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AN ANALYSIS OF MINIMAX FACILITY LOCATION PROBLEMS WITH AREA DEMANDS

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THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

AN ANALYSIS OF MINIMAX FACILITY LOCATION PROBLEMS WITH AREA DEMANDS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

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By

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Norman, Oklahoma

1984

AN ANALYSIS OF MINIMAX FACILITY
LOCATION PROBLEMS WITH AREA DEMANDS

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This work is dedicated to the memory of my late father,

Ahmed Rahali

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ABSTRACT

Most probabilistic facility location problems investigated to date were variations of the generalized Weber formulation. In this research, several single facility minimax location models are analyzed, where both the weights and the locations of the existing facilities are random variables. The demand points are uniformly distributed over rectangular areas, the rectilinear metric is used and the weights are assumed to be independently distributed random variables. Two unconstrained probabilistic models are analyzed and compared to the centroid formulation, it is seen that the probabilistic models are sensitive to deviations from optimal solutions. An expected value criterion formulation is also presented along with lower and upper bound approximating functions.

A minimax objective function constrained by a bound on the total average cost of servicing all existing facilities (minisum function) is then discussed. Using duality properties, this problem is shown to be equivalent to another model which minimizes the minisum function subject to a bound on the same minimax function. This last problem proves to be easier to solve, and a specialized solution technique is developed. The resulting solutions are nondominated solutions in relation to the two criteria involved. Another way

to generate nondominated solutions is by combining the two functions into a weighted sum. The constrained criterion method is shown to be supericr both analytically and practically.

The unconstrained model, and its solution technique can be easily modified to solve the limiting case where all facilities are fixed points, and also the case when metric constraints are added.

Examples are solved to show the impact of assuming area demands, the conflicting nature of the minimax and minisum criteria and to illustrate the solutions techniques developed.

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CHAPTER I

INTRODUCTION

1.1 Introduction

Facility location problems arise in every industrial and public organization. Some typical problems include locating a hospital, a fire station, a power plant, schools, television relays, police stations, military bases, obnoxious facilities (dump sites, nuclear power plants, water recycling facilities, etc.), manufacturing plants, warehouses, radar stations for civilian or military air traffic control, etc.

The variety of locational problems has resulted in a significant amount of attention in the literature. Researchers from many different disciplines have contributed to the analysis of facility location problems. Among these disciplines are industrial engineering, operations research, management science, geography, regional planning, architecture, transportation science, economics, mathematics, urban development, computer science, etc.

1.2 General Characteristics of the Problems to be Considered

Francis and White (1974) classified facility layout and location problems according to six major elements:

- new facility characteristics
- existing facility location

- new and existing facility interaction
- solution space characteristics
- distance measure
- objective function

A location problem is formulated when each one of the six elements cited is determined. In this research, the major characteristics of the locational models to be investigated will be a number of combinations of the following situations:

- There will be a single new facility to locate, represented by a single point.
- The existing facilities are rectangular regions of known dimensions, more restrictions will be added later.
- The interactions between new and existing facilities are quantitative, deterministic or probabilistic, not location dependent, and to be considered as parameters in the mathematical formulation (as opposed to being variables).
- The solution space is continuous in the two-dimensional real space, with or without constraints.
- The metric used is the rectilinear norm.
- The objective is quantitative. It is either to minimize the total average cost of servicing all existing facilities, or to minimize the maximum cost of servicing any one facility, or some combination of these two single objectives.

The single facility generalized Weber problem can be formulated deterministically as follows:

$$\underset{x \varepsilon S_1 \subset R^2}{\text{Minimize}} \sum_{i=1}^{n} w_i \| X - P_i \|_p \qquad \text{(P1.2.1)}$$

where

S_1 : some given compact, nonempty convex subset of R^2

n : the number of existing facilities

$P_i \equiv (a_i, b_i)$: coordinate location of existing facility i

$X \equiv (x_1, x_2)$: coordinate location of the new facility

$\| X - P_i \|_p$: $p \geq 1$, is the distance between new facility X and existing facility P_i

w_i : cost per unit time per unit distance between the new facility and existing facility i

When p = 1, the distance metric is rectilinear or metropolitan distance. This metric usually offers a better approximation to real distances when traveling along warehouse or factory aisles, or in a densely populated metropolitan area.

Problem P1.2.1 is also called the minisum problem or location problem under the minisum criterion.

The minisum criterion is more appropriate when locating a new facility that provides routine services (warehouses, schools, shopping centers, office buildings, etc.). When locating emergency facilities, such as police or fire stations, and ambulance services, the focus is on individual service. The new facility is to be located such that the weighted distance to the furthermost existing facility is minimized. Mathematically, such a model in a continuous space can be formulated as follows:

$$\underset{x \in S_2 \subset R^2}{\text{Minimize}} \left(\max_{1 \le i \le n} \{ w_i \| X - P_i \|_p \} \right) \qquad \text{(P1.2.2)}$$

where S_2 is some given compact, convex and nonempty set in R^2. Among the several models to be investigated, and deriving from P1.2.1 or P1.2.2, more emphasis will be given to cases where S_1 is defined by the points in R^2 that satisfy a given upper bound on the minimax function, and to formulations where S_2 is defined as the set in R^2 satisfying a given upper bound on the minisum function.

1.3 Application of the Research

1.3.1 Rectangular Regions

When large populations are on hand, modeling the demand set as a finite number of points can be computationally impractical because of the number of points which would be involved. A common practice in such a case has been to partition the total populated area under consideration into rectangularly shaped subareas, with uniformly distributed population in each one. This modeling practice can also be useful when representing the probabilistic nature of certain demand facilities such as the occurrence of a fire, accident or crime in a densely populated urban area. Insurance companies for example, subdivide an area of interest into several rectangular regions with respective weights representing some historically justified risk levels. When locating a new fire station in an urban area, it is generally assumed that a fire can erupt anywhere within the total area. The probability of occurrence could of course vary from one neighborhood to another, depending on socio-economic and other factors. A subdivision into rectangular areas with associated uniform distribution function can be a very useful and realistic approximation of the real situation. Also, formulation with rectangular regions can be interpreted as a generalization of the centroid approach.

1.3.2 Combination of the Two Criteria

Solving a location problem under the minisum criterion might produce a solution situated too far from some existing facilities. On the other hand, if a new emergency facility is located under the minimax criterion, too many existing facilities could be the maximum distance away, or close to it, from the new facility. Many location problems can be best modeled as a combination of the two criteria, such that the possible extreme effects of evaluating one single criterion can be controlled. For example, when locating a new school, the location should be close to the most densely populated areas, without any single student having to travel over a number of miles. For the location of an ambulance station, one wants to minimize the maximum distance (or maximum response time) to any emergency call, with a constraint on the minisum function, that is, the location of the ambulance station should be close enough to the most heavily populated areas.

1.3.3 Probabilistic Weights

The weights associated with the existing facilities have an important influence on the location of the new facility. For deterministic location problems, the points with higher weights will attract the location of the new facility. For large populations, increasing the number of points in the deterministic model is approximate to using a region with a high population density. If a weight for a region is increased greatly relatively to the other regions, then the center of gravity of that region will attract the optimal location. To circumvent these extreme cases, it is assumed that the weights are random variables with small variances, and expected value criterion are considered.

1.4 Scope and Limitations

The analysis in this research will concentrate on models where the only sources of random variations are the locations of the existing facilities, and then, only uniform distributions are assumed. For the cases with random weights, the normal distribution is assumed, several optimization criteria will be proposed and analyzed, but no computational experience will be performed since the main research effort is geared towards models with deterministic weights. For models which involve both the minisum and minimax criteria, the computational aspect is very important since it supports and illustrates relationships that will be generated in later chapters.

1.5 Order of Presentation

Because of the variety of models to be considered, the related research literature is surveyed in each subsequent chapter as the need for it arises. Possible practical applications of the various formulations are offered, and example problems are solved when appropriate.

Chapter II will treat location problems under the minimax criterion, several formulations will be evaluated and compared. Deterministic and probabilistic weight cases are studied. In Chapter III, problems with deterministic weights are investigated. The minisum function is minimized under a constraint on the minimax function. A duality relationship with a related problem, described in Chapter II, is developed, and an efficient solution procedure is presented. Chapter IV analyzes another location model obtained by forming a weighted sum of the minisum and minimax functions. This model is shown to be closely related to the two "dual" models. Analytical properties that bind all these problems are developed. In the fifth and last chapter, the research effort is summarized, conclusions are drawn and recommendations for further research are made.

CHAPTER II

ANALYSIS OF PROBABILISTIC MINIMAX FACILITY LOCATION PROBLEMS

2.1 Introduction and Principles of Choice

When modeling a real life problem, three main avenues are possible, either to assume decision under certainty (deterministic parameters), decision under risk or decision under uncertainty. Most location problems have been modeled as decision under certainty, the common parameters, interaction between facilities, and the locations of the existing facilities are usually assumed known deterministically. In Chapter II, the weights w_i's are assumed to be random variables with known probability density functions. For example, when locating an emergency service facility, an existing facility may require service randomly in space, and with a frequency that is often random. When the weights represent cost per unit distance traveled, they can be affected by fluctuating gas prices, cost of equipment used, etc. The weights may also represent volumes of goods transported, which are often random. When response times are measured, they very often are modeled as random variables (Larson (1972) and Volz (1971)). Since it is assumed that all probability density functions are known, the resulting models require decision under risk. When modeling a deterministic location problem, several possible optimization criteria are available (minisum, minimax, maximin, etc), but when considering probabilistic parameters, another choice has to be made on how to incorporate the probabilistic nature of these elements into the formulation and optimization steps. The following five optimization criteria under risk are the most frequently used,

- expected value criterion
- portfolio criterion
- aspiration criterion
- fractile criterion
- chance constrained programming.

Suppose some new facility X is to be located such that it minimizes some appropriately defined cost function Z(X) (or Z), then when risk conditions exist, Z is itself a random variable. The expected value criterion requires finding the location that will minimize the expected value of the random variable Z.

The portfolio criterion seeks the location that minimizes the variance of costs, subject to a constraint on the expected cost generated by that location. Since the location problems to be investigated are minimax problems, the worst cases possible are of interest. Only those realizations near one tail of the probability density function are relevant, and therefore, the portfolio criterion will not be utilized. The aspiration criterion maximizes the probability of cost being less than some given value γ (aspiration level):

$$\max_{X \in R^2} F_Z(\gamma)$$

where Z is the cost function, with distribution function $F_Z(.)$.

The fractile criterion minimizes the α-fractile of the distribution of cost as follows:

$$\underset{\delta, X}{\text{minimize}}\ \delta$$

$$\text{subject to}$$

$$P_r(Z \leq \delta) \geq \alpha$$

where α is a predetermined probability level, δ is a decision variable and Z = Z(X) is the cost function for location X. The fractile criterion is specially appropriate for emergency facility location problems.

2.2 Overview of Previous Research

Until recently, the bulk of the probabilistic location research had been directed to the solution of generalized Weber problems. With the renewed interest in locating emergency service type facilities, the deterministic minimax criterion has received increasing attention.

Hakimi (1964) has studied the problem of finding a minimax solution on a graph, and suggested possible applications to the location of police and fire stations. Smallwood (1965) investigated related problems regarding the placement of detection stations. Groenewoud and Eusanio (1965) studied a problem derived from an investigation of multiple airborne target tracking with a ground based radar. Given a fixed set of points, a smallest covering cone or sphere is found using an iterative algorithmic approach. Francis (1967) derived some properties of a single facility location problem with a ℓ_p norm. A good lower bound on the value of the minimax solution is given, and some geometrical characteristics are discussed.

Francis (1972) geometrically solved a minimax rectilinear distance problem where the solution is constrained within a given nonempty compact set. The procedure basically consists of enclosing the solution set by the smallest diamond possible.

Elzinga and Hearn (1972) proposed geometrical solution procedures to several minimax location problems with Euclidean and rectilinear distances, which translated into finding a minimum covering sphere and diamond. Wesolowsky (1972) proposed a parametric linear programming method for the multifacility case with rectilinear distances. Love, et al. (1973) presented a nonlinear programming technique to find a solution to the multifacility case with Euclidean distances. Dearing and Francis (1974) proposed a network flow solution to a rectilinear multifacility problem. The method is based on a network flow solution for the single facility case by Cabot et al. (1970).

Elzinga et al. (1976) considered a multifacility formulation with Euclidean distances, and applied nonlinear programming duality theory in the development of the solution procedure. Drezner and Wesolowsky (1978) used numerical integration of ordinary differential equations to solve the multifacility problem with ℓ_p norm. Jacobsen (1981) presented an algorithm for solving a single facility Euclidean model. He used an iterative procedure based on the method of feasible directions.

Charalambous (1981) presented an iterative method for the multifacility Euclidean distance problem. Chandrasekaran and Pacca (1980) generalized some solution method developed by Elzinga and Hearn (1972). Hearn and Vijay (1982) classified available techniques for solving the single facility problem with Euclidean metric and proposed some extensions and new versions of solution methods.

Shamos (1975) and Shamos and Hoey (1975) proposed several fast algorithms for a number of problems in computational geometry. For the smallest circle enclosing a given set of points in two dimensions, they proposed a method based on generating the Voronoi polygons associated with the given points.

Chatelon, Hearn, and Lowe (1979) used a subgradient algorithm for optimizing certain types of minimax problems, and applied it to the Euclidean minimax location problem. The technique was based on methods of successive approximations for solving minimax functions by Dem'yanov and Malozemov (1974), and on convexity results by Rockafellar (1970) which will be often used in this research effort.

Even though literature on facility location problems is plentiful, models with probabilistic weights have received only limited attention. Seppälä (1975) studied a multi-facility Weber problem where the weights are assumed to be normal random variables, and the fractile approach is chosen. Seppälä's (1972) CHAPS algorithm is used to solve the deterministic equivalent problem. Aly and White (1978) considered a multifacility location problem when both the weights between the facilities and the location of existing facilities are random variables. Distances are Euclidean and the expected value criterion is used. Unconstrained and chance constrained cases are investigated. Equivalent deterministic problems are derived and solution procedures are proposed. They also noted that the fractile criterion for probabilistic location problems is an analogue of the minimax criterion for the deterministic case.

Another approach when evaluationg probabilistic location problems with random weights is to compute the expected value of perfect information EVPI. The objective is not to find the location that optimizes some given criterion, the main goal is to find the expected cost difference between the actual best location (without knowing the outcome of the w_i's in advance) and the best location resulting from exact knowledge of the outcome of the w_i's (using expected weights). EVPI is thus defined as the upper limit one should pay for information about weights when an expected value criterion is adopted.

Wesolowsky (1977) investigated a one dimensional single facility location problem with normally distributed weights, and he derived an analytical expression for the EVPI. Drezner and Wesolowsky (1980) extended the previous study for the two-dimensional space problem. Both the rectilinear distance and the gravity models are considered. Normal distribution for the weights are also assumed.

In this chapter, models will be investigated that depend on the principle of choice, on the interpretation of the rectangular regions and of the weights w_i's. Consider the following general model,

$$\underset{X \in S_2 \subset R^2}{\text{minimize}} \quad \max_{1 \le i \le n} \{w_i \|X - P_i\|\} \qquad \text{(P2.2.1)}$$

where

S_2 : is a given compact, nonempty convex subset of R^2.

n : number of existing facilities

$P_i \equiv (a_i, b_i)$ coordinate location of existing facility i; P_i is a bivariate uniformly distributed random variable over rectangular region R_i, and with joint density function $\frac{1}{A_i}$

A_i : area of region i.

$X \equiv (x_1, x_2)$: coordinate location of the new facility.

w_i : probabilistic weight associated with existing facility P_i, and with known distribution function.

Aly and White (1978) argued that the occurrence of the w_i's and the P_i's can be interpreted in two different ways. In one case, it is assumed that once the location of P_i

is known, all following trips between new facility X and P_i will share the same distance $\|X - P_i\|$, and the cost of servicing location i is expressed as the product of the random variables w_i and the distance to the new facility.

In the second case, each trip from X to P_i included in w_i (when w_i represents the number of trips) can have different length, that is, in each subsequent trip to region R_i included in w_i, P_i can have a different realization (a_i, b_i). The total distance traveled to facility i can be represented as a random sum of random variables.

For the minisum Weber problem with expected value criterion, the two cases yield identical models. In this analysis, it is assumed that w_i is a random cost associated with servicing facility i, per unit distance traveled, and the cost incurred by facility i is a random variable independent of the location of the existing facility i, and is represented as the product of random variables. Also, in all models to follow, rectangular regions are used to represent existing facilities, and the rectilinear metric is used. The total region under study is partitioned into n rectangular subareas, and the following assumptions are generally accepted:

i) no overlap of the rectangular regions is allowed

ii) the location of a facility requiring service is uniformly distributed over the subarea to which it belongs

iii) no barriers exist within the total area under consideration that would affect interaction between any two points.

If two or more rectangles overlap, then the area they occupy is divided into nonoverlapping rectangles, and the new weights are computed by accumulating weights from the old rectangles as necessary.

2.3 Unconstrained Probabilistic Minimax Location Problems

2.3.1 A Conservative Interpretation of the Rectangular Regions

Depending on the type of problems being modeled, there are several possible interpretations for a demand point uniformly distributed over a rectangle, and the special nature of the minimax criterion allows a particularly interesting and useful formulation.

Problem P2.2.1 reflects the preference of a very conservative decision maker; it is appropriate when modeling for the location of an emergency type facility. When locating a new fire station, it is reasonable to assume that in any rectangular region, the occurrence of a fire is a uniformly distributed event. Since each point in a region is as likely to require service, the extreme value is represented by the distance from the location X of the new facility to the most distant point in the region under consideration.

Let $R_i = [a_{i_1}, a_{i_2}] \times [b_{i_1}, b_{i_2}]$ Cartesian representation of region i and $P_i = (a_i, b_i)$ location of existing facility i; (a_i, b_i) is a bivariate uniformly distributed random variable over region R_i.

The following lemmas will help to obtain a deterministic formulation of problem P2.2.1 when (a_i, b_i) is a random variable.

Lemma 2.3.1: The point(s) furthest away from X in rectangular region i is at an extreme point of the region.

Proof: The function $\|X - P_i\|$ is convex, in the convex polytope R_i, it is optimal for some extreme point of R_i (one of four corner points of the region).

Lemma 2.3.2: The rectilinear distance from X to the most distant point in rectangular region is $\|X - C_i\| + r_i$ where C_i is the centroid of region i, and r_i is one-fourth the perimeter of R_i.

In lemma 2.3.2, there is no need to find the most distant points in each region, since the distance to the new facility depends only on the centroid and dimensions of the region.

2.3.2 Minimax Model with Expected Value of the Weighted Distances: A Conservative Formulation.

If one wants to adopt a conservative attitude, then the rectilinear distance from the new facility to the uniformly distributed location of facility i in region i is replaced by the distance to the most distant point in the region. The expected value criterion model obtained is

$$\underset{X \in R^2}{\text{minimize}} \quad \max_{1 \le i \le n} \{E(w_i)(\|X - C_i\| + r_i)\} \qquad \text{(P2.3.1)}$$

where $C_i = (c_{i_1}, c_{i_2})$ is the centroid of region i.

$$c_{i_1} = \frac{a_{i_2} + a_{i_1}}{2}, \quad c_{i_2} = \frac{b_{i_2} + b_{i_1}}{2},$$

and

$$r_i' = E(w_i) r_i, \; r_i = \frac{a_{i_2} - a_{i_1}}{2} + \frac{b_{i_2} - b_{i_1}}{2}$$

P2.3.1 is mathematically similar to a minimax location problem formulated by Dearing (1972). In that formulation, the term equivalent to r_i' was motivated as follows: an ambulance located at X responds to an emergency at any point C_i, and then travels to the nearest hospital which is r_i' miles away. For simplicity of notation, let $E(w_i) = w_i$.

Francis and White (1974) reviewed several techniques to solve problem P2.3.1. A popular method is to obtain an equivalent linear program by using the following transformation:

$$\underset{X \in R^2}{\text{minimize}} \; Z \qquad \text{(P2.3.2)}$$

subject to

$$w_i(|x_1 - c_{i_1}| + |x_2 - c_{i_2}|) + r_i' \leq Z, \text{ for } i = 1,\ldots,n$$

and then linearizing the absolute values. Network flow techniques have also been used, but a procedure developed by Dearing (1972) is adopted in this research. This method finds all minimax locations and can be used to generate contour lines. A contour line of f(X) for a chosen constant k is the set of all points Y for which f(Y) = k, and it is a rectangle with two parallel sides making a 45° angle with the x_1-axis, and the other two parallel sides making a -45° angle with the

x_1-axis. This method was adopted because it doesn't require any special optimization code, it is fairly easy to program, and the simple construction of contour lines of the minimax function is fully used when solving a related problem in which the minimax function acts as a constraint. Other possible solution procedures could be subgradient based iterative methods, since the functions are not differentiable.

Description of the Dearing procedure for problem P2.3.1

The following linear transformations T and T^{-1} of points in the plane are needed:

$$T(x,y) = (x+y,-x+y)$$

$$T^{-1}(r,s) = \frac{1}{2}(r-s,r+s)$$

Also, let

$$T(c_{i_1},c_{i_2}) = (c_{i_1}+c_{i_2}, -c_{i_1}+c_{i_2}) = (c'_{i_1},c'_{i_2})$$

Step 1: Compute the numbers α_{ij} and β_{ij} where,

$$\alpha_{ij} = \max\left(\frac{w_i w_j |c'_{i_1}-c'_{j_1}| + w_i r'_j + w_j r'_i}{(w_i+w_j)}, r'_i, r'_j\right)$$

$$\beta_{ij} = \max\left(\frac{w_i w_j |c'_{i_2}-c'_{j_2}| + w_i r'_j + w_j r'_i}{(w_i+w_j)}, r'_i, r'_j\right)$$

Step 2: Let p_1 and p_2 be indices for which

$$Z_1 = \max_{1 \le i < j \le n} (\alpha_{ij}) = \alpha_{p_1 p_2}$$

and if $c'_{p_1 1} \le c'_{p_2 1}$

let

$$r^* = \frac{w_{p_1} c'_{p_1 1} + w_{p_2} c'_{p_2 1} - r'_{p_1} + r'_{p_2}}{w_{p_1} + w_{p_2}}$$

otherwise, if $c'_{p_1 1} > c'_{p_2 1}$

let

$$r^* = \frac{w_{p_1} c'_{p_1 1} + w_{p_2} c'_{p_2 1} + r'_{p_1} - r'_{p_2}}{w_{p_1} + w_{p_2}}$$

Step 3: Let q_1 and q_2 be indices for which

$$Z_2 = \max_{1 \leq i < j \leq n} (\beta_{ij}) = \beta_{q_1 q_2}$$

and when $c'_{q_1 2} \leq c'_{q_2 2}$, let

$$s^* = \frac{w_{q_1} c'_{q_1 2} + w_{q_2} c'_{q_2 2} - r'_{q_1} + r'_{q_2}}{w_{q_1} + w_{q_2}}$$

otherwise, if $c'_{q_1 2} > c'_{q_2 2}$, then let

$$s^* = \frac{w_{q_1} c'_{q_1 2} + w_{q_2} c'_{q_2 2} + r'_{q_1} - r'_{q_2}}{w_{q_1} + w_{q_2}}$$

Step 4: then $Z_0 = \max(Z_1, Z_2)$ is the minimum value of P2.3.1, and $T^{-1}(r^*, s^*)$ is a minimax location. In order to find all locations, the following three cases are considered:

<u>Case 1</u>: $Z_0 = Z_1 = Z_2$: $T^{-1}(r^*, s^*)$ is the unique solution.

<u>Case 2</u>: $Z_0 = Z_1 > Z_2$, then compute

$$s_1 = \max_{1 \leq i \leq n} c'_{i_2} - \frac{(Z_0 - r'_i)}{w_i}$$

$$s_2 = \min_{1 \le i \le n} c'_{i_2} + \frac{(Z_0 - r'_i)}{w_i}$$

Any point on the line segment with endpoints $T^{-1}(r^*, s_1)$ and $T^{-1}(r^*, s_2)$ is a minimax location.

Case 3: $Z_0 = Z_2 > Z_1$, then compute

$$r_1 = \max_{1 \le i \le n} c'_{i_1} - \frac{(Z_0 - r'_i)}{w_i} ,$$

$$r_2 = \min_{1 \le i \le n} c'_{i_1} + \frac{(Z_0 - r'_i)}{w_i}$$

Any point on the line segment joining the points $T^{-1}(r_1, s^*)$ and $T^{-1}(r_2, s^*)$ is a minimax location.

2.3.3 Minimax Location Model with Expected Value of the Weighted Distances.

In problem P2.3.1, the distances to the furthest point in each region from the facility are computed in order to evaluate the minimax function. In this case, the average distances from the new facility to each region are computed. The resulting mathematical model is as follows:

$$\underset{X \in R^2}{\text{minimize}} \max_{1 \le i \le n} \{E(w_i \| X - P_i \|)\} \qquad \text{(P2.3.3)}$$

or

$$\underset{X \in R^2}{\text{minimize}} \max_{1 \le i \le n} \{\mu_i \frac{1}{A_i} \iint_{R_i} (|x_1 - a_i| + |x_2 - b_i|) da_i db_i\} \qquad \text{(P2.3.4)}$$

where μ_i is the expected value of the random variable w_i, and $\frac{1}{A_i}$ is the joint probability density function of $P_i \equiv (a_i, b_i)$ defined on R_i.

Lemma 2.3.3: Problem P2.3.4 is a convex programming problem.

