead students to form a wrong view that mathematics is learning ritual while impeding students from rigorous, deductive, logical, and formal thinking in proof construction. Moreover, Knuth (2002) claimed secondary mathematics teachers' perceptions towards proofs and their knowledge of proofs might affect their students' proving skills. Through interviewing 17 secondary mathematics teachers, Knuth observed that all of them accepted their students' empirical arguments. This sort of approach might enhance students' wrong perspectives on proof construction that showing some examples would be enough to prove a statement. Harel and Sowder (1998) warned that if instructors guided their students to get accustomed to justifying a statement based on some examples, they might enhance their students' empirical schemes and prevent their students from developing analytical proof scheme. As Dreyfus (2002) pointed out:

> Students have been taught the products of the activity of scores of mathematicians in their final form, but they have not gained insight into the process that has led mathematicians to create these products (p.28).

To enhance students' conception of proof, Harel (2001) advocated the DNR system – a system of pedagogical principles consisting of Duality, Necessity, and Repeated-reasoning. According to Harel (2000), the duality principle means that students' ways of thinking and their ways of understanding have mutual influences on each other. Students need help with the ways of thinking to enhance their ways of understanding and vice versa. By "ways of understanding," Harel referred to the ways of specific mathematical actions taken in interpreting concepts, solving problems, and justifying an argument. By "ways of thinking," he meant the ways that governed ways of understanding such as beliefs, perceptions, or views of mathematics, problem solving approaches, and proof schemes. The necessity principle means that students understand a learning concept by having an intellectual need for the concept and by eliciting the concept from the solution to the problem involving the concept. By "an intellectual need," Harel referred to a desire to search for a resolution to a problem that a student's existing knowledge cannot cope with. The repeated-reasoning principle means students must practice reasoning while applying the duality and the necessity principles so that they can make their ways of thinking and ways of understanding autonomous and spontaneous.

Harel (2001) suggested that teachers should (1) "form instructional goals in terms of ways of thinking (p.6)," (2) "devise and use appropriate instructional activities through which students can build ways of understanding that can potentially lead the construction of desirable ways of thinking (p. 6)." Harel stressed teachers should pay attention to "how students should come to know facts and procedures and how they should practice those facts and procedures" rather than "whether they need to remember facts or master procedures (p. 9)." Harel claimed the DNR-based instruction might help students develop their transformational proof schemes. Transformational proof scheme is an analytic proof scheme which enables students to reach conclusions through logical deduction while generalizing an idea, applying mental operations that are goal oriented, and transforming images.

Students' ways of thinking can be fostered and enhanced through problem solving. Smith (2006) hypothesized the problem-based structure of the courses may be more effective than the traditional lecture-based teaching in helping students improve their proving-strategies. In her research project, she recruited six students from a problem-based number theory course and conducted a task-based interview session. In the session, she had the student-participants prove some number theory statements. She observed that (1) the participants showed flexibility in shifting the four phases for proving processes: using initial strategies, constructing informal arguments, constructing a formal proof, and validating on the final argument, and (2) they showed a variety of proving strategies while the literature often discussed students' static tendencies for proving.

Ball, Hoyles, Jahnke, and Movshovitz-Hadar (2002) noted more environmental factors for helping students more effectively learn proof construction. They claimed that three areas must be considered with regard to the teaching of proof: (1) the role and the function of proof; (2) the gradual processes and complexities involved in proving;

and (3) effective teaching strategies. Their research led to the following three domains of work as the factors for teachers to help their students learn mathematical reasoning: (1) selecting mathematical tasks; (2) making mathematical knowledge public and scaffolding the use of language and knowledge; (3) creating positive learning environment in a classroom.

This study considers analytical proof scheme to be an ideal proof scheme students should develop. Different researchers suggested different pedagogical approaches to help students with analytical proof scheme. Most of them provided a teaching method but did not show specifically what should be taught. There seems to be a need for the study that explores specific and practical knowledge that helps students advance a reasoning process based on logical deduction.

2.6 Summary

Different researchers have examined students' difficulties with proving from different angles. Much of the research illuminated a particular aspect of proof construction: mathematical language (Finlow-Bates, 1994; Thurston, 1994; Selden & Selden, 1995; Dreyfus, 1999); students' understanding and usage of definitions (Tall, 1991; Vinner 1991; Frid, 1994; Moore, 1994; Edward & Wards, 2004; Zaslavsky & Shir, 2005; Knapp, 2006; Alcock, 2007; Selden & Selden, 2007; Edward & Wards, 2008; Paramerswaran, 2010); logic (Weber, 2002; Stylianides & Stylianides, 2007; Selden & Selden, 2009; Savic, 2011); informal representations (Weber & Alcock, 2004; Weber & Alcock, 2009; Lew, Mejia-Ramos, & Weber, 2013); and proving strategies (Weber, 2001). Although some aspects of proof have been well-researched, there seems to be only a handful of studies that investigated various sources of student's difficulties from a comprehensive point of view (Moore, 1994; Gibson, 1998; Selden & Selden, 2003, Weber, 2006). In addition, in my knowledge, few studies have discussed sources of students' difficulties through various problems from multiple mathematical subjects.

Logical deduction is a central aspect of proof construction. Harel and Sowder (1998) stressed the necessity of fostering students' skills for logical deduction. With several studies related to proof schemes (Harel & Sowder, 2007; Stylianou, Chae,& Blanton, 2006; Zaslabsky & Shir, 2005; Weber & Alcock, 2004), many studies have not explored the knowledge for enhancing students' skills for logical deduction. Weber and Alcock (2004) indicated that both syntactic and semantic approaches must concur to construct a successful proof based on logical deduction. Moreover, Ayalon and Even (2008) claimed that views and approaches to deductive reasoning should be given more attention. There is a strong need and demand for providing an effective way to help students practice both syntactic and semantic approaches.

CadwalladerOlsker, Miller, and Hartmann (2013) attributed a source of students' difficulties with proving to the students' incomplete understanding of what makes a mathematical proof. Knuth (2002) suggested how to view the nature of proof and what constitutes proofs should be clarified. It is important to describe a view of proof construction in the form of a model. Kieran (1998) suggested both empirical and theoretical research should involve explicitly formulated models to describe observed phenomena. Papaleontiou-Louca (2003) stressed the importance of providing metacognition (knowledge of one's processes and cognitive states) by modeling task completion for students' effective learning. Selden and Selden (2007) created a proof framework as an aid of constructing a proof, focusing on the way to produce the *formal-rhetorical part*. However, their method may not be effective in helping students write a proof from the top down. There seems to be room for exploring the way to help students write a proof from the top down.

Considering the existing gaps in the literature, this study created a framework for modeling the structure of proof construction, which can (a) encompass the aspects, factors, patterns, and features involved in cognitive processes in proof construction across mathematical subjects, (b) explain sources of students' difficulties with proving in a clear, organized, and systematic way from a comprehensive perspective, and (c) help students enhance their skills for logical deduction by providing metacognitive and methodological knowledge. This study addresses the following research questions: (1) What is a suitable model for characterizing the structure of proof construction? (2) What difficulties do students have with proof construction and what are the sources of their difficulties in light of the structure of proof construction? (3) How useful is the model of the structure of proof construction? (4) What pedagogical suggestions can be derived to help students with proof construction?

Chapter 3: Theoretical Framework

3.1 Introduction

In order to examine students' difficulties with proof construction, it was necessary and important to have a framework for analyzing students' difficulties in an organized and systematic way. Brown (1998) suggested that the results should be analyzed with theorization. Kieran (1998) stressed the significance of involving a model for describing results in order to better understand the observed phenomena. CadwalladerOlsker, Miller, and Hartmann (2013) indicated students' wrong perspectives on proofs may cause their difficulties with proof construction. Knuth (2002) stressed the importance of clarifying the nature and components of proofs so that they might learn proof construction effectively. In my knowledge, a suitable framework for examining students' difficulties in light of the structure of proof construction was lacking in the literature. Therefore, it was significant for this research to build a framework for providing a comprehensive view that can encompass the aspects, factors, patterns, and features of proof construction.

The following theoretical perspectives led me to the creation of a model. Harel and Sowder (1998) discussed three major types of proof schemes: external, empirical, and analytical. Among the three types of proof schemes, this study considered analytical proof scheme to be the desired proof scheme that students should acquire and develop. The framework was created so that it could agree with the characteristics of analytical proof scheme: setting a goal for subsequent activities; symbolizing definition; gaining new information; and linking new information to the initial problem. Weber and Alcock (2004) discussed two major ways to realize analytical proof scheme:

syntactic and semantics approaches. The framework was designed to be metacognitive knowledge for enabling students to practice syntactic and semantic approaches. The exploration of a framework clarified types of cognitive actions contributing to syntactic and semantic approaches.

In this chapter, I will present a model of the structure of proof construction and a framework for analyzing students' proofs. The analysis framework was built based on the model of the structure of proof construction. I will describe the process of the creation of the model, detail the contents of the model, introduce types of proofs, and discuss inter-rater reliability for the model.

3.2 Creation of a Model of the Structure of Proof Construction

In order to create a model of the structure of proof construction, the think-aloud method was applied to the researcher's cognitive processes in his proving activities. According to Charters (2003), think-aloud is "a research method, in which participants speak aloud any words in their mind as they complete a task (p. 68)". Think-aloud is considered to be a valid and effective research method to understand individual's thinking process (Van Someren, Barnard, & Snadberg, 1994; Olson, Duffy, & Mack, 1984). This study aimed to examine students' difficulties with proof construction across mathematical subjects. I proved 40 theorems, propositions, and lemmas collected from a variety of mathematical subjects such as undergraduate Analysis, Algebra, and Topology, Discrete Mathematics, and Calculus.

Through self-monitoring, I investigated and categorized possible aspects and factors that might be involved in cognitive processes in proof construction. In particular, I noted the operations used to generate a new statement from the previous statement in a proof. Those operations were carefully observed, described, abstracted, and organized to model the structure of *the reasoning activity*. This study will use the expression "*the reasoning activity*" to represent one's cognitive actions for advancing a reasoning process in proof construction. While going through 40 theorems and propositions, the model was adjusted and refined until it explained every operation used to generate a statement from the previous one for every step in each of those proofs. As a result, a framework for modeling the structure of *the reasoning process* was created. The model consisted of four types of operations for advancing a reasoning process (*rephrasing an object*, *combining objects*, *creating a cue,* and *exploring and checking*). Considering the importance of a start of proof construction, two stages were set in the framework (*the opening stage* and *the body construction stage*). *The reasoning activity* focuses on how a reasoning process is advanced in proof construction. The operations used in *the reasoning activity* were considered to be the tools for constructing a proof. Then, the contents or the material on which the operations work can be another important aspect that decides the degree of success in students' performances on proof construction. Those are the knowledge of concepts, definitions, notations, properties, facts, rules, and techniques. This study uses the expression "*the background knowledge*" to represent the knowledge students are required to be equipped with in order to solve a given proving problem.

In order to view proof construction in a comprehensive way, besides the aspects of *the reasoning activity* and *the background knowledge*, some psychological aspects were also considered. Harel and Sowder (2007, p. 4) asserted "a single factor usually is not sufficient to account for students' behaviors with proof." Furinghetti and Morselli

(2009) observed that mathematical thinking was not dominated by purely cognitive behavior but might be influenced by another factor such as affect. In addition to the aspects of *the reasoning activity* and *the background knowledge*, the model was set to include two other psychological aspects: *mental attitudes* (tenacity, persistency, flexibility, carefulness, alertness, and precision); and *affect and beliefs* (emotions, self-confidence, and beliefs toward logic, proofs, and mathematics). The table 3.1 shows the aspects of proof construction.

Table 3.1

Aspects of Proof Construction

Reasoning Activity (See Table 3.2.)	
Main	Rephrasing an object
	Combining objects
	Creating a cue
Supporting	Checking and Exploring
Background Knowledge	
	definitions, properties, theorems, propositions, mathematical laws, and problem solving techniques
Mental Attitudes	
	Tenacity (persistency and patience)
	Flexibility
	Carefulness (alertness and precision)
Affect and Beliefs	
	Affect (emotions, moods, feelings, self-confidence)
	Beliefs (schemes, beliefs toward mathematics, proof, and logic)

3.3 Model of the Structure of Proof Construction

This section first introduces and defines terms used in the model of the structure of proof construction. Then, it presents a model of the structure of *the reasoning activity* (Table 3.2) and a comprehensive view of proof construction in a 3D figure (Figure 3.1) with detailed explanation.

3.3. 1 Terms

Mathematical language

I referred to mathematical language that is fine-grained enough to help students make a clear distinction from everyday language, to advance a reasoning process rigorously, and convince others without leaving any ambiguity as *mathematical language*. The definition of a concept is a representative *mathematical language*. For example, the word "limit" in the statement "There is a speed limit" is everyday language. The statement "The limit of $f(x)$ as x approaches 1 is 2" may remind some students who have not learned Calculus of the everyday language "limit." Although the "limit" in the statement is mathematical language, the difference between the everyday language and the mathematical language should not be clear to those students until they learn the definition of the mathematical concept "limit." In constructing a proof, students are able to advance a reasoning process rigorously when they rephrase the statement "The limit of $f(x)$ as x approaches 1 is 2" with the *mathematical language* "For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-1| < \delta$, then $|f(x)-2| < \varepsilon$."

For another example, the statement of " $f(x): X \to Y$ is continuous" is mathematical language. However, it may be "coarse" for topology students to advance a reasoning argument in constructing a proof. When they rephrase the statement by applying the definition, which is "for an open set *V* in *Y*, $f^{-1}(V)$ is open in *X*," they can make the given statement a "fine-grained" enough to further advance a reasoning process. I called this type of rephrasing "translation of an object into *mathematical language*." Rephrasing a statement containing a mathematical concept by applying the definition of the concept is the most representative example of translation of an object into *mathematical language*. A statement gains a power that enables students to advance a reasoning process rigorously when it is translated into *mathematical language*.

I made a distinction between translating an object into *mathematical language* and unpacking, which Selden and Selden (1995) introduced. They gave the following example for "unpacking." "A function is continuous whenever it is differentiable" is unpacked into "For all functions f, if f is differentiable, then f is continuous." After "unpacking" the informal statement, the new statement still contains the words "differentiable" and "continuous." The unpacked statement "a function is continuous whenever it is differentiable" can be still "coarse." It may not be "fine-grained" enough to advance a reasoning process rigorously because the mathematical terms "differentiable" and "continuous" still remain in the new statement. In order to further advance a reasoning process, the term "continuous" may need to be translated into a "finer" object: $\lim_{x \to a} f(x) = f(a)$ for all $a \in X$." Similarly, the term "differentiable" may need to be translated into " $\lim_{x \to x^{*-}} \frac{f(x) - f(x^*)}{x - x^*} = \lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*}$ for all $x, x^* \in X$ with $x \neq x^*$."

The characteristic of the operation of translating an object into *mathematical language* is that the *mathematical language* contains variables. Variables are crucial and indispensable elements of *mathematical language*. Variables serve as fundamental elements for making a mathematical argument formal and rigorous. In the above example that Selden and Selden gave, the new statement obtained by "unpacking" did not include any variables. In my study, their translation is considered to be "rephrasing an object through interpretation," but translation of an object into *mathematical language* is not just rephrasing an object. The operation of translating an object into *mathematical language* infuses a motive power into the original object so that the new object may enable students to develop a further argument.

There is another difference between 'unpacking" and "translation of an object into *mathematical language*." A statement "a vector *u* and a vector *v* are orthogonal" can be translated into *mathematical language* "$u \cdot v = 0$." However, this translation may not be considered to be "unpacking" because the new object does not involve any logical terms. For another example, students may be asked to find the dimensions of the rectangle with the area 15 cm^2 such that its length is 2 *cm* greater than its width. The sentence can be translated into the following *mathematical language*: "$lw = 15$" and "$l = w + 2$." However, this may not be considered to be "unpacking" because they do not involve logical terms. "Translating into *mathematical language*" includes "unpacking," but not vice versa. For these reasons, I will use the expression "translation of an object into *mathematical language*" throughout this study instead of using "unpacking."

There may be a confusion in the way that the operation is named "translating an object into *mathematical language*." For example, although the statement "a function $f : X \to Y$ is continuous" is mathematical language, it is not considered to be *mathematical language* because it may be still too coarse for students to further advance their reasoning arguments. In order to avoid the confusion, this study uses italicized "*mathematical language*" to represent mathematical language that is "fine-grained" enough to allow students to further develop their arguments.

Ignition phrases

In my study, I called the phrase "for every …," "for any …," or "for all …." an ignition phrase. The phrase "if …" can be an ignition phrase when it is rephrased with "for every …," "for any …," or "for all …." I also considered "for some …" to be an ignition phrase. The phrase "There exists …." can be an ignition phrase when it is rephrased with "for some …." Ignition phrases can be the marks from which students can derive and set a variable in the process of proof construction. However, all ignition phrases are not useful in deriving and setting a variable. I made a distinction between an ignition phrase and an *ignition phrase*. I called the ignition phrase from which a variable must be necessarily derived and set for advancing a reasoning process "an *ignition phrase*." In the above example "for an open set *V* in *Y*, if it is given in the conclusion of a given statement, $f^{-1}(V)$ is open in *X*," the phrase "for an open set *V*" is *an ignition phrase* if it is in the *mathematical language* for the conclusion of a given statement. More specifically, if a topology students is asked to prove " $f(x): X \to Y$ is continuous," the student can start a proof by setting a variable "Suppose *V* is an open set in *X*."

3.3.2 Structure of Reasoning Activity

There are two stages in which the *reasoning activity* occurs: *opening stage* and *body construction stage*. There are four types of operations that compose the reasoning activities: *rephrasing an object*; *combining objects*; *creating a cue*; and *checking and exploring*. This section first describes the model of the structure of the *reasoning activity* (Table 3.2) and detail the stages and the operations.

Table 3.2
Structure of the Reasoning Activity

<table>
<tr><td rowspan="2">Reasoning Activity in Proof Construction</td><td>ACTIONS ⇨</td><td colspan="9">MAIN</td><td colspan="2">SUPPORTING</td></tr>
<tr><td>Roles of the Actions ⇨</td><td colspan="4">Transforming objects</td><td colspan="5">Igniting processes</td><td colspan="2">Supporting a proving argument</td></tr>
<tr><td rowspan="2">STAGES ⇩</td><td rowspan="2">Operations ⇨
Steps ⇩</td><td colspan="3">R
Rephrasing Objects</td><td>CO
Combining Objects</td><td colspan="5">C
Creating a Cue</td><td rowspan="2">Ch
Checking (observe, review, reflect, test, adjust, modify, correct)</td><td rowspan="2">Ex
Exploring (search, try, illustrate, experi-mennt, intuit)</td></tr>
<tr><td>R1</td><td>R2</td><td>R3</td><td>CO(S,T)R</td><td>C1</td><td>C2</td><td>C3</td><td>C4</td><td>C5</td></tr>
<tr><td rowspan="5">Opening Stage</td><td>Z: Choose a major proving strategy.</td><td colspan="9">Decide which proving strategy to use, a direct proof, an indirect proof, or a mathematical induction.</td><td></td><td></td></tr>
<tr><td>X: Set a goal.</td><td colspan="9">Given (Find the conclusion of the given statement)</td><td></td><td></td></tr>
<tr><td>Y: Make the goal clearer.</td><td colspan="9">R1, R2, R3 (Translate the goal into mathematical language. For a contrapositive case, negate the given statement in mathematical language.)</td><td></td><td></td></tr>
<tr><td>P: Make sure of the hypotheses.</td><td colspan="9">Given (Find all the hypotheses of the given statement and translate them into mathematical language if necessary.)</td><td></td><td></td></tr>
<tr><td>S: Set a variable</td><td colspan="9">C1, C2, C3, C4, C5 or given (S can be the same as P.)
In most cases, this step corresponds to Step 1.</td><td></td><td></td></tr>
<tr><td rowspan="4">Body Construction Stage</td><td>Step 1</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 2</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>…</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Conclusion</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Actions	
Main Actions	The operations applied to a step to generate the next step, whose outcome must be explicitly expressed to convince others
Supporting Actions	The operations to produce side work, whose outcome does not necessarily have to appear in the proof to convince others

Rephrasing an object	
R1	Rephrasing an object by translating a concept, a theorem, or a property of concept into mathematical language mainly through applying its definition.
R2	Rephrasing an object through formal interpretation, informal interpretation, or common sense.
R3	Rephrasing an object through algebraic manipulation or calculation, including solving an equation.

Combining objects	
CO(S, T)R	Connect and combine different pieces of objects (S and T) to create a new object. This action is always accompanied by an operation of rephrasing.

Creating a cue	
C1	Set a variable.
C2	Recall prior knowledge, including a theorem, a proposition, a property of concept, or a mathematical law.
C3	Set some cases.
C4	Make a claim or a new object
C5	Consider an object.

The opening stage. The *opening stage* is a preparation stage at which students decide a major proving strategy (a direct proof, or a proof by contrapositive, by contradiction, or by mathematical induction), make the goal of the proof clearer by translating the conclusion of a given statement, and set a starting variable by finding an *ignition phrase*. Here are steps that students can take in the *opening stage*. Students first decide whether to use a direct proof or an indirect proof, then note the conclusion of the given statement, translate it into *mathematical language*, find *an ignition phrase* contained in *the mathematical language*, derive a starting variable from *the ignition phrase*, set the starting variable to start the *body construction stage*. If there is no *ignition phrase* in the *mathematical language* for the conclusion, students translate a hypothesis into *mathematical language* to derive a starting variable. In some proofs, students do not need to derive a starting variable because it is already given in the problem. Although the work in the *opening stage* is useful, the work does not necessarily have to be expressed to convince others.

The body construction stage. T*he body construction stage* is the main part of a proof, in which students advance their reasoning process by making good use of the four operations (*rephrasing an object, combining objects*, *creating a cue,* and *checking and exploring*).

Cognitive actions. The first row in the top table shows there are two main actions to be taken in proof construction: (a) *main actions* and (b) *supporting actions*. The difference between them is the necessity of explicitly writing the work obtained through the actions in convincing others of the validity of the proving argument. Students' work that is performed through the *main actions* must be explicitly written out while students' work done through the *supporting actions* do not so that students can convince others of the validity of their proving arguments. The former actions are the tools directly used to generate a statement for a reasoning process while the latter actions are the tools used to explore and prepare for an idea or a thought to advance a reasoning process.

The main actions have two roles. One is transforming an object and the other is igniting a reasoning process. Transforming an object means changing an expression. An object can be a word, a phrase, and a sentence. "Igniting a process" means sparking a process or triggering an argument. The main actions include three types of operations: *rephrasing an object*, *combining objects*, and *creating a cue*. *Rephrasing an object* and *combining objects* play a role to *transform* an object. *Creating a cue* plays a role to *ignite* a process. *Rephrasing an object* has three types: by applying definitions, properties, and theorems; through interpretation; and through algebraic manipulation. Creating a cue has 5 types: setting a variable; recalling and applying prior knowledge; setting some cases; making a claim or creating a new object; and considering an object. Those operations are direct cognitive actions taken to generate the next step from the previous step. Among those the operations (*rephrasing an object*; *combining objects*; and *creating a cue*), *rephrasing an object* is considered to be the primary operation. *Combining objects* is the next level of operation, which students can try when the operation of *rephrasing an object* does not work for advancing a reasoning process. *Creating a cue* is the highest level of operation, which students can try when neither *rephrasing an object* nor *combining objects* work in advancing a reasoning process. In a sense, *creating a cue* can be considered as the most difficult operation of all those three operations.

The supporting actions are the cognitive actions that support and work behind the *main actions*. The *supporting actions* include *checking* and *exploring*. For example, when students have impasses, they may check, adjust, modify, evaluate, and correct what they have done or explore a method to overcome the difficulty through using a diagram, creating an example, intuiting, or doing trial and error. The word "supporting" does not mean "less important." The function of s*upporting actions* is as important as that of *main actions*. In having others evaluate the proof, the work performed through *main actions* must be explicitly written out while the work performed through *supporting actions* does not necessarily have to be stated.

Illustrations of the reasoning activity

In order to give concrete examples of the components of the *reasoning activity*, the analysis of a proof problem from Analysis is presented.

> Example (Analysis)
>
> *Prove that if* $f'(x)=0$ *for all* x *in an interval* (a,b), *then* $f(x)$ *is constant on* (a,b), *using the Mean Value Theorem.*

In the *opening stage*, first decide which proving strategy to use. In this case, choose a direct proof. Then, note the conclusion of the given statement . The goal of this proof is to show that $f(x)$ is constant on (a,b). Then, translate the conclusion into *mathematical language*. By doing that, students can make the goal of the proof clearer and can prepare for setting a starting variable. The mathematical language for the conclusion " $f(x)$ is constant on (a,b) " is " for $x_1, x_2 \in (a,b)$ " by applying the definition of a function being constant (R1). Then, find *an ignition phrase* "for $x_1, x_2 \in (a,b)$" to derive starting variables. *An ignition phrase* is a phrase containing "for ~" or "if ~," which plays an important role to help students find and set a variable. Then, start the *body construction stage* with "Let x_1, x_2 be arbitrary numbers found in (a,b) with $x_1 < x_2$" (C1). Call this "object 1." An "object" means a statement or sentence for each step. To advance a reasoning process, first apply *rephrasing an object*. However, there is no way to rephrase it this time. Then, try *combining objects*. In order to combine objects, look for a given condition. Since this problem requires the use of the Mean Value Theorem, recall the Mean Value Theorem and translate it into *mathematical language* "Suppose that a function $f(x)$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β). Then, there must exist a value $c \in (\alpha, \beta)$ such that $f'(c) = \dfrac{f(\beta) - f(\alpha)}{\beta - \alpha}$" (R1). Call this "object 2." Then, combine the objects 1 and 2. To combine different objects, find a connection between them. The object 1 states "Let $x_1, x_2 \in (a,b)$ with $x_1 < x_2$." The object 2 states "Suppose that a function $f(x)$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β). Then, there must exist a value $c \in (\alpha, \beta)$ such that $f'(c) = \dfrac{f(\beta) - f(\alpha)}{\beta - \alpha}$". In order to apply the theorem to the given problem, look for the variables in the given problem that correspond to α and β in the theorem. Then, let $\alpha = x_1$ and $\beta = x_2$ (C1). Then, check the validity of the claim by noting $f'(x) = 0$ on (a,b) (CE). By combining the given condition "$f'(x) = 0$ on (a,b)" and the object1 "$x_1, x_2 \in (a,b)$," obtain $f'(x) = 0$ on $[x_1, x_2]$ (CO). Call this "object 4."

Moreover, rephrase " $f'(x)=0$ on $[x_1,x_2]$ " with " $f(x)$ is differentiable on $[x_1,x_2]$ " through formal interpretation (R2). Furthermore, by recalling the fact that differentiability guarantees continuity, apply that to " $f(x)$ is differentiable on $[x_1,x_2]$ " to draw " $f(x)$ is continuous on $[x_1,x_2]$. (R2)" Now, by the Mean Value Theorem, there exists a value $c \in (x_1,x_2)$ such that $f'(c)=\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$. Call this "object 5."

Since *rephrasing an object* does not work on the object 5, try *combining objects*. Look for the information still available to find the hypothesis " $f'(x)=0$ *for all* x *in an interval* (a,b)." Then, combine the object 5 and the hypothesis to obtain $0=f'(c)=\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$ (CO). Since $x_1<x_2$, which means $x_2-x_1\neq 0$, multiply by x_2-x_1 to both sides to obtain $f(x_1)=f(x_2)$ for $x_1,x_2\in(a,b)$ through *algebraic manipulation* (R3).

The following table (Table 3.3) illustrates a possible proof for the above problem with each step being coded based on the framework table (See Table 3.2).

Table 3.3

Analysis Table Type A (Example 1)

	The Opening Stage	Operations
X	Show $f(x)$ is constant on (a,b).	
Y	Show that for every $x_1,x_2\in(a,b)$ with $x_1<x_2$, $f(x_1)=f(x_2)$	R1
P1	$f'(x)=0$ on (a, b).	Given
P2	The Mean Value Theorem says "Suppose $f(x)$ is continuous on $[p,q]$ and differentiable on (p,q). Then, there exists a number $c\in(p,q)$ such that $f'(c)=\dfrac{f(q)-f(p)}{q-p}$."	R1

S	Let $x_1, x_2 \in (a,b)$ with $x_1 < x_2$.	C1
	The Body Construction Stage	
1	Let $x_1, x_2 \in (a,b)$ with $x_1 < x_2$.	C1
2	Note $f(x)$ is differentiable on $[x_1, x_2]$ because $f'(x) = 0$ on (a,b) and $[x_1, x_2] \subset (a,b)$.	R2
3	Recall the theorem that says that differentiability implies continuity.	C2
4	Then, $f(x)$ is continuous on $[x_1, x_2]$.	CO(2,3)R2
5	Note $f(x)$ is differentiable on (x_1, x_2) .	R2
6	Then, there exists $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.	CO (P2, 4,5) R1
7	Then, $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$ because $f'(x) = 0$ on (a, b) and $c \in (x_1, x_2) \subset (a,b)$.	CO (6, P1) R2
8	Then, $f(x_1) = f(x_2)$ because $x_1 \neq x_2$.	R3
9	Therefore, for every $x_1, x_2 \in (a,b)$ with $x_1 < x_2$, $f(x_1) = f(x_2)$.	CO(1.8)R2

The letter X in the first column stands for the conclusion of the give statement. The letter Y stands for *the mathematical language* that students obtain by translating the conclusion of the given statement. P1 and P2 stand for the given hypotheses. The numbers in the first column show the order of the steps to be taken in advancing a reasoning process. The order of the steps shown above does not have to be the only way for advancing a reasoning process for the proof. The second column shows a statement for each step. The third column shows the codes for the operations that are used to obtain the corresponding steps.

3.3.3 Aspects of Proof Construction

This study viewed proof construction from the following four aspects: *reasoning activity*; *background knowledge*; *mental attitudes*; and *affect and emotions*. The previous section detailed the structure of the *reasoning activity*. This section details the other aspects.

The second aspect, *background knowledge,* means the knowledge necessary for solving a given proving problem. It includes definitions, notations, properties, theorems, lemmas, propositions, and proving techniques. If the operations in the *reasoning activity* are compared to the tools for constructing a mathematical argument, *background knowledge* can be compared to the material that is necessary for forming the contents of the argument.

While the aspects of the *reasoning activity* and the *background knowledge* more directly involve proof construction, students' psychological aspects such as *mental attitudes* and *affect and beliefs* should not be ignored as major aspects of proof construction. The *mental attitudes* are the traits that everyone is required to have for a proving activity while the *affect and beliefs* are the traits that depend on each individual.

M*ental attitudes* include *tenacity*, *flexibility*, and *carefulness*. *Tenacity* is the source for sustaining students' cognitive activities during proof construction. If students do not have enough *tenacity* while proving, their proofs end at that point. In the model of the structure of proof construction (Figure 1), *tenacity* is considered as the primary factor for the *mental attitudes* because students cannot advance their reasoning process without *tenacity*.

The second fundamental factor for the *mental attitudes* is *flexibility*. *Flexibility* is required for students to have in addition to *tenacity* especially when they have impasses in a proving activity. Here, "having impasses" means "getting stuck." For example, students may apply a property of a concept and find out that it does not help them. Then, they need to be flexible enough to give up the property and to try another property of the concept. For another example, they may try to advance a reasoning process by rephrasing an object but find out it does not work. Then, they need to be flexible enough to try combining objects or creating a cue. For another example, students may first try a direct proof but find out it does not help them. Then, they need to be flexible enough to try an indirect proof. Thus, *flexibility* is an important mental attitude that enables students to overcome their impasses by changing their ideas and trying a new idea. In the model of the structure of proof construction (Figure 1), flexibility is considered as the second primary factor for the *mental attitudes* because *flexibility* does not occur without *tenacity* in proof construction.

C*arefulness* and *alertness* are considered as the third primary *mental attitudes*. They are psychological traits that enable students to be accurate, precise, and rigorous in dealing with objects in their reasoning process. In this study, they are interpreted to stem from *flexibility* but not vice versa. In order for students to avoid making an error, they need to be flexible and pliable enough to stop to think or to be alert to any variation in a given situation. There may be more factors for the *mental attitudes*. However, in order to avoid making the model of the structure of proof construction complex, only those three factors (*tenacity*; *flexibility*; and *carefulness* and *alertness*) are considered as the major factors for the m*ental attitudes*.

Affect and beliefs are psychological traits that can affect students thinking activities in proving. *Affect* means emotions, moods, and feelings, including easiness, willingness, calmness, anxiety, nervousness, and fear. For example, students may face an event that may greatly affect their emotions in their everyday lives, which may lower their thinking abilities in proving. In another example, students' test-anxieties may obstacle their proving performances. *Beliefs* include perceptions and views toward sense-making, proofs, mathematics, and mathematical abilities. For example, students may believe that learning mathematics is a matter of memorizing and applying definitions and formulas, that may obstruct their deductive and logical thinking. For another example, students' lack of self-confidence may discourage them to think in proving. Although *affect and beliefs* influence students' proving performances, it was not a major focus in this study for the following reasons. First, the former three aspects (*reasoning activity*, *background knowledge*, *mental attitude*) are more general attributes that can be applied to all individuals who engage in proof construction while the fourth aspect (*affect and beliefs*) depends on each individual. Also, while the first three aspects influence students' proving activities more directly and more explicitly, the fourth aspect influence their proving activities more indirectly and more implicitly.

Focusing on the first three aspects, a model of the structure of proof construction is created in the form of a 3D figure (Figure 3.1) as an aid to make it easier and simpler to grasp the view of the structure of proof construction that this study adopted. The fourth aspect is not described in the figure, but one may imagine it exists as a sphere that envelops the whole cuboid.

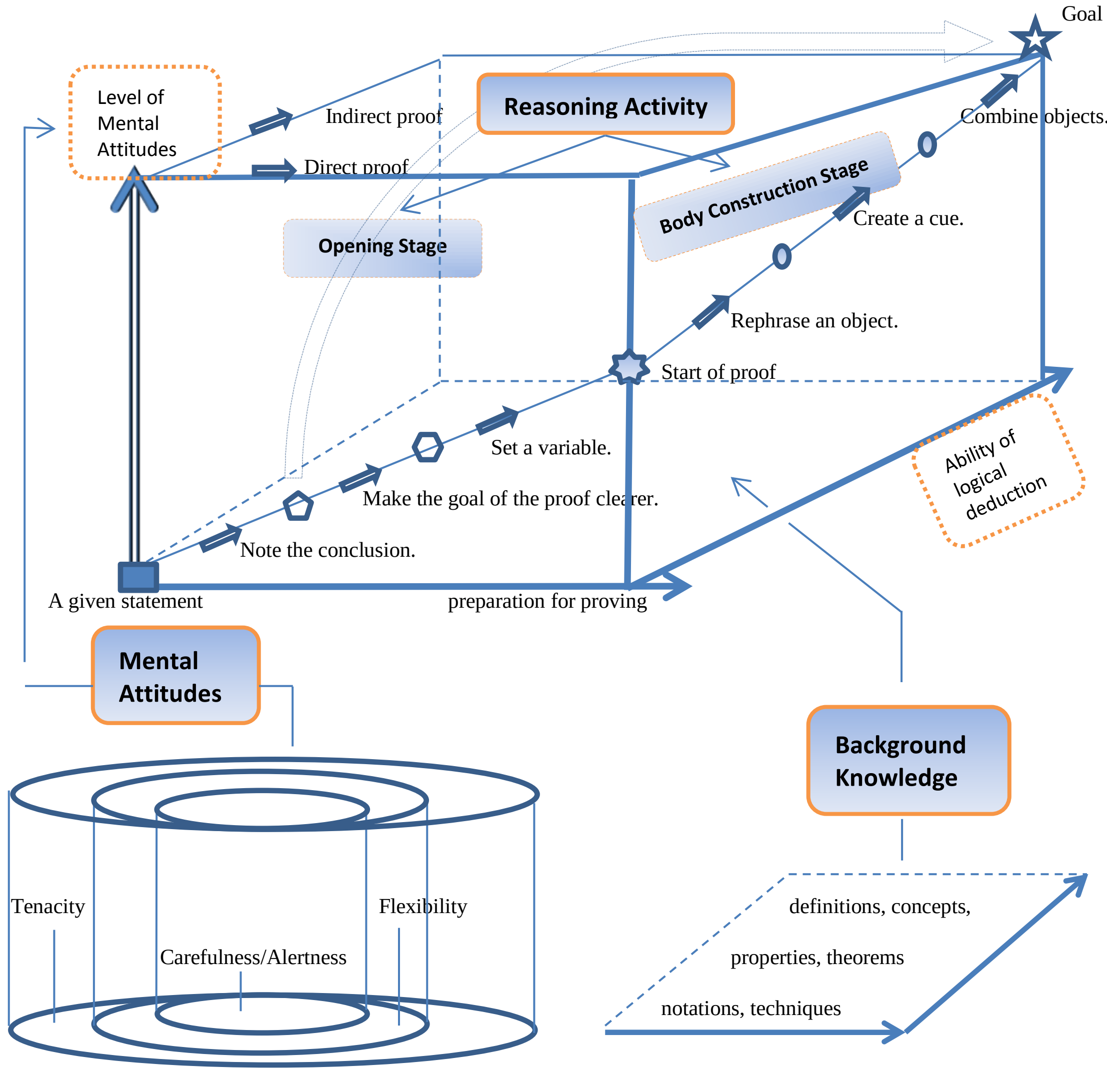


Figure 3.1. 3D Model of the Structure of Proof Construction.

In the above cuboid, the length of the vertical line segments represents the degree of one's mental attitudes necessary for solving a given proof problem. The base area represents one's amount of knowledge necessary for solving the proof. The left front lateral side represents the opening stage and the right front lateral side represents the body construction stage. The line on the both lateral sides represents the reasoning activity.

[1] Mental Attitudes

(1) Tenacity: willingness to persist in, the source of sustaining and continuing one's thinking activity
(2) Flexibility: willingness to change ideas not working and to try new or different methods
(3) Carefulness /Alertness: willingness to be cautious, precise, accurate, and watchful in dealing with objects

[2] Background Knowledge

The knowledge necessary to prove a given statement, such as definitions, notations, and properties of concepts, theorems, propositions, and proving techniques

3D model of proof construction. In Figure 3.1, the lateral sides of the cuboid represent the stages in the *reasoning activity*. There are mainly two major proving strategies: a direct proof and an indirect proof. *The reasoning activity* by a direct proof is expressed with the line with arrows on the front two lateral sides facing to a reader. *The reasoning activity* by an indirect proof can be expressed with a line on the two back lateral sides that are not visible. The front left lateral side of the cuboid shows *the opening stage*, in which students note the conclusion of the given statement, translate it into *mathematical language* to make the goal of the proof clearer, derive a starting variable from *an ignition phrase* contained in *the mathematical language* for the conclusion of the given statement, and set a starting variable to start a proving argument. "Set a goal" in the opening stage means that students make sure of the goal of the proof by noting the conclusion of the given statement. "Set a direction" in *the opening stage* means that students translate the conclusion of the given statement into *mathematical language* so that they can make it easier and clearer to see the direction to reach the goal of the proof. "Set a variable" in *the opening stage* means that students set a variable with which to develop a proving argument.

The right side of the front lateral sides shows *the body construction stage*, at which students advance their reasoning process by *rephrasing an object*, *creating cue*, and *combining objects*. A point on the line drawn on the two lateral sides shows where a student stands in the process of proof construction. In reality, a student's reasoning process may not be expressed with a straight line as it is seen in the 3D figure. A line for representing a student's reasoning process may be curved, winding, and fluctuated in moving from a statement to the next statement. The straight line shown in the 3D figure represents an ideal reasoning process. Although the operations of *exploring* and *checking* are not the main focus in this study, those operations may be expressed with a curve penetrating the inside of the cuboid of the 3D figure.

The base area represents the amount of the knowledge that students are required to have in order to prove the given proof problem. The knowledge includes definitions, properties, notations, theorems, propositions, and proving techniques which are necessary for solving the given proof problem. In the 3D figure, a student's knowledge can be expressed with the area of a base quadrilateral determined by the starting point, a line segment along with the left front base line segment, which may be shorter than or equal to the left front line segment, and a line segment along with the back diagonal base line segment of the cuboid, which may be again shorter than or equal to the whole diagonal base line segment of the cuboid. For example, suppose that a student has as much knowledge as is represented by the quadrilateral determined by the starting point, the point shared by the line segments on the two base line segments of the cuboid, which may be shorter or at most equal to the base line segments of the cuboid. Then, the student's proof stops at the point on "the reasoning process line" in the *body construction stage*, which is the intersection of the "reasoning process line" and the line segment drawn from the upper right corner of the quadrilateral whose area represents the amount of the students' knowledge. If the area of the quadrilateral that represents the amount of a student's knowledge is smaller than the base area of the cuboid, the student cannot reach the goal of the proof.

The vertical distance from the point on the "reasoning process line" seen in the two lateral sides, at which a student stands in the process of proving, to the base of the

3D figure represents the degree of a student's mental attitudes such as tenacity, flexibility, and carefulness. If the degree of a student's mental attitudes is not enough for the level that is required by a given proof problem, the student cannot reach the goal. For example, suppose that a student's argument stops at a point on "the reasoning process line" which is short of the goal. Then, the degree of the mental attitudes that the student has is expressed with the distance from the point at which he stands and the intersection point of the base line segment and the vertical line segment drawn from the point at which he stands. Thus, if the degree of a student's mental attitude is shorter than the one expressed with the height of the 3D figure, the student cannot reach the goal of the proof.

3.4 Frameworks for Analyzing Students' Proofs

Based on the model of the structure of proof construction, two types of frameworks were created for analyzing students' proofs. One was created according to each proof students worked on. It contained all the steps to be taken for the proof. It also included what type of operation was applied to generate a statement from the previous statement for each step in a proof. Each of the operations used in the proof was coded. The purpose of this framework was to detect where students have difficulties or impasses and which operation they fail to apply. I called this type of analysis framework Type A. The framework Type A was created for each problem examined in this study. Table 3.4 is an example of the analysis framework Type A.

> Example (Topology):
>
> *Let* $q: X \to Y$ *be a quotient map and* $f: Y \to Z$ *be a map.*
> *Suppose* $f \circ q$ *is continuous. Show* $f: Y \to Z$ *is continuous.*

Table 3.4

Analysis Table Type A (Example 2)

Step	Statement	Operation	Students	
			A	B
Opening Stage				
X	$f : Y \to Z$ is continuous.	Given		
Y: OTC	For any open set W in Z, $f^{-1}(W)$ is open in Y.	R1		
P1: hypothesis	$q : X \to Y$ is a quotient map.	Given		
P2: hypothesis	$f \circ q$ is continuous.	Given		
S: OSV	Let W be an open set in Z.	C1		
Body Construction Stage				
1	Let W be an open set in Z.	C1		
2	Consider $(f \circ q)^{-1}(W)$.	C5		
3	Note $(f \circ q)^{-1}(W) = q^{-1}(f^{-1}(W))$.	R1		
4	Since $f \circ q$ is continuous, $q^{-1}(f^{-1}(W))$ is open in X.	CO(3,P2)R1		
5	Recall the property of a quotient map.	C2		
6	Since q is a quotient map, $f^{-1}(W)$ must be open in Y.	CO(4,5)R1		
7	Therefore, $f : Y \to Z$ is continuous.	R1		

The other type of analysis framework (Table 3.5) was created based on the model of the structure of proof construction. It was built in order to identify students' difficulties and possible sources of their difficulties. I called this framework Type B. Moore (1994) provided seven major sources of students' cognitive difficulties with proof construction. The framework Type B helps to compare the components of Moore's model and the model created in this study for finding correspondences. The possible sources listed in the framework Type B covers Moore's seven sources from different angles. The correspondences are shown on the fourth column of the table (Table 3.5). The framework Type B was applicable to any proof while the framework Type A (Table 3.4) changes according to a proof that students worked on.

This analysis framework Type B was converted into a framework (Table 3.6) through negation of its components, which turned out to be a list of the types of abilities and proving skills that were necessary for students to construct a successful proof. I called this framework Type C. There is no substantial difference between the framework Type B (Table 3.5) and the framework Type C (Table 3.6), but the former informs more directly of the types of students' difficulties and the sources of their difficulties while the latter informs more directly of the types of students' abilities and skills necessary for proof construction.

Table 3.5

Analysis Framework Type B

the Opening Stage	OTC	unable to pay attention to the conclusion of a given statement, unable to translate it into mathematical language correctly	D7
	OSV	unable to set a right starting variable	D7
Knowledge	KDF	Do not know definitions	D1,D3
	KPR	Do not know properties, related theorems, and propositions	D3
	KTC	Do not know a proving technique	
	KNT	Do not know notations of a concept	D6
Mental Attitudes	MT	Lack of tenacity (give up thinking halfway through)	
	MC	Lack of carefulness, precision, or alertness (including cases of skipping steps)	D5
	MF	Lack of flexibility (stick to a wrong idea or an idea that does not work without trying a different way)	
Rephrasing an Object	R1	Unable to rephrase through applying definitions, properties, and theorems	D1 D2
	R2	Unable to rephrase by formal or informal interpretation	
	R3	Unable to rephrase by algebraic manipulation or computation	
Combining Objects	CO	Unable to connect and combine the objects Unable to use all the given conditions	
Creating cue	C1	Unable to set a variable for cue for advancing a reasoning process	
	C2	Unable to recall definitions, properties, theorems, propositions, mathematical laws, proving techniques from their prior knowledge	
	C3	Unable to set some cases	
	C4	Unable to make a claim	
	C5	Unable to consider an object	
Supporting Actions	CH	Fail to check what has been done	D4
	EX	Unable to explore an idea to advance a reasoning process, do experiment, create and use an example, or intuit an innovative idea	
Beliefs and Emotions	B	Fail to have a sound and appropriate beliefs toward logic, proof, and mathematics.	D5
	E	Get emotional factors affect proving performances.	

Note. The framework for identifying possible sources of students' difficulties with proof construction and for showing the correspondences of Moore's seven sources of students' difficulties, 1994, p. 251-252)

Moore's seven sources of students' cognitive difficulties with proof construction are as follows.

D1. The students did not know the definitions, that is, they were unable to state the definitions.

D2. The students had little intuitive understanding of the concepts.

D3. The students' concept images were inadequate for doing the proofs.

D4. The students were unable, or unwilling, to generate and use their own examples.

D5. The students did not know how to use definitions to obtain the overall structure of proofs.

D6. The students were unable to understand and use mathematical language and notation.

D7. The students did not know how to begin proofs.

Table 3.6

Analysis Framework Type C

Manage the Opening Stage Well	OTC	note the conclusion of the given statement and translate it into mathematical language correctly
	OSV	Set a right starting variable
Have Solid Background Knowledge	KDF	Know definitions
	KPR	Know properties and theorems
	KST	Know proving or solving techniques
	KNT	Know the notations
Have Positive Mental Attitudes	MT	Have Tenacity
	MC	Have Carefulness and Alertness
	MF	Have Flexibility
Rephrase an Object	R1	Rephrase an object through applying definition of a concept
	R2	Rephrase an object through formal or informal interpretation
	R3	Rephrase an object through algebraic manipulation or computation
Combine Objects	CO	Connect and combine objects Use all the given conditions
Create a Cue	C1	Set a variable
	C2	Recall definitions, properties, theorems, lemmas, techniques from their prior knowledge
	C3	Set some cases
	C4	Make a claim
	C5	Consider an object
Practice Supporting Actions	EX	Explore an idea to advance a reasoning process, do experiment, create and use an example, intuit an innovative idea
	CH	Check what has been done.

Note. The framework for showing the abilities and skills necessary for proof construction.

3.5 Types of Proofs

In the process of creation of the structure of proof construction, proofs were classified into three types according to the ways to manage the *opening stages*, more specifically, the ways to set a starting variable. For the first type, students derive a starting variable from an *ignition phrase* in the *mathematical language* for the conclusion of the given statement. For the second type, students derive a starting variable from an *ignition phrase* in the *mathematical language* for a hypothesis of the given statement. A proof by contradiction belongs to this type. A proof that requires students to construct an object seems to belong to this type. The Tube Lemma was such an example, in which students are asked to construct an open set that satisfies certain conditions. For the third type, students do not have to derive a starting variable and can directly work on the conclusion of the given statement. Mathematical induction and proofs of trigonometric identities belong to this type.

3.5.1 Examples of Types of Proofs

In the following pages, some examples for each type of proofs are presented (Tables 3.7 - 3.13). Those proofs were collected from a variety of mathematical subjects: Algebra (Table 3.7), Analysis (Table 3.8, 12), Topology (3.9, 10, 11), Discrete Mathematics (Table 3.13). The first three examples (Table 3.7, 8, 9) belong to the first type, in which students derive a starting variable from the conclusion. The next two examples (Table 3.10, 11) belong to the second type, in which students are required to derive a starting variable from the hypothesis. The remaining two examples (Table 3.12, 13) belong to the third type, in which students are not required to set a variable, and can start to directly work on the variables provided in the conclusion of the given statement. To show the types of proofs do not depend on mathematical subjects, some proofs from a variety of mathematical subjects are presented for each type of proof.

The table below (Table 3.7) is an example of the proofs belonging to the first type, in which students derive a starting variable from *an ignition phrase* in the *mathematical language* for the conclusion of the given statement. The following is the question of the problem.

> Example (Algebra)
>
> *Suppose that* $G/Z(G)$ *is cyclic. Prove that G is abelian.*

Table 3.7

Example of Type I (1)

	Opening Stage	Operation
X	Show G is abelian.	
Y	Show that for any $g,h \in G$, $gh = hg$.	R1
P1	$G/Z(G)$ is cyclic.	Given
P2	$G/Z(G) = \langle xZ(G) \rangle$ for some $x \in G$.	R1
S	Suppose $g,h \in G$.	C1
	Body Construction Stage	
1	Suppose $g,h \in G$.	C1
2	Note that each of $g,h \in G$ is in some coset.	C2
3	Then, $g = x^n a$ and $h = x^m b$ for some $a,b \in Z(G)$ for some $n,m \in Z$	CO(2, P2)R2
4	Consider gh.	C5
5	Then, $gh = x^n a x^m b = x^{n+m} ab = x^{m+n} ba = x^m b x^n a = hg$	CO(3,4)R3
6	Therefore, for any $g,h \in G$, $gh = hg$.	R2
7	Thus, G is abelian.	R1

The table below (Table 3.8) shows another representative example of Type I, which is from Analysis. Students derive a starting variable from the *ignition phrase* in the *mathematical language* for the conclusion of the given statement. The question is the following.

Example (Analysis)

Suppose each of s_n *and* t_n *is a convergent sequence such that* $\lim_{n\to\infty} s_n = L$ *and* $\lim_{n\to\infty} t_n = M$. *Prove* $\lim_{n\to\infty}(s_n + t_n) = L + M$.

Table 3.8

Example of Type I (2)

	Opening Stage	
X	Prove $\lim_{n\to\infty}(s_n + t_n) = L + M$.	Given
Y	Show that for every $\varepsilon > 0$, there exists an $N \in Z^+$ such that for every $n \geq N$, $\lvert (s_n + t_n) - (L + M) \rvert < \varepsilon$.	R1
P1	$\lim_{n\to\infty} s_n = L$	Given
P2	$\lim_{n\to\infty} t_n = M$	Given
S	Suppose $\varepsilon > 0$.	C1
	Body Construction Stage	
1	Suppose $\varepsilon > 0$.	C1
2	Since $\lim_{n\to\infty} s_n = L$, there exists an $N_1 \in Z^+$ such that for every $n \geq N_1$, $\lvert (s_n - L) \rvert < \frac{\varepsilon}{2}$.	CO(P1,1)R1
3	Since $\lim_{n\to\infty} t_n = M$, there exists an $N_2 \in Z^+$ such that for every $n \geq N_2$, $\lvert (t_n - L) \rvert < \frac{\varepsilon}{2}$.	CO(P2,1)R1
4	Let $N = \max\{N_1, N_2\}$.	C1
5	Consider $\lvert (s_n + t_n) - (L + M) \rvert$.	C5
6	Note $\lvert (s_n + t_n) - (L + M) \rvert = \lvert (s_n - L) + (t_n - M) \rvert$.	R3
7	Recall the triangle inequality.	C2
8	Then, $\lvert (s_n - L) + (t_n - M) \rvert \leq \lvert s_n - L \rvert + \lvert t_n - M \rvert$.	CO(6,7)R1
9	Note that for every $n \geq N$, $\lvert s_n - L \rvert + \lvert t_n - M \rvert < \varepsilon$.	CO(1,2,3)R3
10	Thus, for every $\varepsilon > 0$, there exists an $N \in Z^+$ such that for every $n \geq N$, $\lvert (s_n + t_n) - (L + M) \rvert < \varepsilon$.	CO(8,9)R2
11	Therefore, $\lim_{n\to\infty}(s_n + t_n) = L + M$.	R1

The next example (Table 3.9) is another example of Type I, which is from Topology. In this case, students derive more than one variable from the *ignition phrase* in the *mathematical language* for the conclusion of the given statement.

Example (Topology)
Suppose $f : X \to Y$ *is continuous. Then, prove that* $f(\overline{A}) \subset \overline{f(A)}$.

Table 3.9

Example of Type I (3)

	Opening Stage	
X	Prove $f(\overline{A}) \subset \overline{f(A)}$.	
Y	Show that if $y \in f(\overline{A})$, then $y \in \overline{f(A)}$.	R2
Y'	Show that if $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$.	R2
Y"	Show that if $x \in \overline{A}$, for any neighborhood U of $f(x)$, $U \cap f(A) \neq \phi$.	R1
P	f is continuous.	Given
S	Let $x \in \overline{A}$ and let U be a neighborhood of $f(x)$.	C1
	Body Construction Stage	
1	Let $x \in \overline{A}$.	C1
2	Let U be a neighborhood of $f(x)$.	C1
3	Since *f* is continuous, $f^{-1}(U)$ is an open neighborhood of *x*.	CO(P,2)R 1
4	Since $x \in \overline{A}$, there exists an element *z* such that $z \in f^{-1}(U) \cap A$.	CO(1,3)R 1
5	Consider $f(z)$.	C5
6	Note, $f(z) \in U \cap f(A)$.	CO(4,5)R 2
7	Thus, if $x \in \overline{A}$, there exists $z \in Y$ such that $f(z) \in U \cap f(A)$, that is, $U \cap f(A) \neq \phi$.	CO(1, 6)R2
8	Therefore, $f(\overline{A}) \subset \overline{f(A)}$.	R1

The great majority of proofs examined in this study belonged to Type I, in which students were required to set a starting variable from the conclusion of the given statement. However, there were a few proofs in which students were required to derive a starting variable from a hypothesis of the given statement. The followings (Tables 3.10 and 3.11) are such examples (Type II).

Example (Topology)

Let X, Y be topological spaces, Y compact, $x_0 \in X$ *, N an open set containing* $\{x_0\} \times Y$ *in the product space* $X \times Y$.
Prove that there exists an open neighborhood $W \subset X$ *of* x_0 *such that* $W \times Y \subset N$.

