THE ROLE OF THE LANGUAGE AND OPERATIONS OF SETS IN THE ELEMENTARY ARITHMETIC PROGRAM

By

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PREFACE

The purpose of this study was to identify the purpose and role of the language and operations of "sets" in the elementary arithmetic program. It was the sincere opinion of the author that there existed a felt need by both the student and the teacher in the current transition period of elementary mathematics.

Special gratitude is expressed to Dr. W. Ware Marsden, Dr. J. Pascal Twyman, Dr. O. H. Hamilton, Mrs. Helen M. Jones, and Dr. John Hoffman for their assistance and guidance as members of the advisory committee.

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CHAPTER I

THE PROBLEM

The sheer bulk of mathematics available today is overwhelming. The individual who chooses to make its study a career is finally faced with specialization in a particular area, or proficiency in some and only a passing acquaintance with others. However, the mathematician who chooses to look at similarities and contrasts within and among various topics often succeeds in identifying common structures and properties. In this way he succeeds in adding to his own understanding of other areas with a minimum of effort merely by drawing analogies to previously mastered concepts. At times one of the greatest difficulties has been the creation of a language that satisfactorily exhibits the similarities and relationships. Even at the grade school level, frequently the child, and sometimes the teacher views arithmetic as a collection of more or less disconnected topics each having its own little domain in the world of arithmetic. Occasionally there may be an intuitive awareness of connections, but the available language may be inadequate to convey these relations.

To alleviate this difficulty, an increasing number of contemporary mathematics programs for the elementary student are utilizing some of the language and operations taken from elementary set theory.

Therefore, this investigation is to identify the necessary language, operations, and applications of sets that may be utilized in teaching the concepts and manipulative skills of elementary school mathematics. It must serve both as an introduction and a resource for the teacher of arithmetic.

Need for the Study

Many recently written texts allot an ample portion of material to the introduction of set terminology and operations and then fail to utilize this foundation in subsequent material. It is difficult to defend this expenditure of time and effort on a topic which does not seem to be an integral part of the course.

What is and ultimately will be taught depends upon available texts, public pressures, recommendations of groups influential in education, and most important, the preparation and convictions of the teacher.

Of the many texts and books available, the author is unaware of any written primarily for the elementary teacher which exhibits the extensive utility of sets and presents their relative position in elementary arithmetic. A careful reading of abstracts of theses from 1955 to 1962 has failed to reveal any study of a comparable nature.

Although many elementary teachers have participated in mathematics workshops designed to help them understand and appreciate the shift in emphasis and language changes, there are many who have not. Also, an initial introduction needs reinforcement and a ready reference source for reassurance.

The author has had the opportunity to direct many teacher study groups interested in the contemporary mathematics programs at the elementary and junior high school levels. There are always frequent requests by the involved individuals for materials that they can study at their leisure that are particularly pertinent to their situations. Additionally, many parents have expressed a desire for material that would enable them to see the utility of the ideas of sets in the arithmetic program.

Scope of the Study

Since the study is directed to the elementary teacher, its reading and understanding should presuppose mathematical knowledge no greater than that expected of the elementary teacher with a traditional background. Although the author feels that the development will increase the reader's understanding and appreciation of the structure, patterns, and interrelations of arithmetic, this is not the primary purpose. Also, the reader is expected to draw upon his or her own knowledge to contrast the examples given with the methods of a traditionally orientated course. Hence, the scope of the study is limited to the preparation of an ample resource and guide on the language, operations, and applications of sets in elementary arithmetic. An annotated list of related books appropriate to the personal library of the teacher or that of the elementary school will be included.

Procedure

The author proposes to study currently available material, consult with interested professors, individuals, and elementary teachers, and apply the results of these activities to creating an informative guide on "sets" in the elementary arithmetic program. The study will not be a mere "collection" of rephrased opinions of others, but rather a coherent presentation expressly designed for the elementary teacher.

Personal knowledge and experience in working with the elementary teacher will be relied upon heavily, but when especially pertinent, material from other available sources will be included.

The author's committee is particularly well qualified for such a study since it consists of mathematicians and experts in the field of curriculum and instruction.

The completed material will be submitted to a jury for evaluation and criticism. Wherever there is sufficient indication that revision and corrections are necessary, then these changes will appear in the final presentation. The jury will be composed of individuals who are authorities in the appropriate mathematical area and persons who are actively involved in the contemporary arithmetic program. Authors of current books on "sets" or arithmetic texts, professors responsible for teacher preparation and supervisors of elementary teachers of arithmetic will be considered as authorities. A chapter on the evaluation and its implications will be included as an integral part of the study.

Summary and Preview

The author's proposal to identify, organize, and summarize for the elementary teacher a reference booklet on sets is the result of a number of experiences in directing workshops on elementary mathematics. Most workshops are too brief to give the time and preparation necessary to this topic, and the lack of publications with proper orientation has presented problems.

While many of the ideas have only minor effect on actual mathematical processes at this level, they influence greatly the methods of presentation and contribute extensively to the continuity of content. A cursory look at some texts may leave the uninitiated with the false impression that "sets" are something thrown in to "modernize" or add novelty to an otherwise traditional course. This is unfortunate. If the content is to be introduced at all, it should be to clarify concepts, unify and relate segmented topics, and simplify complex situations.

The amount of language and operations involved should not appreciably increase nor decrease the bulk of material presented in a course since the limited number of terms introduced, for the most part, have meanings in mathematics not too far divorced from the meanings attached in everyday usage.

CHAPTER II

LANGUAGE AND OPERATIONS OF SETS

Communication is not always easy. An exposition must first of all be understandable; and, equally important, the material must be meaningful. There must be an agreement between author and reader upon the definitions of basic terms.

Definitions

The reader should be aware that for reasons peculiar to a particular study, one author may choose to modify a definition. In general, however, modifications will be easily acceptable, and the reasons for the changes will be obvious. It should be pointed out that since every word cannot be defined without a circularity of definitions, some authors choose to leave the word "set" as an undefined term. For instance, suppose the dictionary is used to define "quantity." One definition refers to "portion," then the definition of "portion" leads to "extent," continuing, "amount" is found, and finally, "quantity." This completes a circuit of words. If one had no previous understanding of at least one of these, he would be no better off than before. This is the basis for beginning with "undefined" terms. "Set" in this writing will be a primitive term. It is easy, however, to gain an intuitive concept of its meaning by thinking of familiar synonyms. Some appropriate ones are: aggregate, assembly, and collection. Collection of what? Surprisingly enough, the collection can be of anything at all which can somehow be identified. It might be a covey, or set, of quail. It could be a class, or set, of students. One might consider a gang, or set, of boys. The interest could be in a set of points, a set of numbers, or a set of letters.

Mathematicians are symbol-minded people. One of the great strengths of mathematics lies in the preciseness, clarity, and conciseness of its symbolic language. One needs only to translate a single mathematical sentence, such as $5 + 9 = 14$ into words to appreciate these values.

The symbol $\{ \}$, called braces or "curly-brackets," is used to mean "the set of." Upper-case letters are frequently employed to name a set as a whole. Thus, $A = \{1,2,3\}$ is read, "A is the set whose members are one, two, and three." Items in a set will be referred to as <u>elements</u> or <u>members</u>. The number two is an element of the set A, but the number four is not an element. The Greek letter "$\in$" (Epsilon) is used to abbreviate this statement (i.e., $2 \in A$, read "Two is an element of the set A"; a denial can be written $4 \notin A$, read "Four is not an element of the set A." A slash mark through a symbol, in general, implies the negative of the meaning.)

Many times properties inherent to a problem or discussion force restrictions on the elements eligible for consideration. For example, suppose the reader is going to purchase a pair of shoes from the set of all shoes. First, those not of the appropriate size must be disregarded. Secondly, the individual will be interested in those having certain styles, colors, and price ranges. Also, a further limitation is present because it would not be possible to visit every distributor of shoes. It is this type of situation to which the word "universe" applies.

> The <u>universe</u> is the fixed set of elements eligible for consideration in a given discussion.

The universe may vary with the situation. In the example of the shoes, the universe of the consumer would be decidedly different from that of a retailer. Also, the individual's universe would change from season to season and as his feet changed in size.

The universe of numbers used in the first grade would not ordinarily include a solution to a problem such as "$3 - 10 = n$." This problem, however, certainly has a solution in a universe which includes the negative integers. The reader may, at this point, wonder how set notation can be used to indicate the members of an infinite set, such as the negative integers. One convenient way, in some cases, is to list enough members so the pattern is obvious, and then use the ellipsis (three dots meaning "and so on"). $A = \{0,1,2,3,\ldots\}$ means "A is the set of numbers 0,1,2,3, and so on, indefinitely."

There is another set which is just as unique as the universal set.

> The <u>empty</u> or <u>null</u> set is the set which contains no elements. The Greek letter "Ø"

(Phi) will be used to refer to the empty set.

For example, the set of all states in the United States which are larger than Alaska is the null set.

Previously, individual elements of a set were identified as members, but a term is also needed to denote a set totally contained in another set. For example, if the set M = {r,a,g,m,o,p} and N = {o,a,r}, then, N is totally contained in M. The term "subset" is used to describe this relation, and the symbol "⊂" means "is a subset of" or "is contained in." Thus, N ⊂ M.

A is a <u>subset</u> of set B if every element of A is also an element of B. This may be written A ⊂ B or A ⊆ B. (A ⊄ B or A ⊈ B means "A is not a subset of B.")

The definition permits B to be a subset of itself, because certainly every element of B is an element of B. In this case, B is an <u>improper</u> subset of B. This is the reason for introducing the line below the symbol for subset. The symbol "⊆" is sometimes read "is contained in or equal to." Since, the empty set satisfies the definition, then the empty set is a subset of every set.

Every subset of B, such that B contains at least one element not in the given subset, is called a <u>proper</u> subset of B.

Compare the following sets:

G = {2,4,8,6}, H = {4,8}, J = {3,1,2}, Ø = { },

and let U = {1,2,3,4,5,6,7,8} where U is the universe. Then: H ⊂ G, J ⊄ G, Ø ⊂ G, G ⊆ G, H ⊂ U,

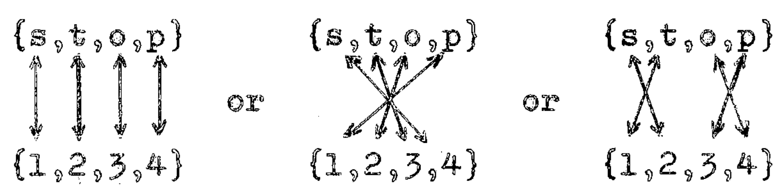
$J \subset U$, $G \not\subset H$, $H \not\subset J$, $\emptyset \subset H$, and $\emptyset \subset H \subset G \subset U$.

There are other statements that could be made about this example. Every other set, for instance, is a subset of the universal set, U; and $\emptyset$, the empty set, is a proper subset of every other set. Notice that the members of sets G and J are not listed in the order one might expect. When elements are listed in the braces, order is not necessarily implied. Hence, $A = \{2,4,6\} = \{6,2,4\} = \{4,2,6\}$. An exception can be agreed upon. For example, it is convenient to agree to list the set $C = \{1,2,3,\ldots\}$ in the customary order of the counting numbers.

This seems an appropriate place to introduce the ideas of equal sets, equivalent sets, and one-to-one correspondence.

> Two sets are said to be in one-to-one correspondence, if each element of one set can be paired with a single element of the other set and every element in both sets used.

Let $D = \{s,t,o,p\}$, $F = \{p,o,t,s\}$, $S = \{1,2,3,4\}$. Then sets D and S may be paired in any one of the following ways:



It is not important which two elements are paired, but rather that every element be used once and only once.

> When two sets can be placed into one-to-one correspondence, they are said to be equivalent.
> The symbol for "is equivalent to" is "$\sim$".
> In the example above, $D \sim S$, read "D is equivalent to S."

Also, $F \sim S$ and $D \sim F$.

> Two sets are said to be <u>equal</u> if they have identically the same elements.