Possible Solution Techniques for P2.3.4:

P2.3.4 can be written as minimize $\max_{X \in R^2 \; 1 \le i \le n} \{f_i(X)\}$ where each function $f_i(X)$ is continuously differentiable, but not $\max_{1 \le i \le n} \{f_i(X)\}$, gradient based techniques are therefore not applicable. Since P2.3.4 is convex and unconstrained direct methods can be very efficient. The pattern search by Hooke and Jeeves (1961) is used to solve P2.3.4. Also, as can be seen in Figure 2.1, which shows several iso-curves of such a function, $f_i(X)$, the complex shapes of these curves do not invite an efficient geometrical solution (such as the smallest covering sphere problem, for example). Problem P2.3.4 can be rewritten as

$$\min_{X \in R^2} Z \qquad \text{(P2.3.5)}$$

subject to

$$f_i(X) \le Z \quad , \quad i = 1,\ldots,n$$

and the following Lagrangian dual problem is derived:

$$\max_{u \ge 0} \min_{X \in R^2} \sum_{i=1}^{n} u_i f_i(X) = \max_{u \ge 0} \theta(u) \qquad \text{(P2.3.6)}$$

$$\text{subject to } \sum_{i=1}^{n} u_i = 1$$

An alternative for solving P2.3.4 is to solve P2.3.6.

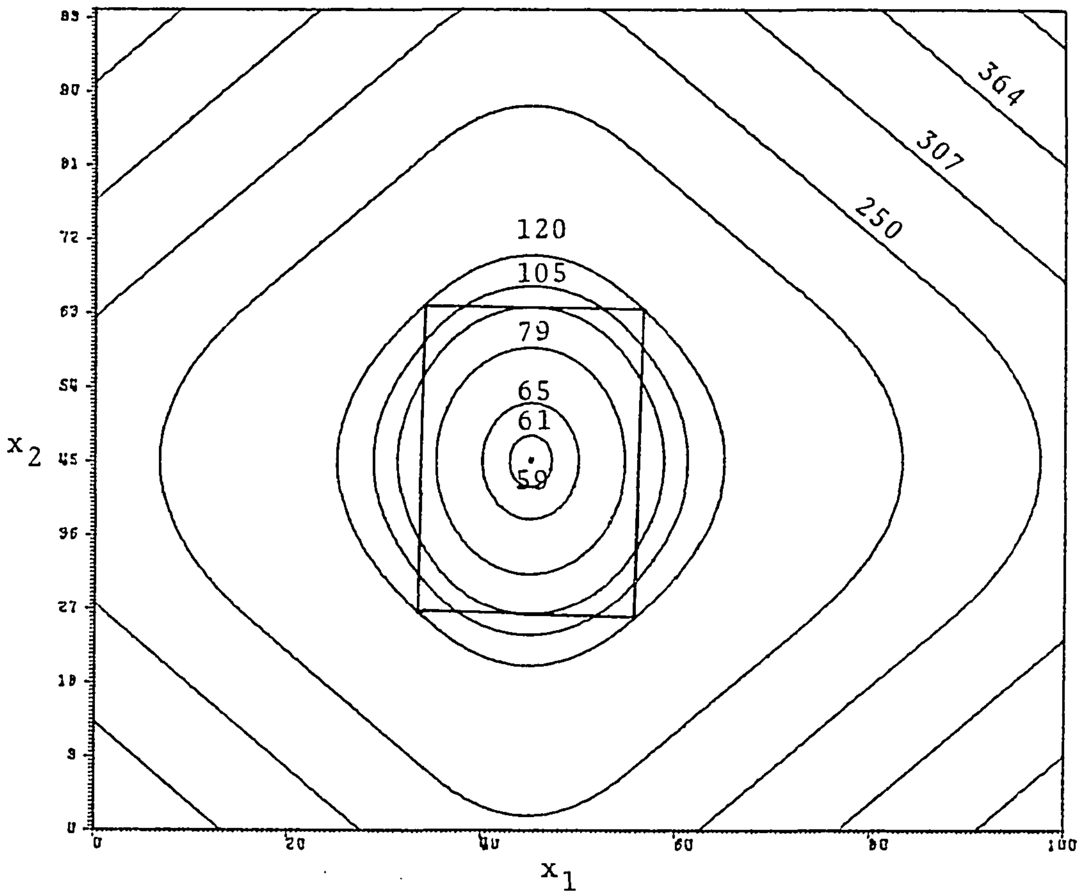


Figure 2.1 Isocurves for the expected weighted distance to a region.

Differentiability of $\theta(u)$:

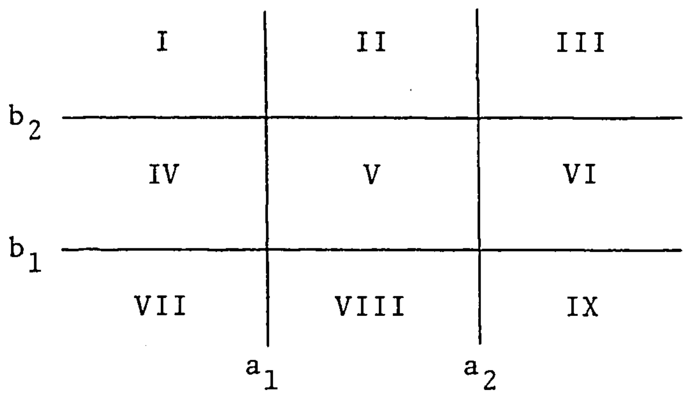
Let $X(\bar{u}) = \{Y/Y \text{ minimizes } \sum_{i=1}^{n} \bar{u}_i f_i(X) \text{ over } R^2\}$, if $X(\bar{u})$ is a singleton $\bar{X}$, then θ is differentiable at $\bar{u}$ with gradient $\nabla\theta(\bar{u}) = (f_1(\bar{X}),\ldots,f_n(\bar{X}))$ (Bazaraa and Shetty (1978)). This is not necessarily true for all u's, and a subgradient based method is recommended for solving P2.3.6.

It is necessary at this point to investigate the nature of the surface of the function

$$f_i(X) = \frac{w_i}{A_i} \iint_{R_i} (|x_1-a_i| + |x_2-b_i|)da_i \; db_i \quad \text{for any given } i.$$

Rectangular region R partitions the plane into nine subareas in the manner illustrated in Table 2.1.

Table 2.1 Partitioning of the plane by region R.



Subarea I, for example, is defined as $\{X \in R^2 / x_1 \leq a_1, b_2 \leq x_2\}$. Once the location of the point X is known, then the rectilinear distance $||X - P||$ for a point $P \in R$ can be evaluated without the absolute values, and the resulting function values

$$f(X) = \frac{w_i}{A} \iint_R (|x_1 - a| + |x_2 - b|)\, da\, db$$

can be exactly evaluated as a quadratic function:

for subarea I: $$f(X) = \frac{w}{2(a_2-a_1)} \times [(a_2-x_1)^2 - (a_1-x_1)^2] + \frac{w}{2(b_2-b_1)} \times [(b_1-x_2)^2 - (b_2-x_2)^2]$$

for subarea II: $$f(X) = \frac{w}{2(a_2-a_1)} \times [(a_1-x_1)^2 + (a_2-x_1)^2] + \frac{w}{2(b_2-b_1)} \times [(b_1-x_2)^2 - (b_2-x_2)^2]$$

for subarea III: $$f(X) = \frac{w}{2(a_2-a_1)} \times [(a_1-x_1)^2 - (a_2-x_1)^2] + \frac{w}{2(b_2-b_1)} \times [(b_1-x_2)^2 - (b_2-x_2)^2]$$

for subarea IV: $$f(X) = \frac{w}{2(a_2-a_1)} \times [(a_2-x_1)^2 - (a_1-x_1)^2] + \frac{w}{2(b_2-b_1)} \times [(b_1-x_2)^2 + (b_2-x_2)^2]$$

for subarea V: $$f(X) = \frac{w}{2(a_2-a_1)} \times [(a_1-x_1)^2 + (a_2-x_1)^2] + \frac{w}{2(b_2-b_1)} \times [(b_1-x_2)^2 + (b_2-x_2)^2]$$

for subarea VI: $$f(X) = \frac{w}{2(a_2-a_1)} \times [(a_1-x_1)^2 - (a_2-x_1)^2] + \frac{w}{2(b_2-b_1)} \times [(b_1-x_2)^2 + (b_2-x_2)^2]$$

for subarea VII: $$f(X) = \frac{w}{2(a_2-a_1)} \times [(a_2-x_1)^2 - (a_1-x_1)^2] + \frac{w}{2(b_2-b_1)} \times [(b_2-x_2)^2 - (b_1-x_2)^2]$$

for subarea VIII: $$f(X) = \frac{w}{2(a_2-a_1)} \times [(a_1-x_1)^2 + (a_2-x_1)^2] + \frac{w}{2(b_2-b_1)} \times [(b_2-x_2)^2 - (b_1-x_2)^2]$$

for subarea IV: $$f(X) = \frac{w}{2(a_2-a_1)} \times [(a_1-x_1)^2 - (a_2-x_1)^2] + \frac{w}{2(b_2-b_1)} \times [(b_2-x_2)^2 - (b_1-x_2)^2]$$

It is clear that the function f is continuous everywhere and in fact, it also is continuously differentiable. Table 2.2 shows the partial derivatives of f(X) in every subarea:

Table 2.2 Partial Derivatives of $\frac{f(X)}{w}$ in each subarea.

	$\frac{\partial f(X)}{\partial x_1} = a_1-a_2$	$2x_1-(a_1+a_2)$	a_2-a_1
	$\frac{\partial f(X)}{\partial x_2} = b_2-b_1$	b_2-b_1	b_2-b_1
b_2			
	a_1-a_2	$2x_1-(a_1+a_2)$	a_2-a_1
	$2x_2-(b_1+b_2)$	$2x_2-(b_1+b_2)$	$2x_2-(b_1+b_2)$
b_1			
	a_1-a_2	$2x_1-(a_1+a_2)$	a_2-a_1
	b_1-b_2	b_1-b_2	b_1-b_2
	a_1	a_2	

When X is in a definite subarea, the function f(X) can be developed into a simple polynomial of first or second degree. For example, if

$$X \varepsilon \{(x_1,x_2)/a_1 \le x_1 \le a_2,\ b_1 \le x_2 \le b_2\}$$

then

$$\frac{f(X)}{w} = \frac{1}{(a_2-a_1)}\ [x_1^2 - x_1(a_2+a_1) + \frac{a_1^2 + a_2^2}{2}] + \frac{1}{(b_2-b_1)}\ [x_2^2 - x_2(b_2+b_1) + \frac{b_1^2 + b_2^2}{2}]$$

$$= \frac{1}{(a_2-a_1)}\{[x_1 - \frac{(a_2+a_1)}{2}]^2 - \frac{a_1a_2}{2} + \frac{a_1^2}{4} + \frac{a_2^2}{4}\} + \frac{1}{(b_2-b_1)}\{[x_2 - \frac{(b_2+b_1)}{2}]^2 - \frac{b_1b_2}{2} + \frac{b_1^2}{4} + \frac{b_2^2}{4}\}$$

$$\frac{f(X)}{w} = \frac{1}{(a_2-a_1)}\{[x_1 - \frac{(a_2+a_1)}{2}]^2 + \frac{(a_2-a_1)^2}{4}\} + \frac{1}{(b_2-b_1)}\{[x_2 - \frac{(b_2+b_1)}{2}]^2 + \frac{(b_2-b_1)^2}{4}\}$$

$$\frac{f(X)}{w} = \frac{[x_1 - \frac{(a_2+a_1)}{2}]^2}{(a_2-a_1)} + \frac{[x_2 - \frac{(b_2+b_1)}{2}]^2}{(b_2-b_1)} + \frac{(a_2-a_1)}{4} + \frac{(b_2-b_1)}{4}$$

which is the analytical expression for an ellipse. This suggests that in region V of Table 2.1 (i.e., inside the rectangular region) the isocurves of f(X) will be ellipses centered at $(\frac{a_2+a_1}{2}, \frac{b_2+b_1}{2})$ which is the center of gravity of the rectangular region.

Marucheck and Aly (1982) noted that if X is such that $x_1 \notin (a_1,a_2)$

and $x_2 \notin (b_1, b_2)$ (i.e., X lies in region I, III, VII, or IX of Table 2.1), then f(X) is equal to the rectilinear distance from X to the center of gravity

$$\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$$

of the rectangle. Thus, in those regions the isocurves will be linear, making a 45° or -45° angle with the x_1-axis.

In the remaining subareas II, VI, VIII and IV, it can easily be shown that the analytical equations are those of parabolas. For example, in subarea II

$$f(X) = \frac{x_1^2}{(a_2-a_1)} - x_1 \frac{(a_2+a_1)}{a_2-a_1} + x_2 + \frac{a_2^2 + a_1^2}{2(a_2-a_1)} - \frac{(b_1+b_2)}{2}$$

for a chosen constant K_0 the isocurve defined by $f(X) = K_0$ in subarea II, is a parabola with a vertical axis and turned upside down.

In subarea VIII, it will be a straight-up parabola with a vertical axis, and so on.

These properties of the function f(X) can be visually observed for isocurves f(X) = K as shown in Figure 2.1. The function f(X) is minimized at the centroid (45,45) of the rectangle, and its value is 59.

Breaking down f(X) into nine possible quadratic expressions allows the exact evaluation of the function without computing the integrations.

2.3.4 Deterministic vs. Probabilistic Minimax Formulations

Two interpretations of the rectangular regions for minimax locations have been presented. One model considered the average distance a customer must travel in each region. Knowing that any point in a given region is equally likely to require service, the second probabilistic model covers the worst case possible, that is, when the most distant point in any region requires service. This last interpretation seems to be the most appropriate for locating emergency type facilities since it evaluates the effects of the worst situation, when service is required at the furthest point away from the new facility in any region.

The three minimax models on hand are:

$$f_1(X) = \max_{1 \leq i \leq n} \{w_i \| X - C_i \| \}$$

deterministic formulation

$$f_2(X) = \max_{1 \leq i \leq n} \{\frac{w_i}{A_i} \int\int_{R_i} \| X - P_i \| dP_i \}$$

probabilistic model I (expected distances)

$$f_3(X) = \max_{1 \leq i \leq n} \{w_i \| X - C_i \| + r_i' \}$$

probabilistic model II (most distant point in region)

$f_1(X)$ is the cost function for the centroid approach, it incorporates the least amount of information on the rectangular regions. $f_2(X)$ is the expected value formulation. $f_3(X)$ is the most conservative interpretation of the probabilistic approaches, and since an emergency type facility is to be located, $f_3(X)$ appears to be the most meaningful model.

In Table 2.3 the deterministic model and the two

Table 2.3 A Comparison Between the Three Minimax Models.

Problem	Optimization Model	Optimal (†) Location	Optimal Objective Function Value	Objective Function value of Deterministic Solution	% Deviation in objective function values between deterministic and probabalistic solutions (††)
A1	Determ.	(13.47,2.25) (11.098,4.62)	21.09	21.09	—
	Prob. I (Exp. dist)	(11.956,3.77)	21.128	24.22 22.036	9.4%
	Prob. II (most distant point)	(12.813,2.25) (10.68,4.38)	32.81	36.094	10%
A2	Determ.	(4.55,4.85) (4.69,5)	21.54	21.54	—
	Prob. I	(3.96,4.74)	23.86	26.71 29.53	17.8%
	Prob. II	(3.67,4.67) (4.15.)	40.	45.54	14%
A3	Determ.	(77.86,20.) (47.86,50.)	317.14	317.14	—
	Prob. I	(67.82,30.04)	317.14	332.14 397.14	14.5%
	Prob. II	(84.28,20.) (54.28,50.)	445.71	497.14	12%

(†) When two points are shown, the segment joining them is optimal.

(††) Average deviation for end points.

probabilistic models are compared for problems A_1, A_2 and A_3 given in Appendix A. Problem A_1 is from Steffen (1978), A_2 comes from Aly (1975). Problem A2 is solved for the above three problems. Figure 2.2 shows the corresponding optimal solutions. (M is the solution of probabilistic model I, M_1', M_2' and M_1'', M_2'' are the extreme points for the deterministic and probabilistic model II, respectively).

The deterministic formulation and the probabilistic model II were solved with the Dearing procedure described in section 2.3.2. Probabilistic model I was solved with Hooke and Jeeves' pattern search.

From Table 2.3 it appears that the objective functions of the probabilistic models are rather sensitive to shifts from the optimal. This observation seems to justify the analysis of the two probabilistic models. The centroid approach results in deviations in costs that cannot be ignored.

Another important observation is that for the same example problem, the optimal function values f_1^*, f_2^* and f_3^* are such that $f_1^* \le f_2^* \le f_3^*$. The following theorem confirms the inequalities.

<u>Theorem 2.3.1</u>: $w_i||X - C_i|| \le \frac{w_i}{A_i}\int\int_{R_i} ||X - P_i|| dP_i \le w_i||X - C_i|| + r_i'$, where all symbols are as defined before and $dP_i = da_i db_i$.

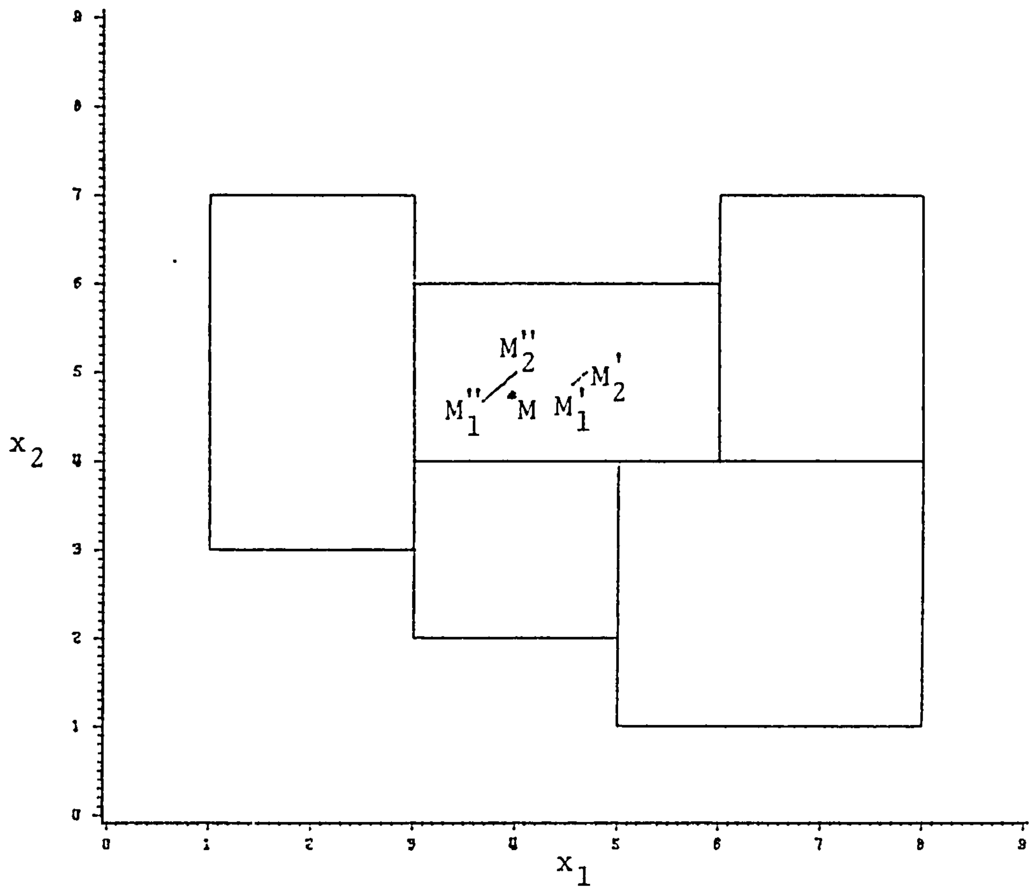


Figure 2.2 Solutions for the three minimax formulations: sample problem A2.

Proof: 1) it is first shown that

$$\frac{w_i}{A_i} \int\int_{R_i} ||X - P_i|| dP_i \leq w_i||X - C_i|| + r_i'. \qquad (1)$$

To simplify the notation, the subscript i is deleted for the remainder of the proof.

$||X - C|| + r'$ is the rectilinear distance from X to the furthest point in the rectangular region R under consideration, then $||X - P|| \leq ||X - C|| + r$ for any point $P \in R$. Integrating both sides over the region R, the inequality is kept since both sides are positive numbers,

$$\int\int_R w||X - P|| dP \leq \int\int_R (w||X - C|| + r')dP$$

$$\leq (w||X - C|| + r') \times \int\int_R dP$$

$$\int\int_R w||X - P|| dP \leq (w||X - C|| + r') \times A$$

where $A = (a_2 - a_1) \times (b_2 - b_1)$ is the area of the region, dividing both sides by A

$$\int\int_R \frac{w}{A} ||X - P|| dP \leq w||X - C|| + r',$$

which proves inequality (1).

2) Inequality

$$w ||X - C|| \leq \frac{w}{A} \int\int_R || X - P|| dP \qquad (2)$$

is more difficult to prove, referring to Table 2.1, one way to show inequality (2) is to verify it for each of the nine subareas defined by the rectangular region. It has been shown that when X is in subareas I, III, VII or IX then

$\int\int_R \frac{1}{A} ||X - P|| \, dP$ is equal to the rectilinear distance from X to the centroid C of region R: $||X - C||$, thus, when X is in any one subarea I, III, VII or IX then inequality (2) holds.

The isocurves in areas II, VI, VIII and IV have been shown to be parabolic in shape, if it can be proven that inequality (2) holds for one of these subareas, a similar proof will hold true for the other three subareas.

Assume X is a point in subarea VI, then let

$$\Delta = \int\int \frac{1}{A} ||X - P_i|| \, dP_i - ||X - C_i||$$

$$= \frac{1}{(a_2 - a_1)} \int_{a_1}^{a_2} |x_1 - a| \, da + \frac{1}{(b_2 - b_1)} \int_{b_1}^{b_2} |x_2 - b| \, db$$

$$- |x_1 - \frac{(a_1 + a_2)}{2}| - |x_2 - \frac{(b_1 + b_2)}{2}|$$

$$= \frac{1}{(a_2 - a_1)} [(a_2 - a_1)x_1 + \frac{a_1^2 - a_2^2}{2}]$$

$$+ \frac{1}{(b_2 - b_1)} [x_2^2 - (b_1 + b_2)x_2 + \frac{b_1^2 + b_2^2}{2}] - x_1 + \frac{(a_1 + a_2)}{2}$$

$$- |x_2 - \frac{(b_1 + b_2)}{2}|$$

$$\Delta = x_1 - \frac{(a_1 + a_2)}{2} + \frac{1}{(b_2 - b_1)} [x_2^2 - (b_1 + b_2)x_2 + \frac{b_1^2 + b_2^2}{2}]$$

$$- x_1 + \frac{(a_1 + a_2)}{2} - |x_2 - \frac{(b_1 + b_2)}{2}|$$

Two cases are possible, either $x_2 \geq \frac{b_1 + b_2}{2}$ or $x_2 \leq \frac{b_1 + b_2}{2}$.

(i) Assume $x_2 \leq \frac{b_1 + b_2}{2}$, then

$$(b_2-b_1) \times \Delta = x_2^2 - (b_1+b_2)x_2 + \frac{b_1^2 + b_2^2}{2} - (b_2-b_1)(x_2 - \frac{(b_1 + b_2)}{2}$$

$$= x_2^2 - 2b_2x_2 + b_2^2 = (x_2-b_2)^2 \geq 0$$

$$\Rightarrow \Delta \geq 0$$

(ii) If $x_i \leq \frac{b_1+b_2}{2}$ then $\Delta = \frac{(x_2-b_1)^2}{(b_2-b_1)} \geq 0$. The proof for subareas II, IV and VIII is similar.

It remains to be shown that inequality (2) holds in subarea V; in this case, four possibilities can occur:

i) $x_1 \geq \frac{a_1+a_2}{2}$ and $x_2 \geq \frac{b_1+b_2}{2}$

ii) $x_1 \geq \frac{a_1+a_2}{2}$ and $x_2 \leq \frac{b_1+b_2}{2}$

iii) $x_1 \leq \frac{a_1+a_2}{2}$ and $x_2 \geq \frac{b_1+b_2}{2}$

iv) $x_1 \leq \frac{a_1+a_2}{a}$ and $x_2 \leq \frac{b_1+b_2}{2}$

only case i) will be investigated since the proof is similar for all cases.

Assume $x_1 \geq \frac{a_1+a_2}{2}$ and $x_2 \geq \frac{b_1+b_2}{2}$, then

$$\Delta = \int \int \frac{1}{A} ||X - P|| \, dP - ||X - C|| = C + D$$

where

$$C = \frac{1}{2(a_2-a_1)} [(a_1-x_1)^2 + (a_2-x_1)^2] - x_1 + \frac{a_1+a_2}{2}$$

and

$$D = \frac{1}{2(b_2-b_1)} [(b_1-x_2)^2 + (b_2-x_2)^2] - x_2 + \frac{b_1 + b_2}{2}$$

It is sufficient to show that $C \geq 0$.

$$2(a_2-a_1) \times C = a_1^2-2a_1x_1 + x_1^2 + a_2^2 - 2a_2x_1$$
$$+ x_1^2 - 2a_2x_1 + 2a_1x_1 + a_2^2 - a_1^2$$
$$= 2x_1^2 - 4a_2x_1 + 2a_2^2 = 2(x-a_2)^2 \geq 0$$

which completes the proof.

Note: In subarea V equality holds at the four corners for the deterministic and expected value cases. For the preceding situation, corner (a_2,b_2) is where the equality holds.

The results in this section have confirmed the need for a probabilistic formulation of the minimax location problem with regions. In the rest of this research effort, every minimax formulation investigated will be one of the two probabilistic models given earlier.

Furthermore, computational experience is developed for only the minimax models with distances to the most distant points in the regions, since it covers the worst realizations of an event in any region which is a main goal in emergency facility location problems. Also, problem P2.3.1 can be solved completely with all optimal solutions generated, and the isocurves can be easily constructed.

