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A GENERALIZED PULSE TESTING TECHNIQUE FOR LINEAR SYSTEM IDENTIFICATION

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A GENERALIZED PULSE TESTING TECHNIQUE

FOR LINEAR SYSTEM IDENTIFICATION

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ABSTRACT

Certain results from the theory of ordinary differential equations have been combined with the concepts of that area of systems work concerned with the determination of mathematical models for physical processes. The result is a coherent structure which defines a novel method by which the characterization of linear systems, regardless of their particular time dependent nature, may be achieved.

The relationship between the system characterization achieved by means of the technique and the familiar impulse response and (where applicable) the transfer function is elucidated through examples dealing with the identification of hypothetical systems.

The somewhat obscure expression resulting from an error analysis of the matrix operations involved in the estimation of the system weighting function has been replaced with a practical approximation which is shown to be valid.

The technique is applied to the determination of the heat transfer dynamics of a jacketed backmix chemical reactor. Results of this study indicate that the cost of obtaining a mathematical model is greater than those associated with pulse testing or frequency response testing. However, the

relative costs reverse if the reactor is considered as a multiple input/output system.

The technique is applied to the identification of the dynamic response of a hypothetical system which is highly time-varying. Results of this study indicate that the cost of obtaining a model for such a process are no higher than those for a similar time invariant system.

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Kenneth Allan Bishop

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CHAPTER I

INTRODUCTION

The acquisition of information concerning physical entities, phenomena, or processes by any means other than direct measurement involves the use of a model of the situation. The veracity of this statement becomes obvious upon reflection on the methods for determining such diverse items as the height of a tree, the time of the next equinox, or the temperature increase resulting from isentropic compression of an ideal gas. The laws of Euclidian geometry serve as the model in the first example. These laws are combined with Newton's laws of motion to model the second situation. In the third example, the equation of state for an ideal gas together with the first and second laws of thermodynamics form the basis for calculating the required temperature rise.

Other common forms of models include the use of an electronic circuit (analog computer) to model the behavior of an automobile suspension system following a bump in the road, the use of iron filings on a sheet of paper to elucidate

two dimensions of the force field surrounding a permanent magnet, and the use of an actual physical model in a wind tunnel to study aircraft performance. While very little imagination is needed to extend this list indefinitely, certainly the most common form of model is the mathematical statement of the physical laws which apply to a given situation.

These mathematical expressions generally consist of combinations of algebraic, differential and integral equations, together with appropriate boundary conditions and/or constraints. Such a model must, of course, be specialized to the particular situation through the inclusion of measured physical parameters as coefficients of the model.

The intelligent application of such a model rests on two fundamental assumptions.

1. The mathematical model is truly descriptive of the physical phenomena or process.
2. The cost of obtaining and employing the model is less than that of making the required measurements.

In general, physical phenomena and processes can be described in terms of differential equations which have the familiar mass, energy, and momentum balances as their bases. These equations commonly involve non-linear functions of time and one or more space coordinates, as well as multiple sources of excitation. While such equations can be derived, their

usefulness may be severely limited by the cost associated with specializing and employing them. This cost includes the difficulties of measuring the particular physical parameters involved and of actually solving the model equations.

Traditionally the path around the implied dilemma has been to sacrifice some of the descriptive power of the model in a trade for a lessening of the cost of its attainment and subsequent use. This trade is accomplished by adopting a more tractable model of the physical situation, through the introduction of simplifying assumptions, and establishing a "goodness" criterion to limit the range of its application.

Common examples of this approach include such assumptions as the Navier-Stokes constraints on a flowing fluid, perfect mixing of the liquid on the trays of a distillation column, Newton's cooling law as a description of the heat flux across a solid-fluid boundary, and a truncated Taylor series as a "linearization" of product type non-linearities (see Stewart (32)).

While notable exceptions, such as the description of turbulent velocity profiles, certainly do exist, the adoption of relatively simple mathematical models has facilitated the acquisition of a vast amount of information on the behavior of physical phenomena and processes. It seems doubtful that the great technological advances of recent years would have been possible without the information gained through the use of models.

Mathematical Models in Automatic Control

One area of application for mathematical models which has received major emphasis in the last few years is that area of systems work commonly designated as process dynamics or system identification. The precise control of a physical process, regardless of its nature, depends on the satisfaction of two requirements:

1. A thorough knowledge of the cause and effect relationships which operate, within the process, to determine its response to a specific excitation.
2. A well defined control strategy, the application of which results in optimal (in some sense) behavior of the process.