Table 3.10

Example of Type II (1)

	Opening Stage	
X	Show there exists an open neighborhood $W \subset X$ of x_0 such that $W \times Y \subset N$.	
Y	Find an open neighborhood $W \subset X$ of x_0 such that $W \times Y \subset N$.	R2
P1	N is an open set containing $\{x_0\} \times Y$ in the product space $X \times Y$	C4
P2	Y is compact.	
	Body Construction Stage	
1	Since N is open in $X \times Y$, for each $(x_0, y) \in \{x_0\} \times Y$, there exists a basis open set $U_y \times V_y \in T_X \times T_Y \subset N_Y$ containing (x_0, y).	C1
2	Then, $\{\{x_0\} \times V_y \in \{x_0 \times T_Y\}$ is an open cover of $\{x_0\} \times Y$.	C4
3	Since $\{x_0\} \times Y$ is homeomorphic to Y, $\{x_0\} \times Y$ is compact.	C4
4	Then, there exists a finite open subcover $\{\{x_0\} \times V_{y_i} \in \{x_0\} \times T_Y\}\}$,where $i \in \{1,\ldots,n \| n \in Z^+\}$and $\bigcup_{i=1}^{n} V_{y_i} = Y$.	C1
5	Note that $\{x_0\} \times Y \subset \{U_{y_i} \times V_{y_i}\} \subset N$. . ,where $i \in \{1,\ldots,n \| n \in Z^+\}$.	C1
6	Let $W = \bigcap_{i=1}^{n} U_i$, where W is an open neighborhood of x_0, where $\{x_0\} \times Y \subset W \times Y \subset N$.	CO

The following (Table 3.11) shows an example of the proof in which students are to derive multiple starting variable from a hypothesis of the given statement.

Example (Topology)

Prove that if $f(x_n)$ converges to $f(x_0)$ for every sequence x_n that converges to x_0, then $f:(X,\lambda)\to(Y,\rho)$ is continuous at x_0, where λ,ρ are metrics for X and Y respectively.

Table 3.11

Example of Type II (2)

	Opening Stage	
Z	Use an indirect proof by showing the contrapositive. Show that if $f(x)$ is not continuous, then $f(x_n)$ does not converge to $f(x_0)$ for some sequence x_n that converges to x_0.	
X	$f(x_n)$ does not converge to $f(x_0)$ for some sequence x_n that converges to x_0,	R2
X′	Construct a sequence x_n that converges to x_0 but $f(x_n)$ does not converge to $f(x_0)$.	R2
Y	Construct a sequence x_n that satisfies the following conditions: (1) $\exists\ \varepsilon>0$, for all $n\in Z^+$, $\rho(f(x_n),f(x_0))\geq\varepsilon$, (2) for any $\kappa>0$, there exists an $N\in Z^+$ such that for any $n\geq Z^+$, $\lambda(x_n,x_0)<\kappa$.	R1
P	Suppose $f(x)$ is not continuous at x_0.	Given
S	Let $\varepsilon>0$ be fixed. Suppose $\delta>0$ and $\kappa>0$.	C1
	Body Construction Stage	
1	Since $f(x)$ is not continuous at x_0, there exists $\varepsilon>0$ such that for any $\kappa>0$, $\lambda(x,x_0)<\kappa\Rightarrow\rho(f(x),f(x_0))\geq\varepsilon$.	C1
2	Claim that $\{x_{n\in Z^+} \vert \lambda(x_n,x_0)<\frac{1}{n}\}$ is a desired sequence.	C4
3	Consider $\{x_n \vert \lambda(x_n,x_0)<\frac{1}{n}\}$.	C5
4	Let $N\in Z^+$ such that $\frac{1}{N}<\kappa$.	C1
5	Then, for any $n\geq N$, $\lambda(x_n,x_0)<\kappa$.	R2
6	Therefore, $\{x_n \vert \lambda(x_n,x_0)<\frac{1}{n}\}$ converges to x_0.	R1
7	Consider $\{f(x_n) \vert \lambda(x_n,x_0)<\frac{1}{n}\}$.	C5
8	Note $\{f(x_n)\}$ does not converge to $f(x_0)$.	CO(1,7)R2

9	Note the sequence $\{x_1, x_2, x_3, ...\}$ converges to x_0 while $f(x_n)$ does not converge to $f(x_0)$.	R2

There was another type of a proof, in which students did not have to derive a starting variable and were able to start a proving argument with directly working on the conclusion of the given statement. The followings (Table 3.12, 13) are such examples.

> Example (Analysis)
>
> *Suppose* $f(x) = \int_{a(x)}^{b(x)} g(t)dt$ *where* $a(x)$ *and* $b(x)$ *are* C^1 *functions and* $g(t)$ *is continuous. Prove that* $f'(x) = b'(x)g(b(x)) - a'(x)g(a(x))$.

Table 3.12

Example of proof for Type III (1)

	Opening	
X	Show $f'(x) = b'(x)g(b(x)) - a'(x)g(a(x))$	
Y	Consider $f'(x) = m'(x) + k'(x)$, where $m(x) = \int_{a(x)}^{0} g(t)dt$ and $k(x) = \int_{0}^{b(x)} g(t)dt$	R2
P1	$a(x)$ and $b(x)$ are C^1 functions.	Given
P2	$g(t)$ is continuous.	Given
S	Consider $f'(x)$.	Given
	Body Construction Stage	
1	Consider $\frac{d}{dx} f(x) = \frac{d}{dx} \int_{a(x)}^{b(x)} g(t)dt$.	C5
2	Note $\frac{d}{dx} f(x) = \frac{d}{dx}\left[\int_{a(x)}^{0} g(t)dt + \int_{0}^{b(x)} g(t)dt\right] = \frac{d}{dx}\left[\int_{0}^{b(x)} g(t)dt - \int_{0}^{a(x)} g(t)dt\right]$	R3
3	Recall and apply the Fundamental Theorem of Calculus.	C2
4	Then, $f'(x) = g(b(x))b'(x) - g(a(x))a'(x)$	CO(2,3)R1

The problem in the next example (Table 3.13) is a mathematical induction. A mathematical induction is another example showing that students start their proving argument with directly working on the conclusion of the given statement. The proofs for showing trigonometry identities also belong to this type.

> Example (Discrete Mathematics)
>
> *Prove that for every positive integer n, show* $1^2+2^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

Table 3.13

Example of Type III (2)

	Opening Stage	
X	Show $1^2+2^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.	
Z	Use mathematical induction. Let P(*n*) be the given statement.	
X1	Show that P(1) holds.	
Y1 (=S)	Show that $1^2=\dfrac{1(1+1)\{2(1)+1\}}{6}$.	R2
	Body Construction Stage	
1	Note that the left hand side is 1.	R3
2	Note that the right hand side is $=\dfrac{1(1+1)\{2(1)+1\}}{6}$.	R3
3	Note, $1=\dfrac{1(1+1)\{2(1)+1\}}{6}$.	CO(1,2) R2
	Opening Stage	
X2	Show that if P(k) holds, then P(k+1) holds.	
Y2 (=S)	Show that if $1^2+2^2+\ldots+k^2=\dfrac{k(k+1)(2k+1)}{6}$ holds, then $1^2+2^2+\ldots+k^2+(k+1)^2=\dfrac{(k+1)(k+2)\{2(k+1)+1\}}{6}$.	R2
P	Assume that $1^2+2^2+\ldots+k^2=\dfrac{k(k+1)(2k+1)}{6}$.	Given
	Body Construction Stage	
1	Consider the left hand side of P(k+1).	C5
2	Note the left hand side is $1^2+2^2+\ldots+k^2+(k+1)^2$.	R3
3	Note that $1^2+2^2+\ldots+k^2+(k+1)^2=\dfrac{k(k+1)(2k+1)}{6}+(k+1)^2$.	CO(P,2) R3
4	Note that $\dfrac{k(k+1)(2k+1)}{6}+(k+1)^2=\dfrac{(k+1)(2k^2+k+6k+6)}{6}=$	R3

5	$=\frac{(k+1)(2k^2+7k+6)}{6}=\frac{(k+1)(2k+3)(k+2)}{6}=$	R3
6	$=\frac{(k+1)\{(k+1)+1)\}\{2(k+1)+1\}}{6}$	R3

3.6 Reliability of the Model of the Structure of Proof Construction

In order to get a sense for the reliability of the model, I had six mathematics professors to review the model of the structure of proof construction. The meetings took place in each professor's office. I explained the model to them, using the proofs I made and the proofs I had them make. They agreed that the model was applicable to those proofs. Two professors gave me minor suggestions. One professor suggested the researcher should avoid using the expression "linking information," which led me to the use the expression "combining objects." Another professor expressed a minor preference of using the notation CO(A, B)R1 instead of using CO(AB)R1 to represent the operation of combining objects A and B. The same professor also posed the question: Is this model applicable to non-proof regular problems? This would be another research question to be examined for a future project.

The model of the structure of *the reasoning activity* was compared with the standard theory for problem solving, which Newell and Simon (1958, 1976) presented. The theory provided four major characteristics pertinent to problem solving: (1) representation, interpretation, and manipulation of symbolic structures; (2) search through a set of available information; (3) selective search through heuristics; (4) reduction of the differences between current and desired states. R*ephrasing an object* may play a major role of (1). *Combining objects* and *creating a cue* can function as (2). C*hecking and exploring* may correspond to (3). The first three actions (*rephrasing an object*, *combining objects*, and *creating a cue*) can realize (4).

The model of the structure of proof construction was compared with the theoretical framework for understanding problem-solving success or failure, which Schoenfeld (2010) presented. The followings are the categories he included in this theory.

- Knowledge base (what students know)
- Problem solving strategies (the tools or techniques students have for solving problems)
- Monitoring and self-regulation (the metacognition concerned with how well students manage the problem solving resources)
- Beliefs (students' sense of mathematics, of themselves, of the context and more, which shape what they perceive and what they choose to do.

There seems to be a correspondence between the above categories and the aspects of proof construction presented in the model of the structure of proof construction (Figure 3.1). The first category "knowledge base" corresponds to *the background knowledge*. The second category "problem solving strategies" can correspond to the operations of the reasoning activity (*rephrasing an object*, *combining objects*, and *creating a cue*). The third category "Monitoring and self-regulation" corresponds to *the mental attitudes* (*tenacity*, *flexibility*, and *carefulness and alertness*). Finally, the fourth category "Beliefs" corresponds to *emotions and beliefs*. The framework and the model for the structure of proof construction may play a role to help students solve proof problems.

Flavell (1979) classified metacognitive knowledge into three types in terms of the following variables.

- Person variable: one's knowledge about one's learning abilities;
- Task variable: one's knowledge about the information available for proof construction;
- Strategy variable: one's knowledge about strategies.

The model of the structure of proof construction may play a role in the metacognitive knowledge that is necessary for proof construction. The aspect of *affect and beliefs* in the model can be considered to correspond to the person variable. For example, one's self-confidence is a person variable. The aspect of *the background knowledge* in the model may correspond to the task variable. For example, students' background knowledge may be abundant or meager, and well or poorly organized. According to their knowledge, they recognize and decide the difficulty of a given proof problem and predict their success in solving the problem. The aspect of *the reasoning activity* together with *the mental attitudes* may correspond to the strategy variable. The model of the structure of proof construction may help students develop their metacognition that is useful for proof construction.

Polya (1957) suggested a framework for problem-solving. His framework consisted of four phases: *orientation* (understanding the problem), *planning* (developing a plan), *executing* (carrying out the plan), and *checking* (looking back). His orientation and planning correspond to the operations conducted in *the opening stage*, including the step of making sure of the goal of the proof, which is often done by translating the conclusion of the given statement into *mathematical language*. His "*checking*" corresponds exactly to *exploring* and *checking* in this study. His "*executing*" corresponds to *the reasoning activity*. This study detailed the thinking actions of his “*executing*” and categorized them into *rephrasing an object*, *combining objects*, and *creating a cue*. Carlson and Bloom (2005) further developed Polya’s problem-solving framework into a multidimensional framework for problem-solving. In their framework, they included *resources*, *affect*, *heuristics*, and *monitoring*, still keeping the four phases that Polya’s framework included. Their *resources*, *affect*, *heuristics*, and *monitoring* correspond to *the background knowledge*, *affect and beliefs*, the action of *exploring and checking*, and *the mental attitudes* in this study, respectively.

3.7 Summary

Being led by frameworks of Harel & Sowder (1998), Weber & Alcock (2004), and Selden and Selden’s (2007), this study created a model of the structure of proof construction. Through proving dozens of theorems and propositions from multiple mathematical subjects, a comprehensive view of proof construction was built in the form of a model. The model clarified the aspects, factors, patterns, and features involved in cognitive process of proof construction. In particular, the model elucidated the types of cognitive actions to realize each of syntactic and semantic approaches while providing the way to classify proofs into three types. The model directly contributed to the creation of the frameworks for analyzing students’ proofs, which also served as a framework for describing the abilities and skills necessary for proof construction. The model of the structure of proof construction earned an agreement from six mathematics professors as inter-rater reliability. The analysis framework was created based on the model of the structure of proof construction is to explain and understand the sources of students’ difficulties with proof construction in a clear, organized, and systematic way.

Chapter 4: Methods

4.1 Introduction

One of the main purposes of this study is to identify the sources of students' difficulties with proof construction. I sought the research method that best fitted for detecting sources of students' difficulties and deriving the patterns and features seen in their difficulties in a variety of types of proofs. This chapter presents the description of the methods used in this study for collecting and analyzing the data.

4.2. Methodology

I used document analysis for a research method for this study. In this section, I am going to explain why I chose the research method.

4.2.1 Document analysis

Document analysis is a qualitative research method for delineating and interpreting phenomena through examining documents (Bowen, 2009). Labuschagne (2003) claimed that document analysis was effective for organizing texts into themes, categories, and case examples. Bowen (2005, 2009) also elucidated the roles of documents as follows: (a) providing research data; (b) allowing researchers to organize information, verify findings, and corroborate evidence; (c) helping researchers practice a thorough examination of the target phenomena; and (d) allowing researchers to take a variety of forms, including books, journals, newspapers, scripts, and public records.

This study aimed to investigate the sources of students' difficulties with proof construction from multiple angles. The target documents were students' proofs from their exams and problem-solving sessions. In order to generalize the patterns and features seen in the difficulties students had, it was necessary to collect as many students' proofs as possible from different individuals and from different mathematical subjects. These proofs provided the data for this study. Document analysis was effective in meeting the demands of this study in the following ways: First, students' proofs directly provided the data as reliable sources. Students' proofs also enabled the researcher to effectively practice a close investigation into and careful interpretation on the data. In addition, students' proofs were strong evidence to support the findings. Moreover, with the use of the analysis framework, students' proofs helped the researcher organize the collected information into categories and to recognize patterns.

4.2.2 Sampling Method

For the sampling method, this study adopted criterion sampling. According to Creswell (2007), criterion sampling is a sampling strategy in which researchers establish criteria for the source of data. This study set the criteria for the participants to be those students enrolled in proof-based courses, more specifically, undergraduate Algebra, Analysis, and Topology, at a large comprehensive research university in the middle Southern United States in 2013. There were some unique characteristics about those participants. Since the chosen school was one of a few research schools in the state, students' academic levels were expected to be relatively high among all the undergraduate students in the same state. In addition, since the target courses were one of the highest undergraduate mathematics course usually taken by mathematics majors, the target participants' mathematics abilities were expected to be relatively high among all the undergraduate students in the same university. The difficulties they might encounter can be representative of those of most undergraduates, in particular, most mathematics majors in other universities. For the same reason, the target participants were also expected to have higher motivation and more positive learning attitudes toward mathematics. Moreover, the target participants were expected to have already completed Calculus sequence and Discrete Mathematics as they were prerequisite courses. Another characteristic of those participants was found in their courses. All the three courses had relatively a small size of students ranging from 10 to 15. Analysis and Algebra classes was a 50-minute class held three times a week while Topology class was a 75 minute class held twice a week.

4.3 Participants

The participants were those students who were enrolled in Introduction to Algebra I, Introduction to Algebra II, Introduction to Analysis I, or Introduction to Topology in 2013 Spring.

Table 4.1

Participants of Algebra I Students

Introduction to Algebra I	Male	Female	Total
In-class problem solving session	6	2	8
Exams	10	3	13
Individual problem solving session	1	0	1

Algebra I students had already taken Linear Algebra, Discrete Mathematics, and Calculus I, II, and III courses. They had a class meeting twice a week and each class was 75 minutes long.

Table 4.2.

Participants of Algebra II Students

Introduction to Algebra II	Male	Female	Total
In-class problem solving session	7	3	10
Exams	7	3	10
Individual problem solving session	3	1	4

Algebra II students had already taken Algebra I in addition to Linear Algebra, Discrete Mathematics, Calclulus I, II, and III courses. They had a class meeting three times a week and each class was 50 minutes long.

Table 4.3

Participants of Analysis I Students

Introduction to Analysis I	Male	Female	Total
Exams	6	1	7

Analysis I students had already taken Discrete Mathematics, and Calculus I, II, and III courses. They had a class meeting twice a week, and each class was 75 minutes long.

Table 4.4.

Participants of Topology I Students

Introduction to Topology I	Male	Female	Total
Exams	3	4	7
Individual problem solving session	1	0	1

Topology I students had already taken Discrete Mathematics, and Calculus I, II, and III. They had a class meeting twice a week. Each class was 75 minutes long.

4.4 Data Collection

The data used for this study was a collection of students' written proofs. There were three types of instruments for collecting students' proofs: (1) in-class mid-term and final examination scripts; (2) in-class problem solving sessions; and (3) individual problem solving sessions. Under the permission of IRB (Institutional Review Board), I visited each target class to recruit participants for each research activity. I obtained consent forms from the students in each course, and upon their agreements, used their proofs as data for this research.

4.4.1 Students' in-class exams

Students' mid-term and final examination scripts were collected from undergraduate Algebra I and II, Analysis I, and Topology courses. The examination questions were made by each instructor of those courses. In sorting out the problems to be analyzed from among many problems in the exams, first, some problems were eliminated. Those problems included the following types: (1) those problems that were irrelevant to proof construction; and (2) problems that asked for construction of a counter example. Then, among the rest of the problems, the priority was given to those problems whose solutions the researcher was confident about. The following table (Table 4.5) shows the number of the students whose examination scripts were collected as data for this study.

Table 4.5.

Population Sizes for Examination Scripts

	Exam 2	Exam 3	Final
Topology	7	7	7
Algebra I	13	0	13
Algebra II	10	0	0
Analysis I	7	6	6

4.4.2 In-class problem solving sessions

An in-class problem solving session was conducted in each of Algebra I and II courses under permission of the instructor of each course. The time length of the session was decided by the instructor of each course. The researcher made a pool of problems to be given in the session in advance. Then, the instructor of each course chose the actual problems that were given to the students in the session. The problems given to the students were all from the materials that the participant students had already learned in their classes (See Appendix).

The problems were designed in the following way. All the problems were for proof construction. Three stages were set for each problem. On the first page, students were asked to solve the problem with no hints. If they were not able to solve the problem, they were led to the next page, where the definitions of the concepts involved in the problem were provided. If they were still not able to solve the problems, they were led to the next page for more hints. Hints included properties of concepts and directions of the proof construction. Students were asked not to use an eraser. If they needed to erase what they wrote, they were asked to cross them out with a straight line. They were also required not to go back to a previous page once they moved to a new page. These directions were written in the worksheets and given to the students orally as well. The following table shows the population of the students who participated in the in-class problems sessions as well as the length of each session.

Table 4.6

Population Sizes for In-Class Problem Solving Sessions

Course	Population of participants	Time Length
Algebra I	7	25 minutes
Algebra II	8	50 minutes

4.4.3 Individual problem solving sessions

In total, nine students participated in individual problem solving sessions. The problem-solving sessions took place in the researcher's office. Each participant came to the office in different times. Each session was 50 minutes long. The researcher prepared a pool of problems for proof construction. Before having the participants solve the problems, he researcher made sure if those problems were from the material they had already learned. When they finished, I went through their work and asked questions if there was any. The form of the problems used in the sessions were similar to the ones used in the in-class problem solving sessions. Each problem had three stages. In the first stage, they were asked to solve the problem with no hints. If they needed help, they were led to the next page which provided definitions of concepts involved in the problem. If they needed more help, they were led to the next page which provided more hints including a direction of proof construction as well as the definitions of the concepts. Once they moved to a new page, they were not allowed to go back to a previous page. They were also not allowed to use an eraser to erase what they had written and were required to cross out with a straight line what they wanted to erase. The table below shows the courses the student participants were enrolled in.

Table 4.7

Population Sizes for Individual Problem Solving Sessions

	Topology	Algebra I	Algebra II	Analysis
population	2	1	4	0

4.5 Data Analysis

For analyzing students' proofs, two types of framework were created. The first type of table (Table 3.3 and Table 3.4) was used for detecting students' impasses and the operation they failed to apply. This type of analysis table (Type A) was created for each problem the participants worked on. Each analysis table showed step-by-step proof and the coded operation used to generate a corresponding statement for each step. The other type of framework (Table 3.5) was used for identifying the sources of students' difficulties. It was created based on the framework for modeling the structure of proof construction.

First, I de-identified participants' names and gave a pseudonym to each student. Then, based on the analysis table Type A, the researcher examined each step of the proof that each student made. Then, each of their mistakes and difficulties was analyzed from the three perspectives based on the framework Type B: *the reasoning activity*; *background knowledge*; and *mental attitudes*). Concerning the last aspect "mental attitudes," there was no way to measure the degrees of students' mental attitudes. Therefore, the decision of if a student's difficulty or failure was due to his or her lack of tenacity, flexibility, carefulness and precision, was subjective and peripheral because it depended on the researcher's interpretation to some extent.

4.6 Summary

In conjunction with the use of the analysis frameworks, the document analysis was a suitable method for gathering and analyzing the data for this study. This method allowed me to examine various difficulties that students confronted in the same proof problem. In particular, with the analysis frameworks, document analysis allowed me to collect a sufficient number of students' proofs to generalize the patterns and features of their difficulties with proof construction across mathematical subjects. In total, the researcher analyzed 81 proofs which were collected from students' examinations and in-class and individual problem solving sessions. The next chapter presents the findings obtained through analyzing students' proofs in light of the structure of proof construction.

Chapter 5: Results

5.1 Introduction

This chapter presents representative results from the analysis of students' proofs. The poofs ranged over introductory Algebra, Analysis, and Topology. The students' proofs were analyzed based on the two types of frameworks as described in Chapter 3. This chapter first gives a few examples of how students' proofs were analyzed with the use of the analysis frameworks. Then, it presents representative results from the analysis of students' proofs. The results will be presented according to the aspects and factors of the structure of proof construction: *the opening stage*, *rephrasing an object*, *combining objects*, *creating a cue*, *background knowledge*, and *mental attitudes*.

5.2 Examples of Analysis Table (Type A)

This section presents some examples that show how a student's proof was analyzed based on the analysis table and framework. In order to show as many factors for possible causes of students' difficulties as possible, which are listed in the framework (Table 3.5), three students' proofs are used. These proofs were all on the same problem. In each example, first, the problem is introduced. Then, the analysis table Type A is presented to show every step of the proof, the coded operation used to generate each statement for each step, and the degree of student's success in obtaining each statement. Then, I will give a comprehensive analysis of the proof students made, using the analysis framework Type B (Table 3.5).

Analysis [9] (Final Exam)

$f:(a,b)\to R$ has a global maximum at some $x^* \in (a,b)$ and is differentiable at $x^* \in (a,b)$. Prove that $f'(x^*)=0$. Note that a function $f:(a,b)\to R$ is said to have a global maximum at $x^* \in (a,b)$ if and only if For all $x\in(a,b)$, $f(x)\le f(x^*)$.

Table 5.1

Example of Analysis Table (Type A)

	Opening Stage	Code	U	Z	C
X	Show $f'(x^*)=0$.	Given			
Y	Show $\lim_{x\to x^*}\frac{f(x)-f(x^*)}{x-x^*}=0$.	R1			
Y′	Show $\lim_{x\to x^{*-}}\frac{f(x)-f(x^*)}{x-x^*}=0=\lim_{x\to x^{*+}}\frac{f(x)-f(x^*)}{x-x^*}$.	R1	I	I	N
P1	$f:(a,b)\to R$ has a global maximum at $x^*\in(a,b)$.	Given			
P1'	For all $x\in(a,b)$, $f(x)\le f(x^*)$.	Given			
P2	$f:(a,b)\to R$ is differentiable at $x^*\in(a,b)$.	Given			
P2'	$\lim_{x\to x^{*-}}\frac{f(x)-f(x^*)}{x-x^*}=\lim_{x\to x^{*+}}\frac{f(x)-f(x^*)}{x-x^*}$.	R1	I		
	Body Construction Stage				
1	Consider the right hand side limit $\lim_{x\to x^{*+}}\frac{f(x)-f(x^*)}{x-x^*}$.	C5	N	S	N
2	Since $f(x)\le f(x^*)$ for all $x\in(a,b)$ and $x-x^*>0$, $\lim_{x\to x^{*+}}\frac{f(x)-f(x^*)}{x-x^*}\le 0$.	CO (1, P1') R2	N	I	N
3	Consider the left hand side limit $\lim_{x\to x^{*-}}\frac{f(x)-f(x^*)}{x-x^*}$.	C5	N	S	N
4	Since $f(x)\le f(x^*)$ for all $x\in(a,b)$, and $x-x^*<0$ $\lim_{x\to x^{*-}}\frac{f(x)-f(x^*)}{x-x^*}\ge 0$.	CO (3, P1') R2	N	I	N
5	Since $f:(a,b)\to R$ is differentiable at $x^*\in(a,b)$, $0\le\lim_{x\to x^{*+}}\frac{f(x)-f(x^*)}{x-x^*}=\lim_{x\to x^{*-}}\frac{f(x)-f(x^*)}{x-x^*}\le 0$.	CO (P2',2,4) R2	N	N	N
6	Since $0\le\lim_{x\to x^{*+}}\frac{f(x)-f(x^*)}{x-x^*}=\lim_{x\to x^{*-}}\frac{f(x)-f(x^*)}{x-x^*}\le 0$,	R2	N	N	N

	$\lim_{x \to x^*} \frac{f(x) - f(x^*)}{x - x^*} = 0$.				
7	Since $\lim_{x \to x^*} \frac{f(x) - f(x^*)}{x - x^*} = 0$, $f^{'}(x^*) = 0$.	R1	N	N	N

The letter "X" in the first column represents the conclusion or the goal of the given statement. The letter "Y" represents *the mathematical language* that X is translated into. As mentioned in Chapter 3, "translating of an object into *mathematical language*" means transforming an object into a mathematical expression "fine" enough to makes it possible for students to further advance a reasoning process. The letter "P" in the first column represents the hypothesis, the assumption, or the condition in the given statement. The letter "P '" represents the *mathematical language* into which "P" is translated. The numbers in the first column represent the order of the steps to be taken for the proof. The second column shows a specific statement necessary for the proof: the conclusion of the proof (X), a given condition or hypothesis (P), and a statement for each step of the proof. The third column shows a code of the operations used to produce each statement in the proof. The list of the codes of the operations for advancing a reasoning process was presented in Table 3.2.

The letters U (Eugene), Z (Zachery), and C (Caleb) in the first row represent the codes of the names of the students whose proofs were analyzed. The letters "I", "N", and "S" stand for "Incomplete", "Not successful", and "Successful" respectively to describe the degree of success in their performance at each step in the proof. Next, the examples of the analysis of each student's proof based on the analysis framework Type B (Table 3.5) are given. For each of wrong or incomplete statements, a possible cause is chosen from the framework Type B (Table 3.5).

Example 1 (Eugene)

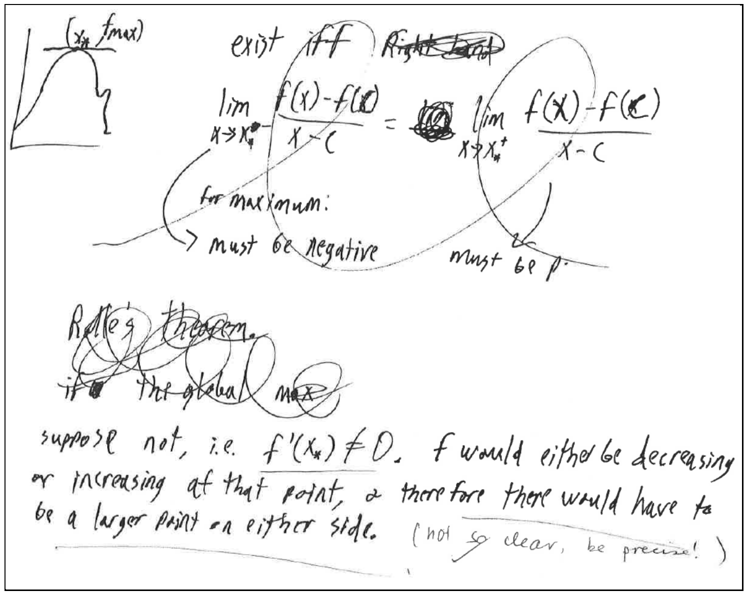


Figure 5.1. Eugene's Proof.

Eugene paid attention to the conclusion of the given statement and tried translating it into mathematical language. However, his notations had a defect. He had $\lim\limits_{x \to x^{*-}} \dfrac{f(x)-f(c)}{x-c}$ and $\lim\limits_{x \to x^{*+}} \dfrac{f(x)-f(c)}{x-c}$ without specifying what *c* represented though the left hand limit was supposed to be $\lim\limits_{x \to x^{*-}} \dfrac{f(x)-f(x^*)}{x-x^*}$ and the right hand limit was supposed to be $\lim\limits_{x \to x^{*+}} \dfrac{f(x)-f(x^*)}{x-x^*}$ (KNT). KNT stands for "students' mismanagement of a notation" (See Table 3.5). He noted and translated the conclusion of the given statement "$f^{'}(x^*)=0$" into *mathematical language*, but it was not perfect (OTC). The code OTC stands for "the mismanagement of the opening stage by failing to translate the conclusion of the given statement into *mathematical language*" (See Table 3.5). He should have had "$\lim\limits_{x \to x^{*-}} \dfrac{f(x)-f(x^*)}{x-x^*} = 0 = \lim\limits_{x \to x^{*+}} \dfrac{f(x)-f(x^*)}{x-x^*}$," but missed "= 0 =" part in his equation. His was right in considering each side of the equality, in particular, the sign of each limit. However, he was not completely right in claiming that the right hand limit was positive and the left hand limit was negative (MC). The code MC stands for "lack of carefulness or alertness" (Table 3.5). He wanted to have

$\lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*} \leq 0$ and $\lim_{x \to x^{*-}} \frac{f(x) - f(x^*)}{x - x^*} \geq 0$. He missed the equality " = 0 " for both sides mainly because he did not note and use the given information about the definition of a global maximum of a function (CO). The code "CO" stands for "students' failure to combine objects to create a new object" (See Table 3.5). He missed the equality " = 0 " for both sides partly because he was not careful in making his claim that the left hand limit was negative and the right hand limit was positive (MC). Thus, possible sources of his difficulties were that he missed translating the conclusion of the given statement completely (OTC) and that he missed combining an object and the given condition (CO), both of which might have involved lack of carefulness as well (MC).

Example 2 (Zachery)

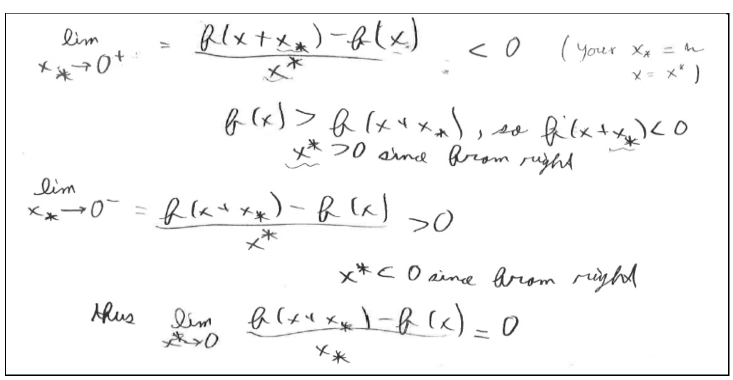
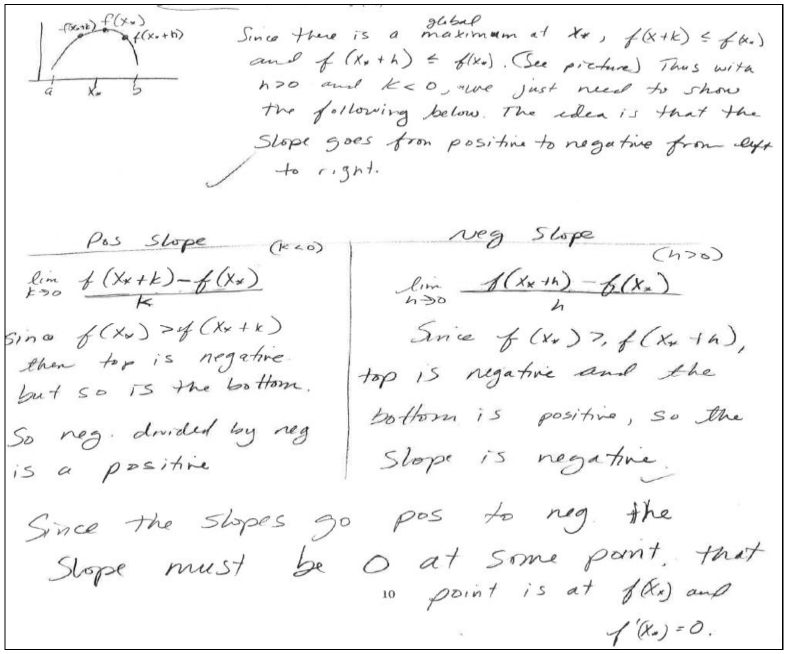


Figure 5.2. Zachery's Proof.

Zachery rightly considered the left and right hand limits while his notations for them were awkward (KNT). He had $\lim_{x^* \to 0^-} \frac{f(x+x^*)-f(x)}{x^*}$ and $\lim_{x^* \to 0^+} \frac{f(x+x^*)-f(x)}{x^*}$ though he needed to have $\lim_{h \to 0^-} \frac{f(x^*+h)-f(x^*)}{h}$ and $\lim_{h \to 0^+} \frac{f(x^*+h)-f(x^*)}{h}$, respectively. The analysis of his proof follows based on the assumption that he meant to show $f'(x)=0$. It was good that he considered the signs for the left and right limits. However, he was not careful enough in using the given condition that $f(x) \le f(x^*)$ (MC). He applied $f(x) < f(x+x^*)$ with $x^* > 0$ and $f(x) > f(x+x^*)$ with $x^* < 0$ to his argument instead of applying $f(x) \le f(x+x^*)$ and $f(x) \ge f(x+x^*)$, respectively. He had $\lim_{x^* \to 0} \frac{f(x+x^*)-f(x)}{x^*} = 0$ at the end, but lacked rigor in making the conclusion because $\lim_{x^* \to 0^+} \frac{f(x+x^*)-f(x)}{x^*} < 0$ and $\lim_{x^* \to 0^-} \frac{f(x+x^*)-f(x)}{x^*} > 0$ would not lead him to $\lim_{x^* \to 0^-} \frac{f(x+x^*)-f(x)}{x^*} = 0$. His lack of rigor might have resulted from the following facts. He did not translate the conclusion of the given statement completely (OTC and R1). He knew that he needed to show that $\lim_{x \to 0} \frac{f(x+x^*)-f(x)}{x^*} = 0$. However, he did not thoroughly transform it into $\lim_{x^* \to 0^-} \frac{f(x+x^*)-f(x)}{x^*} = 0 = \lim_{x^* \to 0^+} \frac{f(x+x^*)-f(x)}{x^*}$ (MC).

Example 3 (Caleb)


Since there is a global maximum at x_*, $f(x+k) \le f(x_*)$ and $f(x_*+h) \le f(x_*)$. (See picture) Thus with $h>0$ and $k<0$, we just need to show the following below. The idea is that the slope goes from positive to negative from left to right.

Pos Slope ($k<0$)

$\lim_{k \to 0} \frac{f(x_*+k)-f(x_*)}{k}$

Since $f(x_*) > f(x_*+k)$ then top is negative but so is the bottom. So neg. divided by neg is a positive

neg Slope ($h>0$)

$\lim_{h \to 0} \frac{f(x_*+h)-f(x_*)}{h}$

Since $f(x_*) \geq f(x_*+h)$, top is negative and the bottom is positive, so the slope is negative.

Since the slopes go pos to neg the slope must be 0 at some point, that point is at $f(x_*)$ and $f'(x_*)=0$.

Figure 5.3. Caleb's Proof.

Caleb's proof was almost complete. He successfully translated the differentiability of a given function into mathematical language and came up with the idea of examining the left and right hand limits of the difference quotient. He had $\frac{f(x^*+k)-f(x^k)}{k} > 0$ for $k<0$ and $\frac{f(x^*+k)-f(x^k)}{k} < 0$ for $k>0$, which were right. However, when he concluded $f'(x^*)=0$ based on his above observations, his argument was incomplete because it might happen $f'(x^*) \neq 0$ even when $\frac{f(x^*+k)-f(x^k)}{k} > 0$ for $k<0$ and $\frac{f(x^*+k)-f(x^k)}{k} < 0$ for $k>0$ (MC). For example, when $f(x)$ has a vertical asymptote at $x = x^*$, $f'(x^*) \neq 0$ even when $\frac{f(x^*+k)-f(x^k)}{k} > 0$ for $k<0$ and $\frac{f(x^*+k)-f(x^k)}{k} < 0$ for $k>0$. In particular, he missed having $\lim_{k\to 0^-}\frac{f(x^*+k)-f(x)}{k}=0=\lim_{k\to 0^+}\frac{f(x+x^*)-f(x)}{k}$ (MC, OTC, and R1).

The code R1 stands for rephrasing an object by applying definitions, properties, or theorems. The above incompleteness of his argument resulted mainly from the following factors. First, he lost his tenacity for completing examining $\lim_{k\to 0^-}\frac{f(x^*+k)-f(x)}{k}$ and $\lim_{k\to 0^+}\frac{f(x+x^*)-f(x)}{k}$ thoroughly (MT) while he came up with the idea of considering them at first. Second, he missed using the given information about a global maximum of a function " $f:(a,b)\to R$ is said to have a global maximum at $x^*\in(a,b)$ if and only if for all $x\in(a,b)$, $f(x)\le f(x^*)$" (CO). This might have helped him have $\frac{f(x^*+k)-f(x^k)}{k}\ge 0$ for $k<0$ and $\frac{f(x^*+k)-f(x^k)}{k}\le 0$ for $k>0$ and therefore, $\lim_{k\to 0^-}\frac{f(x^*+k)-f(x^k)}{k}\ge 0$ and $\lim_{k\to 0^+}\frac{f(x^*+k)-f(x^k)}{k}\le 0$. Third, he did not thoroughly translate the conclusion of the given statement " $f'(x^*)=0$" into " $\lim_{k\to 0^-}\frac{f(x^*+k)-f(x)}{k}=0=\lim_{k\to 0^+}\frac{f(x+x^*)-f(x)}{k}$" (OTC, R1), which might have helped him having $\lim_{k\to 0^-}\frac{f(x^*+k)-f(x^k)}{k}\ge 0$ and $\lim_{k\to 0^+}\frac{f(x^*+k)-f(x^k)}{k}\le 0$. His proof might be considered as an example showing that the skill of combining objects and rephrasing an object can support his tenacity to continue his reasoning process.

The following section presents some examples for each of the possible sources of students' difficulties. Each example is presented in the following order:

(i) A possible factor that can cause students' difficulties

(ii) the possible cause of the difficulty the student encountered;

(iii) the proof problem;

(iv) the analysis table Type A;

(v) a detailed analysis of the proof based on the analysis framework Type B;

(vi) the whole or partial proof produced by the student;

(vii) the analysis based on the analysis framework Type B (Table 3.5).

5.3 Difficulties with Opening Stage

The opening stage is a crucial stage that can determine the degree of the success of one's proof construction. *The opening stage* plays mainly two important roles in proof construction: One is to make the goal of the proof clearer, which is achieved by noting the conclusion of the given statement and translating it into *mathematical language*. The other role is to derive and set a variable as the start of a proof. A starting variable is the key variable with which students open *the body construction stage*. A starting variable is found in *the ignition phrase* contained in *the mathematical language* for the conclusion of the given statement. For the type of proof in which students construct an object, they derive a starting variable from a hypothesis.

There are model steps for students to take in *the opening stage*. For example, students may (1) pay attention to the conclusion of the given statement, (2) translate it into *mathematical language*, (3) find a variable to be set as a start of a proving argument in *the mathematical language*, usually, for the conclusion of the given statement, (4) set a starting variable for developing *the body construction stage*, (5) make sure of the hypotheses of the given statement, and (6) translate them into *mathematical language* if necessary. The steps (1) and (2) are the operations for making the goal of the proof clear. The steps (3) and (4) are the operations for setting a starting variable. Although it is minor, there is a type of proof in which students need to derive a starting variable from a hypothesis of the given statement. In that case, the step (6) can be the operation for setting a starting variable. For another minor type of proof, students do not have to derive a starting variable in the opening stage because a starting variable may be explicitly provided in the given proof problem, in particular, in the conclusion of the given statement. The following examples show how greatly students' managements of *the opening stage* can affect their whole proving arguments.

Students' difficulties with the opening stage are analyzed in the following two terms (Table 3.5): paying attention to the conclusion of the give statement and translating it into *mathematical language* (OTC) and deriving and setting a starting variable for *the body construction stage* (OSV). "O" represents the opening stage. "TC" stands for "translating the conclusion" of the given statement into *mathematical language* and "SV" stands for "setting a variable."

5.3.1 Translating a Conclusion into *Mathematical Language*

It is important for students to make sure of the goal of a proof. Awareness of the goal of a proof keeps them on the right track and helps them avoid going astray in their proof construction. Students can make the goal of a proof clear by translating the conclusion of the given statement. The following are representative proofs that show how crucial it is for students to translate the conclusion of the given statement into *mathematical language*. I will give the following three examples showing students' difficulties with translating the conclusion into *mathematical language* while showing how their difficulties affected their proofs: Frank lost the goal of the proof (Example 1);

Cade set a wrong direction of proof construction (Example 2); and Daniel failed to practice logical deduction and resorted to pictorial proof scheme (Example 3).

Example 1: Frank (Algebra I)

Failing to clarify the goal of a given proof can lead students to wander vaguely and produce a confusion during their proof construction. Frank's proof is such an example. His case shows that students may fail to make the destination of the proof clear for two reasons. One is that students tend to start to work on a hypothesis of the given statement and to derive a starting variable from it. The other is that they do not pay careful attention to the conclusion of the given statement.

> Question [4] (In-class problem solving session)
>
> *Suppose that* $G/Z(G)$ *is cyclic. Prove that G is abelian.*

Table 5.2 shows a possible proof for Questions [4] and shows the difficulties Frank had in the proof construction.

Table 5.2

Analysis (Type A) of Frank's proof

	Proof	Code	Frank
X	Show G is abelian.	Given	
Y	Show $ab = ba$ for any $a, b \in G$.	R1	N
P	$G/Z(G)$ is cyclic.	Given	
1	Let $a, b \in G$.	C1	N
2	Recall $a, b \in G$ are in some cosets.	C2	N
3	Then, $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$.	CO(P, 2)R1	N
4	Let $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$.	R1	N
5	Then, $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$.	R3	N

The conclusion of the given statement is "*G is abelian*." The *mathematical language* for the conclusion is "$ab = ba$ for any $a, b \in G$." The phrase in the *mathematical language*, “for any $a,b \in G$,” is the *ignition phrase*. Deriving starting variables from the *ignition phrase*, start a proof with “Suppose $a,b \in G$.” This proof is the type of the proof of showing A = B. A proving strategy for that type of proof is to work on either A or B to change it into B or A. In this case, try to rephrase the left side of the equation “ab” until it changes into “ba.” To rephrase “ab,”the given hypothesis “$G/Z(G)$ *is cyclic*” can be considered. Finding the connection between the starting variables “a,b” and the hypothesis “$G/Z(G)$ *is cyclic*” and recalling the property that an element of *G* belongs to some coset, $a \in x^m Z$ and $b \in x^n Z$ can be produced for some $x \in G$. Then, rephrasing ab with $x^{m+n} z_1 z_2$ for some $z_1, z_2 \in Z$ and using the property of the center of a group, $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$ can be derived. The following figure shows Frank’s proof (Figure 5.4).

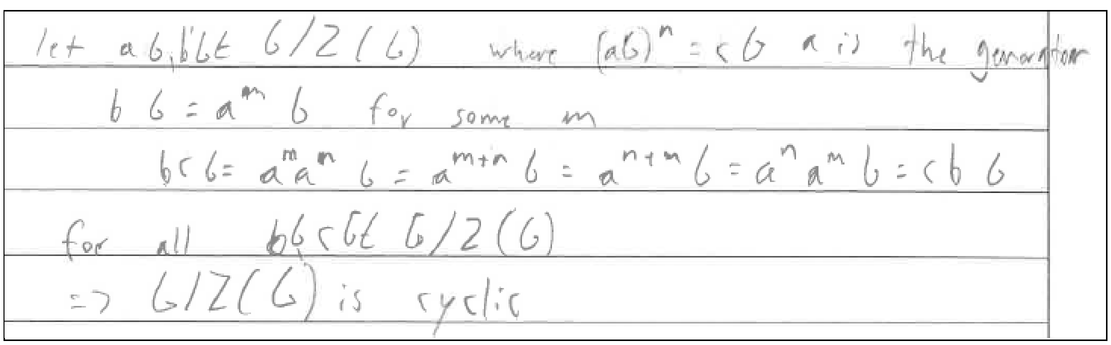


Figure. *5.4*. Frank’s Proof.

Frank first paid attention to the hypothesis of the given statement “$G/Z(G)$ *is cyclic*” instead of paying attention to the conclusion of the given statement (OTC). Then, he started his argument by deriving starting variables from the hypothesis instead of deriving a starting variable from the conclusion (OSV). In expressing the elements of the coset *G*/(*Z*) as his starting variables, he mistakenly had $aG, bG, cG \in G(Z)$, which were supposed to be $aZ, bZ, cZ \in G(Z)$ (KNT, MC). Then, he translated the hypothesis into *mathematical language*, meaning $bZ = a^m Z$ and $cZ = a^n Z$. While working on the hypothesis, his proof started to go astray, ending up with showing $bZcZ = cZbZ$, which he did not have to show because it was obvious that a cyclic group was abelian. Finally, though he showed that "$G/(Z)$ is abelian," he mistakenly concluded "$G/(Z)$ is cyclic," which was a hypothesis already given at the beginning.

There were two main factors that might have caused his proving argument to be unsuccessful. The first factor was that he was unable to pay attention to the conclusion of the given statement "G is abelian" (OTC). This resulted in two problems. One problem was that since he was not attentive to the goal (MC), he did not realize his argument was going astray while showing "$G/(Z)$ is abelian" and that his argument went wrong while concluding "$G/(Z)$ is cyclic." The other problem was that he was unable to derive right starting variables "$a, b \in G$" (OSV), which might have been obtained from *an ignition phrase* of *the mathematical language* for the conclusion of the given statement. The second major factor was that he was, as many other students did, first focused on the hypothesis of the given statement "*G*/(*Z*) is cyclic," derived his starting variables from the hypothesis, which did not help him reach the goal of the proof, and translated the hypothesis into *mathematical language*, which created a confusion in his argument. The second factor resulted in diverting his attention from the conclusion of the given statement and leading him to miss the goal of the proof.

Example 2: Cade (Algebra I)

This example showed that students' failure to accurately translate the whole sentence of the conclusion into *mathematical language* might hinder them from developing their proving arguments.

> Question [5] (In-class problem solving session)
>
> *Suppose that the order of G is a prime number. Prove that G is cyclic.*

Table 5.3 shows a possible proof for Question [5] and shows where Cade had a difficulty in the proof construction.

Table 5.3

Analysis (Type A) of Cade's proof

	Proof	Code	Cade
X	Show G is cyclic.		
Y	Show $G = \langle g \rangle$ for some $g \in G$ with $g \neq 1$.	R1	I
P	The order of G is a prime number.	Given	
S	Let $g \in G$ with $g \neq 1$.	C1	N
1	Let $g \in G$ with $g \neq 1$.	C1	N
2	Consider $\langle g \rangle$.	C5	N
3	Note $\langle g \rangle$ is a subgroup of G.	C2	N
4	Recall the Lagrange's THM and apply it to $\langle g \rangle$.	C2	N
5	Then, by the Lagrange's THM, $\lvert\langle g \rangle\rvert = 1, p$.	CO(3,4,P)R1	N
6	Since $\lvert\langle g \rangle\rvert \neq 1$, $\lvert\langle g \rangle\rvert = p$.	CO(1,5)R2	N
7	Since $\lvert G\rvert = p$, $G = \langle g \rangle$.	CO(6, P)R2	N

The conclusion of the given statement is that "*G is cyclic*." The conclusion "*G is cyclic*" can be translated into *mathematical language* "$G = \langle g \rangle$ for some $g \in G$ with $g \neq 1$." The given proof is the type of the proof of showing A = B. One can work on either A or B through rephrasing it until A becomes B or B becomes A. In this problem, consider and work on $\langle g \rangle$. Recalling Lagrange's Theorem and combining it with the property that a cyclic group generated by an element in *G* is a subgroup of *G*, one may obtain $\lvert\langle g \rangle\rvert = 1, p$. Noting $\lvert\langle g \rangle\rvert \neq 1$, one may decide $\lvert\langle g \rangle\rvert = p$.

Combining the hypothesis $G = < g >$ and $|< g >| = p$, one may conclude $G = < g >$. The following is Cade's proving strategy for the given problem (Figure 5.5).

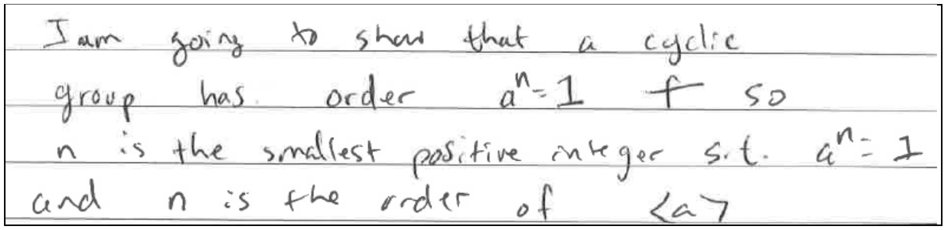


Figure 5.5. Cade's Strategy.

When Cade was asked to state his proving strategy, he successfully noted the conclusion of the given statement, which was "*G* is cyclic". However, he made his statement sound awkward when he put "I am going to show a cyclic group has order $a^n = 1$…" He probably meant to state "$G = < a >$, in which $a^n = 1$ with *n* being the smallest positive integer," but was unable to accurately rephrase the whole sentence of the conclusion of the given statement in *mathematical language*. This may have affected his proof construction. The following figure shows Cade's proof (Figure 5.6).

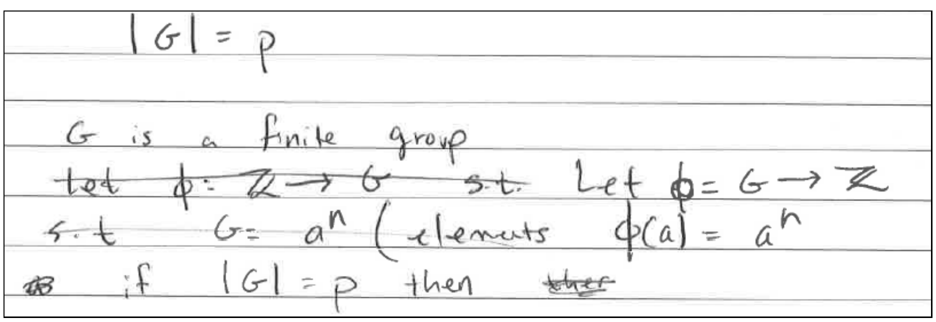


Figure 5.6. Cade's Proof.

In addition, Cade was unable to develop his proving argument mainly because he did not translate the whole sentence of the conclusion of the given statement accurately. Although he noted the conclusion when he thought about the goal of the proof, he focused on only the predicate "cyclic" of the conclusion and missed translating the subject part "*G* is." As a result, he provided the definition of a cyclic group and was not able to have "$G = < g >$ for some $g \in G$ with $g \neq 1$," which might have hindered him from opening up his proving argument. If he had "$G = < g >$ for some $g \in G$ with $g \neq 1$," that might have led him to set a starting variable "$< g >$" for some $g \in G$ with $g \neq 1$," focus on "$< g >$", recall the property of "$< g >$" being a subgroup of *G*, and come up with the idea of using Lagrange's Theorem.

Since he missed the starting variable "$< g >$" for some $g \in G$ with $g \neq 1$, he was unable to open up his argument. Then, he depended on the hypothesis of the given statement "$|G| = p$" for starting the body construction stage. However, that was not helpful, so he further attempted to apply a proving technique of creating a function, which was not helpful, either. Finally, he gave up proving. His example shows how crucial it is to rephrase the whole sentence of the conclusion of the given statement accurately. His example also shows that once students miss setting a variable from the conclusion of the given statement, no matter what they may attempt, that would not help them advance their reasoning process.

Example 3: Daniel (Analysis)

Daniel's case is another example showing how important it is for students to be able to translate the conclusion of the given statement into *mathematical language* so that they can develop a proving argument.

Question [9] (Final Exam)

$f:(a,b)\to R$ has a global maximum at some $x^* \in (a,b)$ and is differentiable at $x^* \in (a,b)$. Prove that $f'(x^*)=0$. Note that a function $f:(a,b)\to R$ is said to have a global maximum at $x^* \in (a,b)$ if and only if for all $x \in (a,b)$, $f(x)\le f(x^*)$.

Table 5.4 shows a possible proof for Question [9] and shows where Daniel had a difficulty in the proof construction.

Table 5.4

Analysis (Type A) of Daniel's Proof

Step	Statement	Code	D
X	Show $f'(x^*)=0$		N
Y	Show $\lim_{x\to x^*}\frac{f(x)-f(x^*)}{x-x^*}=0$	R1	N
P1	$f:(a,b)\to R$ is said to have a global maximum at $x^* \in (a,b)$ *if and only if for all* $x\in(a,b)$, $f(x)\le f(x^*)$. $x^*\in(a,b)$.	Given	
P2	$f:(a,b)\to R$ is differentiable at $x^*\in(a,b)$.	Given	
1	Claim that $\lim_{x\to x^{*-}}\frac{f(x)-f(x^*)}{x-x^*}=0=\lim_{x\to x^{*+}}\frac{f(x)-f(x^*)}{x-x^*}$.	C1	
2	Consider the right hand side limit.	C1	N
3	Note that since $f(x)\le f(x^*)$ for all $x\in(a,b)$, and $x-x^*>0$ $\lim_{x\to x^{*+}}\frac{f(x)-f(x^*)}{x-x^*}\le 0$.	CO (A, P1) R2	N
4	Consider the left hand side limit.	C1	N
5	Note that since $f(x)\le f(x^*)$ for all $x\in(a,b)$, and $x-x^*<0$ $\lim_{x\to x^{*+}}\frac{f(x)-f(x^*)}{x-x^*}\ge 0$	CO (C, P1) R2	N
6	Since $f:(a,b)\to R$ is differentiable at $x^*\in(a,b)$, the right hand limit is the same as the left hand limit.	R1	N
7	Therefore, since $0\le\lim_{x\to x^{*+}}\frac{f(x)-f(x^*)}{x-x^*}=$	CO (B, D) R2	N

	$\lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*} \leq 0$, $\lim_{x \to x^*} \frac{f(x) - f(x^*)}{x - x^*} = 0$.		
8	Since $\lim_{x \to x^*} \frac{f(x) - f(x^*)}{x - x^*} = 0$, $f^{'}(x^*) = 0$.	R1	N

The given proof problem is one of a few examples of the type of the proof in which students do not have to derive and set a starting variable at the beginning of the proof. The conclusion of the given statement is “ $f^{'}(x^*) = 0$.” The translation of the conclusion is “ $\lim_{x \to x^{*-}} \frac{f(x) - f(x^*)}{x - x^*} = 0 = \lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*}$.” One of the way to prove this statement is to work on both $\lim_{x \to x^{*-}} \frac{f(x) - f(x^*)}{x - x^*}$ and $\lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*}$ separately until both sides turn out to be the same statement. Then, one may separately combine each of them and the given condition “ $f : (a,b) \to R$ *has a global maximum at some* $x^* \in (a,b)$ ” to obtain $\lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*} \leq 0$ and $\lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*} \geq 0$. Using the other condition “ $f : (a,b) \to R$ *is differentiable at* $x^* \in (a,b)$,” one may obtain $0 \leq \lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*} = \lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*} \leq 0$. Then, one may conclude $\lim_{x \to x^{*-}} \frac{f(x) - f(x^*)}{x - x^*} = 0 = \lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*}$. The following figure is Daniel’s proof (Figure 5.7).