2.3.5 Minimax Location Models with Expected Value of the Maximum of the Weighted Distances

In P2.3.1 and P2.3.4, some aspects of the random nature of the elements involved were incorporated into the deterministic formulations. This was achieved by using one important statistical parameter of the random variables, the mean. If more probabilistic insight is to be introduced into the modeling of the problem, then the behavior of the maximum operand of the optimization criterion can be evaluated by minimizing the average value of the maximum of the weighted distances.

This objective can be formulated as follows

$$\underset{X \in R^2}{\text{minimize}}\ E[\max_{1 \le i \le n} \{w_i \|X - P_i\|\}]. \qquad \text{(P2.3.7)}$$

The random variable $\max_{1 \le i \le n} w_i \|X - P_i\|\}$ has a distribution function which is very complex to derive analytically. Instead of investigating P2.3.7 two approximating problems will be studied.

The general formulation for both problems is

$$\underset{X \in R^2}{\text{minimize}}\ E[\max_{1 \le i \le n} \{w_i f_i X)\}] \qquad \text{(P2.3.8)}$$

where $f_i(X)$ will be defined accordingly in each following case.

(i) In this case a conservative attitude is adopted, $f_i(X)$ will be the distance from the new facility to the most distant point in rectangular region i. (Since P_i is uniformly distributed over region i, P_i can occur with equal probability anywhere in the region and the extreme values of the random variable $\|X - P_i\|$ will happen for the most distant point in R_i) and the corresponding mathematical model will be

$$\underset{X \in R^2}{\text{minimize}}\ E[\max_{1 \le i \le n} \{w_i (\|X - C_i\| + r_i)\}] \qquad \text{(P2.3.9)}$$

(ii) Each region is assumed densely populated, and when region i requires service, all facilities situated within the region travel to the new facility. The total distance traveled by the customers in region i can be approximated by the following function:

$$f_i(X) = m_i \int\int_{R_i} (|x_1 - a_i| + |x_2 - b_i|)\, da_i\, db_i$$

where m_i is the population density over region i and the resulting mathematical model is

$$\underset{X \in R^2}{\text{minimize}}\ E[\max_{1 \leq i \leq n} \{w_i m_i \int\int_{R_i} \|X - P_i\| dP_i\}] \qquad \text{(P2.3.10)}$$

where w_i is the cost per unit distance to travel from region i to the new facility, and is probabilistic in nature.

Problems P2.3.9 and P2.3.10 can be written as in P2.3.8 with the function $f_i(X)$ appropriately defined for each case. Therefore, the analysis will concentrate on problem 2.3.8 and the result will apply for both P2.3.9 and P2.3.10.

Recall that it was assumed that the random variables w_i's are independently distributed, the following theorem (Mood, et al. (1974)) is useful for the rest of this analysis.

<u>Theorem 2.3.2</u>: If $X_1, \ldots, X_k$ are independent random variables and $g_1(.), \ldots, g_k(.)$ are k functions such that $Y_j = g_j(X_j), (j=1,\ldots,k)$ are random variables, then $Y_1, \ldots, Y_k$ are independent.

<u>Note</u>: Let $W_i = w_i g_i(X)$, if w_i is a normal random variable with mean μ_i and variance of σ_i^2, then W_i is a normal

random variable with mean $\mu_i' = \mu_i \cdot f_i(X)$ and variance $\sigma_2'^2 = \sigma_i^2 f_i^2(X)$, and the random variables W_i are independent.

Theorem 2.3.3: If the w_i's are positively valued random variables, then $E[\max_{1 \leq i \leq n} \{w_i f_i(X)\}]$ is a convex function of X.

Proof: Let

$$F:(w_1 \ldots w_n) \longrightarrow F(w_1,\ldots,w_n) = \max_{1 \leq i \leq n} \{w_i f_i(X)\}$$

the n-dimensional random variable $(w_1,\ldots,w_n)$ has a joint probability density function $g_{w_1,\ldots,w_n}(\cdot,\ldots,\cdot)$ (since the w_i are independently distributed, then $g_{w_1,\ldots,w_n}(\cdot,\ldots,\cdot) = \Pi_{i=1}^n g_{w_i}(\,.\,)$) and

$$E[\max_{1 \leq i \leq n} \{w_i f_i(X)\}]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \max_{1 \leq i \leq n} \{w_i f_i(X)\} g_{w_1,\ldots,w_n}(w_1,\ldots,w_n) dw_1 \ldots dw_n .$$

$$f_i(X) = \|X - C_i\| + r_i \text{ or } f_i(X) = \iint_{R_i} (|x_1 - a_i| + |x_2 - b_i|) da_i db_i$$

are convex functions, thus for the cases considered, $f_i(X)$ is convex.

The weights w_i represent parameters that are positive in nature such as volume of goods transported, or time per unit distance, or frequencies, etc. Thus, it is perfectly legitimate to assume that the random variables w_i are restricted to only positive outcomes (possible such random variables are exponentially distributed or with truncated density functions).

Then for $w_i \geq 0$, $w_i f_i(X)$ is also convex for all i and which implies that

$$\max_{1 \leq i \leq n} \{w_i f_i(X)\} \text{ is convex,}$$

for X_1 and X_2 points in R^2 and some real number $1 < \alpha < 0$, then $\alpha X_1 + (1-\alpha)X_2 \varepsilon R^2$, and the following inequality holds:

$$\alpha \max_{1 \leq i \leq n} \{w_i f_i(X_1)\} + (1-\alpha) \max_{1 \leq i \leq n} \{w_i f_i(X_2)\}$$
$$- \max_{1 \leq i \leq n} \{w_i f_i(\alpha X_1 + (1-\alpha)X_2)\}$$
$$\geq 0$$

multiplying both sides by $g_{w_1,\ldots,w_n}$ and integrating

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [\alpha \max_i \{w_i f_i(X_1)\} + (1-\alpha)\max_i \{w_i f_i(X_2)\}$$
$$-\max_i \{w_i f_i(\alpha X_1 + (1-\alpha)X_2\}]$$
$$\times g_{w_1 \ldots w_n}(w_1, w_2, \ldots, w_n) dw_1\, dw_2 \ldots dw_n$$
$$\geq 0$$

The previous inequality can be rewritten as:

$$\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \max_i \{w_i f_i(X_1)\} g_{w_1 \ldots w_n}(w_1, \ldots, w_n) dw_1\, dw_2 \ldots dw_n$$
$$+(1-\alpha) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \max_i \{w_i f_i(X_2)\} g_{w_1 \ldots w_n}(w_1, \ldots, w_n) dw_1\, dw_2 \ldots dw_n$$
$$- \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \max_i \{w_i f_i(\alpha X_1 + (1-\alpha)X_2)\} g_{w_1 \ldots w_n}(w_1, \ldots, w_n) dw_1 \ldots dw_n$$
$$\geq 0$$

which means that $E[\max_i \{w_i f_i(X)\}]$ is a convex function in R^2.

2.3.6 Evaluation of the Expected Value of the Maximum of the Weighted Distances.

In the previous theorem the expected value is computed by evaluating a multiple integral. When optimizing $E[\max \{w_i f_i(X)\}]$, the multiple integration may be repeated for a possibly great number of times, which could severely handicap the efficiency of any methodology to solve problem P2.3.8.

There exists another way to obtain $E[\max_i \{w_i f_i(X)\}]$; set $W = \max_{1 \leq i \leq n} \{w_i f_i(X)\}$ then

$$E[W] = \int_{-\infty}^{\infty} w g_W(w) dw$$

which involves a single integration, but on the other hand, it requires the probability density function of W, which needs to be obtained before integrating.

Let $W_i = w_i f_i(X)$ for all i; by a previous theorem, the random variables W_i $1 \leq i \leq n$ are independently distributed with distribution function $G_{W_i}(.)$ such that

$$Pr(W_i \leq t) = Pr(w_i f_i(X) \leq t) = G_{W_i}(t)$$

$$= Pr(w_i \leq \frac{t}{f_i(X)}) = G_{W_i}(\frac{t}{f_i(X)})$$

then,

$$g_{W_i}(t) = \frac{1}{f_i(X)} g_{W_i}(\frac{t}{f_i(X)})$$

and

$$E(W) = E[\max_{1 \leq i \leq n} \{W_i\}] = \int_{-\infty}^{\infty} w g_W(w) dw$$

<u>Lemma 2.3.4</u> If $W = \max_{1 \leq i \leq n} \{W_i\}$, where W_i are independent random variables with density function $g_{W_i}(.)$, then

$$g_W(w) = \sum_{k=1}^{n} g_{W_k}(w) \times \left(\prod_{j \neq k} G_{W_i}(w) \right)$$

<u>Proof</u>:

$$G_W(w) = \Pr(W \leq w) = \Pr\left(\max_{1 \leq i \leq n} \{W_i\} \leq w \right)$$

$$= \Pr(W_1 \leq w, \ldots, W_n \leq w)$$

from the independence of the W_i, $1 \leq i \leq n$:

$$G_W(w) = \prod_{i=1}^{n} \Pr(W_i \leq w) = \prod_{i=1}^{n} G_{W_i}(w)$$

now:

$$g_W(w) = \frac{d}{dw} G_W(w) = \frac{d}{dw} \left(\prod_{i=1}^{n} G_{W_i}(w) \right)$$

and

$$g_W(w) = \sum_{k=1}^{n} \frac{d}{dw} G_{W_k}(w) \left(\prod_{j \neq k} G_{W_i}(w) \right) = \sum_{k=1}^{n} g_{W_k}(w) \left(\prod_{j \neq k} G_{W_i}(w) \right)$$

and

$$E(W) = \int_{-\infty}^{\infty} w g_W(w)\, dw$$

$$= \int_{-\infty}^{\infty} w \sum_{k=1}^{n} g_{W_k}(w) \left(\prod_{j \neq k} G_{W_j}(w) \right) dw$$

$$= \sum_{k=1}^{n} \int_{-\infty}^{\infty} w g_{W_k}(w) \left(\prod_{j \neq k} G_{W_j}(w) \right) dw.$$

If the density function of $W_j(.)$ (for all j) can be easily or directly evaluated, then it is preferable to compute the single integral representation of $E[\max_{1 \leq i \leq n} \{w_i f_i(X)\}]$. But if an efficient numerical method for computing the multiple integrations is used, then either method is acceptable.

2.3.7 Computing Lower and Upper Bound Approximations for the Expected Value of the Weighted Distances

(i) Lower Bound: $w = (w_1, \ldots, w_n)$. Let

$$F: w \rightarrow F(w) = \max_{1 \leq i \leq n} \{w_i f_i(X)\}$$

where F is a convex function and using Jensen's inequality, the following lower bound is generated:

$$F(E(w)) \leq E(F(w)) \text{ or}$$

$$\max_{1 \leq i \leq n} \{E(w_i) f_i(X)\} \leq E[\max_{1 \leq i \leq n} \{w_i f_i(X)\}]$$

thus if P2.3.8 is too difficult to solve explicitly, a lower bound approximation can be generated by solving

$$\underset{X \in R^2}{\text{minimize}} \max_{1 \leq i \leq n} \{E(w_i) f_i(X)\} \qquad \text{(P2.3.11)}$$

which is equivalent to problem P2.3.4 if $f_i(X) = \int\int_{R_i} ||X - P_i|| \, dP_i$, or to problem P2.3.1 if $f_i(X) = ||x - C_i|| + r_i$.

(ii) Upper Bound: An upper bound has been generated by Madansky (1959) for the case of independent multivariate random variables, it generalized an upper bound developed by Edmundson (1957) which was for a univariate random variable. This type of upper bound is generally known as Edmundson-Madansky inequality.

It is first assumed that each random variable w_i is defined over a finite interval $v_{i1} \leq w_i \leq v_{i2}$ where $v_{i1} < v_{i2}$ for all i, then I is the bounded n-dimensional rectangle such that $w \in I$ and for all i: $v_{i1} \leq w_i \leq v_{i2}$. I is the bounded n-dimensional rectangle defined by the 2^n vertices of the form $(v_{1\phi_1}, v_{2\phi_2}, \ldots, v_{n\phi_n})$ where ϕ_i takes on the values 1 and 2 (for all i). Then the Edmundson-Madansky inequality is defined by

$$E(F(w)) = E(F(w_1,\ldots,w_n)) \le \sum_{\phi} \prod_{j=1}^{n} (-1)^{\phi_j} \frac{(v_{j\phi_j} - E(w_j))}{(v_{j2} - v_{j1})}$$

$$\times\ F(v_{1\bar{\phi}_1},\ldots,v_{n\bar{\phi}_n}), \text{ where } \bar{\phi}_i = 3 - \phi_i$$

or more explicitly for the function F(.) investigated in this chapter.

$$E[\max_i \{w_i f_i(X)\}] \le \sum_{\phi} \prod_{j=1}^{n} (-1)^{\phi_j} \frac{(v_{j\phi_j} - E(w_j))}{(v_{j2} - v_{j1})}$$

$$\times \max_{1 \le i \le n} \{v_{i\bar{\phi}_i} f_i(X)\}$$

One important result is that

$$\sum_{\phi} \prod_{j=1}^{n} (-1)^{\phi_j} \frac{(v_{j\phi_j} - E(w_j))}{(v_{j2} - v_{j1})} = 1$$

and

$$\prod_{j=1}^{n} (-1)^{\phi_j} \frac{(v_{j\phi_j} - E(w_j))}{(v_{j2} - v_{j1})} \ge 0$$

therefore the upper bound is defined as a convex combination of the functions

$$\max_{1 \le i \le n} \{V_{i\phi_i} f_i(X)\}$$

for all combinations of ϕ and each one of these functions is a convex function of X (since w_i are assumed positive then $0 \le v_{i1} \le w_i \le v_{i2}$ for all i. And the $v_{i\phi}$, $E(w_i)$ for all i's are known values and the upper bound is a relatively simple convex function of X which can be minimized by a number of available nonlinear programming codes.

2.4 Constrained Probabilistic Minimax Location Problems

2.4.1 Introduction

In the previous formulations in this chapter no constraints were imposed. In this section, restrictions are imposed on the location of the new facility. Very little work has been done in probabilistic location theory with constraints. Contributions have been made by Hurter and Prawda (1972), who solved the Euclidean, single facility location problem with random weights independently distributed. The problem was formulated as a chance constrained programming problem. Seppälä (1975) used the fractile criterion for a probabilistic multifacility Weber problem, and converted the resulting chance constraint into deterministic constraints. Aly (1974) did an extensive study of probabilistic facility location problems when both the weights and the locations of the existing facilities are assumed probabilistic. Aly and White (1978) investigated emergency service location problems with existing facilities randomly distributed over rectangular regions. The models formulated are set cover problems. Chance constraints on the response times are also added.

2.4.2 A Conservative Minimax Location Problem with a Constraint on the Total Average Cost.

When a service call in any region is a random and discrete event (fire, crime, accident), then an important model is:

$$\underset{X \in R^2}{\text{minimize}} \quad \max_{1 \le i \le n} \{E(w_i)||X - C_i|| + r_i'\} \qquad \text{(P2.4.1)}$$

subject to

$$\sum_{i=1}^{n} \frac{E(w_i)}{A_i} \iint_{R_i} ||X - P_i|| \, dP_i \le \mu$$

This model covers the worst cases possible (travel to the furthest points in any region), but it also sets a limit μ on the total average cost of servicing all facilities. For now, it is assumed that μ is some upper limit on total cost, chosen by the decision maker (in later chapters, a thorough analysis of these bounds will be performed).

Model P2.4.1 can also be applicable for the following situation: in this case, the number of existing facilities is too large to be represented as a discrete model, and an accurate approximation of the system is obtained by a continuous model. Love (1972) described a continuous location model for rectangular areas with Euclidean distances. In that model, the population is distributed uniformly over each of several rectangular areas. The population density over region i is m_i units per unit area, each member of the population of region i has an expected trip frequency f_i to the new facility over a time period t. Let c_i be cost per unit distance traveled from region i to the new facility, the resulting mathematical model is as follows:

$$\underset{X \in R^2}{\text{minimize}} \quad \max_{1 \le i \le n} \{c_i f_i ||X - C_i|| + r_i''\} \qquad \text{(P2.4.2)}$$

subject to

$$\sum_{i=1}^{n} m_i f_i c_i \int\int_{R_i} \|X - P_i\| dP_i \le \mu$$

where $r_i'' = c_i f_i r_i$.

P2.4.1 and P2.4.2 are very similar, but they reflect the two types of populations being modeled as rectangular regions. They will have the same analytical properties and will share the same solution procedures. To simplify the analysis, only one formulation will be investigated and it will be P2.4.1.

2.4.3 General Properties of P2.4.1

Lemma 2.4.1: Problem P2.4.1 is a convex programming problem.

Proof: The function $g_i(X) = w_i\|X - C_i\| + r_i'$ is convex for all i, therefore $\max_i \{g_i(X)\}$ is also convex. The constraint can be written:

$$\sum_{i=1}^{n} f_i(X) \le \mu$$

where

$$f_i(X) = \frac{w_i}{A_i} \int\int_{R_i} \|X - P_i\| dP_i$$

$$= \frac{w_i}{(a_{i_2} - a_{i_1})} \int_{a_{i_1}}^{a_{i_2}} |x_1 - a_i| da_i$$

$$+ \frac{w_i}{(b_{i_2} - b_{i_1})} \int_{b_{i_1}}^{b_{i_2}} |x_2 - b_i| db_i$$

to show that the feasible set S_2 is a convex set, it is sufficient to prove that

$$f(x) = \int_{a_1}^{a_2} |x - a| da$$

is convex for $x \in R$, which is simple.

But the objective function is not differentiable and the feasible set doesn't have favorable geometrical properties that could help in developing an efficient solution procedure. In a following chapter, a related problem will be presented which is equivalent to P2.4.1 in many ways, and which on the contrary, offers advantageous geometrical properties.

It can also be observed that the constraint in problem P2.4.1 is active for only a range $[\mu_1, \mu_2]$ of values for μ. μ_1 is the absolute minimum value of the minisum function, it is obvious that if $\mu < \mu_1$ the feasible set is empty. μ_2 is the smallest value of μ that will allow P2.4.1 to be optimal at an absolute minimax solution. If $\mu > \mu_2$, a minimax solution will always solve P2.4.1.

2.4.4 Constrained Minimax with Expected Distances Traveled

In problem P2.4.1, the most distant point in each region from the new facility is the concern of the decision maker. The worst possible situation is under consideration, this is usually the case when human lives are endangered or when valuable properties are threatened by a fire. A less radical attitude is to evaluate the average weighted distances to each rectangular region and to locate the new facility such that the largest of the resulting weighted average distances is minimized.

The resulting constrained minimax problem is

$$\underset{X \in R^2}{\text{minimize}} \quad \max_{1 \leq i \leq n} \left\{ \frac{E(w_i)}{A_i} \iint_{R_i} \| X - P_i \| \, dP_i \right\} \qquad \text{(P2.4.3)}$$

subject to

$$\sum_{i=1}^{n} \frac{E(w_i)}{A_i} \iint_{R_i} \| X - P_i \| \, dP_i \leq \mu$$

where all parameters and variables are as defined before.

2.4.5 General Properties of P2.4.3

P2.4.3 is a convex programming problem since each function

$$f_i(X) = \frac{w_i}{A_i} \iint_{R_i} \| X - P_i \| \, dP_i$$

is convex. Also, each $f_i(X)$ is continuously differentiable, but the objective function $\max_{1 \leq i \leq n} \{f_i(X)\}$ is not, thus it precludes the use of a gradient based method.

Other equivalent formulations of P2.4.3 can be derived that will reveal new properties. Consider the following equivalent formulation

$$\underset{\substack{Z \in R \\ X \in R^2}}{\text{minimize}} \; z = z_0 \qquad \text{(P2.4.4)}$$

subject to

$$f_i(X) \leq z, \; \forall i$$

$$\sum_{i=1}^{n} f_i(X) \leq \mu$$

P2.4.2 can be transformed into an unconstrained problem by developing its Lagrangian dual.

$$\max_{\substack{u \geq 0 \\ v \geq 0}} \theta(u,v) \qquad \text{(P2.4.5)}$$

where $u = (u_1,\ldots,u_n)^t$ and

$$\theta(u,v) = \min_{Z,X}[z(1 - \sum_{i=1}^{n} u_i) + \sum_{i=1}^{n} (u_i+v)f_i(X) - v\mu]$$

where v and u_i are the Kuhn-Tucker multipliers. If μ is such that $\mu_1 < \mu$, then Slater's constraint qualification holds, and by the "Strong Duality Theorem" there exist optimal multipliers $\bar{u}_i$ and $\bar{v}$ such that

$$\min_{Z,X}[z(1-\sum_{i=1}^{n} \bar{u}_i) + \sum_{i=1}^{n} (\bar{u}_i+\bar{v})(f_i(X)-\bar{v}\mu)] = z_0$$

furthermore, for $\theta(u,v)$ to exist, the coefficient of z must be zero, otherwise the minimum would not exist if $z \to \pm\infty$, therefore

$$z_0 = \min_{X \varepsilon R^2} [\sum_{i=1}^{n} (\bar{u}_i+\bar{v})f_i(X) - \bar{v}\mu] \qquad \text{(P2.4.6)}$$

with $\sum_{i=1}^{n} \bar{u}_i - 1 = 0$.

This means that P2.4.1 is equivalent to solving an unconstrained minisum location problem, where the weights w_i are adjusted by a factor $\bar{u}_i + \bar{v}$ (optimal multipliers).

2.4.6 Chance Constrained Minimax Location Problems

In the previous formulations with random weights, no constraints were imposed. In this section new restrictions will be added on the location of the new facility. The restrictions will be chance constraints. Chance constraints programming has been a very popular modeling tool for probabilistic problems in many areas of application; farming problems (1971), capital budgeting (1975), etc. This popularity has led to many abuses, and recently, detractors have criticized the use of chance constraints programming. Blau (1975), Hogan et al. (1981) noted "important problems" concerning the modeling of decision problems under risk as chance constraints programs. They backed their arguments by comparing chance constraints programming to stochastic programming with recourse. They concluded that chance constraints programs is generally not used with the extra care it requires.

In this paper chance constraint formulations were chosen over stochastic programming with recourse because for the problems investigated, recourse strategies would have to be modeled and computed for all possible outcomes of the random variables. This process will result in a very large problem (even for simpler linear problems). Also, recourse actions for the type of facility location problems under consideration, are not obvious and since the location of the new facility is over a continuous space, a possible recourse model could not be numerically solved. The cost of such modeling would outweigh its benefits.

It is reasonable to assume that when chance constraints are violated, a cost will result. In most situations this cost is very subjectively evaluated and depends partly on the decision maker's values and needs. Through chance constraint programming modeling, these needs are represented by two factors, the cost incurred as measured by the objective function, and the aspiration level α_i (for constraint i). These two types of objectives are usually conflicting in nature, higher α_i (which means higher reliability) would cause higher cost.

The aspiration levels α_i indicate some tolerance measure for admitting constraints violations. To ensure equal service over the n regions, all the α_i's could be set equal.

The following two models (depending on the definition of $f_i(X)$), P2.3.1 and P2.3.4 were studied in sections 2.3.2 through 2.3.4:

$$\underset{X \in R^2}{\text{minimize}} \quad \max_{1 \leq i \leq n} \{E(w_i) f_i(X)\}$$

where

$$f_i(X) = \int\int_{R_i} \|X - P_i\| \, dP_i \quad \text{for P2.3.4,}$$

or

$$f_i(X) = \|X - C_i\| + r_i \qquad \text{for P2.3.1}$$

These models used a little probabilistic aspect of the w_i's since only the expected values of the random variables are included in the formulations. This shortcoming can be compensated by the use of chance constraints as follows:

$$\underset{X \in R^2}{\text{minimize}} \quad \max_{1 \leq i \leq n} \{E(w_i) f_i(X)\} \qquad \text{(P2.4.7)}$$

subject to

$$\Pr(w_i f_i(X) \leq \beta_i) \geq \alpha_i, \; i = 1, \ldots, n$$

where $f_i(X)$ are as defined in P2.3.1 or P2.3.4. The parameters

β_i's are some preassigned upper bounds on the cost of servicing region i from the new facility. α_i is the aspiration level (or confidence level). It is usually assumed that $0.5 \leq \alpha_i \leq 1$ since it is reasonable to want to increase the probability of some objective to be satisfied (and for other reasons that will be given later). The chance constraint i in P2.4.7 expresses a constraint on the probability of satisfying the goal:

$$w_i f_i(X) \leq \beta_i.$$

It is assumed that the random variable w_i is normally distributed with mean μ_i and variance σ_i^2, then using the theory from Charnes and Cooper (1963), the following deterministic constraints are obtained:

$$f_i(X) \leq \frac{\beta_i}{\sigma_i \ \Phi^{-1}(\alpha_i) + \mu_i}, \quad i = 1,\ldots,n$$

The assumption that $0.5 < \alpha_i < 1.0$ leads to $\Phi^{-1}(\alpha_i) > 0$ and with $\mu_i > 0$ the above constraint is well defined and since $f_i(X)$ is a convex function then the set

$$\{X \in R^2 | f_i(X) \leq \frac{\beta_i}{\alpha_i \Phi^{-1}(\alpha_i) + \mu_i} \ \forall i \quad \text{is a convex set.}$$

Problem P2.4.7 is equivalent to problem

$$\underset{X \in R^2}{\text{minimize}} \ \max_{1 \leq i \leq n} \{\mu_i f_i(X)\} \qquad \text{(P2.4.8)}$$

subject to

$$f_i(X) \leq \frac{\beta_i}{\sigma_i \ \Phi^{-1}(\alpha_i) + \mu_i}, \quad i = 1,\ldots,n$$

2.4.7 General Properties of Problem P2.4.8

P2.4.8 is a convex programming problem but the objective function is not differentiable which precludes the use of some gradient based solution method, but gradient free search methods exist that work very well for convex problems. The method of successive approximation for constrained minimax problems as described in Dem'yanov and Malozemov (1974) can be adopted.