It is clear that without a relatively complete model of the system's dynamic behavior, the spectrum of possible control strategies which may be employed is very narrow, hence the interest in techniques for obtaining mathematical models on the part of the control engineer.

In the area of system identification, the approach of adopting a more tractable model through the introduction of simplifying assumptions has usually been tantamount to the assumption that the response of the process in question may be adequately described by the solutions of linear, constant coefficient, ordinary differential equations. Models of this simple form are eminently usable, generally consistent with the quality of measurements available, and therefore appropriate for most applications.

Throughout the first decade of application of feedback control theory to chemical processes the emphasis was on the solution to the so-called regulator problem. The reasons for this emphasis lay in the fact that the simple model's validity was limited to narrow ranges of the process variables, and in the economic structure of the chemical industry itself.

The economic structure of the industry was such that while the savings in operating costs attainable by smoothing out the operation of the plant were certainly large enough to warrant automatic control, it was not clear that the investment required to develop truly optimal control of the plant could be justified in terms of additional profit return.

The economic picture has changed rapidly over the last decade. Today it is mandatory that the last few per cent of yield (of which a plant is capable) be obtained merely to hold an economic position. Consequently the goals of process control have changed.

A few years ago manual operation of plants was employed during start-up and operating level changes, automatic control being employed only when "steady state" had been established. Today processes are controlled automatically through minor level changes and one would like to consider control through major changes--control that would minimize the time during which off specification product is produced and accomplish the change at minimum operating cost. Another approach to the problem of maximizing profit involves innovations in operating

procedures wherein the plant is operated on a transient basis. McWirter and Lloyd (26) have discussed this concept in connection with operation of distillation and extraction units in a controlled cycle mode.

Today the situation is that most large companies find themselves replete with optimal control strategies, or rather with methods for defining them, i.e. the Maximum Principle (29), the Principle of Invariance (21), Dynamic Programing techniques (4), etc. Unfortunately this wealth of optimization techniques is balanced by a dearth of plant applications. This insufficiency does not imply that the control problems of interest are all solved; rather it is evidence of the mismatch that exists among the descriptive powers of the usual process model, the complexity of the physical processes of interest, and the class of systems to which the optimization techniques may be applied. Further, it is not clear that the cost of correcting the mismatch is even finite, let alone reasonable.

In order to make the inevitable decision as to whether or not it is possible to justify the expenditure necessary to develop practical optimal control policies for complex physical processes one requires two types of information:

1. An estimate of the improvement in operation and hence economic position which would result from truly optimal plant operation.
2. An estimate of the increased cost associated with obtaining and using a more complex mathematical

model in the implementation of some optimal control strategy.

It is clear than an investigation to determine this information in a general context would be nearly impossible due to its complete dependence upon the specific physical process involved and the specific optimization technique and model generalization envisioned.

However, one small step toward acquisition of the necessary information is possible through investigation of the increased cost associated with a specific generalization of the class of equations used as models.

Basis for This Investigation

The investigation reported herein deals with the removal of the constant coefficient constraint on the differential representation of the process model. This particular constraint was chosen for attack for two reasons:

1. There exists a well established and complete theory pertaining to the solution of linear differential equations, thus permitting a general approach to the problem of obtaining a model (12, 22, 28, 33, 36).
2. There are many important physical systems which are characterized by linear time-varying dynamic behavior; therefore the results of the investigation are of considerable practical interest (8, 14, 19, 26, 31).

The intent to evaluate the cost of obtaining models for physical processes whose behavior is linear but not necessarily time invariant implies the existence of suitable techniques for specializing linear differential equations to a particular system. A review of the capabilities of the standard plant testing techniques indicates that:

1. Frequency response techniques are applicable only if the system to be identified is time invariant (1, 17, 18, 35).
2. Statistical techniques for the analysis of non-stationary systems (time-varying) are enormously complex due to the loss of the ergodic property (9, 23, 24).
3. Pulse testing techniques are perfectly applicable to the identification of linear systems but the requirement for forcing large amounts of "energy" across the boundaries of a physical system in a short time often presents experimental difficulties (16, 23).