Hence, $D = F$, as well as being equivalent. D is not equal to S, even though they are equivalent. Symbolically this can be written: $D \neq S$, but $D \sim S$. It should be apparent that any sets which are equal are also equivalent. Two sets may be equivalent, however, without being equal.

Lest the reader becomes concerned that this discussion of sets be too self-centered, a brief deviation will be made. (The bulk of applications to arithmetic are reserved for the next chapter.)

One of the basic problems of whole number arithmetic is to find out how many members are in a given set. "This problem can always be solved, at least in principle, by counting."[1] (Whole number arithmetic means arithmetic whose universe is the set $W = \{0,1,2,3,\ldots\}$.) What is counting? One might choose to answer by saying that for each item in the set to be counted, a number is recited in order. The last number recited, then, identifies a number with the set, say three. But what is three, or two, or one hundred? The concept of number is an abstraction. No one has ever really seen a two or a three. What is written is a symbol or "numeral" for the property of "twoness." "Twoness" is an abstract property enjoyed by sets. For example, {a,b}, {*,@}, {5,4}, and {John, Mary} are each sets which enjoy the prop-

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[1] Howard Levi, "Why Arithmetic Works," <u>The Mathematics Teacher</u>, January, 1963, p. 2.

erty of "twoness." The sets are not equal because they are not identical. They are, however, all equivalent since they can be placed into one-to-one correspondence. In the same manner the sets {a}, {*}, and {9} enjoy the property of "oneness." When a child counts on his fingers, he is recognizing the one-to-one correspondence of equivalent sets.

In order to communicate, it is necessary for man to agree upon language and symbols to represent these number abstractions. Suppose the symbol "0" is chosen to identify the number of elements in the empty set, { }. The symbol will be named "zero." Now, since this symbol is known, it can be enclosed in braces: {0}. This is a set which is not empty (because it contains the symbol "0") and a new name is needed. "One" will suffice and the numeral "1" will satisfy the symbolic need. Next, enclose both symbols {0,1} and adopt the name "two" and numeral "2." The question "What is two?" can now be answered by saying it is a property enjoyed by any set which is equivalent (can be placed into one-to-one correspondence) to the set {0,1}. One can continue in this manner with {0,1,2}, {0,1,2,3}, {0,1,2,3,4},... . This is the <u>cardinal</u> use of numbers. Thus, for comparison purposes, a "standard" set is associated with each cardinal number (or equivalence class).

Cardinal Number (or equivalence class)	Standard Set
0	{ }
1	{0}
2	{0,1}
3	{0,1,2}
.	.
.	.
.	.

The cardinal number of $A = \{r, a, g, m, o, p\}$ is six because the set A is equivalent to the standard set $\{0, 1, 2, 3, 4, 5\}$. Place value, of course, will be utilized to assign cardinal names to sets having more than nine elements.

To be concise, when specifically concerned with the cardinal number of a set, the braces or identifying letter will be preceded by a lower case letter. $n(B)$ and $m\{x, y, z\}$ mean "the cardinal number of set B" and "the cardinal number of the set whose elements are x, y, and z," respectively. $m\{x, y, z\} = 3$ and if $B = \{a, b, c, d\}$, then, $n(B) = 4$ or $n\{a, b, c, d\} = 4$.

What is counting? Counting is establishing a one-to-one correspondence between some ordered subset $\{1, 2, 3, \ldots n\}$ of the set $\{1, 2, 3, \ldots\}$ of nonzero whole numbers and the set to be counted. The last element "n" necessary for the correspondence is said to be the count of the set. Notice that only the set $C = \{1, 2, 3, \ldots n\}$ is required to be ordered. No order is necessary for the set to be counted. For example, if the students in a class are counted, it is unimportant which child is counted first.

The ordinal concept has always developed along with the cardinal.[2] The cardinal has been used to answer "How many?" and the ordinal will be used to answer "In what order?" "Tots often learn to count in rote fashion--one, two, three, ...-- before they are able to relate these number words to

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[2]Foster E. Grossnickle et. al., Instruction in Arithmetic, Twenty-Fifth Yearbook, The National Council of Teachers of Mathematics, (Washington, D. C., 1960), p. 285.

sets. Do some children first connect *three* with a position in a sequence, and others first connect *three* with a group of objects?"[3] Whatever the sequence of learning, it is necessary for children to learn both, and to mix them freely. If the set S = {2,4,6,8} is considered, then its cardinal is four. Each of its members have an ordinal. From left to right, two is the first, four is the second, and eight the fourth. First, second, and fourth are ordinal uses of numbers. An agreement can be made to label the set {1, 2,3,...} as the set of *counting numbers* and represent it with the letter "C". The set C, by definition will always be ordered in this indicated pattern.

> If a one-to-one correspondence is established between C = {1,2,3,...}, or some subset of C, and a given set, then, the *ordinal* of any member of the given set is the corresponding element of C.

Thus, for set S:

{2,4,6,8}		{8,2,4,6}		{6,8,2,4}		{4,6,8,2}
↕ ↕ ↕ ↕	or	↕ ↕ ↕ ↕	or	↕ ↕ ↕ ↕	or	↕ ↕ ↕ ↕
{1,2,3,4}		{1,2,3,4}		{1,2,3,4}		{1,2,3,4}

In the original correspondence, two is said to be the "first," while in the last four is the "first." In counting, the ordinal idea of number is used and the last, or highest, ordinal reached is always the cardinal number of the set. The cardinal number of a set is independent of the order in which the elements are counted; but the

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[3]Ibid.

ordinal number of an element is dependent upon the chosen order. So even though the two processes are distinct, they are subtly related; it is almost impossible to think of one without the other.

Operations with Sets

It was shown that elements of sets need not be numbers or, in fact, even mathematical symbols. Hence, ways of constructing new sets from given sets must be more general than the operations used for numbers.

One way to construct new sets from given sets is to join or unite sets; e.g., if A = {l,o,n,g}, and B = {h,a, n,d,l,e}, A joined to B would be the set D = {l,o,g,h,a,n,d, e}. This operation will be called "union" and the symbol "∪" will be its symbol. Thus, A ∪ B = D. Note that the elements n and l were listed only once in the union, since these elements were common to both sets. In fact, new sets can be derived by listing such common elements. This operation is named "intersection," and symbolized by "∩". A ∩ B = {l,n}. If E = {l,n}, then, A ∩ B = E.

> If A and B are sets, then, A <u>union</u> B (written A ∪ B) is the set of all elements belonging to A or B (or both).
>
> If A and B are sets, then, A <u>intersection</u> B (written A ∩ B) is the set of all elements belonging to both A and to B.

An aid to understanding of the properties of sets are Venn diagrams such as shown in Fig. 2-1.

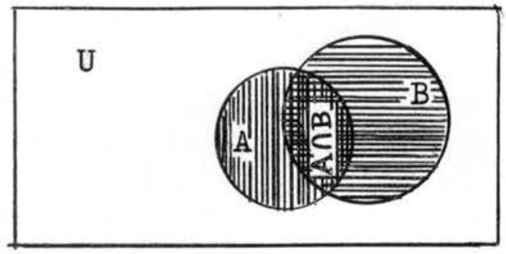
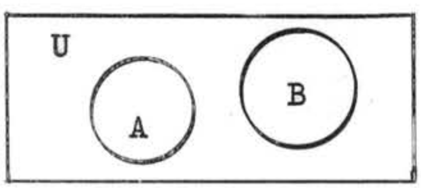


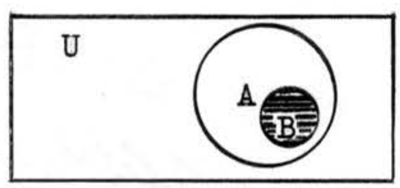
Fig. 2-1

Here the rectangle represents the universe U; the elements of U are the points in the rectangle. A and B are subsets of U. The members of A are the points in the vertically lined circle; the elements of B are the points in the horizontally lined circle. $A \cup B$ is the set of points which are inside either or both of the lined circles. $A \cap B$ is the set indicated by the overlapping parts of the circles having both vertical and horizontal lines. Fig. 2-1 shows only one possibility. Some others are indicated below:



$A \cap B = \emptyset$. $A \cup B$ is the set of all points in A or B.

Fig. 2-2



$B \subset A$. $A \cap B = B$. $A \cup B = A$.

Fig. 2-3

In Fig. 2-2, A and B have no elements in common. In other words, the set which is their intersection is the empty set.

If the intersection of sets is the empty set, then, the sets are <u>disjoint</u>. (A and B are disjoint if $A \cap B = \emptyset$.)

Just as sets are formed from given sets by taking the union, a set may be separated into subsets. If these subsets are disjoint this operation is given a name.

> If a given set is separated into subsets so that the subsets have no elements in common, and every element of the given set is included in some subset, then, this is said to be a <u>partition</u> of the set.

Let V = {o,h,u,m}, then, one partition of V is W = {o,h}, X = {u,m}. It is necessary for $W \cap X = \emptyset$, and $W \cup X = V$, because the definition requires the subsets to be disjoint and yet every element be used. There are other partitions just as acceptable; e.g., {o}, {m}, {u,h}. A common partition of a class of students could be the set of boys and the set of girls. Another partition for the same class might be made according to grades earned in a particular subject. The class could be partitioned by classifying hair colors. There are many other ways, but each must satisfy the two stipulations for partitioning.

A fourth operation is "complementation." The complement of a set can be considered only relative to another set. For example, the complement of the set of boys in the seventh grade, relative to the whole grade, is the set of girls in that grade. The complement of the set of seventh grade boys, relative to the whole school, is the set of seventh grade girls unioned with all other grades in the school. The symbol that will be used for the complement of the set A is A' (read "A complement" or "not A).

> The complement, A', of set A relative to set B, is the set of all elements in B and not in A.

In Fig. 2-4, A', relative to B, is the subset of the interior of B which has the diagonal lines. A', relative to U, is the dotted subset of the interior of the rectangle.

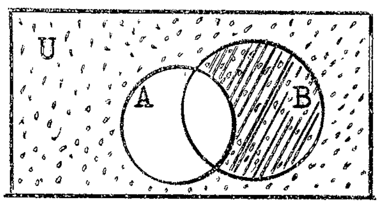


Fig. 2-4

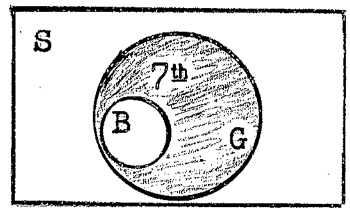


Fig. 2-5

The previous illustration for seventh grade boys could be pictured by Fig. 2-5. The rectangular universe is the whole school and the interior of the smallest circle is the set of seventh grade boys. The set of seventh grade girls is represented by the shaded portion of the interior of the larger circle, and the whole interior of the larger circle designates the seventh grade.

Another way of symbolizing the complement of a set A, relative to a set B, is B - A. This second notation has some advantage in analogies involving number operations.

Finally, consider the "Cartesian product" set. It is first convenient to agree upon the meaning of "ordered pair."

> An ordered pair of elements, say, a and b, means that the paired elements have a fixed sequence. Parentheses will be used to distinguish

the ordered pair (a,b) from the set {a,b}.

The set symbol {a,b} does not imply an order. Hence, $(a,b) \neq (b,a)$, ($\neq$ means "is not equal to"), but $\{a,b\} = \{b,a\}$. In indicating money, for example, one might consider dollars and cents as an ordered pair, with the elements separated by the decimal. Thus, \$12.34 $\neq$ \$34.12, or $(12,34) \neq (34,12)$.

> The <u>Cartesian</u> <u>product</u> of two sets A and B is the set of all possible ordered pairs (a,b), such that a is an element of A and b an element of B. Using notation, A X B is the set of all pairs (a,b), such that $a \in A$ and $b \in B$. Similarly, B X A is the set of all ordered pairs (b,a), such that $b \in B$ and $a \in A$. ("a" represents <u>any</u> of the elements of A and "b" <u>any</u> of the elements of B.)