Note that as α_i increases, the right-hand side of constraint i decreases, which means that the feasible set defined by the constraints shrinks, and therefore, the optimal value of the objective function deteriorates (increases) as the feasible set shrinks. On the other hand, for α_i fixed, if one wants to increase the upper bounds β_i, then the feasible set of P2.4.8 becomes larger and possibly the optimal objective function will decrease. The analysis of P2.4.8 as α_i or β_i are varied can be simplified by considering a new parameter

$$\gamma_i = \frac{\beta_i}{\sigma_i \, \Phi^{-1}(\alpha_i) + \mu_i} \quad \text{for all } i,$$

and analyzing the following problem

$$\underset{X \in R^2}{\text{minimize}} \; \max_{1 \le i \le n} \{\mu_i f_i(X)\} \qquad \text{(P2.4.9)}$$

subject to

$$f_i(X) \le \gamma_i \quad i = 1,\ldots,n$$

for various values of $\gamma_i (\forall i)$.

Similarly to the analysis done for problem P2.4.3, P2.4.9 can be rewritten as:

$$\underset{X \in R^2}{\text{minimize}}\ Z \qquad \text{(P2.4.10)}$$

subject to

$$f_i(X) - \frac{Z}{\mu_i} \leq 0, \quad i = 1,\ldots,n$$

$$f_i(X) - \gamma_i \leq 0, \quad i = 1,\ldots,n$$

and taking the Lagrangian dual:

$$\underset{\substack{u \geq 0 \\ v \geq 0}}{\max} \psi(u,v) = \underset{X \in R^2}{\min} \left(\sum_{i=1}^{n} (u_i + v_i) f_i(X) - \sum_{i=1}^{n} v_i \gamma_i \right) \qquad \text{(P2.4.11)}$$

subject to

$$1 - \sum_{i=1}^{n} \frac{u_i}{\mu_i} = 0$$

u_i and v_i for all i are the Lagrange multipliers. For optimal $\bar{u}_i$ and $\bar{v}_i$ P2.4.11 is equivalent to solving a positively weighted sum of the $f_i(X)$'s, where the weights are related to the parameters γ_i's of P2.4.9.

Therefore P2.4.9 can be seen as a multiple objectives problem (see Appendices B and C) where one objective is to minimize a cost function such that the other objectives (defined by the $f_i(X)$'s) satisfy given upper bounds. As the upper bounds are changed different solutions are obtained which are efficient solutions to the following vector optimization problem:

$$\underset{X \in R^2}{\min} \left(\underset{1 \leq i \leq n}{\max} \{\mu_i f_i(X)\},\ f_1(X),\ \ldots,\ f_n(X) \right).$$

In turn, the variation in the value of the γ_i can easily be interpreted in terms of the parameter α_i and β_i defined earlier.

2.4.8 Fractile Formulations of Minimax Location Problems

Still another criterion of optimization under risk is the fractile criterion where the α-fractile of the distribution of cost is minimized as follows:

$$\text{minimize } \gamma \qquad \text{(P2.4.12)}$$

$$\text{subject to}$$

$$P_r(Z \leq \gamma) \leq \alpha$$

where α is a predetermined probability, γ is a decision variable and $Z = \max_{1 \leq i \leq n}\{w_i f_i(X)\}$ is the cost function adopted for the location problem under investigation.

Geoffrion (1967) considered the fractile and aspiration criteria for a stochastic linear program, he proved a close relationship between the two criteria and solved both problems by considering a bicriteria optimization problem where one objective is expressed as the expected value of a derived random variable, and the other objective comes from its variance.

Sengupta and Portillo-Campbell (1970) investigated the fractile approach to stochastic linear programs. They assumed normality and used a numerical method developed by Kataoka (1963) to solve an equivalent deterministic profit function. They applied the theory to farming problems.

P2.4.12 can be written as

$$\text{minimize } \delta \qquad \text{(P2.4.13)}$$

subject to

$$Pr[\max_{1 \le i \le n} \{w_i f_i(X)\} \le \delta] \ge \alpha$$

where δ is the cost below which the cost function occurs with at least a probability of α. Since the w_i's are independent then

$$Pr(\max_{1 \le i \le n} \{w_i f_i(X)\} \le \delta) = \prod_{i=1}^{n} Pr(w_i f_i(X) \le \delta)$$

and P2.4.12 is equivalent to

$$\underset{\delta, X}{\text{minimize}}\ \delta \qquad \text{(P2.4.14)}$$

subject to

$$\prod_{i=1}^{n} Pr(w_i f_i(X) \le \delta) \ge \alpha$$

It is clear that if the left-hand side of the constraint is a concave function, then P2.4.14 would be a convex program, and global optimal solution can be found by any one of many algorithms. But

$$\prod_{i=1}^{n} P_r(w_i f_i(X) \le \delta) = \prod_{i=1}^{n} G_i(\delta)$$

where $G_i(.)$ is the distribution function of $w_i f_i(X)$, is generally not concave for most commonly adopted distributions. P2.4.14 is most likely not a convex program, and only local optimal solutions can be guaranteed.

Miller and Wagner (1965) investigated some situations where additional restrictions could result in convex programs. In particular, they studied the equivalent relation obtained by taking the natural log of each side of the constraint in

P2.4.14.

$$\underset{\delta, X}{\text{minimize}}\ \delta \qquad \text{(P2.4.15)}$$

subject to

$$\sum_{i=1}^{n} \ln G_i(\delta) \geq \ln \alpha$$

Some special distribution functions have been developed that could achieve convexity for P2.4.15.

2.4.9 A Pseudo-fractile Criterion of Minimax Location Problems

Consider the following problem

$$\underset{X \in R^2}{\text{minimize}}\ \max_{1 \leq i \leq n} \{w_i f_i(X)\} \qquad \text{(P2.4.16)}$$

or equivalently

minimize δ

subject to

$$w_i f_i(X) \leq \delta \quad i = 1,\ldots,n$$

where w_i is a random variable for all i.

P2.4.16 is an ill-defined stochastic problem, since if it is optimized for some realization of the w_i's, the corresponding solution $\overline{X}$ may not stay optimal for another realization of the w_i. To circumvent this problem the following chance constrained problem is defined:

$$\underset{\delta, X}{\text{minimize}}\ \delta \qquad \text{(P2.4.17)}$$

subject to

$$P_r(w_i f_i(X) \leq \delta) \geq \alpha_i \quad i = 1,\ldots,n$$

which means that the i^{th} constraint may be violated, but at most $\beta_i = (1-\alpha_i)$ percent of the time. The α_i are predetermined probabilities. For a uniform quality of service over all rectangular regions, the α_i can be set equal to α.

Assuming that $w_i \sim N(\mu_i,\sigma_i^2)$ for all i and that $0.5 < \alpha < 1.0$ then P2.4.17 is equivalent to

$$\underset{\delta,X}{\text{minimize}}\ \delta \qquad \text{(P2.4.18)}$$

subject to

$$z_i f_i(X) \leq \delta \quad i = 1,\ldots,n$$

and P2.4.17 is a convex programming problem which is equivalent to

$$\underset{X \in R^2}{\text{minimize}}\ \max_{1 \leq i \leq n} \{z_i f_i(X)\} \qquad \text{(P2.4.19)}$$

which is a deterministic minimax criterion single facility location problem with weights $z_i = \sigma_i \Phi^{-1}(\alpha) + \mu_i$. Also P2.4.19 is similar to P2.3.1 or P2.3.4 depending on which of $f_i(X)$ is adopted. In problems P2.3.1 and P2.3.4 only the expected value of the random variable w_i is used. But in P2.4.19 more information about the mean and the variance are used, as well as a factor α which permits to set different safety level of servicing the existing facilities.

2.5 Summary

In this chapter, probabilistic formulations of the single facility minimax location problem have been analyzed. In section 2.4 unconstrained formulations were considered. Two minimax models were investigated and numerically compared to the deterministic centroid approach. The first one covered cases when the furthest point in each region to the new facility is the location requiring service, and the expected value of each function inside the resulting maximand was computed. The second model involved evaluating the expected values of each function inside the maximand. It was shown that the objective functions of the three minimax models on hand satisfied specific inequalities.

Another unconstrained model involved minimizing the expected value of the random variable defined by the maximand. Two cases were considered, depending on two interpretations of the rectangular regions. The resulting mathematical programs were found to be convex, but the complexity of the objective function proved to be appreciable. Lower and upper bounds approximating functions were derived. The lower bound from Jensen's inequality, and the upper bound is derived using Edmundson-Madansky's inequality. Both bounding functions were shown to be convex, which would ensure that any local optimal solutions found is also a global optimum. Constrained models with the minisum function were also formulated and found to be convex mathematical programs. They set the stage for the analysis in the following chapters, when both minisum and minimax criteria are simultaneously active in a model. Some chance constrained and fractile formulations are also studied.

CHAPTER III

CONSTRAINED MINISUM AND MINIMAX PROBLEMS

3.1 Introduction and Overview of Related Research

Traditionally, location problems involve locating one or several new facilities among a set of existing facilities such that some cost function is minimized. The most commonly used optimization criteria are the minisum and minimax. In many situations, neither criterion can best model the problem on hand by itself, and a combination of both criteria is preferable. The minisum criterion is appropriate when the interest of many is considered, whereas the minimax criterion serves the interest of individuals. These two goals are more than often conflicting. To illustrate this point, consider problem A3 in Appendix A; Figure 3.1 is a graphical representation of the rectangular regions. The following notation will be used in the rest of this research effort: let $F_1(.)$ represent the minisum function, and $F_2(.)$ be the minimax function,

$$F_1(X) = \sum_{i=1}^{n} \frac{w_i}{A_i} \int\int_{R_i} ||X - P_i|| \, dP_i$$

and

$$F_2(X) = \max_{1 \le i \le n} \{w_i ||X - C_i|| + r_i'\}$$

then F_1^* and F_2^* are the respective unconstrained optimal

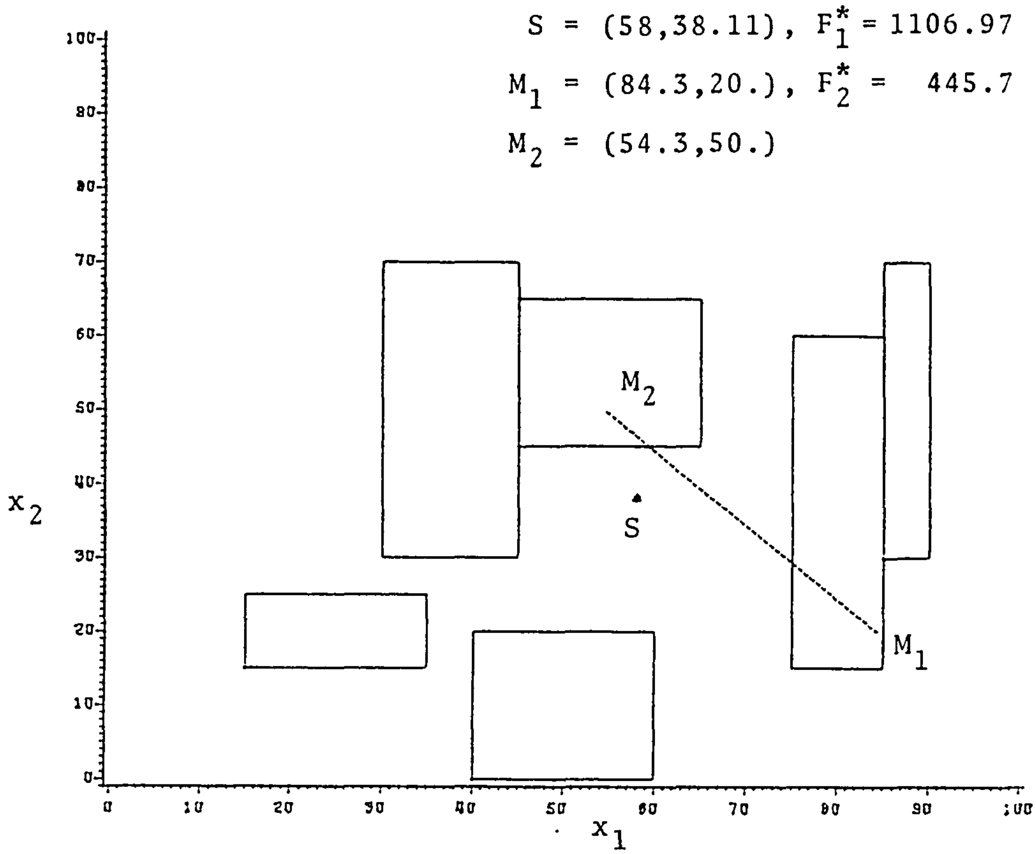


Figure 3.1 A graph of sample problem A_3, with the minisum and minimax solutions.

functions values. The optimal minimax solution is M, if it is a singleton, or it will be represented by the endpoints M_1 and M_2. Similarly, let S, or S_1 and S_2 represent the unconstrained minisum solution set.

In Figure 3.2, some isocurves of the minisum function are plotted. The dotted curve represents the set of points such that the minisum function evaluated at these points is 5% from the optimal (this illustrates the "flatness" of the minisum function around the optimal).

If the optimal minisum location is not available as a location site, then any point inside the dotted isocurve will be within 5% of optimum, and is therefore acceptable as an alternate choice for locating the new facility. Consider the points P_1 and P_2 as shown in Figure 3.2. They are inside the dotted line and are "equivalent" in terms of the minisum problem. However, their respective minimax function values show a variation greater than 20%. Similarly, even though M_1 and M_2 are alternate minimax solutions, their performance under the minisum function is very disparate (respectively 19% and 6% variation from optimal minisum). In fact, there is an alternate minimax solution M_3 which is only 0.8% from the optimal minisum function value.

This example illustrated and confirmed the need for a better modeling approach, where both the minisum and minimax criteria are evaluated concurrently. It showed that, even when the minimax and minisum solutions are relatively "near"

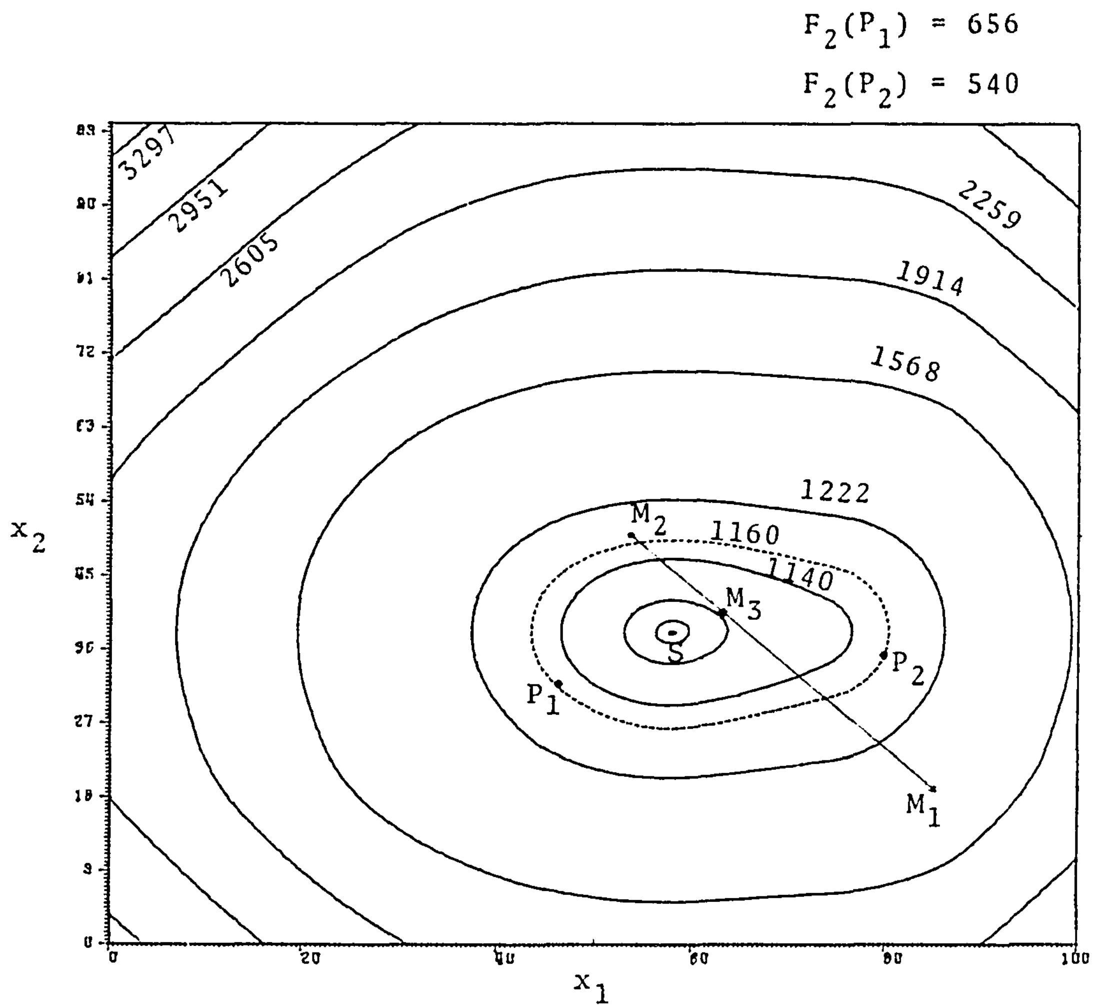


Figure 3.2 Isocurves of the minisum function for sample problem A3.

each other, there is a possibility of high variations from a point to another, and there is a need to be able to control the conflicts and find compromise points.

Consider example problem A4. It has been constructed to illustrate a situation where the minisum and minimax solutions are not approximate. Problem A4 is graphically shown in Figure 3.3. The minimax function evaluated at S is 66% from the optimal minimax value, while the minisum function evaluated at M_1 and M_2, respectively, show 30% and 17% variations. In the next chapter, the constrained minisum (or minimax) location problem with rectangular regions will be compared to equivalent formulation for the centroid approach in order to show the relevance of using region when modeling the existing facilities.

In recent years, many papers have dealt with minimax location problems, but only a few allowed constraints in the models. Brady and Rosenthal (1980) introduced interactive computer graphical methods to solve a constrained single facility case. Brady et al. (1983) extended the interactive graphical methods to the multifacility case. Drezner (1983) investigated cases where the solution is limited to be inside some circles and outside some other circles. Other related problems which have received attention, are the deterministic Weber problems with locational constraints. Schaefer and Hurter (1974), and Hurter et al. (1975) investigated a case where the solution is constrained to be within given distances from each existing facility. A Lagrangean

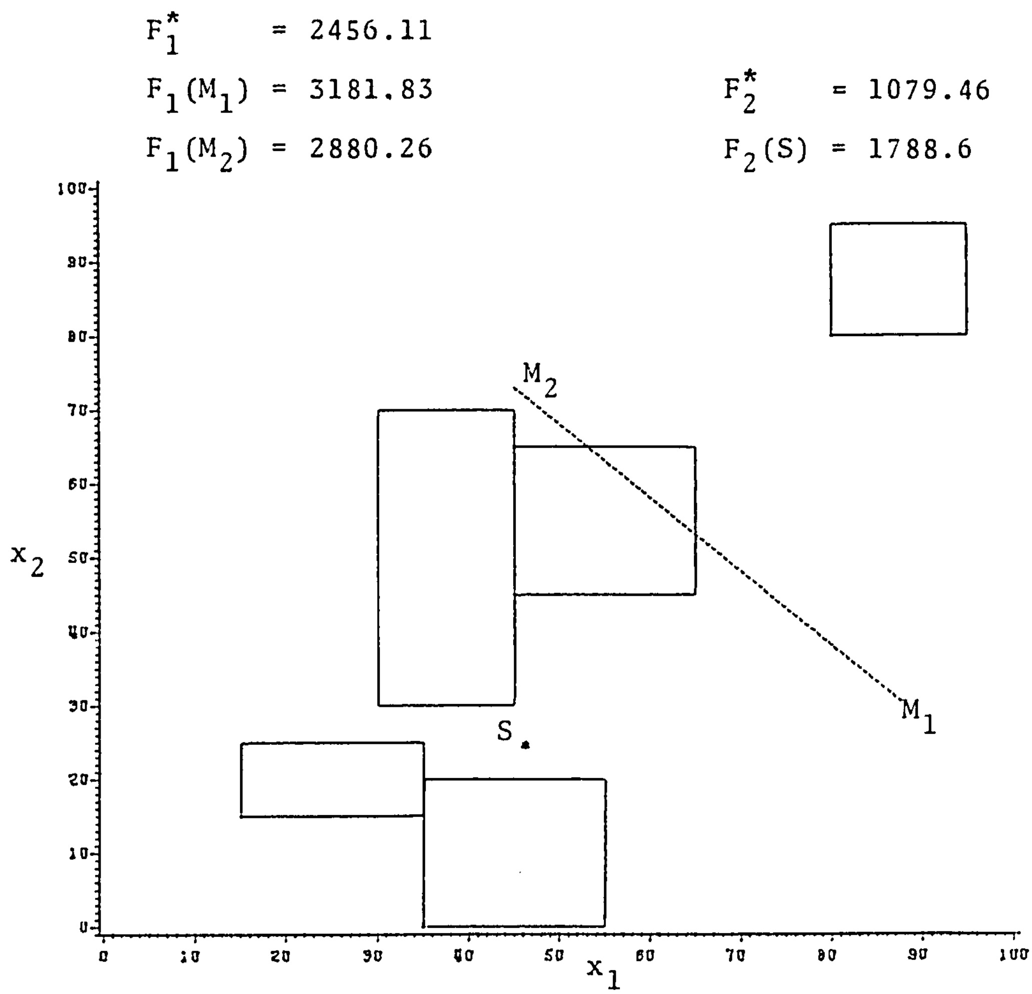


Figure 3.3 Graphical representation of sample problem A4.

interpretation is given, and a dual solution procedure is proposed. Examples with Euclidean distances are solved. Katz and Cooper (1981) solved a Weber problem with a given restricted area, in which no location nor transportation is allowed. Hansen et al. (1982) solved a problem when the feasible set is a union of a finite number of convex polygons. The polygons are ranked following a dominance rule, and the objective function is minimized successively over each polygon, and not all the polygons need to be considered.

3.2 Terminology

Let S represent a nonempty compact and convex subset of $R^p (P \geq 2)$ and let f_i: $R^p \longrightarrow R$ be real valued functions (for $i = 1, \ldots, n$).

$$\text{set } f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

then

$$f : R^p \longrightarrow R^n$$

and the vector minimization problem is:

$$\underset{x \,\varepsilon\, S}{\text{minimize}}\ f(x) \qquad \text{(P3.2.1)}$$

where S is the feasible set for the decision variable x. An optimal solution that simultaneously minimizes all criteria almost never exists. Usually, the criteria are conflicting, a solution that improves one criterion could very well worsen another. Solving problem P3.2.1 reduces to finding the set of all efficient solutions. The following definition is from Geoffrion (1968)

<u>Definition 3.1</u> A point $x^0 \,\varepsilon\, S$ is called efficient if

there exists no other feasible point x such that $f(x) \leq f(x^0)$ and $f(x) \neq f(x^0)$. x^0 is also called pareto-optimal, admissible, nondominated, noninferior.

Recall that $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ then for x_1 and x_2:

$$(f(x_1) \leq f(x_2)) \Longleftrightarrow (f_i(x_1) \leq f_i(x_2) \text{ for all } i)$$

The set $E = \{x \in S | x \text{ is efficient points}\}$ is called the efficient set. Kuhn and Tucker (1950) observed that some efficient solutions can have an undesirable property; they called these solutions improper solutions. Geoffrion (1968) generalized the definition of proper efficiency as follows.

Definition 3.2 x^0 is called a properly efficient solution of P3.2.1 if it is efficient and if there exists a strictly positive scalar M such that for each i the following holds:

$$\frac{f_i(x^0) - f_i(x)}{f_j(x) - f_j(x^0)} \leq M$$

for some j such that $f_j(x) > f_j(x^0)$ whenever $x \in S$ and $f_i(x) < f_i(x^0)$.

Let $E^p = \{x \in S | x \text{ is properly efficient point}\}$, then $E^p \subseteq E$.

If n = 2, P3.2.1 is called a bicriteria minimization problem. More results on bicriteria optimization can be found in Appendix B.

3.3 Constrained Minisum Location Problem

3.3.1 Analysis and Development of a Solution Technique

In Chapter II the following problem was introduced

$$\underset{X \varepsilon R^2}{\text{Minimize}} \ (\max_i \{w_i \|X - C_i\| + r_i'\}) \qquad \text{(P3.3.1)}$$

subject to

$$\sum_{i=1}^{n} \frac{w_i}{A_i} \int\int_{R_i} \|X - P_i\| dP_i \leq \mu$$

It was shown to be a convex programming problem, and justifications were given regarding the practical and beneficial aspects of such models. According to results summarized in Appendix B, problem P3.3.1 is equivalent to the following problem:

$$\underset{X \varepsilon R^2}{\text{Minimize}} \ \sum_{i=1}^{n} \frac{w_i}{A_i} \int\int_{R_i} \|X - P_i\| dP_i \qquad \text{(P3.3.2)}$$

subject to

$$\max_{1 \leq i \leq n} \{w_i \|X - C_i\| + r_i'\} \leq \lambda$$

Formulation P3.3.2 could apply when locating a new school, then the total average distance is minimized, without any student having to travel over some maximum distance λ.

When λ is large enough to make the constraint redundant, the resulting problem is similar to one formulated by Wesolowski and Love (1972), a gradient reduction solution procedure is used to solve the problem. Marucheck and Aly (1982) used a direct search technique to solve the multi-facility case. The method by Wesolowsky and Love (1972) can be summarized for the x_1-subproblem as follows: (the technique applies similarly to the x_2-problem).