The theory of ordinary differential equations provides the basis for a novel technique for system identification which is recognized as a generalization of the pulse testing method. This technique involves measurement of the unforced transient response of the physical system and provides characterization of the system in terms of weighting functions. The transition from these weighting functions to the differential

representation of the model is seen to be a relatively simple one.

The technique possesses a number of desirable features. Four of these are of prime importance.

1. General applicability to linear systems.
2. Ease of application to multiple input/output systems.
3. Relatively mild system perturbation required.
4. Very simple calculations required.

There are, of course, important undesirable features of the technique:

1. Severe requirements on the measurability of process variables.
2. High precision measurements required, since
3. Calculations involve numerical matrix operations.

Purpose and Scope of the Investigation

This investigation was undertaken for the purposes of elucidating the generalized pulse testing technique, assessing its value as a practical tool for the identification of linear process models, and, in so far as practical, comparing the cost of its use with those of the standard techniques.

The elucidation of the generalized pulse testing technique is accomplished by means of a parallel presentation of the formal theory and pertinent example. Three examples are presented which detail the application of the technique to hypothetical systems of up to the third order.

Two of these examples involve numerical calculations and thereby introduce the necessity for consideration of the effect of small imprecisions in the response data on the quality of the weighting function estimate. This consideration leads to the development of a formal expression for the error growth in the combined operations of matrix inversion and multiplication required to estimate the weighting functions.

The usefulness of this expression is obscure since it involves the knowledge of the true system response. Therefore a practical method for estimating the maximum error associated with the computational procedure is developed. This method is used, in connection with the estimation of weighting functions for a hypothetical system from systematically corrupted response data, to gain an appreciation for the magnitude and nature of weighting function quality degeneration due to types of error which are considered pertinent to the experimental phase of the investigation.

The technique is applied to the identification of a linear model for the heat transfer dynamics of a backmix chemical reactor. This study was made in order to determine whether or not the data obtainable by experimental measurement of the response of a physical system may be successfully used in the calculation of weighting functions by the matrix methods, and to assess the cost of applying the technique to a time-invariant physical system.

The application of the technique to a hypothetical time-varying system is made in order to assess the cost of

computing estimates of weighting functions for a time-varying system and to introduce the concept of redundant testing of physical processes.

While the relationship of the coefficient matrix of the differential equation representation of the process model is employed to establish the validity of the model for the backmix reactor, the specific problems of obtaining the differential representation from the weighting functions have been arbitrarily placed outside the scope of this investigation. Similarly, practical considerations of the cost of obtaining unforced response information from time-varying systems have arbitrarily been excluded.

Literature on Time-Varying Systems

The vast majority of reported effort in the area of time-varying linear systems is to be found in the electrical engineering and Russian automatic control literature.

Because of the fact that the electrical engineer is primarily interested in either the analysis problem (calculation of the response of a system for which a model is available) or the synthesis problem (design of a circuit which possesses a desired response) there has been very little attention given to the problem of identifying a mathematical model for an existing physical system. Consequently the literature which has been found beneficial to this investigation deals with the peripheral (although important) problems of making the transition between various forms of mathematical models and their subsequent employment.

Bennett (5) has provided a survey of the effort, in this country, prior to 1950. Zadeh (38) has surveyed the area and has compiled a very complete bibliography of publications through 1960. An entire issue of the IRE Transactions (Circuit Theory) (11, 18, 28), published in 1955, was devoted to the "State of the Art". These three sources represent an excellent introduction to the literature of time-varying systems.

Pipes (28) has described the four basic techniques for the analysis of time-varying systems: Classical solution of the differential equation, Matrix theory application (essentially the state space concept, see Zadeh (34)), use of the Brillouin-Wentzel-Kramers approximation of Quantum Mechanics, and the use of Laplace transforms and integral equations in the solution of variable parameter problems.

A variation, (using the concept of the adjoint system) of the classical approach is given by Matyash (25) and by Aseltine and Faurear (2) in papers which discuss the application of analog computer techniques to the solution of linear non-constant coefficient differential equations.

Aseltine (1), Gerardi (17), Gerlach (18) and Gilbert (20) have done an excellent job of describing the analysis of systems which may be described in terms of Bessel's or Euler-Cauchy differential equations, as well as those for which an admissible solution may be expressed in terms of an infinite summation of functionally weighted solutions to constant coefficient equations.