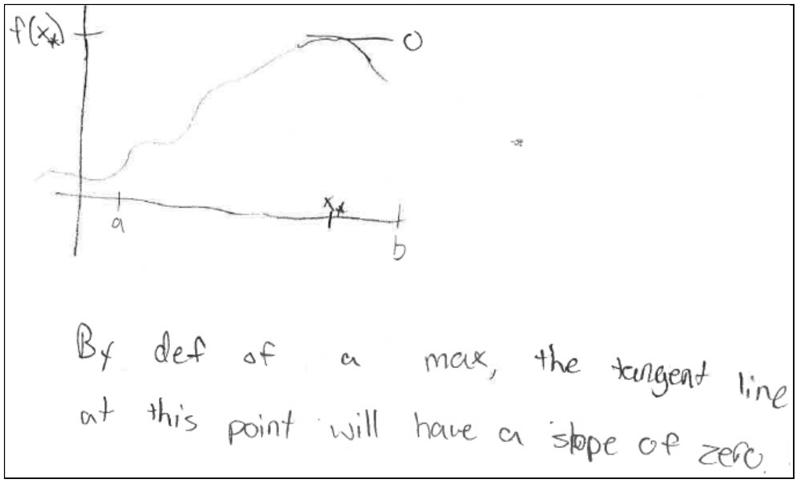


Figure 5.7. Daniel's Proof.

Daniel's proof was not successful because he was unable to construct a proving argument based on logical deduction. Instead, he depended on a graphical explanation for his argument, which resulted in his lack of rigor. He resorted to the fact that the tangent line at the maximum point had a slope of zero. He did not realize what he used was the very thing that he was asked to prove. He might have avoided the mistake by not only paying attention to the conclusion of the given statement but also translating it into mathematical language. The goal of the proof was to show $f'(x^*) = 0$. He might have translated it into " $\lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*} = 0 = \lim_{x \to x^{*-}} \frac{f(x) - f(x^*)}{x - x^*}$," which might have led him to consider $\lim_{x \to x^{*+}} \frac{f(x) - f(x^*)}{x - x^*}$ and $\lim_{x \to x^{*-}} \frac{f(x) - f(x^*)}{x - x^*}$. By considering them, he might have developed a proving argument based on logical deduction.

Thus, the above three examples show that noting the conclusion of the given statement, clarifying the goal of the proof, and translating it into mathematical language might help students to construct a more successful proving argument.

5.3.2 Setting a Variable

Mathematical ideas or thoughts in a proving argument are often conveyed by way of a variable. A variable is a key unit for advancing a reasoning process. It is crucial for students to set a correct starting variable to construct a proving argument. In order to be successful in setting a right starting variable, there are three steps that students can take. The first step is to note the conclusion of the given statement. The second is to translate it into *mathematical language* accurately. The third is to pay attention to *an ignition phrase* contained in *the mathematical language* and derive a starting variable from it. A possible major obstacle that may hinder students from setting a right starting variable is that they are tempted to pay attention to a hypothesis of the given statement to derive a starting variable.

There are mainly two ways for students to derive a starting variable in a proving argument. One is to derive a variable from the conclusion of the given statement. In particular, students often derive a starting variable from an *ignition phrase* contained in *the mathematical language* for the conclusion. In most cases, *an ignition phrase* comes from a definition of concept contained in *the mathematical language* for the conclusion. The other way is to derive a starting variable from anything other than the conclusion of the given statement, including a hypothesis of the given statement, a claim that students make, or a property of concept or a theorem that students have to bring in. In any case, deriving a right starting variable by noting *an ignition phrase* contained in *the mathematical language* for the conclusion of the given statement can be a key factor for constructing a successful proving argument. However, it can be difficult for some students to set a starting variable correctly. I will show five examples of students' difficulties with setting a variable while showing how their difficulties occurred and how their difficulties with setting a starting variable affected their proofs: Alex derived a starting variable from a hypothesis (Example 4); Quincy failed to note an *ignition phrase* (Example 5); Matthew missed deriving a variable from a hypothesis (Example 6); Natalie was unable to make a proving argument in *mathematical language* (Example 7); Anthony ruined his whole proving argument (Example 8).

Example 4: Alex (Algebra I)

Alex's proof is a representative example showing that students' failure to derive a right starting variable can spoil their whole proving arguments. His case also shows students may fail to derive a right starting variable because they tend to start to work on a given condition or hypothesis instead of the conclusion of the given statement.

> Question [5] (In-class problem solving session)
>
> *Suppose that the order of G is a prime number. Prove that G is cyclic.*

Table 5.5 shows a possible proof for Question [5] and shows where Alex had difficulties in the proof construction.

Table 5.5

Analysis (Type A) of Alex's Proof

	Proof	Code	Alex
X	Show G is cyclic.		
X'	Show $G = \langle g \rangle$ for some $g \in G$ with $g \neq 1$.	R1	
P	$\lvert G \rvert = p$.	Given	
1	Let $g \in G$ with $g \neq 1$.	C1	N
2	Consider $\langle g \rangle$.	C5	N
3	Recall that a cyclic group generated an element in *G* is a subgroup of *G*.	C2	N
4	Note $\langle g \rangle$ is a subgroup of *G*.	CO(2,3)	N
5	Recall the Lagrange's Theorem.	C2	N
6	By the Lagrange's THM, $\lvert \langle g \rangle \rvert = 1$ or *p*.	CO(4,5)R1	N

7	Since $\lvert < g > \rvert \neq 1$, $\lvert < g > \rvert = p$	CO(1,6)R2	N
8	Since $\lvert G \rvert = p$, $G =< g >$	CO(P,7)R2	N

Question [5] can be proved in the following way. The conclusion of the given statement is that "*G is cyclic*." The conclusion "*G is cyclic*" can be translated into *mathematical language* "$G =< g >$ for some $g \in G$ with $g \neq 1$." The given proof is the type of the proof of showing A = B. One can work on either A or B through rephrasing it until A becomes B or B becomes A. In this problem, work on $< g >$. Recalling and combining Lagrange's Theorem and the property that a cyclic group generated by an element in *G* is a subgroup of *G*, one may obtain $\lvert < g > \rvert = 1, p$. Noting $\lvert < g > \rvert \neq 1$, decide $\lvert < g > \rvert = p$. Combining the hypothesis $G =< g >$ and $\lvert < g > \rvert = p$, conclude $G =< g >$. The following figure shows Alex's whole proof (Figure 5.8).

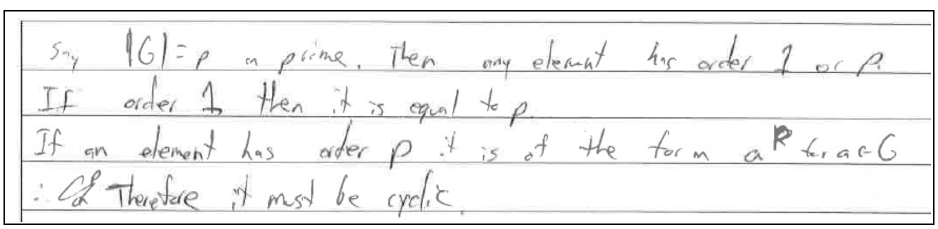

Say $|G| = p$ a prime. Then any element has order 1 or p.
If order 1 then it is equal to p.
If an element has order p it is of the form a^p for $a \in G$
∴ Therefore it must be cyclic.

Figure 5.8. Alex's Proof.

Alex started his proving argument with working on the hypothesis "the order of *G* is prime." He rephrased the given condition with "any element of *G* has order 1 or (some prime number) *p*". Then, he followed "it (an element of *G* with order *p*) is of the form of a^p for some $a \in G$" without any explanation, which lacked rigor of logic. There were a few factors that might have caused his lack of rigor in the argument. He was unable to consider a subgroup of *G* generated by an element *g* in *G* with $g \neq 1$ and to use Lagrange's Theorem. In particular, he was not able to start his argument with translating the conclusion of the given statement "G is cyclic" into "$G = < g >$ for some $g \in G$ with $g \neq 1$." If he had $G = < g >$ with $g \neq 1$, that might have led him to consider $< g >$, which could have led him to use Lagrange's Theorem.

Example 5: Quincy (Topology)

Students' success in managing *the opening stage* is an imperative factor for making their proving arguments successful. In particular, being able to derive a right starting variable from *an ignition phrase* in the mathematical language for the conclusion of the given statement can be a major factor. Quincy's case is an example showing students' mismanagement of deriving a starting variable may damage their whole proving arguments.

> Question [7] (Exam II)
>
> *Let X be a Hausdorff space. Let $\{x_n : n \in Z^+\}$ be a sequence in X converging To a point x_0. Prove that the set $K = \{x_n : n = 0,1,2,...\}$ is compact.*

Table 5.6 shows a possible proof for Question [7] and shows where Quincy had a difficulty in the proof construction.

Table 5.6

Analysis (Type A) of Quincy's Proof

Object	Proof	Code	Q
X	Show that $K = \{x_n : n = 0,1,2,...\}$ is compact.		
Y	Show that for any open cover of *K*, *K* has a finite open subcover.		
P	$\{x_n : n \in Z^+\}$ be a sequence in *X* converging to x_0.	Given	
1	Let $U = \{U_\alpha \in T_X\}$ be an open cover of *X*.	C1	S
2	Construct an open cover of *K* by letting $V = \{V_\alpha = U_\alpha \cap K\}$.	C1	N
3	Since $U = \{U_\alpha \in T_X\}$ is an open cover of *X*, $\exists U_{\alpha_0} \in U$ such that $x_0 \in U_{\alpha_0}$.	R1	S

4	Since x_n converges to x_0, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in U_{\alpha_0}$.	CO(P,3)R1	S
5	Let $V_{\alpha_0} = U_{\alpha_0} \cap K$, where $V_{\alpha_0} \in V$.	C1	N
6	For each x_i with $i < N$, find an open set $V_{x_i} \in V$ such that $x_i \in V_{x_i}$.	C1	N
7	Note that $\{V_{x_1}, V_{x_2}, \ldots V_{x_{N-1}}, V_{\alpha_0}\}$ is a desired finite open subcover of *K*.	CO(5,6)R2	N

One way to prove Question [7] is as follows. The conclusion of the given statement is "$K = \{x_n : n = 0,1,2,\ldots\}$ is compact." It can be translated into the *mathematical language* "For any open cover of *K*, *K* has a finite open subcover." By paying attention to *the ignition phrase* "For any open cover of *K*," one may explore the way to construct an open cover of *K*. Recalling the property of a subspace topology, one can set a starting variable by having "Let $U = \{U_\alpha \in T_X\}$ as an open cover of *X*". Then, one may construct an open cover of *K* by having "$V = \{V_\alpha = U_\alpha \cap K\}$." To further advance a reasoning process, one may note and consider the given hypothesis "$\{x_n : n \in Z^+\}$ converges to a point x_0." Then, the given hypothesis can be translated into "For an open set $V_{\alpha_0} = U_{\alpha_0} \cap K$ in the open cover of *K*, in which $x_0 \in U_{\alpha_0}$, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in V_{\alpha_0}$. Finally, they may create a finite open subcover $\{V_{x_1}, V_{x_2}, \ldots V_{x_{N-1}}, V_{\alpha_0}\}$ by setting $V_{x_i} \in V$ such that $x_i \in V_{x_i}$ for $n < N$. Quincy's proof is shown in the following figure (Figure 5.9).

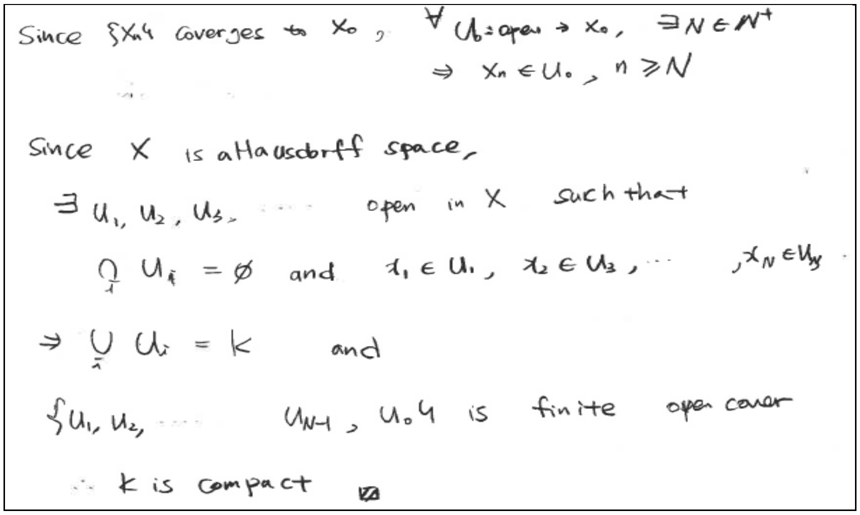
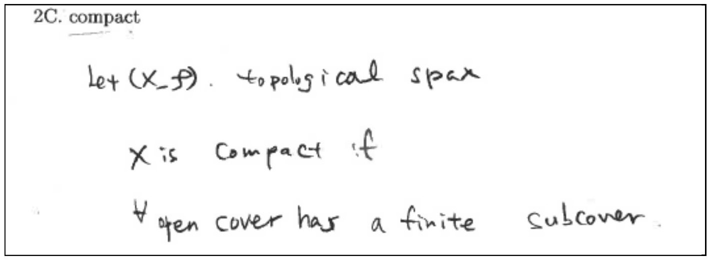


Figure 5.9. Quincy's Proof.

There were mainly two problems with his proving argument. One was that he was unable to bring in the concept of a subspace topology and apply it to *K*. He seemed to use an open cover of *X* as a substitute for an open cover of *K*. He might have lacked the knowledge of a subspace topology. Another was that he constructed a finite open subcover of *K* without specifying an open cover of *K*, from which a finite subcover was supposed to come. He missed noting *the ignition phrase* "For every open cover of *K*" in the definition of compactness, which led him to fail to set an open cover of *K*.

His case is also an example showing students' knowledge of the definition of a concept does not necessarily mean they can make good use of it in their proving arguments. When he was asked to define "compactness" in a problem given prior to the above proof problem on the same exam, he was able to answer the question correctly with some minor awkward expressions. As the following figure (Figure 5.10) shows, he stated the definition of compactness as "For every open cover (of *X*, it) has a finite open subcover."


2C. compact

Let (X, T). topological space

X is compact if

∀ open cover has a finite subcover.

Figure 5.10. Quincy's Statement.

However, in the given proof problem, he did not pay attention to *the ignition phrase* "For every open cover." As a result, he constructed a finite open subcover without setting an open cover of *K* from which the finite subcover could have been derived. His case implies students should know the role of *an ignition phrase* and how to utilize it for advancing a reasoning process.

Example 6: Matthew (Topology)

There is a type of proof for which students have to derive a starting variable from a given condition or hypothesis of the given statement. This type of proof was rare among the proofs examined in this study while the great majority of the proofs required students to derive a starring variable from the conclusion of the given statement. Matthew's case is an example showing students' failure to set a starting variable from a hypothesis of the given statement can cause their proving arguments to be unsuccessful.

> Question [6] (Exam II)
>
> *Let X, Y be topological spaces; Y be compact;* $x_0 \in X$ *; and N be an open set containing* $\{x_0\} \times Y$ *in the product space* $X \times Y$. *Prove there exists an open neighborhood* $W \subset X$ *of* x_0 *such that* $W \times Y \subset N$.

Table 5.7 shows a possible proof for Question [6] and shows where Matthew had a difficulty in the proof construction.

Table 5.7

Analysis (Type A) of Matthew's Proof

	Proof	Code	M
X	Construct an open neighborhood $W \subset X$ of x_0 such that $W \times Y \subset N$.		
P1	N is an open set containing $\{x_0\} \times Y$.		
P2	Y is compact.		
1	Since N is open in $X \times Y$, for each $(x_0, y) \in \{x_0\} \times Y$, there exists a basis open set $U_y \times V_y \subset N$ containing (x_0, y) for each $y \in Y$ where U_y is open in X and V_y is open in Y.	C1	N
2	Then, $\{\{x_0\} \times V_y\}$ is an open cover of $\{x_0\} \times Y$.	R2	N
3	Note $\{x_0\} \times Y$ is homeomorphic to Y.	C2	S
4	Then, $\{x_0\} \times Y$ is compact.	CO(P2,3) R2	N
5	Then, there exists a finite open subcover $\{\{x_0\} \times V_{y_i} \in \{x_0\} \times T_Y\}\}$, where $i \in \{1,\ldots,n \mid n \in Z^+\}$ and $\bigcup_{i=1}^{n} V_{y_i} = Y$.	CO(2,4) R1	N
6	Note that $\{x_0\} \times Y \subset \{U_{y_i} \times V_{y_i}\} \subset N$. . , where $i \in \{1,\ldots,n \mid n \in Z^+\}$.	C1	N
7	Let $W = \bigcap_{i=1}^{n} U_i$, where W is an open neighborhood of x_0, where $\{x_0\} \times Y \subset W \times Y \subset N$.	C1	N

The following shows how to obtain the above proof for Question [6]. The conclusion of the given statement is "*there exists an open neighborhood* $W \subset X$ *of* x_0 *such that* $W \times Y \subset N$." Noting the given condition "*N is an open set containing* $\{x_0\} \times Y$"and recalling the property of an open set, one can set a starting variable $U_y \times V_y \subset N$ as an open neighborhood of (x_0, y) for each $y \in Y$." Further noting another given condition "*Y* is compact" and realizing $\{x_0\} \times Y$ is homeomorphic to *Y*, one can construct an open cover of $\{x_0\} \times Y$ by having $\{ U_y \times V_y \mid U_y \times V_y \subset N$ and $y \in Y \}$. Then, since $\{x_0\}\times Y$ is compact, there must exist a finite open subcover $\{\{x_0\}\times V_{y_i} \in \{x_0\}\times T_y\}$,

in which $i \in \{1,\ldots,n | n \in Z^+\}$ and $\bigcup_{i=1}^{n} V_{y_i} = Y$. Then, one may construct $W = \bigcap_{i=1}^{n} U_i$ so that $\{x_0\}\times Y \subset W \times Y \subset N$. The following (Figure 11) shows Matthew's proof.

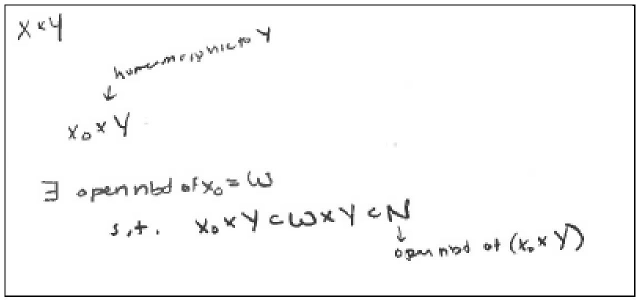


Figure 5.11. Matthew's Proof.

Matthew's proof was not successful because he was unable to construct an open neighborhood $W \subset X$ *of* x_0 *such that* $W \times Y \subset N$. A direct cause of his failure may be that he was unable to set a starting variable $U_y \times V_y \in T_X \times T_Y \subset N$ of (x_0, y) for each $y \in Y$. In this proof problem, a starting variable can be derived from the hypothesis of the given statement "*N is an open set containing* $\{x_0\}\times Y$." The hypothesis can be translated into "there exists an open neighborhood of (x_0, y) contained in *N* for each $y \in Y$." Then, an open neighborhood of (x_0, y) can be expressed with $U_y \times V_y \in T_X \times T_Y$ as a starting variable. Using this object, students may construct an open cover $\{\{x_0\}\times V_y\}$ of $\{x_0\}\times Y$ so that they can use another hypothesis "*Y* is compact." Thus, a reasoning process cannot be advanced without setting a starting variable.

Example 7: Natalie (Topology)

Natalie's proof is another representative example showing students' failure to set a starting variable can hinder them advancing a reasoning process based on logical deduction.

> Question [3b] (Exam II)
>
> *Let $q : X \to Y$ be a quotient map and $f : Y \to Z$ be a map. Suppose $f \circ q$ is continuous. Show $f : Y \to Z$ is continuous.*

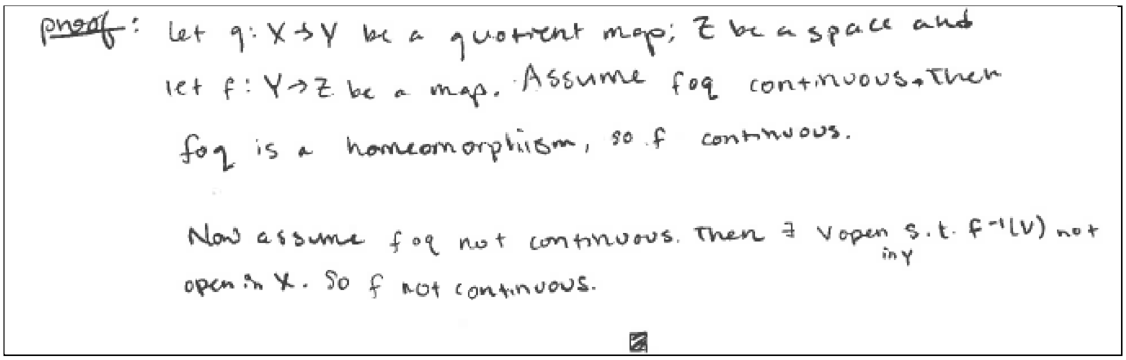
Table 5.8 shows a possible proof of the statement of Question [3b] and shows where Natalie had a difficulty in the proof construction.

Table 5.8

Analysis (Type A) of Natalie's Proof

Step	Statement	Operation	N
Opening Stage			
X	$f : Y \to Z$ is continuous.	Given	
Y: OTC	For any open set W in Z, $f^{-1}(W)$ is open in Y.	R1	N
P1: hypothesis	$q : X \to Y$ is a quotient map.	Given	
P2: hypothesis	$f \circ q$ is continuous.	Given	
S: OSV	Let W be an open set in Z.	C1	
Body Construction Stage			
1	Let W be an open set in Z.	C1	N
2	Consider $(f \circ q)^{-1}(W)$.	C5	N
3	Note $(f \circ q)^{-1}(W) = q^{-1}(f^{-1}(W))$.	R1	N
4	Since $f \circ q$ is continuous, $q^{-1}(f^{-1}(W))$ is open in X.	CO(3,P2)R1	N
5	Recall the property of a quotient map.	C2	N
6	Since q is a quotient map, $f^{-1}(W)$ must be open in Y.	CO(4,5)R1	N
7	Therefore, $f : Y \to Z$ is continuous.	R1	N

The above proof can be obtained in the following way. The conclusion of the given statement is "$f : Y \to Z$ is continuous." The translation of the conclusion into *mathematical language* is "For any open set W in Z, $(f^{-1}(W))$ is open in Y." Noting *the ignition phrase* “For an open set W in Z ,” one can set a starting variable by having “$W \in T_Z$.” Combining the other given condition “$q : X \to Y$ is a quotient map” and the property of a quotient map “If $(q^{-1}(H))$ is open in Z for a quotient map $q : Y \to Z$ and for $H \subset Z$, then H is open in Y,” one may conclude $(f^{-1}(W)) \in T_Y$. The following figure shows Natalie’s proof (Figure 5.12).


proof: let $q: X \to Y$ be a quotient map; Z be a space and let $f: Y \to Z$ be a map. Assume $f \circ q$ continuous. Then $f \circ q$ is a homeomorphism, so f continuous.

Now assume $f \circ q$ not continuous. Then $\exists$ V open in Y s.t. $f^{-1}(V)$ not open in X. So f not continuous.

Figure 5.12. Natalie’s Proof.

Natalie was unable to prove the statement partly because she was unable to set a starting variable properly. She attempted to prove the given statement in two ways. In her first attempt, she claimed $f \circ q$ was homeomorhism for no reason. Since she did not provide any supporting explanations for her claim, her argument was not valid. In her second attempt, she tried the contrapositive. She was right in negating the conclusion of the given statement when she had “There exists an open set V (in Y) such that $(f^{-1}(V)$ is not open.” This implies that she did pay attention to the conclusion of the given statement and that she was capable of translating it into *mathematical language* by applying the definition of a continuous function. However, she did not set a starting variable correctly. If she had set a starting variable $W \in T_Z$ by paying attention to *the ignition phrase* in *the mathematical language* for the conclusion of the given statement, she might have been able to more rigorously advance a reasoning process with the given conditions.

Example 8: Anthony (Algebra I)

Anthony's case is a representative example showing that students' setting a starting variable from a hypothesis of the given statement may damage their whole proving arguments.

Question [4] (In-class problem solving session)
Suppose that $G/Z(G)$ *is cyclic. Prove that G is abelian.*

Table 5.9 shows a possible proof for Question [4] and shows where Carlos had difficulties in the proof construction.

Table 5.9

Analysis (Type A) of Anthony's Proof

Object	Proof	Code	A
X	Show G is abelian.		S
Y	Show $ab = ba$ for any $a, b \in G$.	R1	S
P	$G/Z(G)$ is cyclic.	Given	
1	Let $a, b \in G$.	C1	N
2	Note $a, b \in G$ are in some cosets.	C2	N
3	Let $a \in x^m Z$ and $b \in x^n Z$.	CO(2,P)R1	N
4	Let $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$.	R2	N
5	Then, $ab = x^{m+n} z_1 = x^{n+m} z_2 = ba$.	R3	N

The following is one way to obtain the above proof. The conclusion of the given statement is that "*G is cyclic*." The conclusion "*G is cyclic*" can be translated into *mathematical language* "$G = < g >$ for some $g \in G$ with $g \neq 1$." The given proof is the type of the proof of showing A = B. One can work on either A or B through rephrasing it until A becomes B or B becomes A. In this problem, one may consider and work on $<g>$. Recalling and combining Lagrange's Theorem and the property that a cyclic group generated by an element in *G* is a subgroup of *G*, one may obtain $|<g>| = 1, p$. Noting $|<g>| \neq 1$, one may decide $|<g>| = p$. Combining the hypothesis $G =< g >$ and $|<g>| = p$, one may conclude $G =< g >$. The following figure (Figure 5.13) shows Anthony's proof.

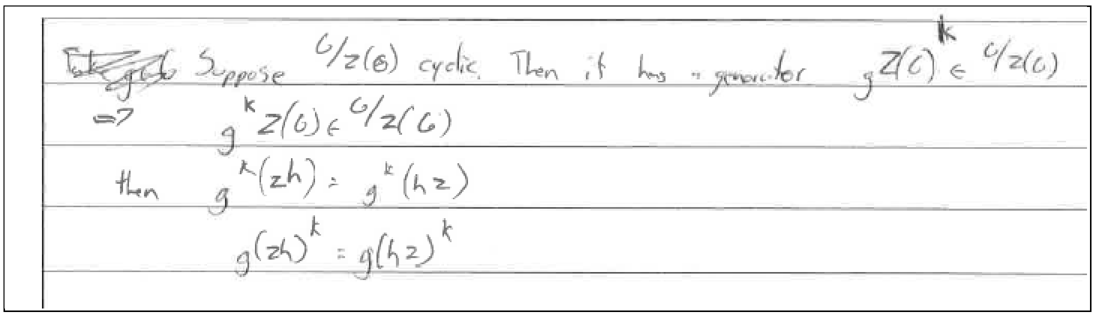


Figure 5.13. Anthony's Proof.

Anthony started with working on the given hypothesis "*G*/*Z*(*G*) is cyclic." While many other students struggled with expressing an element of the cyclic group of group {*G*/*Z*(*G*)}, he was one of a few students who successfully expressed it by $g^k Z(G)$. However, his proving arguments did not make sense partly because he was unable to set a right goal of the proof "$ab = ba$ for any $a, b \in G$" and partly because he was unable to set right starting variables "$a, b \in G$" from the conclusion of the given statement.

His case was a representative example showing that students were tempted to note the hypothesis to set a starting variable. He started to work on the hypothesis "*G*/*Z*(*G*) is cyclic" to introduce the variables "*z* and *k*" to consider a generator of *G*/*Z*(*G*). With his lack of the knowledge of the fact that every element in *G* belonged to some coset, his argument turned out to be incomplete.

5.4 Difficulties with Rephrasing an Object

A proving process involves a sequence of transformations of statements. *Rephrasing an object* is a major operation for transforming statement in a proof. *Rephrasing* an object is done by applying a definition of concept, a property of concept, through formal or informal interpretation, and through algebraic computation. Students' failure to make good use of the rephrasing operation may greatly affect their proving arguments. I will present six examples showing students' difficulties with rephrasing an object while showing how their difficulties occurred and affected their proving arguments: Katherine made a wrong start of a proof (Example 9); Natalie was unable to make a proving argument in *mathematical language* (Example 10); Bill was unable to interpret an object for rephrasing an object (Example 11); Eric failed to rephrase an object because of his lack of knowledge (Example 12); and Berkeley (Example 13) missed trying algebraic manipulation for rephrasing an object.

Example 9: Katherine (Topology)

Translating a concept into *mathematical language* is a crucial operation of rephrasing an object by applying the definition of the concept. Katherine's case is a representative example showing the importance of students' being able to translate a given statement into *mathematical language*.

> Question [7] (Exam II)
>
> *Let X be a Hausdorff space. Let $\{x_n : n \in Z^+\}$ be a sequence in X converging to a point x_0. Prove that the set $K = \{x_n : n = 0,1,2,\ldots\}$ is compact.*

Table 5.10 shows a possible proof for Question [7] and shows where Katherine had difficulties in the proof.

Table 5.10

Analysis (Type A) of Katherine's Proof

	Proof	Code	K
X	Show that $K = \{x_n : n = 0,1,2,...\}$ is compact.		
Y	Show that for any open cover of *K*, *K* has a finite open subcover.		
P	$\{x_n : n \in Z^+\}$ be a sequence in *X* converging to x_0.	Given	
1	Let $U = \{U_\alpha \in T_X\}$ be an open cover of *X*.	C1	S
2	Construct an open cover of *K* by letting $V = \{V_\alpha = U_\alpha \cap K\}$.	C1	N
3	Since $U = \{U_\alpha \in T_X\}$ is an open cover of *X*, $\exists U_{\alpha_0} \in U$ such that $x_0 \in U_{\alpha_0}$.	R1	N
4	Since x_n converges to x_0, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in U_{\alpha_0}$.	CO(P,3)R1	N
5	Let $V_{\alpha_0} = U_{\alpha_0} \cap K$, where $V_{\alpha_0} \in V$.	C1	N
6	For each x_i with $i < N$, find an open set $V_{x_i} \in V$ such that $x_i \in V_{x_i}$.	C1	N
7	Note that $\{V_{x_1}, V_{x_2}, ...V_{x_{N-1}}, V_{\alpha_0}\}$ is a desired finite open subcover of *K*.	CO(5,6)R2	N

The conclusion of the given statement is "$K = \{x_n : n = 0,1,2,...\}$ is compact." It can be translated into the following *mathematical language*: "For any open cover of *K*, *K* has a finite open subcover." By paying attention to *the ignition phrase* "For any open cover of *K*," one may explore the way to construct an open cover of *K*. Recalling the property of a subspace topology, one can set a starting variable by having "Let $U = \{U_\alpha \in T_X\}$ as an open cover of *X*". Then, one may construct an open cover of *K*, providing "$V = \{V_\alpha = U_\alpha \cap K\}$." Noting the given hypothesis "$\{x_n : n \in Z^+\}$ converges to a point x_0," one may translate it into the following mathematical language: "For an open set $V_{\alpha_0} = U_{\alpha_0} \cap K$ in the open cover of *K*, in which $x_0 \in U_{\alpha_0}$, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in V_{\alpha_0}$. Then, one can construct a finite open subcover $\{V_{x_1}, V_{x_2}, \ldots V_{x_{N-1}}, V_{\alpha_0}\}$ by setting $V_{x_i} \in V$ such that $x_i \in V_{x_i}$ for $n < N$. The following (Figure 5.14) is Katherine's proof.

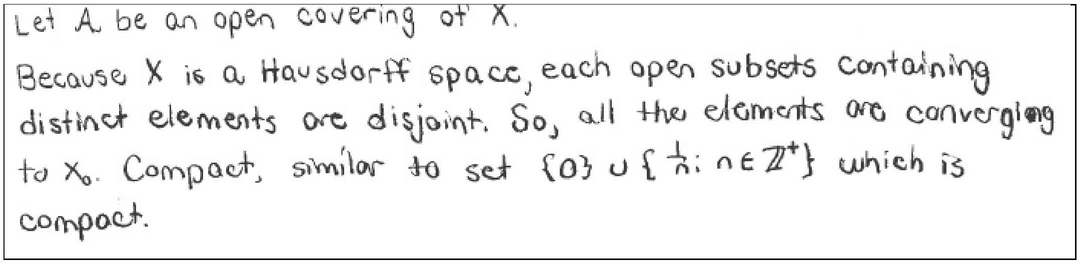

Let A be an open covering of X.
Because X is a Hausdorff space, each open subsets containing distinct elements are disjoint. So, all the elements are converging to x_0. Compact, similar to set $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ which is compact.

Figure 5.14. Katherine's Proof.

Katherine seemingly made a good start when setting an open cover of *X*. However, it seems that she set the open cover not because she intended to use it to derive an open cover of *K* but because she tried to substitute the open cover of *X* itself for an open cover of *K*. If she had noted the conclusion of the given statement "*K* is compact" and translated it into "For every open cover of *K*, there exists a finite open subcover of *K*," she might have at least mentioned an open cover of *K* in her argument. Students' inability to rephrase an object can result from their lack of knowledge of, especially, the definition of a concept. However, Katherine knew the definition of compactness. She correctly stated the definition of compactness in the problem given prior to the above proof problem in the same exam, as seen in the following figure (Figure 5.15).

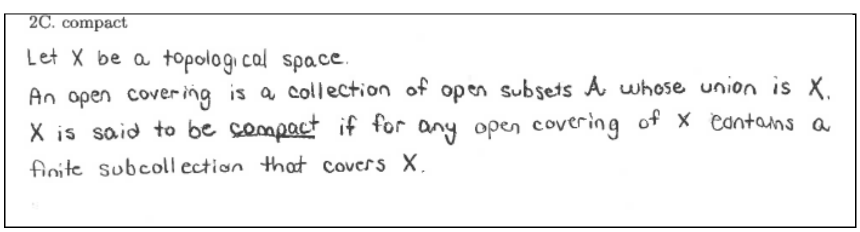

2C. compact
Let X be a topological space.
An open covering is a collection of open subsets A whose union is X.
X is said to be compact if for any open covering of X contains a finite subcollection that covers X.

Figure 5.15. Katherine's Statement.

Katherine's example shows even when students know the definition of a concept, that does not necessarily mean they can apply it. This problem may occur because they are not aware of the importance of precisely translating an object, especially a mathematical key concept such as "compact," "differentiable," or "abelian," into *mathematical language*. In particular, they may not be aware of the role of *an ignition phrase* playing in proof construction: *an ignition phrase* can provide a variable with which students can start, develop, and advance their reasoning process. If Katherine had had the knowledge of the role of *an ignition phrase*, she might have noted *the ignition phrase* "for every open cover of *K*" to consider how to set an open cover of *K* as a starting variable.

Katherine also did not try to translate the given hypothesis "$\{x_n : n \in Z^+\}$ is a sequence in *X* converging to x_0" into *mathematical language* "$\exists N \in Z^+$ such that for every $n \geq N$, $x_n \in U_{\alpha_0}$," which might have hindered her from advancing her reasoning process. What is crucial in translating a statement into *mathematical language* lies in understanding, remembering, and accurately expressing the definition of a concept involved in a statement. She showed her mental image about a sequence converging to a point in mentioning "$\{0\} \cup \{\frac{1}{n} : n \in Z\}$." However, she was unable to express the definition of a sequence converging to a point in a formal way. Students' inability to rephrase an object can be directly caused by their incomplete knowledge of a concept, in particular, the definition of the concept.

Example 10: Natalie (Topology)

A formal proving argument can be realized by way of *mathematical language*.

M*athematical language* makes a proving argument rigorous, logical, and convincing. The operation of r*ephrasing an object* plays a central role to translate an object into *mathematical language*. Natalie's proof (Figure 5.16) is an example showing students' inability to rephrase an object may produce a proving argument that lacks logic and rigor.

Question [3.b] (Exam II)

> *Let $q : X \to Y$ be a quotient map and $f : Y \to Z$ be a map. Suppose $f \circ q$ is continuous. Show $f : Y \to Z$ is continuous.*

Table 5.11 shows a possible proof for Question [3.b] and where Natalie had a difficulty in the proof construction.

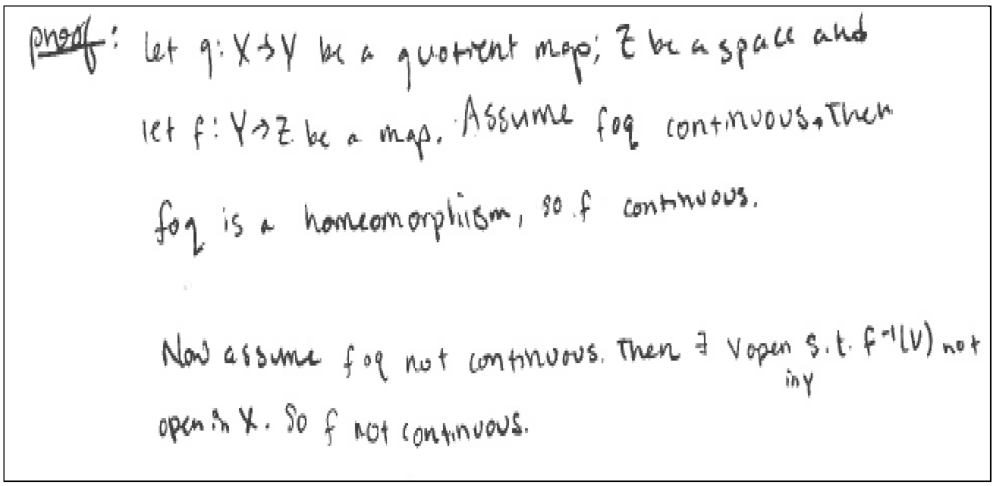
Table 5.11

Analysis (Type A) of Natalie's Proof

Step	Statement	Operation	N
Opening Stage			
X	$f : Y \to Z$ is continuous.	Given	
Y: OTC	For any open set W in Z, $f^{-1}(W)$ is open in Y.	R1	S
P1: hypothesis	$q : X \to Y$ is a quotient map.	Given	
P2: hypothesis	$f \circ q$ is continuous.	Given	
S: OSV	Let W be an open set in Z.	C1	
Body Construction Stage			
1	Let W be an open set in Z.	C1	S
2	Consider $(f \circ q)^{-1}(W)$.	C5	N
3	Note $(f \circ q)^{-1}(W) = q^{-1}(f^{-1}(W))$.	R1	N
4	Since $f \circ q$ is continuous, $q^{-1}(f^{-1}(W))$ is open in X.	CO(3,P2)R1	N
5	Recall the property of a quotient map.	C2	N
6	Since q is a quotient map, $f^{-1}(W)$ must be open in Y.	CO(4,5)R1	N
7	Therefore, $f : Y \to Z$ is continuous.	R1	N

The conclusion of the given statement is "$f : Y \to Z$ is continuous." The translation of the conclusion into *mathematical language* is "For any open set W in Z,

$(f^{-1}(W))$ is open in *Y*." Noting *the ignition phrase* "For an open set W in Z ," one can set a starting variable by having "Let $W \in T_Z$." Combining the other given condition "$q : X \rightarrow Y$ is a quotient map" and the property of a quotient map, which says "If $(q^{-1}(H))$ is open in *Z* for a quotient map $q : Y \rightarrow Z$ and for $H \subset Z$, then *H* is open in *Y*," one may conclude $(f^{-1}(W)) \in T_Y$. The following (Figure 5.16) is Natalie's proof.


proof: let q: X→Y be a quotient map; Z be a space and let f: Y→Z be a map. Assume f∘q continuous. Then f∘q is a homeomorphism, so f continuous.

Now assume f∘q not continuous. Then ∃ V open in Y s.t. f⁻¹(V) not open in X. So f not continuous.

Figure 5.16. Natalie's Proof.

Natalie's proof was not convincing because she did not advance her reasoning process in *mathematical language*. She claimed "$f \circ q$ is homeomorphic" without showing the reason. She concluded f was continuous but did not provide the reason. She resorted to the abuse of a property of homeomorphism to prove the given statement. She might have avoided her incomplete argument if she had translated "$f \circ q$ is continuous" into "$q^{-1}(f^{-1}(W)) \in T_X$ " for $W \in T_Z$. As introduced in Chapter 3 , in this study, mathematical concepts, for example, "continuous" and "homeomorphic," are mathematical language but not treated as *mathematical language*. *Mathematical language* is a rigorous expression of a concept involving a variable, which empowers students to advance a reasoning process logically. Natalie was not able to rephrase "continuous" and "a quotient map " by translating them into *mathematical language*, which hindered her from making a formal proving argument.

Example 11: Bill (Algebra I)

Bill's proof is a representative case showing students' failure to rephrase an object through interpretation can be a cause of hindering them from advancing their reasoning process.

> Question [5] (In-class problem solving session)
>
> *Suppose that the order of G is a prime number. Prove that G is cyclic.*

Table 5.12 shows a possible proof for the given problem and where Bill had difficulties in the proof construction.

Table 5.12

Analysis (Type A) of Bill's Proof

Object	Proof	Code	B
X	Show G is cyclic.		
Y	Show $G = <g>$ for some $g \in G$ with $g \neq 1$.	R1	
1	Consider $G = <g>$ for some $g \in G$ with $g \neq 1$.	C1	N
2	Note $<g>$ is a subgroup of G.	C2	N
3	Recall the Lagrange's THM.	C2	N
4	Then, by the Lagrange's THM, $\|<g>\| = 1, p$	CO(2,3)R1	N
5	Since $\|<g>\| \neq 1$, $\|<g>\| = p$	CO(1,4)R2	N
6	Since $\|G\| = p$, $G = <g>$	R1	N

The following is an explanation of the above proof. The conclusion of the given statement is that "*G is cyclic*." The conclusion "*G is cyclic*" can be translated into *mathematical language* "$G = <g>$ for some $g \in G$ with $g \neq 1$." The given proof is the type of the proof of showing A = B. One can work on either A or B through rephrasing it until A becomes B or B becomes A. In this problem, one may consider and work on $< g >$ for some $g \in G$ with $g \neq 1$. Recalling Lagrange's Theorem and combining it with the property that a cyclic group generated by an element in *G* is a subgroup of *G*, one may obtain $|< g >| = 1, p$. Noting $|< g >| \neq 1$, one may decide $|< g >| = p$. Combining the hypothesis $G =< g >$ and $|< g >| = p$, one can conclude $G =< g >$. The following (Figure 5.17) shows Bill's proof.

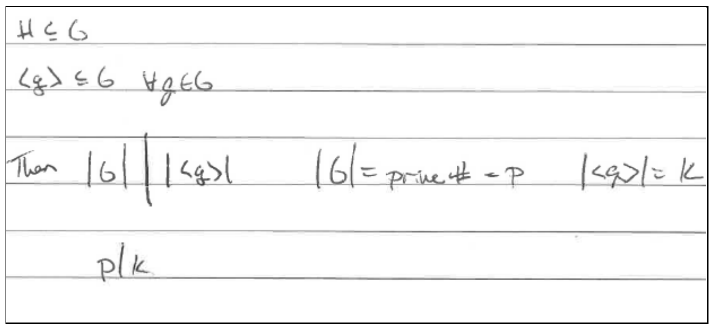


Figure 5.17. Bill's Proof.

Bill noted that $< g >$ was a subgroup of *G* and tried applying the Lagrange's Theorem. His notation was wrong when he had $|G| \mid |g|$. He probably meant that $|g| \mid |G|$ by that. After that, he was unable to advance his reasoning process mainly because he was unable to interpret $|g| \mid |G|$ as $|< g >|$ = 1 or *p*. If he had obtained $|< g >|$ = 1 or *p*, he might have obtained $|G| = p =< g >$ with $|< g >| \neq 1$.

Example 12: Anthony (Algebra I)

Anthony's case is another example of showing that students' failure to rephrase an object can cause their proving arguments to be unsuccessful. In particular, his case is a representative example showing students' lack of knowledge may affect their ability of rephrasing an object.

> Question [6] (In-class problem solving session)
>
> *Suppose that* $|G| = pq$ *for some primes p and q. Prove that G is either abelian Or* $Z(G) = \{e\}$ *and* $|Z(G)| \neq p, q$.

The following (Table 5.13) is a possible proof for the given proof problem and shows where Anthony had difficulties in the proof construction.

Table 5.13

Analysis (Type A) on Anthony's Proof

	Proof	Code	A
X	Show G is abelian or $Z(G) = \{e\}$ and $|Z(G)| \neq p, q$.		
Y	Show $Z(G) = G$ or $|Z| = 1$ and $|Z| \neq p, q$	R1	N
P	$|G| = pq$ *for some primes p and q.*	Given	
1	Consider $Z(G)$.	C5	S
2	Note that $Z(G)$ is a subgroup of G.	C2	S
3	Recall the Lagrange's THM.	C2	S
4	Then, $|Z(G)| = pq, 1, p$, or q.	CO(2,3)R 2	N
5	Case 1: Suppose $|Z(G)| = pq$.	C3	N
6	Since $|G| = pq = |Z|$, $G = Z$.	CO(5,P)R 2	N
7	Since Z is abelian, G is abelian	R2	N
8	Case 2: Suppose $|Z(G)| = 1$.	C3	N
9	Then, $Z(G) = \{e\}$.	R2	N
10	Case 3: For a contradiction, suppose $|Z(G)| = p$.	C3	N
11	Consider the order of the quotient group $|G/Z|$.	C5	N
12	Since $|G| = pq$ and $|Z(G)| = p$, $|G/Z| = q$.	CO(P,11) R2	N
13	Recall that if $|K|$ is prime, K is cyclic.	C2	N
14	Therefore, $G/Z(G)$ is cyclic.	CO(12,13) R1	N
15	Recall that if the order of the quotient group $|K/H|$ is cyclic, then K is abelian.	C2	N
16	Therefore, G is abelian.	CO(14,15) R1	N

17	Then, $G = Z$.	R2	N
18	Then, $pq = \|G\| = \|Z\| = p$, which is a contradiction.	CO(P,15) R2	N

The above proof cab be obtained in the following way. The conclusion of the given statement is "*G is either abelian or* $Z(G) = \{e\}$ *and* $|Z(G)| \neq p, q$." The translation of the conclusion into *mathematical language* can be "$G = Z(G)$" or $|Z(G)| = 1$ and $|Z(G)| \neq p, q$. One may further rephrase "$G = Z(G)$" with $|Z(G)| = pq$. in terms of $|Z(G)|$. Students may rephrase "*G is either abelian or* $Z(G) = \{e\}$" with "$|Z(G)| = pq$" or "$|Z(G)| = 1$." Noting the given condition "$|G| = pq$"and recalling the relationship between $|Z(G)|$ and $|G|$, which is "*Z*(*G*) is a subgroup of *G*," and Lagrange's Theorem, one can set the following three cases: $|Z(G)| = pq$; $|Z(G)| = 1$; and $|Z(G)| = p, q$. For the first case, one may notice $|Z(G)| = pq = |G|$ and conclude that $G = Z$, which means that *G* is abelian. For the second case, one may note that $Z(G) = \{e\}$ so that $|Z(G)| = 1$. For the third case, one may use a contradiction assuming $|Z(G)| = p$. Considering the quotient group $G / Z(G)$ and recalling the fact that if $|G / Z(G)|$ is a prime number, $G / Z(G)$ is cyclic, one may realize that if $G / Z(G)$ is cyclic. Moreover, recalling the fact that if $G / Z(G)$ is cyclic, *G* must be abelian, one may realize *G* = *Z*(*G*). However, it is a contradiction because one would get $pq = |G| = |Z(G)| = p$. The following (Figure 5.18) is Anthony's proof .

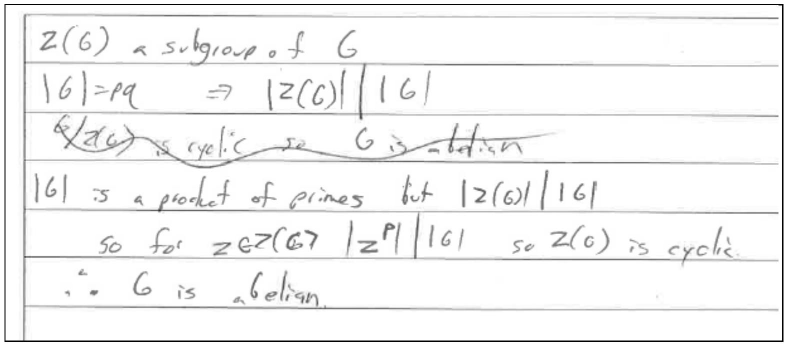

Z(G) a subgroup of G
|G|=pq ⇒ |Z(G)| | |G|
~~G/Z(G) is cyclic so G is abelian~~
|G| is a product of primes but |Z(G)| | |G|
so for z∈Z(G) |z^p| | |G| so Z(G) is cyclic
∴ G is abelian

Figure 5.18. Anthony's Proof.

Although Anthony had "$|Z|$ divides $|G| = pq$," he was unable to advance his reasoning process partly because he was unable to interpret the meaning of "$|Z|$ divide $|G| = pq$" and translate it into "$|Z| = 1, p, q$ or pq." If Anthony had obtained "$|Z| = 1, p, q$ or pq," he might have considered those three cases in which $|Z| = pq$, 1, and *p* (or *q*). Another difficulty he had was that he was unable to rephrase "*G* is abelian" with "*G* = *Z*(*G*)." rephrase His lack of knowledge of that "*G* is abelian" is equivalent to saying "*G* = *Z*(*G*)" might have directly hindered him from applying the operation of *rephrasing an object*.

Example 13 Eric (Algebra II)

Eric's proof shows that students' failure to rephrase an object though algebraic manipulation may cause them to have impasses. In particular, his case showed that students' failure to rephrase a whole sentence or a whole equation can be a factor of hindering them from advancing a reasoning process.

Question [9] (4) (In-class problem solving session)

Let R and S be rings. Let $\phi : R \to S$ is a ring homomorphism.
$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a well-defined ring homomorphism.
Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective.

Table 5.14 shows a possible proof and where Eric had difficulties in the proof construction.

Table 5.14

Analysis (Type A) on Eric's Proof

	Proof	Code	E
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective.	Given	
Y	Show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$.	R1	S
P1	$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a well-defined ring homomorphism.	Given	
P2	$\phi : R \to S$ is a ring homomorphism.	Given	
1	Suppose that $\psi([r]) = \psi([s])$.	C1	S
	Then, $\phi(r) = \phi(s)$.	CO(1,P1)R1	I
2	Then, $0_S = \phi(r) - \phi(s) = \phi(r - s)$.	CO(2,P2)R3	N
3	Then, $r - s \in Ker(\phi)$.	R2	N
4	Then, $r = s + k$ for some $k \in Ker(\phi)$.	R2	N
5	Then, $r = s + k \in [s]$.	R2	N
6	Then, [r]=[s].	R2	N
	Another Proof		
X	Show $Ker(\psi) = 0_{R/Ker(\phi)}$.		S
1	Consider $Ker(\psi) = \{[r] \in R / Ker(\phi)$, where $\psi([r]) = 0_S \}$.	C1	N
2	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $Ker(\psi) = \{[r] \in R / Ker(\phi)$, where $\phi(r) = 0_S \}$.	CO(1,P1)R1	N
4	Then, $Ker(\psi) = \{[r] \in R / Ker(\phi), r \in Ker(\phi) \}$.	R1	N
5	Then, $Ker(\psi) = \{[r] \in R / Ker(\phi), [r] = Ker(\phi)$.	R1	N
6	Therefore, $Ker(\psi) = 0_{R/Ker(\phi)}$.	R1	N

The following explains how to obtain the above proof. One of the ways to show the function $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective is to show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$. Then, paying attention to the ignition phrase "$\psi([r]) = \psi([s])$," one may start a proving argument with "Suppose that $\psi([r]) = \psi([s])$." Noting the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r]) = \psi([s])$ with $\phi(r) = \phi(s)$ based on the way for $\psi : R/Ker(\phi) \to S$ to be defined. One can further rephrase $\phi(r) = \phi(s)$ with $0_S = \phi(r) - \phi(s) = \phi(r-s)$ to obtain $r - s \in Ker(\phi)$, which can lead them to conclude that $[r] = [s]$.

There is another way to prove the given proof problem. There is a property of an injective homomorphism that $\phi : R \to S$ is an injective ring homomorphism if and only if $Ker(\phi) = 0_R$. Therefore, in order to prove that $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective, one can show $Ker(\psi) = 0_{R/Ker(\phi)}$. Then, one may start with considering $Ker(\psi)$. Applying the definition of $Ker(\psi)$, one may translate it into $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $\psi([r]) = 0_S\}$. Combining the given condition $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, one can further rephrase it with $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $\phi(r) = 0_S\}$. Furthermore, one can rephrase it with $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $r \in Ker(\phi)\}$. Then, they can conclude that $Ker(\psi) = \{[r] = Ker(\phi)\}$, namely, $Ker(\psi) = 0_{R/Ker(\phi)}$. The following figure shows Eric's proof (Figure 5.19).

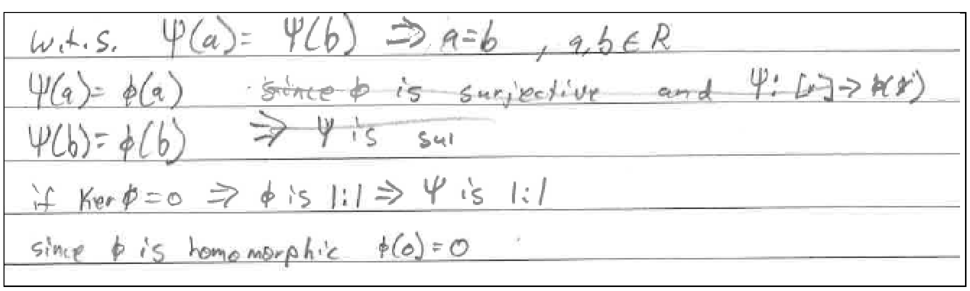


Figure 5.19. Eric's Proof.

Eric had a right proving strategy, trying to show that if $\psi([a]) = \psi([b])$, then $[a] = [b]$. Moreover, he was able to rephrase $\psi([a])$ and $\psi([b])$ with $\phi(a)$ and $\phi(b)$, respectively. However, he was unable to advance his reasoning process after that mainly because he was unable to rephrase the whole equation $\psi([a]) = \psi([b])$ with $\phi(a) = \phi(b)$. He missed the equal sign of the equation. If he had carefully rephrased the whole equation $\psi([a]) = \psi([b])$ with $\phi(a) = \phi(b)$, he might have obtained $0_S = \phi(a) - \phi(b) = \phi(a-b)$ and realized that $a - b \in Ker_{\varphi}$.