I. Initialization Step

Compute $w_i' = \dfrac{w_i}{a_{i_2} - a_{i_1}}$, for each i. Sort the intervals $[a_{i_1}, a_{i_2}]$ by increasing a_{i_1}, then decompose the intervals $[a_{i_1}, a_{i_2}]$ into nonoverlapping intervals $[r_j, s_j]$ with corresponding weights w_j' accumulated as needed.

II. Gradient Reduction Step

1) Compute $M = \sum_{i=1}^{n} (s_j - r_j)w_i'$.

2) Let k = 1.

3) Compute $t_k = w_k'(s_k - r_k)$ and $d(s_k) = -M + 2\sum_{t=1}^{k} t_j$.

4) If $d(s_k) < 0$ let $k = k+1$, go to step 3.

5) If $d(s_k) = 0$ then $s_k \leq x^* \leq r_{k+1}$; stop.

6) If $d(s_k) > 0$ then $x^* = r_k - d(s_k - 1)\ \dfrac{(s_k - r_k)}{2t_k}$

For example, consider example problem A2 in Appendix A, the optimal unconstrained minisum solution is $X^* = (4.65, 4.42)$. If tighter restrictions need to be set on the value of the minimax function, then smaller values of λ are chosen. Below a specific value λ_2, problem P3.3.2 will not be optimal at a minisum solution, and the value of the objective value deteriorates (increases). λ_2 is the smallest value of λ for which a minisum solution still solves P3.3.2. If tighter restrictions on the maximum weighted traveling distance are needed, λ can be reduced as low as λ_1, which is the optimal minimax objective function value. It is clear that if $\lambda < \lambda_1$, then the feasible set of P3.3.2 is empty. Then for problem P3.3.2 only values $\lambda \varepsilon [\lambda_1, \lambda_2]$ should be considered. Similarly, for the constrained minimax problem P3.3.1 only values $\mu \varepsilon [\mu_1, \mu_2]$ are of interest. In Appendix B, a relationship between the intervals $[\lambda_1, \lambda_2]$ and $[\mu_1, \mu_2]$ is described, it is also shown that P3.3.2 and P3.3.1 with λ and μ in the given intervals, will generate the same solutions.

The following procedures explain how λ_1 and λ_2 are found,

i) Finding λ_1: λ_1 is determined by solving the following problem:

$$\underset{X \varepsilon R^2}{\text{minimize}} \quad \max_{1 \leq i \leq n} \{w_i \|X - C_i\| + r_i'\}$$

λ_1 is the resulting optimal objective function value. (This problem is solved in Chapter II)

ii) Finding λ_2: Step 1: Find the solution set B of the unconstrained minisum problem:

$$B = \{X \varepsilon R^2 | X \text{ minimizes } \sum_{i=1}^{n} \frac{w_i}{A_i} \iint \|X - P_i\| dP_i\}$$

let F_1^* be the resulting optimal minisum function value then

$$B = \{X \varepsilon R^2 | F_1(X) = F_1^*\}$$

The set B is found using the gradient reduction technique described earlier. For sample problem A_2, B is the singleton $\{(4.65,4.42)\}$. Other possible geometrical shapes are: line segment parallel to either coordinate axis, rectangle with sides parallel to the axis.

Step 2: i) If B is a singleton $\{\overline{X}\}$, then

$$\lambda_2 = F_2(\overline{X}) = \max_{1 \le i \le n} \{w_i \| \overline{X} - C_i \| + r_i'\}$$

ii) If B is not a singleton, then the following problem is solved:

$$\underset{X \in R^2}{\text{minimize}} \max_{1 \le i \le n} \{w_i \| X - C_i \| + r_i'\} \quad \text{(P3.3.3)}$$

Subject to $X \in B$

or similarly

$$\underset{X \in R^2}{\text{minimize}} \max_{1 \le i \le n} \{w_i \| X - C_i \| + r_i'\} \quad \text{(P3.3.4)}$$

subject to

$$\sum_{i=1}^{n} w_i \iint \| X - P_i \| dP_i \le F_1^*$$

and λ_2 is the optimal objective function of P3.3.4.

Note: The constraint in P3.3.4 is a less or equal type, since there exists no point $\overline{X}$ such that $F_1(\overline{X}) < F_1^*$, then P3.3.4 is exactly P3.3.3. Also, let A be the solution set of the unconstrained minimax problem, then if $A \cap B \neq \emptyset$, the solution sets of the unconstrained minimax and minisum problems intersect, and the resulting constrained problems P3.3.1 and P3.3.2 are trivial, since points in the intersection $A \cap B$ will minimize either unconstrained problem. Thus, in any solution procedure to solve P3.3.2 (or P3.3.1) it is necessary to include a test routine that checks whether $A \cap B$ is empty or not. An efficient solution procedure has been developed in this research effort that generates the sets A and B, evaluates $A \cap B$, if it is empty, then it proceeds to find λ_1 and λ_2 and then generates efficient points as λ is varied over the range $[\lambda_1, \lambda_2]$. The algorithm is summarized as follows.

3.3.2 Description of the Solution Procedure

The mathematical model is

$$\underset{X \in R^2}{\text{minimize}} \sum_{i=1}^{n} w_i \int\int_{R_i} || X - P_i || dP_i \qquad \text{(P3.3.2)}$$

subject to

$$\max_{1 \le i \le n} \{ || X - C_i || + r_i' \} \le \lambda$$

Find λ_1

Step 1: Solve the unconstrained minimax problem. Let λ_1 equal the optimal function value, and A be the solution set.

Find λ_2

Step 2: Solve the unconstrained minisum problem. Let B be the resulting solution set.

Step 3: Verify that the solution sets of the minisum and minimax problems do not intersect. If the intersection is not empty, the case is trivial; identify intersection points and stop. Otherwise go to step 4.

Step 4: Minimize the minimax objective function over the minisum solution set. (Geometrical properties of the solutions sets are utilized to speed up the optimization process.) Let λ_2 equal the resulting optimal minimax function value.

Generate Efficient Solutions

Step 5: For $\lambda \varepsilon [\lambda_1, \lambda_2]$ find the extreme points of the diamond defined by the constraint

$$\max_{1 \leq i \leq n} \{w_i \| X - C_i \| + r_i'\} \leq \lambda$$

Step 6: Using the Golden section line search, optimize the minisum function over the four arcs connecting the extreme points of the feasible set. Increase λ and go to step 5.

The solution technique has been coded in Fortran, and verified by testing it with examples from Appendix A, among others.

3.3.3 Computational Results

Consider example problem A2 from Appendix A, λ_1 is the optimal unconstrained minimax function and $\lambda_1 = 40$. The set B of optimal minisum solutions is the singleton {(4.65, 4.42)} and

$$\begin{aligned} \lambda_2 &= F_2(4.65, 4.42) \\ &= \max_{1 \leq i \leq n} \{w_i(|4.65 - c_{i1}| + |4.42 - c_{i2}|) + r_i'\} \\ &= 49.87 \end{aligned}$$

In Appendix B it is shown that if $\lambda \varepsilon [\lambda_1, \lambda_2]$ then the constraint in problem P3.3.2 is tight at the optimal. Also, if P3.3.2 solved for some $\lambda_0 \varepsilon\ [\lambda_1, \lambda_2]$, then the resulting solution X_0 is an efficient solution.

Let

$$F_2(X_0) = \max_i \{w_i \| X_0 - C_i \| + r_i'\} = \lambda_0$$

$$F_1(X_0) = \sum_{i=1}^{n} w_i \iint \| X_0 - P_i \| dP_i = \mu_0$$

then X_0 is also a solution of problem P3.3.1 for $\mu = \mu_0 = F_1(X_0)$, that is

$$F_2(X_0) = \underset{X \varepsilon R^2}{\text{minimize}} \max_{1 \le i \le n} \{w_i \| X - C_i \| + r_i'\}$$

subject to

$$\sum_{i=1}^{n} \frac{w_i}{A_i} \iint_{R_i} \| X - P_i \| dP_i \le \mu_0 = F_1(X_0)$$

Figure 3.4 illustrates the regions defined by sample problem A2, as well as the efficient set E generated by solving P3.3.2 for $\lambda \varepsilon [\lambda_1, \lambda_2]$. Z_1 is the only unconstrained minimax solution which is efficient. Z_2 is the unconstrained minisum function, all other points on the dotted line are efficient points.

Figure 3.4 illustrates the set of efficient solutions of P3.3.2, in the decision space R^2. The objective space refers to the set $T = \{(F_2(X), F_1(X)), X \varepsilon R^2\}$, then $T \subseteq R^2$.

Let (λ_0, μ_0) be the pair corresponding to an efficient point X_0, if all such pairs are plotted in the objective

$F_2(Z_1) = \lambda_1 = 40$

$F_2(Z_2) = \lambda_2 = 49.87$

$F_2(Z_0) = \lambda_0 = 43.95$

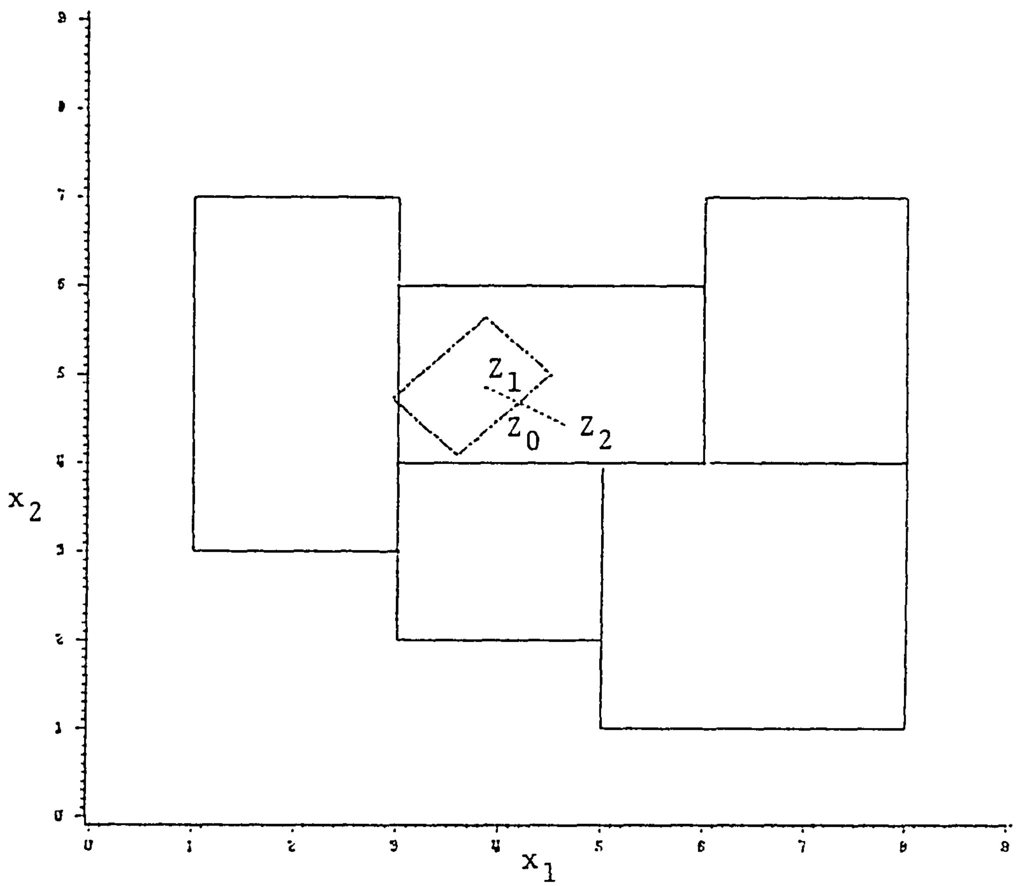


Figure 3.4 Efficient set in the decision space (sample problem A_2).

space, then Figure 3.5 is a representation of that set (for sample problem A2). The curve in Figure 3.5 is also called efficient frontier (of the set T), it shows the conflicting nature of the two location criteria under investigation.

An improvement in the minimax function (represented by λ) is achieved at the expense of the minisum function (μ) and vice versa. The curve is continuous, which can be explained by the stability of the two constrained problems under study and which result in strong dual optimality (no duality gap). It is clear in Figure 3.5 that if $\lambda_0 > \lambda_2$, the corresponding minisum function value μ_0 will be μ_1, but the resulting points are not efficient (the pair (λ_2,μ_1) dominate all pairs $(\lambda_0\mu_1)$ where $\lambda_0 > \lambda_2$).

When $\lambda = 40$, the curve is a vertical line segment that represents all possible minisum function values over the set A. (A is the set of all optimal unconstrained minimax points.) The image of R^2 under the (F_2,F_1) map is unbounded. A Lagrangian duality interpretation of Figure 3.5 will be given later.

With the help of Figures 3.4 and 3.5, a decision maker can choose a location for the new facility somewhere along the efficient set (or near to it), and check the tradeoff resulting in the two cost functions. If the minimax cost desired is greater than λ_2, then the unconstrained minisum solution with the lowest minimax function value is the best location for the new facility. For a chosen value λ_0 such

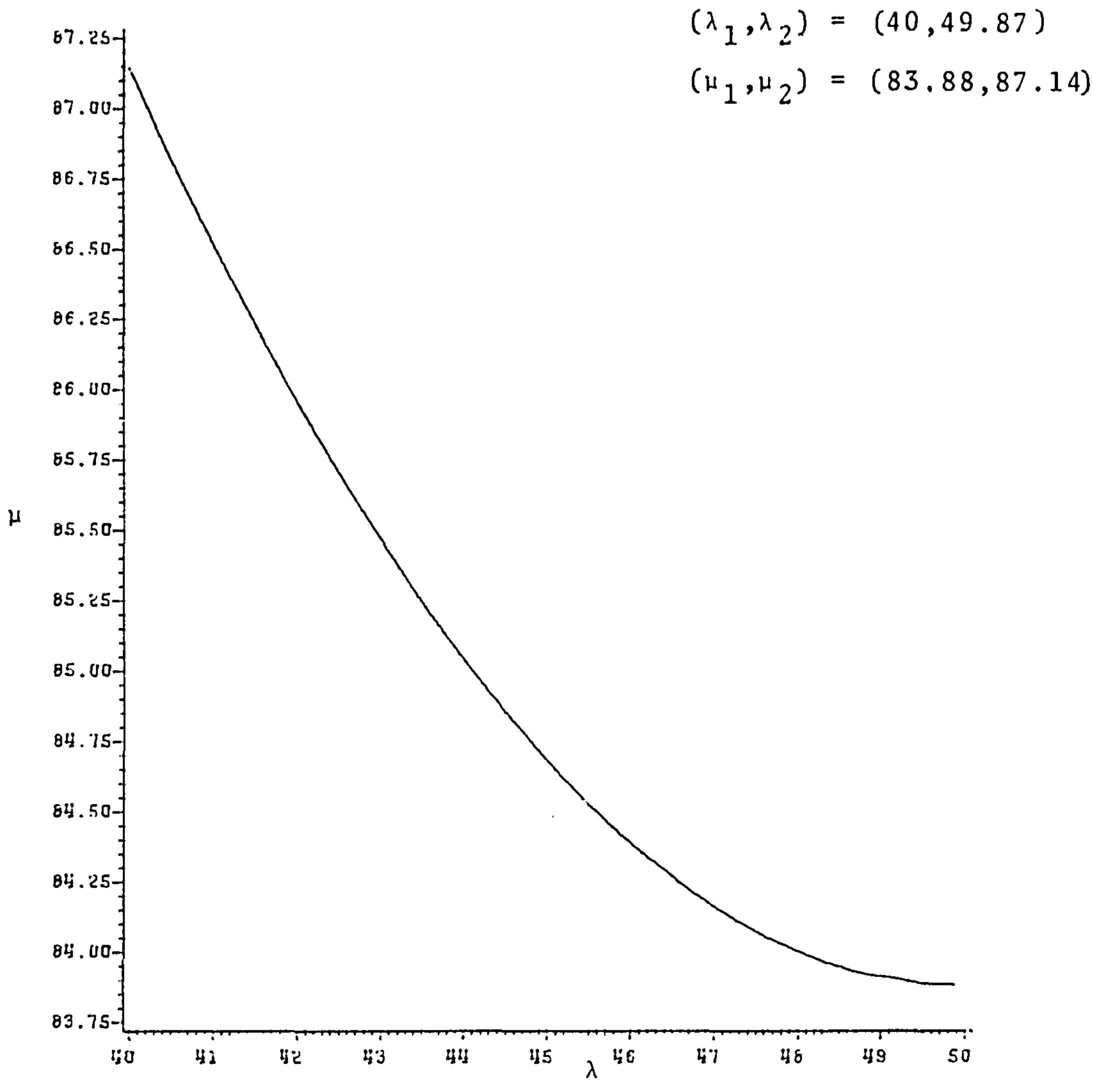


Figure 3.5 Efficient set in the objective space (sample problem A_2).

that $\lambda_1 \leq \lambda_0 \leq \lambda_2$, there exists a point in the feasible set defined by the constraint $F_2(X) \leq \lambda_0$, that cannot be dominated by any other point. This efficient point is the intersection of the efficient set with the diamond defined by the constraint. For example, in Figure 3.4, if $\lambda_0 = 43.95$ then the resulting diamond defined by $F_2(X) \leq 43.95$, intersects with the efficient set at $Z_0 = (4.17,4.68)$. The mapping (F_2,F_1) only defines a partial ordering of the plane R^2 (see Appendix B). The set of all nondominated points in R^2 (for the partial ordering given) have been generated. If one wishes to compare any two points in the plane, it can be achieved by using and combining the geometrical properties of the two dual constrained problems P3.3.1 and P3.3.2. When comparing two points, any of three possible situations can occur:

(1) The two points are on the same minimax isocurve.

(2) The two points are on the same minisum isocurve.

(3) None of the above.

If either case (1) or (2) occurs, then one point dominates the other one. For case (3), either one point is on minimax and minisum isocurves that are both inside the isocurves of the other point, which it then dominates. Or, the points cannot be ordered as one performs better in one criterion and worse in the other. To illustrate these comparison rules, consider sample problem A3, in Figure 3.6 are superimposed several isocurves for the minisum function and the minimax function. Instead of looking at the total area [0,100] X

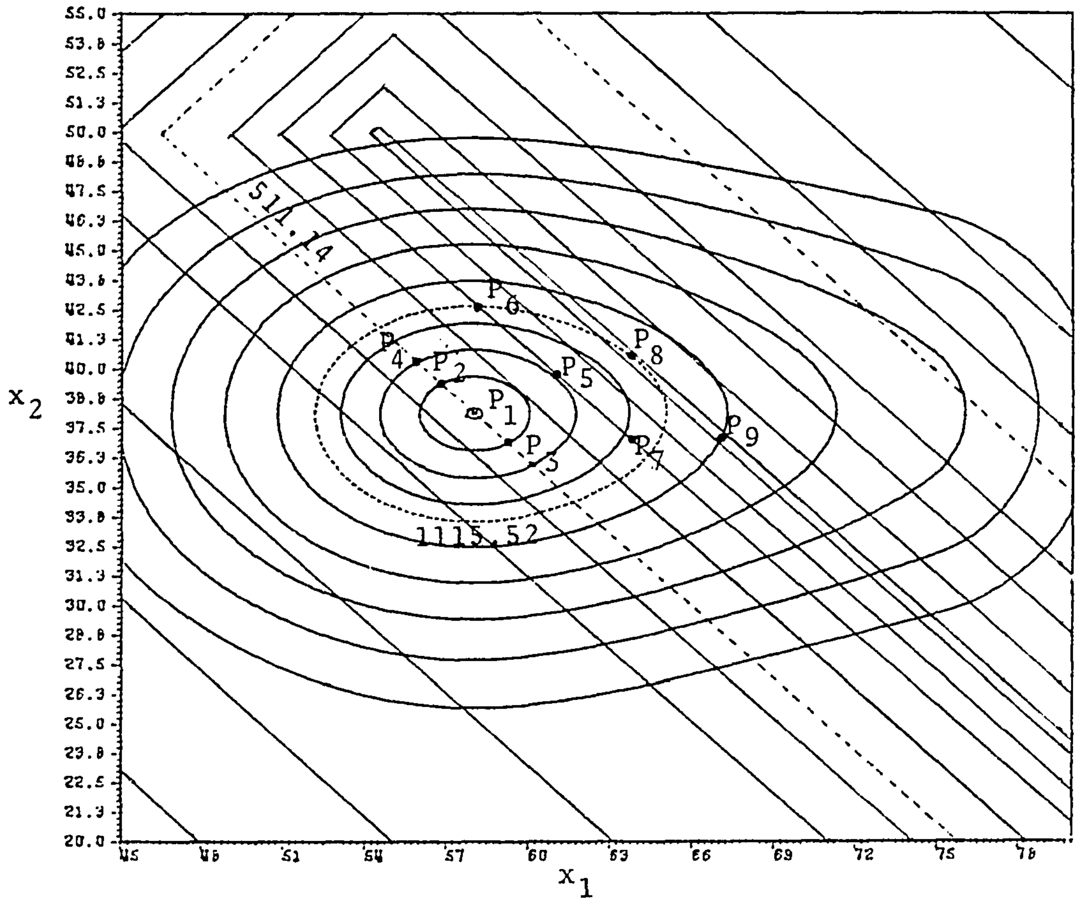


Figure 3.6 Graphical ranking of points for sample problem A3 (overlaid minimax and minisum isocurves).

[0,100], a focus is made on the area [45.,80] X [20,55] which covers the most sensitive and relevant location area. The dotted minisum and minimax isocurves represent the critical values λ_2 and μ_2 in this case λ_2 = 511.14 and μ_2 = 1115.52. It is clear that P_1 dominates P_2, P_3 and P_4, since P_1 has the lowest minisum function value (optimal) and all four points are on the same minimax isocurve. However, P_2 and P_3 are equivalent since they lay on the same minisum isocurve. P_2 and P_3 dominate P_4. Also $P_5 > P_7 > P_6$ because they lay on the same minimax isocurve and on different minisum isocurves. P_5 is an efficient point, no other point dominates it. P_8 and P_9 are two minimax solutions, but P_8 has a lower minisum function value. But P_7 and P_9 cannot be compared, P_9 is superior in the minimax function but worse in the minisum function. This simple graphical and visual technique can also be used to generate efficient points; just set the minimax function at a value $\lambda_1 \leq \lambda_0 \leq \lambda_2$ (or set the minisum function at a value $\mu_1 \leq \mu_0 \leq \mu_2$) and travel along the resulting isocurve $F_2(X) = \lambda_0$ (or $F_1(X) = \mu_0$) until the lowest possible minisum (or minimax) isocurve is reached.

3.4 Conclusion

In this chapter, the importance of considering the minisum and minimax criteria together was shown. Minimizing the minisum function subject to a bound λ on the minimax function is equivalent to minimizing the minimax function subject to a bound μ on the minisum function. A solution for one problem can be obtained by solving the other problem for specific parameters λ_0 and μ_0. Based on this equivalence only one problem needs to be solved, and because of the simple feasible set defined by the minimax function, the constrained minisum problem was solved. A powerful solution procedure was developed that generates efficient solutions. Two initial steps required solving the unconstrained minisum and minimax problems, the techniques by Dearing (1972) and Wesolowsky and Love (1971) are used because they are exact and fast.

The set of all nondominated (or efficient) points was generated and plotted in the decision space and in the objective space. The efficient set as plotted in the decision space gives a spatial representation in relation to the existing facilities, allowing the decision maker to visually evaluate the alternatives. On the other hand, the efficient set as plotted in the objective space gives a quantitative representation. Using both representations, a final decision can be made, based on cost tradeoff between the two criteria, and locational preferences. A graphical approach for comparing points has also been discussed, it can be used to find efficient points.

CHAPTER IV

A BICRITERIA LOCATION MODEL AND RELATIONSHIP TO THE CONSTRAINED MODELS

4.1 Literature Review

Multicriteria facility location problems have received increasing attention, which follows recent developments in multicriteria mathematical programming.

Important results were introduced by Kuhn and Tucker (1950) as they discussed the vector minimization problem, and derived necessary and sufficient conditions to obtain solutions with a special property, and which are called properly efficient solutions. Their theory was based on differentiability arguments. Geoffrion (1967) addressed an interactive bicriteria maximization problem, and showed how it can be solved as parametric subproblems. Klinger (1967) extended previous work by Kuhn and Tucker (1950) regarding some solutions which were found to possess an undesirable property. These solutions were called improper, and it was shown that only properly efficient solutions are relevant when solving a vector optimization problem.

Geoffrion (1968) generalized the concept and definition of proper solutions in order to exclude efficient solutions that allow for a first order gain in one criteria at the expense of but a second order loss in another. Proper solutions are characterized by necessary and sufficient conditions.

Iserman (1974) showed that for the linear vector optimization problem, all efficient solutions are properly efficient. Benson and Morris (1977) gave necessary and sufficient conditions for an efficient solution to be properly efficient. These conditions relate the proper efficiency of a solution to the stability of a single objective optimization problem.

Wendell and Lee (1977) generalized several results on efficiency for linear problems to nonlinear cases. Their results are based on duality theory. Bacopoulos and Singer (1977) proved that the bicriteria convex minimization problem can be solved by considering either one of two constrained single objective convex programs. Benson (1979) extended these results and developed a parametric procedure for generating the set of efficient points for the convex bicriteria maximization problem. Gearhart (1979) generalized the characterization of efficient points for some nonconvex functions. Sadagopan and Ravindran (1982) gave more results on efficient solutions for concave maximization and developed some interactive methods for solving bicriteria problems.

The earliest multicriteria location problems investigated in the literature involved trees and graphs. Halpern

(1976) considered a weighted sum of the minimax and minisum criteria cost functions on a tree. The solution was found to lie either on the center of the tree or on the path connecting the center and the median points of the tree. The exact location depending on the weights attributed to two objectives. Lowe (1976) and Handler (1976) independently studied the same problem and obtained comparable results.