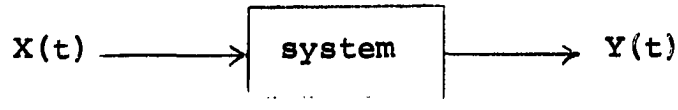
A number of authors (3, 8, 9, 11, 12, 27, 35, 38) have written on the problems associated with the transitions between the differential, integral, and frequency domain representations of the model for physical systems.

Emel'yanov and Taran (14) and Shigin (31) have published papers which consider the application of automatic control systems which contain variable physical parameters to the control of time-varying physical systems. Gibson and Meditch (19) have discussed the problems of real time control of variable parameter processes.

CHAPTER II

THEORETICAL ASPECTS OF THE GENERALIZED PULSE TESTING TECHNIQUE

Consider the operation of the linear system indicated schematically below. This operation may be described, implicitly, in terms of a linear differential equation or, explicitly, in terms of an integral equation.



The implicit description of the system's operation in terms of a differential equation consists of a linear vector differential equation, a value taken on by the output vector at some point in time and the statement that the equation is valid at that time:

$$\dot{Y}(t) = A(t)Y(t) + X(t) \qquad Y(T) = Z, \quad T \in t$$

$\dot{Y}(t)$ is the time derivative of the system's N-dimensional output vector, Y(t). X(t) is an N-dimensional vector which represents the system's input. A(t) is an N by N matrix of system parameters which describes the internal structure of

the system's operation. A restriction on the class of systems covered is that no variation in the system parameter matrix, A(t), may be dependent on the nature of the input vector, X(t).

The differential description of the system as given here is generally referred to as the "state space" formulation (34). In this formulation, the state of the system is regarded as being described by the point, in an N-dimensional vector space, defined by the system's output vector, Y(t), at any instant of time. The path taken by the state vector, Y(t), in moving from one state to another is to be regarded as a state trajectory. The point, Y(T) equals Z, is a point on the particular trajectory being considered, thereby completing the definition of the state of the system at any time. The symbolic statement: $T \in t$, is an assurance that time T is included in the range of time for which the formulation is applicable.

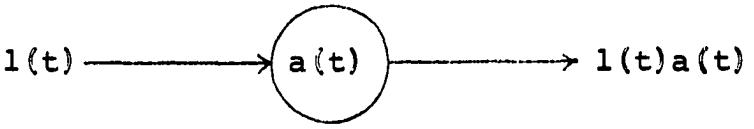
The discussion which follows is more lucid if a second order linear system, the operation of which is described by the vector differential equation:

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \times \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} , \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

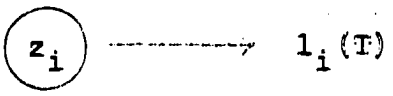
is considered. A detailed schematic of this system may be constructed if three kinds of operational elements are defined:

1. Energy Storage Elements: operational elements which perform analogously to the mathematical operations of combined summation and integration; symbolically:

2. Multiplication Elements: operational elements which perform analogously to the mathematical operation of multiplication by a function of time; symbolically:



3. Initial Value Elements: elements which perform analogously to the mathematical operation of provision of an instantaneous value; symbolically:



The schematic of this second order linear system is given on the following page. Referring to the schematic, the interpretation of the passage of signal $y_j(t)$ through an element $a_{ij}(t)$ is multiplicative modification of the output of the jth energy storage element prior to its entry to the ith energy storage element. Recognizing that the sum of the signals entering the ith energy storage element must, by