Example 14 Berkeley (Algebra II)

Berkeley's case is another representative example showing that failing to rephrase an object through algebraic manipulation can harm their proving arguments. His case also showed that flexibility might be required in rephrasing an object.

> Question [9] (4) In-class problem solving session)
>
> *Let R and S be rings. Let* $\phi: R \to S$ *is a ring homomorphism.*
> $\psi: R / Ker(\phi) \to S : [r] \to \phi(r)$ *is a well-defined ring homomorphism.*
> *Show* $\psi: R / Ker(\phi) \to S : [r] \to \phi(r)$ *is injective.*

Table 5.15 shows a possible proof for the given proof problem and shows where Berkeley had difficulties in the proof construction.

Table 5.15

Analysis (Type A) of Berkeley's Proof

	Proof	Code	B
X	Show $\psi: R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective.	Given	
Y	Show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$.	R1	S
P1	$\psi: R / Ker(\phi) \to S : [r] \to \phi(r)$ is a well-defined ring homomorphism.	Given	
P2	$\phi: R \to S$ is a ring homomorphism.	Given	
1	Suppose that $\psi([r]) = \psi([s])$.	C1	S
	Then, $\phi(r) = \phi(s)$.	CO(1,P1)R1	I
2	Then, $0_S = \phi(r) - \phi(s) = \phi(r - s)$.	R3	N
3	Then, $r - s \in Ker(\phi)$.	R2	N
4	Then, $r = s + k$ for some $k \in Ker(\phi)$.	R2	N
5	Then, $r = s + k \in [s]$.	R2	N

6	Then, [r] = [s].	R2	N
	Another Proof		
X	Show $Ker(\psi) = 0_{R/Ker(\phi)}$.		S
1	Consider $Ker(\psi) = \{[r] \in R/Ker(\phi)$, where $\psi([r]) = 0_S\}$.	C1	N
2	Since $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$, $Ker(\psi) = \{[r] \in R/Ker(\phi)$, where $\phi(r) = 0_S\}$.	CO(1,P1)R1	N
4	Then, $Ker(\psi) = \{[r] \in R/Ker(\phi),\ r \in Ker(\phi)\}$.	R1	N
5	Then, $Ker(\psi) = \{[r] \in R/Ker(\phi),\ [r] = Ker(\phi)$	R1	N
6	Therefore, $Ker(\psi) = 0_{R/Ker(\phi)}$.	R1	N

One of the ways to show the function $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$ is injective is to show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$. Then, paying attention to the ignition phrase "$\psi([r]) = \psi([s])$," one may start a proving argument with "Suppose that $\psi([r]) = \psi([s])$." Noting the given condition "$\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r]) = \psi([s])$ with $\phi(r) = \phi(s)$ based on the way for $\psi : R/Ker(\phi) \to S$ to be defined. One can further rephrase $\phi(r) = \phi(s)$ with $0_S = \phi(r) - \phi(s) = \phi(r - s)$ to obtain $r - s \in Ker(\phi)$, which can lead them to conclude that $[r] = [s]$.

There is another way to prove the given proof problem. There is a property of an injective homomorphism that $\phi : R \to S$ is an injective ring homomorphism if and only if $Ker(\phi) = 0_R$. Therefore, in order to prove that $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$ is injective, one can show $Ker(\psi) = 0_{R/Ker(\phi)}$. Then, one may start with considering $Ker(\psi)$. Applying the definition of $Ker(\psi)$, one may translate it into $Ker(\psi) = \{[r] \in R/Ker(\phi)$, in which $\psi([r]) = 0_S\}$. Combining the given condition $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$, one can further rephrase it with $Ker(\psi) = \{[r] \in R/Ker(\phi)$, in which $\phi(r) = 0_S\}$. Furthermore, one can rephrase it with

$Ker(\psi) = \{[r] \in R/Ker(\phi)$, in which $r \in Ker(\phi)$ }. Then, they can conclude that $Ker(\psi) = \{[r] = Ker(\phi)$ }, namely, $Ker(\psi) = 0_{R/Ker(\phi)}$. The following figure is Berkeley's proof (Figure 5.20).

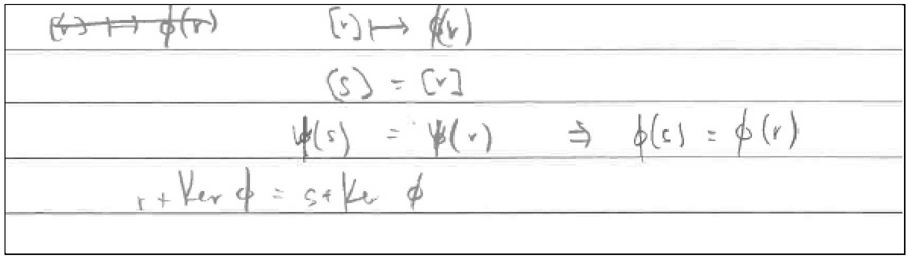


Figure 5.20. Berkeley's Proof.

Providing that if "$\psi(r) = \psi(s)$," then $\phi(r) = \phi(s)$, Berkeley made a good start apart from a minor mistake on his notations. He wanted to have $\psi([r]) = \psi([s])$ instead of having "$\psi(r) = \psi(s)$." Judging from his statement "$r + Ker(\phi) = s + Ker(\phi)$," he seemed to intend to show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$, which would be correct. Moreover, assuming that he meant $\psi([r]) = \psi([s])$ by $\psi(r) = \psi(s)$, he successfully rephrased $\psi([r]) = \psi([s])$ with $\phi(r) = \phi(s)$ by using the given condition "$\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$." However, assuming that he meant $[r] = [s]$ by $r + Ker(\phi) = s + Ker(\phi)$, what he missed was that he was unable to show the process to obtain $[r] = [s]$ from $\phi(r) = \phi(s)$ clearly. He was required to have flexibility to rephrase $\phi(r) = \phi(s)$ with $\phi(r) - \phi(s) = 0$ through algebraic manipulation and to further rephrase $\phi(r) - \phi(s) = 0$ with $r - s \in Ker(\phi)$ to derive $[r] = [s]$.

5.5 Difficulties with Combining Objects

Combining objects is one of the main operations for advancing a reasoning process. An object can be a phrase, a term, part of a sentence, or a whole sentence.

There are several ways to combine objects. One is to combine an object with a given condition or assumption. Another is to combine the objects obtained in the process of advancing a reasoning process. The other is to combine an object with a theorem, a lemma, a proposition, or a property of concept that they are required to recall in the process of reasoning. Failing to combine objects can result in students' having impasses during the process of proof construction and making their proofs incomplete. I will present three examples of students' difficulties with combining objects while showing how their difficulties occurred and affected their proofs: Edward missed using a given hypothesis to make his proof incomplete (Example 15); Berkeley missed using part of given hypotheses (Example 16); and Dominique missed using all the given hypotheses (Example 17).

Example 15 Edward (Topology)

Edward's proof was a representative example showing that students failed to make their proving arguments complete because they failed to combine objects. Among some possible causes of students' failure to combine objects, the cause that was frequently seen was that they missed using a given condition or hypothesis. His case was also such an example, too.

> Question [3.b] (Exam II)
>
> *Let* $q : X \to Y$ *be a quotient map and* $f : Y \to Z$ *be a map. Suppose* $f \circ q$ *is continuous. Show* $f : Y \to Z$ *is continuous.*

Table 5.16 shows a possible proof for Question [3.b] and where Edward had difficulties in the proof construction.

Table 5.16

Analysis Table (Type A) of Edward's Proof

Step	Statement	Operation	E
Opening Stage			
X	$f : Y \to Z$ is continuous.	Given	
Y: OTC	For any open set W in Z, $f^{-1}(W)$ is open in Y.	R1	S
P1: hypothesis	$q : X \to Y$ is a quotient map.	Given	
P2: hypothesis	$f \circ q$ is continuous.	Given	
S: OSV	Let W be an open set in Z.	C1	
Body Construction Stage			
1	Let W be an open set in Z.	C1	S
2	Consider $(f \circ q)^{-1}(W)$.	C5	S
3	Note $(f \circ q)^{-1}(W) = q^{-1}(f^{-1}(W))$.	R1	N
4	Since $f \circ q$ is continuous, $q^{-1}(f^{-1}(W))$ is open in X.	CO(3,P2)R1	N
5	Recall the property of a quotient map.	C2	N
6	Since q is a quotient map, $f^{-1}(W)$ must be open in Y.	CO(4,5)R1	N
7	Therefore, $f : Y \to Z$ is continuous.	R1	N

The following shows a way to obtain the above proof. The conclusion of the given statement is "$f : Y \to Z$ is continuous." The translation of the conclusion into *mathematical language* is "For any open set W in Z, $(f^{-1}(W))$ is open in Y." Noting *the ignition phrase* "For an open set W in Z," one can set a starting variable by having "$W \in T_Z$." Combining the other given condition "$q : X \to Y$ is a quotient map" and the property of a quotient map "If $(q^{-1}(H))$ is open in Z for a quotient map $q : Y \to Z$ and for $H \subset Z$, then H is open in Y," one may conclude $(f^{-1}(W)) \in T_Y$. The following (Figure 5.21) is Edward's proof.

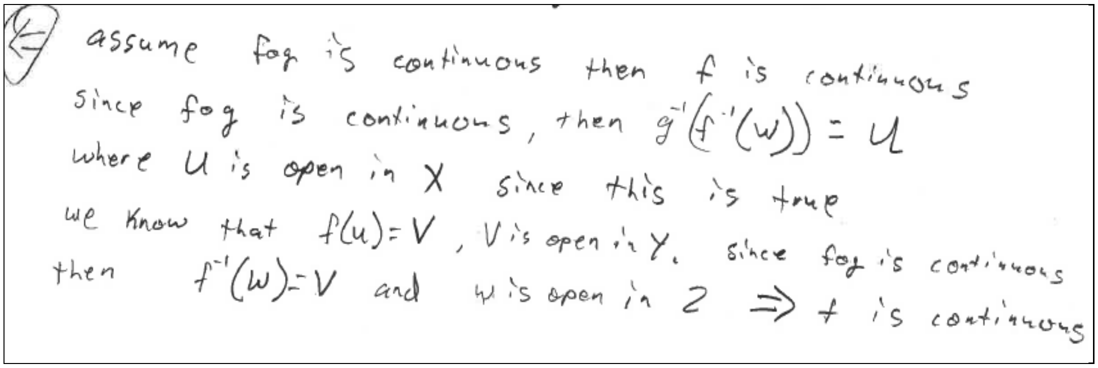

assume fog is continuous then f is continuous
since fog is continuous, then $g^{-1}(f^{-1}(W)) = U$
where U is open in X since this is true
we Know that f(u) = V , V is open in Y. since fog is continuous
then $f^{-1}(W) = V$ and W is open in Z ⇒ f is continuous

Figure 5.21. Edward’s Proof.

Edward was unable to complete the proof mainly because he missed using the given condition that $q : X \to Y$ was a quotient map. He dealt with the opening stage successfully by setting an open set $W \in T_Z$ and by trying to show $f^{-1}(W) \in T_Y$. He further successfully combined the starting variable $W \in T_Z$ with the given condition “$f \circ q$ is continuous” to obtain $q^{-1}(f^{-1}(W)) \in T_X$. Then, the only thing that was left for him to show was $f^{-1}(W) \in T_Y$. To show $f^{-1}(W) \in T_Y$, he had to combine $q^{-1}(f^{-1}(W)) \in T_X$ and the other given condition “$q : X \to Y$ is a quotient map,” which he missed. It is important for students to make sure if they have used all the given conditions.

Example 16 Berkeley (Algebra II)

Berkeley’s case was another example showing that students’ failure to use a given condition can cause weak or incomplete proofs. Her example also implied carefulness and flexibility were required in practicing the operation of combining objects.

Question [9] (3) (In-class problem solving session)

Let R and S be rings. Let $\phi : R \to S$ *is a ring homomorphism.*
Consider a map $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.
Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ *is a ring homomorphism.*

Table 5.17 shows a possible proof for the given proof problem and shows where Berkeley had difficulties in the proof construction.

Table 5.17

Analysis (Type A) of Berkeley's Proof

Object	Proof	Code	B
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism.	Given	
Y	Show (i) $\psi([r]+[s]) = \psi([r]) + \psi([s])$ and (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$.		
P1	$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.		
P2	$\phi : R \to S$ is a ring homomorphism.	Given	
	(i)		
1	Let $[r],[s] \in R / K(\phi)$.	C1	S
2	Consider $\psi([r]+[s])$.	C1	S
3	Note that $\psi([r]+[s]) = \psi([r+s])$.	R2	S
4	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\psi([r+s]) = \phi(r+s)$.	CO(3,P1)R1	N
5	Since $\phi : R \to S$ is a homomorphism, $\phi(r+s) = \phi(r) + \phi(s)$.	CO(4, P2)R1	N
6	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r) + \phi(s) = \psi([r]) + \psi([s])$.	CO(5,P1)R1	N
7	Then, $\psi([r]+[s]) = \psi([r]) + \psi([s])$.	CO(2-6)R2	I
	(ii)		
8	Consider $\psi([r][s])$.	C1	S
9	Note $\psi([r][s]) = \psi([rs])$.	R1	S
10	Then, $\psi([rs]) = \phi(rs)$.	CO(9,P1)R1	N
11	Then, $\phi(rs) = \phi(r)\ \phi(s)$.	CO(10, P2)R1	N
12	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$.	CO(11, P1)R1	N
13	Therefore, $\psi([r][s]) = \psi([r])\ \psi([s])$.	CO(8-12)R2	I

The goal of the proof is "$\psi : R / Ker(\phi) \rightarrow S : [r] \rightarrow \phi(r)$ is a ring homomorphism." There are two things to show: (i) $\psi([r]+[s]) = \psi([r+s])$; (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$. For (i), one can rephrase $\psi([r]+[s])$ with $\psi([r+s])$ through algebraic manipulation. Using the given condition "$\psi : R / Ker(\phi) \rightarrow S : [r] \rightarrow \phi(r)$," one can rephrase $\psi([r+s])$ with $\phi(r+s)$. Using another given condition "$\phi : R \rightarrow S$ is a ring homomorphism," they can rephrase $\phi(r+s)$ with $\phi(r)+\phi(s)$. Using the given condition "$\psi : R / Ker(\phi) \rightarrow S : [r] \rightarrow \phi(r)$" again, one can derive that $\phi(r)+\phi(s) = \psi([r]) + \psi([s])$ to conclude that) $\psi([r]+[s]) = \psi([r]) + \psi([s])$.

Similarly, for (ii), one starts with considering $\psi([r][s])$. One can rephrase $\psi([r][s])$ with $\psi([rs])$ through algebraic manipulation. Using the given condition "$\psi : R / Ker(\phi) \rightarrow S : [r] \rightarrow \phi(r)$," one can further rephrase $\psi([rs])$ with $\phi(rs)$. Using the other condition "$\phi : R \rightarrow S$ is a ring homomorphism," one can rephrase $\phi(rs)$ with $\phi(r)\ \phi(s)$. Using the condition "$\psi : R / Ker(\phi) \rightarrow S : [r] \rightarrow \phi(r)$" again, one can derive $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$ to conclude that $\psi([r][s]) = \psi([r])\ \psi([s])$. The following (Figure 5.22) is Berkeley's proof.

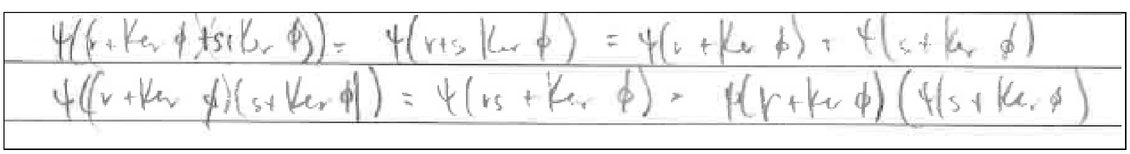


Figure 5.22. Berkeley's Proof.

Berkeley was unable to prove the given statement successfully mainly because he was unable to use the given condition that $\psi : R / Ker(\phi) \rightarrow S : [r] \rightarrow \phi(r)$, where $\phi : R \rightarrow S$ was a ring homomorphism. He knew what he needed to show in order to prove $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ was a ring homomorphism. However, he was unable to show why $\psi([r+s]) = \psi([r]) + \psi([s])$. In particular, he missed using a given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" to rephrase $\psi([r+s])$ with $\phi(r+s)$. He needed to be careful enough to realize that $\psi([r+s]) = \psi([r]) + \psi([s])$ was the very statement that he needed to prove and was not what he was able to obtain for free. He also needed to have flexibility to combine $\psi([r+s])$ and the given condition $\phi : R \to S$ and $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ so that he might rephrase $\psi([r+s])$ with $\psi([r]) + \psi([s])$.

Similarly, he was unable to show why $\psi([rs]) = \psi([r])\psi([s])$. He missed combining the object $\psi([rs])$ and the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" in order to rephrase $\psi([rs])$ with $\phi(rs)$. When he had $\psi([rs]) = \psi([r])\psi([s])$, he needed to be careful enough to question himself why the equality of the equation was able to hold and to look for another information that might lead him to $\psi([rs]) = \psi([r])\psi([s])$. Having carefulness to check what has been done and having flexibility to make good use of all the given conditions might play an important role in *combining objects*.

Example 17 Dominique (Algebra II)

Every single information given as a hypothesis or condition is important and necessary in constructing a proof. Example 16 showed students' failure to use part of the given conditions made their proving arguments weaker or incomplete. Students' missing using all the given conditions can lead to their complete failure to make a proof. Dominique's proof is such an example.

Question [9] (3) (In-class problem solving session)

Let R and S be rings. Let $\phi: R \to S$ *is a ring homomorphism. Consider a map* $\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$. *Show* $\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$ *is a ring homomorphism.*

Table 5.18 shows a possible proof for the given proof problem and shows where Louis had difficulties in the proof construction.

Table 5.18

Analysis (Type A) of Dominique's Proof

Object	Proof	Code	D
X	Show $\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$ is a ring homomorphism.	Given	
Y	Show (i) $\psi([r]+[s]) = \psi([r]) + \psi([s])$ and (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$		
P1	$\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$.		
P2	$\phi: R \to S$ is a ring homomorphism.	Given	
	(i)		
1	Let $[r],[s] \in R/K(\phi)$	C1	
2	Consider , $\psi([r]+[s])$	C1	
3	Note that $\psi([r]+[s]) = \psi([r+s])$.	R2	
4	Since $\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$, $\psi([r+s]) = \phi(r+s)$.	CO(3,P1)R1	
5	Since $\phi: R \to S$ is a homomorphism, $\phi(r+s) = \phi(r) + \phi(s)$.	CO(4, P2)R1	
6	Since $\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$, $\phi(r) + \phi(s) = \psi([r]) + \psi([s])$.	CO(5,P1)R1	
7	Then, $\psi([r]+[s]) = \psi([r]) + \psi([s])$.	CO(2-6)R2	
	(ii)		
8	Consider $\psi([r][s])$.	C1	
9	Note $\psi([r][s]) = \psi([rs])$.	R1	
10	Then, $\psi([rs]) = \phi(rs)$.	CO(9,P1)R1	
11	Then, $\phi(rs) = \phi(r)\ \phi(s)$.	CO(10, P2)R1	
12	Since $\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$,	CO(11, P1)R1	

	$\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$.		
13	Therefore, $\psi([r][s]) = \psi([r])\ \psi([s])$.	CO(8-12)R2	

The goal of the proof is "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism." There are two things to show: (i) $\psi([r]+[s]) = \psi([r+s])$; (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$. For (i), one can rephrase $\psi([r]+[s])$ with $\psi([r+s])$ through algebraic manipulation. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r+s])$ with $\phi(r+s)$. Using another given condition "$\phi : R \to S$ is a ring homomorphism," they can rephrase $\phi(r+s)$ with $\phi(r)+\phi(s)$. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" again, one can derive that $\phi(r)+\phi(s) = \psi([r])+\psi([s])$ to conclude that) $\psi([r]+[s]) = \psi([r])+\psi([s])$.

Similarly, for (ii), one starts with considering $\psi([r][s])$. One can rephrase $\psi([r][s])$ with $\psi([rs])$ through algebraic manipulation. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can further rephrase $\psi([rs])$ with $\phi(rs)$. Using the other condition "$\phi : R \to S$ is a ring homomorphism," one can rephrase $\phi(rs)$ with $\phi(r)\ \phi(s)$. Using the condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" again, one can derive $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$ to conclude that $\psi([r][s]) = \psi([r])\ \psi([s])$. The following is Dominique's proof (Figure 5.23).

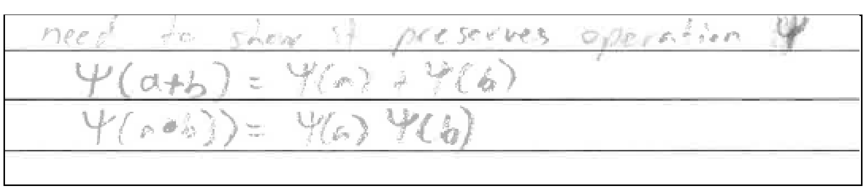


Figure 5.23. Dominique's Proof.

Dominique knew exactly what he needed to show, but his proof was not successful partly because his notations were incorrect, partly because he lacked his alertness in advancing his reasoning process, and partly because he failed to combine objects, and. He provided $\psi(a+b)$ though he was supposed to provide $\psi([a]+[b])$. His use of incorrect notations may be attributed to his lack of carefulness in making sure of how the homomorphism $\psi : R/Ker(\phi) \to S$ was defined. Also, he was not alert enough to realize what he showed was exactly what he was asked to prove. He needed to ask himself why he might say $\psi([a]+[b]) = \psi([a])+\psi([b])$. However, the most crucial source of his incomplete argument might be that he was not able to note and utilize the given condition $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$ when he was trying to change $\psi([a]+[b])$ into $\psi([a])+\psi([b])$. By using the condition, he might have had $\psi([r]+[s]) = \psi([r+s]) = \phi[r+s] = \phi[r]+\phi[s] = \psi([r])+\psi([s])$.

5.6 Difficulties with Creating a Cue

Creating a cue is another major operation for advancing a reasoning process in proof construction. There are four ways to create a cue: (1) to set a variable; (2) to recall a theorem, a lemma, a proposition, and a property of concept, and engage it in a proving argument; (3) to set some cases; (4) make a claim; (5) and consider an object. *Creating a cue* can be considered as the highest level of operation of the three main operations (*rephrasing an object*, *combining objects*, and *creating a cue*) in terms of the difficulty. While the operations of *rephrasing an object* and *combining objects* allow students to directly use the object that is already given or obtained, the operation of *creating a cue* requires students to come up with a new object without having them directly depend on the objects that have already existed. The results also implied that the factors of students' *background knowledge* and their *mental attitudes* can be closely related to their use of the operation of *creating a cue*. I will present three examples of students' difficulties with creating a cue while showing how their difficulties occurred and affected their proofs: Eliot was unable to create a new object that helped him further advance a reasoning process (Example 18): Elgar failed to derive a right starting variable from a hypothesis of the given statement (Example 19); and Kyle failed to recall and apply prior knowledge (Example 20).

Example 18 Eliot (Analysis)

Eliot's case is a representative example showing that students' difficulty with *creating a cue* can cause students to produce an incomplete proof. In particular, he had a difficulty to create a new function to be considered. His case also implied flexibility might be an important factor that allowed students to create a cue.

> Question [7] (Final Exam)
>
> *Let* $f:[0,1]\to[0,1]$ *be continuous. Prove that there exists a number* $x\in[0,1]$ *such that* $f(x)=x$.

Table 5.19 shows a possible proof for Question [7] and where Eliot had a difficulty in the proof construction.

Table 5.19

Analysis (Type A) of Eliot's Proof

	Proof	Code	E
X	Prove that there exists a number $x\in[0,1]$ *such that* $f(x)=x$.		
P	$f:[0,1]\to[0,1]$ is continuous.	Given	
1	Define $g:[0,1]\to[-1,1]$ by $g(x)=f(x)-x$.	C1	S
2	Consider $g(x)=f(x)-x$.	C5	N
3	Recall the Intermediate Value Theorem.	C2	N

4	Since $0 \in (-1,1)$, there exists $c \in (0,1)$ such that $g(c) = f(c) - c = 0$.	CO(2.3)R1	N
5	Therefore, there exists $c \in (0,1)$ such that $f(c) = c$.	R2	N

Considering using the Intermediate Value Theorem, one may create a continuous function $g:[0,1] \to [-1,1]$ by defining $g(x) = f(x) - x$. Noting that $0 \in [-1,1]$ and applying the theorem to the function $g(x)$, one may derive $c \in (0,1)$ such that $g(c) = f(c) - c = 0$. Then, one can conclude there exists $c \in [0,1]$ such that $f(c) = c$. The following (Figure 5.24) is Eliot's proof.

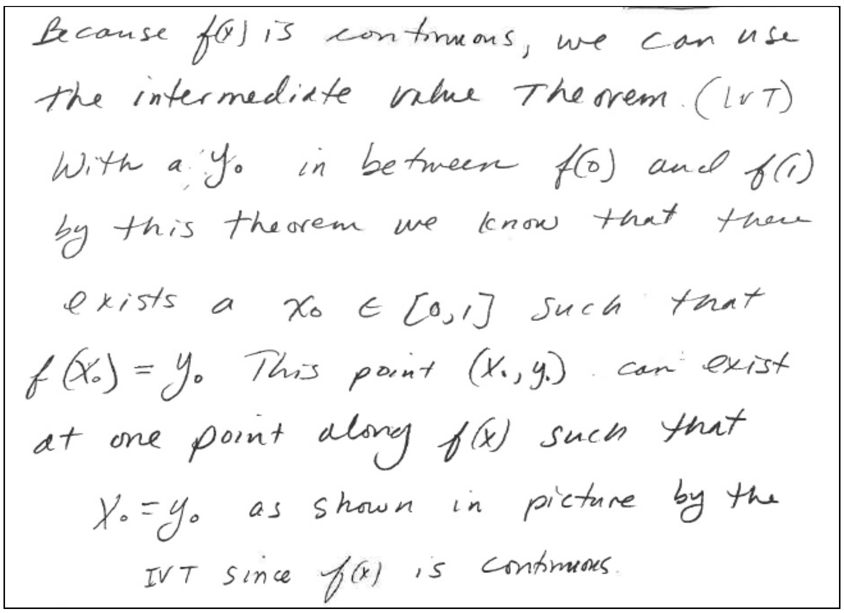

Because f(x) is continuous, we can use
the intermediate value Theorem. (IVT)
With a y₀ in between f(0) and f(1)
by this theorem we know that there
exists a x₀ ∈ [0,1] such that
f(x₀) = y₀ This point (x₀, y₀) can exist
at one point along f(x) such that
x₀ = y₀ as shown in picture by the
IVT since f(x) is continuous.

Figure 5.24. Eliot's Proof.

Eliot thought about applying Intermediate Value Theorem to the given function $f:[0,1] \to [0,1]$. Then, he set y_0 which is between $f(0)$ and $f(1)$. Then, as he stated, there existed $x_0 \in (0,1)$ such that $f(x_0) = y_0$. However, he was unable to show $y_0 = x_0$ clearly. One of the causes of his difficulties was that he was required to have flexibility to create and consider a new continuous function $g:[0,1] \to [-1,1]$ by defining

$g(x) = f(x) - x$ so that he might apply the theorem to the function. If he had set $g:[0,1] \to [-1,1]$ by defining $g(x) = f(x) - x$, he might have derived $c \in (0,1)$ such that $g(c) = f(c) - c = 0$ by applying the intermediate value theorem.

Example 19 Elgar (Topology)

Setting a variable is one of the major types for *creating a cue*. A variable is a key unit for advancing a reasoning process. Without a variable, students cannot construct a rigorous proving argument. In addition, students are often required to create a variable in their proofs. It is crucial for them to be able to set a variable. However, it can be difficult. Elgar's case is such a representative example.

> Question [6] (Exam II)
>
> *Let X, Y be topological spaces; Y be compact;* $x_0 \in X$ *; N be an open set containing* $\{x_0\} \times Y$ *in the product space* $X \times Y$. *Prove that there exists an open neighborhood* $W \subset X$ *of* x_0 *such that* $W \times Y \subset N$.

The following (Table 5.20) is a possible proof for the given problem and shows where Elgar had difficulties in the proof construction.

Table 5.20

Analysis (Type A) of Elgar's Proof

	Proof	Code	E
X	Construct an open neighborhood $W \subset X$ of x_0 such that $W \times Y \subset N$.		
P1	N is an open set containing $\{x_0\} \times Y$.		
P2	Y is compact.		
1	Since N is open in $X \times Y$, for each $(x_0, y) \in \{x_0\} \times Y$, there exists a basis open set $U_y \times V_y \in T_X \times T_Y \subset N$ containing (x_0, y) for each $y \in Y$.	C1	N
2	Then, $\{\{x_0\} \times V_y\}$ is an open cover of $\{x_0\} \times Y$.	R1	N
3	Note $\{x_0\} \times Y$ is homeomorphic to Y.	C2	S

4	Then, $\{x_0\}\times Y$ is compact.	CO(P2,3)R2	S
5	Then, there exists a finite open subcover $\{\{x_0\}\times V_{y_i} \in \{x_0\}\times T_Y\}\}$, where $i \in \{1,\ldots,n \mid n \in Z^+\}$ and $\bigcup_{i=1}^{n} V_{y_i} = Y$.	CO(2,4) R1	N
6	Note that $\{x_0\}\times Y \subset \{U_{y_i} \times V_{y_i}\} \subset N$ . . , where $i \in \{1,\ldots,n \mid n \in Z^+\}$.	C1	N
7	Let $W = \bigcap_{i=1}^{n} U_i$, where W is an open neighborhood of x_0 , where $\{x_0\}\times Y \subset W\times Y \subset N$.	C1	N

The conclusion of the given statement is "*there exists an open neighborhood* $W \subset X$ *of* x_0 *such that* $W\times Y \subset N$." Noting the given condition "*N is an open set containing* $\{x_0\}\times Y$ "and recalling the property of an open set, one can set a starting variable $U_y \times V_y \subset N$ as an open neighborhood of (x_0, y) for each $y \in Y$." Noting another given condition "*Y* is compact" and realizing $\{x_0\}\times Y$ is homeomorphic to *Y*, one can construct an open cover of $\{x_0\}\times Y$ by having $\{ U_y \times V_y \mid U_y \times V_y \subset N$ and $y \in Y \}$. Since $\{x_0\}\times Y$ is compact, there must exist a finite open subcover $\{\{x_0\}\times V_{y_i} \in \{x_0\}\times T_y\}$, in which $i \in \{1,\ldots,n \mid n \in Z^+\}$ and $\bigcup_{i=1}^{n} V_{y_i} = Y$. Then, one may construct $W = \bigcap_{i=1}^{n} U_i$ so that $\{x_0\}\times Y \subset W\times Y \subset N$. The following (Figure 5.25) is Elgar's proof.

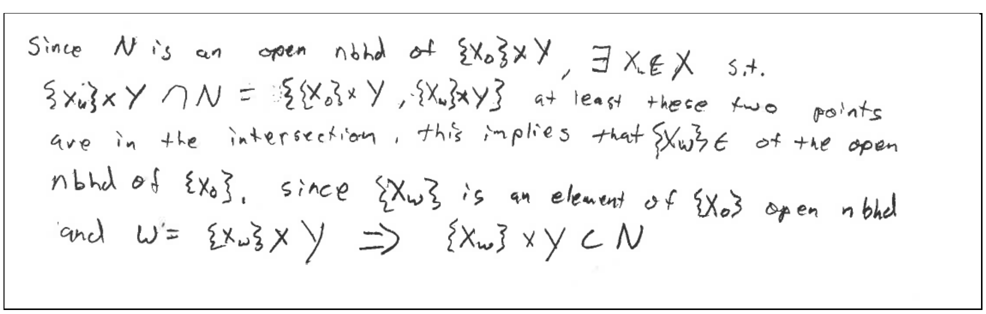

Since N is an open nbhd of {X₀}×Y , ∃ X ∈ X s.t.
{Xw}×Y ∩ N = {{X₀}×Y , {Xw}×Y} at least these two points
are in the intersection , this implies that {Xw} ∈ of the open
nbhd of {X₀}. since {Xw} is an element of {X₀} open nbhd
and W = {Xw}×Y ⇒ {Xw}×Y ⊂ N

Figure 5.25. Elgar's Proof.

Elgar had some difficulties in his proof. He stated that there existed $x_w \in X$ such that $\{x_0\}\times Y \cap N = \{\{x_0\}\times Y, \{x_w\}\times Y\}$, which was wrong because $\{x_0\}\times Y \cap N = \{x_0\}\times Y$. Also, he was not very careful about what he provided at the end of his proving argument. He was supposed to provide an open neighborhood *W* of x_0 such that $\{x_0\}\times Y \subset W \times Y \subset N$, but what he provided was not an open neighborhood of x_0. A more serious factor that made his proving argument unsuccessful might have lain in his difficulty with creating a variable for developing a proving argument. Unlike most other proofs examined in this, this proof problem required students to derive a starting variable from not the conclusion but the hypothesis of the given statement. Although Elgar noted the given hypothesis "*N* is an open set containing $\{x_0\}\times Y$ in the product space $X \times Y$" at the beginning, he was unable to set a variable from the hypothesis, which was a basis open set $U_y \times V_y \in T_X \times T_Y \subset N$ containing (x_0, y) for each $y \in Y$. A possible cause of his difficulty with creating the open basis might be that he might not have translated "*N* is an open set containing $(x_0, y) \in \{x_0\}\times Y$" into *mathematical language* "there exists a basis open set $U_y \times V_y \in T_X \times T_Y \subset N$ containing (x_0, y) for each $y \in Y$." Students were required to derive "an open neighborhood $U_y \times V_y \in T_X \times T_Y$ of each point $(x_0, y) \in \{x_0\}\times Y$" by noting "$(x_0, y) \in \{x_0\}\times Y$ for each $y \in Y$." If Elgar had $U_y \times V_y \in T_X \times T_Y \subset N$, he might have thought of making an open cover of $\{x_0\}\times Y$ so that he might have used the condition of *Y* being compact.

Example 20 Kyle (Analysis I)

Recalling a relevant theorem, proposition, or property and bringing it in to a proving argument is one of the ways to advance a reasoning process in proof construction. The theorem, proposition, or property brought in from outside can work as a cue that helps students to move on. Students' failure to create a cue through recalling and applying a theorem, proposition, or property can be a crucial factor for hindering them from advancing a reasoning process. Kyle's proof is such a representative example.

In the majority of proofs analyzed in this study, a starting variable was drawn from the conclusion of the given statement. However, there were a few proofs in which students had to derive a starting variable from a phrase or a statement other than the conclusion of the given statement. For example, there was a type of proof in which students had to derive a starting variable from a given hypothesis of the given statement. There was also a type of proof in which students had derive a starting variable from a proposition or a theorem that students were required to recall at the beginning of the proof. Kyle's proof problem belonged to this type. He had to derive a starting variable from a property he needed to recall. Kyles' proof showed, however, Although the operation can be imperative, that it might be difficult for students to hit on, recall, and choose a right one from their prior knowledge.

> Question [1] (Exam III)
>
> *Let* $a < b$ *is fixed. Suppose that* $g_n \geq 0$ *is a sequence of Riemann integrable functions such that* $\lim_{n\to\infty}\int_a^b g_n(x)dx = 0$. *Prove that if f is Riemann integrable on (a, b), then* $\lim_{n\to\infty}\int_a^b f(x)g_n(x)dx = 0$.

Table 5.21 shows a possible proof for Question [1] and shows where Kyle had difficulties in the proof construction.

Table 5.21

Analysis (Type A) of Kyle's Proof

	Proof	Code	K
X	Show $\lim_{n\to\infty}\int_a^b f(x)g_n(x)dx = 0$.	Given	
Y	Show $\lim_{n\to\infty}\left\|\int_a^b f(x)g_n(x)dx\right\| = 0$.	C5	N
P1	$g_n \geq 0$ is a sequence of Riemann integrable functions such that $\lim_{n\to\infty}\int_a^b g_n(x)dx = 0$.	Given	
P2	$f(x)$ is Riemann integrable on (a, b).	Given	
1	Recall $\left\|\int_s^t h(x)dx\right\| \leq \int_s^t \|h(x)\|dx$.	C2	N
2	Then, $\left\|\int_a^b f(x)g_n(x)dx\right\| \leq \int_a^b \|f(x)g_n(x)\|dx$.	CO(1,Y)R2	N
3	Since $g_n(x) \geq 0$, $\int_a^b \|f(x)g_n(x)\|dx = \int_a^b \|f(x)\|g_n(x)dx$.	CO(2, P1)R2	N
4	Since *f* is Riemann integrable, *f* is bounded, namely, $\|f(x)\| \leq M$ for some $M \in R$.	C2	I
5	Then, $\int_a^b \|f(x)\|g_n(x)dx \leq \int_a^b Mg_n(x)dx$.	CO(3, 4)R2	I
6	Note $\int_a^b Mg_n(x)dx = M\int_a^b g_n(x)dx$.	R3	N
7	Therefore, we have now $0 \leq \left\|\int_a^b f(x)g_n(x)dx\right\| \leq M\int_a^b g_n(x)dx$.	CO(2, 6)R2	N
8	Consider $M\int_a^b g_n(x)dx$	C5	N

9	Since $\lim_{n\to\infty}\int_a^b g_n(x)dx = 0$, $\lim_{n\to\infty} M\int_a^b g_n(x)dx = 0$.	CO(P1, 8)R2	N
10	Therefore, $\lim_{n\to\infty}\left|\int_a^b f(x)g_n(x)dx\right| = 0$ as is desired.	CO(7, 9)R2	N

One of the ways to solve this problem is to bring in and apply the proposition that if $\lim_{n\to\infty}|A_n| = 0$, then $\lim_{n\to\infty} A_n = 0$. The conclusion of the given problem "Show $\lim_{n\to\infty}\int_a^b f(x)g_n(x)dx = 0$" can be translated into "Show $\lim_{n\to\infty}\left|\int_a^b f(x)g_n(x)dx\right| = 0$."

Recalling and applying "$\left|\int_s^t h(x)dx\right| \le \int_s^t |h(x)|dx$"to $\lim_{n\to\infty}\left|\int_a^b f(x)g_n(x)dx\right|$, one can obtain $\left|\int_a^b f(x)g_n(x)dx\right| \le \int_a^b |f(x)g_n(x)|dx$. Then, noting the given condition "$g_n(x) \ge 0$," one can obtain $\int_a^b |f(x)g_n(x)|dx = \int_a^b |f(x)|g_n(x)dx$. Noting another given condition "$f(x)$ is Riemann integrable" and recalling the property of a Riemann integrable function, which is "if $f(x)$ is Riemann integrable, $f(x)$ is bounded," one can obtain "$0 \le \left|\int_a^b f(x)g_n(x)dx\right| \le M\int_a^b g_n(x)dxf$ for some $M \in R$." By using the other condition "$\lim_{n\to\infty}\int_a^b g_n(x)dx = 0$" and applying the squeeze theorem, one can conclude that $\lim_{n\to\infty}\left|\int_a^b f(x)g_n(x)dx\right| = 0$. The following (Figure 5.26) is Kyle's proof.

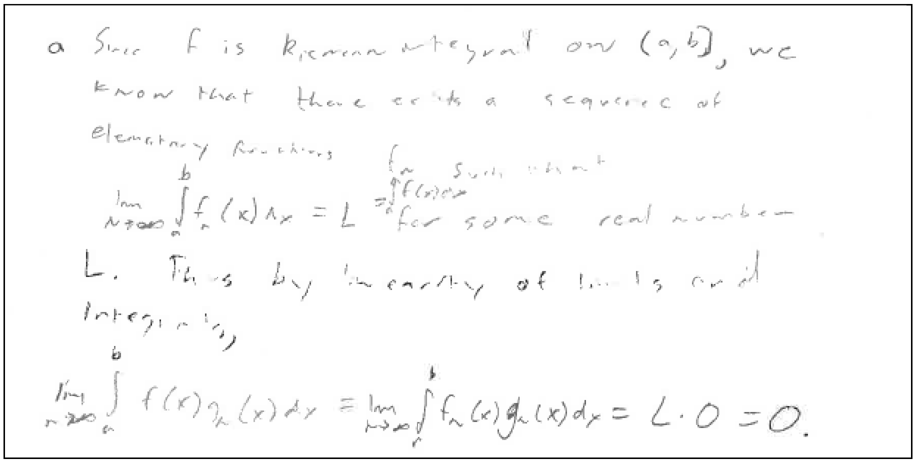


Figure 5.26. Kyle's Proof.

One of the difficulties that Kyle had was that he was unable to think of using the proposition "$\lim_{n\to\infty} |A_n| = 0$, then $\lim_{n\to\infty} A_n = 0$." If Kyle had considered $\lim_{n\to\infty} \left| \int_a^b f(x)g_n(x)dx \right|$, he could have advanced his reasoning process by coming up with the idea that $\left| \int_a^b f(x)g_n(x)dx \right| \le \int_a^b |f(x)g_n(x)|dx$. Another difficulty that he had was that he was unable to recall the proposition "if $f(x)$ is Riemann integrable, $f(x)$ is bounded, namely, $|f(x)| \le M$ for some $M \in R$." If he had known that, he could have considered $0 \le \lim_{n\to\infty} \left| \int_a^b f(x)g_n(x)dx \right| \le \lim_{n\to\infty} M \int_a^b g_n(x)dx = 0$ to conclude that $\lim_{n\to\infty} \left| \int_a^b f(x)g_n(x)dx \right| = 0$. This example also showed Kyle was required to have flexibility to recall and choose a necessary theorem, proposition, or property from their prior knowledge and apply it to a given proof problem.

5.7 Difficulties with Checking and Exploring

Although this study did not focus on the fourth operation of *Checking and Exploring* so much as others, it is a crucial, indispensable, and essential operation for making proving arguments successful. For instance, it is important for students to check what they have done, adjust or correct their ideas, and make another attempt if necessary. Practicing the operation of checking can be closely related to students' *mental attitudes*, especially carefulness and alertness. Two examples will be presented to show students' difficulties with checking and exploring: Curt failed to check what he came up with (Example 21); Ryan tried a property that was not helpful but did not check the effectiveness of the property (Example 22).

Example 21 Curt (Algebra I)

Curt's case is a representative example showing that students' failure to check what they have done can cause them to produce unsuccessful proofs.

> Question [5] (In-class problem solving session)
>
> *Suppose that the order of G is a prime number. Prove that G is cyclic.*

Table 5.22 shows a possible proof for Question [5] and shows where Curt had difficulties in the proof construction.

Table 5.22

Analysis (Type A) of Curt's Proof

	Proof	Code	
X	Show G is cyclic.		
Y	Show $G=<g>$ for some $g \in G$ with $g \neq 1$.	R1	I
P	The order of G is a prime number.	Given	
S	Let $g \in G$ with $g \neq 1$.	C1	N
1	Let $g \in G$ with $g \neq 1$.	C1	N
2	Consider $<g>$.	C5	N

3	Note $< g >$ is a subgroup of G.	C2	N
4	Recall the Lagrange's THM and apply it to $< g >$.	C2	N
5	Then, by the Lagrange's THM, $\lvert < g > \rvert = 1, p$.	CO(3,4)R2	N
6	Since $\lvert < g > \rvert \neq 1$, $\lvert < g > \rvert = p$.	CO(1,5)R2	N
7	Since $\lvert G \rvert = p$, $G =< g >$.	R1	N

The conclusion of the given statement is "*G is abelian*." The translation of it into *mathematical language* provides "$ab = ba$ for any $a, b \in G$." The phrase in the *mathematical language*, "for any $a, b \in G$," is the ignition phrase. One can derive starting variables from the *ignition phrase* and provide "Suppose $a, b \in G$." This problem is the type of the proof of showing A = B. One can work on either A or B until A changes into B or B changes into A while making good use of the given conditions. In this case, one can attempt to rephrase the left side of the equation "ab" until it changes into "ba." To rephrase "ab," one may consider the given hypothesis "$G / Z(G)$ *is cyclic*" and look for the connection between the starting variables "a, b" and the hypothesis "$G / Z(G)$ *is cyclic*." Recalling the property that an element of G belongs to some coset, one may produce $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$. Then, one can rephrase ab with $x^{m+n} z_1 z_2$ for some $z_1, z_2 \in Z$. Using the property of the center of a group, one may derive $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$. The following are Curt's proving strategy (Figure 5.27) and his proof (Figure 5.28).

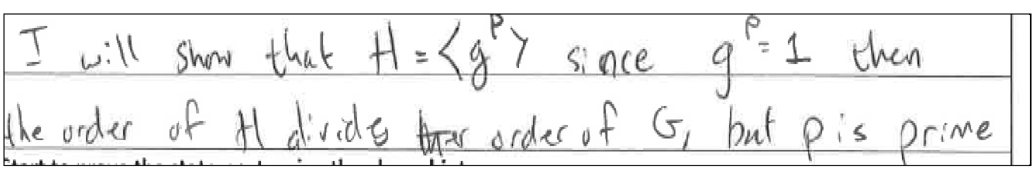


Figure 5.27. Curt's Strategy.

His statement for his proving strategy had two problems. First, he had a difficulty with his notation. He stated "I will show that $H = < g^p >$." He probably meant $H = < g >$ by $H = < g^p >$. He needed to be careful to realize that $H = < g^p > = \{e\}$, which would be a trivial case. Another problem was that even when he showed $H = < g >$, that would not have led him to reach the conclusion "*G* is cyclic."

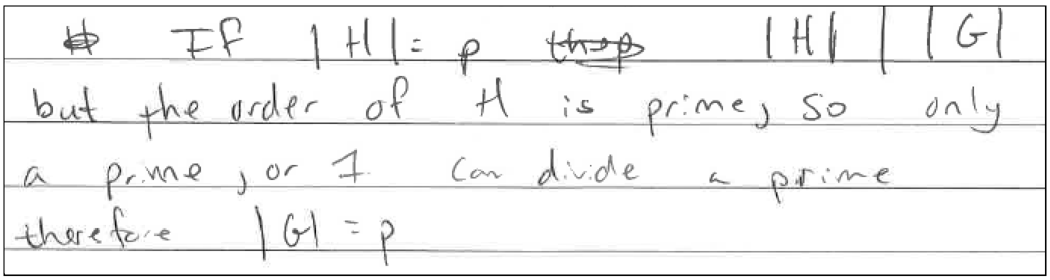

~~#~~ If $|H| = p$ ~~then~~ $|H| \mid |G|$
but the order of H is prime, so only
a prime, or 1 can divide a prime
therefore $|G| = p$

Figure 5.28. Curt's Proof.

Curt further argued that since $|H| = p$ divided $|G|$, $|G| = p$, which was false. He again needed carefulness to realize that $|G|$ might be multiple of *p*. Moreover, he ended up with $|G| = p$ as his conclusion, which was not the goal of the proof he was supposed to show. By making sure of the goal, he might have changed his proving arguments.

Example 22 Ryan (Topology)

Rayan's case was a representative example showing that students failed to make a successful proving argument because they tired applying their prior knowledge that was not necessary nor helpful for solving the given proof problems.

Question [7] (Exam II)

Let X be a Hausdorff space. Let $\{x_n : n \in Z^+\}$ be a sequence in X converging to a point x_0. Prove that the set $K = \{x_n : n = 0,1,2,...\}$ is compact.

Table 5.23 shows a possible proof for the given proof problem and shows where Ryan had difficulties in the proof construction.

Table 5.23

Analysis (Type A) of Ryan's Proof

Object	Proof	Code	R
X	Show that $K = \{x_n : n = 0,1,2,...\}$ is compact.		
Y	Show that for any open cover of *K*, *K* has a finite open subcover.		
P	$\{x_n : n \in Z^+\}$ be a sequence in *X* converging to x_0.	Given	
1	Let $U = \{U_\alpha \in T_X\}$ be an open cover of *X*.	C1	N
2	Construct an open cover of *K* by letting $V = \{V_\alpha = U_\alpha \cap K\}$.	C1	N
3	Since $U = \{U_\alpha \in T_X\}$ is an open cover of *X*, $\exists U_{\alpha_0} \in U$ such that $x_0 \in U_{\alpha_0}$.	R1	N
4	Since x_n converges to x_0, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in U_{\alpha_0}$.	CO(P,3)R1	N
5	Let $V_{\alpha_0} = U_{\alpha_0} \cap K$, where $V_{\alpha_0} \in V$.	C1	N
6	For each x_i with $i < N$, find an open set $V_{x_i} \in V$ such that $x_i \in V_{x_i}$.	C1	N
7	Note that $\{V_{x_1}, V_{x_2}, ..., V_{x_{N-1}}, V_{\alpha_0}\}$ is a desired finite open subcover of *K*.	CO(5,6)R2	N

The conclusion of the given statement is "$K = \{x_n : n = 0,1,2,...\}$ is compact." It can be translated into the *mathematical language* "For any open cover of *K*, *K* has a finite open subcover." By paying attention to *the ignition phrase* "For any open cover of *K*," one may explore the way to construct an open cover of *K*. Recalling the property of a subspace topology, one can set a starting variable by having "Let $U = \{U_\alpha \in T_X\}$ as an open cover of *X*". Then, one may construct an open cover of *K* by having "$V = \{V_\alpha = U_\alpha \cap K\}$." To further advance a reasoning process, one may note and consider the given hypothesis "$\{x_n : n \in Z^+\}$ converges to a point x_0." Then, the given hypothesis can be translated into "For an open set $V_{\alpha_0} = U_{\alpha_0} \cap K$ in the open cover of *K*, in which $x_0 \in U_{\alpha_0}$, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in V_{\alpha_0}$. Finally, they may create a finite open subcover $\{V_{x_1}, V_{x_2}, \ldots V_{x_{N-1}}, V_{\alpha_0}\}$ by setting $V_{x_i} \in V$ such that $x_i \in V_{x_i}$ for $n < N$. The following (Figure 5.29) is Ryan's proof.

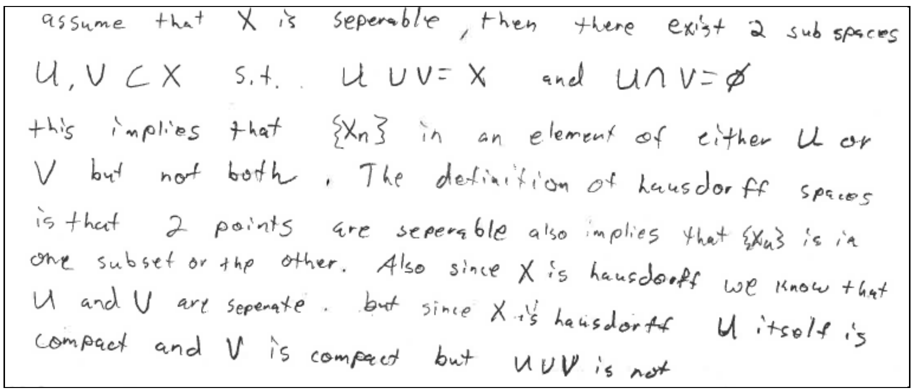

assume that X is seperable, then there exist 2 sub spaces $U, V \subset X$ s.t. $U \cup V = X$ and $U \cap V = \emptyset$ this implies that $\{X_n\}$ in an element of either U or V but not both. The definition of hausdorff spaces is that 2 points are seperable also implies that $\{X_n\}$ is in one subset or the other. Also since X is hausdorff we know that U and V are seperate. but since X is hausdorff U itself is compact and V is compact but $U \cup V$ is not

Figure 5.29. Ryan's Proof.

Ryan's proof was not successful partly because he used a property that was not helpful for solving the given proof problem. He noted the hypothesis of the given statement "*X* is Hausdorff." It seems he got the concepts of a Hausdorff space and a disconnected space mixed. He moved on his proving argument, using the concept of connectedness without realizing it was not helping him. When he made a conclusion, he seemed not to realize that his proving argument was fruitful. What he needed was his carefulness to realize the concept he applied was not helpful and his flexibility to try a different method by noting the other given condition "*K* is a convergent sequence." The problem with their attempt was not the fact that they explored the solution by applying their prior knowledge but the fact that they did not realize that the object they used was not helpful and the fact that they did not question themselves of the effectiveness of the object. The operation of *exploring* and *checking* may be closely related to students' mental attitudes such as flexibility, carefulness, and alertness.

Other multiple factors might also have affected his proof. He did not note and translate the conclusion of the given statement in order to make sure of the goal of the proof. In addition, he started his argument with assuming that the given space was separable, which was not good because he was not supposed to define a given space to be some specific one at his discretion. Moreover, it seems he got the concepts of a separable space and of a disconnected space mixed. A separable space is a space that contains a countable dense subset. Namely, a separable space must contain a sequence of elements of the space such that every open subset of the space contains at least one of the elements of the sequence. Above all, he missed paying a close attention to the conclusion of the given statement and translating it into *mathematical language* to make sure of the goal and to derive a starting variable, which resulted in an invalid proving argument.

5.8 Lack of Background Knowledge

The *background knowledge* is the knowledge necessary for solving a given proof problem, including definitions, properties, notations, theorems, lemmas, propositions, mathematical laws, and proving techniques. Students' lack of *background knowledge* can directly affect and damage their proof construction. Students' lack of knowledge can impede, hinder, and disable them from practicing those operations such as *rephrasing an object* and *creating a cue*. It is imperative for students have, recall, and apply the knowledge necessary for a given problem correctly so that they can make their proving arguments successful. I will present 13 examples while showing what type of knowledge they lacked and how their lack of knowledge affected their proofs: Billy mixed different concepts (Example 23); Savanna lacked elementary and basic knowledge of concepts (Example 24); Davis was unable to recall a relevant property correctly (Example 25); Carlos created a wrong notation (Example 26); Elias had an incomplete understanding of a concept (Example 27); Savanna lacked knowledge of the basics of concepts (Example 28); Donald did not know the definition of a concept (Example 29); Dayton created a wrong property of a concept (Example 30); Anthony did not know a proving strategy (Example 31); Zack did not know a relevant fact (Example 32); Carlos produced a wrong notation (Example 33); and Ben produced a wrong notation due to an incomplete understanding of concepts (Example 34).

Example 23 Billy (Algebra II)

Billy's case is another representative example showing that students' lack of knowledge can directly damage their proving arguments.

> Question [9] (3) (In-class problem solving session)
>
> *Let R and S be rings. Let* $\phi : R \to S$ *is a s ring homomorphism.*
> *Consider a map* $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.
> *Show* $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ *is a ring homomorphism.*

Table 5.24 shows a possible proof for the given proof problem and shows where Louis had difficulties in the proof construction.

Table 5.24

Analysis (Type A) of Billy's Proof

Object	Proof	Code	L
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism.	Given	
Y	Show (i) $\psi([r]+[s]) = \psi([r]) + \psi([s])$ and		I

	(ii) $\psi([r][s]) = \psi([r])\ \psi([s])$		
P1	$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.		
P2	$\phi : R \to S$ is a ring homomorphism.	Given	
	(i)		
1	Let $[r],[s] \in R / K(\phi)$	C1	S
2	Consider , $\psi([r]+[s])$	C1	S
3	Note that $\psi([r]+[s]) = \psi([r+s])$.	R2	S
4	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\psi([r+s]) = \phi(r+s)$.	CO(3,P1)R1	S
5	Since $\phi : R \to S$ is a homomorphism, $\phi(r+s) = \phi(r) + \phi(s)$.	CO(4, P2)R1	S
6	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r) + \phi(s) = \psi([r]) + \psi([s])$.	CO(5,P1)R1	I
7	Then, $\psi([r]+[s]) = \psi([r]) + \psi([s])$.	CO(2-6)R2	I
	(ii)		
8	Consider $\psi([r][s])$.	C1	N
9	Note $\psi([r][s]) = \psi([rs])$.	R1	N
10	Then, $\psi([rs]) = \phi(rs)$.	CO(9,P1)R1	N
11	Then, $\phi(rs) = \phi(r)\ \phi(s)$.	CO(10, P2)R1	N
12	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$.	CO(11, P1)R1	N
13	Therefore, $\psi([r][s]) = \psi([r])\ \psi([s])$.	CO(8-12)R2	N

One of the ways to show the function $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective is to show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$. Then, paying attention to the ignition phrase "$\psi([r]) = \psi([s])$," one may start a proving argument with "Suppose that $\psi([r]) = \psi([s])$." Noting the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r]) = \psi([s])$ with $\phi(r) = \phi(s)$ based on the way for $\psi : R / Ker(\phi) \to S$ to be defined. One can further rephrase $\phi(r) = \phi(s)$ with $0_S = \phi(r) - \phi(s) = \phi(r-s)$ to obtain $r - s \in Ker(\phi)$, which can lead them to conclude that $[r] = [s]$.