Halpern (1977), (1978) and (1980) extended the bicriteria location problem on tree to graphs. He concentrated on three problems, the weighted sum of the minisum and minimax objective functions, and the single objective minimization of one of the criteria with the other objective acting as a constraint. The three problems are shown to be related, and that a special duality exists between the two constrained problems.

Tansel et al. (1983) considered a bicriteria multi-facility minimax location problem on a tree network. The two objectives involved are the maximum weighted distance between pairs of new and existing facilities, and the maximum weighted distance between pairs of new facilities. Necessary and sufficient conditions are developed for a solution to be efficient. Another class of multicriteria optimization with application to location problems involves problems with binary variables; Ross and Soland (1980), Burkard et al. (1982).

Continuous multicriteria location problems have also been investigated. Kuhn (1967) investigated a problem with Euclidean distances and where the objectives are the distances between facilities. Wendell and Hurter (1973) proved for a general ℓ_p norm that the points in the convex hull of the existing facilities are dominant, and only those points need to be considered for the minisum criterion. Wendell, et al. (1977) generated the efficient set for a single facility rectilinear case where the objectives are the weighted distances. Using an approach depending more on geometrical considerations, Chalmet et al. (1981) improved the algorithms developed by Wendell et al. McGinnis and White (1976) studied a single facility rectilinear problem with a weighted sum combination of the minisum and minimax criteria. A linear programming formulation is proposed but a direct search procedure is developed. Rahali and Aly (1980) studied a weighted sum approach for the minisum approach for the minisum and minimax multifacility Euclidean criteria. A subgradient iterative procedure is proposed, but a direct search approach is used to solve an example and obtain properly efficient solutions.

4.2 Characterization of Proper Efficient Solutions

In Chapter III, two location models were shown to be equivalent, and generated the same set of nomdominated solutions. They were

$P_1(\lambda)$: $\underset{X \varepsilon R^2}{\text{minimize}}\ F_1(X)$ and $P_2(\mu)$: $\underset{X \varepsilon R^2}{\text{minimize}}\ F_2(X)$

subject to $F_2(X) \leq \lambda$; subject to $F_1(X) \leq \mu$

where

$$F_1(x) = \sum_{i=1}^{n} w_i \int\int R_i \| X - P_i \| dP_i$$

and

$$F_2(x) = \max_{1 \leq i \leq n} \{w_i \| X - C_i \| + r_i'\}$$

for $\lambda \varepsilon [\lambda_1, \lambda_2]$ and $\mu \varepsilon [\mu_1, \mu_2]$.

Geoffrion (1968) proved for convex function $F_1(x)$ and $F_2(x)$ that a point X is a properly efficient point if and only if X_0 is a solution of the following problem.

$$\underset{X \varepsilon R^2}{\text{minimize}} (\gamma F_1(x) + (1-\gamma) F_2(x)) \qquad \text{(P4.2.1)}$$

for some $0 < \gamma < 1$

(see Theorem B.1 in Appendix B).

Benson and Morris (1977) have characterized properly efficient solutions by verifying the "stability" of the associated constrained single objective optimization problems for maximization problems with concave functions.

Definitions and results on characterization of properly efficient solutions have been summarized in Appendix C. Based on these results, the following theorem can be stated.

Theorem 4.1: All efficient solutions X_0 such that $F_2(X_0) \notin \{\lambda_1, \lambda_2\}$ (or $F_1(X_0) \notin \{\mu_1, \mu_2\}$) are properly efficient solutions.

Proof: Recall that

$$\lambda_1 = \inf \{F_2(X) | X \varepsilon R^2\}$$

$$\mu_1 = \inf \{F_1(X) | X \varepsilon R^2\}$$

If

$$A = \{X \varepsilon R^2 | F_2(X) = \lambda_1\}$$

and

$$B = \{X \varepsilon R^2 | F_1(X) = \mu_1\}$$

then

$$\mu_2 = \inf\{F_1(X) | X \varepsilon A\}$$

$$\lambda_2 = \inf\{F_2(X) | X \varepsilon B\}$$

First, it is shown that only the cases $F_2(X_0) \notin \{\lambda_1, \lambda_2\}$ are to be considered since they imply $F_1(X_0) \notin \{\mu_1, \mu_2\}$:

if $F_2(X_0) > \lambda_1$ then $X_0 \notin A$ and $F_1(X_0) < \mu_2$ (otherwise if $F_1(X_0) \geq \mu_2$ then $F_2(X_0) = \lambda_1$ and $X_0 \varepsilon A$ which contradicts the hypothesis $F_2(X_0) > \lambda_1$). Also, since $F_2(X_0) < \lambda_2$ then $X_0 \notin B$ and $F_1(X_0) > \mu_1$.

Therefore, let X_0 be an efficient solution such that $\lambda_1 < F_2(X_0) < \lambda_2$ and consider the following two problems.

$P_1(\lambda_0)$ and $P_2(\mu_0)$ where $\lambda_0 = F_2(X_0)$ and $\mu_0 = F_1(X_0)$. X_0 is an efficient point such that $X_0 \notin A$, then there exists a feasible point X_2^* for problem $P_1(\lambda_0)$ such that $X_2^* \varepsilon A$ and

$F_2(X_2^*) = \lambda_1 < \lambda_0 = F_2(X_0)$ which shows that the convex feasible region defined by problem $P_1(\lambda_0)$ satisfies Slater's constraint qualification (see Appendix B), which also implies that problem $P_1(\lambda_0)$ is stable (Geoffrion (1971)).

Similarly, since $X_0 \notin B$ then there exists a feasible point X_1^* for problem $P_2(\mu_0)$ such that $X_1^* \in B$ and

$$F_1(X_1^*) = \mu_1 < \mu_0 = F_1(X_0)$$

which implies that $P_2(\mu_0)$ is stable. Applying Theorem C.1 (from Appendix C) this shows that efficient solution X_0 is also properly efficient.

Theorem 4.1 states that all efficient solutions obtained by solving $P_1(\lambda)$ for $\lambda \in (\lambda_1,\lambda_2)$(respectively $P_2(\mu)$ for $\mu \in (\mu_1,\mu_2)$) are properly efficient solutions of the vector minimization problem. Combining this result with the implications from Theorem B.1 (Geoffrion characterization of proper solutions) shows that there also exists a scalar $\gamma_0 \in (0,1)$ such that X_0 solves problem P4.2.1.

In the next section, for a given properly efficient solution X_0 the corresponding scalars λ_0, μ_0 and γ_0 will be computed, such that X_0 solves $P_1(\lambda_0)$, $P_2(\mu_0)$ and P4.2.1.

Note: If $\lambda_0 = F_2(X_0) \in \{\lambda_1,\lambda_2\}$ then either $P_1(\lambda_0)$ or $P_2(\mu_0)$ will not satisfy Slater's conditions and no conclusion can be made whether X_0 is proper or not, unless the corresponding scalar $\lambda_0 \in (0,1)$.

4.3 Relationship Between the Bicriteria and Constrained Location Problems

In the previous section, it was shown that properly efficient solutions can be generated by solving problem P3.2.3 for $\lambda \varepsilon (\lambda_1, \lambda_2)$ or problem P4.2.1 where $\gamma \varepsilon (0,1.$ Problem P4.2.1 can be rewritten as

$$\underset{X \varepsilon R^2}{\text{minimize}} \; \gamma\Big(\sum_{i=1}^{n} \frac{w_i}{A_i} \iint_{R_i} \|X - P_i\| dP_i \Big) \qquad \text{(P4.3.1)}$$

$$+ (1 - \gamma)\Big(\max_{1 \leq i \leq n} \{w_i \|X - C_i\| + r_i'\}\Big)$$

for some $\gamma \varepsilon (0,1)$

Suppose X_0 is a properly efficient solution obtained by solving the weighted sum problem P4.3.1 for $0 < \gamma_0 < 1$, then there exist parameters λ_0 and μ_0 such that X_0 solves the constrained minisum and minimax problem. Since X_0 is efficient, there exists a parameter $\lambda_0 \varepsilon [\lambda_1, \lambda_2]$ such that X_0 is a solution of

$$\underset{X \varepsilon R^2}{\text{minimize}} \; \Big(\sum_{i=1}^{n} \frac{w_i}{A_i} \iint \|X - P_i\| dP_i \Big) \qquad \text{(P4.3.2)}$$

subject to

$$\max_{1 \leq i \leq n} \{w_i \|X - C_i\| + r_i'\} \leq \lambda_0$$

In Appendix B it is proven that the constraint is tight at optimal, then set

$$\lambda_0 = \max_{1 \leq i \leq n} \{w_i \|X_0 - C_i\| + r_i'\}$$

and similarly, for the constrained minimax problem, set

$$\mu_0 = \sum_{i=1}^{n} \frac{w_i}{A_i} \int \int_{R_1} \|X_0 - P_i\| dP_i .$$

Conversely, if X_0 solves problem P4.3.2 for $\lambda_1 < \lambda_0 < \lambda_2$, then P4.3.2 satisfies the K.T. saddlepoint necessary optimality theorem as formulated by Mangasarian (1969), and therefore, there exists a multiplier u_0 strictly positive (see proof of Theorem B.2 in Appendix B) such that X_0 is the minimizer of the function

$$\theta(u_0) = \min_{X \varepsilon R^2} \{F_1(X) + u_0(F_2(X) - F_2(X_0))\}$$

since $u_0F_2(X_0)$ is a constant, then X_0 is a minimizer of $F_1(X) + u_0F_2(X)$ with $u_0 > 0$. After normalizing, X_0 is a minimizer of

$$\frac{1}{1+u_0}F_1(X) + \frac{u_0}{1+u_0}F_2(X), \quad \text{set } \gamma_0 = \frac{1}{1+u_0} .$$

In summary, in order to find the weight γ_0 for which problem P4.3.1 has the same optimal solution X_0 obtained from solving P4.3.2 (for $\lambda_0 = F_2(X_0)$, the following problem is solved:

$$\max_{u > 0} \{ \min_{X \varepsilon R^2} [F_1(X) + u(F_2(X) - \lambda_0)]\}. \qquad \text{(P4.3.3)}$$

This problem is the Lagrangian dual of P4.3.2.

4.4 Description of the Solution Technique for Solving P4.3.3

P4.3.3 can be rewritten as $\max_{u > 0} \theta(u)$ where

$$\theta(u) = \min_{X \varepsilon R^2} [F_1(X) + u(F_2(X) - \lambda_0)],$$

each function evaluation of $\theta(.)$ requires the minimization of a convex function in X, and since $\theta(u)$ is concave in u, a line search over $[o, \hat{u}]$ will find the optimal u^0(u is chosen large enough for u^0 to be in $[o, \hat{u}]$). The most efficient linear search for unimodal functions is the golden section method, which is used on $\theta(u)$. Each function evaluation of $\theta(u)$ is an optimization of a convex unconstrained and not differentiable function in X. Direct search methods are very efficient for this type of problem, and Hooke and Jeeves (1961) pattern search is used because it converges quickly to the optimal.

4.5 An Example Problem

Consider problem A2 in Appendix A, Figure 4.1 shows the relationship between the bicriteria problem P4.3.1 and the constrained minisum problem $P_1(\lambda)$. For any pair (λ,γ) on the graph, corresponds an efficient solution X such that X solves $P_1(\lambda)$ and X solves P4.3.1. For example, (λ_0,γ_0) shown in Figure 4.1 corresponds to the efficient point $X_0 = (4.17, 4.68)$. Similarly, Figure 4.2 illustrates the relationship between problem P4.3.1 and the constrained minimax problem $P_2(\mu)$. The same efficient point $X_0 = (4.17, 4.68)$ corresponds in this case to the pair (μ_0,γ_0) where μ_0 is the value of the minisum function value evaluated at X_0 (and X_0 solves problem $P_2(\mu_0)$), and γ_0 is such that X_0 solves P4.3.1 for $\gamma = \gamma_0$.

It can be shown that all efficient solutions are generated for $0.6 < \gamma < 1$. If $0 < \gamma < 0.6$, then problem P4.3.1

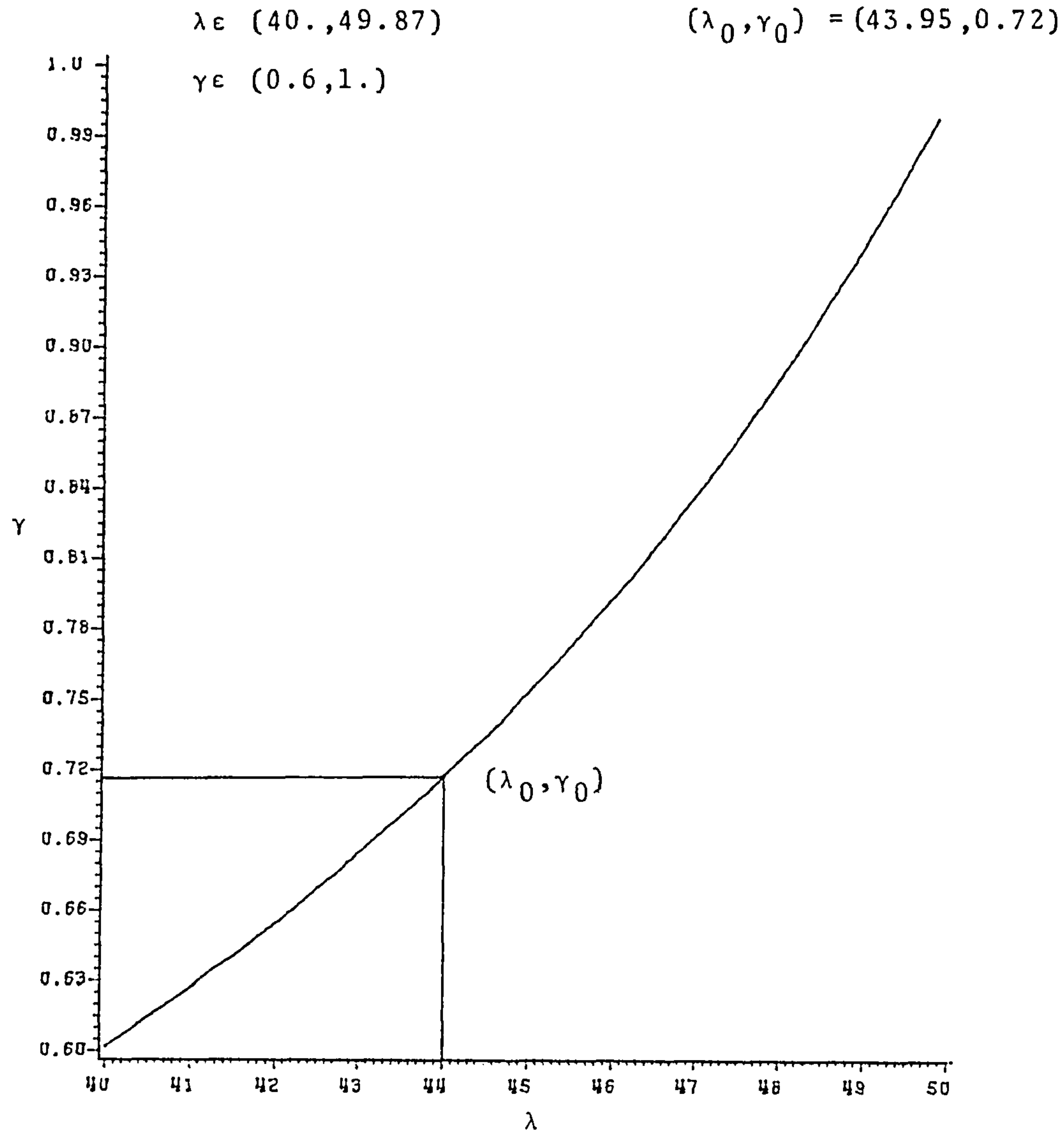


Figure 4.1 Relationship between constrained minisum and weighted sum problems (sample problem A2).

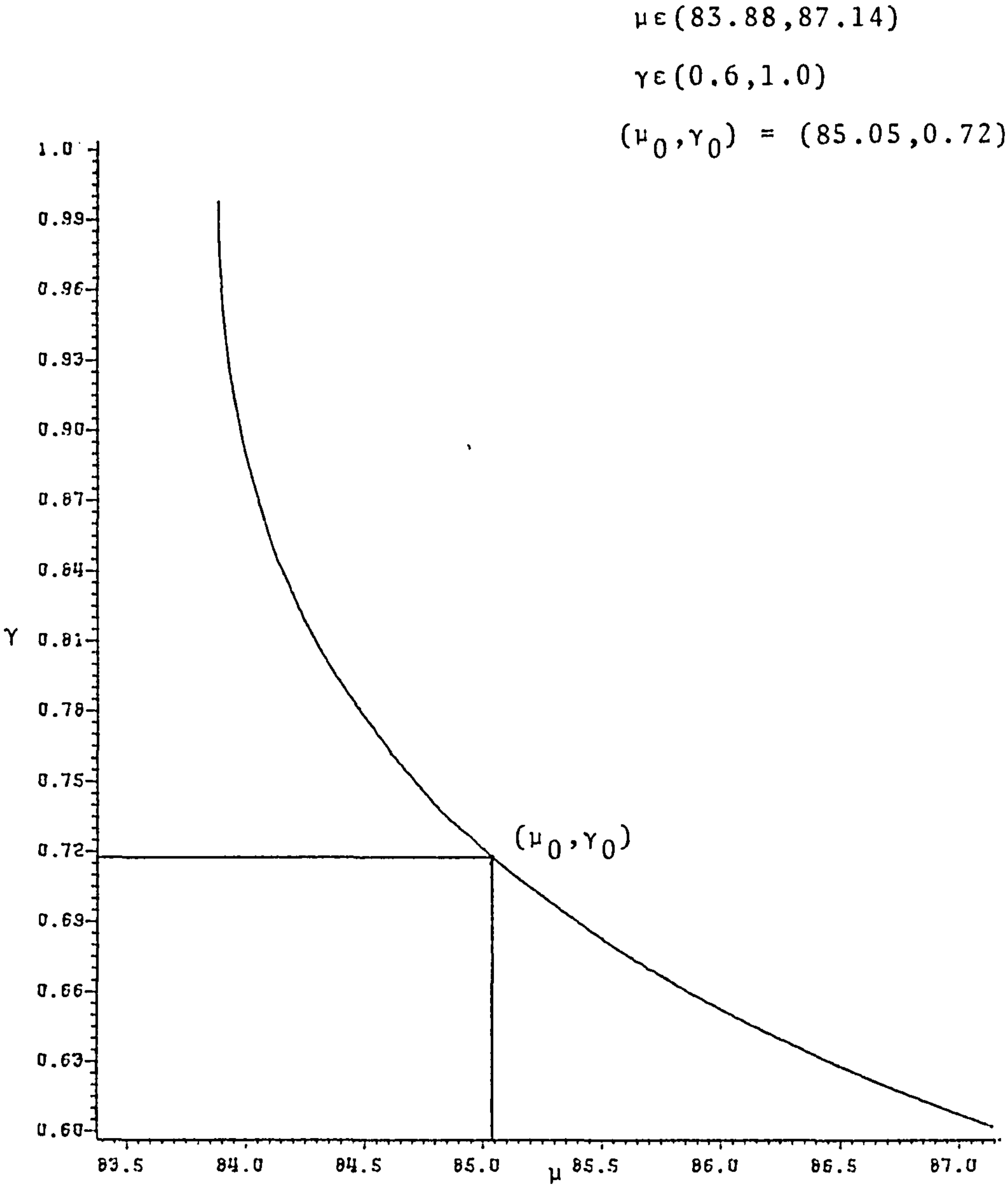


Figure 4.2 Relationship between constrained minimax problem and weighted sum problem (sample problem A2)

is optimized at X_0 = (3.86,4.86) which is a minimax solution that can be obtained by solving either $P_1(40)$ or $P_2(87.14)$. At $\gamma = 0.6$, the weight is such that the solution of P4.3.1 starts to move away from the minimax point, when $\gamma = 0.99$, the solution is at the optimal minisum location. This process is well illustrated in Figures 4.1 and 4.2. Recall that λ represents a value of the minimax function and μ the minisum function. As λ increases (respectively, μ decreases), the corresponding efficient solution moves towards the minisum solution (respectively, minisum solution) and the weight γ increases, which creates a shift of the bicriteria problem P4.3.1 closer to the unconstrained minisum problem.

4.6 The Constrained Approach vs. the Weighted Sum Approach

In the previous chapter, efficient solutions were generated by minimizing the minisum function such that the minimax function satisfied an upper bound λ. When λ was varied between two specific values λ_1 and λ_2, the solutions obtained are efficient and the constraint is tight, that is,

$$\max_{1 \leq i \leq n} \{w_i \|X - C_i\| + r_i'\} = \lambda .$$

This property allows the decision maker to choose a minimax or (minisum) cost and then find the optimal efficient solution that corresponds to these "desirable" costs. Whereas, for the weighted sum problem P4.3.1 it is hard to give a practical meaning to the weight γ. The objective function of problem 4.3.1 behaves as a new function compared to the minisum or minimax function. When a weight is chosen, the decision maker has a general feeling about which criteria is being favored over the other, but he will not know until a solution is obtained, what kind of solution he will obtain.

Another advantage of adopting the constrained minisum problem is the relatively simple optimization technique needed to completely solve the problem. In order to find λ_1 and λ_2, the unconstrained minisum and minimax problems have to be solved first. The techniques chosen are exact, easy to implement and only require simple arithmetic and data structuring techniques (sorting of vectors). Solving the constrained minisum problem was reduced to line searches over the four sides of a diamond defined by the minimax constraint. On the other hand, the weighted sum function of problem P4.3.1 is nonlinear, doesn't offer any favorable geometrical properties and is not differentiable. To solve P4.3.1, requires finding a saddlepoint which is more difficult. Iterative subgradient-free or subgradient methods could be used (other possible methods are simulation, approximation techniques, etc). These methods are more difficult to develop and to implement.

Another benefit from adopting the constrained approach over the weighted sum approach is derived from the use of the isocurves of the associated functions. It was seen earlier how the two constrained problems can usually be interpreted by overimposing the isocurves of the minimax function over those of the minisum function for the ranges (λ_1,λ_2) and

(μ_1,μ_2) respectively. But for problem P4.3.1, a set of iso-curves corresponds to only one weight γ and a specific efficient solution. This limitation does not allow a single graphical analysis of cost tradeoffs resulting from alternate locations.

4.7 Lagrangian Duality Interpretation of the Efficient Set

Let a point in the objection space (see Figure 3.5) be (λ,μ), then there exists $X \varepsilon R^2$ such that

$$\lambda = \max_{1 \leq i \leq n} \{w_i \|X - C_i\| + r_i'\} = F_2(X),$$

and

$$\mu = \sum_{i=1}^{n} \frac{w_i}{A_i} \int\int_{R_i} \|X - P_i\| dP_i = F_1(X).$$

For a specific $u_0 > 0$,

$$\theta(u_0) = \min_{X \varepsilon R^2} \{F_1(X) + u_0(F_2(X) - \lambda_0)\}$$

or, it is the minimization of $\mu + \mu_0\lambda - \mu_0\lambda_0$ over points of the objective space, which can be interpreted as finding the supporting hyperplane at the point (λ_0,μ_0) where λ_0, μ_0 relate to u_0 in the way described earlier. $-u_0$ is the slope of this supporting hyperplane at (λ_0,μ_0).

For example, if $(\lambda_0,\mu_0) = (45.,84.7)$ then from Figure 4.1 or 4.2 $\gamma_0 \approx 0.75$ and since

$$\gamma_0 = \frac{1}{1+u_0} \Rightarrow u_0 = \frac{1}{\gamma_0} - 1, \quad u_0 = 1.33 - 1 = 0.33$$

and the slope of the supporting hyperplane at (45.,84.7) is -0.33, which means that a gain in the minisum function results in a loss in the minimax function at a rate of 33%, Figure 4.3 illustrates this example.

4.8 Constrained Deterministic Problem

This method developed to solve the following problem

$$\underset{X \in R^2}{\text{minimize}} \sum_{i=1}^{n} \frac{w_i}{A_i} \int \int_{R_i} ||X - P_i|| dP_i \qquad \text{(P4.8.1)}$$

subject to

$$\max_{1 \leq i \leq n} \{v_i ||X - C_i|| + r_i'\} \leq \lambda$$

can be used to solve the deterministic constrained version obtained by considering the centroids of the rectangular regions

$$\underset{X \in R^2}{\text{minimize}} \sum_{i=1}^{n} w_i ||X - C_i|| \qquad (4.8.2)$$

subject to

$$\max_{1 \leq i \leq n} \{v_i ||X - C_i||\} \leq \lambda$$

It is clear that problem P4.8.2 is the limiting case of problem P4.8.1 as the areas of the rectangular regions are reduced to zero. From Figure 2.3, if the area of the rectangular region is monotonically reduced to zero, the isocurves of the function

$$f_i(X) = \frac{w_i}{A_i} \int \int_{R_i} ||X - P_i|| dP_i$$

converge toward the isocurves of the weighted rectilinear distance to the centroid C_i, i.e., $w_i ||X - C_i||$.