If A = {1,2,3} and B = {4,2,6}, then A X B = {(1,4), (1,2), (1,6), (2,4), (2,2), (2,6), (3,4), (3,2), (3,6)}. If B and G are the sets of boys and girls at a dance, then, B X G is the set of all possible dancing partners, such that the first member of a pair is a boy and the second a girl. Pairing is a simple process if a rectangular array is used to organize the work:

A X B

	6 →	(1,6)	(2,6)	(3,6)
B	2 →	(1,2)	(2,2)	(3,2)
	4 →	(1,4)	(2,4)	(3,4)
		↑	↑	↑
		1	2	3

A

The introduction of union, intersection, partition, complementation, and Cartesian product completes the list of operations with sets which will be used. From time to time, it will be necessary to discuss properties of these operations. It is the feeling of the author, however, that such discussion will be more meaningful when associated with a more familiar situation.

Summary

Much of the confusion in the learning of mathematics arises not from the depth of content, but rather the casual attitude toward basic definitions. Regardless of the language or symbolism used, agreement upon definitions is of prime importance.

The language and operations of sets have been tried and found to be especially lucid. Although set theory is interesting in itself, its permeation of the many areas of mathematics is unparalleled by any other topic. It has fundamentals applicable to elementary arithmetic which are models of simplicity and directness. Most of these same ideas are equally acceptable at the more complex levels of mathematics. Hence, this is a desirable way to present fundamentals in a fashion that will be a consistent and integrating factor throughout a student's education.

For convenience, and to further stress the importance of language agreement, the terms and symbols are collected here with page locations of definitions indicated. This is not intended to be a complete list, rather it is intended to be a convenience for those who desire to devote only casual attention to Chapter II. The terminology and symbolism will be used extensively in the development of Chapter III, and it may be necessary for the reader to refer to the definitions from time to time.

Symbol	Meaning	Page
$\{\ \}$	Notation for sets	7
$\in$	Is an element of	7
$\notin$	Is not an element of	7
U	Universal set	8
$\emptyset$	Empty set	8
$\subset$	Is a proper subset of	9
$\not\subset$	Is not a subset of	9
$\subseteq$	Is a subset of or equal to	9
$=$	Equal	10
$\neq$	Is not equal to	10
$\sim$	Equivalent	10
1-1	One-to-one	10
n(A)	Cardinal of set A	12
no symbol	Ordinal	14
C	Set of counting numbers, $\{1,2,3,\ldots\}$	14
$\cup$	Union	15
$\cap$	Intersection	15
no symbol	Venn Diagram	15
no symbol	Partition	16
no symbol	Disjoint	16
A'	Complement of set A	17
$B - A$	Complement of A, relative to B	18

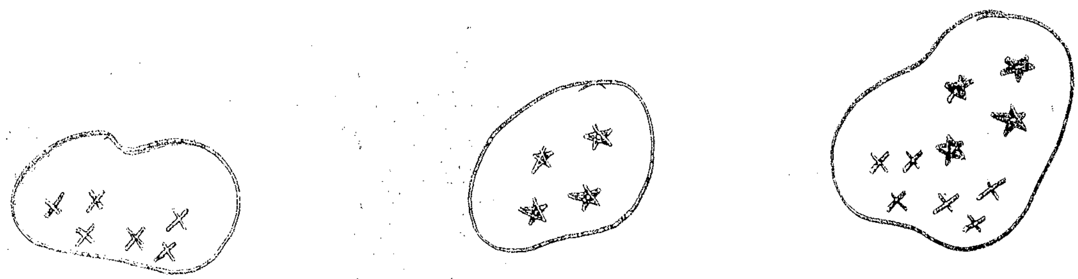
(a,b)	Ordered pair	18
A X B	Cartesian product of A and B	18

CHAPTER III

APPLICATION TO ELEMENTARY ARITHMETIC

One might suppose that the formal idea of sets did not appear in the previous elementary arithmetic program because it is too difficult. On the contrary, the fundamental ideas about sets are easily understood and the basic operations which put these to work are only slightly more difficult. Set theory is not, however, a mathematical panacea. It lends itself well to many analogies, but care should be taken that it not be used just for the sake of being "modern." Each application should be purposeful and in keeping with the appropriate level of the material concerned.

Just as the concept of a set is more primitive than number, the idea of union precedes that of addition. If 2 and 3 are defined in terms of standard sets, then, 2 + 3 can be interpreted in terms of the union of disjoint sets which have these numbers as their cardinals. Let $A = \{r,s\}$ and $B = \{o,z,a\}$, then, $A \cup B = \{z,o,s,a,r\}$. $n(A) = 2$, $n(B) = 3$, and $n(A \cup B) = 5$. It is not difficult to see that this is a relative of the operation addition. In fact, $n(A) + n(B) = n(A \cup B)$ or $2 + 3 = 5$, which is what was expected. It is very important, however, that one notices that the sets chosen are disjoint sets (sets which have no elements in common). If the sets used had been D = {m,r} and E = {m,p,s}, then D ∪ E = {m,r,p,s} and n(D) + n(E) ≠ n(D ∪ E). This happened because, if the sets overlap (are not disjoint), then the elements common to both sets must be counted only once. Frequently sets are constructed, such as the ones below, in which the elements may look alike but are nevertheless distinct.



A
n(A) = 6

B
n(B) = 4

A ∪ B
n(A ∪ B) = 10

Fig. 3-1

Here A ∪ B = B ∪ A and n(B ∪ A) = 10. Since order is not implied in these sets, it makes no difference whether the elements of B are listed first or those of A in the set A ∪ B. Thus, a + b = b + a follows easily from the analogy of sets. This is called the "commutative property" for addition.

Let a and b be any two counting numbers.
Then, the <u>commutative property</u> states a + b = b + a.

The reader may wonder if a cardinal is always available for the union of two disjoint sets. In Chapter II, page 12, it was shown that the definition of each cardinal number always leads to a definition of the next one, or a successor. This is an assurance that no matter what counting number is considered there is always a next larger one. If the sets are finite, then let "1" correspond to a "first" element, 2 to the second, 3 (the successor of 2) correspond to the third, and so on, until an element of the set of counting numbers is paired with one and only one element of the set to be counted. Although the operation of addition is interpreted in terms of the union of sets, it is still necessary to <u>count</u> the new set which is the union. Addition is obviously just an abbreviated form of counting! It is beneficial in time and effort saved to memorize certain addition facts and utilize patterns for addition of larger sums. However, if time and effort were of no consequence, then the process of addition might only be a novelty!

Always being able to find a next ordinal is a powerful result. This is an assurance of a solution to the sum of any two counting numbers and that the solution number will be just another counting number. This property is called "closure." Since it also holds for other operations, a rather general definition is useful.

> A set of numbers has the property of <u>closure</u> for an operation (such as addition) if upon performing the operation upon members of the set, the resulting number is a member of the original set.

For example, because the counting numbers are closed under addition, then the answer to 7 + 19 is available in the counting set. The number 26 is a member of the original universe. If subtraction were considered for the same set, no solution would be found for 3 - 9. The counting numbers are not closed under subtraction, and to find a solution would require a different universe.

Previous discussion has avoided the union of more than two sets or the addition of more than two numbers. Addition is a "binary" operation (i.e., combines <u>two</u> numbers). To consider a sum of three or more numbers, they must be combined in pairs until finally a one-number solution is found. In adding 5 + 2 + 9, the five and two may be associated or the two and nine, and then the result combined with the remaining third number.

$$(5 + 2) + 9 = 7 + 9 = 16 \quad \text{or} \quad 5 + (2 + 9) = 5 + 11 = 16$$

This illustrates the <u>associative</u> property for addition. Note that 5 + 9 was not considered in this example. This would be permissable by the <u>commutative</u> property, but is <u>not</u> a consequence of the associative property. Thus,

$$\begin{aligned}(5 + 2) + 9 &= (2 + 5) + 9 \text{ by the commutative property.}\\ &= 2 + (5 + 9) \text{ by the associative property.}\\ &= 2 + 14\\ &= 16\end{aligned}$$

Suppose the cardinal number of the union of several <u>disjoint</u>, <u>equivalent</u> sets is to be found. Let the sets be $A = \{a,b,c\}$, $B = \{d,e,f\}$, $C = \{g,h,i\}$, and $D = \{j,k,l\}$. Then, $A \cup B \cup C \cup D = \{a,b,c,d,e,f,g,h,i,j,k,l\}$. The cardinal of the union is 12. If the addition facts are known, one procedure could be

$$n(A \cup B \cup C \cup D) = n(A) + n(B) + n(C) + n(D)$$

$= 3 + 3 + 3 + 3$

$= (3 + 3) + (3 + 3)$

$= 6 + 6$

$= 12$

Finding the cardinal of the union of a number of equivalent, disjoint sets occurs sufficiently often to distinguish the process with a name. This process is called multiplication and a sufficient number of the facts are memorized to expedite computation.

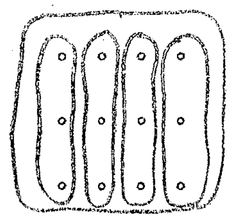


Fig. 3-2

The rectangular array of Fig. 3-2 illustrates the union of four equivalent, disjoint sets each having three elements. Thus, three multiplied by four is twelve, a multiplication fact which the student is asked to memorize. "Multiplication" of b by a is symbolized by writing a x b, a ∘ b, a(b), or ab (where no confusion arises). Hence, the problem may be written 3 x 4, 3∘4, and 3(4). (34 would <u>not</u> be acceptable in this case.)

Obviously, multiplication is just an efficient procedure for adding equivalent, disjoint sets. It follows also that multiplication must be founded in the process of counting and be a special case of an operation with sets.

The Cartesian product of sets is sometimes a more appealing analogy to multiplication. For example, consider the set D = {a,r,f} and K = {m,e,o,w}. The Cartesian product D X K can be illustrated by Fig. 3-3 or Fig. 3-4.

m → (a,m)(r,m)(f,m)
K e → (a,e)(r,e)(f,e)
o → (a,o)(r,o)(f,o)
w → (a,w)(r,w)(f,w)
↑ ↑ ↑
a r f

D

Fig. 3-3

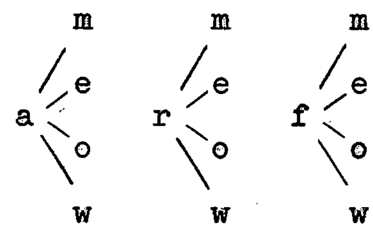
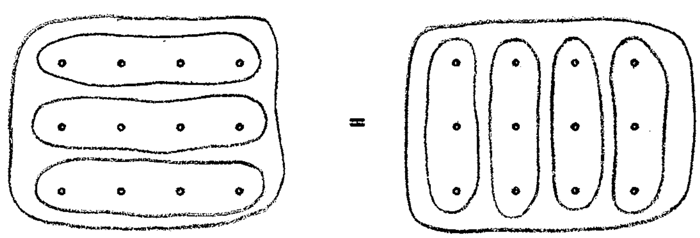


Fig. 3-4

D X K = {(a,m), (a,e), (a,o), (a,w),
(r,m), (r,e), (r,o), (r,w),
(f,m), (f,e), (f,o), (f,w)}

Now, n(D) = 3, n(K) = 4, and n(D X K) = 12. The binary operation defined in this way then is multiplication. Since a solution can always be found for finite sets, the counting numbers are <u>closed</u> under multiplication.

A slightly different arrangement of Fig. 3-2 and Fig. 3-3 reassures one that the commutative property also holds.



3 x 4 (3 sets, 4 elements) = 4 x 3 (4 sets, 3 elements)

Fig. 3-5

No new arrangement is necessary in Fig. 3-5 to see the commutative property.

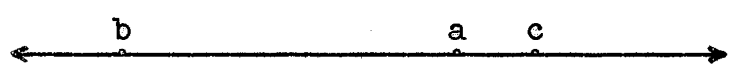
Multiplication is also a *binary* operation, and from its relation to addition, it is easy to see that multiplication of the counting numbers is *associative*.