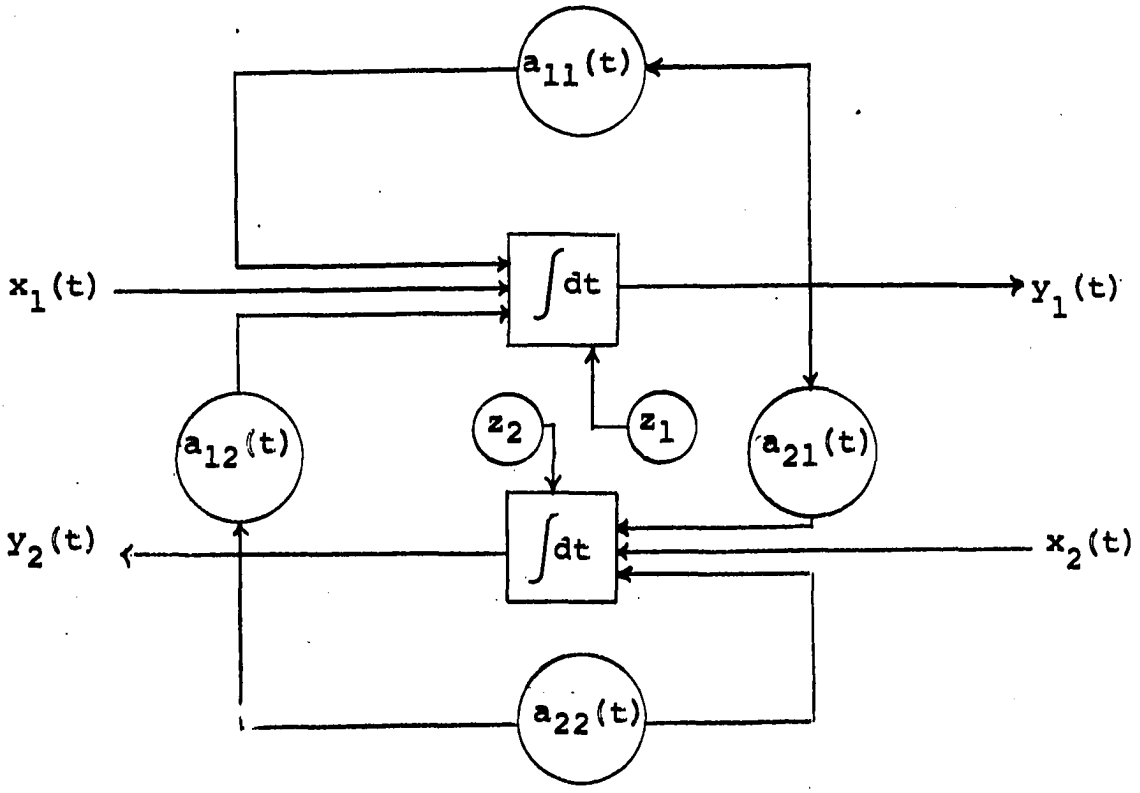
definition, be the time derivative of the output of that element permits writing:

$$\dot{y}_1(t) = a_{11}(t)y_1(t) + a_{12}(t)y_2(t) + x_1(t), \quad y_1(T) = z_1$$

and

$$\dot{y}_2(t) = a_{21}(t)y_1(t) + a_{22}(t)y_2(t) + x_2(t), \quad y_2(T) = z_2$$

which is precisely the operation described by the vector differential equation.



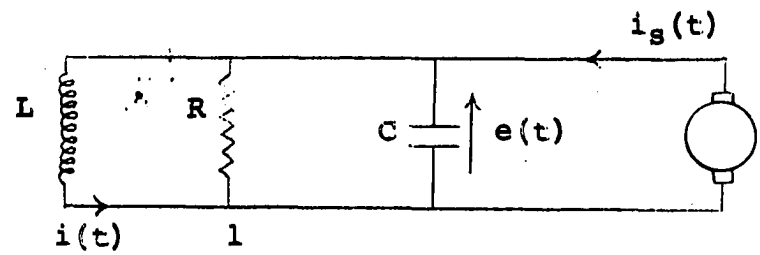
Amplifying the definition of the energy storage elements, it is clear that if the interpretation of the output vector, Y(t), is as a point in an N-dimensional vector space which describes the state of the system, then the interpretation of an energy storage element is as being definitive of

one of the coordinates of that point. Physically, the interpretation of an energy storage element is as an entity, such as a capacitor (in an electrical network) or a mass (in a mechanical network) having associated with it a measure (potential or velocity) of the state of the system.

The interpretation of the multiplication elements in the state space formulation is that they act as functional descriptions of the system interactions. Physically, they describe constraints on the rate of energy transfer between the energy storage elements.

The interpretation of the initial value elements in the state space formulation is obvious from the definition. The interpretation of these elements physically is nonexistant; one may say only that at some time, T, the measure of the system's state associated with each energy storage element must be defined.

At this point an example, which will serve to illustrate the discussion to follow, becomes pertinent. Consider the electrical network consisting of a parallel arrangement of an inductor, a resistor and a capacitor, driven by a current source given schematically below.