There is another way to prove the given proof problem. There is a property of an injective homomorphism that $\phi : R \to S$ is an injective ring homomorphism if and only if $Ker(\phi) = 0_R$. Therefore, in order to prove that $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective, one can show $Ker(\psi) = 0_{R/Ker(\phi)}$. Then, one may start with considering $Ker(\psi)$. Applying the definition of $Ker(\psi)$, one may translate it into $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $\psi([r]) = 0_S\}$. Combining the given condition $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, one can further rephrase it with $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $\phi(r) = 0_S\}$. Furthermore, one can rephrase it with $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $r \in Ker(\phi)\}$. Then, they can conclude that $Ker(\psi) = \{[r] = Ker(\phi)\}$, namely, $Ker(\psi) = 0_{R/Ker(\phi)}$. The following (Figure 5.30) is Billy's proof.

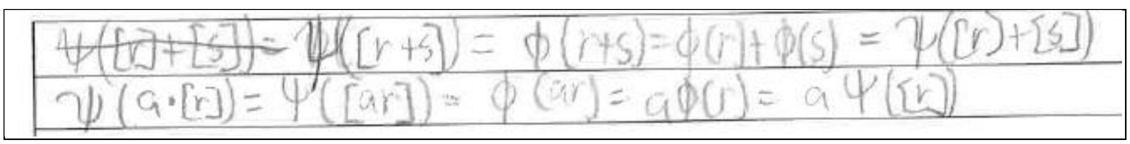


Figure 5.30. Billy's Proof.

Billy mistakenly got the concept of ideal involved as the second property of a ring homomorphism "$\psi([r][s]) = \psi([r])\ \psi([s])$." He showed $\psi(a[r]) = a\psi([r])$ though he was supposed to show $\psi([r][s]) = \psi([r])\psi([s])$. His incomplete knowledge of the second property of a ring homomorphism directly damaged his proof.

Example 24 Savanna (Algebra II)

Savanna's proof is a representative example showing that students' lack of knowledge of definitions directly affects their proving arguments.

Question [9] (4) (In-class problem solving session)

Let R and S be rings. Let $\phi : R \to S$ is a ring homomorphism.
$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a well-defined ring homomorphism.
Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective.

Table 5.25 shows a possible proof for Question [9] (4) and shows where Savanna had difficulties in the proof construction..

Table 5.25

Analysis (Type A) of Savanna's Proof

	Proof	Code	S
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective.	Given	
Y	Show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$.	R1	I
P1	$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a well-defined ring homomorphism.	Given	
P2	$\phi : R \to S$ is a ring homomorphism.	Given	
1	Suppose that $\psi([r]) = \psi([s])$.	C1	N
	Then, $\phi(r) = \phi(s)$.	CO(1,P1)R 1	N
2	Then, $0_S = \phi(r) - \phi(s) = \phi(r - s)$.	R3	N
3	Then, $r - s \in Ker(\phi)$.	R2	N
4	Then, $r = s + k$ for some $k \in Ker(\phi)$.	R2	N
5	Then, $r = s + k \in [s]$.	R2	N
6	Then, [r]=[s].	R2	N
	Another Proof		
X	Show $Ker(\psi) = 0_{R/Ker(\phi)}$.		N
1	Consider $Ker(\psi) = \{[r] \in R / Ker(\phi)$, where $\psi([r]) = 0_S \}$.	C1	N
2	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $Ker(\psi) = \{[r] \in R / Ker(\phi)$, where $\phi(r) = 0_S \}$.	CO(1,P1)R 1	N
4	Then, $Ker(\psi) = \{[r] \in R / Ker(\phi)$, $r \in Ker(\phi) \}$.	R1	N
5	Then, $Ker(\psi) = \{[r] \in R / Ker(\phi)$, $[r] = Ker(\phi)$	R1	N
6	Therefore, $Ker(\psi) = 0_{R/Ker(\phi)}$.	R1	N

The following is an explanation of the above proof. One of the ways to show the function $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective is to show that if

$\psi([r]) = \psi([s])$, then $[r] = [s]$. Then, paying attention to the ignition phrase "$\psi([r]) = \psi([s])$," one may start a proving argument with "Suppose that $\psi([r]) = \psi([s])$." Noting the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r]) = \psi([s])$ with $\phi(r) = \phi(s)$ based on the way for $\psi : R / Ker(\phi) \to S$ to be defined. One can further rephrase $\phi(r) = \phi(s)$ with $0_S = \phi(r) - \phi(s) = \phi(r - s)$ to obtain $r - s \in Ker(\phi)$, which can lead them to conclude that $[r] = [s]$.

There is another way to prove the given proof problem. There is a property of an injective homomorphism that $\phi : R \to S$ is an injective ring homomorphism if and only if $Ker(\phi) = 0_R$. Therefore, in order to prove that $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective, one can show $Ker(\psi) = 0_{R/Ker(\phi)}$. Then, one may start with considering $Ker(\psi)$. Applying the definition of $Ker(\psi)$, one may translate it into $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $\psi([r]) = 0_S\}$. Combining the given condition $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, one can further rephrase it with $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $\phi(r) = 0_S\}$. Furthermore, one can rephrase it with $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $r \in Ker(\phi)\}$. Then, they can conclude that $Ker(\psi) = \{[r] = Ker(\phi)\}$, namely, $Ker(\psi) = 0_{R/Ker(\phi)}$. The following (Figure 5.31) is Savanna's proof.

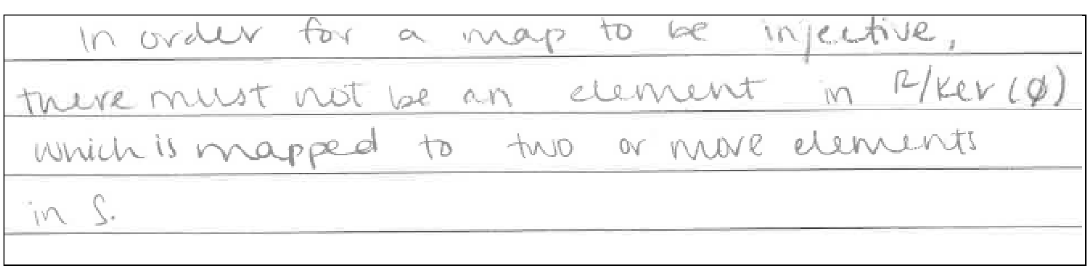

In order for a map to be injective, there must not be an element in R/Ker(ϕ) which is mapped to two or more elements in S.

Figure 5.31. Savanna's Proof.

She was unable to prove the given proof problem and only showed her concept image about an injective function. However, her concept image of an injective function was wrong. Her statement "In order for a map to be injective, there must not be an element in $R/Ker(\phi)$ which is mapped to more than one element in *S*" was not the definition of an injective function but that of a function. Since she did not know the definition of an injective function, there was no way for her to prove the given statement. Savanna also gave an example showing that even those students who had already exposed themselves to some proof-based courses may forget a basic and elementary proving skill of proving, for example, that a map is injective.

Example 25 Davis (Analysis)

> Question [2] (Exam II)
> *Using the Inverse Function Theorem, show that* $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$,
> *for all* $x \in (-1,1)$.

Table 5.26 shows a possible proof for Question [2] and shows where Davis had difficulties in the proof construction.

Table 5.26

Analysis (Type A) of Davis's Proof

	Proof	Code	D
X	Show that $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$ for all $x \in (-1,1)$		
1	Consider the left hand side of the equation.	C5	
2	Recall the Inverse Function Theorem, which is $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$, where $f(y) = x$.	C2	N
3	Note $\frac{d}{dx}\arcsin x = \frac{1}{\cos\arcsin x}$ with $\arcsin x \in (\frac{-\pi}{2}, \frac{\pi}{2})$.	CO(1,2)R1	N

4	Recall that $\cos\phi = \sqrt{1-\sin^2\phi}$ on $\phi \in (\frac{-\pi}{2}, \frac{\pi}{2})$.	C2	N
5	Then, $\cos\arcsin x = \sqrt{1-\sin^2(\arcsin x)}$.	CO(3, 4)R3	N
6	Note $\sin(\arcsin x) = x$ for $\arcsin x \in (\frac{-\pi}{2}, \frac{\pi}{2})$.	C2	N
7	Then, $\sqrt{1-\sin^2(\arcsin x)} = \sqrt{1-x^2}$.	CO(5, 6)R3	N
8	Therefore, $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$.	CO(3,4,7)R3	N

The following is a possible way to prove the above problem. One can first recall the Inverse Function Theorem "$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$" as suggested in the question. The denominator of the right hand side of the equation becomes $\cos(\arcsin x)$ for $\arcsin x \in (\frac{-\pi}{2}, \frac{\pi}{2})$. Recalling $\cos\varphi = \sqrt{1-\sin^2\varphi}$ for $\varphi \in (\frac{-\pi}{2}, \frac{\pi}{2})$ and applying it to $\cos(\arcsin x)$, one can get $\sqrt{1-\sin^2(\arcsin x)} = \sqrt{1-x^2}$ for $\cos(\arcsin x)$ with $\arcsin x \in (\frac{-\pi}{2}, \frac{\pi}{2})$. Finally, students can conclude $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$. The following (Figure 5.32) is Davis's proof.

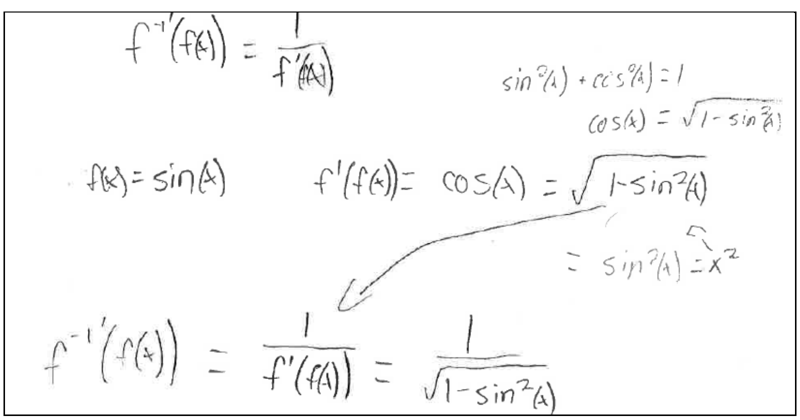


Figure 5.32. Davis's Proof.

Davis was able to come up with the idea of using the theorem for the derivative of an inverse function, but did not recall it correctly, which damaged his whole proving argument. He applied $(f^{-1})'(x) = \frac{1}{f'(f(x))}$ though he was supposed to apply $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

Example 26 Carlos (Algebra I)

Carlos's case is another representative example showing that students' lack of knowledge of a concept can cause of their producing of an unsuccessful proof.

> Question [4] (In-class problem solving session
>
> *Suppose that* $G/Z(G)$ *is cyclic. Prove that G is abelian.*

Table 5.27 shows a possible proof for the given proof problem and shows where Carlos had difficulties in the proof construction.

Table 5.27

Analysis (Type A) of Carlos's Proof

Object	Proof	Code	C
X	Show G is abelian.		S
Y	Show $ab = ba$ for any $a, b \in G$.	R1	S
P	$G/Z(G)$ is cyclic.	Given	
1	Let $a, b \in G$.	C1	N
2	Note $a, b \in G$ are in some cosets.	C2	N
3	Let $a \in x^m Z$ and $b \in x^n Z$.	CO(2,P)R1	N
4	Let $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$.	R2	N
5	Then, $ab = x^{m+n} z_1 = x^{n+m} z_2 = ba$.	R3	N

The conclusion of the given statement is "*G is abelian*." The translation of it into *mathematical language* provides "$ab = ba$ for any $a, b \in G$." The phrase in the *mathematical language*, "for any $a, b \in G$," is the ignition phrase. One can derive starting variables from the *ignition phrase* and provide "Suppose $a, b \in G$." This proof is the type of the proof of showing A = B. One can work on either A or B until A changes into B or B changes into A while making good use of the given conditions. In this case, one can attempt to rephrase the left side of the equation "ab" until it changes into "ba." To rephrase "ab," one may consider the given hypothesis "$G / Z(G)$ *is cyclic*" and look for the connection between the starting variables "a, b" and the hypothesis "$G / Z(G)$ *is cyclic*." Recalling the property that an element of G belongs to some coset, one may produce $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$. Then, one can rephrase ab with $x^{m+n} z_1 z_2$ for some $z_1, z_2 \in Z$. Using the property of the center of a group, one may derive $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$. The following (Figure 5.33) is Carlos's proof.

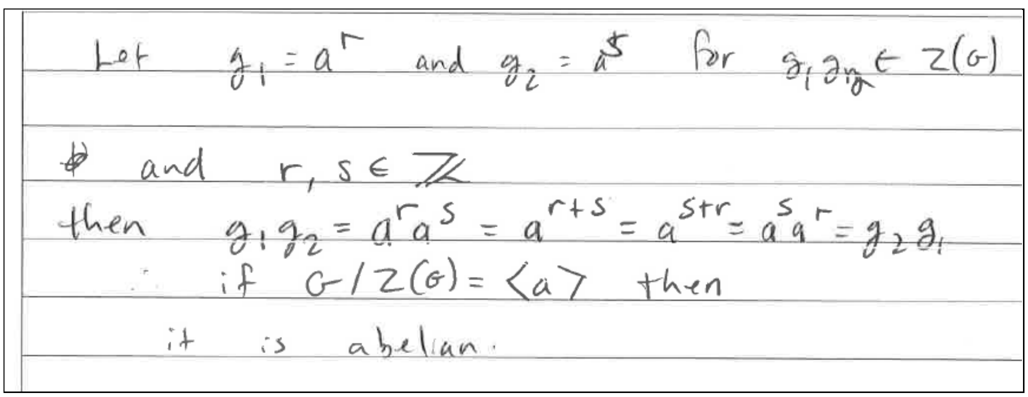


Figure 5.33. Carlos's Proof.

Carlos's proof was not successful partly because he lacked the knowledge of the way to express a coset of *Z*(*G*) and of the relationship between an element of *G* and a coset of *Z*(*G*). He provided $< a >$ to express $G / Z(G)$ as a cyclic group though he was supposed to provide $G/(Z) =< x^m Z >$ for some $x \in G$. He lacked the knowledge of the notation for a coset of G/(Z). The relationship between an element of *G* and a coset of

$Z(G)$ was that an element of G belonged to some coset of $Z(G)$. If he had known the relationship, he might have rephrased $g_1 = x^m z_1$ and $g_2 = x^n z_2$ for some $z_1, z_2 \in Z(G)$, which might have led him to obtain $g_1 g_2 = x^m z_1 x^n z_2 = x^n z_2 x^m z_1 = g_2 g_1$.

Example 26 Elias (Topology)

Elias's case is another representative example showing students' lack of solid understanding of a concept and a theorem can cause them to produce incomplete proofs.

> Question [7] (Exam II)
>
> *Let X be a Hausdorff space. Let $\{x_n : n \in Z^+\}$ be a sequence in X converging to a point x_0. Prove that the set $K = \{x_n : n = 0,1,2,...\}$ is compact.*

Table 5.28 shows a possible proof for Question [7] and shows where Elias had difficulties in the proof construction.

Table 5.28

Analysis (Table A) of Elias's Proof

Object	Proof	Code	E
X	Show that $K = \{x_n : n = 0,1,2,...\}$ is compact.		
Y	Show that for any open cover of *K*, *K* has a finite open subcover.		
P	$\{x_n : n \in Z^+\}$ be a sequence in *X* converging to x_0.	Given	
1	Let $U = \{U_\alpha \in T_X\}$ be an open cover of *X*.	C1	S
2	Construct an open cover of *K* by letting $V = \{V_\alpha = U_\alpha \cap K\}$.	C1	N
3	Since $U = \{U_\alpha \in T_X\}$ is an open cover of *X*, $\exists U_{\alpha_0} \in U$ such that $x_0 \in U_{\alpha_0}$.	R1	S
4	Since x_n converges to x_0, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in U_{\alpha_0}$.	CO(P,3)R1	S
5	Let $V_{\alpha_0} = U_{\alpha_0} \cap K$, where $V_{\alpha_0} \in V$.	C1	N
6	For each x_i with $i < N$, find an open set $V_{x_i} \in V$ such that $x_i \in V_{x_i}$.	C1	N

7	Note that $\{V_{x_1}, V_{x_2}, \ldots V_{x_{N-1}}, V_{\alpha_0}\}$ is a desired finite open subcover of *K*.	CO(5,6)R2	N

The conclusion of the given statement is "$K = \{x_n : n = 0,1,2,\ldots\}$is compact." It can be translated into the following *mathematical language*: "For any open cover of *K*, *K* has a finite open subcover." By paying attention to *the ignition phrase* "For any open cover of *K*," one may explore the way to construct an open cover of *K*. Recalling the property of a subspace topology, one can set a starting variable by having "Let $U = \{U_\alpha \in T_X\}$ as an open cover of *X*". Then, one may construct an open cover of *K*, providing "$V = \{V_\alpha = U_\alpha \cap K\}$." Noting the given hypothesis "$\{x_n : n \in Z^+\}$ converges to a point x_0," one may translate it into the following mathematical language: "For an open set $V_{\alpha_0} = U_{\alpha_0} \cap K$ in the open cover of *K*, in which $x_0 \in U_{\alpha_0}$, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in V_{\alpha_0}$. Then, one can construct a finite open subcover $\{V_{x_1}, V_{x_2}, \ldots V_{x_{N-1}}, V_{\alpha_0}\}$ by setting $V_{x_i} \in V$ such that $x_i \in V_{x_i}$ for $n < N$. The following (Figure 5.34) is Elias's proof.

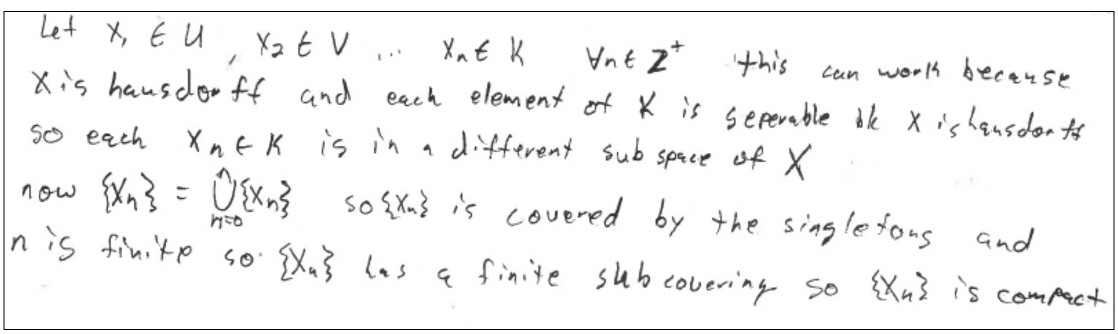
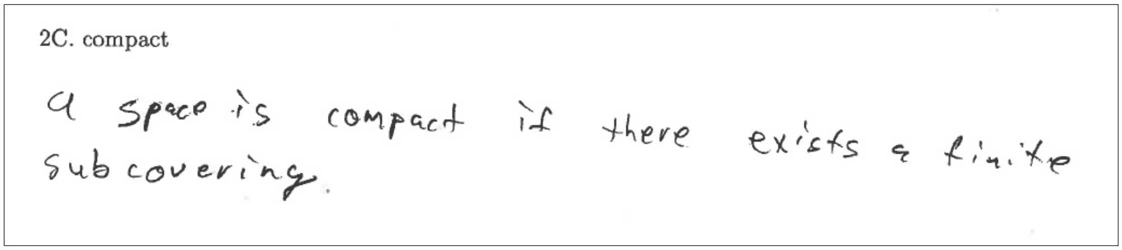

Let $x_1 \in U$, $x_2 \in V$ ... $x_n \in K$ $\forall n \in Z^+$ this can work because
X is hausdorff and each element of K is seperable b/c X is hausdorff
so each $x_n \in K$ is in a different subspace of X
now $\{x_n\} = \bigcup_{n=0}^{\infty} \{x_n\}$ so $\{x_n\}$ is covered by the singletons and
n is finite so $\{x_n\}$ has a finite subcovering so $\{x_n\}$ is compact

Figure 5.34. Elias's Proof.

Elias was unable to solve the problem partly because he was unable to construct an open finite subcover of *K* properly. He did not understand the definition of compactness precisely. As is shown in the following figure (Figure 5.35), he stated in a previous problem "A space is compact if there exists a finite sub-covering." He missed including in it "For every open cover of *X*." His incomplete understanding of compactness might have led him to fail to set an open cover of *K*, from which a finite subcover might have been derived.


2C. compact

a space is compact if there exists a finite subcovering.

Figure 5.35. Elias's Statement.

Example 28 Savanna (Algebra II)

Students' ignorance of definitions severely damages their proving arguments. Savanna's case was such a representative example.

> Question [9] (3) (In-class problem solving session)
>
> *Let R and S be rings. Let* $\phi : R \to S$ *is a ring homomorphism.*
> *Consider a map* $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.
> *Show* $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ *is a ring homomorphism.*

Table 5.29 shows a possible proof for the given proof problem and shows where Savanna had difficulties in the proof construction.

Table 5.29

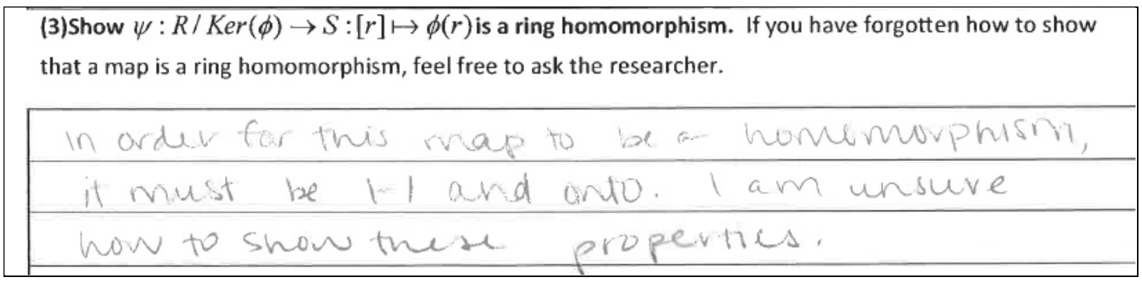
Analysis (Type A) of Savanna's Proof

Object	Proof	Code	S
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism.	Given	
Y	Show (i) $\psi([r]+[s]) = \psi([r]) + \psi([s])$ and (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$		
P1	$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.		

P2	$\phi : R \to S$ is a ring homomorphism.	Given	
	(i)		
1	Let $[r],[s] \in R / K(\phi)$	C1	N
2	Consider , $\psi([r]+[s])$	C1	N
3	Note that $\psi([r]+[s]) = \psi([r+s])$.	R2	N
4	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\psi([r+s]) = \phi(r+s)$.	CO(3,P1)R1	N
5	Since $\phi : R \to S$ is a homomorphism, $\phi(r+s) = \phi(r) + \phi(s)$.	CO(4, P2)R1	N
6	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r) + \phi(s) = \psi([r]) + \psi([s])$.	CO(5,P1)R1	N
7	Then, $\psi([r]+[s]) = \psi([r]) + \psi([s])$.	CO(2-6)R2	N
	(ii)		
8	Consider $\psi([r][s])$.	C1	N
9	Note $\psi([r][s]) = \psi([rs])$.	R1	N
10	Then, $\psi([rs]) = \phi(rs)$.	CO(9,P1)R1	N
11	Then, $\phi(rs) = \phi(r)\ \phi(s)$.	CO(10, P2)R1	N
12	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$.	CO(11, P1)R1	N
13	Therefore, $\psi([r][s]) = \psi([r])\ \psi([s])$.	CO(8-12)R2	N

The goal of the proof is "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism." There are two things to show: (i) $\psi([r]+[s]) = \psi([r+s])$; (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$. For (i), one can rephrase $\psi([r]+[s])$ with $\psi([r+s])$ through algebraic manipulation. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r+s])$ with $\phi(r+s)$. Using another given condition "$\phi : R \to S$ is a ring homomorphism," they can rephrase $\phi(r+s)$ with $\phi(r) + \phi(s)$. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" again, one can derive that $\phi(r) + \phi(s) = \psi([r]) + \psi([s])$ to conclude that) $\psi([r]+[s]) = \psi([r]) + \psi([s])$.

Similarly, for (ii), one can start with considering $\psi([r][s])$. One can rephrase $\psi([r][s])$ with $\psi([rs])$ through algebraic manipulation. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can further rephrase $\psi([rs])$ with $\phi(rs)$. Using the other condition "$\phi : R \to S$ is a ring homomorphism," one can rephrase $\phi(rs)$ with $\phi(r)\ \phi(s)$. Using the condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" again, one can derive $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$ to conclude that $\psi([r][s]) = \psi([r])\ \psi([s])$. The following is Savanna's proof (Figure 5.36).



(3)Show $\psi : R / Ker(\phi) \to S : [r] \mapsto \phi(r)$ is a ring homomorphism. If you have forgotten how to show that a map is a ring homomorphism, feel free to ask the researcher.

In order for this map to be a homomorphism, it must be 1-1 and onto. I am unsure how to show these properties.

Figure 5.36. Savanna's Proof.

Savanna was unable to prove the above statement mainly because she had not remembered the definition of a ring homomorphism correctly. She mistakenly believed the definition of a ring homomorphism had to be a one to one and onto function. In addition, she did not know how to show a function is one to one and onto. She lacked the knowledge of the basics of some concepts.

Example 29 Donald (Topology)

Donald's case was another example showing students' lack of knowledge gave a flaw to their proving arguments.

Question [7] (Exam II)

Let X be a Hausdorff space. Let $\{x_n : n \in Z^+\}$ be a sequence in X converging to a point x_0. Prove that the set $K = \{x_n : n = 0,1,2,...\}$ is compact.

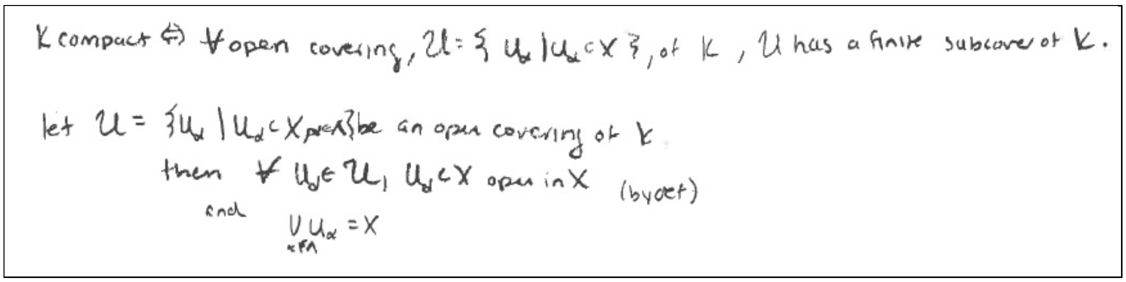
Table 5.30 shows a possible proof for the given proof problem and shows where Matt had difficulties in the proof construction.

Table 5.30

Analysis (Type A) of Donald's Proof

Object	Proof	Code	Q
X	Show that $K = \{x_n : n = 0,1,2,...\}$ is compact.		
Y	Show that for any open cover of *K*, *K* has a finite open subcover.		
P	$\{x_n : n \in Z^+\}$ be a sequence in *X* converging to x_0.	Given	
1	Let $U = \{U_\alpha \in T_X\}$ be an open cover of *X*.	C1	S
2	Construct an open cover of *K* by letting $V = \{V_\alpha = U_\alpha \cap K\}$.	C1	N
3	Since $U = \{U_\alpha \in T_X\}$ is an open cover of *X*, $\exists U_{\alpha_0} \in U$ such that $x_0 \in U_{\alpha_0}$.	R1	S
4	Since x_n converges to x_0, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in U_{\alpha_0}$.	CO(P,3)R1	S
5	Let $V_{\alpha_0} = U_{\alpha_0} \cap K$, where $V_{\alpha_0} \in V$.	C1	N
6	For each x_i with $i < N$, find an open set $V_{x_i} \in V$ such that $x_i \in V_{x_i}$.	C1	N
7	Note that $\{V_{x_1}, V_{x_2}, ...V_{x_{N-1}}, V_{\alpha_0}\}$ is a desired finite open subcover of *K*.	CO(5,6)R2	N

The conclusion of the given statement is "$K = \{x_n : n = 0,1,2,...\}$ is compact." It can be translated into the *mathematical language* "For any open cover of *K*, *K* has a finite open subcover." By paying attention to *the ignition phrase* "For any open cover of *K*," one may explore the way to construct an open cover of *K*. Recalling the property of a subspace topology, one can set a starting variable by having "Let $U = \{U_\alpha \in T_X\}$ as an open cover of *X*". Then, one may construct an open cover of *K* by having "$V = \{V_\alpha = U_\alpha \cap K\}$." To further advance a reasoning process, one may note and consider the given hypothesis "$\{x_n : n \in Z^+\}$ converges to a point x_0." Then, the given hypothesis can be translated into "For an open set $V_{\alpha_0} = U_{\alpha_0} \cap K$ in the open cover of *K*, in which $x_0 \in U_{\alpha_0}$, $\exists N \in Z^+$ such that for all $n \geq N$, $x_n \in V_{\alpha_0}$. Finally, they may create a finite open subcover $\{V_{x_1}, V_{x_2}, \ldots V_{x_{N-1}}, V_{\alpha_0}\}$ by setting $V_{x_i} \in V$ such that $x_i \in V_{x_i}$ for $n < N$. The following (Figure 5.37) is Donald's proof.


K compact ⇔ ∀ open covering, $\mathcal{U} = \{U_\alpha | U_\alpha \subset X\}$, of K, $\mathcal{U}$ has a finite subcover of K.

let $\mathcal{U} = \{U_\alpha | U_\alpha \subset X, \alpha \in A\}$ be an open covering of K

then $\forall U_\alpha \in \mathcal{U}$, $U_\alpha \subset X$ open in X (by def)

and $\bigcup_{\alpha \in A} U_\alpha = X$

Figure 5.37. Donald's Proof.

Donald's proof was close to a completed proof, but it seems he did not know the concept of a subspace topology, which gave a flaw to his proof. He set $U = \{U_\alpha \in T_X\}$ as an open cover for *K* and used it to derive a finite open subcover of *K* though he was supposed to have $V = \{V_\alpha = U_\alpha \cap K\}$ as an open cover for *K*. It seems he assumed that an open covering for *X* might be used as an open cover for *K*.

Example 30 Dayton (Algebra I)

Dayton's case is a representative example showing that students' lack of knowledge hinders them from rephrasing an object, which causes their proofs to be unsuccessful.

Question [4] (In-class problem solving session)
Suppose that $G/Z(G)$ is cyclic. Prove that G is abelian.

Table 5.31 shows a possible proof for Question [4] and where Dayton had difficulties in the proof construction.

Table 5.31

Analysis (Type A) of Dayton's Proof

Object	Proof	Code	A
X	Show G is abelian.		S
Y	Show $ab = ba$ for any $a, b \in G$.	R1	S
P	$G / Z(G)$ is cyclic.	Given	
1	Let $a, b \in G$.	C1	N
2	Note $a, b \in G$ are in some cosets.	C2	N
3	Let $a \in x^m Z$ and $b \in x^n Z$.	CO(2,P)R1	N
4	Let $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$.	R2	N
5	Then, $ab = x^{m+n} z_1 = x^{n+m} z_2 = ba$.	R3	N

The conclusion of the given statement is "*G is abelian*." The translation of it into *mathematical language* provides "$ab = ba$ for any $a, b \in G$." The phrase in the *mathematical language*, "for any $a, b \in G$," is the ignition phrase. One can derive starting variables from the *ignition phrase* and provide "Suppose $a, b \in G$." This proof is the type of the proof of showing A = B. One can work on either A or B until A changes into B or B changes into A while making good use of the given conditions. In this case, one can attempt to rephrase the left side of the equation "ab" until it changes into "ba." To rephrase "ab," one may consider the given hypothesis "$G / Z(G)$ *is cyclic*" and look for the connection between the starting variables "a, b" and the hypothesis "$G / Z(G)$ *is cyclic*." Recalling the property that an element of G belongs to some coset, one may produce $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$. Then, one can rephrase ab with $x^{m+n} z_1 z_2$ for some $z_1, z_2 \in Z$. Using the property of the center of a group, one may derive $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$. The following (Figure 5.38) is Dayton's proof.

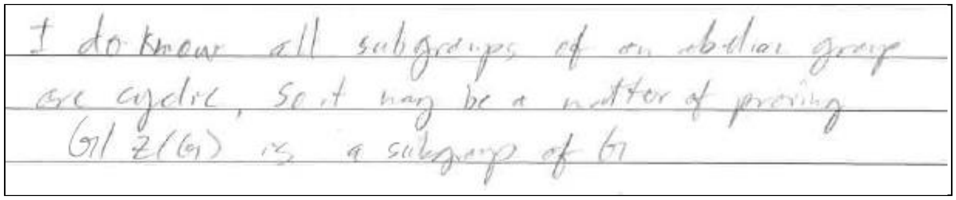

I do know all subgroups of an abelian group
are cyclic, So it may be a matter of proving
G/Z(G) is a subgroup of G

*Figure 5.3*8. Dayton's Proof.

Dayton was unable to prove the given proof problem mainly because he had a wrong concept image of an abelian group. He proclaimed "I do know all subgroups of an abelian group are cyclic," which was wrong. He might have been right if he had stated "all subgroups of an abelian group are normal," though. His wrong concept image hindered him from translating "*G is abelian*" into "$ab = ba$ for any $a, b \in G$," and from advancing a reasoning process. His proof was an example showing students' lack of knowledge might directly affect their use of the operation of rephrasing an object, which produced an incomplete proof.

Example 31 Anthony (Algebra I)

Anthony's case is an example showing students' lack of knowledge of a proving technique can cause them to get astray in advancing a reasoning process and to produce an incomplete proof. In particular, Anthony's proof was a representative example showing students' difficulties with proving might be caused by their lack of the knowledge for dealing with the type of proof of showing A = B.

Question [4] (In-class problem solving session)

Suppose that $G/Z(G)$ is cyclic. Prove that G is abelian.

Table 5.32 shows a possible proof for the given proof problem.

Table 5.32

Analysis (Type A) of Anthony's Proof

Object	Proof	Code	A
X	Show G is abelian.		S
Y	Show $ab = ba$ for any $a, b \in G$.	R1	S
P	$G / Z(G)$ is cyclic.	Given	
1	Let $a, b \in G$.	C1	N
2	Note $a, b \in G$ are in some cosets.	C2	N
3	Let $a \in x^m Z$ and $b \in x^n Z$.	CO(2,P)R1	N
4	Let $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$.	R2	N
5	Then, $ab = x^{m+n} z_1 = x^{n+m} z_2 = ba$.	R3	N

The conclusion of the given statement is "*G is abelian*." The translation of it into *mathematical language* provides "$ab = ba$ for any $a, b \in G$." The phrase in the *mathematical language*, "for any $a, b \in G$," is the ignition phrase. One can derive starting variables from the *ignition phrase* and provide "Suppose $a, b \in G$." This proof is the type of the proof of showing A = B. One can work on either A or B until A changes into B or B changes into A while making good use of the given conditions. In this case, one can attempt to rephrase the left side of the equation "ab" until it changes into "ba." To rephrase "ab," one may consider the given hypothesis "$G / Z(G)$ *is cyclic*" and look for the connection between the starting variables "a, b" and the hypothesis "$G / Z(G)$ *is cyclic*." Recalling the property that an element of G belongs to some coset, one may produce $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$. Then, one can rephrase ab with $x^{m+n} z_1 z_2$ for some $z_1, z_2 \in Z$. Using the property of the center of a group, one may derive $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$. As is shown in the figure (Figure 5.39), Anthony was successful in making the goal of the proof clear to himself by translating the conclusion of the given statement "G is abelian" into "$gh = hg$ for any $g, h \in G$."

Figure 5.39. Anthony's Strategy.

The following figures (Figure 5.39 and 5.40) is Anthony's proof.

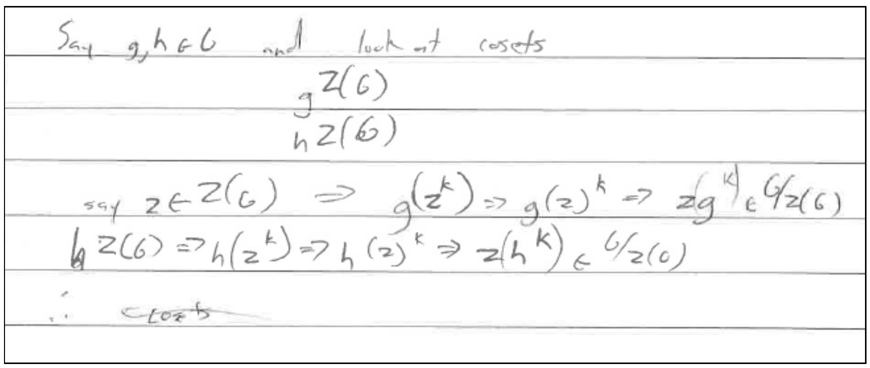


Figure 5.40. Anthony's Proof.

Although Anthony successfully set starting variables by providing $g, h \in G$, his proving argument was not successful partly because he did not have a clear strategy for solving the given proof problem. In particular, he was unable to apply the technique for solving the type of proof of showing A = B. The goal of the proof was to show G was abelian. Namely, he needed to show "for any $g, h \in G$, $gh = hg$." This is the type of proof for showing A = B. There were mainly two ways to deal with this type of proof. One is to work on either A or B and rephrase it until they get the other, which is B or A. Another way is to work on and rephrase each side separately until they can change the expressions of A and B into the same expression C. It seems Anthony was unable to make the goal of the proof clear to himself. As a result, he tried combining his starting variables $g, h \in G$ with the given hypothesis "$G / Z(G)$ is cyclic" to create $gZ(G)$ and

$hZ(G)$, which caused his proving argument to be unsuccessful. If Anthony had known the proving strategy for the type of proof of showing A = B, he might have worked on gh and rephrased it until he might obtain hg while recalling and using the fact that every element in G belonged to some coset.

Example 32 Zach (Analysis I)

Zach's proof is a representative example showing that students' lack of knowledge hinders them from advancing a reasoning process. In particular, students' lack of knowledge of a relevant fact such as a property, a theorem, and a mathematical law shuts down their proving process if a given proof problem requires them to use it.

Question [1] (Exam III) *Let* $a < b$ *is fixed. Suppose that* $g_n \geq 0$ *is a sequence of Riemann integrable functions such that* $\lim_{n\to\infty} \int_a^b g_n(x)dx = 0$. *Prove that if f is Riemann integrable on (a, b), then* $\lim_{n\to\infty} \int_a^b f(x)g_n(x)dx = 0$.

Table 5.33 shows a possible proof for Question [1] and shows where Zach had difficulties in the proof construction.

Table 5.33

Analysis (Type A) of Zach's Proof

	Proof	Code	Z
X	Show $\lim_{n\to\infty} \int_a^b f(x)g_n(x)dx = 0$.	Given	
Y	Show $\lim_{n\to\infty} \left| \int_a^b f(x)g_n(x)dx \right| = 0$.	C5	S
P1	$g_n \geq 0$ is a sequence of Riemann integrable functions	Given	

	such that $\lim_{n\to\infty}\int_a^b g_n(x)dx = 0$.		
P2	$f(x)$ is Riemann integrable on (a, b).	Given	
1	Consider $\lim_{n\to\infty}\int_a^b f(x)g_n(x)dx$.	C5	S
2	Note that $\int_a^b f(x)g_n(x)dx \le \left\|\int_a^b f(x)g_n(x)dx\right\|$.	C2	S
3	Then, $\lim_{n\to\infty}\int_a^b f(x)g_n(x)dx \le \lim_{n\to\infty}\left\|\int_a^b f(x)g_n(x)dx\right\|$.	CO(1,2)R2	S
4	Recall $\left\|\int_s^t h(x)dx\right\| \le \int_s^t \|h(x)\|dx$.	C2	N
5	Then, $\lim_{n\to\infty}\left\|\int_a^b f(x)g_n(x)dx\right\| \le \lim_{n\to\infty}\int_a^b \|f(x)g_n(x)\|dx$.	CO(3,4)R2	N
6	Since $g_n(x) \ge 0$, $\lim_{n\to\infty}\int_a^b \|f(x)g_n(x)\|dx = \lim_{n\to\infty}\int_a^b \|f(x)\|g_n(x)dx$.	CO(5, P1)R2	N
7	Since f is Riemann integrable, f is bounded, namely, $\|f(x)\| \le M$ for some $M \in R$.	R1	N
8	Then, $\lim_{n\to\infty}\int_a^b \|f(x)\|g_n(x)dx \le \lim_{n\to\infty}\int_a^b Mg_n(x)dx = \lim_{n\to\infty} M\int_a^b g_n(s)dx$.	CO(6,7)R2	N
9	Note $\lim_{n\to\infty} M\int_a^b g_n(x)dx = 0$	CO(8, P1)R2	N
10	Then, $\lim_{n\to\infty}\left\| \int_a^b f(x)g_n(x)dx \right\| = 0$	CO(5,6,8,9) R2	N
11	Recall that if $\lim_{n\to\infty}\|A_n\| = 0$, then $\lim_{n\to\infty} A_n = 0$.	C2	N
12	Since $\lim_{n\to\infty}\left\| \int_a^b f(x)g_n(x)dx \right\| = 0$, $\lim_{n\to\infty}\int_a^b f(x)g_n(x)dx = 0$.	CO(10,11)R2	N
	Alternative proof		

1	Recall $\left\|\int_s^t h(x)dx\right\| \le \int_s^t \|h(x)\|dx$.	C2	N
2	Then, $\left\|\int_a^b f(x)g_n(x)dx\right\| \le \int_a^b \|f(x)g_n(x)\|dx$.	CO(1,Y)R2	N
3	Since $g_n(x) \ge 0$, $\int_a^b \|f(x)g_n(x)\|dx = \int_a^b \|f(x)\|g_n(x)dx$.	CO(2, P1)R2	N
4	Since *f* is Riemann integrable, *f* is bounded, namely, $\|f(x)\| \le M$ for some $M \in R$.	C2	S
5	Then, $\int_a^b \|f(x)\|g_n(x)dx \le \int_a^b Mg_n(x)dx$.	CO(3, 4)R2	I
6	Note $\int_a^b Mg_n(x)dx = M\int_a^b g_n(x)dx$.	R3	I
7	Therefore, we have now $0 \le \left\|\int_a^b f(x)g_n(x)dx\right\| \le M\int_a^b g_n(x)dx$.	CO(2, 6)R2	S
8	Consider $M\int_a^b g_n(x)dx$	C5	S
9	Since $\lim_{n\to\infty}\int_a^b g_n(x)dx = 0$, $\lim_{n\to\infty} M\int_a^b g_n(x)dx = 0$.	CO(P1, 8)R2	S
10	Therefore, $\lim_{n\to\infty}\left\|\int_a^b f(x)g_n(x)dx\right\| = 0$ as is desired.	CO(7, 9)R2	I

One of the ways to solve this problem is to bring in and apply the proposition that if $\lim_{n\to\infty}|A_n| = 0$, then $\lim_{n\to\infty} A_n = 0$. The conclusion of the given problem "Show $\lim_{n\to\infty}\int_a^b f(x)g_n(x)dx = 0$" can be translated into "Show $\lim_{n\to\infty}\left|\int_a^b f(x)g_n(x)dx\right| = 0$." Recalling and applying "$\left|\int_s^t h(x)dx\right| \le \int_s^t |h(x)|dx$"to $\lim_{n\to\infty}\left|\int_a^b f(x)g_n(x)dx\right|$, one can obtain $\left|\int_a^b f(x)g_n(x)dx\right| \le \int_a^b |f(x)g_n(x)|dx$. Then, noting the given condition

"$g_n(x) \ge 0$," one can obtain $\int_a^b |f(x)g_n(x)|dx = \int_a^b |f(x)|g_n(x)dx$. Noting another given condition "$f(x)$ is Riemann integrable" and recalling the property of a Riemann integrable function, which is "if $f(x)$ is Riemann integrable, $f(x)$ is bounded," one can obtain "$0 \le \left|\int_a^b f(x)g_n(x)dx\right| \le M\int_a^b g_n(x)dxf$ for some $M \in R$." By using the other condition "$\lim_{n\to\infty}\int_a^b g_n(x)dx = 0$" and applying the squeeze theorem, one can conclude that $\lim_{n\to\infty}\left|\int_a^b f(x)g_n(x)dx\right| = 0$. The following (Figure 5.41) is Zach's proof.

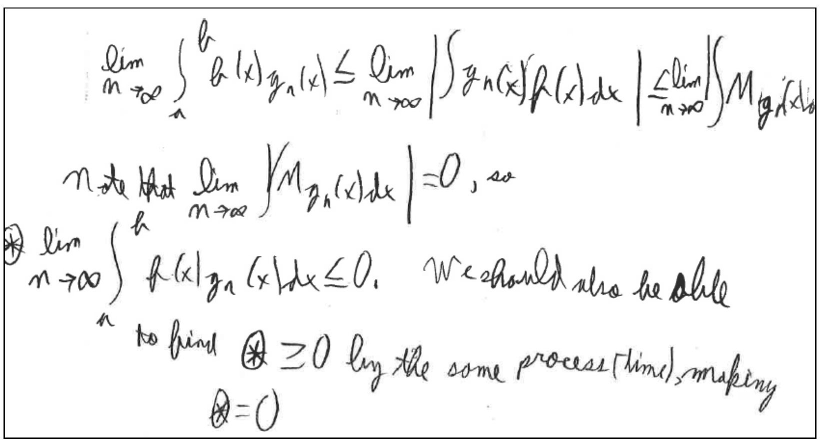


Figure 5.41. Zach's Proof.

This problem was so difficult that nobody was able to make a proper argument. Among them, Zach was the only student who was able to make his argument relatively good. He was able to consider $\lim_{n\to\infty}\left|\int_a^b f(x)g_n(x)dx\right|$. Unfortunately, he was unable to take the next step, which was that $\left|\int_a^b f(x)g_n(x)dx\right| \le \int_a^b |f(x)g_n(x)|dx$. He was unable to recall the triangle inequality for integration and apply it to the object. Since he was unable to consider $\int_a^b |f(x)g_n(x)|dx$, he was unable to have a chance to use the given two conditions "$\lim_{n\to\infty}\int_a^b g_n(x)dx = 0$" and "$f(x)$ is Riemann integrable on (a, b)."

Example 33 Carlos (Algebra I)

Students' incomplete understanding and knowledge of concepts might be reflected in their notations of those concept. Their use of wrong notations can cause their proofs to be unsuccessful. Carlos showed such an example.

> Question [4] (In-class problem solving session)
>
> *Suppose that $G/Z(G)$ is cyclic. Prove that G is abelian.*

Table 5.34 shows a possible proof for Question [4] and shows where Carlos had a difficulty in the proof construction.

Table 5.34

Analysis (Type A) of Carlos's Proof

	Proof	Code	C
X	Show G is abelian.	Given	
Y	Show $ab = ba$ for any $a, b \in G$.	R1	N
P	$G/Z(G)$ is cyclic.	Given	
1	Let $a, b \in G$.	C1	N
2	Recall $a, b \in G$ are in some cosets.	C2	N
3	Then, $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$.	CO(P, 2)R1	N
4	Let $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$.	R1	N
5	Then, $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$.	R3	N

The conclusion of the given statement is "*G is abelian*." The translation of it into *mathematical language* provides "$ab = ba$ for any $a, b \in G$." The phrase in the *mathematical language*, "for any $a, b \in G$," is the ignition phrase. One can derive starting variables from the *ignition phrase* and provide "Suppose $a, b \in G$." This proof is the type of the proof of showing A = B. One can work on either A or B until A changes into B or B changes into A while making good use of the given conditions. In this case, one can attempt to rephrase the left side of the equation "ab" until it changes into "ba." To rephrase "ab," one may consider the given hypothesis "$G / Z(G)$ *is cyclic*" and look for the connection between the starting variables "a, b" and the hypothesis "$G / Z(G)$ *is cyclic*." Recalling the property that an element of *G* belongs to some coset, one may produce $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$. Then, one can rephrase ab with $x^{m+n} z_1 z_2$ for some $z_1, z_2 \in Z$. Using the property of the center of a group, one may derive $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$. As is shown in the figure (Figure 5.39), Anthony was successful in making the goal of the proof clear to himself by translating the conclusion of the given statement "*G* is abelian" into "$gh = hg$ for any $g, h \in G$." The following (See Figure 5.42) is Carlos's proof .

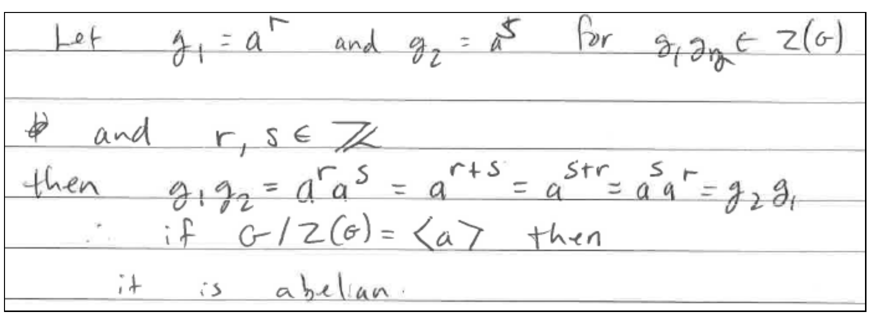

Let $g_1 = a^r$ and $g_2 = a^s$ for $g_1, g_2 \in Z(G)$
and $r, s \in \mathbb{Z}$
then $g_1 g_2 = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = g_2 g_1$
∴ if $G/Z(G) = \langle a \rangle$ then
it is abelian.

Figure 5.42. Carlos's Proof.

Carlos had a problem with his notation of a coset of *Z*(*G*). It seems that he did not have a solid understanding of a coset of *Z*(*G*), which might have affected his notation. In this problem, he might have thought that a coset was generated by an element of *G*, which resulted in his having $G / Z(G) =< a >$. The generator of a coset of *Z*(*G*) might have been expressed with $[a] = aZ(G)$ for some $a \in G$, which was an equivalence class, and $G / Z(G)$ might have been expressed with $<[a]>$. Another problem with his argument was that his notation was inconsistent. He had $g_1, g_2 \in Z(G)$ but he concluded that *G* was abelian by showing $g_1 g_2 = g_2 g_1$, which implied that he meant that $g_1, g_2 \in G$. His inconsistency might have been avoided if he had made sure of the starting variable through translating the conclusion of the given statement into *mathematical language* so that he might have $g_1, g_2 \in G$. Another possible problem might be that he might not have known that an element of *G* belonged to some coset of *Z*(*G*).

Example 34 Ben (Algebra II)

Ben's proof is an example showing that students' lack of precision in notations can damage their proving arguments.

> Question [9] (3) (in-class problem solving session)
>
> *Let R and S be rings. Let* $\phi : R \to S$ *is a ring homomorphism.*
> *Consider a map* $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.
> *Show* $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ *is a ring homomorphism.*

Table 5.35 shows a possible proof for Question [9] (3) and shows where Ben had difficulties in the proof construction.

Table 5.35

Analysis table (Type A) of Ben's Proof

Object	Proof	Code	B
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism.	Given	

Y	Show (i) $\psi([r]+[s])=\psi([r])+\psi([s])$ and (ii) $\psi([r][s])=\psi([r])\ \psi([s])$		
P1	$\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$.		
P2	$\phi : R \to S$ is a surjective ring homomorphism.	Given	
	(i)		
1	Let $[r],[s] \in R/K(\phi)$	C1	N
2	Consider , $\psi([r]+[s])$	C1	N
3	Note that $\psi([r]+[s])=\psi([r+s])$.	R2	I
4	Since $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$, $\psi([r+s])=\phi(r+s)$.	CO(3,P1)R1	S
5	Since $\phi : R \to S$ is a homomorphism, $\phi(r+s)=\phi(r)+\phi(s)$.	CO(4, P2)R1	S
6	Since $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r)+\phi(s)=\psi([r])+\psi([s])$.	CO(5,P1)R1	I
7	Then, $\psi([r]+[s])=\psi([r])+\psi([s])$.	CO(2-6)R2	I
	(ii)		
8	Consider $\psi([r][s])$.	C1	I
9	Note $\psi([r][s])=\psi([rs])$.	R1	N
10	Then, $\psi([rs])=\phi(rs)$.	CO(9,P1)R1	S
11	Then, $\phi(rs)=\phi(r)\ \phi(s)$.	CO(10, P2)R1	S
12	Since $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r)\ \phi(s)=\psi([r])\ \psi([s])$.	CO(11, P1)R1	N
13	Therefore, $\psi([r][s])=\psi([r])\ \psi([s])$.	CO(8-12)R2	N

The goal of the proof is "$\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism." There are two things to show: (i) $\psi([r]+[s])=\psi([r+s])$; (ii) $\psi([r][s])=\psi([r])\ \psi([s])$. For (i), one can rephrase $\psi([r]+[s])$ with $\psi([r+s])$ through algebraic manipulation. Using the given condition "$\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r+s])$ with $\phi(r+s)$. Using another given condition "$\phi : R \to S$ is a ring homomorphism," they can rephrase $\phi(r+s)$ with $\phi(r)+\phi(s)$. Using the given condition "$\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$"

again, one can derive that $\phi(r)+\phi(s)=\psi([r])+\psi([s])$ to conclude that) $\psi([r]+[s])=\psi([r])+\psi([s])$.

Similarly, for (ii), one can start with considering $\psi([r][s])$. One can rephrase $\psi([r][s])$ with $\psi([rs])$ through algebraic manipulation. Using the given condition "$\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$," one can further rephrase $\psi([rs])$ with $\phi(rs)$. Using the other condition "$\phi : R \to S$ is a ring homomorphism," one can rephrase $\phi(rs)$ with $\phi(r)\ \phi(s)$. Using the condition "$\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$" again, one can derive $\phi(r)\ \phi(s)=\psi([r])\ \psi([s])$ to conclude that $\psi([r][s])=\psi([r])\ \psi([s])$.

The following is Ben's proof (Figure 5.43).

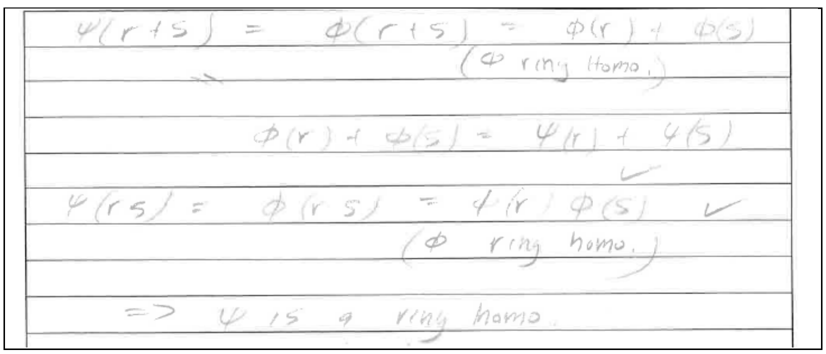


Figure 5.43. Ben's Proof.