To illustrate:

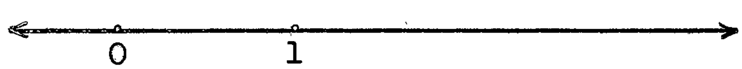
$$(5 \times 7) \times 2 = 35 \times 2 = 70 \quad \text{and} \quad 5 \times (7 \times 2) = 5 \times 14 = 70$$

The association of sets of points and sets of numbers has much to contribute. It offers an excellent visual aid for exhibiting certain relations among the sets of numbers, and it helps to integrate geometric concepts with number ideas.

One such association is that of a *number line*. It is assumed that the reader has an intuitive concept of what a straight line is. Every straight line (or curved line) will be considered to be a set of points. Also, line does not mean line segment. Below is a model of a straight line with points a, b, and c indicated. The arrows indicate that the line continues indefinitely in both directions.



Suppose a similar model is used, but some points labeled with a specific purpose in mind. An *arbitrary* point is selected and paired, or labeled, with the symbol 0. An



arbitrary unit segment is chosen, extending from zero to the right, the right end point labeled with the numeral 1. The word "arbitrary" is underlined because the purposes can be served equally well by choices such as those below.

The zero point (or origin) and unit segment are utilized to establish a relation between the set of whole numbers (W = {0,1,2,3,...}) and points on the number line. By letting 1

correspond to the left end point of the unit segment, a point is located to the right of 1 which is labeled 2. Continuing this process sets up a 1-1 correspondence between the set of whole numbers and a set of points on the number line. (Notice, however, only a relatively small subset of the points available to the right of zero have been utilized, and those to the left have not been mentioned.

Previous use has been made of the word "order" and the reader's experience was relied upon to interpret this term. It seems beneficial here to discuss this briefly. Order, as it will be used, is concerned with the intuitive interpretations of greater than or less than. The symbols " $>$ " (greater than) and " $<$ " (less than) will be the respective symbols for these ideas. Thus, the mathematical statement $9 > 5$ means "nine is greater than five" and $5 < 9$ means "five is less than nine." (Observe the similarity between the symbols " $<$ " and "$\subset$.")

The <u>order</u> property will mean that if a and b are different numbers, then, either $a > b$ (a is greater than b) or b > a (b is greater than a).

On the number line it is seen that if a > b then the point corresponding to a lies to the right of b. Also, if A and B are sets, then, n(A) > n(B) if the standard set (page 12) which corresponds to n(B) is a proper subset of the standard set which corresponds to n(A).

The number line also provides a convenient model for addition and multiplication of the whole numbers. Consider 3 + 2 = n.

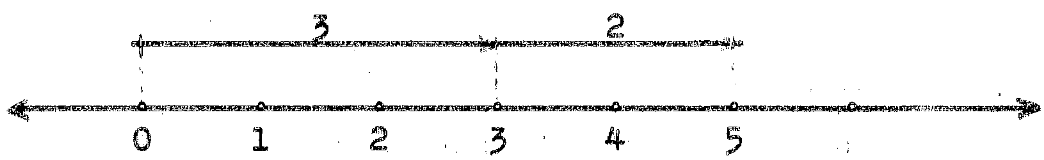


Fig. 3-6

The sets corresponding to the cardinals, three and two, are the set of three units and the set of two units, respectively. Since the operation is addition, the cardinal of the union is five units. The word "join" might serve better than "union," although this added burden will not be imposed on the vocabulary. (In fact, English mathematicians frequently use join instead of union.)

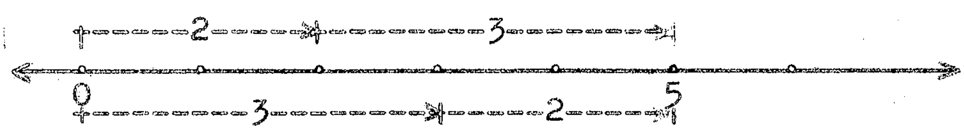


Fig. 3-7

This can serve equally well for 2 + 3 = n, and, in fact, doing both on the same line serves as a reinforcement of the commutative property.

The model for 3 x 2 = n and 2 x 3 = n might be interpreted as three sets of two units each, and two sets of three units each, respectively.

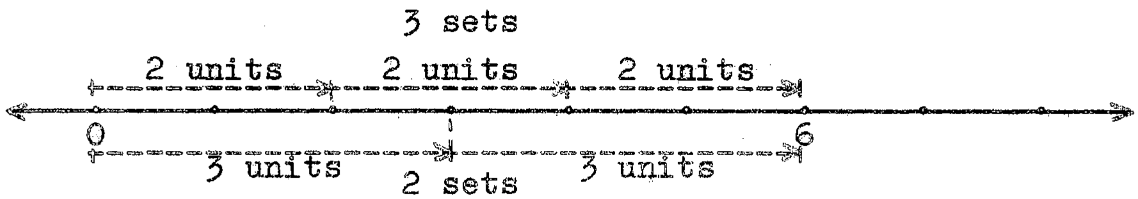


Fig. 3-8

The associative properties are similarly shown.

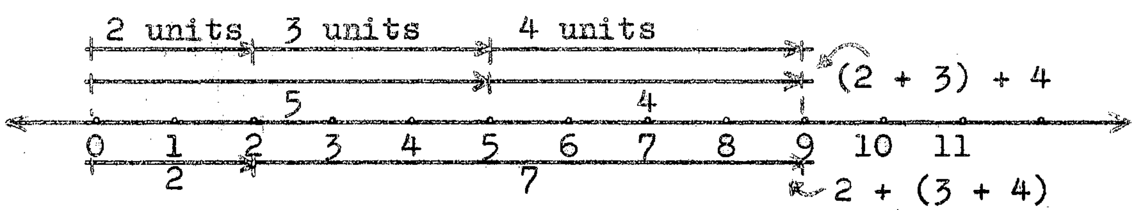


Fig. 3-9

Sets have offered a foundation for the understanding of the operations of addition and multiplication of the counting numbers. However, can they also clarify subtraction and division in a similar way?

Throughout mathematics the notion of inverse elements and inverse operations prevails. Two disjoint sets are unioned to get a single set containing the elements of both, and this operation is "undone" by considering one of the unioned sets and its complement relative to the union. For example, if A and B are disjoint (i.e., $A \cap B = \emptyset$) and $A \cup B = C$, then $C - B = A$ (the complement of A relative to C is B). Another notation for $C - B$ is B', (page 18). B' (or the complement of B) is exactly the set A. This merely means remove the set B from the union and what is left is the set A. Thus, using cardinal numbers, if $n(A) + n(B) = n(C)$, then, n(C) − n(B) = n(A).

Let A = {*,@,x} and B = {s,y,m,b,o,l}, then,

C = {*,@,x,s,y,m,b,o,l} and C − B = {*,@,x}.

B' (or C − B)   B   A (or B')

This procedure is acceptable for problems such as 9 − 4 = n, where 9 > 4, but not for a problem like 4 − 9 = m. (There is no whole number m such that m + 9 = 4. The whole numbers are not closed under subtraction.) A closer examination of the process of complementation reveals that really all that is done is to partition the union, to find what set must be unioned to a given set (or cardinal number added to a given number) to yield the union (or sum). Thus, 5 − 2 = n means to find the number n such that n + 2 = 5. Then, the problem 2 − 5 = m must mean find the number m such that m + 5 = 2. A satisfactory introduction to this problem can be developed on the number line.

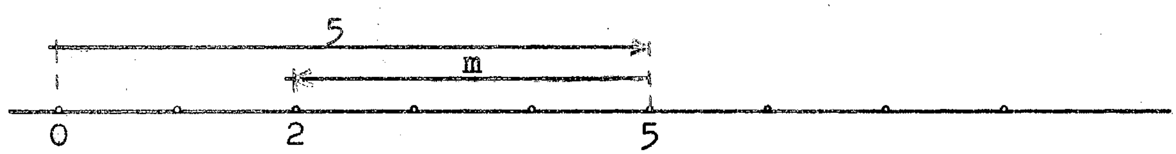


Fig. 3-10

Here m is directed to the left rather than the right (in the sense of "undoing"). Another method is to introduce "inverse" elements for addition. That is, introduce a set of elements which can "undo" an added set. An analogy might be "putting your coat on" as opposed to "taking your coat off." Thus, for every element a ∈ W, create some element, "opposite a", so that a + (opposite a) = 0. On the number line:

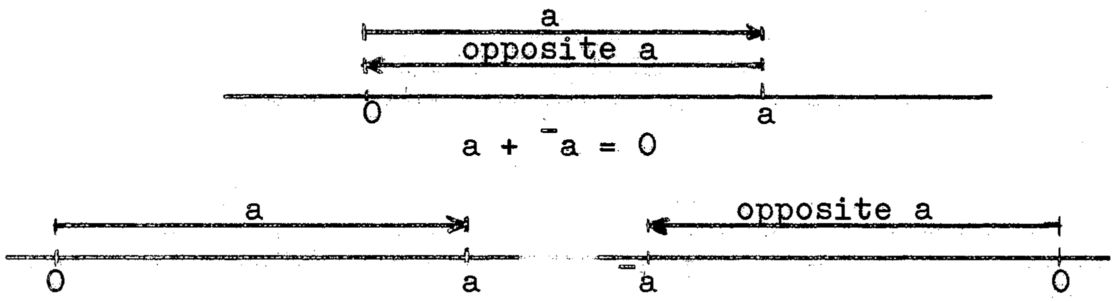


Fig. 3-11

"Opposite a" is symbolized as ^-a and read as "negative a" or "opposite a." If a is the cardinal number of a set having a elements, then ^-a is an equal cardinal but of a set of elements which are inverse elements.

The primary concern in the introduction of the numbers which are less than zero, has been to motivate the extension of the number system (in this special case, to "undo" addition). Since operations with these elements are rarely treated in the elementary school, undue emphasis should be avoided here. It should be pointed out, however, that once these elements are introduced a much larger universe is available. This set will be named the <u>integers</u> and identified as $I = \{\ldots, ^-3, ^-2, ^-1, 0, 1, 2, 3, \ldots\}$. The problem $2 - 5 = m$ has a solution in this universe, namely $m = {}^-3$. In fact, the set enjoys the property of closure under subtraction. Comparison of the subtraction problem $2 - 5 = m$ with the addition problem $2 + (^-5) = m$ on the number line shows that the results are equal. That is, it is not absolutely necessary to introduce subtraction if one prefers to elaborate on the use of inverse elements. The author does not insist that this is a better method for elementary school

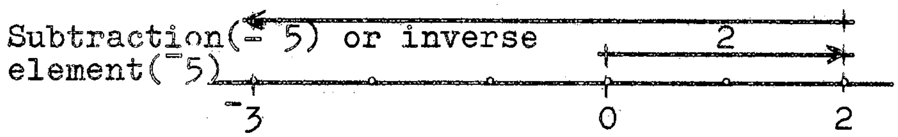


Fig. 3-12

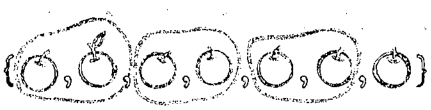
arithmetic, but the teacher should be aware of such an interpretation.

Very little has been said previously about the element zero. Zero occupies a very unique place in the set of integers. It is the cardinal number of the empty set. If it is added to any other integer, (i.e., union the empty set with any other set) the sum is identically the original integer (or the same set). For this reason, it is called the additive identity. If any integer is added to its inverse, the result is zero. $(8 + (^-8) = 0$ or $(^-3) + 3 = 0.)$ Also, zero is its own additive inverse! Because the sum of an integer and its opposite is zero we have a precise method for identifying the additive inverse.

An accepted name for the set of integers which are less than zero is the <u>negative</u> integers. One partition of the set of integers is the set of negative integers, zero, and the counting numbers. Suppose the set of negative integers is identified by $I_n = \{\ldots, {}^-3, {}^-2, {}^-1\}$ and the set containing zero by $Z = \{0\}$. Then, $\{\ldots, {}^-3, {}^-2, {}^-1\} \cup \{0\} \cup \{1, 2, 3, \ldots\} = I_n \cup Z \cup C = I$.