Defining the current, i(t), flowing in the inductor, L, and the voltage rise, e(t), across the capacitor, C, as state variables (base vectors for the two dimensional vector space used to describe the state of the system at any instant of time), the transient behavior of the network may be expressed as follows:

Kirchoff's first law (Sum of currents at a node equals zero) at node 1 gives:

$$-i_s(t) + C\dot{e}(t) + e(t)/R + i(t) = 0$$

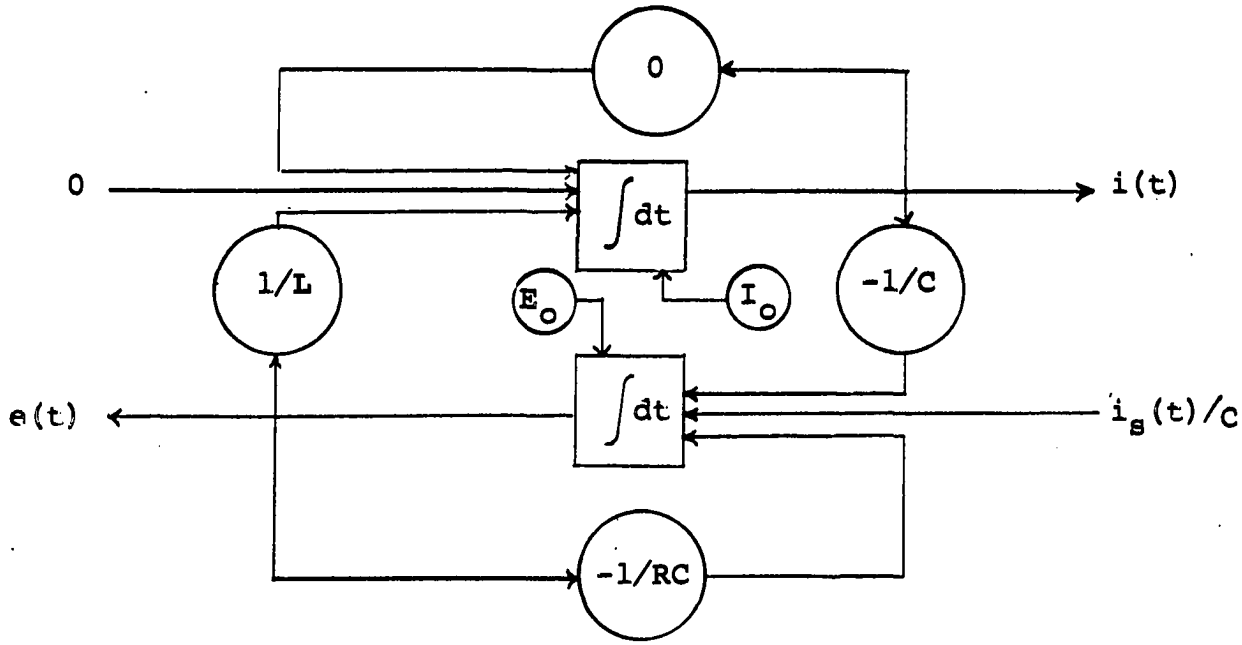
Kirchoff's second law (sum of voltage drops around a loop equals zero) around the loop containing the capacitor and the inductor gives:

$$L\dot{i}(t) - e(t) = 0$$

These expressions may be rearranged to give, with the specification of initial conditions:

$$\begin{bmatrix} \dot{i}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/RC \end{bmatrix} \times \begin{bmatrix} i(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} 0 \\ i_s(t)/C \end{bmatrix}, \begin{bmatrix} I_o \\ E_o \end{bmatrix}$$

The schematic representation of this system is then:



The operation performed by energy storage element #1 is:

$$i(t) = 1/L \int_0^t e(t)dt + I_o$$

which is the definition of the current flowing in an inductor. The operation performed by energy storage element #2 is:

$$e(t) = -1/C \int_0^t (e(t)/R + i(t) - i_s(t))dt + E_o$$

which is the definition of the voltage rise across a capacitor. Notice that the multiplicative modification of the state variables by system parameters does describe the system interactions (state space interpretation) and does describe

constraints on the rate of energy transfer (physical interpretation).

Returning now to the general development, since the state of the system is described by the solution to the vector differential equation, it is of interest to consider that solution. It is well known that, by the method of "Variation of Constants", the solution to a non-homogeneous differential equation is expressible as a functionally (with the independent variable) weighted combination of the solutions to the homogeneous form of the differential equation.