Ben's proof was close to a successful proof. However, there were some defects. First, his notations were not precise. He had $\psi(r+s)=\phi(r+s)=\phi(r)+\phi(s)=\psi(r)+\psi(s)$ though he wanted to have $\psi([r]+[s])=\psi([r+s])=\phi(r+s)=\phi(r)+\phi(s)=\psi([r])+\psi([s])$. There was a clear difference in the meanings between $[r]$ and r. Because of the way ψ was defined, Ben's notations such as $\psi(r+s)$, $\psi(r)$, and $\psi(s)$ did not make sense though he might have meant $\psi([r+s])$, $\psi([r])$, and $\psi([s])$ by them. Such a small element of *mathematical language* as a mathematical notation can play an important role and have a great power to covey mathematical thoughts in a reasoning process. Moreover, students' understanding of a concept may be reflected by their use of the notation of the concept. Students' incomplete understanding of a concept may lead to their wrong use of the notation, which can result in a weak or incomplete argument.

5.9 Influence of Mental Attitudes

The *mental attitudes* include *tenacity* (persistence), *flexibility*, and *carefulness and alertness* (precision). *Tenacity* is the most basic factor for the *mental attitudes*. Students are required to have tenacity or persistence to try to figure things out and move forward. As soon as they stop thinking, their proofs will end at that point. *Flexibility* is the second primary factor for the *mental attitudes*, which includes changing ideas if necessary, trying a different method, paying attention to a different object, and recalling and applying a new object. Flexibility plays an important role when students have impasses. They may overcome their impasses through *flexibility*. *Carefulness* and *alertness* are the third primary factors for the mental attitudes, which involves dealing with an object precisely and accurately, and checking what has been done. *Carefulness* and *alertness* are important psychological traits for proof construction as a small careless mistake can ruin the whole proving argument.

As there was no definite way to decide the degrees of students' *mental attitudes*, the analysis involved the researcher's subjective interpretation. The results from the analysis of students' proofs implied that the *mental attitudes* were not independent of but closely related to the other aspects, including *the reasoning activity* and *the background knowledge*. I will present 7 examples while showing how students' mental attitudes affected their proofs: Students' lack of flexibility may affect their practice of rephrasing an object (Example 36); their lack of knowledge may affect their tenacity (Example 37); their lack of knowledge may affect their carefulness (Example 38); their lack of knowledge can influence their precision in the use of notations (Example 39); Tenacity and persistency can be factors for advancing a reasoning process (Example 41); and their lack of knowledge and precision or carefulness and ability to use the operation of rephrasing and combining objects may affect one another (Example 42).

Example 36 Eric (Algebra II)

Eric's case is a representative example showing that students' lack of flexibility, knowledge, and carefulness and alertness might be intertwined to produce an incomplete proof.

> Question [9] (4) (In-class problem solving session)
>
> *Let R and S be rings. Let $\phi : R \to S$ is a ring homomorphism.*
> $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ *is a well-defined ring homomorphism. Show*
> $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ *is injective.*

Table 5.36 shows a possible proof for Question [9] and Eric's difficulties in the proof construction (Table 5.36).

Table 5.36

Analysis (Type A) of Eric's Proof

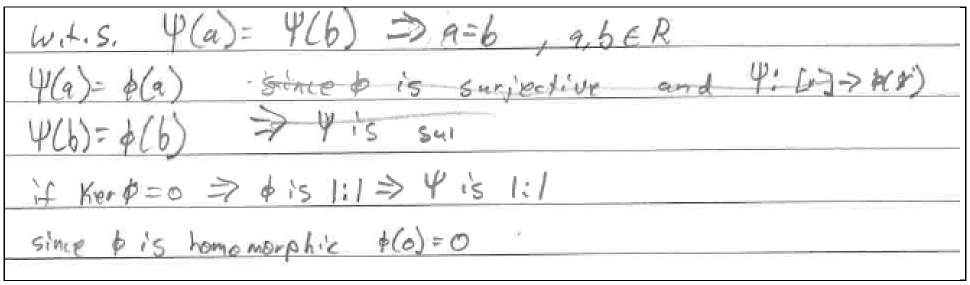
	Proof	Code	E
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective.	Given	
Y	Show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$.	R1	S
P1	$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a well defined ring homomorphism.	Given	
P2	$\phi : R \to S$ is a surjective ring homomorphism.	Given	
1	Suppose that $\psi([r]) = \psi([s])$.	C1	S
	Then, $\phi(r) = \phi(s)$.	CO(1,P1)R1	I

2	Then, $0_S = \phi(r) - \phi(s) = \phi(r-s)$.	R3	N
3	Then, $r - s \in Ker(\phi)$.	R2	N
4	Then, $r = s + k$ for some $k \in Ker(\phi)$.	R2	N
5	Then, $r = s + k \in [s]$.	R2	N
6	Then, [r] = [s].	R2	N
	Another Proof		
X	Show $Ker(\psi) = 0_{R/Ker(\phi)}$.		S
1	Consider $Ker(\psi) = \{[r] \in R/Ker(\phi)$, where $\psi([r]) = 0_S\}$.	C1	N
2	Since $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$, $Ker(\psi) = \{[r] \in R/Ker(\phi)$, where $\phi(r) = 0_S\}$.	CO(1,P1)R1	N
4	Then, $Ker(\psi) = \{[r] \in R/Ker(\phi),\ r \in Ker(\phi)\}$.	R1	N
5	Then, $Ker(\psi) = \{[r] \in R/Ker(\phi),\ [r] = Ker(\phi)$	R1	N
6	Therefore, $Ker(\psi) = 0_{R/Ker(\phi)}$.	R1	N

One of the ways to show the function $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$ is injective is to show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$. Then, paying attention to the ignition phrase "$\psi([r]) = \psi([s])$," one may start a proving argument with "Suppose that $\psi([r]) = \psi([s])$." Noting the given condition "$\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r]) = \psi([s])$ with $\phi(r) = \phi(s)$ based on the way for $\psi : R/Ker(\phi) \to S$ to be defined. One can further rephrase $\phi(r) = \phi(s)$ with $0_S = \phi(r) - \phi(s) = \phi(r-s)$ to obtain $r - s \in Ker(\phi)$, which can lead them to conclude that $[r] = [s]$.

There is another way to prove the given proof problem. There is a property of an injective homomorphism that $\phi : R \to S$ is an injective ring homomorphism if and only if $Ker(\phi) = 0_R$. Therefore, in order to prove that $\psi : R/Ker(\phi) \to S : [r] \to \phi(r)$ is injective, one can show $Ker(\psi) = 0_{R/Ker(\phi)}$. Then, one may start with considering $Ker(\psi)$. Applying the definition of $Ker(\psi)$, one may translate it into

$Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $\psi([r]) = 0_S \}$. Combining the given condition $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, one can further rephrase it with $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $\phi(r) = 0_S \}$. Furthermore, one can rephrase it with $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $r \in Ker(\phi) \}$. Then, they can conclude that $Ker(\psi) = \{[r] = Ker(\phi) \}$, namely, $Ker(\psi) = 0_{R/Ker(\phi)}$. The following (Figure 5.44) is Eric's proof.


w.t.s. $\Psi(a) = \Psi(b) \Rightarrow a = b$, $a, b \in R$

$\Psi(a) = \phi(a)$ ~~since ϕ is surjective and $\Psi : [r] \to \phi(r)$~~

$\Psi(b) = \phi(b)$ ~~$\Rightarrow \Psi$ is sur~~

if $Ker\,\phi = 0 \Rightarrow \phi$ is 1:1 $\Rightarrow \Psi$ is 1:1

since ϕ is homomorphic $\phi(0) = 0$

Figure 5.44. Eric's Proof.

Eric was knowledgeable enough to come up with two methods to prove the given statement, both of which were right strategies. He also showed flexibility to try the second strategy when he had an impasse in proving with the first strategy. Although he made his argument close to a complete proof, he was unable to do that. He first tried to show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$. He was able to rephrase $\psi([r]) = \phi(r)$ and $\psi([s]) = \phi(s)$ by using a given condition. However, he was unable to advance his reasoning process after that, at which he lost his tenacity. If he had had flexibility to rephrase $\psi([r]) = \psi([s])$ with $\phi(r) = \phi(s)$, he might have advanced his reasoning process to obtain $[r] = [s]$.

However, when he was not successful with his first strategy, he showed his flexibility to try another method to prove the given statement. Although he had a right idea, he was unable to apply it to the given problem successfully partly because he missed carefulness in expressing his idea accurately. He probably thought about showing $Ker(\psi) = 0_{R/Ker(\phi)}$ so that he might show that $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ was injective. However, he missed precision in showing his strategy by $Ker(\phi) = 0 \Rightarrow \phi$ is one to one. It was not ϕ but ψ that he wanted to work on. This carelessness might have led him to a confusion seen in his argument after that. Then, he had to forcibly lead his reasoning to the conclusion that ψ was injective. The direct cause of his failure in his second attempt was that he missed working on $Ker(\psi)$. If he had been careful enough to realize that he wanted to work on $Ker(\psi)$ and rephrased it by applying the definition of kernel and combining it with the given hypothesis, he might have obtained $[r] = Ker(\phi)$.

Example 37 Dustin (Algebra I)

The model of the structure of proof construction included tenacity and persistence as the most basic psychological factors for the mental attitudes necessary in advancing a reasoning process in proof construction. Students' lack of tenacity and persistence ends their proving argument halfway through. Students' lack of tenacity and persistence might be enhanced by their lack of knowledge. Donald's proof was such an example.

> Question [6] (In-class problem solving session)
>
> *Suppose that* $|G| = pq$ *for some primes p and q. Prove that G is either abelian or* $Z(G) = \{e\}$ *and* $|Z(G)| \neq p, q$.

The following table is a possible proof for Question [6] and shows where Aaron had difficulties in the proof construction (Table 5.37).

Table 5.37

Analysis (Type A) of Dustin's Proof

	Proof	Code	A
X	Show G is abelian or $Z(G)=\{e\}$ and $\lvert Z(G)\rvert \neq p,q$.		
Y	Show $Z(G)=G$ or $\lvert Z\rvert =1$ and $\lvert Z\rvert \neq p,q$	R1	N
P	$\lvert G\rvert = pq$ *for some primes p and q.*	Given	
1	Consider $Z(G)$.	C5	S
2	Note that $Z(G)$ is a subgroup of *G*.	C2	S
3	Recall the Lagrange's THM.	C2	S
4	Then, $\lvert Z(G)\rvert = pq, 1$, *p*, or *q*.	CO(2,3)R2	N
5	Case 1: Suppose $\lvert Z(G)\rvert = pq$.	C3	N
6	Since $\lvert G\rvert = pq = \lvert Z\rvert$, *G* = *Z*.	CO(5,P)R2	N
7	Since *Z* is abelian, *G* is abelian	R2	N
8	Case 2: Suppose $\lvert Z(G)\rvert = 1$.	C3	N
9	Then, $Z(G)=\{e\}$.	R2	N
10	Case 3: For a contradiction, suppose $\lvert Z(G)\rvert = p$.	C3	N
11	Consider the order of the quotient group $\lvert G/Z\rvert$.	C5	N
12	Since $\lvert G\rvert = pq$ and $\lvert Z(G)\rvert = p$, $\lvert G/Z\rvert = q$.	CO(P,11)R2	N
13	Recall that if $\lvert K\rvert$ is prime, *K* is cyclic.	C2	N
14	Therefore, $G/Z(G)$ is cyclic.	CO(12,13)R1	N
15	Recall that if the order of the quotient group $\lvert K/H\rvert$ is cyclic, then *K* is abelian.	C2	N
16	Therefore, *G* is abelian.	CO(14,15)R1	N
17	Then, *G* = *Z*.	R2	N
18	Then, $pq = \lvert G\rvert = \lvert Z\rvert = p$, which is a contradiction.	CO(P,15)R2	N

The conclusion of the given statement is "*G is either abelian or* $Z(G)=\{e\}$ *and* $\lvert Z(G)\rvert \neq p,q$." The translation of the conclusion into *mathematical language* can be "$G = Z(G)$ or $\lvert Z(G)\rvert = 1$ and $\lvert Z(G)\rvert \neq p,q$. "$G = Z(G)$ " can be rephrased with $\lvert Z(G)\rvert = pq$. Also, "*G is either abelian or* $Z(G)=\{e\}$ " can be rephrased with

"$|Z(G)| = pq$ or $|Z(G)| = 1$." Noting the given condition "$|G| = pq$" and recalling the relationship between $|Z(G)|$ and $|G|$, which is "*Z*(*G*) is a subgroup of *G*," and Lagrange's Theorem, one can set the following three cases: $|Z(G)| = pq$; $|Z(G)| = 1$; and $|Z(G)| = p, q$. For the first case, one may notice $|Z(G)| = pq = |G|$ and conclude that $G = Z$, which means that *G* is abelian. For the second case, one may note that $Z(G) = \{e\}$ so that $|Z(G)| = 1$. For the third case, one may use a contradiction assuming $|Z(G)| = p$. Considering the quotient group $G / Z(G)$ and recalling the fact that if $|G / Z(G)|$ is a prime number, $G / Z(G)$ is cyclic, one may realize that if $G / Z(G)$ is cyclic, *G* must be abelian and *G* = *Z*(*G*). However, it is a contradiction because one would get $pq = |G| = |Z(G)| = p$. The following figure shows Dustin's proof (Figure 5.45).

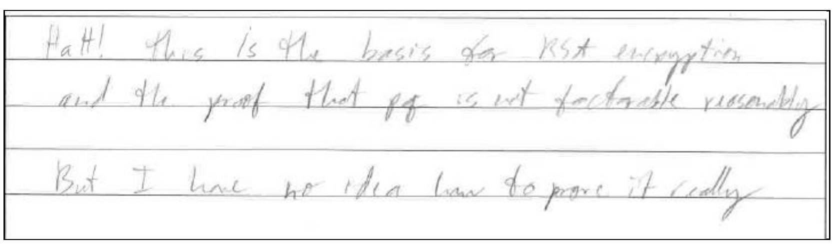
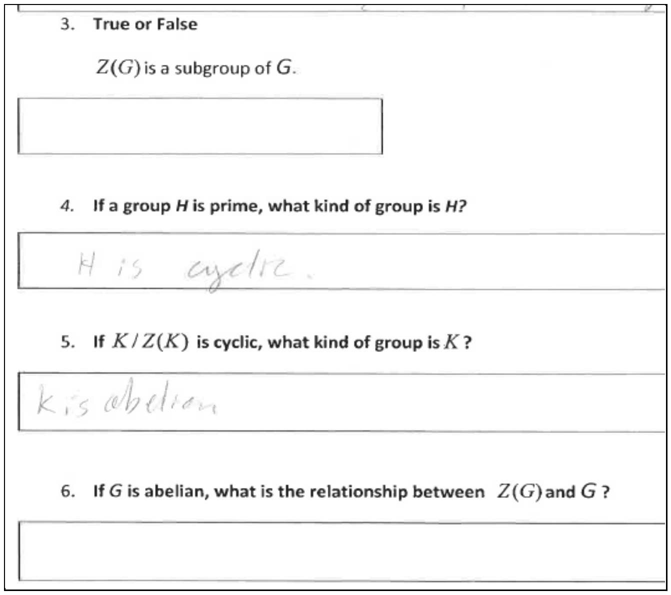

HaH! This is the basis for RSA encryption
and the proof that pq is not factorable reasonably
But I have no idea how to prove it really

Figure 5.45. Dustin's Proof.

Dustin had no idea how to prove the given problem. After providing the above statement, he was given some questions as a guide. However, as shown in the following figure (Figure 5. 45) he was not able to tell that *Z*(*G*) was a subgroup of *G* and that if *G* is abelian, $G = Z(G)$. This problem required students to consider the possible orders of *Z*(*G*) by realizing that *Z*(*G*) was a subgroup of *G* and by applying Lagrange's theorem to *Z*(*G*), which might have led them to the three cases:

$|Z(G)| = pq, 1$, or p (or q.) Dustin's lack of knowledge might have less motivated him to tackle the given problem. The following (Figure 5.46) shows Dustin's statements.


3. True or False

$Z(G)$ is a subgroup of G.

4. If a group *H* is prime, what kind of group is *H*?

H is cyclic.

5. If $K / Z(K)$ is cyclic, what kind of group is K?

K is abelian

6. If *G* is abelian, what is the relationship between $Z(G)$ and G ?

Figure 5.46. Dustin's Statement.

Example 38 Caleb (Algebra I)

Caleb's proof was an example implying that students' lack of carefulness and alertness might give a flaw to their proving arguments and that students' lack of knowledge and their lack of carefulness and alertness might be intertwined to influence their proving arguments.

Question [5] (In-class problem solving session)
Suppose that the order of G *is a prime number. Prove that G is cyclic.*

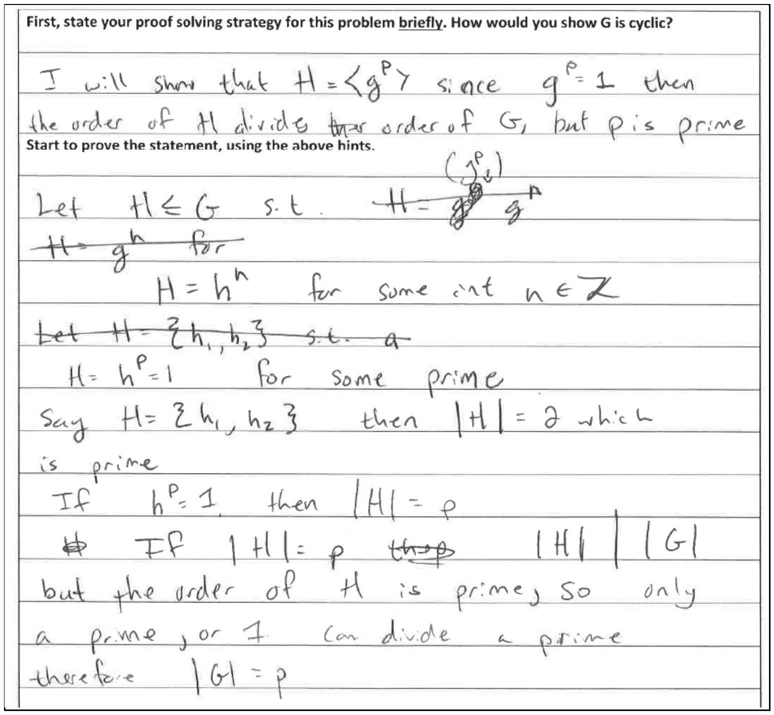
Table 5.38 shows a possible proof for Question [5] and shows where Caleb had difficulties in the proof construction.

Table 5.38

Analysis (Type A) of Caleb's Proof

	Proof	Code	Caleb
X	Show G is cyclic.		
Y	Show $G = < g >$ for some $g \in G$ with $g \neq 1$.	R1	I
P	The order of G is a prime number.	Given	
S	Let $g \in G$ with $g \neq 1$.	C1	N
1	Let $g \in G$ with $g \neq 1$.	C1	N
2	Consider $< g >$.	C5	N
3	Note $< g >$ is a subgroup of G.	C2	N
4	Recall the Lagrange's THM and apply it to $< g >$.	C2	N
5	Then, by the Lagrange's THM, $\|< g >\| = 1, p$.	CO(3,4)R2	N
6	Since $\|< g >\| \neq 1$, $\|< g >\| = p$.	CO(1,5)R2	N
7	Since $\|G\| = p$, $G = < g >$.	CO(P, 6)R2	N

The conclusion of the given statement is that "*G is cyclic*." The conclusion "*G is cyclic*" can be translated into *mathematical language* "$G = < g >$ for some $g \in G$ with $g \neq 1$." The given proof is the type of the proof of showing A = B. One can work on either A or B through rephrasing it until A becomes B or B becomes A. In this problem, one may consider and work on $< g >$ for some $g \in G$ with $g \neq 1$. Recalling Lagrange's Theorem and combining it with the property that a cyclic group generated by an element in *G* is a subgroup of *G*, one may obtain $|< g >| = 1, p$. Noting $|< g >| \neq 1$, one may decide $|< g >| = p$. Combining the hypothesis $G = < g >$ and $|< g >| = p$, one can conclude $G = < g >$. The following figure (Figure 5.47) is Caleb's proof.



First, state your proof solving strategy for this problem briefly. How would you show G is cyclic?

I will show that $H = \langle g^p \rangle$ since $g^p = 1$ then the order of H divide ~~that~~ order of G, but p is prime

Start to prove the statement, using the above hints.

Let $H \leq G$ s.t. ~~$H = g^p$~~ (g^p_0)

~~$H = g^n$ for~~

$H = h^n$ for some int $n \in \mathbb{Z}$

~~Let $H = \{h_1, h_2\}$ s.t. a~~

$H = h^p = 1$ for some prime

Say $H = \{h_1, h_2\}$ then $|H| = 2$ which is prime

If $h^p = 1$ then $|H| = p$

If $|H| = p$ ~~then~~ $|H| \mid |G|$

but the order of H is prime, so only a prime, or 1 can divide a prime

therefore $|G| = p$

Figure 5.47. Caleb's Proof.

One of the problems Caleb made in his proof was that he started his argument with $H = < g^p >$. One of the possible causes of the problem was that he was not careful enough to realize $< g^p > = < e > = \{e\}$. He might have lacked the knowledge that g^p turned out to be e. There is a possibility for both his lack of carefulness and lack of knowledge were intertwined to influence each other, which caused him to produce a non-useful statement. Another problem was that it seemed he did not make the goal of the proof clear to himself. As a result, he expressed his strategy, stating "I will show that $H = < g^p >$," and concluded his argument with "$|G| = p$." Neither of them made sense. He might have avoided those mistakes if he had paid close attention to the conclusion of the given statement "*G* is cyclic."

Example 38 Billy (Algebra II)

Billy's case is another example showing that students' lack of precision can get a proving argument blemished. His case is also representative example showing students' lack of knowledge can damage their proving arguments.

> Question [9] (3) (In-class problem solving session)
>
> *Let R and S be rings. Let $\phi : R \to S$ is a s ring homomorphism. Consider a map $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$. Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism.*

Table 5.39 shows a possible proof for Question [9] (3) and shows where Billy had difficulties in the proof construction.

Table 5.39

Analysis (Type A) of Billy's Proof

Object	Proof	Code	L
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism.	Given	
Y	Show (i) $\psi([r]+[s]) = \psi([r]) + \psi([s])$ and (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$		
P1	$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.		
P2	$\phi : R \to S$ is a ring homomorphism.	Given	
	(i)		
1	Let $[r],[s] \in R / K(\phi)$	C1	
2	Consider , $\psi([r]+[s])$	C1	
3	Note that $\psi([r]+[s]) = \psi([r+s])$.	R2	
4	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\psi([r+s]) = \phi(r+s)$.	CO(3,P1)R1	
5	Since $\phi : R \to S$ is a homomorphism, $\phi(r+s) = \phi(r) + \phi(s)$.	CO(4, P2)R1	
6	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r) + \phi(s) = \psi([r]) + \psi([s])$.	CO(5,P1)R1	
7	Then, $\psi([r]+[s]) = \psi([r]) + \psi([s])$.	CO(2-6)R2	
	(ii)		
8	Consider $\psi([r][s])$.	C1	
9	Note $\psi([r][s]) = \psi([rs])$.	R1	
10	Then, $\psi([rs]) = \phi(rs)$.	CO(9,P1)R1	

11	Then, $\phi(rs) = \phi(r)\ \phi(s)$.	CO(10, P2)R1	
12	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$.	CO(11, P1)R1	
13	Therefore, $\psi([r][s]) = \psi([r])\ \psi([s])$.	CO(8-12)R2	

The goal of the proof is "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism." There are two things to show: (i) $\psi([r]+[s]) = \psi([r+s])$; (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$. For (i), one can rephrase $\psi([r]+[s])$ with $\psi([r+s])$ through algebraic manipulation. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r+s])$ with $\phi(r+s)$. Using another given condition "$\phi : R \to S$ is a ring homomorphism," they can rephrase $\phi(r+s)$ with $\phi(r)+\phi(s)$. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" again, one can derive that $\phi(r)+\phi(s) = \psi([r])+\psi([s])$ to conclude that) $\psi([r]+[s]) = \psi([r])+\psi([s])$.

Similarly, for (ii), one can start with considering $\psi([r][s])$. One can rephrase $\psi([r][s])$ with $\psi([rs])$ through algebraic manipulation. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can further rephrase $\psi([rs])$ with $\phi(rs)$. Using the other condition "$\phi : R \to S$ is a ring homomorphism," one can rephrase $\phi(rs)$ with $\phi(r)\ \phi(s)$. Using the condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" again, one can derive $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$ to conclude that $\psi([r][s]) = \psi([r])\ \psi([s])$. The following (Figure 5.48) is Billy's proof.

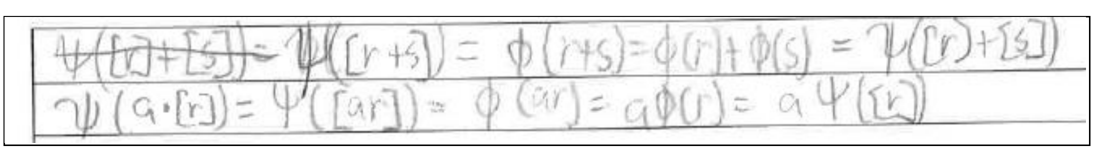


Figure 5.48. Billy's Proof.

Billy was close in showing the first property of a ring homomorphism "$\psi([r]+[s])=\psi([r])+\psi([s])$." His first attempt was right when he started with $\psi([r]+[s])$ though he crossed it out later. He was successful in rephrasing $[r]+[s]$ with $[r+s]$ and in using a given hypothesis to rephrase $\psi([r+s])$ with $\phi(r+s)$. However, he missed precision when he translated $\phi(r)+\phi(s)$ into $\psi([r]+[s])$. He was supposed to translate $\phi(r)+\phi(s)$ into $\psi([r])+\psi([s])$ by applying the given hypothesis $\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$.

Billy mistakenly got the concept of ideal involved as the second property of a ring homomorphism "$\psi([r][s]) = \psi([r])\ \psi([s])$." He showed $\psi(a[r]) = a\psi([r])$ instead of showing $\psi([r][s]) = \psi([r])\psi([s])$. This resulted from his incomplete understanding of the property of a ring homomorphism.

Example 40 Collin (Algebra II)

Collins' case is an example showing that students' lack of tenacity and their lack of knowledge might be intertwined to cause them to produce an incomplete proof.

> Question [9] (4) In-class problem solving session)
>
> *Let R and S be rings. Let* $\phi: R \to S$ *is a surjective ring homomorphism.* $\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$ *is a well defined ring homomorphism. Show* $\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$ *is injective.*

Table 5.40 shows a possible proof for Question [9] (4)and where Collin had difficulties in the proof construction.

Table 5.40

Analysis (Type A) of Collin's Proof

	Proof	Code	C
X	Show $\psi: R/Ker(\phi) \to S:[r] \to \phi(r)$ is injective.	Given	

Y	Show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$.	R1	S
P1	$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a well defined ring homomorphism.	Given	
P2	$\phi : R \to S$ is a surjective ring homomorphism.	Given	
1	Suppose that $\psi([r]) = \psi([s])$.	C1	S
	Then, $\phi(r) = \phi(s)$.	CO(1,P1)R1	I
2	Then, $0_S = \phi(r) - \phi(s) = \phi(r - s)$.	R3	N
3	Then, $r - s \in Ker(\phi)$.	R2	N
4	Then, $r = s + k$ for some $k \in Ker(\phi)$.	R2	N
5	Then, $r = s + k \in [s]$.	R2	N
6	Then, [r]=[s].	R2	N
	Another Proof		
X	Show $Ker(\psi) = 0_{R/Ker(\phi)}$.		N
1	Consider $Ker(\psi) = \{[r] \in R / Ker(\phi)$, where $\psi([r]) = 0_S \}$.	C1	N
2	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $Ker(\psi) = \{[r] \in R / Ker(\phi)$, where $\phi(r) = 0_S \}$.	CO(1,P1)R1	N
4	Then, $Ker(\psi) = \{[r] \in R / Ker(\phi),\ r \in Ker(\phi) \}$.	R1	N
5	Then, $Ker(\psi) = \{[r] \in R / Ker(\phi),\ [r] = Ker(\phi)$	R1	N
6	Therefore, $Ker(\psi) = 0_{R/Ker(\phi)}$.	R1	N

One of the ways to show the function $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective is to show that if $\psi([r]) = \psi([s])$, then $[r] = [s]$. Then, paying attention to the ignition phrase "$\psi([r]) = \psi([s])$," one may start a proving argument with "Suppose that $\psi([r]) = \psi([s])$." Noting the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r]) = \psi([s])$ with $\phi(r) = \phi(s)$ based on the way for $\psi : R / Ker(\phi) \to S$ to be defined. One can further rephrase $\phi(r) = \phi(s)$ with $0_S = \phi(r) - \phi(s) = \phi(r - s)$ to obtain $r - s \in Ker(\phi)$, which can lead them to conclude that $[r] = [s]$.

There is another way to prove the given proof problem. There is a property of an injective homomorphism that $\phi : R \to S$ is an injective ring homomorphism if and only if $Ker(\phi) = 0_R$. Therefore, in order to prove that $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is injective, one can show $Ker(\psi) = 0_{R / Ker(\phi)}$. Then, one may start with considering $Ker(\psi)$. Applying the definition of $Ker(\psi)$, one may translate it into $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $\psi([r]) = 0_S \}$. Combining the given condition $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, one can further rephrase it with $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $\phi(r) = 0_S \}$. Furthermore, one can rephrase it with $Ker(\psi) = \{[r] \in R / Ker(\phi)$, in which $r \in Ker(\phi) \}$. Then, they can conclude that $Ker(\psi) = \{[r] = Ker(\phi) \}$, namely, $Ker(\psi) = 0_{R / Ker(\phi)}$. The following (Figure 5.49) shows Collin's proof.

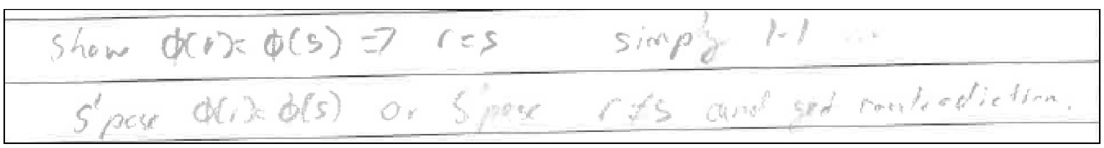

Show $\phi(r) = \phi(s) \Rightarrow r = s$ simply 1-1 ...
S'pose $\phi(r) \neq \phi(s)$ or S'pose $r \neq s$ and get contradiction.

Figure 5.49. Collin's Proof.

Judging from his statement "$\phi(r) = \phi(s) \Rightarrow r = s$," Collin seemed to have a right proving strategy to show that a function was injective. However, he was unable to apply it to $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ and to start with $\psi([r]) = \psi([s])$ in order to show $[r] = [s]$. His tenacity to try to apply his knowledge to the conclusion of the given statement might not have been strong enough. Also, he might have been confused by $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ because his understanding of a coset of $Ker(\phi)$ might not have been strong enough.

Example 40 Louis (Algebra II)

Louise's case is a representative example showing that multiple factors are intertwined together to cause students to produce incomplete proofs. In his case, lack of precision, lack of knowledge, and failure to combine objects contributed to his incomplete reasoning process.

> Question [9] (3) (In-class problem solving session)
>
> *Let R and S be rings. Let* $\phi : R \to S$ *is a ring homomorphism.*
> *Consider a map* $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.
> *Show* $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ *is a ring homomorphism.*

Table 5.41 shows a possible proof for Question [9] (3) and shows where Louis had difficulties in the proof construction.

Table 5.41

Analysis (Type A) of Louis's Proof

Object	Proof	Code	L
X	Show $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism.	Given	
Y	Show (i) $\psi([r]+[s]) = \psi([r]) + \psi([s])$ and (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$		
P1	$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$.		
P2	$\phi : R \to S$ is a surjective ring homomorphism.	Given	
	(i)		
1	Let $[r],[s] \in R / K(\phi)$	C1	N
2	Consider , $\psi([r]+[s])$	C1	N
3	Note that $\psi([r]+[s]) = \psi([r+s])$.	R2	N
4	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\psi([r+s]) = \phi(r+s)$.	CO(3,P1)R1	N
5	Since $\phi : R \to S$ is a homomorphism, $\phi(r+s) = \phi(r) + \phi(s)$.	CO(4, P2)R1	N
6	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r) + \phi(s) = \psi([r]) + \psi([s])$.	CO(5,P1)R1	N
7	Then, $\psi([r]+[s]) = \psi([r]) + \psi([s])$.	CO(2-6)R2	N
	(ii)		
8	Consider $\psi([r][s])$.	C1	N
9	Note $\psi([r][s]) = \psi([rs])$.	R1	N
10	Then, $\psi([rs]) = \phi(rs)$.	CO(9,P1)R1	N

11	Then, $\phi(rs) = \phi(r)\ \phi(s)$.	CO(10, P2)R1	N
12	Since $\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$, $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$.	CO(11, P1)R1	N
13	Therefore, $\psi([r][s]) = \psi([r])\ \psi([s])$.	CO(8-12)R2	N

The goal of the proof is "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism." There are two things to show: (i) $\psi([r]+[s]) = \psi([r+s])$; (ii) $\psi([r][s]) = \psi([r])\ \psi([s])$. For (i), one can rephrase $\psi([r]+[s])$ with $\psi([r+s])$ through algebraic manipulation. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can rephrase $\psi([r+s])$ with $\phi(r+s)$. Using another given condition "$\phi : R \to S$ is a ring homomorphism," they can rephrase $\phi(r+s)$ with $\phi(r)+\phi(s)$. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" again, one can derive that $\phi(r)+\phi(s) = \psi([r])+\psi([s])$ to conclude that) $\psi([r]+[s]) = \psi([r])+\psi([s])$.

Similarly, for (ii), one can start with considering $\psi([r][s])$. One can rephrase $\psi([r][s])$ with $\psi([rs])$ through algebraic manipulation. Using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," one can further rephrase $\psi([rs])$ with $\phi(rs)$. Using the other condition "$\phi : R \to S$ is a ring homomorphism," one can rephrase $\phi(rs)$ with $\phi(r)\ \phi(s)$. Using the condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" again, one can derive $\phi(r)\ \phi(s) = \psi([r])\ \psi([s])$ to conclude that $\psi([r][s]) = \psi([r])\ \psi([s])$.

The following is Louis's proof (Figure 5.50).

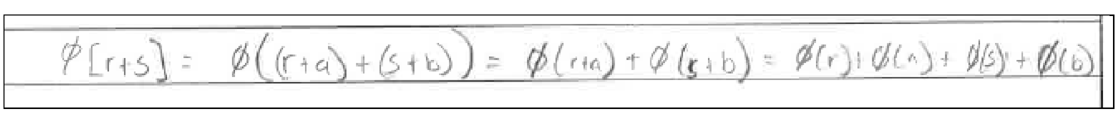


Figure 5.50. Louis's Proof.

Louis made two mistakes in starting his proof with $\phi([r+s])$. The first mistake was that he had ϕ instead of ψ . He might have avoided this mistake if he had been careful in making sure of the goal of the proof through the conclusion of the given statement "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$ is a ring homomorphism." Or, he wanted to be careful enough to realize that he made a domain error when he had $\phi([r+s])$.

The second mistake was that he started with $[r+s]$ instead of $[r]+[s]$. This mistake might have resulted from his incomplete understanding of the property of a homomorphism. He wanted to start with "$\psi([r]+[s])$" and to rephrase it with "$\psi([r+s])$."

Next, he rephrased $\phi([r+s])$ with $\phi((r+a)+(s+b))$. He probably meant an element of $[r]$ and $[s]$ by $(r+a)$ with $a \in Ker(\phi)$ and $(s+b)$ with $b \in Ker(\phi)$ respectively. Assuming that he meant $\psi([r+s])$ by $\phi([r+s])$, he was unable to apply the given hypothesis "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" so that he might have rephrased $\psi([r+s])$ with $\phi(r+s)$. He might have avoided the mistake if he had been careful to notice that he should not have rephrased $[r+s]$ with $(r+a)+(s+b)$ because $[r+s]$ was an equivalence class while $(r+a)$ with $a \in Ker(\phi)$ and $(s+b)$ with $b \in Ker(\phi)$ were elements of $[r]$ and $[s]$.

5.10 Influence of Affect and Beliefs

The model of the structure of proof construction included psychological traits as the major factors that might influence students' cognitive processes in proof construction. The model categorized those traits into two aspects: *mental attitudes* and *emotions and beliefs*. The former are the traits everyone is required to have while the latter varies according to individuals. *Affect* included emotions, moods, feelings, and self-confidences. *Beliefs* included one's proof schemes, perceptions and perspectives on logic, proofs, and mathematics. *Affect* and *beliefs* are not independent of but intertwined with the other aspects and factors of the structure of proof construction. For example, *affect* may directly influence *mental attitudes*. *Beliefs* and *background knowledge* may be related to each other. What students may believe can be the knowledge they may use in proof construction. A distinction between them is that the *beliefs* is more like a cognitive environment while the *background knowledge* is specific elements used in the cognitive environment. The beliefs is one's thinking habit or tendency while the *background knowledge* is more specific mathematical contents that are necessary for students to have to solve a given proof problem. This study did not focus on *affect* and *beliefs* as much as the other aspects. However, the analysis of students' proofs encountered some cases in which their difficulties might be related to the aspects of *beliefs*. I will present two examples that imply students' beliefs in logic for advancing a reasoning process can affect their whole proving arguments.

Example 41 Dillon (Algebra I)

Dillon's case is an example showing that student's wrong beliefs toward their logic might cause them to produce unsuccessful proofs.

> Question [4] (In-class problem solving session)
>
> *Suppose that* $G/Z(G)$ *is cyclic. Prove that* G *is abelian.*

Table 5.42 shows a possible proof for Question [4] and where Dillon had difficulties in the proof construction.

Table 5.42

Analysis (Type A) on Dillon's Proof

	Proof	Code	A
X	Show G is abelian.	Given	
Y	Show $ab = ba$ for any $a,b \in G$.	R1	N
P	$G/Z(G)$ is cyclic.	Given	
1	Let $a,b \in G$.	C1	N
2	Recall $a,b \in G$ are in some cosets.	C2	N
3	Then, $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$.	CO(P, 2)R1	N
4	Let $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$.	R1	N
5	Then, $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$.	R3	N

The conclusion of the given statement is "*G is abelian*." The translation of it into *mathematical language* provides "$ab = ba$ for any $a,b \in G$." The phrase in the *mathematical language*, "for any $a,b \in G$," is the ignition phrase. One can derive starting variables from the *ignition phrase* and provide "Suppose $a,b \in G$." This proof is the type of the proof of showing A = B. One can work on either A or B until A changes into B or B changes into A while making good use of the given conditions. In this case, one can attempt to rephrase the left side of the equation "ab" until it changes into "ba." To rephrase "ab," one may consider the given hypothesis "$G/Z(G)$ *is cyclic*" and look for the connection between the starting variables "a, b" and the hypothesis "$G/Z(G)$ *is cyclic*." Recalling the property that an element of G belongs to some coset, one may produce $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$. Then, one can rephrase ab with $x^{m+n} z_1 z_2$ for some $z_1, z_2 \in Z$. Using the property of the center of a group, one may derive $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$. The following (Figure 5.51) is Dillon's proof.

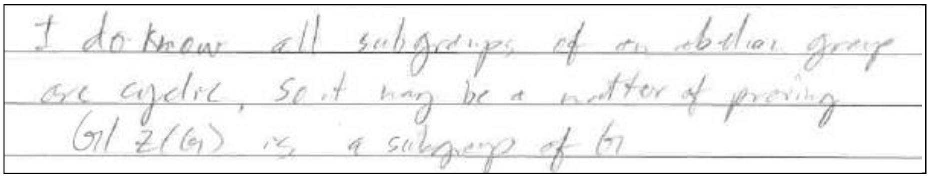


Figure 5.51. Dillon's Proof.

Dillon was unable to prove the given statement partly because he applied a wrong reasoning logic. He went "Since all subgroups of an abelian group are cyclic, we have only to show that $G/Z(G)$ was a subgroup of G so that I might prove G is abelian." To make his logic simpler, "Suppose X is S and G is A, then X is C. Suppose X is C. In order to prove G is A, you have only to show X is S." Namely, he believed that the converse of a conditional statement was true, which was not always true. He also made a wrong assumption that all subgroups of an abelian group are cyclic. His wrong belief in his logic and incomplete understanding of a concept resulted in producing a barren argument.

Example 42 Anthony (Algebra I)

Anthony's proof is another example showing students' wrong conception in logic can affect their proofs.

Question [4] (In-class problem solving session)

Suppose that $G/Z(G)$ is cyclic. Prove that G is abelian.

Table 5.43 shows a possible proof for Question [4] and where Anthony had difficulties in the proof construction.

Table 5.43

Analysis (Type A) on Anthony's Proof

	Proof	Code	A
X	Show G is abelian.	Given	

Y	Show $ab = ba$ for any $a, b \in G$.	R1	N
P	$G/Z(G)$ is cyclic.	Given	
1	Let $a, b \in G$.	C1	N
2	Recall $a, b \in G$ are in some cosets.	C2	N
3	Then, $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$.	CO(P, 2)R1	N
4	Let $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$.	R1	N
5	Then, $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$.	R3	N

The conclusion of the given statement is "*G is abelian*." The translation of it into *mathematical language* provides "$ab = ba$ for any $a, b \in G$." The phrase in the *mathematical language*, "for any $a, b \in G$," is the ignition phrase. One can derive starting variables from the *ignition phrase* and provide "Suppose $a, b \in G$." This proof is the type of the proof of showing A = B. One can work on either A or B until A changes into B or B changes into A while making good use of the given conditions. In this case, one can attempt to rephrase the left side of the equation "ab" until it changes into "ba." To rephrase "ab," one may consider the given hypothesis "$G/Z(G)$ *is cyclic*" and look for the connection between the starting variables "a, b" and the hypothesis "$G/Z(G)$ *is cyclic*." Recalling the property that an element of *G* belongs to some coset, one may produce $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$. Then, one can rephrase ab with $x^{m+n} z_1 z_2$ for some $z_1, z_2 \in Z$. Using the property of the center of a group, one may derive $ab = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = ba$. The following (Figure 5.52) is Anthony's proof.

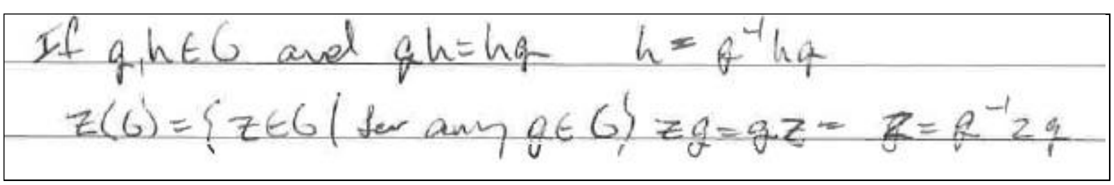

If $g, h \in G$ and $gh = hg$ $h = g^{-1}hg$
$Z(G) = \{z \in G \mid$ for any $g \in G\}$ $zg = gz$ → $z = g^{-1}zg$

Figure 5.52. Anthony's Proof.

Anthony noted the conclusion of the given statement "*G* is abelian" and translated it into mathematical language "$gh = hg$." The problem of his argument was that he tried using the conclusion as a hypothesis. He was not careful enough to realize that he was not allowed to use it as a condition with which to advance his reasoning process. Anthony was unable to prove the statement mainly because he did not understand the logical argument to advance a reasoning process. The greatest problem of his was that he tried to advance a reasoning process by assuming that the conclusion of the given statement that was to be proved. His logical flaw might have been avoided if he had made it clear that $ab = ba$ for any $a, b \in G$ was the goal.

5.11 Summary

This chapter has detailed various types of students' difficulties with proof construction. Students' proofs were systematically analyzed through the frameworks created based on the model of the structure of proof construction. The results were presented in terms of each component of the structure of proof construction. The analysis revealed that each of students' difficulties, mistakes, and impasses were often caused by multiple factors being intertwined. Although students' difficulties seemed to occur in a complex way, the analysis based on the model of the structure of proof construction helped to make the mechanism of the occurrences of students' difficulties clear. The model let the analysis to sum up the sources of students' difficulties in three types of sources: students' lack of knowledge; their lack of tenacity, persistence, flexibility, carefulness, alertness, and precision; and their lack of metacognitive knowledge for advancing a reasoning process. Although students' emotions, feelings, self-confidences, and beliefs were considered to influence their proving performances, this study did not focus on those factors. The next chapter will answer the research questions based on the results of the analysis of students' proofs presented in this chapter, while referring back to the existing literature.

Chapter 6: Discussion

6.1 Introduction

In Chapter 5, I presented the results from the analysis of students' proofs in light of the structure of proof construction. This chapter is dedicated to addressing the research questions while highlighting the findings and referring to the supporting literature. In this chapter, I will discuss the followings: (1) significance of the model of the structure of proof construction; (2) sources of students' difficulties with proof construction, (3) specific pedagogical suggestions to help students with proof construction, and (4) usefulness of the model of the structure of proof construction. I end this chapter with the limitations of this study and possible future research.

6.2 Model of the Structure of Proof Construction

Kieran (1998) reported the significance of models in research, stating "the current reporting of research suggests that both (empirical and theoretical researches) involve the description of observed phenomena by means of models (p. 213)". She also reported that research results were frequently described and explained in the form of explicitly formulated models. Moreover, she claimed that results without explanatory models would not help us understand the observed phenomena. Brown (1998) also indicated that results themselves would not be effective in understanding the phenomena and implied that results should be analyzed with theorization for a better understanding of the phenomena. This study started with modeling the structure of proof construction for the purpose of analyzing, describing, explaining, and interpreting students' difficulties with proof construction in a clear and organized way.

First, I clarify the significance of having created a model of the structure of proof construction. Then, I will answer the first research question while referring to how the model was created. Lastly, I mention the types of proofs that were obtained in the process of modeling the structure of proof construction.

6.2.1. Significance of the model of proof construction

Before examining students' difficulties with proof construction, it had to be specified in what terms and from what angles their proofs should be analyzed. These demands motivated me to create a framework for analyzing students' difficulties in a systematic and organized way. In order to create such a framework, I explored a model of the structure of proof construction. The structure of proof construction meant a comprehensive picture of proof construction that can describe and encompass the aspects, factors, patterns, and features seen in thinking processes in a proving activity.

The motivation of creating a model of the structure of proof construction also arose from the major goals of this study. This study aimed not only to clarify the sources of students' difficulties but also to make practical suggestions to help students. In particular, I focused on exploring the patterns and features that might help them with a syntactic approach.

Different researchers have studied students' difficulties with proof construction from different angles. Some of them focused on detecting students' difficulties with proving on a particular subject. For example, Selden and Selden (1995) pointed out students' difficulties with translating informal language to formal language through a calculus proof. Weber (2001) noted students' lack of proving strategy through an abstract algebra proof. Edward and Wards (2004) observed students' misuse of definitions through a real analysis proof. Savic (2011) showed that logic rarely occurred in students' proofs through the "chunk-by-chunk" analysis. Other studies categorized students' difficulties with proof construction. For example, Moore (1994) examined students' difficulties with proving and categorized them into seven types. Selden and Selden (2009) categorized students' difficulties with proving into more types but from a different angle than Moore's. Other studies examined students' proofs schemes. Harel and Sowder (1998) examined students' proofs in terms of *external*, *empirical*, and *analytical* proof schemes. Weber and Alcock (2004) studied students' proofs in terms of semantic or syntactic proof schemes. However, there seemed to be no studies that attempted to analyze students' proofs in light of the structure of proof construction. Selden and Selden (2007) offered a proof framework as a method to teach students, in which they suggested students should write a proof from the beginning and the end towards the middle. There was room for exploring an effective method to help students to write a proof from the top down.

The creation of a model of the structure of proof construction was important and necessary for the following reasons: (1) to clarify in what terms and from what angles to examine students' difficulties with proof construction; (2) to analyze, understand, and explain their difficulties in a clear and organized way; (3) to help students grasp what proof construction was like and how they should advance a reasoning process; and (4) to shed light on students' difficulties with proof construction from a new perspective, which past studies may not have.

6.2.2 Model of the structure of proof construction

The first research question this study raised was the following:

"*What is a suitable model for characterizing the structure of proof construction*?"

This section answers the above research question, describes how the model was obtained, presents the model, and refers to the related literatures. This study started with examining the researcher's thinking processes in proof construction in order to explore what might be involved in cognitive processes in proof construction. A think aloud method was adopted for the method to examine the researcher's cognitive processes. In order to explore and generalize the patterns and features seen in proof construction across multiple mathematical subjects, more than 40 proofs of theorems and propositions were analyzed, which were collected mainly from undergraduate Algebra, Analysis, and Topology. A few proofs came from Calculus, Discrete Mathematics, and Linear Algebra, respectively. The researcher solved those proofs, while carefully self-monitoring, observing, describing, organizing, and categorizing the factors involved in the researcher's thinking processes. In particular, the researcher focused on how each step can be obtained for successful proof construction. The researcher explored what types of thinking actions or what types of operations might be applied to generate the next statement from a previous one.

As a result, all the observed operations for advancing a reasoning process were categorized into four types: *rephrasing an object*; *combining objects*; *creating a cue*; and *checking and exploring* (Table 3.2). In this study, an object was meant to represent a statement or part of a statement for each step in a proof. The operations of *rephrasing an object*, *combining objects*, and *creating a cue* were considered as *main thinking actions* while *checking and exploring* were considered as *supporting actions*. The difference between *the main actions* and *the supporting actions* was that in order to convince others of the validity of a proof, the processes or the steps obtained through the former had to be explicitly expressed while the processes or the steps obtained through the latter did not necessarily have to be expressed explicitly. For example, the operation of *checking and exploring* included observing, reviewing, reflecting, searching, intuiting, trying, illustrating, and experimenting. The processes taken through *checking and exploring* did not necessarily have to be explicitly expressed in order to convince others of the validity of their proofs. Since different individuals *check* what they have done and *explore* the methods to figure things out in different ways, this study did not set a main focus on the operation of *checking and exploring*. Therefore, a possible structure for *checking and exploring* was not scrutinized. The analysis through the think-aloud method also found two major roles in *the main actions*: *transforming objects*; and *igniting processes*. R*ephrasing an object* and *combining objects* were categorized into the operations for *transforming objects*. C*reating a cue* was categorized into the operations for igniting a reasoning process. *Rephrasing an object* was further categorized into the following three types.

- R1: Rephrasing an object by applying definitions, properties, theorems, propositions, and negations
- R2: Rephrasing an object through formal interpretations and informal interpretations such as common sense
- R3: Rephrasing an object through algebraic manipulation such as calculation, computation, and solving equations

C*reating a cue* was also further categorized into five types:

- C1: Set a variable;

- C2: Recall prior knowledge such as relevant properties, theorems, propositions, and mathematical laws;
- C3: Set some cases;
- C4: Make a claim or set a new object;
- C5: Consider an object.

Moreover, the relationships among the operations were clarified. The analysis revealed that there was an order of the operations that students should try in advancing a reasoning process. *Rephrasing an object* is the primary and basic operation. *Combining objects* is the secondary operation that should be tried when the operation of *rephrasing an object* does not work. *Creating a cue* is the final operation to be used. The operation of *creating a cue* can be considered as the highest level of operation in terms of difficulty because students have to generate a new object while they can depend on what they are given for the operations of *rephrasing an object* and *combining objects*. Moreover, the analysis observed that the operation of *combining objects* is accompanied by the operation of *rephrasing an object*. That is, after students combine objects, they produce a new object through *rephrasing an old object*. The operation of *creating a cue* (see C2 in Table 3.2 or 6.1) is accompanied by the operations of *combining objects* and *rephrasing an object*.

In the process of exploring the features of the *reasoning activity*, noting the importance of how to get started on a proof for successful proof construction, two major stages (*the opening stage* and *the body construction stage*) were set for proof construction: *the opening stage* and *the body construction stage*. *The opening stage* is a crucial stage for students to make their proving arguments successful though the amount of the contents contained in *the opening stage* may be much smaller than those contained in *the body construction stage*. In *the opening stage*, students note the conclusion of the given statement, translate it into *mathematical language*, set a starting variable from *the mathematical language*, make sure of the given hypotheses, and translate those hypotheses into *mathematical language*, if necessary. In *the body construction stage*, one advances a reasoning process by the four actions (*rephrasing an object*, *combining objects*, *creating a cue*, and *checking and exploring*).

The significance of *the opening stage* cannot be overemphasized. There are three important roles in *the opening stage*: making the goal of the proof clearer; setting a starting variable; and making sure of what conditions are available. In particular, the first two roles are crucial. Both roles are realized usually through translating the conclusion of the given statement into *mathematical language*. Translation of the conclusion of the given statement into *mathematical language* not only makes the goal of the proof clearer but also makes the distance or the proving process to the goal shorter. Translation of an object into *mathematical language* gives a motive power for students to develop a further reasoning argument in *the body construction stage* as well as in *the opening stage*. Newell and Simon (1972) presented the standard theory of problem solving, in which they viewed a problem solving process as a process of reducing the differences between the desired and current states by applying operators or creating sub-problems. Translation of an object into *mathematical language* can be considered to be a key factor for problem solving in proof construction. Considering all the above features observed in the *reasoning activity*, the following framework (Table 3.1) was created as a model of the structure of *the reasoning activity*.

The *reasoning activity* focused on students' cognitive or thinking actions, in particular, their operations for advancing a reasoning process. The operations can be compared to the "tools" used for proof construction. Then, the "materials" students manipulate with those tools correspond to the elements of their *background knowledge*, including concepts, definitions, properties, notations, theorems, and propositions. Therefore, the background knowledge was considered to be another major factor for proof construction. The *reasoning activity* and the *background knowledge* can be considered as main factors involved in cognitive processes in proof construction.

However, in order to capture the aspects for proof construction comprehensively, this study further added two more aspects: *the mental attitudes*, and *affect and beliefs*. Schoenfeld (1983) pointed out that purely cognitive behavior was extremely rare. He discussed a variety of factors that might shape pure cognition, including the environment, one's affect, feeling, and perception of self and the environment. The mental attitudes are the abilities related to self-regulation. *The mental attitudes* included tenacity, existence, flexibility, carefulness, alertness, and precision. *Tenacity* is one's ability of persistence to try to continue to think, work on, and tackle a proof, and not to give up. Once students lose their tenacity in proving, their proofs end at that point. *Flexibility* is a mental attribute that allows students to discard the idea that does not work and to explore and try a new idea. It is a motive power to help students overcome their impasses. *Carefulness and alertness* is a mental skill that allows students to deal with an object precisely and correctly. There may be more factors for *the mental attitudes* necessary for proof construction, but this study set those three attributes as the main factors of *the mental attitudes*.

This study further considered *affect and beliefs* for the aspect of proof construction. *Affect and beliefs* include one's emotions, moods, feelings, beliefs, and self-confidence. For example, students' worries from everyday life, test anxieties, or low self-confidence in mathematics may influence their proving performances and any mathematics problem-solving. Students' beliefs may also affect their performances. For example, some students may believe that mathematics learning is a matter of memorization of the formulas and of application of those formulas to given problems. This belief may lead students to depend on their memorization instead of thinking deeply and critically by themselves. Students' proof scheme is another example of student's *beliefs*. Their beliefs that a statement can be proved by showing examples or diagrams may cause them to develop their empirical proof schemes.