One interpretation of multiplication was in terms of the union of equivalent, disjoint sets. The "undoing," or inverse of such a union, is the <u>partitioning</u> of the union back into equivalent, disjoint subsets, which include every element. For instance, if A = {a,b}, C = {c,d}, E = {e,f}, and G = {g,h}, then, A $\cup$ B $\cup$ C $\cup$ D = {a,b,c,d,e,f,g,h}. The cardinal of the union (8) is found to be the product of the number of sets (4) and the number of elements in each set (2). Suppose, instead, the cardinal of the union is known to be eight and the number of elements in each disjoint, equivalent subset to be two. How many subsets are involved? If the union is <u>partitioned</u> into two element subsets, then the number of subsets is four. {a,b,c,d,e,f,g,h} = {a,b} $\cup$ {c,d} $\cup$ {e,f} $\cup$ {g,h}. Thus, "division" of cardinals can be interpreted in this manner. 8 $\div$ 2 = 4 is a special case, then, of the example. Consider the following instance where the subsets do not work out so nicely.

> A set of seven apples are to be distributed among three boys. How many apples will each receive?

Partition A = {}. The results are three subsets each containing two apples and a subset containing one apple. Since the universe for the moment is restricted to the set of integers, then, one can hope for a worm in one and only one apple. If all apples are good, however, to find a solution would require a different universe. (Obviously, the set of integers are <u>not closed</u> under division, since 7 $\div$ 2 = n does not have an integer solution.) Consider the subset of one apple. If this subset can be partitioned into three disjoint, equivalent subsets, then the dilemma is solved. These partitions of the apple must, however, have a notation different from those of set A. One might agree that a division of <u>one element</u> into <u>n</u> <u>parts</u> be symbolized by 1/n. In this case, one apple divided into three parts is written 1/3. A partitioning of set A into three equivalent, disjoint subsets would be . The solution can be written as 2 + 1/3 (or by agreement as 2 1/3).

How about three boys and eight apples? This time the remaining subset contains two apples. Will the previous interpretation still hold? Yes. All one need do is consider the two sets (apples) partitioned into three, disjoint, equivalent subsets and symbolize each by 2/3. Hence, .

This introduced a new symbol not in the previous universe, I. Since the elements were "fractured" in the process, a connotative term would be "fractions."

A greater insight can be obtained if a different viewpoint is taken. A set of twenty-four ping pong balls are boxed in a number of different ways.

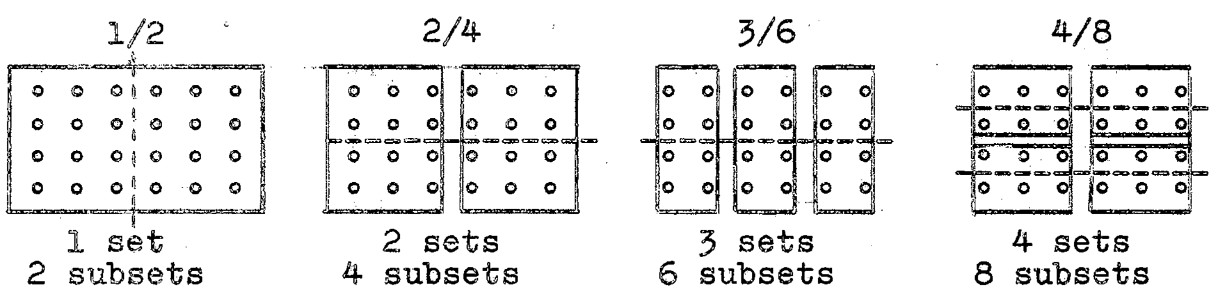


Fig. 3-13

It appears that if a multiple of the cardinal of the set or sets to be partitioned is taken, and the same multiple of the number of partitions, then the resulting fractions are equivalent. This is indeed the case.

> If the numerator <u>and</u> denominator of a fraction are each multiplied by the same nonzero number, then the value of the fraction is unchanged.

The "ratio" of two numbers is a comparison by division. In the preceding paragraphs, the fractions satisfy this property. Let this, then, serve as motivation for renaming these fractions "rational numbers."

> A <u>rational number</u> is defined as any number which can be represented as a ratio of the form a/b, where a and b are integers and $b \neq 0$.

The set of whole numbers satisfies this definition (e.g., four can be represented by 4/1, 12/3, 16/4, etc.). The set of all integers, of course, are members of the set of rationals, but integers such as $^{-}6/^{-}3$ involve explanation to which the language of sets does not particularly contribute.

The set of rationals does not lend itself to a listing, or an indicated listing, as was done with the preceding sets of numbers. To even attempt to list the set of rationals between zero and one involves an unwieldy array. A more efficient approach is to introduce the "set builder" notation. For A = {Alaska, Hawaii}, one writes A = {s | s became a state of the United States since 1950.}. The vertical mark is read "such that," and the statement is read "A is equal to the set of all states s, such that s became a state of the United States since 1950." The symbol "s" can represent either of the elements, Alaska or Hawaii. The set builder notation consists of two parts, the variable (s, in this example), and a descriptive statement following the symbol for "such that." Hence, the set of rationals can be described as R = {n | n can be written in the form a/b, where a and b belong to the set I and b ≠ 0.} Using more symbols, one could write R = {n | n is of the form a/b, a and b ∈ I, and b ≠ 0}. Also, the set of equivalent symbols for a particular rational number is usually thought of as an equivalence class or family. For example, {1/3, 2/6, 3/9, 4/12,...} is an equivalence class in which any symbol is a valid representative.

It is assumed that the reader is familiar with the usual operations of rationals. The language of sets is particularly applicable to finding the least common denominator and greatest common factor. Since the idea of "prime" factors is needed, a prime number is defined as follows:

> A counting number is said to be prime if it has no factors other than itself and one and is > 1.

To add or subtract fractions, it is necessary to find common denominators. It is ordinarily desirable to find the "least" common denominator. The least common denominator can be found by taking the product of the elements in the union of the sets of prime factors of each denominator. An agreement on interpretation must be reached, however. If a prime factor is repeated in the set for any particular denominator, then, these are to be regarded as distinct factors. A factor which is merely repeated from set to set will be listed only once in the union. Consider the following problem:

$$1/6 + 1/4 + 7/15 + 1/27 = n$$

$$6 = 2 \times 3,\ 4 = 2 \times 2,\ 15 = 3 \times 5,\ 27 = 3 \times 3 \times 3$$

The two and three appear as repeated factors in a set of factors and, hence, the second two in four and the second and third three in the factors of twenty-seven must be considered as distinct elements in the union. For purposes of clarity, the sets might be listed $\{2_1, 3_1\}$, $\{2_1, 2_2\}$, $\{3_1, 5\}$, and $\{3_1, 3_2, 3_3\}$ where the subscripts are used only to reinforce the previous statements. Thus, the union would appear as $\{2_1, 2_2, 3_1, 3_2, 3_3, 5\}$ and the product $2 \times 2 \times 3 \times 3 \times 3 \times 5 = 540$, the least common denominator. It can now be seen that each denominator is a factor of 540 by examining it in factored form.

$$\overbrace{2 \times 2}^{4} \times \overbrace{3 \times 3 \times 3}^{27} \times 5 = 540, \quad 2 \times \underbrace{2 \times 3}_{6} \times 3 \times \underbrace{3 \times 5}_{15}$$

The appropriate factors to multiply each fraction by are easily determined if the l.c.d. (least common denominator) set is left in factored form. For instance, twenty-seven has the set of factors $\{3_1, 3_2, 3_3\}$. Hence, to express it with a denominator of 540, multiply the product of its set of factors by the product of the factors in its complement, relative to the union.

$$\frac{1}{27} = \frac{1}{3 \times 3 \times 3} = \frac{(1) \times (2 \times 2 \times 5)}{(3 \times 3 \times 3) \times \underbrace{(2 \times 2 \times 5)}_{\text{complement}}} = \frac{20}{540}$$

Similarly,

$$\frac{1}{4} = \frac{1}{2 \times 2} = \frac{(1) \times (3 \times 3 \times 3 \times 5)}{(2 \times 2) \times (3 \times 3 \times 3 \times 5)} = \frac{135}{540}$$

It is obviously a waste of effort to write the factors of 540 each time. A more desirable procedure is to write them once to permit easy identification of the various complements.

$$\frac{1}{6} = \frac{1}{2 \times 3} = \frac{1}{(2 \times 3) \times (2 \times 3 \times 3 \times 5)} \frac{90}{} = \frac{90}{540}$$

$$\frac{1}{4} = \frac{1}{2 \times 2} = \frac{45}{540} = \frac{45}{540}$$

$$\frac{7}{15} = \frac{7}{3 \times 5} = \frac{7 \times 36}{540} = \frac{252}{540}$$

$$\frac{1}{27} = \frac{1}{3 \times 3 \times 3} = \frac{20}{540}$$

With a little practice much of the writing will be unnecessary.

To find the <u>greatest common factor</u> of two or more numbers, a simple procedure is to take the <u>product</u> of the <u>intersection</u> of the sets of prime factors.

$$420 = 2 \times 2 \times 3 \times 5 \times 7$$

$$126 = 3 \times 3 \times 7 \times 2$$

$$330 = 3 \times 5 \times 2 \times 11$$

The intersection $\{2,2,3,5,7\} \cap \{3,3,7,2\} \cap \{3,5,2,11\}$

$$= \{2,3\}$$

Therefore, the g.c.f. (greatest common factor) = 6.

This procedure is especially pertinent to ratio expressions. For example, 420:126:330 as 70:21:55. Also, the reduction of fractions, $\frac{126}{420} = \frac{6 \times 21}{6 \times 70} = \frac{6 \times 7 \times 3}{6 \times 7 \times 10} = \frac{3}{10}$.

Return briefly to the relations of the various sets of numbers. One can now write $C \subset W \subset I \subset R$. Also, note that the rationals may be partitioned in a manner similar to the integers into a set of positive rationals, negative rationals, and zero. One might think that he has a number available which corresponds to each point on the number line. This is not the case.

Consider one more set of numbers called the irrationals. The term "irrational" represents a set, the elements of which cannot be represented as a ratio of two integers. These numbers arise in many ways and a few examples are: $\sqrt{2}$, $\sqrt{7}$, and π (which is only approximately 3.14159). Notice that $\sqrt{4}$ is not an irrational since it is just another symbol for the number two. If the reader will recall the Pythagorean Theorem, (The sum of the squares on the legs of a right triangle is equal to the square on the hypotenuse.) then, this will motivate an introduction to a relation between some of the irrationals and points on the number line.

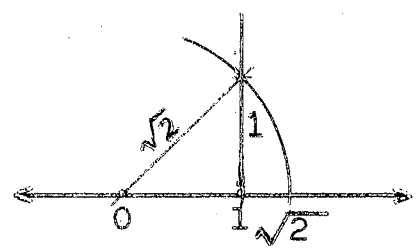
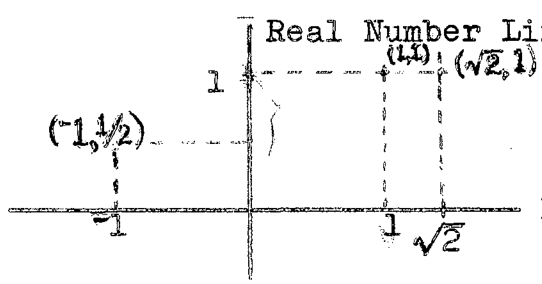


Fig. 3-14

If the legs of a right triangle are each one unit in length, then, by the Pythagorean Theorem, the area of the square on the hypotenuse is two ($1^2 + 1^2 = 1 + 1 = 2$). Therefore, the hypotenuse must have length $\sqrt{2}$. Now, if a segment of equal measure is marked on the number line, the right-end point corresponds to $\sqrt{2}$. Since $\sqrt{2}$ is irrational, and cannot be represented by a ratio of two integers, then, none of the set of rationals can correspond to this same point. Mathematicians can prove that between every two rationals, there exists an irrational and, therefore, there are many infinitely many such correspondences.[1]

With the introduction of this new set of numbers, there is now a unique rational or irrational which corresponds to each point on the number line. Suppose the set of irrationals is designated by I_r. Then the union of the sets of rationals and irrationals gives a new universe, which mathematicians call the <u>real numbers</u>. Symbolize the set of reals by R_e, to distinguish it from the set of rationals. One may, then, write $R \cup I_r = R_e$. The set of rationals has ordinarily been considered adequate for elementary arithmetic, but current trends may soon modify this. In any event, the set of real numbers are certainly a necessary part of the background of an elementary teacher. The "number line" is frequently called the "real number line," because of the one-to-one correspondence between the set of real numbers and the set of points on the line. If the Cartesian product of the set of real numbers is taken with the set of real numbers ($R_e \times R_e$), the result is a unique, ordered pair of real numbers corresponding to each point in the plane!