It is equally well known that the solution of the homogeneous differential equation:

$$\dot{Y}(t) = A(t)Y(t) \qquad Y(T) = Z$$

depends upon the "initial" vector $Y(T)$. The question of how many linearly independent solutions, $Y(t)$, to the homogeneous equation exist is answerable in terms of how many linearly independent vectors, $Y(T)$, exist in the N-dimensional vector space which is spanned by solution vectors. The definition of linear independence is, essentially, that if a set of quantities are linearly independent of one another, then they are not linearly dependent on one another. If a set of quantities are linearly dependent, then there exists a set of N constants, $c_1, c_2, \ldots, c_N$, not all zero, such that the linear equation:

$$\sum_{i}^{N} c_i Q_i = 0$$

holds. Suppose that for an N-dimensional vector space one chooses N initial vectors, $Z_i(T)$, such that:

$$Z_i = \begin{bmatrix} z_1 \\ \vdots \\ z_i \\ z_{i+1} \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ \vdots \end{bmatrix} \qquad i = 1,2,\ldots,N$$

The sum of these vectors, each multiplied by a constant, c_i, is:

$$\sum_i^N c_i z_i = \begin{bmatrix} c_1 \\ \vdots \\ c_i \\ \vdots \\ \vdots \end{bmatrix}$$

which is clearly not zero unless every constant, c_i, is identically zero; thus demonstrating that there are at least N linearly independent vectors $Z_i(T)$. Suppose now that one additional initial vector, $Z_{N+1}(T)$, is added to the set, where:

$$Z_{N+1} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{N-1} \\ z_N \end{bmatrix} = \begin{bmatrix} z' \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

The sum of these vectors each multiplied by a constant,

c_i, is:

$$\sum_{i}^{N+1} c_i z_i = \begin{bmatrix} c_1+c_{N+1}z' \\ \vdots \\ c_i+c_{N+1}z' \\ \vdots \end{bmatrix}$$

Clearly, there are an infinite number of choices of the N+1 constants, c_i, not all zero, such that $c_1z_1+c_2z_2+\ldots+c_{N+1}z_{N+1}$ is zero; thus demonstrating that there are no more than N linearly independent initial vectors, Z(T), in an N-dimensional vector space.

Defining $\phi_j(t)$ to be the solution to the linear homogeneous differential equation, $\dot{Y}(t) = A(t)Y(t)$, $Y(T) = Z_j$, it is clear that one could equally well write the complete set of solutions in the matrix form $\dot{\underline{\phi}}(t) = A(t)\underline{\phi}(t)$, $\underline{\phi}(T) = \underline{Z}$. In this formulation, $\underline{\phi}(t)$ is an N by N matrix created by adjoining the N linearly independent solutions, $\phi_j(t)$, and $\underline{Z}$ is an N by N matrix created by adjoining N linearly independent initial vectors Z_j. The matrix of solution vectors, $\underline{\phi}(t)$, is called the fundamental matrix of solutions as any particular solution may be formed as the product of the fundamental matrix and some appropriate column vector, C.

Considering the second order case, the matrix differential equation above becomes:

$$\begin{bmatrix} \dot{\phi}_{11}(t) & \dot{\phi}_{12}(t) \\ \dot{\phi}_{21}(t) & \dot{\phi}_{22}(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix}$$

$$\underline{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

It should be noted that $\phi_{ij}(t)$ indicates the output of the ith energy storage element in the trajectory which corresponds to the jth set of initial values. Similarly, z_{ij} refers to the initial state of the ith energy storage element in the jth set of initial values.

The solution of the non-homogeneous differential equation by the method of "Variation of Constants" is:

$$Y(t) = \underline{\overline{\phi}}(t)P(t)$$

where P(t) is an N-dimensional column vector of weighting functions for the fundamental matrix of solutions, $\underline{\overline{\phi}}(t)$. Differentiation of this expression with respect to time gives:

$$\dot{Y}(t) = \underline{\overline{\phi}}(t)\dot{P}(t) + \underline{\dot{\overline{\phi}}}(t)P(t)$$