Considering all these, this study viewed proof construction as a process of advancing a reasoning process by way of *mathematical language* through the four main operations of *rephrasing an object*, *combining objects*, *creating a cue*, and *checking and exploring* while making applying their *background knowledge* and practicing their *mental attitudes* (*tenacity*, *flexibility*, and *carefulness and alertness*) under the influence of *affect and beliefs*. Thus, the model of the structure of proof construction was created and described in a 3D model (Figure 3.1).

I found out that the following literature was in agreement with the structure of proof construction. For example, Funke (2010) provided four elements to describe a problem-solving situation: givens, goals, operators and barriers. Givens is the knowledge that a students has about the problem. The operators are the actions that a student applies to reach the goal of a given problem. Barriers are the difficulties a student has to overcome in the process of achieving the goal of the problem. *The background knowledge* in the model of the structure of proof construction may correspond to Funke's knowledge. The operations in the model (*rephrasing an object*, *combining objects*, *creating a cue*, and *exploring and checking*) detail Funke's operators and tools. Various difficulties occurring each aspect of proof construction in the model (Figure 3.1) extend Funke's barriers. Funke pointed out that motivational and affective means as well as cognition can be factors to help students overcome their difficulties. The aspects of *the mental attitudes* and *affect and beliefs* in the model (Figure 3.1) can be related to Funke's motivational and affective means.

Other researchers noted the affective aspect, including beliefs, feelings, and moods, as a significant factor, as well as the cognitive aspect, in mathematical thinking (Fringhetti and Morselli, 2004; Goldin, 2002; Leron and Hazzan, 1997; Schoenfeld, 1983). More specifically, Fringhetti and Morselli (2004) explored the way to integrate the cognitive and affective aspects through a student's proof construction. Golding (2002) even indicated that the affective aspect was central to cognition in mathematical activity. Leron and Hazzan (1997) gave some specific mental forces as factors for the affective aspect which may influence mathematical thinking. Those mental forces have some similarities with the factors in the structure of proof construction. For example, students' mental force to try to make things sense may be related to their *mental attitude*, in particular, *tenacity*. Their mental force to stick to something familiar to them may be related to their *flexibility* as the counter-mental force.

6.2.3 Types of proofs

Proofs were classified into three types according to the ways to set a starting variable in *the opening stage*. The first type of proofs had *the opening stage* in which students derived a starting variable from the conclusion of the given statement. The second type of proofs had *the opening stage* in which students derived a starting variable from a hypothesis of the given statement. The third type of proofs had *the opening stage* in which students did not have to derive a starting variable and had only to directly work on the conclusion of the given statement. The majority of the proofs analyzed in this study belonged to the first type of proof. The model of the structure of proof construction (Figure 3.1) was created based on the first type of proofs. The proofs that required students to construct an object such as a sequence and an open set belonged to the second type. The proofs asking students to prove A = B belonged to the third type. This type included mathematical induction and proofs of trigonometric identities.

6.3 Sources of Students' Difficulties with Proof Construction

This section highlights the findings obtained from the analysis of students' proofs in light of the structure of proof construction while answering the second research question:

> "*What difficulties do students have with proof construction and what are the sources of their difficulties in light of the structure of proof construction*?"

The components in the structure of proof construction are the main factors contributing to the sources of students' difficulties with proof construction. For example, students' inabilities to apply the operations of *rephrasing an object*, *combining objects,* and *creating a cue* can be the direct sources of students' difficulties with advancing a reasoning process. Moreover, students' difficulties with advancing a reasoning process are directly caused by their lack of knowledge, tenacity, persistence, flexibility, carefulness, alertness, and precision. The model of the structure of proof construction also suggests that students' self-confidence and beliefs about logic and proof construction can affect their proving performances. Finally, the analysis of students' proofs also suggested that students' lack of knowledge of metacognitive and methodological knowledge for advancing a reasoning process caused their difficulties.

6.3.1 Opening Stage

How to start a proving argument is a key factor for constructing a successful proof. Moore (1994) observed, as one of the major sources of students' difficulties with proof construction, that students did not know how to begin proofs. He pointed out that many factors might affect their inability to start their proof construction, including their difficulties with language and notation, and with definitions. This study scrutinized how students difficulties with opening proof construction occurred and explored how instructors would help students to start proof construction.

Out of the 81 proofs that were analyzed, 59 proofs (73.4%) were incomplete. Out of those unsuccessful proofs, 39 proofs (66.1%) had defects in their opening stages. Overall, almost 50% of the proofs analyzed for this study were unsuccessful because students may have mismanaged their opening stages. The analysis of the data showed that about 67% of the unsuccessful proofs resulted from students' mismanagement of the opening stage. This showed how important it was for students to manage the opening stage successfully. The degree of success in managing the opening stage may well decide the degree of success in proof construction. The results also revealed two major roles of the opening stage: (a) having students make sure of the goal of the proof;

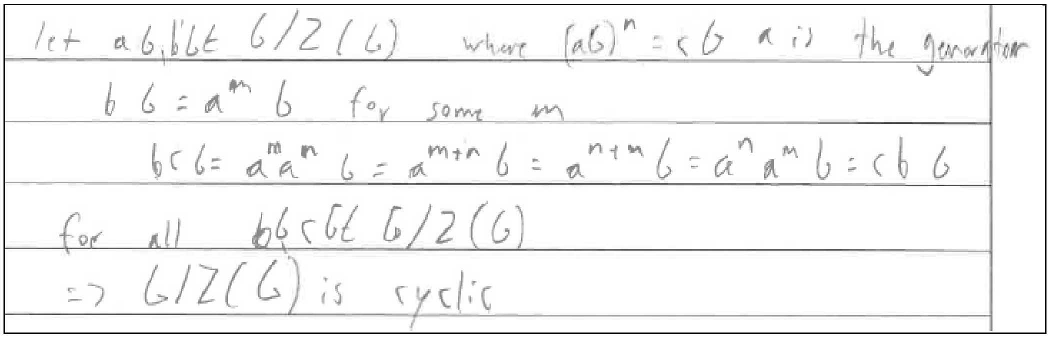
(b) having students set a starting variable with which to develop a reasoning process.

The opening stage helps students to make sure of the goal and set a direction to reach the goal. The results showed that students' failure to have the goal of a proof in *mathematical language* was a major source for the mismanagement of the opening stage. Having a goal in *mathematical language* gives students a right direction. The goal has a potential to lead students to advance their reasoning process and to get students back on the right track when they go astray or have impasses. The goal of the proof comes from the conclusion of the given statement. In most proofs used for this study, the conclusion of the given statement contains some concept such as abelian, compact, continuous, cyclic, homomorphism, or injective, etc. When these concepts are translated into *mathematical language*, namely, their definitions, they acquire a power to drive students to advance their reasoning processes.

The opening stage gets students to set a variable necessary to develop *the body construction stage*. The great majority of the proofs used in this study required students to set a variable in *the opening stages*. A proving argument is advanced by using variables. A variable is a key element in making a rigorous, formal, and logical mathematical argument. Students cannot convey their mathematical ideas rigorously, formally, and logically without using variables in their proving arguments.

Then, a question arises: How do students set a variable correctly? The great majority of the proofs used in this study showed that it was *the mathematical language* for the conclusion of the given statement that a starting variable were derived from. In particular, in most cases, the starting variable was directly found in *the ignition phrase* such as "for…" or "if …," which was contained in the definition of a concept involved in the conclusion of the given statement. Therefore, the key to setting a starting variable is to pay attention to the conclusion of the given statement, translate it into mathematical language while applying the definition of a concept involved in it, and find an *ignition phrase*, and derive a starting variable. There were a few cases in which students had to derive a variable from *the mathematical language* for a hypothesis of the given statement. However, those types of proofs were the type of proofs in which students were required to construct a certain object such as an open set. In any case, the variable is found in *the ignition phrase* in *the mathematical language*.

For example, Frank's proof was a representative example of showing students' mismanagement in the opening stage. Frank (Figure 6.1) showed "$G/(Z)$ is cyclic" though he was asked to show "G is abelian." If he had made sure of the goal of the proof by noting the conclusion of the given statement "*G is abelian*," he may have at least avoid ending up with "$G/(Z)$ is cyclic" as his conclusion. Also, he started his proving argument by setting variables $aG \in G/Z(G)$ and $bG \in G/Z(G)$, which were not correct notations. It is most likely that he derived those variables from the given hypothesis "$G/(Z)$ is cyclic," which caused his proving argument to be unsuccessful. If he had derived a starting variable from *the mathematical language* for the conclusion of the given statement, he could have been able to avoid the confusion that was found in his proving argument.



let $aG, bG \in G/Z(G)$ where $(aG)^n = cG$ a is the generator

$bG = a^m G$ for some m

$bcG = a^m a^n G = a^{m+n} G = a^{n+m} G = a^n a^m G = cbG$

for all $bG\, cG \in G/Z(G)$

$\Rightarrow G/Z(G)$ is cyclic

Figure 6.1. Frank's Proof.

In summary, the findings indicated that the students' inability to get started on a proof might result from their lack of knowledge of the features of the opening stage, including the representative steps to take in *the opening stage*: paying attention to the conclusion of the given statement; translating it into *mathematical language*; deriving a starting variable from an *ignition phrase* in *the mathematical language* for the conclusion.

6.3.2 Rephrasing an Object

Based on my framework, rephrasing an object is the primary operation for advancing a reasoning process among the three (*rephrasing an object*, *combining objects*, and *creating a cue*) in the sense that the operations of *combining objects* and *creating a cue* are often accompanied by the operation of *rephrasing an object*. There are three types of the operation of *rephrasing an object*: (a) rephrasing an object by applying a definition, a theorem, or a property; (b) rephrasing an object through interpretation; and (c) rephrasing an object through algebraic manipulation. This section highlights the sources of students' difficulties with *rephrasing an object* based on the following findings.

First, students' lack of knowledge of an object, including the definition of a concept and a property of concept, can be a central cause for malfunctioning the operation of *rephrasing an object*. There are two cases for "students' lack of knowledge of a definition or a property of concept." The first case is that students simply do not know or have forgotten what a definition or property of concept is. Then, students cannot rephrase an object by applying the definition or a property of concept. The second case is that students have an incomplete or a weak understanding of a definition or a property of concept. This can cause problems with their notations or expressions of mathematical terms, which affects their practice of the operation of *rephrasing an object*.

For example, Billy (Figure 6.2) had an incomplete understanding of a property of a ring homomorphism, which misled him to rephrase the property of homomorphism in a wrong way and consequently resulted in an unsuccessful proof.

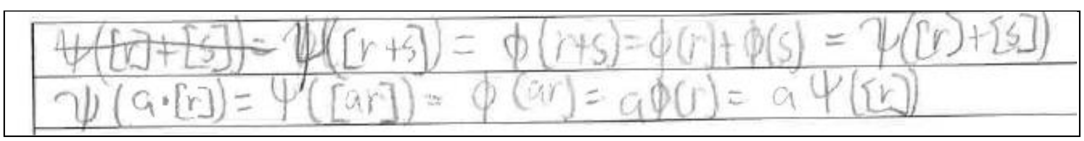
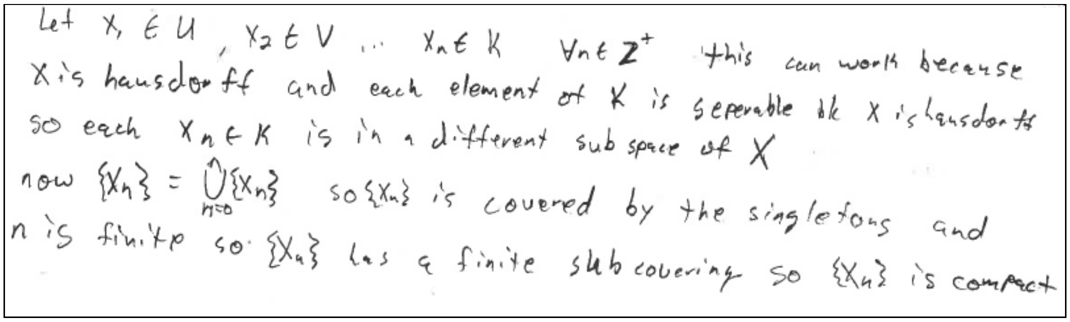


Figure 6.2. Billy's Proof.

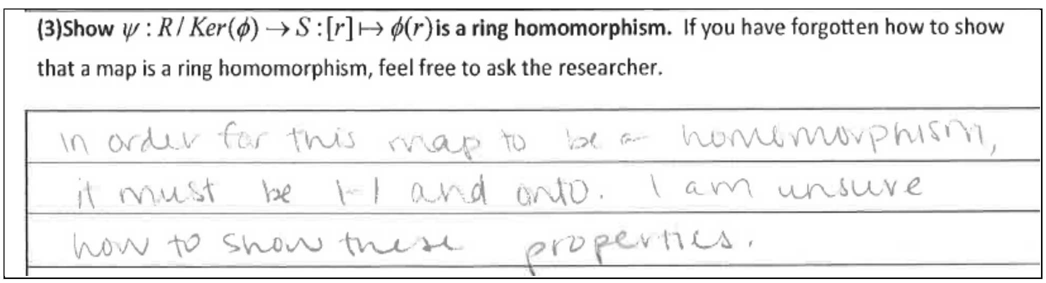
In another example, Elias (Figure 6.3) had an incomplete understanding of the concept of compactness. He did not have the notion that a finite subcover of a space had to be derived from an open cover of the space and that he had to set an open cover of a space before talking about a finite open subcover of the space. His incomplete understanding of the concept of compactness hindered him from translating the conclusion into *mathematical language* to set a right starting variable, an open cover of *K* in his case.



Let $x_1 \in U$, $x_2 \in V$ ... $x_n \in K$ $\forall n \in \mathbb{Z}^+$ this can work because X is hausdorff and each element of K is seperable b/c X is hausdorff so each $x_n \in K$ is in a different sub space of X now $\{x_n\} = \bigcup_{n=0}^{n} \{x_n\}$ so $\{x_n\}$ is covered by the singletons and n is finite so $\{x_n\}$ has a finite subcovering so $\{x_n\}$ is compact

Figure 6.3. Elias's Proof.

Similarly, Savanna (Figure 6.4) was unsure about how to show a given homomorphism was one to one and onto. Instructors cannot overemphasize the importance of students having a solid understanding of the basics of concepts, including the definitions, properties, and meanings, relevant theorems, and proving techniques.



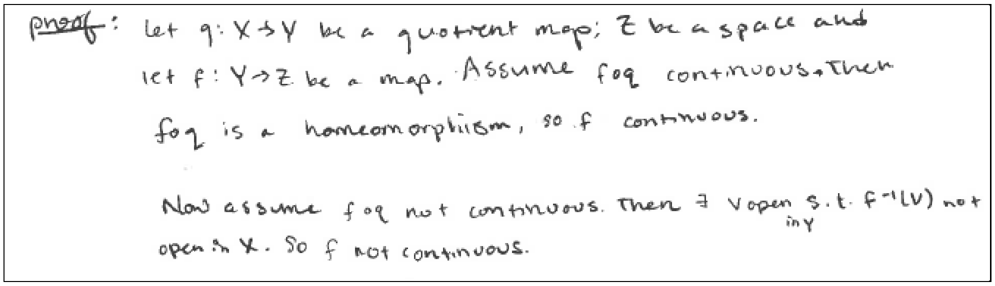
(3)Show $\psi : R / Ker(\phi) \rightarrow S : [r] \mapsto \phi(r)$ is a ring homomorphism. If you have forgotten how to show that a map is a ring homomorphism, feel free to ask the researcher.

in order for this map to be a homomorphism, it must be 1-1 and onto. I am unsure how to show these properties.

Figure 6.4. Savanna's Proof.

Another cause for students' failure to *rephrase an object* may be that they have not fully acquired or established the skill of using *mathematical language* to advance their reasoning process. They may not fully understand the usefulness, effectiveness, necessity, and importance of using *mathematical language* as the most fundamental element for constructing a proving argument. Even when they know the definition of a concept, they are sometimes unable to apply it to an object.

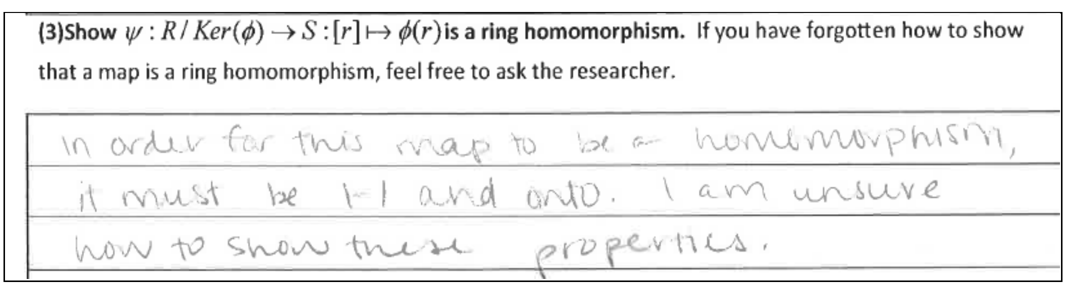
For example (Figure 6.5), Natalie knew the definition of a continuous function in a topological sense, but was unable to apply it to the composite function $f \circ q$, in which $f: Y \to Z$ and $q: X \to Y$, being continuous and to translate it into *the mathematical language* "$(f \circ q)^{-1}(W)$ is open in X."


proof: let $q: X \to Y$ be a quotient map; Z be a space and let $f: Y \to Z$ be a map. Assume $f \circ q$ continuous, then $f \circ q$ is a homeomorphism, so f continuous.

Now assume $f \circ q$ not continuous. Then $\exists$ V open in Y s.t. $f^{-1}(V)$ not open in X. So f not continuous.

Figure 6.5. Natalie's Proof.

In another example (Figure 6.6), when Savanna was asked to show a given homomorphism was injective, she illustrated what an injective map was but did not translate it into *mathematical language* "if $\psi([r]) \neq \psi([s])$, then $[r] \neq [s]$" or "$Ker(\psi) = 0$." It is crucial for students to be equipped with the skill necessary for translating a concept into *mathematical language*, especially, definitions of concepts, which are the most representative *mathematical language*. A definition of concept is not just an explanation or a description of a concept but is a structure, which has a potential to motivate students to advance their reasoning process in proof construction.


(3)Show $\psi: R/Ker(\phi) \to S: [r] \mapsto \phi(r)$ is a ring homomorphism. If you have forgotten how to show that a map is a ring homomorphism, feel free to ask the researcher.

In order for this map to be a homomorphism, it must be 1-1 and onto. I am unsure how to show these properties.

Figure 6.6. Savanna's Proof.

Some researchers noted the importance of advancing a proving process with mathematical language for constructing a proof. For example, Selden and Selden (1995) observed students' difficulties with "unpacking" informal statements into formal statements. In this study, their terms "*unpacking*" and "formal statement" are included in "translating an object into *mathematical language*" and "*mathematical language,*" respectively. Instructors can emphasize to their students that *mathematical language* has a motive power for advancing a reasoning process. In particular, instructors must train students to get into the habit of converting a mathematical concept into *mathematical language* whenever they see a mathematical concept in a statement.

Third, students' lack of *flexibility* can affect their skill of rephrasing an object. For example, as it was seen in Eric's proof (Figure 6.7), although he realized that $\psi([r]) = \phi(r)$ and $\psi([s]) = \phi(s)$ for $\psi([r]) = \psi([s])$, he was not flexible enough to convert $\psi([r]) = \psi([s])$ into $\phi(r) = \phi(s)$. If he had flexibility enough to manipulate it into $\phi(r) - \phi(s) = 0$ by moving the term $\phi(s) = 0$ to the left hand side, he could have further converted it into $\phi(r - s) = 0$ to reach $r - s \in Ker(\phi)$.

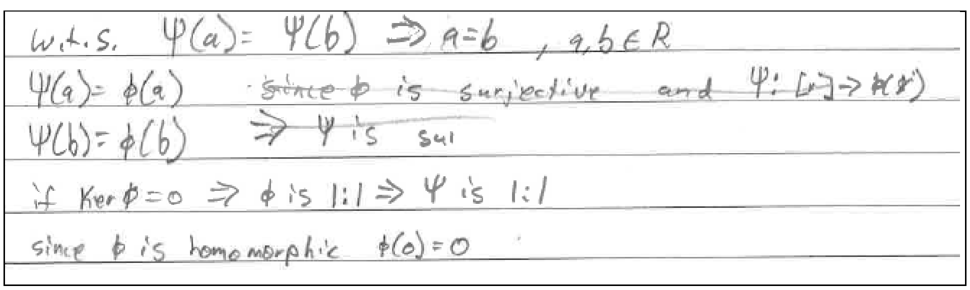


Figure 6.7. Eric's Proof.

In summary, students' inability to *rephrase an object* may be directly influenced by their lack of *background knowledge*, and directly or indirectly by their lack of *flexibility*, *carefulness and alertness*. The findings also indicates students' difficulties with rephrasing an object may result from their lack of awareness of the importance of *mathematical language* as a means of advancing a reasoning process.

6.3.3 Combining Objects

Combining objects is the second primary operation for advancing a reasoning process. Based on the findings, this section discusses two possible sources of students' difficulties with *combining objects* based on the results of the analysis of students' proofs. One is that students miss using a hypothesis in a given statement. The other is that students' difficulties with finding a connection between the two objects that are supposed to be combined. The latter source is directly related to their lack of knowledge.

Students have impasses because they fail to use the operation of combining objects. For example, although Berkeley (Figure 6. 8) knew what she needed to show to prove the given statement, which was "$\psi([r][s]) = \psi([r])\psi([s])$, she was not able to show why the equality held. This was because she missed noting and using the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$" to apply it to $\psi([rs])$. If she had come up with the idea of combining $\psi([rs])$ and the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$," she might have rephrased $\psi([rs])$ with $\phi(rs)$ to obtain $\phi(rs) = \phi(r)\phi(s) = \psi([r])\psi([s])$. The source of her failure to *combine the objects* can be considered to be her lack of carefulness and alertness. She needed to be careful enough to realize she was not allowed to rephrase $\psi([rs])$ with $\psi([r])\psi([s])$ for free. She also needed to be careful to ask herself how she should transform $\psi([rs])$ into $\psi([r])\psi([s])$.

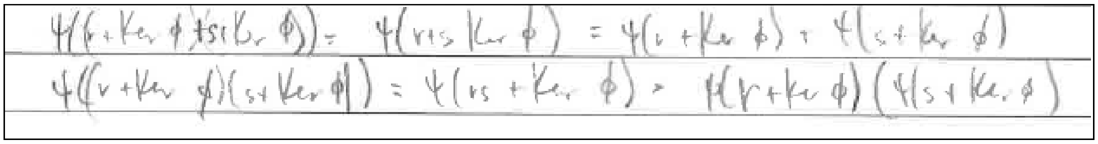


Figure 6.8. Berkeley's Proof.

Similarly, Edward's proof (Figure 6.9) was not successful because he missed using the given hypothesis "$q: X \to Y$ is a quotient map." He had $q^{-1}(f^{-1}(W)) = U$ for $U \in T_X$ and $W \in T_Z$. He needed to apply the property of a quotient map, which is "For $q: X \to Y$, if $q^{-1}(W) = U \in T_X$ for $W \subset Y$, then $W \in T_Y$" to conclude that $W \in T_Y$. Instead, he made a stretch to reach the conclusion $W \in T_Y$ out of $q^{-1}(f^{-1}(W)) = U$ with $U \in T_X$ and $W \in T_Z$. A possible source that hindered him from *combining objects* was that he might not be careful and alert enough to notice he was not allowed to move from $q^{-1}(f^{-1}(W)) = U$ to $W \in T_Y$ for free. There were some factors that might have helped him apply *combining objects*.

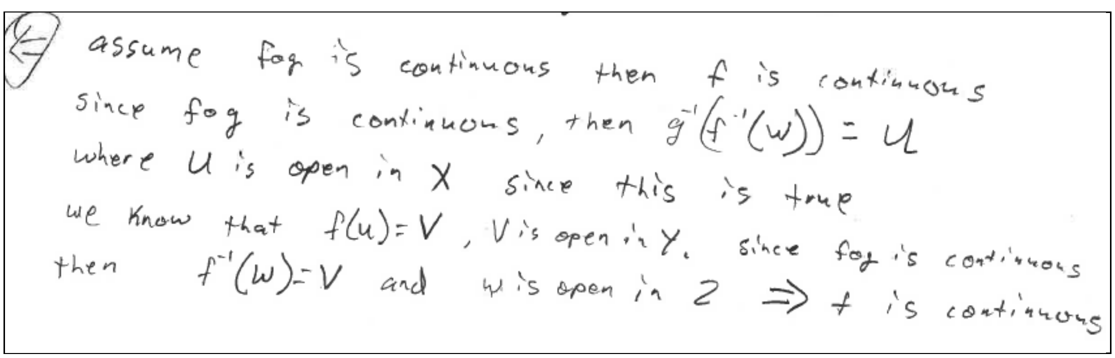


Figure 6.9. Edward's Proof.

Finally, the analysis of students' proofs implied that students' difficulty with *combining objects* might have resulted from their difficulties with finding a connection between the objects to be combined. Neither Frank (Figure 6.10.) nor Carlos (Figure 6.11) was able to combine the given condition "$G / Z(G)$ is cyclic" with an element

$g \in G$ properly. They had difficulties with combining the objects because they had to think about the relationship between an element $g \in G$ and a coset $G/Z(G)$ and to recall the fact that "every element $g \in G$ belongs to some coset $xZ(G)$ with $x \in G$."
They needed to ask themselves what would be a connection between an element $g \in G$ and a coset $G/Z(G)$. In addition, they needed to be equipped with the knowledge that "every element $g \in G$ belongs to some coset $xZ(G)$ with $x \in G$."

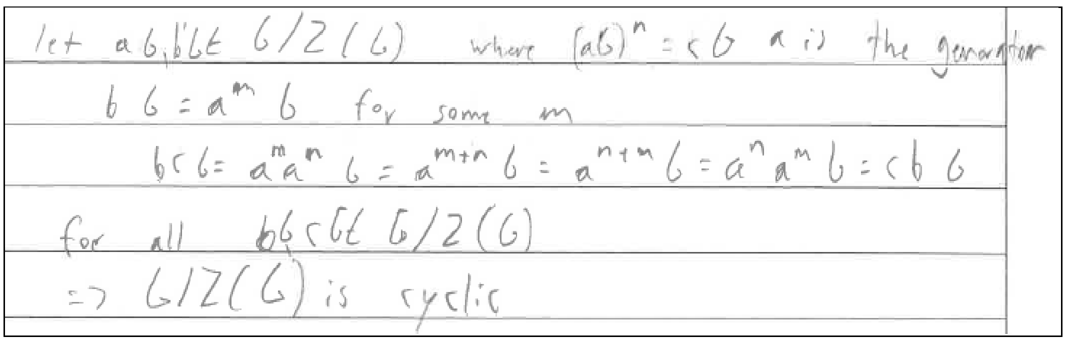


Figure 6.10. Frank's Proof.

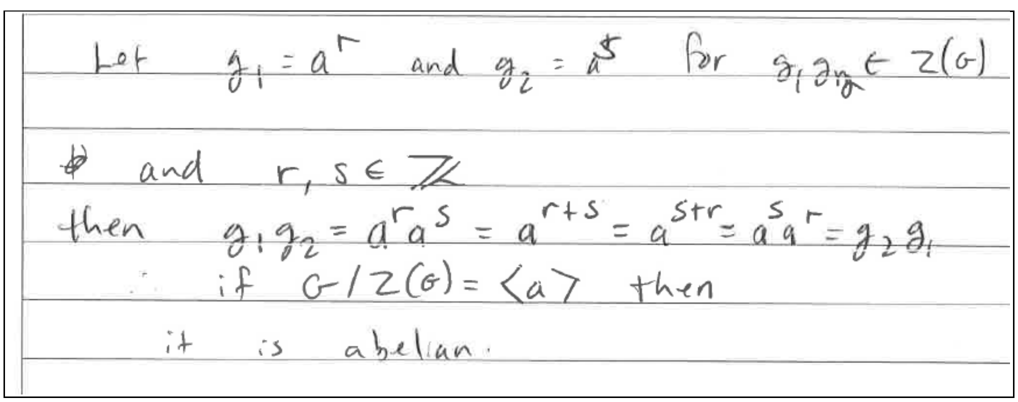


Figure 6.11. Carlos's Proof.

For another example, Carlos (Figure 6.12) was unable to combine the object $< g >$ and the given condition "the order of *G* is a prime number." The difficulty with combining these two objects was that he had to seek the relationship between $< g >$ and G, to realize that $< g >$ is a subgroup of *G*, to recall Lagrange's theorem, and to apply it to G and $< g >$. The operation of *combining objects* that was required in the problem was difficult because it involved the operation of *creating a cue* (C2) by recalling and applying prior knowledge. Students had to be equipped with the knowledge related to the concepts of the order of a group, including "Lagrange's theorem." Students' stronger *background knowledge* should help them operate combining objects more successfully.

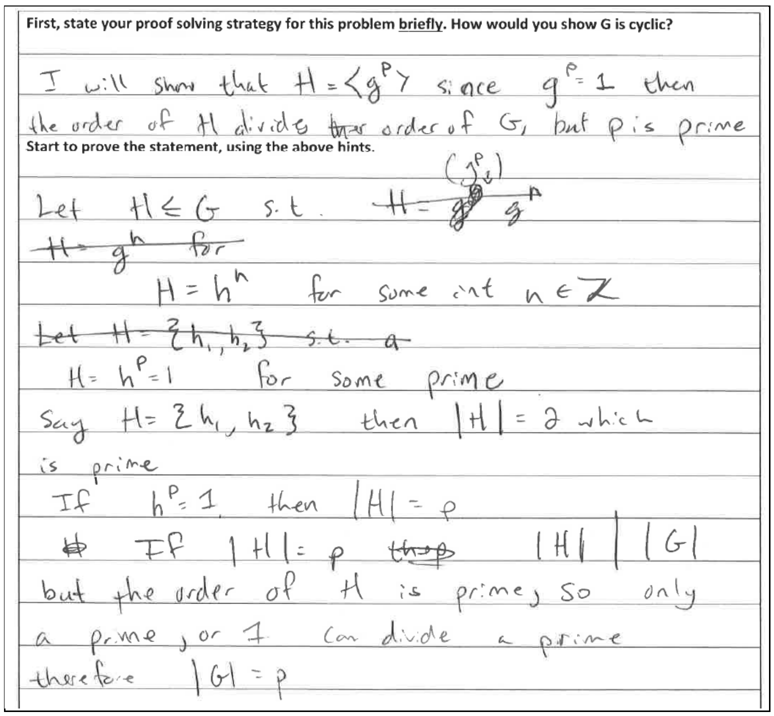


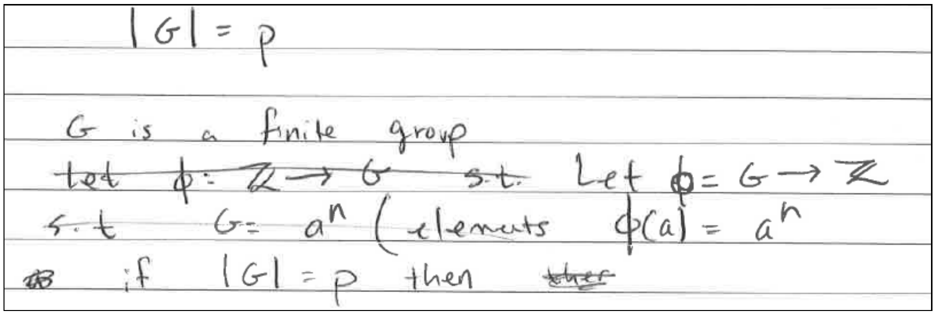
Figure 6.12. Carlos's Proof.

Thus, the analysis of students' proofs indicated the following in order for students to practice the operation of *combining objects* successfully. First, students need to be careful and alert to make sure if they have used all the given conditions. Second, students need to be equipped with broad and strong knowledge so that they can recognize a connection between two objects to be combined.

6.3.4 Creating a Cue

Creating a cue is the third primary operation for advancing a reasoning process. This operation can be considered among the three operations (*rephrasing an object*, *combining objects*, and *creating a cue*) as the most difficult. The first two operations can depend on the objects that are explicitly given while *creating a cue* sometimes requires students to literally "create" a new object through their prior knowledge, intuition, an innovative idea. The following are the highlights of the findings of students' difficulties with creating a cue: (a) Students' lack of knowledge of *the opening stage* may hinder them from creating a cue by setting a variable; (b) Students' lack of *flexibility*, *carefulness*, and *alertness* may affect their use of *creating a cue*; and (c) Students' lack of *background knowledge* may hinder them from applying the operation of *creating a cue*.

Creating a cue by setting a variable is a crucial operation especially in *the opening stage*. Many of the students were not able to set a right starting variable in the opening stage because they did not note and translate the conclusion of the given statement into *mathematical language* and also because they tended to note a hypothesis and tried deriving a starting variable from the hypothesis. Cade (Figure 6.13) and Alex (Figure 6.14) showed representative examples of students' difficulties with creating a cue by setting a right starting variable in the opening stage. Both of them were unable to derive and set a starting variable $x \in G$ by noting the conclusion "G is cyclic," which resulted in their producing incomplete proofs. The analysis of students' proofs indicated that the knowledge of the opening stage, especially the model steps that should be taken in the opening stage, might help them overcome their difficulties.


|G| = p

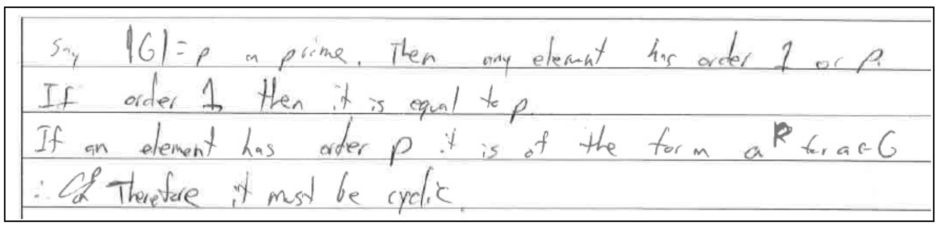
G is a finite group

~~Let φ: ℤ → G s.t.~~ Let φ = G → ℤ

s.t G = a^n (elements φ(a) = a^n

if |G| = p then ~~other~~

Figure 6.13. Cade's Proof.


Say |G| = p a prime. Then any element has order 1 or p

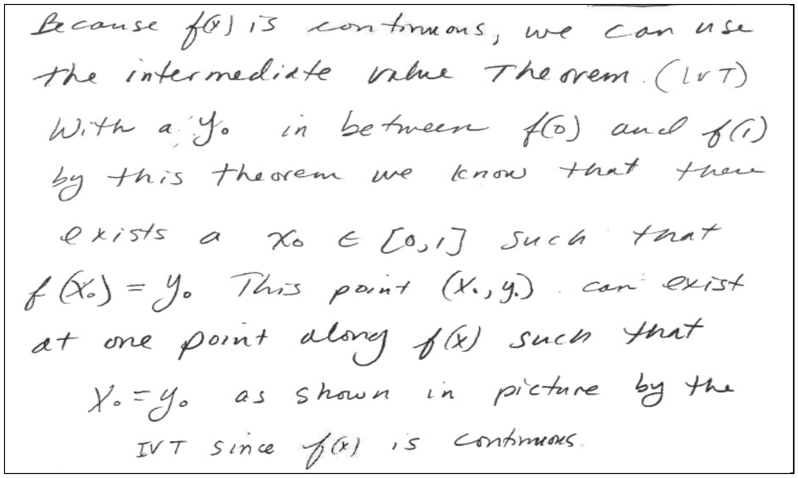
If order 1 then it is equal to p.

If an element has order p it is of the form a^p for a ∈ G

∴ Therefore it must be cyclic.

Figure 6.14. Alex's Proof.

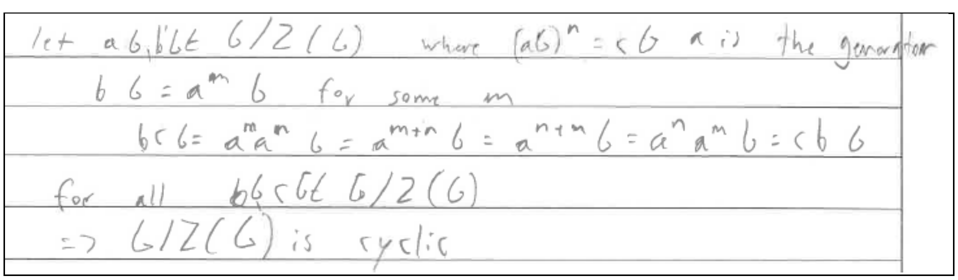
Eliot (Figure 6.15) gave an example showing that *flexibility* was required for students to have in order to practice creating a cue smoothly. He was unable to set a function $g:[0,1]\to[-1,1]: x \to f(x)-x$ in order to show that there existed $x_0 \in (0,1)$ such that $f(x_0)=y_0$. If he had had flexibility to r rephrase $f(x)=x$ with $f(x)-x=0$, that might have helped him consider $g:[0,1]\to[-1,1]: x\to f(x)-x$.



Because $f(x)$ is continuous, we can use the intermediate value Theorem (IVT) With a y_0 in between $f(0)$ and $f(1)$ by this theorem we know that there exists a $x_0 \in [0,1]$ such that $f(x_0) = y_0$ This point (x_0, y_0) can exist at one point along $f(x)$ such that $x_0 = y_0$ as shown in picture by the IVT since $f(x)$ is continuous.

Figure 6.15. Eliot's Proof.

Both Frank (Figure 6.16) and Carlos (Figure 6.17) gave examples of students' lack of *background knowledge*, which had a negative impact on their use of *creating a cue*. They needed to think about a possible relationship between an element $g \in G$ and a coset *G*/*Z*(*G*) and recall and use the property of coset "every element $g \in G$ belongs to some coset $xZ(G)$ with $x \in G$." However, none of them were able to recall and apply the property so that they were able to express $g \in G$ in terms of an element in some coset *G*/*Z*(*G*). Students are required to have broader knowledge around a concept so that they can create a cue by recalling and applying prior knowledge.



let $aG, b'G \in G/Z(G)$ where $(aG)^n = cG$ a is the generator

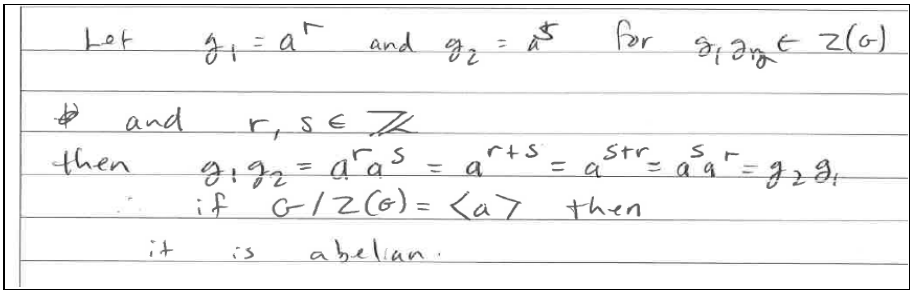
$bG = a^m G$ for some m

$bcG = a^m a^n G = a^{m+n} G = a^{n+m} G = a^n a^m G = cbG$

for all $bG, cG \in G/Z(G)$

$\Rightarrow G/Z(G)$ is cyclic

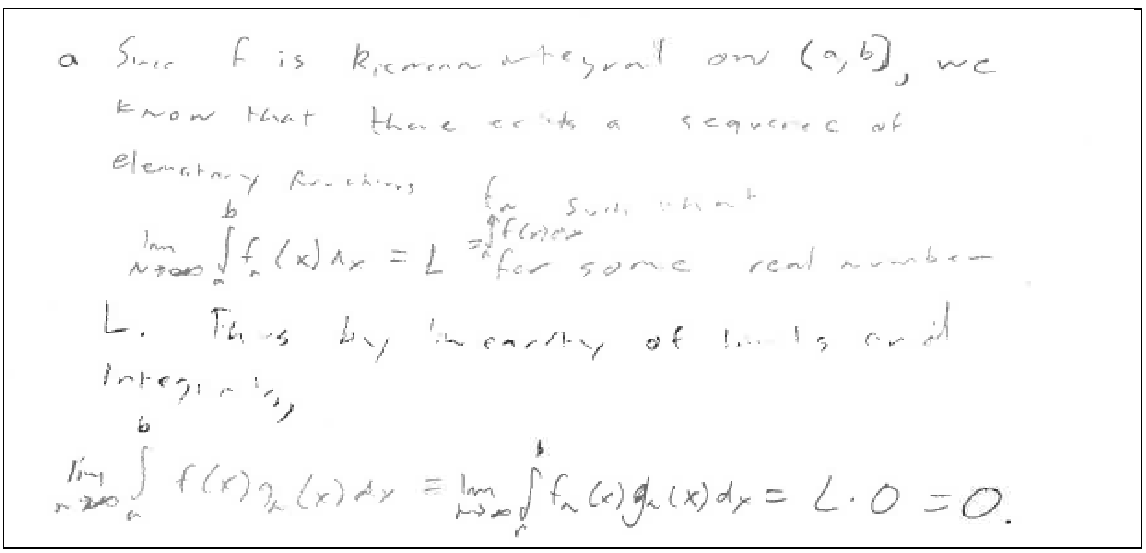
Figure 6.16. Frank's Proof.



Let $g_1 = a^r$ and $g_2 = a^s$ for $g_1 g_2 \in Z(G)$
and $r, s \in \mathbb{Z}$
then $g_1 g_2 = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = g_2 g_1$
∴ if $G/Z(G) = \langle a \rangle$ then
it is abelian.

Figure 6.17. Carlos's Proof.

Kyle (Figure 6.18) also showed an example showing students' background knowledge played a great role in advancing a reasoning process by creating a cue. He was unable to show $\lim_{n\to\infty}\int_a^b f(x)g_n(x)dx = 0$ because he was unable to recall and use $\lim_{n\to\infty}\left|\int_a^b f(x)g_n(x)dx\right| = 0$.



a Since f is Riemann integrable on (a, b], we know that there exists a sequence of elementary functions f_n such that
$\lim_{n\to\infty}\int_a^b f_n(x)dx = L = \int_a^b f(x)dx$ for some real number L. Thus by linearity of limits and integration,
$\lim_{n\to\infty}\int_a^b f(x)g_n(x)dx = \lim_{n\to\infty}\int_a^b f_n(x)g_n(x)dx = L\cdot 0 = 0.$

Figure 6.18. Kyle's Proof.

Creating a cue can be difficult because students are required to create a new object without depending on the given objects unlike the other two operations (*rephrasing an object* and *combining objects*). For example, students are required to recall a proposition, a theorem, a lemma, or a property of concept, choose a right one among several choices, and apply it properly. They are also required to have flexibility to come up with a new object that helps them further advance a reasoning process. In addition, they are required to be able to set a right variable necessary for advancing a reasoning process. The analysis of proofs indicated that the more knowledge and the more flexibility students had, the more smoothly they could advance a reasoning process by *creating a cue*.

6.3.5 Background Knowledge

Moore (1994) found the seven major sources of the students' cognitive difficulties with proof construction. Three of them are directly related to students' lack of knowledge. The following are three of the seven major sources.

- D1: The students did not know the definitions.
- D3: The students' concept images were inadequate.
- D6: The students were unable to understand language and notation.

Proofs involve some concepts. Then, without the knowledge of concepts including their definitions, meanings, and notations, students cannot make their proving arguments successful. D1, D3, and D6 are the issues of students' lack of knowledge of concepts. This study put the above sources together into one category "students' lack of *background knowledge*." Students' knowledge of theorems, propositions, corollaries, properties, mathematical laws, and proving techniques were also categorized as *background knowledge*. This section consists of two parts. The first part discusses how students' lack of background knowledge affected their reasoning activities. The second part highlights the issues of student's use of definitions in proof construction.

Students' lack of background knowledge. First, this section discusses how students' lack of background knowledge can influence the operations of the reasoning activity (*rephrasing an object*, *combining objects*, and *creating a cue*). The analysis of students' proofs strongly indicated that their lack of *background knowledge* was a crucial factor that hindered them from successfully advancing a reasoning process, directly affecting their practices of these three operations. The following presents the highlights of how students' lack of *background knowledge* hindered them from applying the operations of the reasoning activity (a) *rephrasing an object* and (b) *creating a cue* and *combining objects*.

(a) Billy and Savanna provided proofs that illustrated students' lack of knowledge of definitions, properties, and relevant theorems can affect their skills of *rephrasing an object*. Billy (Figure 6.18.) was asked to prove that a given map was a ring homomorphism. He tried showing $\psi(a[r]) = a\psi([r])$ though it was not a property of a ring homomorphism. He was unable to prove the given statement because he did not know the property of a homomorphism correctly.

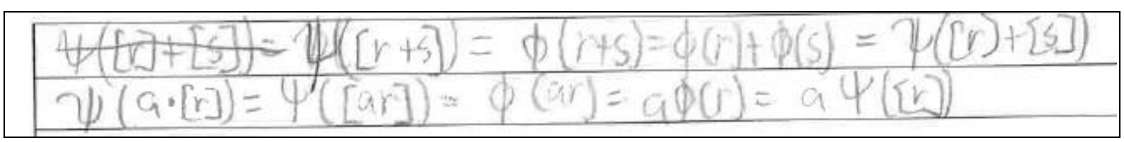
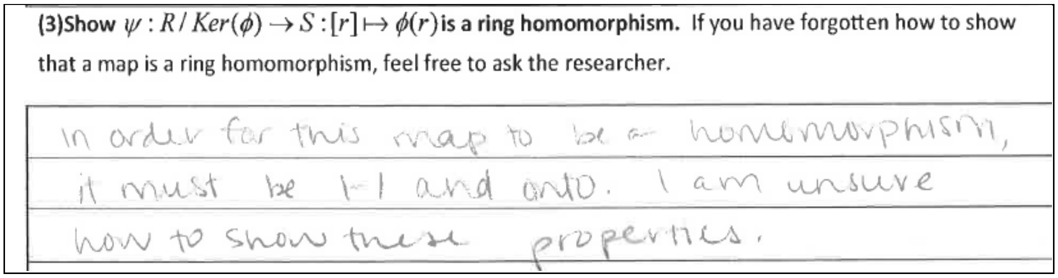


Figure 6.19. Billy's Proof.

Savanna (Figure 6.19) thought that she needed to show that it was one to one and onto, which was an incorrect assumption. In addition, she did not know how to prove that a given map was one to one and onto. Her lack of knowledge of a ring homomorhpihsm directly caused her to fail to rephrase "a ring homomorphism" with $\psi([r]+[s]) = \psi([r]) + \psi([s])$ and $\psi([r][s]) = \psi([r])\psi([s])$.

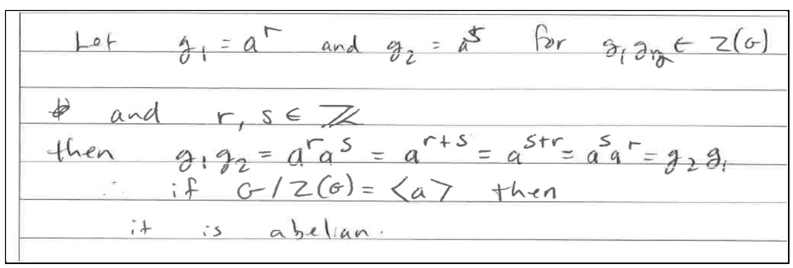


(3)Show $\psi: R/Ker(\phi) \to S : [r] \mapsto \phi(r)$ is a ring homomorphism. If you have forgotten how to show that a map is a ring homomorphism, feel free to ask the researcher.

In order for this map to be a homomorphism, it must be 1-1 and onto. I am unsure how to show these properties.

Figure 6.20. Savanna's Proof.

(b) Students' lack of *background knowledge* of a property of concept can directly and indirectly hinder them from *creating a cue* and *combining objects*, which results in producing an incomplete proof. For example, Carlos (Figure 6.20) seemed to have not known the property of a coset "every element $g \in G$ belongs to some coset $xZ(G)$ with $x \in G$." His lack of the knowledge of the property made it impossible to recall and apply it to combine an element $g \in G$ and a coset $G/Z(G)$.



Let $g_1 = a^r$ and $g_2 = a^s$ for $g_1 g_2 \in Z(G)$

and $r, s \in \mathbb{Z}$

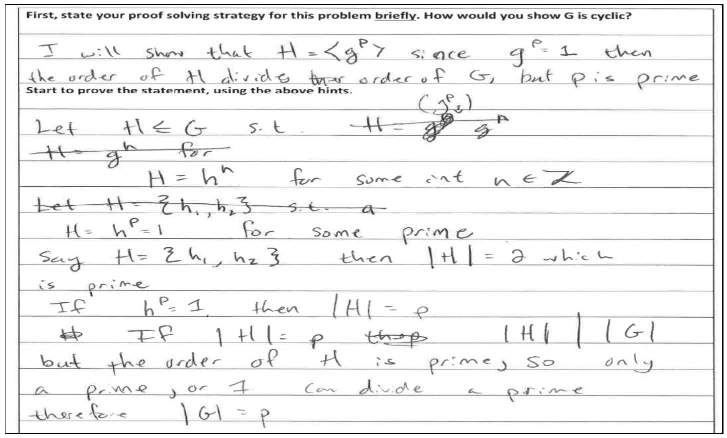
then $g_1 g_2 = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = g_2 g_1$

if $G/Z(G) = \langle a \rangle$ then

it is abelian.

Figure 6.21. Carlos's Proof.

For another example, Caleb (Figure 6.21) seemed not to have known the relationship between $< g >$ and G, namely, the fact that $< g >$ was a subgroup of G. His lack of knowledge of this fact directly hindered him from *creating a cue* by recalling the fact and indirectly hindered him from combining the object $< g >$ and the given condition "the order of G is a prime number," which might have led him to derive $< g > = 1$ or p. The analysis indicated students' stronger *background knowledge* might help them operate *creating a cue* and *combining objects* more successfully.



First, state your proof solving strategy for this problem briefly. How would you show G is cyclic?

I will show that $H = \langle g^p \rangle$ since $g^p = 1$ then the order of H divide ~~the~~ order of G, but p is prime

Start to prove the statement, using the above hints.

Let $H \leq G$ s.t. ~~$H = (g^p_?) g^n$~~

~~$H = g^n$ for~~

$H = h^n$ for some int $n \in \mathbb{Z}$

~~Let $H = \{h_1, h_2\}$ s.t. a~~

$H = h^p = 1$ for some prime

Say $H = \{h_1, h_2\}$ then $|H| = 2$ which is prime

If $h^p = 1$ then $|H| = p$

~~#~~ If $|H| = p$ ~~then~~ $|H| \mid |G|$

but the order of H is prime, so only a prime, or 1 can divide a prime

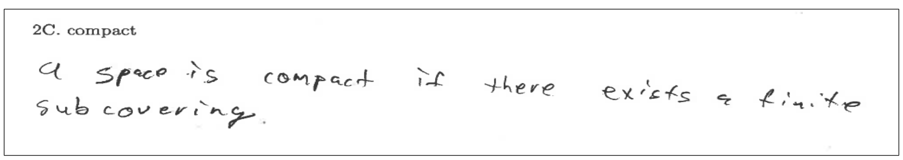
therefore $|G| = p$

Figure 6.22. Caleb’s Proof.

Star and Rittle-Johnson (2008) introduced the concept of flexible knowledge as a key factor for problem solving. They observed two key features of flexibility for problem solving: (a) the knowledge of what strategies were available for a given problem; (b) the knowledge of which of them were more effective. In proof construction , students’ knowledge of properties of concept or theorems and propositions related to a concept can be considered to be the first type of knowledge (a). Students’ ability to decide which property, theorem, or proposition to adopt from among several choices can be considered to be the second type of knowledge (b). As Star and Rittle-Johnson indicated, students’ *background knowledge* and their *flexibility* are intertwined, which directly or indirectly influence their ability to apply the operations of *rephrasing an object*, *combining objects*, and *creating a cue*. The more knowledge about concepts and the more problem solving experiences students have, the greater the degree of their flexibility for problem solving becomes.

Definitions of concepts. The issues of students’ use of definitions are crucial in proof construction because definitions are fundamental elements in *the background knowledge* for proof construction. This subsection consists of two parts: (a) The first part discusses how students' incomplete understanding of definitions can affect their proof construction. (b) The second part discusses the roles that definitions play in constructing a proof.

Students' incomplete understanding of definitions. Moore (1994) listed students' inadequate concept images as one of the seven major sources of students' difficulties with proving. This study also observed cases that supported his observations. Elias (Figure 6.22) provided the following as the definition of compactness: "a space is compact if there exists a finite subcovering." He missed the part "for any open cover of the space" in his statement. His concept image did not have the structure in which a finite open subcovering could be derived from an arbitrary open cover for the space."


2C. compact

a space is compact if there exists a finite subcovering.

Figure 6.23. Elias's Statement.

His incomplete knowledge of compactness directly affected his proof (Figure 6.23). When he was asked to show that a given space was compact, he never set an open cover for the given space in his argument, which made it impossible for him to make his proving argument successful.

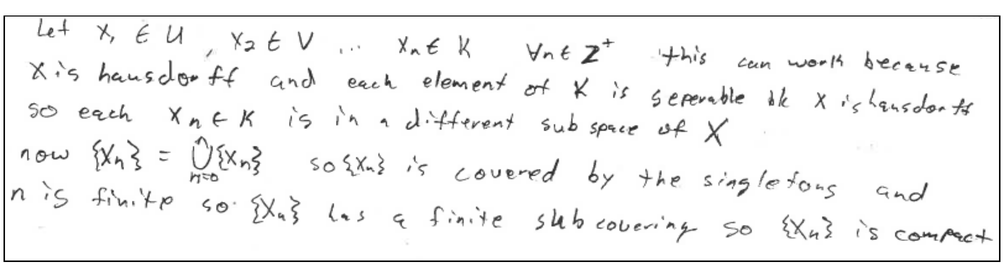

Let $x_1 \in U$, $x_2 \in V$ ... $x_n \in K$ $\forall n \in \mathbb{Z}^+$ this can work because X is hausdorff and each element of K is seperable b/c X is hausdorff so each $x_n \in K$ is in a different sub space of X now $\{x_n\} = \bigcup_{n=0}\{x_n\}$ so $\{x_n\}$ is covered by the singletons and n is finite so $\{x_n\}$ has a finite subcovering so $\{x_n\}$ is compact

Figure 6.24. Elias's Proof.

The definition of a mathematical concept is not just a static description or explanation of a technical word, but it is a logical structure that requires a rigorous understanding of the concept through abstract thinking. A definition consists of some meaningful and indispensable units, each of which must be carefully understood and dealt with. Students' incomplete understanding of even small part of the units can produce a gap between their "definition" derived from their mental picture of the concept (concept image) and the actual definition of the concept (concept definition), which may become a source of an incomplete proving argument. In the above example, Elias missed the small portion of the definition of compactness "for any open cover of *K*," which resulted in his unsuccessful proving argument. Thus, students' incomplete concept images about definitions can directly affect their proofs.