Also, as the area R_i is continuously reduced to zero,

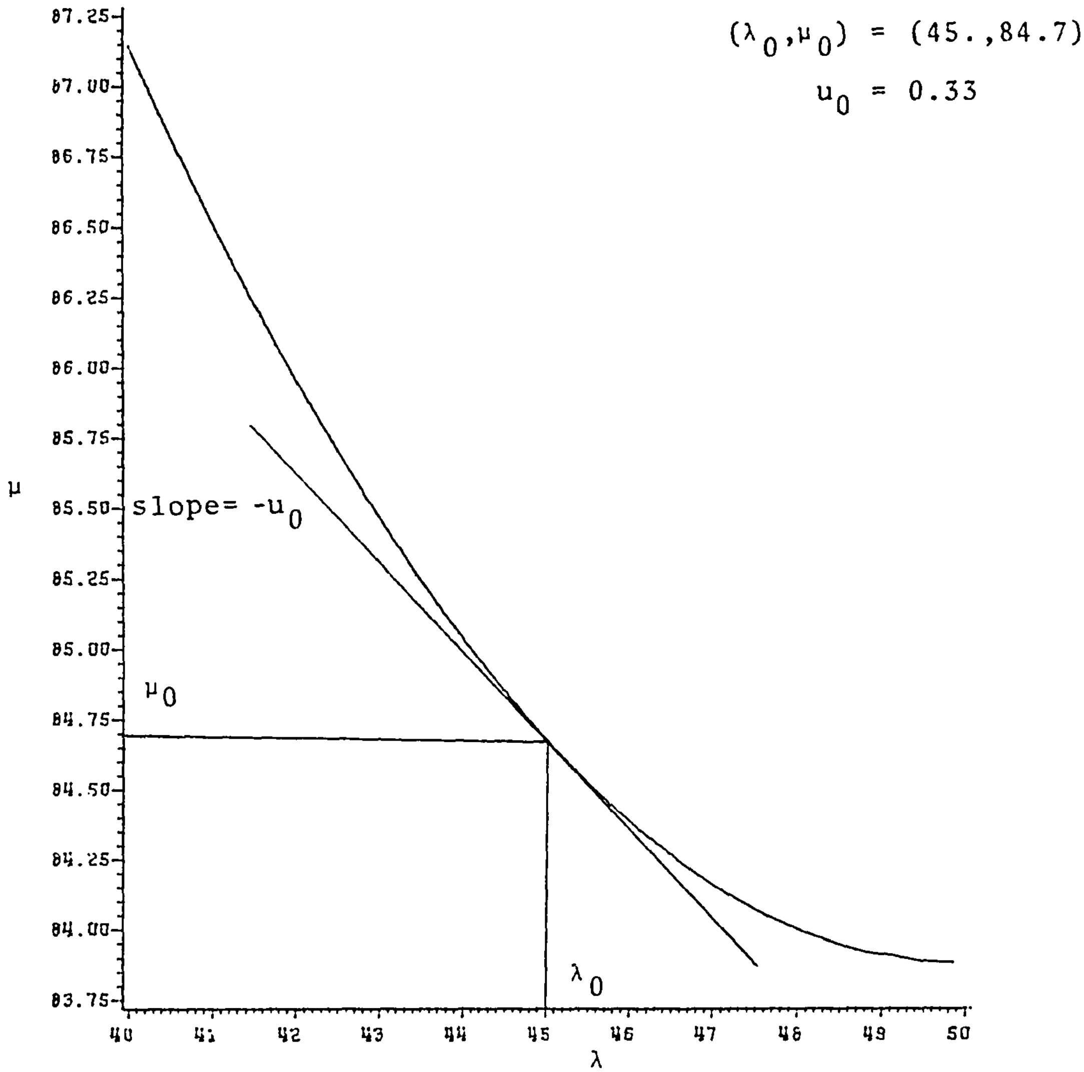


Figure 4.3 A geometrical interpretation of the dual variable.

the quantity

$$r_i = w_i \cdot [\frac{(a_{i_2} - a_{i_1}) + (b_{i_2} - b_{i_1})}{2}]$$

will converge to zero (because $(a_{i_2} - a_{i_1})$ and $(b_{i_2} - b_{i_1})$ converge to zero). By using rectangular region (with centroids C_i) of a very small area, and making minor adjustments to the solution technique (described in section 3.2.2), then P4.8.2 can be readily solved.

Consider the following example problem presented in McGinnis and White (1978)

there are five existing facilities

Table 4.1 Example Problem

i	c_{i_1}	c_{i_2}	w_i	v_i
1	1	14	1	1
2	2	10	1	2
3	3	15	1	1
4	7	9	1	1
5	7	12	1	2

where the weights w_i are used to compute the minisum cost function and the weight v_i is used to compute the minimax cost function.

Let $E = (0.0001)^2$ be the area of each rectangular region, then the unconstrained minisum solution is the point S= (3.,12.). The optimal minimax solution is the line segment defined by the endpoints $M_1 = (3.5,12.)$ and $M_2 = (4.75,10.75)$ (see Figure 4.4).

The following critical interval is computed $[\lambda_1, \lambda_2] = [7,8]$. And as λ varies from 7 to 8, the solution of problem

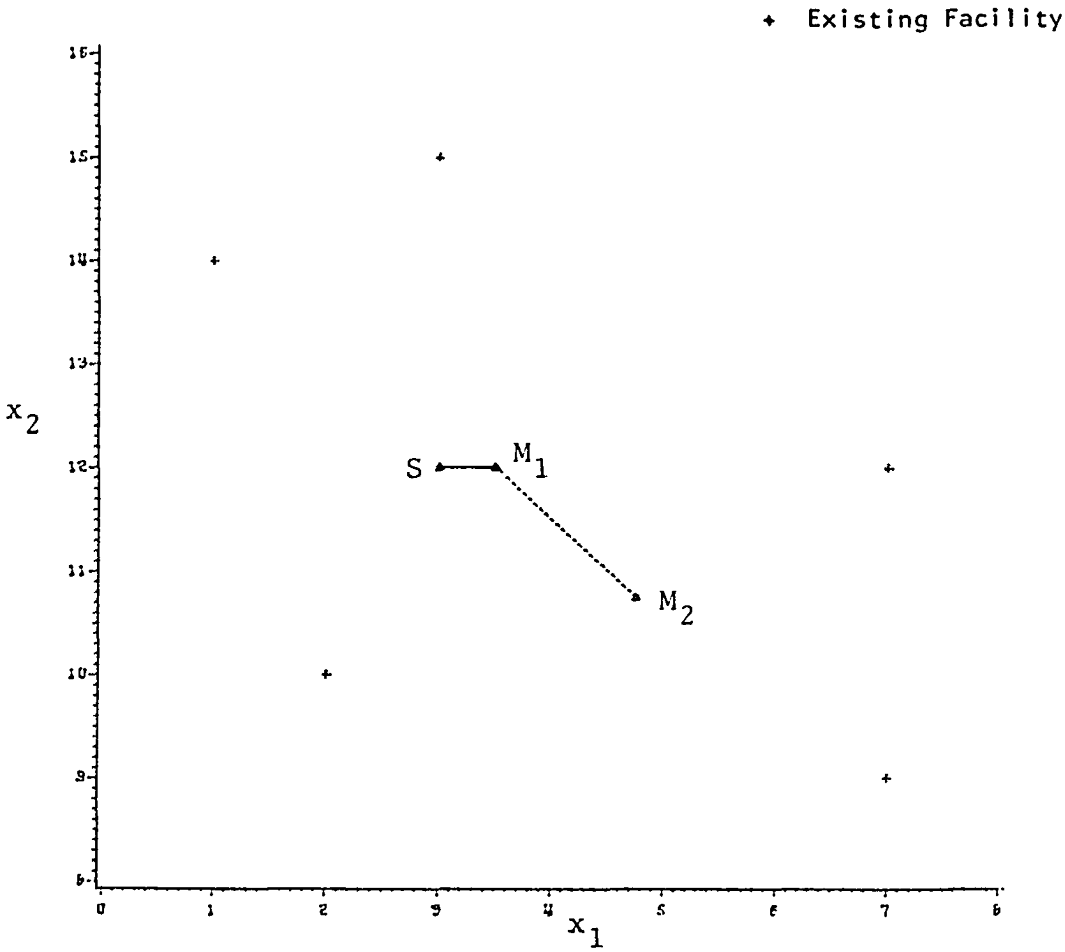


Figure 4.4 Minimax and minisum solutions for sample problem in Table 4.1.

P4.8.2 generates the following efficient set

$$E = \{(x_1,x_2) \in R^2 | 3. \leq x_1 \leq 3.5, \quad x_2 = 12\}.$$

The point (3.5,12.), which is a minimax solution, is generated for $\lambda = 7$. The point (3.,12.), which is the minisum solution, is for $\lambda = 8$. These results confirm the ones by McGinnis and White (1978), which they obtained by solving the following location problem

$$\underset{X \in R^2}{\text{minimize}}[\alpha(\sum_{i=1}^{n} w_i\|X - C_i\|) + (1 - \alpha)(\max_{1 \leq i \leq n} \{v_i\|X - C_i\|\})] \qquad (P4.8.3)$$

But, contrary to their statement, the other minimax solutions are not efficient points.

Another relevant point is that the point (3.5,12.) is optimal for P4.8.3 when $0 \leq \alpha \leq \frac{2}{3}$ and the point (3.,12) is optimal for $\frac{2}{3} \leq \alpha \leq 1$. (The author was able to verify these results by solving the probabilistic version of P4.8.3 with areas = $(0.0001)^2$).

McGinnis and White (1978) state that the points on the segment connecting these two points are also efficient, but they do not give the corresponding weights α for which these other efficient points solve P4.8.3. A closer study of problem P4.8.3 showed that all the efficient points are actually alternate solutions of P4.8.3 for $\alpha_0 = \frac{2}{3}$ (optimal function value 16.6667). These findings give more weight to the arguments given earlier favoring the constrained criterion approach over the bicriteria (or weighted sum) approach for generating efficient solutions. According to problem P4.8.3, all efficient solutions for the given example (of Table 4.1) are equivalent since they result from one single weight $\alpha = \frac{2}{3}$. But for these points, the minimax function varies from 7. to 8. (over 14% variation). The variation for the mimisum function (21. to 21.5) is small (results from flatness of the minisum function around the optimal point).

With this example, it was established that the solution method developed for the constrained minisum function with rectangular regions can be readily used for solving the deterministic version (with centroids). It was also shown that the constrained criterion approach is superior to the bicriteria formulation for generating efficient solutions.

4.9 Applications to Location Problems with Metric Constraints

Schaefer and Hurter (1974) studied the following problem

$$\underset{X \in R^2}{\text{minimize}} \sum_{i=1}^{n} w_i \|X - C_i\| \qquad \text{(P4.9.1)}$$

subject to

$$\|X - C_i| \leq \lambda_i, \quad i = 1,\ldots,n$$

They proposed a dual based algorithm to find the solution to P4.9.1. They also investigated the following special case

$$\underset{X \in R^2}{\text{minimize}} \sum_{i=1}^{n} w_i \|X - C_i\| \qquad \text{(P4.9.2)}$$

subject to

$$\|X - C_i\| \leq \lambda \quad \text{for all } i$$

In this problem the point X is constrained to be within the same distance λ, of the existing facilities C_i. P4.9.2 is equivalent to

$$\underset{X \varepsilon R^2}{\text{minimize}} \; F_1(X) = \sum_{i=1}^{n} w_i \| X - C_i \| \qquad \text{(P4.9.3)}$$

subject to

$$\max_{1 \leq i \leq n} \{ \| X - C_i \| \} \leq \lambda$$

Note that P4.9.3 is similar to P4.8.2 where all v_i's are equal to one.

Therefore, the algorithm used in section 4.8 to solve P4.8.2 can also solve P4.9.3.

It will now be shown how problem 4.9.1 (with general constraint bounds) can be solved by a subtle modification of the general algorithm developed in this research effort.

In Chapter III the algorithm was described in its general form. In section 4.7, the algorithm was slightly modified to solve the centroid formulation. In step 5 in the algorithm, the extreme points of the feasible set are found. The problem solved is:

$$\underset{X \varepsilon R^2}{\text{minimize}} \; \sum_{i=1}^{n} w_i \| X - C_i \| \qquad \text{(P4.9.4)}$$

subject to

$$\max_{1 \leq i \leq n} \{ v_i \| X - C_i \| \} \leq \lambda$$

where $\lambda \varepsilon [\lambda_1, \lambda_2]$.

The convex polyhedron defined by the constraint can be found as follows

$$v_i \| X - C_i \| \leq \lambda \text{ for all } i$$

then the feasible set $S(\lambda)$ can be defined as

$$S(\lambda) = \{X \in R^2 | v_i \| X - C_i \| \le \lambda\}$$
$$= \{(x_1, x_2) \in R^2 | e_2 \le x_1 + x_2 \le e_1, e_4 \le - x_1 + x_2 \le e_3\}$$

where

$$e_1 = \min_{1 \le i \le n} \left(\frac{\lambda}{v_i} + c_{i_1} + c_{i_2} \right)$$
$$e_2 = \max_{1 \le i \le n} \left(- \frac{\lambda}{v_i} + c_{i_1} + c_{i_2} \right)$$
$$e_3 = \min_{1 \le i \le n} \left(\frac{\lambda}{v_i} - c_{i_1} + c_{i_2} \right)$$
$$e_4 = \max_{1 \le i \le n} \left(- \frac{\lambda}{v_i} - c_{i_1} + c_{i_2} \right)$$

and

$$C_i = (c_{i_1}, c_{i_2}).$$

When solving the special case P4.9.1, set $v_i = 1$ for all i's, replace λ by λ_i for each constraint i, and compute the modified e_1', e_2', e_3', e_4'. Therefore, if the calculations of step 5 are modified as explained above, the algorithm will be capable of solving any rectilinear minisum problem with metric constraints.

4.10 Regions vs. Centroids

The bicriteria location problem with regions was defined as follows:

$$\underset{X \in R^2}{\text{minimize}} \sum_{i=1}^{n} \frac{w_i}{A_i} \int\int_{R_i} \| X - P_i \| dP_i \qquad \text{(P4.10.1)}$$

subject to

$$\max_{1 \le i \le n} \{w_i \| X - C_i \| + r_i'\} \le \lambda$$

for $\lambda \in [\lambda_1, \lambda_2]$. The deterministic centroid formulation is

$$\underset{X \varepsilon R^2}{\text{minimize}} \sum_{i=1}^{n} w_i||X - C_i|| \qquad \text{(P4.10.2)}$$

subject

$$\max_{1 \leq i \leq n} \{w_i||X - C_i||\} \leq \lambda'$$

for $\lambda' \varepsilon [\lambda_1', \lambda_2']$.

With two sample problems, this section will illustrate that the centroid formulation is not a good approximation to the probabilistic formulation, by showing the disparity between the efficient sets generated by both models.

Consider sample problems A2 and A3 from Appendix A. Figures 4.5 and 4.6 show the efficient sets for both models. For sample problem A3, the efficient set for the deterministic model is a singleton because the unconstrained minimax and minisum solution intersect at that single point. The respective possible deviations in the minimax function are 14% and 11.5%. These two examples show that the centroid approach does not approximate the probabilistic model well. On the other hand, the probabilistic model can very well approximate the deterministic model by monotonically shrinking the regions.

4.11 Summary

In this chapter, the bicriteria model formed by the weighted sum of the minisum and minimax function was investigated. It is shown that all efficient solutions generated by either constrained models are also properly efficient (if $\lambda \varepsilon (\lambda_1, \lambda_2)$ or $\mu \varepsilon (\mu_1, \mu_2)$). The bicriteria model and the constrained models are theoretically equivalent, but it is shown

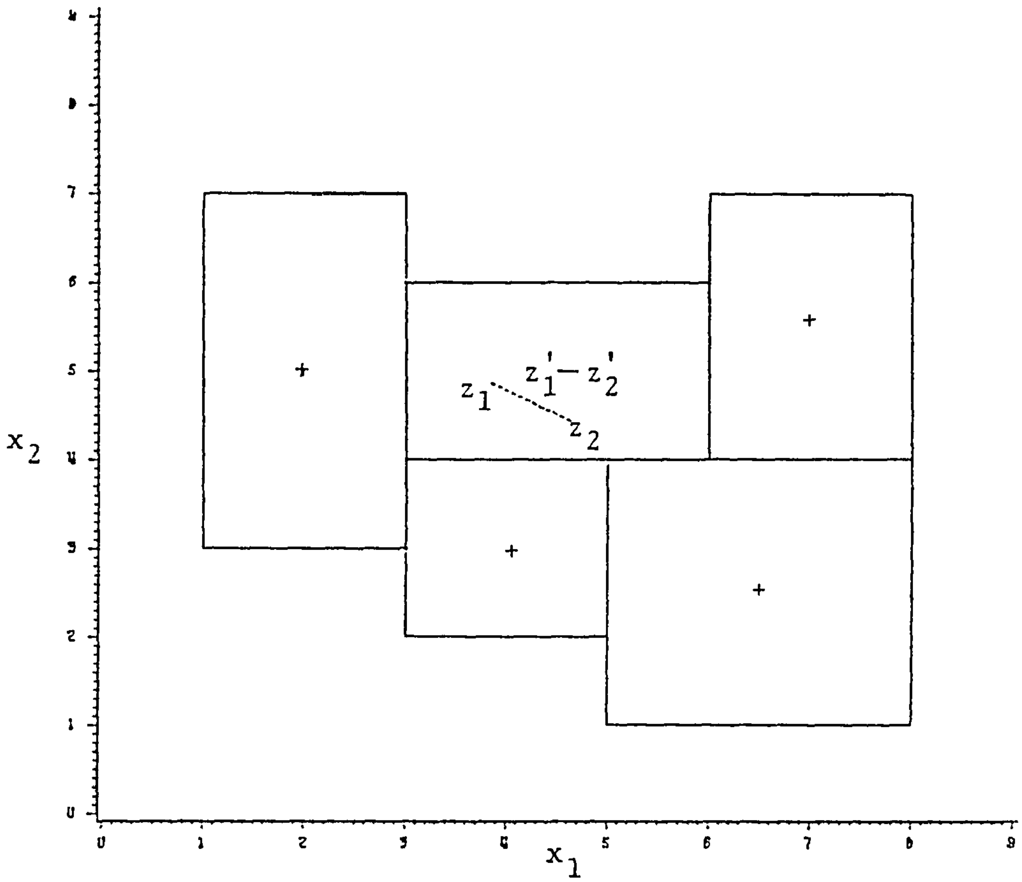


Figure 4.5 Efficient sets for sample problem A2.
(z_1, z_2): area demand formulation.
(z_1', z_2'): point demand formulation.

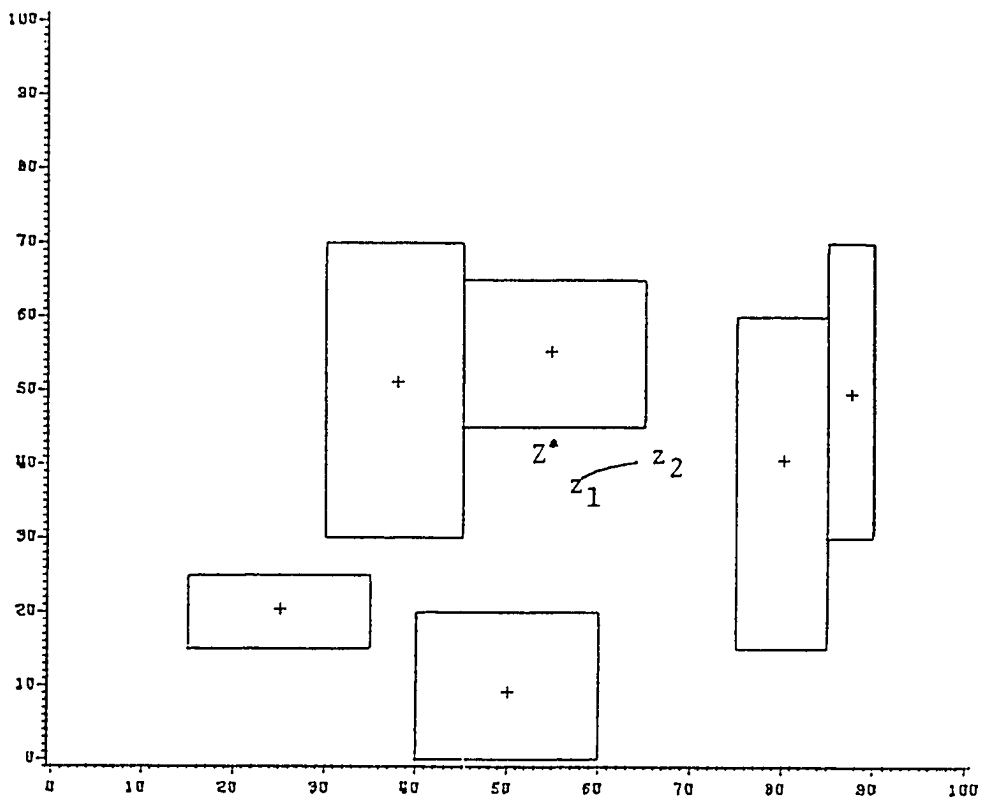


Figure 4.6 Efficient sets for sample problem A3.
(z_1, z_2): area demand formulation
z : point demand formulation

that it is more efficient and simpler to generate nondominated solutions using the constrained criterion approach. When solving the bicriteria model, a critical range $(\gamma_1, \gamma_2) \subseteq (0,1)$ is found for which all properly efficient solutions are generated, and usually $(\gamma_1, \gamma_2) \neq (0,1)$. This result does not contradict developments by Geoffrion (1968) but only gives more insight into the bicriteria model, and its relationship with the two constrained criterion models.

The constrained model with regions can give an excellent approximation of the deterministic version (with centroids). The approximation was verified by solving, among others, an example by McGinnis and White (1978). For a large population, the deterministic model would give a very good solution if a large number of points is taken, but the increased accuracy will be achieved at a greater computational cost.

A deterministic minisum location model with metric constraints proposed by Schaefer and Hurter (1974) can also be efficiently and quickly solved after a few minor changes in the algorithm, as explained above. Schaefer and Hurter's dual algorithm can handle any norm, but it requires solving a series of unconstrained Weber problems for which only approximate algorithms are used, as compared to the method developed in Chapter IV which is simple, straightforward and fast.

CHAPTER V

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Summary

Several single facility location problems with rectangular regions have been investigated in this research effort. The central model involves both the minisum and the minimax cost functions, where one is the optimization criterion, and the other is bounded and acts as a constraint on the location of the new facility.

In Chapter II, two minimax formulations with probabilistically distributed demand points are investigated. One model computes average weighted distances in each rectangular region, and the other one computes maximum weighted distances to any point in each region. Both probabilistic models could be made to solve the deterministic approach (by considering very small rectangular regions centered at each centroid), but the deterministic model did not approximate the probabilistic models well. A relationship between all three minimax cost functions is given, in which the deterministic cost function is a lower bound to the probabilistic cost function with expected distances, which in turn is a lower bound to the other probabilistic minimax cost function.

When weights are also assumed probabilistic in nature, several minimax formulations are analyzed. An expected value criterion model was proven to be a convex problem, but the resulting formulations are too complex to optimize. Lower and upper bound approximation functions are developed as an alternate way to approach the problem. A chance constraints model is also studied, its deterministic equivalent is similar to a deterministic minimax problem with metric constraints. All models were shown to be convex problems (except a fractile formulation), and optimization techniques could be easily developed for most of them. In all the other chapters, the weights are assumed known deterministically.

The central formulation with both the minisum and minimax functions was analyzed, for the two types of probabilistic minimax functions. When the minimax function with expected distances is used, the resulting constrained problem is found to be equivalent to a minisum location problem with regions, with new weights which are functions of the optimal Lagrange multipliers of the dual problem. The other formulation, with maximum distances, is chosen for a thorough analysis because the resulting minimax function better reflects the very conservative approach one has to adopt when considering emergency type location problems (account for the worst possible outcome). In Chapter III, it is first shown that the minimax and minisum criterion investigated independently are antagonistic, and it is therefore realistic and superior as a modeling approach to combine both criteria into one single model.

Minimizing the minimax function subject to a bound on the minisum function is shown to be equivalent to a model where the roles of the two functions are reversed. Using multi-criteria optimization and duality theories, it is also shown that all nondominated solutions can be generated when solving for a specific interval of values for the bound on the constraint. The constrained minisum model is solved, because the feasible set has a simple, geometrical shape (diamond), and can be easily represented analytically. A specialized solution technique was developed, which uses geometrical and analytical properties of both the minisum and minimax cost functions. The solution technique, in addition to solving the constrained criterion model for the appropriate range of bound $[\lambda_1, \lambda_2]$, also solves both unconstrained single criteria. Graphical representations of the efficient set in both the decision and objective spaces are given. Also, it is shown how the isocurves of the minisum and minimax functions can be used to find the unconstrained optimal, and simultaneously to rank points, and to even generate the efficient points graphically.

In Chapter IV, with the aid of duality theory, the constrained criterion approach is proven to be equivalent to a bicriteria model where a weighted sum of the minimax and minisum functions is minimized. But, it is both practically and computationally more advantageous to solve the constrained minisum model.

The algorithm developed to solve the constrained minisum model can easily be altered to solve the deterministic formulation of the problem, as well as to handle the deterministic minisum problem with metric constraints, thus demonstrating its versatility and power in handling several types of location problems. Example problems are solved to illustrate all situations encountered.

5.2 Conclusions

In Chapter II, the three unconstrained minimax models are analyzed, and the most relevant conclusion is that the deterministic model can be used at best as a heuristic for solving either probabilistic formulations. The deterministic model is closer to the model with expected value distances. Solution methods developed for the probabilistic models are not more complicated than the techniques for the deterministic model, and they can both solve the deterministic problem. When using probabilistic weights, the resulting formulations are often more complex, but chance constraints are equivalent to metric constraints and solution techniques similar to those used for deterministic problems can be implemented.

In Chapters III and IV, the bicriteria location problem with regions is investigated. The minimax and minisum criteria are natural choices in such problems, since they measure the interest of a few against the interest of the masses, which often leads to unfair contradictions.

A unified approach is developed where several location problems are linked into one model. When this central model is solved, all the other problems are also solved; it can also be made to handle the deterministic bicriteria problem. The main reason for such versatility is that all bicriteria location problems, or location problems with metric constraints, can be transformed into Weber problems among rectangular regions and discrete points. Another important conclusion is the remarkable ease with which many location problems can be solved using interactive graphics.