Fig. 3-15 

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[1] Norman T. Hamilton and Joseph Landin, <u>Set Theory, The Structure of Arithmetic</u> (Boston, 1961), pp. 229-231.

Fig. 3-15 is a model of this concept with a few such correspondences indicated. The similarity to the x and y axes of ordinary algebra is no accident, because they are identical ideas. This is a powerful tool in mathematics. Not only is it an excellent visual aid in the study of numbers, but it is the very foundation of relations between geometry and algebra. In short, this is basic to the study of analytic geometry and, yet, so direct that it is being introduced in grade school texts.

Geometry is now being introduced at all grade levels by some of the contemporary programs. The language of sets is especially pertinent to much of this material. In the introduction to the number line, the reader was asked to think of a line as a set of points. One can think of the intersection of such sets as points or sets of points which are common to lines. A plane is a set of points and, in fact, space itself is considered the set of all points. Line segments are sets of points and triangles, rectangles, and other polygons are defined as unions of line segments. Circles are defined as the locus of all points in a plane equidistant from a given point. Hence, the language of sets clarifies and simplifies both number and spatial concepts, and, as was just indicated, it unifies the two. It unifies them not only by the common core of language, but also by the correspondences available between sets of points and sets of numbers. Little difficulty is involved in the shift of the concepts from sets of numbers to sets of points. Therefore, the topics of geometry will not be treated here.

There are a number of excellent texts and references on geometry available at grade levels, ranging from primary through the sixth grade. (A number of these are included in the bibliography.)

The solution of problems is sometimes a single answer, other times, several answers and, at times, no solution is available! To speak of solution sets, serves to underline these various possibilities. For example, the solution set to $x + 3 = 5$ is $\{2\}$. The solution set for the set of all whole numbers greater than 5 and less than 10 is $\{6,7,8,9\}$. The solution set for the set of all counting numbers less than zero is the empty set, $\emptyset$.

The discussion of applications in Chapter III is intended primarily as a guide. As the teachers and students become proficient in the language and operations, then, ingenuity and experience will suggest many more. However, content should not be warped to fit the ideas of sets. When a better approach is available, it should be used.

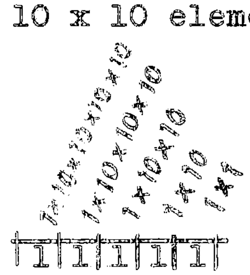
Chapter IV will indicate a few, further applications, but it is primarily intended to show the great utility of sets in clarifying some of the patterns and structures of the real number system.

CHAPTER IV

PATTERN AND STRUCTURE

Much of the fascination of mathematics lies in its orderliness, symmetry, and the simple and complex patterns of its varied systems. The reader may have been intrigued by the underlying patterns in the definitions of cardinal numbers (Page 12). Prior knowledge of place value was assumed, and consequently one of the most useful and fundamental patterns of a numeration system was not discussed.

Consider the meaning of the five-digit, counting number 11,111. Each of the digits is the numeral one, and yet because of the concepts of place and base, each one has a distinct significance. Order the digits from right to left. The first digit signifies one set of one element; the second digit indicates one set of ten elements; and the third digit denotes one set of ten subsets, each of which contains ten elements (1 x 10 x 10 elements). The fourth digit indicates one set of ten subsets, each of which is made up of ten subsets, and each of these ten subsets contains ten elements (i.e., 1 x 10 x 10 x 10 elements).



A less cumbersome explanation can be made by using exponential notation.

> Define the symbol b^n, for $n \in C$, to mean b is to be used as a factor n times. For example, $10^4 = 10 \times 10 \times 10 \times 10$.

This makes the place value pattern obvious. Consider the number 31,724, written in an expanded form, using exponents. $31{,}724 = (3 \times 10^4) + (1 \times 10^3) + (7 \times 10^2) + (2 \times 10^1) + (4 \times 1)$. If more digits are annexed to the left, then, the set of exponents of ten continues in the familiar pattern of the counting numbers. The first digit at the right, however, fails to conform. To be consistent, the one needs to be replaced by 10^0, but the previous definition of exponential notation is not meaningful here. Since 10^0 has no other meaning assigned to it, then, it is just a symbol and can be defined in any way one chooses! Hence, define 10^0 to be another symbol for one. Now the pattern of exponents from the units position to the left follows the pattern of the set of whole numbers in reverse order.

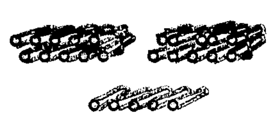
Does the pattern hold for a decimal such as 3,421.4532? 3,421 gives no difficulty but how about .4532? One way of writing this in an expanded form is as follows: $.4532 = .4 + .05 + .003 + .0002 = 4(1/10^1) + 5(1/10^2) + 3(1/10^3) + 2(1/10^4)$. A pattern is apparent, but, if possible, an extension of the previous set is more desirable. Since the exponents decrease from left to right, this would require $1/10^1 = 10^{-1}$, $1/10^2 = 10^{-2}$, etc.

Hence, define b^{-n}, where b and n $\in$ C, to mean $1/b^n$.

Thus, 3,421.4532 can be written $3(10^4) + 4(10^2) + 2(10^1) + 1(10^0) + 4(10^{-1}) + 5(10^{-2}) + 3(10^{-3}) + 2(10^{-4})$. This pattern of exponents turns out to be the set of all integers with the ordering reversed!

Next, lets explore a choice of a base set (or radix) for a numeration system. The decimal system utilizes base ten, but other bases have utility. Also, the study of bases other then ten serves to add interest and to reinforce the understanding of the decimal system. The binary system (base two) has the advantage of requiring a set of only two symbols, {0,1}, while the duodecimal system (base twelve) requires twelve, {0,1,2,3,4,5,6,7,8,9,X,E}. Twelve, however, has more factors than ten or two, and a large number can be written with fewer digits. Set S below has been partitioned to conform to three different bases, ten, two, and twelve. If b represents the base chosen, then, $b^0 = 1$ in every case.

Set S



2 sets of 10 elements

5 sets of 1 element

25_{ten}

Fig. 4-1

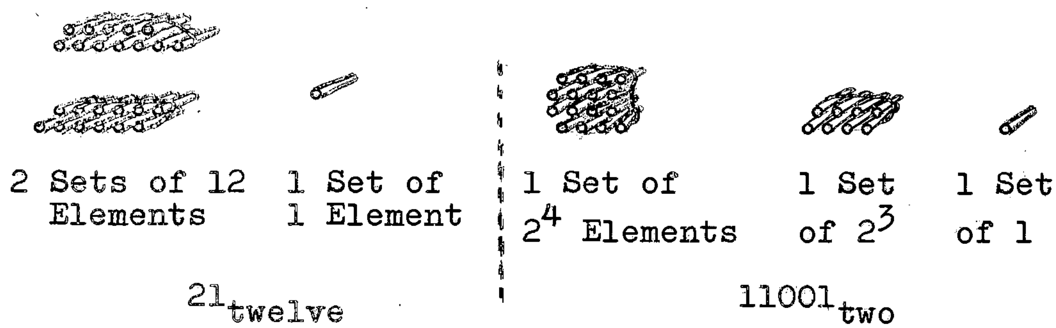


Fig. 4-2

Any number which can be represented in the decimal system can be represented in other bases by appropriate partioning of the set of elements. One needs only understand the pattern of exponents indicated in the diagram.

$$\ldots \; b^4 \;\; b^3 \;\; b^2 \;\; b^1 \;\; b^0 . \; b^{-1} \;\; b^{-2} \;\; b^{-3} \ldots$$

$$| \quad | \; 1 \; | \; 2 \; | \; 1 \; | \; 1 \; | \; 2 \; | \quad |$$

If the base is three, then, the number 121.12_3 has the value

$(1 \times 3^2) + (2 \times 3^1) + (1 \times 3^0) + (1 \times 3^{-1}) + (2 \times 3^{-2})$

$= \; 9 \; + \; 6 \; + \; 1 \; + \; 1/3 \; + \; 2/9$

$= 16\ 5/9$ in base ten.

It should be stressed that the previous discussion was devoted to a numeration system (or naming of numbers) and not a number system. The study of a number system involves the consideration of operations and properties of operations with the elements of the system. (For example, the set of whole numbers enjoys the commutative property under the operation addition.) Also, one studies the relative roles of the elements and various subsets within the system. In this way the structure is revealed and the weaknesses and advantages are discovered. An attempt was made in Chapter III to motivate an extension from the counting numbers to the real numbers. Each extension was made to find solutions which the preceding set did not include. The mathematician speaks of this property of including a solution as "closure under an operation."

The study of the structure is so fundamental that some of its terms should be a part of everyone's vocabulary. First of all, it is necessary to consider a set of elements and one or more operations.

The set of counting numbers, C, enjoys the following properties under the operations of addition and multiplication:

+	x	
closure	closure	(The sum or product of any two counting numbers is a counting number.)
associative	associative	a + (b + c) = (a + b) + c a x (b x c) = (a x b) x c
commutative	commutative	a + b = b + a and a x b = b x a
NO IDENTITY (i.e., no zero)	identity element (one)	a x 1 = a = 1 x a

Multiplication is distributive over addition. a x (b + c) = (a x b) + (a x c).

The distributive property deserves special emphasis here because it has not been mentioned previously. When taking a multiple of a sum, it is sometimes an advantage to distribute the factor over the addends before taking the sum. For example, 4 x (25 + 16) can be handled mentally with greater ease if the four is distributed first.

$$4 \times (25 + 16) = (4 \times 25) + (4 \times 16)$$
$$= 100 + 64$$
$$= 164$$

In contrast, the problem $4 \times (78 + 22)$ is best handled by first finding the sum and then multiplying by four. The distributive property is a formal statement of this choice of procedure. It should be noted that addition is <u>not</u> distributive over multiplication.

If the set of whole numbers is considered, then, the number zero is an identity for the operation addition. All other properties for multiplication and addition are the same as those for the counting numbers.

The set of integers yields additive inverse elements. Thus, for every number a, there is an ^{-}a, such that $a + (^{-}a) = 0$. The set of integers has the following properties under the binary operations of addition and multiplication:

+	x	
closure	closure	(If a and $b \in I$, then, $(a + b)$, and $(a \times b) \in I$).
associative	associative	$a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$
commutative	commutative	$a + b = b + a$ and $a \times b = b \times a$
identity	identity	$a + 0 = a$ and $a \times 1 = a$
inverses	NO INVERSES	$a + (^{-}a) = 0$

Distributive for multiplication over addition $a \times (b + c) = (a \times b) + (a \times c)$

The set of rationals has the advantage of having a multiplicative inverse available for every element <u>except zero</u>. Since the integers are embedded within the rationals, then, the rationals are considered an extension of the set of integers.

The set of rationals is said to form a field under the binary operations of addition and multiplication because it enjoys the following properties:

+	x
closure	closure
associative	associative
commutative	commutative
identity	identity
inverses	inverses (except zero)

Distributive for multiplication over addition

The desire to make possible the operation of finding a root (square root, cube root, etc.) forces further extension of the number system. The irrationals makes this possible for the positive numbers, but an extension to the complex system would be necessary for closure under this process. The complex system is rarely introduced until the student is ready for a formal course in algebra; therefore, it is not a pertinent topic in this discussion.