Role of definitions in proof construction. Definitions of concept are considered to be most representative *mathematical language* in this study, which have a power to enable students to develop and advance a reasoning process. Vinner (1991) claimed that one of the major roles of definitions was proving theorems. Knapp (2006) proposed definitions played a role of giving a structure to proof and warranted to logical implications. The analysis of students' proofs clarified two main roles definitions of concepts played in proof construction, especially, in *the opening stages*: making the direction and goal of the proof clear to them; and helping students set a starting variable through the *ignition phrases*. In particular, definitions of concepts contained in the conclusion of the given statement can provide students with a key element "a variable" on which to convey mathematical thoughts, and keep and guide them on the right track to the goal. Therefore, students' lack of knowledge of a definition or of ability of rephrasing a concept with its definition may spoil their proving arguments. For example, Frank (Figure 6.24) was not successful in translating the conclusion of the given statement "(G is) abelian" into "for any $x, y \in G$, $xy = yx$" by applying the definition of abelian, which might have caused him to lose the direction to reach the goal of the proof.

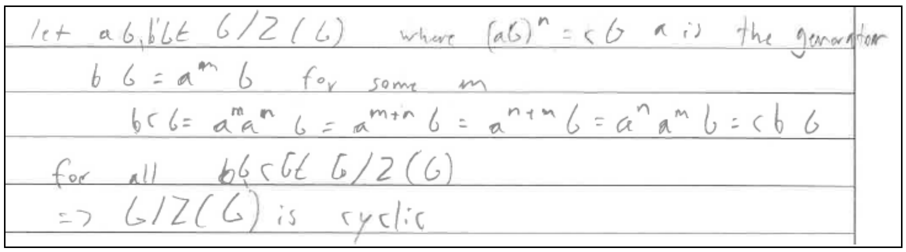


Figure 6.25. Frank's Proof.

The definition of a concept can provide students with a fundamental element "a variable," which is indispensable for advancing a reasoning process through its *ignition phrase*. *The ignition phrase* in the definition of a concept, which is usually found in the conclusion of the given statement, provides students with a great sign for setting a right variable. Quincy (Example 5) was unable to make his proving argument successful because he was unable to derive a right starting variable "an open cover U of K"from *the ignition phrase* contained in the conclusion of the given statement "K is compact."

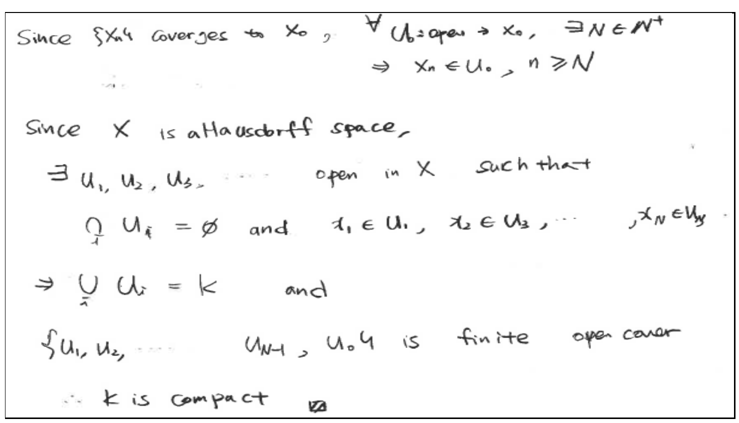


Figure 6.26. Quincy's Proof.

Edwards and Ward (2008) found the characteristic of definitions in the meanings that were explicitly set up. This characteristic of definitions makes it difficult for students to apply a definition of concept in proof construction. Students are required to set and use a variable in a proving argument so that they can make good use of a definition of concept. Students' knowledge of the role of *an ignition phrase* providing a clue for setting a variable may help them overcome their difficulty with starting a proving argument.

Students' knowledge of mathematical concepts is crucial. Especially, their complete knowledge of the definitions of concepts is indispensable in proof construction. A mathematical concept becomes more helpful and powerful when it is translated into *mathematical* language, namely when it is rephrased with its definition. The definition of a concept gives students a motive power to advance a reasoning process, while making the direction of an argument clear and providing necessary variables. In particular, students' knowledge of those roles that definitions play in *the opening stage* may help them start their proof construction and advance their reasoning processes.

6.3.6 Mental Attitudes

Rigelman (2007) described key characteristics of effective problem solvers, which he called flexible and fluent thinkers, as follows: confidence in use of knowledge and processes; willingness to take on a challenge; perseverance in the quest to make sense of a situation and to solve a problem; and reflective thinking. The aspects of *the mental attitudes* and *affect and beliefs* may include these characteristics. For example, students' confidence can be categorized into the aspect of *emotions and beliefs*.

Students' willingness and perseverance may correspond to *tenacity* in *the mental attitudes* and also can be related to *affect and beliefs*. Students' reflective thinking may be related to *carefulness and alertness* in *the mental attitudes*.

The model of the structure of proof construction included four aspects for proof construction: *the reasoning activity*, *the background knowledge*, *the mental attitudes*, and *affect and beliefs*. These aspects are not independent of one another and are intricately intertwined, influencing one another. The model was designed to simplify a complex cognitive activity of proof construction. The analysis of students' proofs also indicated their difficulties with proof construction may be caused by multiple factors in the four aspects.

For example, Berkeley's proof (Figure 6.26) showed that students' lack of *carefulness* and *alertness* might result in their failure to combine objects. Berkeley should have been careful in advancing her reasoning process when she provided $\psi(r + Ker\phi + s + Ker\phi) = \psi(r + s + Ker\phi) = \psi(r + Ker\phi) + \psi(s + Ker\phi)$. Her lack of carefulness seemed to result in her failure to combine the object $\psi(r + s + Ker\phi)$ and the given condition "$\psi : R / Ker(\phi) \to S : [r] \to \phi(r)$."

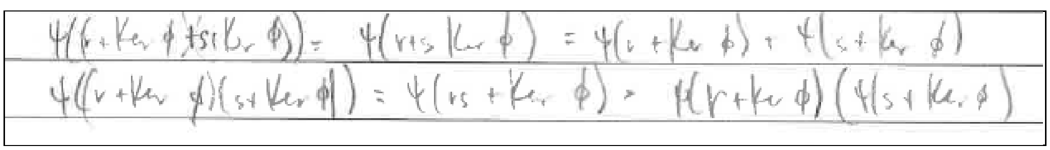


Figure 6.27. Berkeley's Proof.

For another example, Ryan (Figure 6.27) provided an example showing that students' lack of flexibility may result in producing an unsuccessful proof. The direct problem with his proof was that he did not use the given condition "*K* is a convergent sequence." Instead, he was trying to prove the given statement by applying the concept of connectedness, which was irrelevant and not helpful for solving the given problem. His lack of flexibility seems to have hindered him from realizing that what he was doing was not working and that he should look for information still available that could be used for advancing a reasoning process.

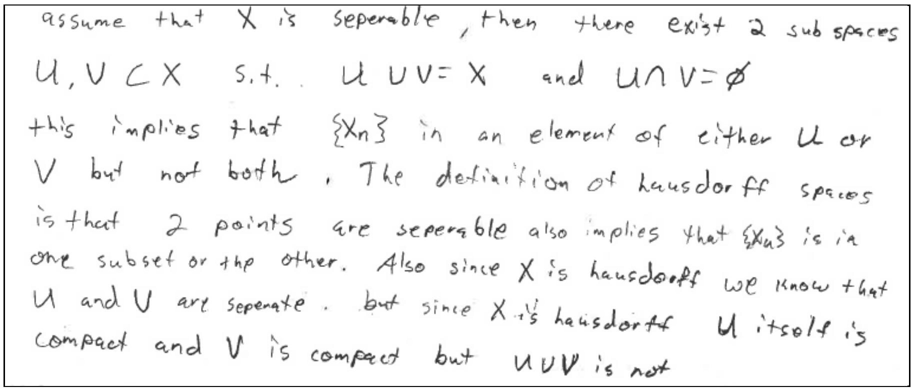

assume that X is seperable, then there exist 2 sub spaces
$U, V \subset X$ s.t. $U \cup V = X$ and $U \cap V = \emptyset$
this implies that $\{X_n\}$ in an element of either U or
V but not both, The definition of hausdorff spaces
is that 2 points are seperable also implies that $\{X_n\}$ is in
one subset or the other. Also since X is hausdorff we know that
U and V are seperate. but since X is hausdorff U itself is
compact and V is compact but $U \cup V$ is not

Figure 6.28. Ryan's Proof.

Hardin (2002) introduced self-monitoring as one of the key skills that makes students' problem-solving successful. As she pointed out, problem-solving experts are more aware when they make errors and check their solutions. In the model of proof construction, *carefulness and alertness* corresponds to self-monitoring. Students' awareness of the necessity of *carefulness and alertness* together with *flexibility* may help them advance their reasoning process more successfully.

6.3.7 Checking and Exploring, and Affect and Beliefs

Lastly, the analysis of students' proofs encountered students' difficulties that fall into the following two categories. One was that students used a property that did not work. The properties they used were not necessarily wrong, but those properties did not work for a given problem. Student's problems were not that they tried a property of a concept that turned out not to work, but that they did not realize the property did not help them advance a reasoning process and did not try a different method. They could have modified their ideas through *exploring and checking*. The source of their difficulties can also be considered to be their lack of flexibility and carefulness. In the other case, students used a logic that did not make sense. This is rather a matter of language ability rather than mathematical ability. This type of students' difficulty was categorized as *affect and beliefs*.

6.4 Usefulness of the Model of the Structure of Proof Construction

This section answers the last research question: "How useful is the model of the structure of proof construction?" This section consists of three parts: (1) the usefulness in creating the framework for analyzing students' proofs; (2) the usefulness of the framework for examining students' difficulties; (3) a possible contribution to theoretical frameworks for proof construction

The model of the structure of proof construction directly contributed to the creation of the framework for analyzing students' proofs (Table 3.6). Since the model viewed proof construction from the three aspects (*the reasoning activity*, *the background knowledge*, *the mental attitudes*), it was natural to consider these aspects to be the categories in which students' difficulties occurred. The components of those three aspects became the factors that decided types of students' difficulties. The model of the structure of proof construction led to the creation of two types of analysis frameworks. One is the comprehensive error list (Table 3.5) that can cover a variety of difficulties students encounter (Type B). The other framework is created for each proof (Type A). It shows every step for a model proof and the operation used to generate the step. The model of the structure of proof construction also provided dual ways to examine students' difficulties. One was to examine students' difficulties in terms of each aspect (*reasoning activity*, *background knowledge*, and *mental attitudes*) separately and independently. The model also suggested another perspective for examining students' difficulties. It was to define students' difficulties to be those with the *reasoning activity* and to consider the other aspects (*background knowledge* and *mental attitudes*) to be the categories for the sources of their difficulties. This perspective enabled me to explain students' difficulties with the *reasoning activities* in terms of the other two aspects (*background knowledge* and *mental attitudes*). Thus, the model of the structure of proof construction directly contributed to the creation of an analysis framework (Table 3.6) and provided the perspectives for examining students' difficulties.

Next, I will discuss the usefulness of the analysis framework in examining students' proofs. The analysis framework Type A (See Table 3.4 as an example) was useful in detecting where students had difficulties in their proofs and what operations they failed to use. The analysis framework Type B (Table 3.5) was useful in deciding the source of each mistake, impasse, and difficulty students made. The analysis framework Type C (Table 3.6) served as the list of the skills and abilities necessary for proof construction. These frameworks enabled me to examine students' difficulties in a clear and organized manner.

This section ends with a possible contribution of the model to theoretical frameworks for proof construction. The model of the structure of proof construction simplified and organized complex nature of logical deduction involved in proof construction. The model also indicated that *rephrasing an object* and *combining objects* were the operations for syntactic approach and that *creating a cue* and *checking* *and exploring* were the operations for semantic approach. The model may contribute to the development of an effective method to help students practice syntactic and semantic approaches.

Brown, Bransford, Ferrara, and Campione (1983) claimed that metacognitive skills are crucial factors for successful problem solving. Papaleontiou-Louca (2003) suggested that teachers should demonstrate metacognition for modeling task completion so that their students can learn effectively. She claimed "modeling offers the vocabulary that students need for thinking and talking about their own thinking (p. 23)." The model of the structure of proof construction may serve as metacognitive and methodological knowledge for helping students advance a reasoning process and overcome their impasses and difficulties.

Quesada, Kintsch, and Gomez (2005) claimed that the theories in the area of complex problem-solving had not been established due to the lack of good definitions and classifications of the tasks. This study considers proof construction as a complex problem-solving task that requires complex cognitive actions. The model (Figure 3.1) and the framework (Table 3.2) attempted to capture and simplify the whole structure of proof construction by exploring, defining, and organizing the aspects and the operations involved in advancing a reasoning process in a proving activity. The model (Figure 3.1) and the framework (Table 3.2) may contribute to developing a theoretical framework for proof construction.

The model and the framework may account for various students' difficulties with proving. Not only that, but also those various examples of students' difficulties seem to verify the aspects, the factors, the patterns, and the features involved in proof construction. Kieran (1998) claimed that there was no theoretical work without empirical research and vice versa, stating "there is no escaping the fact that, in mathematics education, theory building and empirical studies form the vicious circle of research; each requires the other" (p. 223). She implied that research results may refine and develop theoretical models and that research results with theoretical models may explain the phenomena better. I hypothesize that the structure of proof construction (Table 3.2 and Figure 3.1) can contribute not only to the body of knowledge of proof construction but also to the body of knowledge of any mathematical problem-solving.

6.5 Pedagogical Suggestions

This section answers the third research question of this study: "*What pedagogical suggestions can be drawn to help students with proof construction*"

The answer to this research question consists of two parts. The first part presents suggestions for students. The second part presents suggestions for instructors.

6.5.1 Suggestions for students

There are mainly two sources from which this study derives suggestions from: (a) the model of the structure of proof construction (Table 3.1, Table 3.2) (b) the findings obtained through the analysis of students' proofs.

Suggestions based on the model of the structure of proof construction. The model of the structure of proof construction (Figure 3.1) was designed to introduce a comprehensive view of proof construction. It suggests that proof construction can be viewed from the following four aspects: *the background knowledge*; *the reasoning activity*; *the mental attitudes*; and *affect* and *beliefs*.

First, the model suggests that students should be equipped with the knowledge necessary and sufficient for solving a given proof problem. The knowledge includes definitions of concepts, the meaning of the definitions, properties of concepts, notations of concepts, theorems, propositions, mathematical laws, and proving techniques. Without complete knowledge, it is impossible for students to build a complete proving argument. Therefore, it is recommended that students acquire and develop a strong system of knowledge around concepts.

Second, the model of the structure of proof construction gives some guides on how they can advance a reasoning process. There are four types of operations they can apply: *rephrasing an object*; *combining objects*; *creating a cue*; and *checking* and *exploring*. In addition, there is an order of the operations to be tried. Students first should try the operation of *rephrasing an object*. If it does not work, they can try the operation of *combining objects*, looking for a given condition. If the operation of *combining objects* does not work, they can try the operation of *creating a cue*. If the operation of *creating a cue* does not work, they can try the operation of *checking* and *exploring*. The knowledge of these operations can help students especially when they "get stuck" and cannot advance a reasoning process.

The model of the structure of proof construction also suggests that students should be equipped with some psychological traits such as *tenacity*, *flexibility*, and *carefulness and alertness* as well as the knowledge of the operations for advancing a reasoning process and vast strong knowledge centered at mathematical key concepts. The model implies the following: (i) Proof construction requires students to have tenacity and persistence to keep on thinking and not to easily give up thinking; (ii)

Proof construction also requires students to be equipped with flexibility to change their methods, to give up an idea that is not working, and to try a new method; (iii) Proof construction requires students to be careful and alert in dealing with an object accurately, correctly, and precisely.

Suggestions based on the findings of the analysis of students' proofs. The analysis of students' proofs provided various results that would contribute to some specific and practical pedagogical suggestions. The following are highlighted pedagogical suggestions derived from the analysis of students' proofs.

Opening stage. Students tended to note a hypothesis of the given statement when they started their proving arguments. They seemed to be tempted to derive a starting variable from the hypothesis instead of the conclusion of the given statement. When students try to set a starting variable, they should first note the conclusion of the given statement, translate it into mathematical language often by applying the definition of a concept involved in the conclusion, and try to derive and set a starting variable often by noting *an ignition phrase*.

Combining objects. Students had a difficulty with *combining objects* often because they were unable to find a connection between those objects. When students apply the operation of *combining objects*, they should seek a relationship between the key objects contained in the statements to be combined. They are often required to recall and apply prior knowledge that is relevant to the objects. A broader knowledge around the concept involved in those objects would be necessary. Students should be encouraged to widen their knowledge of concepts correctly, including their properties, related theorems, propositions, mathematical laws, and proving techniques.

Flexibility. Students sometimes used a property of a concept or a theorem that did not help, believing it was working. They should have flexibility to change their ideas to try a different theorem or a different property of the concept when they realized the first attempt does not work. They also need to be careful and alert about what they are doing, asking themselves if their method was working.

Given conditions. The analysis saw multiple cases in which students failed to combine objects because they did not note and use all the given conditions. They should be careful and alert in making sure to use all the given conditions to advance a reasoning process.

Knowledge. The analysis of students' proofs found out that students' lack of knowledge of a concept, including its definition, property, notation, and a related theorem, is fatal to proof construction, directly and indirectly affecting their use of the three operations for advancing a reasoning process (*rephrasing an object*, *combining objects*, and *creating a cue*). For example, students' lack of or their incomplete knowledge of a concept affected and ruined their reasoning process in multiple ways: creating a wrong notation that makes their arguments no sense; causing students to fail to rephrase an object correctly; making it difficult for students to combine two objects by missing the connection between them; hindering students from recalling prior knowledge necessary to solve the given problem; and making students' *tenacity*, *flexibility*, carefulness, and *alertness* weaker. It seems to be a matter of course to say that students should be encouraged to deepen and widen their knowledge of the facts around mathematical concepts, including their definitions, their meanings, their properties, their notations, and a related theorem. However, the findings of this study strongly suggests that instructors cannot put too much emphasis on the importance of students' acquisition and construction of their knowledge around basic concepts. The analysis of students' proofs also found multiple cases in which students made their proving argument unsuccessful due to their lack of flexibility, carelessness and alertness. These factors directly and indirectly affect students' reasoning activity (*rephrasing an object*, *combining objects*, and *creating a cue*). Students should be encouraged to keep in mind *tenacity*, *flexibility*, *carefulness*, and *alertness* are also important factors for proof construction.

Hardin (2002) discussed two types of knowledge: declarative knowledge and procedural knowledge. She defined declarative knowledge as knowing of facts, theories, events, and objects, and procedural knowledge as knowing how to do something, which, for example, includes cognitive skills and strategies. The *background knowledge* (See Figure 3.1), apart from the knowledge of proving techniques, helps students construct a proving argument as declarative knowledge. On one hand, not only proving techniques but also the knowledge of the structure of proof construction itself (Table 3.1, Table 3.2, and Figure 3.1) might help them construct more successful proving arguments as the procedural knowledge.

Although the aspect of *affect and beliefs* was not the main focus of this study, it is a crucial aspect for successful proof construction. Bandura (1997) asserted the importance of students' building self-efficacy for a successful learner. Pintrich (1999) and Zimmerman (2000) claimed that students' self-efficacy may influence their attitudes and performances in mathematics. Bandura (1997) suggested that students should have a mastery of experience to build their self-efficacy. Students need to be encouraged to build their mathematical knowledge through practicing proof construction. The suggestions for students can be summarized in the following way.

Opening Stage.

- Translate the conclusion into *mathematical language*.
    - Make sure of the goal of the proof.
    - Find *an ignition phrase* if there is any.
- Set a starting variable through *an ignition phrase*.

Body Construction Stage.

- First, try the operation of *rephrasing an object*.
    - Apply definitions or theorems.
    - Change the expression through interpretation.
    - Manipulate the object algebraically.
- If it does not work, try the operation of *combining objects*.
    - Make sure to use all the given conditions or hypotheses.
    - Combine objects by way of a connection (a common factor).
- If it does not work, try the operation of *creating a cue*.
    - Set a new variable.
    - Recall prior knowledge, including a theorem, a proposition, or a property.
    - Set some cases.
    - Make a claim.
    - Consider an object.

- Translate an object containing mathematical concept into *mathematical language.*
- Be flexible.
    - Review what has been done.
    - Give up an idea that does not work and try a new idea.
    - Try a another property of a concept.
    - Use different operation for the reasoning activity.
- Be careful and alert in dealing with an object.
    - Treat each object carefully.
    - Make sure if all the given conditions are used.
- Be equipped with the knowledge of the basics of a concept.
    - Definitions
    - Notations
    - Properties
    - Relevant theorems

6.5.2 Suggestions for instructors

This section consists of two parts. The first part discusses some suggestions on teaching proofs while referring to the relationship between this study and other studies. The second part provides suggestions that were directly derived from the findings of this study.

Harel and Sowder (1998) placed analytical proof scheme as the highest level of proof scheme among the three (external, empirical, and analytical proof schemes). However, researchers have observed students' difficulties with practicing analytical proof scheme (Ruthven & Coe, 1994; Selden & Selden, 1995). Harel (2000) suggested that the knowledge of specific actions to be taken for solving problems might enhance students' proof schemes. To meet the demands, this study provided the model steps in the *opening stage* and the method for advance a reasoning process with the operations in the *reasoning activity*. Instructors can use the knowledge of those to help their students develop their analytical proof schemes.

Also, Ball, Hoyles, Jahnke, and Movshovitz-Hadar (2002) suggested that instructors should consider the gradual processes and complexities involved in proving as a major factor for teaching proofs. The model of the structure of proof construction clarified the complex nature of proof construction in terms of cognitive processes. The model organized and simplified step-by-step thought processes necessary for constructing a complete proof as well as multi-dimensional aspects of proof construction.

Next, this section provides some suggestions for teaching that were derived from the results of the analysis of students' proofs. There are mainly two sources from which this study derives suggestions for instructors: (1) the findings from the analysis of students' proofs; (2) the model of the structure of proof construction.

Suggestions based on the findings of the analysis. There are three suggestions for instructors, which were derived from the analysis of students' proofs. First, instructors cannot emphasize too much to their students the importance of their building and widening the knowledge solidly and accurately. Students forget, mix, and miss information even when it is elementary and basic. For example, Savanna (Example 27) did not know what a homomorphism meant. She also did not know how to show that a function was one to one and onto. For another example, Carlos (Example 26) and Anthony (Example 31) did not understand the meaning and the notation of coset. As Tall and Vinner (1981) indicated, instructors need to help their students decrease the gap between students' concept images and the concept definitions. Instructors may want to monitor their students' knowledge and understanding levels, especially definitions, properties, and notations, through homework assignments, quizzes, and exams. They may also want to review the basics of concepts which students have learned before according to their learning needs. Repetition should help students narrow the gaps between their concept images and the concept definition. Second, instructors should help their students organize their knowledge of concepts by reviewing all the properties of a concept at the end of the lesson for learning the concept. A concept can have multiple properties. Students may be confused with which property to use and what property is available for a specific type of problem. Students may choose a property that would not work for advancing a reasoning process. For example, Quincy (Example 5) and Elias (Example 27) applied a property of Hausdorff space, which was neither necessary nor helpful for solving a given proof problem. It would be even effective if instructors tell in what situation a property of concept would work and in what situation another property of concept would work. Third, instructors can introduce or review proving techniques, including the one for the proof of showing A = B. The results found multiple cases in which students did not know how to prove the type of proof showing A = B. Instructors can suggest to their students that they should work on either A or B until they get B or A respectively or that they should work on both A and B separately until both become C.

Suggestions based on the model of the structure of proof construction. The model of the structure of proof construction (Figure 3.1 and Table 3.2) may help instructors to guide their students. Selden and Selden (2012) introduced "a proof framework" as an aiding tool of writing a proof. In their "proof framework," students start with the hypotheses, leave a blank space, write the conclusion at the end, and fill in the blank for the remaining work. However, their methods may not help students write a proof from the top down. The model of the structure of proof construction works to help students write a proof from the top down. Through the model of the structure of proof construction (Tables 3.1 and 3.2, and *Figure* 3.1), instructors can help their students know what they need to be equipped with for proof construction and how to get started on a proof and how to advance a reasoning process in proof construction.

The model of the structure of proof construction provides an algorithm for advancing a reasoning process for each type of proof. This section introduces the algorithm and illustrates how the algorithm works.

6.5.3 Algorithm for Proof Construction

A: *Opening Stage*

A0: Read the problem

- If necessary, translate the whole problem into *mathematical language*. (A0.1)

A1: Decide a major strategy.

- Decide which proving strategy to use, a direct proof, by contrapositive, by contradiction, by counter example, or by mathematical induction. (A1.1)
- For a proof by contrapositive or contradiction, rephrase the problem. (A1.2)
- For Type III, skip to B0. (A1.3)

A2: Note the conclusion.

- Do not be tempted to note a hypothesis. (A2.1)

A3: Translate the conclusion into *mathematical language*.

- Rephrase the whole conclusion through R1 (See Table 1). (A3.1)
- Rephrase the conclusion more than once, if necessary. (A3.2)

A4: Find an *ignition phrase* in the *mathematical language* for the conclusion.

A5: Decide the type of the proof.

- If A4 is a primary ignition phrase, the proof belongs to Type I. (A5.1)
- If there is no *ignition phrase*, the proof belongs to Type II or Type III. (A5.2)
- If there is no *ignition phrase* and the problem asks to prove A = B, it belongs to Type III. (A5.3)

A6: Find a starting variable.

- For Type I, derive a starting variable from the *ignition phrase*. (A6.1)
- For Type II, note a hypothesis, translate it into *mathematical language*, and find an *ignition phrase*. (A6.2)
- For Type III, start the *body construction stage* by trying one of the followings: Work on either A or B to change it into B or A, work on both to obtain A = C = B, or show $A \subset B$ and $B \subset A$. This can work for the proofs in Type I (b). (A6.3)

TA: Supporting tips for the *opening stage*

TA1 (Type I): A starting variable should be first found in a primary ignition phrase in the *mathematical language* for the conclusion. However, if a variable in the primary ignition phrase is a trivial variable, it may not be a starting variable. A variable from a second primary ignition phrase in the *mathematical language* for the conclusion cannot be a starting variable. If there is not *ignition phrase* in the conclusion, derive a starting variable from a hypothesis.

TA2: (Type II) A starting variable can be derived from both a primary and a second primary ignition phrases in the *mathematical language* for a hypothesis.

TA3: (Type I.b and Type III) Try one of the following methods. (i) Work on either A or B until you change it into B or A, (ii) Work on both A and B until you get A = C = B, or (iii) Show both A ⊂ B and B ⊂ A. For $A \cong B$, (iv) find an isomorphism between A and B.

B: *Body construction stage*

B0: State the hypothesis (hypotheses).

B1: Set a starting variable.

- For Type I, set a starting variable from the *ignition phrase* obtained in A4.1. (B1.1)
- For Type II, translate the hypothesis into *mathematical language*. (B1.2)
- For Type III, skip this step and start to work on part of the conclusion. (B1.3)

B2: Make sure of the new goal of the proof obtained in A3.

B3: Try *rephrasing an object*, recalling the three sub-types (See Table 3).

- Whenever seeing a sentence containing a mathematical concept, translate it into *mathematical language*, and make it as fine-grained as possible. (B3.1)

B4: If it does not work, try *combining objects*.

- Find a hypothesis and use it (B4.1).

- If there is more than one hypothesis, choose the one that has a connection with the object you would like to combine with. (B4.2)
- When the *mathematical language* for a hypothesis contains a controlling variable, use this operation (*combining objects*) to specify the controlling variable.

B5: If it does not work, try *creating a cue*, recalling the five sub-types (See Table 1).

B6: If it does not work, try *exploring and checking*.

T: Supporting Tips.

TB1: Whenever encountering a statement containing a mathematical concept, translate it into *mathematical language* and make it as fine-grained as possible.

TB2: For Type II, when the *mathematical language* for a conclusion contains a *trivial variable* or when the *mathematical language* for a hypothesis contains a *controlling variable*, confine the variable to some specific object at a certain step.

TB3: For type I(b) and Type III, try one of the followings. (i) work on either A or B until you change it into B or A, (ii) work on both A and B until you get A = C = B, or (iii) show both A ⊂ B and B ⊂ A. For $A \cong B$, (iv) find an isomorphism between A and B.

The following examples show how the above algorithm helps students to construct a proof. To make the algorithm more understandable, I will explain in the form of a dialogue between an instructor and students. In the dialogue, I assume that the students are fully equipped with not only the knowledge of the above algorithm but also the knowledge necessary for solving the given problems.

6.5.4 Examples of the Use of Algorithm

Example 1 (Type I). "Suppose $G/Z(G)$ is cyclic, where $Z(G)$ is the center of G. Prove G is abelian. What should we do first?" "Decide the major proving strategy (A1)." "What strategy would you use?" "A direct proof." "What is the next step?" "Note the conclusion (A2), translate it into *mathematical language* (A3), and find an *ignition phrase* (A4)." "What is the conclusion?" "G is abelian." "What is the *mathematical language*?" "For any $a, b \in G$, $ab = ba$." "What is the *ignition phrase*?" "For any $a, b \in G$." "What is the type of this proof?" "Type I(b)." "How did you figure that out?" "The *mathematical language* for the conclusion contains a primary *ignition phrase* 'for any $a, b \in G$' and the goal of the proof is to show A= B, where A = ab and B = ba." "Let's begin the *body construction stage*. After stating the hypothesis (B0) 'Suppose $G/Z(G)$ is cyclic, where $Z(G)$ is the center of G,' what would you do?" "Set a starting variable from the *ignition phrase* 'for any $a, b \in G$' (B1.2)." "How?" "(1) Let $a, b \in G$." "Then?" "Work on the left hand side (2) 'ab' until it changes into the right hand side 'ba' so that we can show $ab = ba$." "Then?" "First, try r*ephrasing an object* (B2)" "Does that (B2) work for 'ab' or 'a' and 'b'?" "No." "What should we do?" "Try B3 (*combining objects*)." "How?" "Note the hypothesis and use it." "What is the hypothesis?" "(3) $G/Z(G)$ is cyclic." "Are we ready to combine the objects (2) 'ab' and (3) '$G/Z(G)$ is cyclic'?" "No." "Why not?" "Because (3) '$G/Z(G)$ is cyclic' contains a mathematical concept 'cyclic.' "So?" "Translate the object (3) '$G/Z(G)$ is cyclic' into *mathematical language*. (T1)" "What is the *mathematical language*?" "(4) 'A coset of $Z(G)$ is generated by $< xZ >$ for some $x \in G$.'" "Now, are we ready to combine the objects (2) 'ab' (or 'a' and 'b') and (4) 'a coset of $Z(G)$ is generated by $< xZ >$'?" "Not really." "What can we do?" "Since B3 (*combining objects*) does not work, try B4 (*creating a cue*)." "There are five types of *creating a cue* (Table 2). Which would you try?" "C2 (recalling and applying prior knowledge)." "What relevant fact can we use to combine the objects (2) 'ab' and (4) 'a coset of $Z(G)$ is generated by $< xZ >$ for some $x \in G$'?" " (5) 'Every element in a group belongs to some coset.'" "Now, can we combine these three objects (2) 'ab', (4) 'a coset of $Z(G)$ is generated by $< xZ >$', and (5) 'every element belongs to some coset'?" "Yes, we can combine them to obtain (5) $a \in x^m Z$ and $b \in x^n Z$ for some $x \in G$ and for some $m, n \in Z^+$." "Then?" "Since we have finished applying B4 (*creating a cue*), we can resume with B2 (*rephrasing an object*)." "Can we further rephrase the object (5) '$a \in x^m Z$ and $b \in x^n Z$'?" "Yes. $a = x^m z_1$ and $b = x^n z_2$ for some $z_1, z_2 \in Z$." "So?" "Using the commutative property of elements of the center Z of G, we obtain $ab = x^m z_1 x^n z_2 = x^{m+n} z_1 z_2 = x^{n+m} z_2 z_1 = x^n z_2 x^m z_1 = ba$."

Example 2 (Type I). "Suppose that q: $X \to Y$ is a quotient map and that $f : Y \to Z$ is a map such that $f \circ q$: $X \to Z$ is continuous. Prove $f : Y \to Z$ is continuous. Let's start the *opening stage*. What proving strategy would you use? (A1)" "A direct proof." "What is the next step?" "Note the conclusion. (A2)." "What is the conclusion?" "$f : Y \to Z$ is continuous." "Next?" "Translate it into *mathematical language* (A3)." "What is the *mathematical language*?" "For any open set W in Z, $(f^{-1}(W))$ is open in Y." "Then?" "Find an ignition phrase. (A4)" "What is the *ignition phrase*? (A3)" "'For any open set W in Z.'" "What is the starting variable? (A4)" "An open set W in Z." "Let's start the *body construction stage*. After writing the hypothesis, what would you do?" "Set a starting variable (B1)." "So?" "Start with 'Let W be an open set in Z'" "Then?" "Make sure of the new goal." "What is that?" "To show $(f^{-1}(W))$ is open in *Y*." "Next?" "Start to apply the four types of operations while keeping the supporting Tips (T1 – T2) in mind." "We have gotten the object (1) an open set W in Z. What would you do?" "Apply *rephrasing an object* to the objet (1) an open set *W* in *Z*. (B3)" " Does that work?" "No." "Then, what would you do?" "Try the second operation '*combining objects*.'" "How would you do that?" "Find a hypothesis and use it. (B4.1)" "What is the hypothesis?" "There are two. (i) $q: X \to Y$ is a quotient map and (ii) $f \circ q: X \to Z$ is continuous." "Which hypothesis should we use?" "Choose the one which has a connection with the object (1) 'the open set *W* in *Z*.' (B4.2)" Which hypothesis has a connection with the object (1) an open set *W* in *Z*?" "The second hypothesis (ii) $f \circ q: X \to Z$ is continuous." "Why?" "Because both involve the space Z." "Now are we ready to combine (1) '*W* is open in *Z*' and (ii) '$f \circ q: X \to Z$ is continuous'?" "No." "Why not?" "Because the object (ii) '$f \circ q: X \to Z$ is continuous' contains a mathematical concept 'continuous.'" "So?" "By T1, translate the object (ii) into *mathematical language*." "What is the *mathematical language*?" " (2) For any open set V *in* Z, $(f \circ q)^{-1}(V)$ is open in *X*." "What do you observe in the object?" "The object (2) comes from the hypothesis of the given statement and the *mathematical language* for the statement contains a primary *ignition phrase* 'for any open set in Z.' So, By T2, we may want to specify the open set *V* in *Z* later." "Now, are we ready to combine the objects (1) *W* is open in *Z* and (2) *for any open set* V *in* Z, $(f \circ q)^{-1}(V)$ *is open in X*?" "Yes, we can confine *V* by replacing

V with *W* to obtain (3) $(f \circ q)^{-1}(W)$ *is open in X*." "Then, what should we do?" "Try r*ephrasing an object* on the object (3) $(f \circ q)^{-1}(W)$ *is open in X* (B3)." "Does that work?" "Yes, the object (3) '$(f \circ q)^{-1}(W)$ *is open in X*' can be rephrased with the object (4) '$q^{-1}(f^{-1}(W))$ *is open in X*.'" "Can we further rephrase it?" "No" "Then?" "Try *combining objects*. (B4)" "How?" "Find a hypothesis and use it (B4.1)." "Do we have one?" "Yes, we have not used the first hypothesis (i) '$q : X \to Y$ *is a quotient map*' yet." "Can we combine the objects (4) '$q^{-1}(f^{-1}(W))$ *is open in X*' and the hypothesis (i) '$q : X \to Y$ *is a quotient map*'?" "No." "Why not?" "Because (i) '$q : X \to Y$ *is a quotient map*' contains a mathematical concept '*a quotient map*.' "So?" "Translate the hypothesis (i) into *mathematical language*. (T1)" "What is *the mathematical language*?" " (5) 'For any set *H* in *Y* that satisfies $q^{-1}(H)$ is open in *Z* for a quotient map $q : Y \to Z$, *H* is open in *Y*.'" "Now, are we ready to combine the objects (4) '$q^{-1}(f^{-1}(W))$ *is open in X*' and (5) 'For any set *H* in *Y* that satisfies $q^{-1}(H)$ is open in *Z* for a quotient map $q : Y \to Z$, *H* is open in *Y*'" "Yes, since $f^{-1}(W)$ is a set in *Y*, we can specify the *H* by replacing *H* with *W* to obtain (6) '$f^{-1}(W)$ *is open in Y*.'"

Example 3 (Type II). "Suppose that a sequence $\{a_n\}$ is convergent. Show $\{a_n\}$ is bounded." "What major strategy would you use? (A1)" "A direct proof." "How would you start the *opening stage*?" "Note the conclusion (A2), translate it into *mathematical language* (A3), and find an *ignition phrase* (A4)." "What is the conclusion?" "$\{a_n\}$ is bounded ." "What is the *mathematical language*?" "For every $n \in Z^+$, $|a_n| \le M$ for some $M \in R$." "What is the *ignition phrase*?" "None." "Are not

‘For every $n \in Z^+$ ’ and ‘for some $M \in R$’ ignition phrases?” “ The phrase ‘For every $n \in Z^+$ ’ is not an *ignition phrase* because $n \in Z^+$ is a *trivial variable*. A primary ignition phrase that provides a *trivial variable* is not considered as an *ignition phrase*. The phrase ‘for some $M \in R$” is not an *ignition variable* because a phrase ‘for some …” in the conclusion cannot be an *ignition phrase*.” “Then, how would you set a starting variable?” “Since there is no *ignition phrase* in the conclusion, this proof belongs to Type II. So, after stating the hypothesis (B0), translate it into *mathematical language*. (B1.2)” “What is the hypothesis?” “$\{a_n\}$ is convergent .” “What is the *mathematical language*?” “$\lim_{n\to\infty} a_n = L$ for some $L \in R$.” “Then, what would you do?” “We can further rephrase it.” “How?” “For every $\varepsilon > 0$, $\exists N \in Z^+$ such that for every $n \ge N$, $|a_n - L| < \varepsilon$.” “Next?” “ Derive a starting variable (A5).” “How would you do that?” “Find an *ignition phrase* (A6.2)” “What is an *ignition phrase*?” “’For every $\varepsilon > 0$.” “So?” “We can set $\varepsilon > 0$ as a starting variable. However, since the variable is a *controlling variable* derived from a hypothesis, you might want to confine it to certain object by T2.” “How would you do that?” “Let $\varepsilon = 1$.” “What have we gotten so far?” “(1) $\exists N \in Z^+$ and $L \in R$ such that for every $n \ge N$, $|a_n - L| < 1$.” “How would you advance a reasoning process?” “First, try B2 (*rephrasing an object*).” “Does it work?” “Yes. (1) can be rephrased with (2) for every $n \ge N$, $|a_n| < |L| + 1$.” “Can B2 (*rephrasing an object*) still work?” “No.” “Then?” “Try B3 (*combining objects*).” “Would that work?” “No, there is nothing to combine with the object (2) for every $n \ge N$, $|a_n| < |L| + 1$.” “Then, what would you do?” “Try B4 (*creating a cue*).” “There are five types of *creating a cue*. (See Table 2). Which type would you try?” “Create a new object (C4)." "What would you create?" "The M such that $|a_n| \leq M$ for every $n \in Z^+$." "How would you do that?" "(3) Let M = max $\{|a_1|, |a_2|, ..., |a_{N-1}|, |L|+1\}$." "Can you rephrase it? (B2)" "No." "So?" "Combining the objects (2) and (3), conclude that for every $n \in Z^+$, $|a_n| \leq M$."

Example 4 (Type II). "Suppose $a \in Z$. Prove 4 does not divide $a^2 - 3$." "What proving strategy would you use? (A1)" "A proof by contradiction." "Then, what would you do?" "Rephrase the problem. (A1.2)" "What is the new statement?" "Suppose that 4 divides $a^2 - 3$ for every $a \in Z$" "What is next?" "Make sure of an *ignition phrase* in the new statement and start the *body construction stage* by directly working on the new statement to lead it to a contradiction. (A1.2)" "What is an *ignition phrase*?" "For every $a \in Z$, which is a *controlling variable*." "What does that imply?" "Since $a \in Z$ is a controlling variable derived from the *mathematical language* for a hypothesis, it may happen that we may want to confine the variable to a certain object (T2)." "Now, what would you do?" "Since it contains a mathematical concept 'divide,' translate it into *mathematical language* (T1)." "What is the *mathematical language*?" "(1) There exists $n \in Z$ such that $4n = a^2 - 3$." "Next?" "First, try B3 (*rephrasing an object*)." "Can you do that?" "Yes, rephrase the object (1) with, for example, (2) $3 = a^2 - 4n$, but I am not sure if that will work." "OK, then let's keep it to see what will happen. Then, what would you do?" "Since B3 (*rephrasing an object*) does not work anymore, try B4 (*combining objects*)." "Does that work?" "No, there is nothing to combine with (2) $3 = a^2 - 4n$." "Then, what would you do?" "Try B5 (*creating a cue*)." "There are five sub-types for *creating a cue* (See Table 2). Which would you try?" "C3 (set some cases)." "How would you use that?" "Set two cases, in which (i) $a \in Z$ is even and (ii) $a \in Z$ is odd. As expected, confine $a \in Z$ to a certain object (T2)." "Next?" "Consider the case (i). Suppose (3) $a \in Z$ is even." "Then?" "Since the statement contains a mathematical concept 'even,' translate it into *mathematical language* (T1)." "How?" "(4) Let $a = 2m$ for some $m \in Z$." "Then?" "First, try B3 (*rephrasing an object*)." "Does that work?" "Not anymore." "So?" "Try B4 (*combining objects*)." "How would you do that?" "Combine the objects (2) $3 = a^2 - 4n$ and (4) $a = 2m$ to obtain $3 = (2m)^2 - 4n = 4(m^2 - n)$, where $m^2 - n \in Z$." "Then?" "Since 4 does not divide 3, which is a contradiction." "Next?" "Work on the case (2) in a similar way. By letting (5) $a = 2m + 1$, combining the objects (2) $3 = a^2 - 4n$ and (5) $a = 2m + 1$, obtain $3 = (2m+1)^2 - 4n = 4(m^2 + m - n)$, where $m^2 + m - n \in Z$ (R1). It is a contradiction because 4 does not divide 3.

Example 5 (Type III). "Suppose $a \equiv b (\mathrm{mod}\, n)$ for $a, b \in Z$ and $n \in N$. Prove $a^3 \equiv b^3 (\mathrm{mod}\, n)$. What would you do first?" "Decide a proving strategy." "What strategy would you use?" "A direct proof." "Next?" "Note the conclusion (A2) and translate it into *mathematical language* (A3)." "What is the conclusion?" "$a^3 \equiv b^3 (\mathrm{mod}\, n)$." "What is the *mathematical language*?" "$a^3 - b^3 = (a - b)(a^2 + ab + b^2) = nc$ for some $c \in Z$." "Are we going to find an *ignition phrase* (A4)?" "No." "Why not?" "Because this proof belongs to Type III, so you don't need to derive a starting variable. (B1.3)" "Then, after stating the hypothesis, how would you start the *body construction stage*?" "Consider the left hand side (1) $(a - b)(a^2 + ab + b^2)$ and work on it until it can be changed into nc. (A6.3)" "Then, what would you do?" "First, try *rephrasing an object* (B2)." "Does that work for (1) $(a-b)(a^2+ab+b^2)$?" "No." "Then, what would you do?" "Try *combining objects*. (B3) " "How would you do that?" "Find a hypothesis and use it." "What is the hypothesis?" "(2) $a \equiv b (\mathrm{mod}\, n)$ for $a,b \in Z$ and $n \in N$." "Can we combine (1) and (2)?" "No." "Why not?" "Because (2) $a \equiv b (\mathrm{mod}\, n)$ contains a mathematical concept 'mod n.'" "Then?" "Translate (2) $a \equiv b (\mathrm{mod}\, n)$ into *mathematical language*. (T1)" "What is the *mathematical language*?" "(3) $a-b=nd$ for some $d \in Z$." "Are we ready to combine (1) and (3)?" "Yes, we can combine them to obtain (4) $(a-b)(a^2+ab+b^2)=nd(a^2+ab+b^2)=nc$, where $c=a^2+ab+b^2 \in Z$."

Example 6 (Type III). "Suppose $f:(a,b) \to R$ has a global maximum at some $c \in (a,b)$ and is differentiable at $c \in (a,b)$. Prove that $f'(c)=0$. What proving strategy would you use? (A1)" "A direct proof." "Then?" "Note the conclusion. (A2)." "What is the conclusion?" "$f'(c)=0$." "Next?" "Translate it into *mathematical language* (A3)." "What is the *mathematical language*?" "$\lim_{x \to c} \frac{f(x)-f(c)}{x-c}=0$." "What do you observe in the object?" "This proof belongs to Type III." "Then, what would you do?" "Work on the left hand side of the equation $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ until we can change it into the right hand side, which is 0." "So?" "(1) Consider $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$." "Then?" "Apply *rephrasing an object* to the object (1) $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$. (B3)" "Can you do that?" "Yes, considering $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ means considering both (2) $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}$ and (3) $\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$. So, work on each separately." "Next?" "Apply *rephrasing an object* to the object (2) $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}$. (B3)" "Does it work?" "No." "Then?" "Try *combining objects*. (B4)" "How?" "Find a hypothesis and use it. (B4.1)" "What hypothesis is available?" "(4) $f:(a,b)\to R$ has a global maximum at some $c\in(a,b)$." "How would you combine the objects (2) and (4)?" "We are not ready to combine them." "Why not?" "Because the object (4) $f:(a,b)\to R$ has a global maximum at some $c\in(a,b)$ contains a mathematical concept 'a global maximum.'" "So?" "Translate it into *mathematical language*. (T1)" "What is the mathematical language?" "(5) For all $x\in(a,b)$, $f(x)\le f(c)$." "Now, can we combine the objects (2) $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}$ and "(5) For all $x\in(a,b)$, $f(x)\le f(c)$?" " Yes. Since $f(x)\le f(c)$, $f(x)-f(c)\le 0$. Also, since $x\to c^-$, $x-c<0$. So, we can obtain the object (6) $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}\ge 0$." "Then?" "Work on the object (3) $\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$ in a similar way to obtain (7) $\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}\le 0$." "Then?" "Since we cannot rephrase each object anymore, we try *combining objects*. (B4)." "How?" "Find a hypothesis and use it. (B4.1.)" "Do we have one?" "Yes, we have (8) $f:(a,b)\to R$ is differentiable at $c\in(a,b)$." "How would you combine them?" "The object (8) $f:(a,b)\to R$ is differentiable at $c\in(a,b)$" contains a mathematical concept, translate it into *mathematical language* (T1)." "What is the *mathematical language*?" "(9) $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$." "Are we ready to combine the objects (6)

$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$, (7) $\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$, and (9) $\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} =$

$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$?" "Yes." "How?" "$0 \leq \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$."

"Then, what would you do?" "Try *rephrasing an object*. (B3)" "Does it work?" "Yes.

$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = 0 = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$." "So?" "$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$, which means

$f'(c) = 0$."

In reality, students may not advance a reasoning process as smoothly as the above even if they are fully equipped with all the necessary knowledge for solving a given proof problem and the full knowledge of both the algorithm and the model of the structure of proof construction. Moreover, there must be proofs for which the algorithm does not work well. I have shown above that the algorithm has the potential to serve as effective method to help students with proof construction, it still needs refining and improving.

6.6 Conclusion

Proof construction can be a difficult task especially for novice students. Students are often at a loss for how to start and advance a reasoning process in constructing a proof. Struggling with advancing a reasoning process, they resort to external, empirical, or pictorial proof schemes for their proofs. As Harel and Sowder (1998) suggested, I consider analytical proof scheme, which enables students to construct a proof based on logical deduction, to be an ideal proof scheme for students to practice. Weber and Alcock (2004) indicated that both syntactic and semantic approaches must concur to construct a proof based on logical deduction. However, there seems to be little research that provided specific and practical pedagogical suggestions to help students with both approaches. Focusing on the syntactic approach, Selden and Selden (2007) provided procedural knowledge to produce the *formal-rhetorical part*. However, their method may have a limitation in helping students write a proof from the top down. By offering a model of the structure of proof construction, this study attempted to fill those gaps in the current literature. The model can serve as an effective tool for realizing syntactic and semantic approaches to help students practice analytical proof scheme.

There were four goals for this study to achieve: (1) to provide a model of the structure of proof construction; (2) to clarify the sources of students' difficulties with proof construction; (3) to evaluate the usefulness of the model of the structure of proof construction; and (4) to provide practical pedagogical suggestions to help students with proof construction.

6.6.1 Model of the structure of proof construction

Through providing a model of the structure of proof construction, this study presented a comprehensive view of proof construction that can encompass the aspects, factors, patterns, and features involved in cognitive processes in proof construction. The model was created by viewing proof construction from four aspects (*reasoning activity*, *background knowledge*, *mental attitudes*, and *affect and beliefs*). The model suggested those aspects were intertwined to influence one another to affect students' performances in proof construction. Also, the model provided the factors that compose each aspect in an organized way while simplifying a complex nature of the cognitive processes involved in proof construction. Moreover, the model clarified the features of the factors, focusing on the operations used in the *reasoning activity*. Furthermore, the model detected patterns that was used in advancing a reasoning process. The model offered the order of the operations to be tried in advancing a reasoning process, types of proofs classified by the ways to derive a starting variable, the ways to manage variables. The model provided two stages (opening stage and body construction stage), clarifying the roles and features of each stage. The model was found to be applied to proofs across a variety of mathematical subjects. The knowledge of the structure of proof construction can function as metacognitive and methodological knowledge for advancing a reasoning process.

6.6.2 Sources of students' difficulties with proof construction

The analysis of students' proofs found out that multiple factors were intertwined to affect their performances. In light of the model of the structure of proof construction, students' difficulties were identified to be those with practicing the operations in the *reasoning activity* and that the sources of their difficulties were ascribed to their lack of their *background knowledge* and *mental attitudes*. The greatest factor that affected students' proofs were their lack of knowledge. In particular, their lack of knowledge directly hindered them from *rephrasing an object* and *creating a cue*. The analysis also strongly indicated that students' lack of flexibility and carefulness contributed to their inabilities of practicing all the operations in *the reasoning activity* (*rephrasing an object*, *combining objects*, *creating a cue*, and *checking and exploring*). The analysis also identified students' difficulties with starting a proof. Students had difficulties with noting the conclusion, translating the conclusion into *mathematical language*, and preparing a starting variable. The remarkable sources of their difficulties included their tendency to note a hypothesis and their lack of knowledge of definitions.

6.6.3 Usefulness of the model of the structure of proof construction

The model of the structure of proof construction was useful in the following ways. First, the model made it easy to view and understand the complex cognitive processes involved in proof construction. Next, the model directly contributed to the creation of a framework for analyzing students' proofs. The analysis framework helped to identify students' difficulties and to explain the sources of their difficulties in a clear and organized way. The model also contributed to clarification of the skills and abilities necessary for proof construction. Moreover, the model produced algorithm for constructing a proof.

6.6.4 Pedagogical suggestions

Both the model of the structure of proof construction and the findings from the analysis of students' proofs produced a variety of pedagogical suggestions. Students should be encouraged to be equipped with strong understanding and knowledge, including definitions, notations, properties, theorems, relevant facts, and problem-solving techniques. They should be also encouraged to be aware that they must be persistent, patient, flexible, careful, and precise in proof construction. Instructors may need to help students remind and organize their mathematical knowledge in class. However, the most significant suggestion was that the model of the structure of proof construction itself can serve as metacognitive knowledge to help students with proof construction. The model can help students grasp a comprehensive view of proof construction and increase their accessibility to proof construction. Above all, the model gives students specific and practical methods for advancing a reasoning process in proof construction. Finally, the establishment of the model and the analysis of students' proofs culminated in producing the algorithm for proof construction. I expect the algorithm to be an innovating method to help students with proof construction.

6.7 Limitations

There are some limitations with this study. The model of the structure of proof construction was created based on a limited number of proofs. In addition, the majority of the proofs examined were proofs collected from undergraduate mathematics courses. In addition, the creation of the model did not include the proofs that asked to construct a counter example. There is still room for improvement in the model of the structure of proof construction, including the types of proofs and the functions of variable, and above all, the algorithm for proof construction. The algorithm is not an ultimate formula for solving any proof problem. It must have weaknesses or defects in it. In order to refine and improve the model of the structure of proof construction and the algorithm for proof construction, more proofs from a variety of mathematical subjects need to be examined.

Another limitation was that the analysis of students' proof s had to involve my subjective interpretation to some extent. For example, there was no definite way to measure the degrees of *tenacity*, *flexibility*, and *carefulness*. Students' difficulties were analyzed from three perspectives: *reasoning activity*; *background knowledge*; and *mental attitudes*. It was unknown that exactly what factor caused their difficulties to what extent. In addition, the analysis depended on only students' written work.

The model of the structure of proof construction is just one of the ways to view proof construction. It may work only for some students. In addition, in order for them to master the knowledge of the structure of proof construction, they will need to be trained by an expert. Instructors may teach the model in a workshop or in a class while, adjusting and modifying the model based on their insights.

6.8 Future Research

I hypothesize that the model of the structure of proof construction can help students advance a reasoning process more successfully. A possible future study would be to examine the effectiveness of the knowledge of the model of the structure of proof construction. Another possible future research would be to improve the model of the structure of proof construction and to make a stronger algorithm for proof construction by examining more proofs from a wide range of mathematical subjects.

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Appendix A

Problem [4] from Algebra I for the in-class problem solving session

Suppose that $Z(G)/G$ is cyclic. Prove that G is abelian.

1. First, state your problem solving strategy briefly. What would you like to show? How do you prove the statement?

2. Prove the statement. If you need hints, please go to the next page. Once you move to the next page, please don't come back of this page to fill in the blank below.

3. Problem: Suppose $Z(G)/G$ is cyclic. Prove that G is abelian.

Use the following hints.
(1) Show for any $g, h \in G$, $gh = hg$.
(2) If $G = Z(G)$ is abelian, $G = Z(G)$.
(3) Every element in G belongs to some coset of H.

Problem [5] from Algebra I for the in-class problem solving session

1. Suppose that the order of a group G is a prime number. Prove that G is cyclic.
    a. First, state your problem solving strategy briefly. What would you like to show? How do you prove the statement?
    b. Prove the statement. If you need hints, please go to the next page. Once you move to the next page, please don't come back to this page to fill in the blank below.

2. Problem: Suppose that the order of a group G is a prime number. Prove G is cyclic. Use the following hints (1) ~ (3).

(1) Let $g \in G$. Show $G = \langle g \rangle$.
(2) $\langle g \rangle$ is a subgroup of G for any $g \in G$.
(3) Use Lagrange's Theorem: Suppose that G is a finite group and H is a subgroup of G. Then, the order of H divides the order of G.

Problem [6] from Algebra I for the in-class problem solving session

1. Suppose $|G| = pq$ for some primes p and q. Prove, G is either abelian or $Z(G) = \{e\}$. If you need hints, please move to the next page. Once you move to the next page, please don't come back to this page to fill in the following blank.

2. Problem: Suppose $|G| = pq$ for some primes *p* and *q*. Prove, *G* is either abelian or $Z(G) = \{e\}$. Use the following hints.

(1) $Z(G)$ is a subgroup of *G*.
(2) Use Lagrange's Theorem: Suppose that *G* is a finite group and *H* is a subgroup of *G*. Then, the order of *H* divides the order of *G*.
(2) If the order of a group *H* is prime, *H* is cyclic.
(3) If $K / Z(K)$ is cyclic, *K* is abelian.
(4) Show $|Z(G)| = p$ and $|Z(G)| = q$ never happens by contradiction.

Problem [9] from Algebra II for the in-class problem solving session

1. Let *R* and *S* be rings. Suppose $\phi : R \to S$ is a ring homomorphism. Assume $\psi : R / Ker(\phi) \to S : [r] \mapsto \phi(r)$ is a well-defined ring homomorphism. Show $\psi : R / Ker(\phi) \to S$ is a ring homomorphism.

2. Show $\psi : R / Ker(\phi) \to S$ is injective.