5.3 Recommendations for Further Research

Several direct extensions of this research effort are possible, and are listed below:

1) Development of solution procedures for the Euclidean metric cases.
2) Generalize to the multifacility problems with rectilinear or Euclidean metrics.
3) Computational experience for the models with probabalistic weights, especially for the upper and lower bound functions derived for the expected value criterion, and for the chance constraints models.
4) Develop systematic graphical solution procedures for both the single and multifacility cases.
5) Investigate the effects of different probability distributions for the existing population.
6) Use differently shaped regions (discs, hexagons, etc.).

7) Study the bicriteria location problem with minisum and maximin criteria.

8) Development of a solution technique for the location-allocation minimax problem with rectangular regions.

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APPENDIX A

Problem A1 Steffen (1978)

Facility	Rectangular Region	Weight
1	[5.0,7.5] x [7.5,10.]	1
2	[10.,14] x [5.,7.5]	2
3	[16.,18.5] x [3.5,7.5]	3
4	[12.5,15.] x [0.5,3.5]	4
5	[7.5,11.] x [1.0,3.5]	5

Problem A2 Aly (1975)

Facility	Rectangular Region	Weight
1	[1.,3.] x [3.,7.]	8
2	[3.,5.] x [4.,6.]	4
3	[6.,8.] x [4.,7.]	6
4	[3.,5.] x [2.,4.]	4
5	[5.,8.] x [1.,4.]	5

Problem A3

Facility	Rectangular Region	Weight
1	[15.,35.] x [15.,25.]	6
2	[30.,45.] x [30.,70.]	4
3	[45.,65.] x [45.,65.]	2
4	[75.,85.] x [15.,60.]	5
5	[85.,90.] x [30.,70.]	8
6	[40.,60.] x [0.,20.]	3

Problem A4

Facility	Rectangular Region	Weight
1	[15.,35.] x [15.,25.]	12
2	[30.,45.] x [30.,70.]	5
3	[45.,65.] x [45.,65.]	4
4	[80.,95.] x [80.,95.]	15
5	[35.,55.] x [0.,20.]	13

APPENDIX B

NONLINEAR BICRITERIA OPTIMIZATION

$$\text{Let } \Lambda = \{\lambda \in R^n | \lambda_i \geq 0, \quad i = 1,\ldots,n \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}$$

$$\text{and } \Lambda^+ = \{\lambda \in R^n | \lambda_i > 0, \quad i = 1,\ldots,n \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}$$

Geoffrion (1968) studied the following scalar minimization problem

$$P_\lambda : \underset{x \in S}{\text{minimize}} \sum_{i=1}^{n} \lambda_i f_i(x)$$

for some parameter $\lambda = (\lambda_1,\ldots,\lambda_n) \in \Lambda^+$, and he proved the following theorem:

<u>Theorem B.1</u>

Let S be a convex set, and let the f_i's be convex on S, then $x^0 \in S$ is properly efficient if and only if x^0 is optimal in P_λ for some $\lambda \in \Lambda^+$.

This important result gives a parametric procedure for generating the set E^p of all properly efficient solutions. If one is interested in generating all efficient points, the following auxiliary problem, formulated by Wendell and Lee (1977), is to be solved:

$$\psi(\lambda): \underset{x \in S}{\text{minimize}} \sum_{i=1}^{n} f_i(x)$$

$$\text{subject to}$$

$$f(x) \leq \lambda$$

for $\lambda = (\lambda_1, \ldots, \lambda_n)$ and such that $\exists\, \bar{x} \in S$ and $\lambda = f(\bar{x}) = (f_1(\bar{x}), \ldots, f_n(\bar{x}))$.

It is assumed now that n = 2, and the following problem is the focus point of this appendix:

$$\text{PB.1} \quad \underset{x \in S}{\text{minimize}}(f_1(x), f_2(x))$$

where $f_1(x)$ and $f_2(x)$ are assumed convex on S. Solving PB.1 is equivalent to generating the set E (or E^p).

Recall the partial ordering of the plane R^2: for real numbers a_1, a_2, b_1 b_2, then

$(a_1, a_2) \leq (b_1, b_2)$ iff $a_1 \leq b_1$ and $a_2 \leq b_2$ also,

$(a_1, a_2) < (b_1, b_2)$ iff $a_1 \leq b_2$ and $a_2 \leq b_2$ and

$(a_1, a_2) \neq (b_1, b_2)$.

If $(a_1, a_2) < (b_1, b_2)$ then (a_1, a_2) dominates (b_1, b_2)

Let $f = (f_1, f_2)$ define a mapping from S into R^2 then $x^0 \in S$ is an efficient point if there exists no other point $x \in S$ such that f(x) dominates $f(x^0)$, i.e., $\nexists\, x$, $x \in S$ such that $f(x) < f(x^0)$.

The most recent developments in multicriteria optimization have focused mainly on two steps: the first step involves finding the set of efficient solutions E. If E is not a singleton, then the second step consists of defining the preference structure of the decision maker (it is a process involving value judgments), and assuming that this preference structure is characterized by a multiattribute utility function $u(f_1(x),f_2(x))$, then an efficient solution is chosen such that it maximizes u(.). This is equivalent to solving:

$$\underset{x \, \varepsilon \, E}{\text{maximize}} \; u(f_1(x),f_2(x)$$

Some other procedures (for example, Sadagopan and Ravindran (1982)) solve both step one and two iteratively and interactively. These methods use progressively revealed preferences from the decision maker. Hershley et al. (1982) discussed possible sources of bias when assessing procedures for utility functions. They raised several methological and empirical questions regarding the uniqueness of the utility function for a given person. For this research effort, it was decided that the bicriteria location problem is solved when the efficient set is sufficiently generated. A graphical representation of this set among the rectangular region (decision space), and in the objective space, would offer the decision maker more flexibility when making a final choice. Many different preference structures could be constructed by the decision maker, and they could be different for other persons.

Several methods have been developed for generating efficient solutions. Two methods are adopted in this research: the parametric approach of Geoffrion, which involves solving P_λ, and a constrained criteria approach which solves one of the two equivalent subproblems:

$$P_1(\lambda): \quad \underset{x \in S}{\text{minimize}}\ f_1(x)$$

subject to

$$f_2(x) \le \lambda$$

and

$$P_2(\mu): \quad \underset{x \in S}{\text{minimize}}\ f_2(x)$$

subject to

$$f_1(x) \le \mu .$$

For the constrained criterion approach, efficient solutions are generated for definite ranges $[\lambda_1, \lambda_2]$ and $[\mu_1, \mu_2]$, as no other values for λ or μ need to be considered. These intervals are determined as follows:

let $\lambda_1 = \inf\{f_2(x) | x \in S\}$
and define $A = \{x \in S | f_2(x) = \lambda_1\}$
if $A = \emptyset$ then set $\mu_2 = +\infty$
if $A \neq 0$ then $\mu_2 = \inf\{f_1(x) | x \in A\}$.

Similarly, let $\mu_1 = \inf\{f_1(x) | x \in S\}$
and define $B = \{x \in S | f_1(x) = \mu_1\}$.
if $B = \emptyset$ then set $\lambda_2 = +\infty$
if $B \neq 0$ then $\lambda_2 = \inf\{f_2(x) | x \in B\}$.

It is easy to verify that $P_1(\lambda)$ or $P_2(\mu)$ are feasible or nontrivial only for those ranges. For example, consider $P_1(\lambda)$, if $\lambda < \lambda_1$, then the problem is not feasible since there exist no points in S that will give a value of $f_2(.)$ smaller than λ_1. If $\lambda > \lambda_2$ then an unconstrained optimal solution x^* of the criterion f_1 in S will satisfy the constraint $f_2(x^*) \leq \lambda$.

The following theorem proved by Bacopoulos and Singer (1977) and also by Sadagapan and Ravindran (1982), characterizes an efficient solution for problem PB.1.

Theorem B.2

A solution $\overline{x} \varepsilon S$ is efficient for problem PB.1 iff $\overline{x}$ solves $P_1(\overline{\lambda})$ (resp. $P_2(\overline{\mu})$ for some $\overline{\lambda} \varepsilon [\lambda_1, \lambda_2]$ (resp. $\overline{\mu} \varepsilon [\mu_1, \mu_2]$)).

Let $x_1^*(\lambda)$ be a solution of $P_1(\lambda)$ and $x_2^*(\mu)$ be a solution of $P_2(\mu)$. If $\lambda = +\infty$ (resp. $\mu = +\infty$) then $P_1(\infty)$ (resp. $P_2(\infty)$) is the unconstrained minimization of f_1 (resp. f_2) over the set S.

let $x_1^* = x_1^*(\infty)$ and efficient

and $x_2^* = x_2^*(\infty)$ and efficient

(i.e., when problems $P_i(\infty)$, i = 1,2, have alternate optimal solutions, then take x_i^* such that $f_k(x_i^*)$ is minimum $(k \neq i)$.)

For $[\lambda_1, \lambda_2]$ as defined earlier, $f_1(x_1^*(\lambda_1))$ is the maximum achievable value of f_1 without sacrificing f_2, and $f_1(x_1^*(\lambda_2))$ is the lowest value achieved by $f_1(x(\lambda))$ for $\lambda \varepsilon [\lambda_1, \lambda_2]$.

then $f_1(x^*(\lambda)) \varepsilon [\mu_1, \mu_2]$ for $\lambda \varepsilon [\lambda_1, \lambda_2]$

and similarly: $f_2(x^*(\mu)) \varepsilon [\lambda_1, \lambda_2]$ for $\mu \varepsilon [\lambda_1, \lambda_2]$.

Definition: <u>Slater's Constraint Qualification</u>

(Mangasarian (1968))

Let S be a convex nonempty set in R^2, the convex function g on S which defined the convex feasible region $S^\lambda = \{x | x \in S, g(x) \leq \lambda\}$ is said to satisfy Slater's constraint qualification on S if there exists an $\hat{x} \in S$ such that $g(\hat{x}) - \lambda < 0$.

Also, the Lagrangian dual of problem $P_1(\lambda)$ (with respect to the constraint $f_2(x) - \lambda \leq 0$) is

$$D_1(\lambda): \underset{u \geq 0}{\text{maximize}} \{ \inf_{y \in S} (f_1(x) + u(f_2(x) - \lambda)) \}$$

u is the dual variable.

For a pair (x,u) such that x is a feasible point in $P_1(\lambda)$, and u is feasible in $D_1(\lambda)$(i.e., $u \geq 0$) then by the "Weak Duality" theorem

$$f_1(x) \geq \theta(u)$$

where $\theta(u) = \inf_{y \in S} (f_1(y) + u(f_2(y) - \lambda))$

<u>Theorem B.3</u>

Let x^0 be an efficient solution of problem PB.1, then there exist scalars $\lambda^0 \in [\lambda_1 \lambda_2]$ and $\mu^0 \in [\mu_1, \mu_2]$ such that x^0 solves $P_1(\lambda^0)$ and $P_2(\mu^0)$, and $\lambda^0 = f_2(x^0)$ and $\mu^0 = f_1(x^0)$.

<u>Proof</u>: From theorem B.2, x^0 efficient implies that x^0 solves $P(\lambda^0)$ for some $\lambda^0 \in [\lambda_1, \lambda_2]$, and x^0 solves $P_2(\mu^0)$ for some $\mu^0 \in [\mu_1, \mu_2]$; three cases will be considered.

<u>Case 1</u>: $x^0 = x_2^*$, this solution can be obtained by either solving $P_1(\lambda_1)$ or $P_2(\mu_2)$. Also, $\lambda^0 = \lambda_1 = f_2(x_2^*) = f_2(x^0)$ and

$\mu^0 = \mu_2 = f_1(x_2^*) = f_1(x^0)$.

Case 2: $x^0 = x_1^*$, can be obtained by solving $P_1(\lambda_2)$ or $P_2(\mu_1)$ and $\lambda^0 = \lambda_2 = f_2(x_1^*) = f_2(x^0)$ and $\mu^0 = \mu_1 = f_1(x_1^*) = f_1(x^0)$.

Case 3: $x^0 \in \{x_2^*, x_1^*\}$, since x^0 is efficient, then $\exists \lambda^0$, $\lambda^0 \notin (\lambda_1, \lambda_2)$ such that x^0 solves $P_1(\lambda^0)$.

Also, $x^0 \neq x_1^*$ implies $f_2(x) - \lambda^0$ for $x \in S$ satisfies Slater's constraint qualification (take $x = x_1^*$, then $f_2(x_1^*) = \lambda_1 < \lambda^0$).

The necessary conditions for satisfying the "Strong Duality" theorem are verified and therefore, there exists $u^0 \geq 0$ such that (x^0, u^0) solve $D_1(\lambda^0)$ and $f_1(x^0) = f_1(x^0) + u^0(f_2(x^0) - \lambda^0)$ with the complementary slackness conditions holding

$$u^0(f_2(x^0) - \lambda^0) = 0$$

x^0 is the minimizer of $f_1(x) + u^0(f_2(x) - \lambda^0)$ over S. If $u^0 = 0$, then x^0 would be the minimizer of $f_1(x)$ over S and $x^0 = x_1^*$ would be a solution. But $x^0 \neq x_1^*$, and therefore $u > 0$ and the complementary slackness condition implies that $\lambda^0 = f_2(x^0)$. This proves that at the optimum, the constraints of problem $P_1(\lambda)$ and of problem $P_2(\mu)$ are tight if $\lambda \in [\lambda_1, \lambda_2]$ and $\mu \in [\mu_1, \mu_2]$.

In summary, for any efficient solution x^0 of PB.1 there exist a pair $(\lambda^0, \mu^0) = (f_2(x^0), f_1(x^0))$ such that x^0 solves $P_1(\lambda^0)$ and $P_2(\mu^0)$.

APPENDIX C

CHARACTERIZATION OF PROPERLY EFFICIENT SOLUTION

The following definitions and propositions are adapted from the analysis of multicriteria maximization of concave functions of Benson and Morris (1977), to bicriteria minimization of convex functions.

$$\text{let VMP: } \underset{x \,\varepsilon\, R^2}{\text{minimize}} \; (f_1(X), f_2(X))$$

where $f_1(X)$ and $f_2(X)$ are convex functions.

Definition C.1

$X_0 \,\varepsilon\, R^2$ is said to be k^{th} entry efficient solution of VMP where $k \,\varepsilon\, \{1,2\}$, if $f_k(X) < f_k(X_0)$ for some $X \,\varepsilon\, R^2$ implies that $f_j(X) > f_j(X_0)$ for $j \,\varepsilon\, \{1,2\}$ and $j \neq k$.

Definition C.2

X_0 is said to be properly k^{th} entry efficient of VMP, where $k \,\varepsilon\, \{1,2\}$, when it is k^{th} entry efficient for VMP and there exists a scalar $M_k > 0$ such that for each $X \,\varepsilon\, R^2$ satisfying $f_k(X) < f_k(X_0)$ then $f_j(X) > f_j(X_0)$ and

$$\frac{f_k(X_0) - f_k(X)}{f_j(X) - f_j(X_0)} \leq M_k \quad \text{for} \quad j \,\varepsilon\, \{1,2\} \text{ and } j \neq k$$

Proposition C.1

A point X_0 is an efficient solution of VMP if and only if it is a k^{th} entry sufficient solution of VMP for each

$k = \{1,2\}$.

Proposition C.2

A point X_0 is a properly efficient solution of VMP if and only if it is a properly k^{th} entry efficient solution of VMP for each $k \varepsilon \{1,2\}$.

Benson and Morris derived necessary and sufficient conditions for an efficient solution to be properly efficient by studying the following problems:

$$P_k(b_j): \underset{X \varepsilon R^2}{\text{minimize}} \; f_k(X)$$

subject to

$$f_j(X) - b_j \leq 0 \quad \text{for} \quad j \neq k \quad \text{and } j,k \varepsilon \{1,2\}$$

The following definitions are by Geoffrion (1961)

Definition C.3

The perturbation function $v(.)$ associated with $P_k(b_j)$ is defined in R as

$$v(y) = \inf_{x \varepsilon R^2} \{f_k(X) | f_j(X) - b_j \leq y; \; j \neq k \varepsilon \{1,2\}\}$$

Definition C.4

Problem $P_k(b_j)$ is said to be stable if $v(0)$ is finite and there exists a scalar $M < 0$ such that

$$\frac{v(0) - v(y)}{|y|} \leq M \quad \text{for all } y \neq 0$$

If the stability fails to hold, then the ratio of improvement in the optimal value of $P_k(b_j)$ can be made as large as desired. Geoffrion (1968) states that stability is implied by all known constraint qualifications, thus if Slater's constraint qualification holds for $P_k(b_j)$ then it is also stable.

The following theorem is adopted from Benson and Morris (1977) to the bicriteria minimization problem.

Theorem C.1

Assume $f_1(X)$ and $f_2(X)$ are convex functions on the nonempty convex set S. Suppose X_0 is an efficient solution for VMP, then X_0 is a properly efficient solution for VMP if and only if $P_k(b_j^0)$ is stable for $k \varepsilon \{1,2\}$ and $b_j^0 = f_j(X_0)$ and $j \neq k$.

APPENDIX D

PROGRAMS DESCRIPTION AND SAMPLE OUTPUT

D.1 Program SUMCMAX

This program generates the efficient set for the minimax and minisum criteria as described by the algorithm in section 3.3.2. Two main tasks are accomplished; λ_1 and λ_2 are first computed. This is done by subroutines MINMAX, MINSUM and LMBDA. MINMAX solves the unconstrained minimax function (thus, finds λ_1), MINSUM solves the minisum problem and LMBDA computes λ_2 as described in the algorithm. The second task is performed by the subroutine DIAMND, efficient solutions are generated for values of $\lambda \varepsilon (\lambda_1, \lambda_2)$.

The input data consists of the number of regions, the coordinate dimensions of these rectangular regions and the weights associated with them. SUMCMAX allows for different weights when computing the minisum and minimax functions.

For the sample problem given (problem A2), the associated printout is given: SUMCMAX prints the problem's data, the solutions for the unconstrained minimax and minisum criteria, followed by the critical values λ_1 and λ_2. For $\lambda \varepsilon (\lambda_1, \lambda_2)$, efficient solutions are successively generated with their respective functions values.

D.2 Program DETERM

Program SUMCMAX is modified in order to solve the limiting case when all existing facilities are points. This is achieved by replacing subroutines MSRTFL, which executes the gradient reduction step for solving the minisum problem (described in section 3.3.1). MSRTFL in SUMCMAX is replaced by a modified version which finds all alternate solutions for the centroid formulation (when applicable). Also, the existing points are approximated by very small rectangular regions about them.

D.3 Program EXPDUAL

EXPDUAL is used to find the weight for which the corresponding weighted sum function (of the two criteria) yields the same efficient solution as the one generated by the constrained criterion method (for a given RHS value $\lambda \varepsilon (\lambda_1, \lambda_2)$).

A brief description of the solution technique is given in section 4.4. The input data consists of the number of regions, the values λ_1 and λ_2, the coordinates of each region, and its weight.

As illustrated in the sample output, the input data is first printed, then for each value $\lambda \varepsilon (\lambda_1, \lambda_2)$ (bound on the constraint for the constrained criterion formulation) the corresponding Lagrangian dual problem is solved, and all informations about both problems are given. For $\lambda \varepsilon (\lambda_1, \lambda_2)$ the resulting weight for the weighted sum criterion is $\gamma_0 = \frac{1}{1+u_0}$, where u_0 is the optimal dual solution.

D.4 Program NORMCT

This program is also generated from program SUMCMAX. NORMCT solves the single facility rectilinear location Weber problem with point demands and constraints on the distances from the new facility to each existing ones. The subroutine MINSUM (similar to the version in program DETERM) finds the unconstrained minisum solution. With a few moderate changes, DIAMND performs as before for the feasible set defined by the constraints. The input data includes the coordination of the small rectangular regions approximating the point demands (first-fourth columns), the weights used to compute the minisum function are in column five. The numbers in column six are set equal to one for all facilities since the bounds are on the distances (if weighted distances are to be bounded, the appropriate weights can be entered in column six).

Column seven included the bounds on the distances. The output reproduces the input data, then gives the unconstrained and constrained optimal solutions and functions values.

SAMPLE OUTPUT FOR PROGRAM SUMCMAX :

1.00000	3.00000	3.00000	7.00000	8.00000	8.00000
3.00000	6.00000	4.00000	6.00000	4.00000	4.00000
6.00000	8.00000	4.00000	7.00000	6.00000	6.00000
3.00000	5.00000	2.00000	4.00000	4.00000	4.00000
5.00000	8.00000	1.00000	4.00000	5.00000	5.00000

MINIMAX PROBLEM

ALTERNATE OPTIMAL SOLUTIONS
MINIMUM MINIMAX FUNCTION : F = 40.00

ANY POINT ON THE LINE SEGMENT JOINING THE FOLL. TWO POINTS IS OPTIMAL

X1= 3.67 , Y1= 4.67 X2= 4.00 , Y2= 5.00

MINISUM PROBLEM

SINGLE SOLUTION FOR THE X-COORD. PROBLEM : X* = 4.65

SINGLE SOLUTION FOR THE Y-COORD. PROBLEM : Y* = 4.42

LAMBDA1 = 40.00 LAMBDA2 = 49.87

EFFICIENT SOLUTION : 3.6615 4.6307
OBJECTIVE FUNCTION VALUE: 86.9819
BOUND ON THE CONSTRAINT: 40.2467

EFFICIENT SOLUTION : 3.6564 4.5947
OBJECTIVE FUNCTION VALUE: 86.8251
BOUND ON THE CONSTRAINT: 40.4933

EFFICIENT SOLUTION : 3.6512 4.5588
OBJECTIVE FUNCTION VALUE: 86.6723
BOUND ON THE CONSTRAINT: 40.7400

EFFICIENT SOLUTION : 3.6461 4.5228
OBJECTIVE FUNCTION VALUE: 86.5235
BOUND ON THE CONSTRAINT: 40.9867

EFFICIENT SOLUTION : 3.6410 4.4868
OBJECTIVE FUNCTION VALUE: 86.3789
BOUND ON THE CONSTRAINT: 41.2333

EFFICIENT SOLUTION : 3.6358 4.4508
OBJECTIVE FUNCTION VALUE: 86.2383
BOUND ON THE CONSTRAINT: 41.4800

EFFICIENT SOLUTION : 3.6307 4.4149
OBJECTIVE FUNCTION VALUE: 86.1018
BOUND ON THE CONSTRAINT: 41.7267

SAMPLE OUTPUT FOR PROGRAM EXPDUAL :

NUMBER OF REGIONS = 5

DLB = 40.00   DUB = 49.87

1.00	3.00	3.00	7.00	8.00
3.00	6.00	4.00	6.00	4.00
6.00	8.00	4.00	7.00	6.00
3.00	5.00	2.00	4.00	4.00
5.00	8.00	1.00	4.00	5.00

```
RHS VALUE =    43.947
OPTIMAL DUAL SOLUTION =    0.396
OPTIMAL PRIMAL SOLUTION = (4.17,4.68)
OPTIMAL DUAL FUNCTION =   85.057
MINISUM FUNCTION =  85.057
MINIMAX FUNCTION =  43.947
```

```
RHS VALUE =    44.193
OPTIMAL DUAL SOLUTION =    0.380
OPTIMAL PRIMAL SOLUTION = (4.19,4.67)
OPTIMAL DUAL FUNCTION =   84.961
MINISUM FUNCTION =  84.961
MINIMAX FUNCTION =  44.193
```

```
RHS VALUE =    44.440
OPTIMAL DUAL SOLUTION =    0.363
OPTIMAL PRIMAL SOLUTION = (4.21,4.66)
OPTIMAL DUAL FUNCTION =   84.869
MINISUM FUNCTION =  84.869
MINIMAX FUNCTION =  44.440
```

```
RHS VALUE =    44.687
OPTIMAL DUAL SOLUTION =    0.347
OPTIMAL PRIMAL SOLUTION = (4.23,4.65)
OPTIMAL DUAL FUNCTION =   84.782
MINISUM FUNCTION =  84.782
MINIMAX FUNCTION =  44.687
```

```
RHS VALUE =    44.933
OPTIMAL DUAL SOLUTION =    0.330
OPTIMAL PRIMAL SOLUTION = (4.25,4.64)
OPTIMAL DUAL FUNCTION =   84.698
MINISUM FUNCTION =  84.698
MINIMAX FUNCTION =  44.933
```

SAMPLE OUTPUT FOR PROGRAM NORMCT :

0.99995	1.00005	-0.00005	0.00005	1.00000	1.00000	2.75000
0.99995	1.00005	2.99995	3.00005	1.00000	1.00000	4.50000
1.99995	2.00005	-0.00005	0.00005	1.00000	1.00000	5.00000
1.99995	2.00005	-1.00005	-0.99995	1.00000	1.00000	6.00000
0.99995	1.00005	-2.00005	-1.99995	3.00000	1.00000	6.00000
-1.00005	-0.99995	-0.00005	0.00005	5.00000	1.00000	4.00000
-1.00005	-0.99995	0.99995	1.00005	6.00000	1.00000	3.00000
-1.00005	-0.99995	2.99995	3.00005	2.00000	1.00000	5.00000
-2.00005	-1.99995	-2.00005	-1.99995	4.00000	1.00000	7.00000
-3.00005	-2.99995	-1.00005	-0.99995	1.00000	1.00000	7.00000
-3.00005	-2.99995	0.99995	1.00005	2.00000	1.00000	3.75000

UNCONSTRAINED SOLUTION:

SINGLE SOLUTION FOR THE X-COORD. PROBLEM : X * = -1.00

SINGLE SOLUTION FOR THE Y-COORD. PROBLEM : Y * = 0.00

UNCONSTRAINED FUNCTION VALUE: 59.000

CONSTRAINED SOLUTION : -1.0000 0.5000
CONSTRAINED FUNCTION VALUE: 61.500