The real numbers (rationals and irrationals) enjoy all of the field properties. Also, they have the property of completeness. It is sufficient here to say this means that there is a one-to-one correspondence between the points on the number line and the set of real numbers. Each of the sets considered has the property of order and the rationals and reals have the property of "denseness."

A set is said to be dense if for every two distinct elements, there is another element between them.

The integers are not dense since there is no integer between any two consecutive integers.

If some of the language in this chapter, or preceding ones, seem overly demanding to the reader, it is primarily because it is a first introduction. With only a few exceptions, the terms are those used in contemporary, elementary arithmetic texts. The terms "partition" and "complementation" have not appeared in the texts with which the author is acquainted. However, these terms have been used profitably by the author in presentation of similar material to elementary teachers. Mathematical rigor has been sacrificed to permit communication. At the same time, the author attempted to avoid statements not consistent with more advanced mathematics.

The reader will recall that this writing is intended to give the concerned individual a broad view of the role of the language and operations of sets in the elementary arithmetic program.

Only the competent, well-informed teacher can determine the appropriate time to introduce these concepts. This time will depend upon the child's mathematical maturity, rather than grade placement and chronological age. Chapter V, however, includes suggestions based upon the content of current texts.

CHAPTER V

EVALUATION

The purposes of a formal evaluation of this study are twofold. Constructive criticism from qualified, impartial individuals is essential to assure accuracy and readability. Secondly, if the material is to serve its intended purpose, there must be evidence that it is appropriate and desirable. A great deal of skepticism is expected, and frequently desirable, when a large scale revision of an educational area is proposed. Hence, the opinions of experts in the field were solicited to assure credence in the study.

The author's committee suggested that the opinions come from teachers and supervisors of elementary arithmetic, college teachers involved with teacher preparation, mathematicians, and writers of mathematics texts related to the study. An initial list of thirty such individuals was compiled. This list was formulated with information from college catalogues, title pages of books, individuals suggested by the author's committee, and persons known by the author to be actively engaged in arithmetic education.

Each individual was contacted by letter requesting permission to send the material for evaluation. A postal card was included which required only a check to indicate consent. Twenty-four people responded in the affirmative and were sent the material. Eighteen of the twenty-four actually read and completed the evaluation.

Actual evaluation was by means of the questionnaire in the appendix of this study. The questionnaire required responses according to a scale identifying the degree of agreement with given statements. Suggestions were added by some respondents for corrections and improvements. These were primarily grammatical in nature and whenever possible were included in the final presentation. Two obvious errors were detected. One of these was in the original definition of a prime number (on page 39) and the other was a typing error in the expanded form of a number (on page 47). Both of these were corrected. Two individuals (both elementary supervisors in the public schools) asked if they could obtain the material in quantity in a published form.

Approximately two months elapsed between the mailing of the material and the response of a sufficient number of persons to assure some validity to the evaluation. This was not unexpected, especially since these people were all prominent in the field of education and hard-pressed for time in their own busy schedules. Some of the original thirty wrote letters indicating their present obligations did not allow time to participate in the evaluation.

The eighteen who finally responded fell into three general categories: supervisors of elementary teachers, college teachers of mathematics, and authors of recent books related either to "sets" or the contemporary arithmetic program.

There was some overlap in the categories of "authors of texts" and "college teachers". However, where the college teacher had written a text, especially appropriate to the concerned area, he was categorized as an "author".

The writer realizes that the assumption of a "two valued" logic cannot be completely assumed in a question which permits different degrees of opinion. The very design of a "scaled answer" is based upon this consideration. However, if the reader will arbitrarily agree to scale "disagreement" as opposed to "agreement", some evaluation of statements "unfavorable" to the thesis can be made simultaneously with the "favorable" statements. For example, agreeing to a statement such as "The concepts are accurately presented" seems favorable, while "Greater elaboration is needed" seems unfavorable. A scaled agreement of "2" to an "unfavorable" statement can, then, be interpreted as indicative of a response of "3" favoring the thesis. Hence, agreement to statements 4, 5, 14, 15, 19, and 20 of the questionnaire, will be interpreted as "unfavorable" while agreement to other statements are considered "favorable".

Using the approach suggested in the preceding paragraph each scaled value of responses to "unfavorable" statements can be converted by subtracting the tabulated value from five. All other scale values are left unchanged. The converted response averages for these questions appear in the following table:

Question	Average Response	Responses by Category		
		Elem.	Coll.	Auth.
4.	4.1	4.8	3.7	4.0
5.	3.5	4.4	3.0	3.3
14.	4.8	4.8	4.7	4.8
15.	2.6	2.0	3.1	2.3
19.	5.0	5.0	5.0	4.8
20.	4.3	3.6	4.4	4.8

Incorporating these changes permits computation of an average scale value of "agreement" over the complete questionnaire. The arithmetic mean of the "Average Response" column is 3.7, of the "Elementary Teachers" column is 4.1, the "College Teachers" column is 3.7, and the "Authors" is 3.3. Of the three categories, the "Authors" response seemed least favorable, but 3.3 still appears to be an encouragingly high evaluation.

Responses of 0.2 and 3.0 to the opposing statements 14 and 15 indicate that the "Elementary Supervisors" felt more elaboration would be desirable. Responses of 0.2 and 2.7 to the same questions in the "Authors" category agrees with this reaction. The "College Teacher" category, however, responded with 0.3 and 1.9, implying that they felt the presentation was ample. All other questions had favorable scale responses in the two categories of "Elementary Supervisors" and "College Teachers". The "Authors" responded to questions 1, 3, 11, 12, 13, and 16 with scale values ranging from 2.3 to 2.8 (i.e., the responses varied between "some agreement" and "considerable agreement"). While none of these scale values seem particularly unfavorable the author wishes to point out that the sample size permitted one or two questionnaires to skew the mean below "3" in this particular column of these statements. Also, most of these questions are peculiarly pertinent to the appropriateness and adequacy of presentation. It seems reasonable to assume that a college teacher should be well informed in this area and the mean response of the "College Teacher" category to these same questions ranges from 2.9 to 3.7. The "Elementary Supervisors" category responded with 3.0 to 4.2. Initially, more elaboration was intended but careful consideration of suggestions of teachers modified this. Some potential readers would be discouraged by the very bulk of an elaborate presentation. Also, one of the primary purposes is to present an overview or perspective to the role of sets. A profusion of examples and detail in any one area would hamper the achievement of this aim.

The respondents who were supervisory elementary teachers offered surprisingly high support of the study with a mean response of 4.1. If this is indicative of the reaction of the elementary teacher in general, the author has realized a primary goal. Much of the material has been tried by the writer in various workshops with a similar reception.

There were of course varied reactions to every statement, but the overall mean response of 3.7 lends a high degree of support to the presentation.

Although the author did not embark upon this procedure without some apprehension, it seemed appropriate. The material was designed to permit the unassisted reader to gain an introduction to sets and should not require additional explanation.

The tabulated results of the replies are summarized in the table on page 60, and a graphical summary of scaled responses to each question has been added to the questionnaire on page 74.

QUESTIONNAIRE RESPONSES

Question	Answers Per Scale Number 0	1	2	3	4	5	Average Response	Responses by Category: Elem.	Coll.	Auth.
1.	0	1	3	3	7	4	3.5	4.4	3.6	2.8
2.	0	0	1	5	9	3	3.8	4.0	3.7	3.7
3.	1	1	2	8	5	1	3.0	4.0	2.9	2.3
4.	11	4	0	1	1	1	0.9	0.2	1.3	1.0
5.	5	5	4	2	2	0	1.5	0.6	2.0	1.7
6.	0	0	2	2	10	4	3.9	4.2	4.0	3.5
7.	0	0	1	4	7	6	4.0	4.6	3.7	3.7
8.	1	0	1	9	6	1	3.2	3.4	3.6	2.7
9.	0	1	0	6	9	2	3.6	3.8	3.7	3.3
10.	1	0	0	3	11	3	3.8	3.8	4.1	3.3
11.	1	1	3	5	8	1	3.3	4.0	3.6	2.3
12.	0	2	1	2	10	3	3.6	4.2	4.0	2.7
13.	1	0	1	5	6	4	3.4	4.0	3.4	2.8
14.	14	4	0	0	0	0	0.2	0.2	0.3	0.2
15.	3	3	3	4	2	3	2.4	3.0	1.9	2.7
16.	0	1	3	6	6	2	3.3	4.2	3.0	2.8
17.	0	1	3	7	5	2	3.2	3.4	3.1	3.1
18.	0	0	2	6	7	3	3.6	4.0	3.7	3.1
19.	17	1	0	0	0	0	0.0	0.0	0.0	0.2
20.	11	4	2	0	1	0	0.7	1.4	0.6	0.2
21.	1	0	2	2	9	4	3.7	4.0	4.0	3.0
22.	1	0	2	1	8	6	3.8	4.4	4.1	3.0
23.	1	0	1	3	6	7	3.9	4.2	3.9	3.1

CHAPTER VI

CONCLUSIONS

The improvement in mathematics, as in every other area, should be a continuing process that never terminates. To designate and delineate a fixed program at any level would be a mistake. There must always be room to try new approaches and content. The teacher must have sufficient knowledge and freedom to foster the students' curiosity, and to encourage the students' fascination and understanding of the role of mathematics. A broad perspective is desirable to properly direct the individual as he experiences and discovers the realm of mathematics.

"... it is hoped that everyone recognizes good mathematics education to be a sequential experience. Thus, the teacher at any particular level should have an understanding of the mathematics which will confront the student in subsequent courses; and as a consequence, it is desirable that a teacher at a given level be prepared to teach at least some succeeding courses."[1]

"All avenues of investigation of the foundation of mathematics converge toward set theory. In a systematic, deductive

[1] "Recommendation of the Mathematical Association of America for the Training of Teachers of Mathematics," American Mathematical Monthly, Vol. 67, No. 10 (December 1960), p. 72.

development of mathematics, all constructions radiate out from set theory like the spokes of a wheel from a hub. Set theoretic concepts are as necessary to mathematical discourse today as common nouns are to ordinary discourse. The analogy invoked by this remark, by the way, is not a superficial one in view of the fact that every common noun defines a set."[2]

The introductory remarks are intended to stress two points. First, no fixed boundaries can or should be drawn on the role of sets in the study of elementary arithmetic. Rather, limitations should depend on the mathematical maturity of the individual and whether the language and operations of sets contribute to the concepts being introduced. Although the study of set theory, divorced from other areas, is an interesting and important endeavor, it is not appropriate to the elementary arithmetic program. Second, the ideas of sets are <u>unavoidable</u> in elementary arithmetic. Whether the specific term "set" is used or not when attention is called to a collection, then, the idea is present. It seems only logical that if terminology is available appropriate to the elementary program and useful throughout the sequential development of advanced mathematics, then, it should be used.

For the primary grades, both the term "set" and the braces used to symbolize set appear in some of the current texts. Sets are used to aid the student with "greater than" and "less than" and to strengthen the understanding of number. At least

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[2]Irving Adler, "The Changes Taking Place in Mathematics," (Address delivered June 19, 1961, at Conference on Mathematics), U. S. Office of Education, Washington, D.C.

one text,[3] uses the idea of union to define addition and subtraction. Texts which introduce the term "set" in the primary seem to avoid "subset." It is perhaps considered too difficult for the student. The students work with subsets throughout the primary program, however, and the author questions its omission from the content. At least one text presents the word and idea of intersection[4] at the primary level. The students may well have sufficient experience with the concept to grasp this term readily (e.g., consider the "intersection" of two roads or the intersection of the set of boys who like football with the set of those who like basketball). The symbol for union is introduced and used extensively in another text[5] written for grades one and two. If one surveys contemporary texts at various levels, however, it will be found that initial introductions to these terms can be found in intermediate, secondary, undergraduate, and graduate text books. This should not be interpreted to mean that the terminology is inappropriate to the primary level, but rather that the authors are aware that traditional texts have not previously included such an introduction.

Hence, it appears that the language and operations of sets have appropriate uses at every level of mathematics.

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[3]Patrick L. Suppes, Sets and Numbers, Grades 1 and 2 (New York, 1963).

[4]Newton Hawley and Patrick Suppes, Geometry for Primary Grades, Book 1 (San Francisco, 1961), p. 14.

[5]Ibid.

Teachers must make decisions concerning the amount and usefulness of sets in a given situation.

Sets have a highly useful role in the introduction of the abstract idea of "number." They offer a method for explaining the operations of counting, addition, and multiplication. Operations with sets have analogies to the relations of "greater than," "less than," "equal to," and "equivalent to." The relations of the set of real numbers to its various subsets can be expressed by this language. The language and operations of sets serve to unify number and geometric concepts. Finding common factors and multiples are conveniently interpreted by the use of operations of sets. Place value in numeration can be explained by the use of sets. Pattern and structure in mathematics are more conveniently and precisely discussed by the language of sets. The terminology of sets serve to add continuity and unification throughout the domain of mathematics. The teacher should not attempt to bend the mathematics to fit the language, but every effort should be made to use such tools when they can be of service. The power of considering a set of abstract objects and the properties under a given operation must be stressed again and again. In this lies the process of generalization, and the methods of interpreting other mathematical systems on the basis of familiar systems. (For example, if an individual understands that the set of counting numbers are commutative under addition, it should be far less difficult to define for him the same property for the rationals.) The language and operations of sets offer a convenient and efficient avenue for probing the level of mathematical maturity of the student without simultaneously isolating him from familiar ground.

A careful survey of contemporary texts reveals that some fail to consistently make use of the language and operations of sets in situations where they have obvious applications. This weakens the unifying advantages of sets. Familiarity should increase the powers of simplification and clarification.

This study and the authoritative opinions of the evaluating jury offers support to the author's premise that the language and operations of sets have value in the presentation of arithmetic at the elementary level. The writer recommends, however, that actual classroom studies be made to evaluate the immediate and long-range results. Comparisons need to be made between traditional approaches and contemporary approaches to specific topics. Longitudinal studies are needed to determine the influence such presentations have on success in subsequent studies.

SELECTED BIBLIOGRAPHY

Adler, Irving. "The Changes Taking Place in Mathematics." (Address delivered June 19, 1961, at Conference on Mathematics). U. S. Office of Education, Washington, D. C.

Banks, J. Houston. Elements of Mathematics. New York: Allyn and Bacon, 1961.

The material in this book is designed specifically for teachers with no prerequisites beyond elementary arithmetic. It would be an excellent reference book for the teacher's personal library.

Committee on the Undergraduate Program in Mathematics. Elementary Mathematics of Sets with Application. Ann Arbor: Cushing Malloy, 1959.

This is a book written by a committee of the Mathematical Association of America. It is a comprehensive text that would be adequate background for a secondary teacher. An elementary teacher who is particularly motivated to pursue the concepts of sets may well find this appealing.

________. "Recommendation of the Mathematical Association of America for the Training of Teachers of Mathematics." American Mathematical Monthly. Reprint Publication from Volume 67, No. 10. (December, 1960) p. 72.

This pamphlet contains stated recommendations for teacher training in mathematics from primary through secondary.

Educational Research Council of Greater Cleveland. Elementary Mathematics Series. Chicago: Science Research Associates, 1961.

This series of texts has a definite contemporary approach. Teacher's editions are available.

Gray, James F. Sets, Relations, and Functions. New York: Holt, Rinehart and Winston, 1962.

This is a paperback text written specifically for secondary teachers. Most, if not all, of the material is within the reach of a high school student with one year of algebra. It includes answers.

Hamilton, Norman T. and Joseph Landin. Set Theory, The Structure of Arithmetic. Boston: Allyn and Bacon, 1961.

The early parts of this text are readable with only an elementary arithmetic background. Latter parts demand concentrated effort and some mathematical maturity. It is intended primarily for high school teachers.

Gundlach, Bernard H. Glossary of Arithmetical-Mathematical Terms. Dallas: Laidlaw, 1961.
An economical glossary of terms found in contemporary arithmetic texts are provided in this paperback.

Hawley, Newton and Patrick Suppes. Geometry for Primary Grades. San Francisco: Holden-Day, 1961.
This text introduces some language from both set concepts and geometry for grades one through four. Teacher guides are available.

Levi, Howard. "Why Arithmetic Works." The Mathematics Teacher. (January, 1963, p. 2).
Mr. Levi has written an article that should answer some of the questions for individuals who fail to see a need for changes.

Johnson, Donovan A. and William H. Glenn. Exploring Mathematics on Your Own. Dallas: Webster, 1960.
This is a series of enrichment booklets that can be read by some sixth grade students as well as junior and senior high. The booklets: Sets, Sentences and Operations, Understanding Numeration Systems, and Number Patterns are especially pertinent.

McFadden, Myra, William J. Moore, and Windall J. Smith. Sets, Relations and Functions. New York: McGraw-Hill, 1963.
This is a programmed unit which is much more inclusive than Language of Sets. The material is comprehensive enough to include all symbolism, terms, and operations commonly used in the elementary or secondary mathematics program. This would be an excellent self-study for the elementary teacher who desires a broader range of knowledge.

Mechner, Francis, and Donald A. Cook. Language of Sets. New York: Appleton-Century-Crofts, 1963.
This is a very elementary programmed instruction text which includes only the minimum amount of terminology and operations. It is well within reach of the sixth grade child. It would be used primarily for learning the language.

Morton, Robert L. et al. Modern Arithmetic Through Discovery. Dallas: Silver Burdett, 1963.
This is a transitional text with a blending of traditional and contemporary language. It leans toward structure and patterns, but does utilize

extensively the language of sets.

National Council of Teachers of Mathematics. Twenty-Fifth Yearbook, Instruction in Arithmetic. Washington, D. C.: N. C. T. M., 1960.
This is a fine reference book for the teacher's library.

_______. Twenty-Fourth Yearbook. The Growth of Mathematical Ideas, Grades K-12. Washington, D. C.: N. C. T. M., 1959.
Here is an excellent reference book for the elementary teacher that gives the picture from the elementary through the secondary program.

Peterson, John A. and Joseph Hashisaki. Theory of Arithmetic. New York: Wiley, 1963.
This textbook is one of the most readable texts for elementary teachers that utilizes a completely modern approach to the presentation of the various number systems. The unit on geometry included in the text is inadequate and fails to utilize current terminology but the early chapters are very well written.

School Mathematics Study Group. Studies in Mathematics (Volume VI, Number Systems; Volume VII, Intuitive Geometry). California: Stanford University Press, 1960.
These two books are made up of selected materials from the S. M. S. G. texts specifically for the training of elementary teachers.

_______. Mathematics for the Elementary School. California: Stanford University Press, 1960.
The S. M. S. G. texts have probably been the most influential series in current trends in the United States. Texts and excellent teacher's commentaries are presently available from grades four through high school. Texts for the first three grades are being written and tested.

Suppes, Patrick. Sets and Numbers, Books 1-A and 2-A. Dallas: L. W. Singer Co., 1962.
These books are textbooks for grades one and two, which utilize far more language and operations of sets than the majority of contemporary primary books. Teacher's guides are available.

Van Engen, Henry et al. Seeing Through Mathematics. Dallas: Scott, Foresman, 1961.
The teacher's edition of the junior high texts would be a useful reference for the elementary teacher. The seventh and eighth grade texts are completely modern in flavor and the teacher's edition includes background material and answers.

Zehna, Peter W. and Robert L. Johnson. Elements of Set Theory. Boston: Allyn and Bacon, 1962.
This book is recommended for the individual who has some prior knowledge of the language of sets, but wishes to pursue it further. All basic definitions are included, however, and it could be used without previous experience.

APPENDIX

REQUEST FOR JURY PARTICIPATION

August 7, 1963

(Inside Address)

Dear

I am currently doing a study on the role of the language and operations of "sets" in the elementary arithmetic program. This is a part of my thesis toward a degree at Oklahoma State University.

Because of your prominence in the field, my committee has suggested that I ask you to comment on what I have written. The comments will be in the form of a check-type questionnaire, with space for any suggestions you might care to make. It will not be necessary for you to return any material other than the brief questionnaire. The reading consists of approximately fifty-eight, double-spaced pages.

Would you please indicate on the enclosed postal card whether or not I may have the privilege of sending you the material.

Yours very truly,

Raymond McKellips

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QUESTIONNAIRE

(Graph below scale added after summary of responses was made)

The following statements pertain to the material on the role of sets in the elementary arithmetic program. Please indicate by encircling the appropriate number after each statement the extent to which you agree with the statement according to the following scale:

0 Disagree
1 Only slight agreement
2 Some agreement
3 Considerable agreement
4 Highly agree
5 Very highly agree

Example: Too great an emphasis is sometimes placed on rule memorization.
0 1 2 3 4 5

This would mean that you highly agree with the statement.

---

1. The presentation would encourage the use of the language and operations of sets in teaching arithmetic.
0 1 2 3 4 5

2. The scope is adequate for an initial introduction.
0 1 2 3 4 5

3. The material is appropriately written for the elementary teacher.
0 1 2 3 4 5

4. More material of greater depth should be included.
0 1 2 3 4 5

5. The author has overestimated the mathematical understanding of the elementary teacher.
0 1 2 3 4 5

6. The ideas are organized in a sufficiently logical sequence.
0 1 2 3 4 5

7. The concepts are accurately presented.
0 1 2 3 4 5

8. Assuming no prior introduction, the material has enough interest to encourage further reading.
0 1 2 3 4 5

9. There is sufficient stress on the use of sets in advanced study.

0 1 2 3 4 5

10. The thesis gives a sufficiently broad view of the role of sets in arithmetic.

0 1 2 3 4 5

11. Enough examples are included to clarify concepts.

0 1 2 3 4 5

12. The author has shown how sets add unification to arithmetic.

0 1 2 3 4 5

13. The thesis indicates how sets clarify and simplify arithmetic.

0 1 2 3 4 5

14. The development is too "wordy."

0 1 2 3 4 5

15. Greater elaboration is needed.

0 1 2 3 4 5

16. An adequate <u>skeletal</u> outline for an introduction to the real number system is included.

0 1 2 3 4 5

17. The interrelations of the various number systems are sufficiently indicated for the proposed audience.

0 1 2 3 4 5

18. The development has adequate continuity.

0 1 2 3 4 5

19. The author has applied sets to topics where they are not applicable.

0 1 2 3 4 5

20. The presentation should incorporate greater use of symbolism.

0 1 2 3 4 5

21. The use of symbols is adequately defended.

0 1 2 3 4 5

22. The conclusions seem valid.

0 1 2 3 4 5

23. The "format" is appropriate.

0 1 2 3 4 5

VITA

Raymond Leon McKellips

Candidate for the degree of

Doctor of Education

Thesis: THE ROLE OF THE LANGUAGE AND OPERATIONS OF SETS IN THE ELEMENTARY ARITHMETIC PROGRAM

Major Field: Higher Education

Biographical:

Personal Data: Born at Custer City, Oklahoma, October 28, 1927, the son of Ray and Lillian McKellips.

Education: Attended elementary school at Custer City, Terlton, and Fay in Oklahoma; graduated from Fay High School, Fay, Oklahoma in 1945; received the Bachelor of Science degree from Southwestern State College, Weatherford, Oklahoma in August, 1951, with a major in mathematics and a minor in art; received the Master of Science degree in August, 1956, with a major in educational administration, from Oklahoma State University, Stillwater, Oklahoma; completed requirements for the Doctor of Education degree at Oklahoma State University, Stillwater, Oklahoma, December, 1963, with a major in higher education.

Professional Experience: Entered the teaching profession in 1949; was mathematics teacher and principal, in that order, for ten years in the Oklahoma public schools; was teacher of mathematics at Southwestern State College, Weatherford, Oklahoma, for three years; served as staff assistant, Oklahoma State University Mathematics Department, summer of 1963.

Professional Organizations: Oklahoma Council of Teachers of Mathematics, National Council of Teachers of Mathematics, Oklahoma Education Association, Mathematical Association of America.