Substitution for $\underline{\dot{\overline{\phi}}}(t)$ in terms of the homogeneous form of the differential equation gives:

$$\dot{Y}(t) = \underline{\overline{\phi}}(t)\dot{P}(t) + A(t)\underline{\overline{\phi}}(t)P(t)$$

or:

$$\dot{Y}(t) = \underline{\bar{\phi}}(t)\dot{P}(t) + A(t)Y(t)$$

Substitution for $\dot{Y}(t)$ in terms of the non-homogeneous form of the differential equation gives:

$$A(t)Y(t) + X(t) = \underline{\bar{\phi}}(t)\dot{P}(t) + A(t)Y(t)$$

or:

$$X(t) = \underline{\bar{\phi}}(t)\dot{P}(t)$$

Multiplication, from the left, of the differential equation by the inverse of the fundamental matrix gives:

$$\underline{\bar{\phi}}^{-1}(t)X(t) = \underline{\bar{\phi}}^{-1}(t)\underline{\bar{\phi}}(t)\dot{P}(t) = \dot{P}(t)$$

Integration with respect to time between the limits, zero and the current instant, gives:

$$P(t) = \int_0^t \bar{\phi}^{-1}(s)X(s)ds$$

Finally, substitution for $P(t)$ in the statement of the "Variation of Constants" theorem gives:

$$Y(t) = \underline{\bar{\phi}}(t) \int_0^t \underline{\bar{\phi}}^{-1}(s)X(s)ds$$

The linear integral equation derived above describes the state of the system explicitly, requiring information about

the nature of the fundamental matrix of operations performed by the system and the input vector of interest. However, as has been seen earlier, the complete definition of a particular trajectory includes the location of that trajectory at some instant of time. By recalling the linear nature of the system one may immediately write the response of the system from some non-zero set of initial conditions as:

$$Y(t) = \phi_j(t) + \int_T^t \underline{H}(t,s)X(s)ds \qquad (1)$$

where $\underline{H}(t,s) = \underline{\phi}(t)\underline{\phi}^{-1}(s)$ and $\phi_j(t) = \underline{\phi}(t)\underline{\phi}^{-1}(T)Z_j$. Z_j is the particular set of initial conditions of interest.

Equivalence of the Weighting Function and the Impulse Response

The equivalence of the function $\underline{H}(t,s)$ and the familiar impulse response function should be noted at this point. Suppose that at time $t = T$, a system in equilibrium, ($Z_j = 0$) receives a unit impulse to its jth energy storage element:

$$\begin{bmatrix} x_1(t) \\ \vdots \\ x_j(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \delta(t-T) \\ 0 \\ \vdots \end{bmatrix}$$

where: $\delta(t-T)$ is the Dirac Delta function at time $t = T$. Then

the solution of the integral equation becomes:

$$Y(t) = \int_{T}^{t} H_j(t,s) \quad \delta_j(s-S)ds = H_j(t,S)$$

where $H_j(t,s)$ is the jth column of the weighting function $\underline{H}(t,s)$. Suppose, on the other hand, that at time t = T, the state of the system is described by the vector Z_j where:

$$Z_j = \begin{bmatrix} z_1 \\ \vdots \\ z_j \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

Further suppose that the input vector, X(t), is identically zero for all time, t. Then the solution to the integral equation becomes:

$$Y(t) = \emptyset_j(t) = \underline{\emptyset}(t)\underline{\emptyset}^{-1}(T)Z_j = H_j(t,T)$$

which, since s is a dummy variable for t, is identical to the impulse response.

Again, considering a second order system, which for simplicity may be considered to operate from equilibrium; the integral representation is:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \int_0^t \left[\begin{bmatrix} h_{11}(t,s) & h_{12}(t,s) \\ h_{21}(t,s) & h_{22}(t,s) \end{bmatrix} X \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} \right] ds$$

or:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \int_0^t \left[h_{11}(t,s)x_1(s) + h_{12}(t,s)x_2(s) \right] ds \\ \int_0^t \left[h_{21}(t,s)x_1(s) + h_{22}(t,s)x_2(s) \right] ds \end{bmatrix}$$

The elements of the output vector, Y(t) are expressible as:

$$y_i(t) = \int_T^t \left[\sum_{j=1}^{N} h_{ij}(t,s)x_j(s) \right] ds, \quad y_i(T) = 0$$

$$i = 1,2,\ldots,N$$

where the subscript i refers to the energy storage element, the output of which is described and j refers to the energy storage element to which the input is applied. Further, consideration of the subscripts on the elements of the $\underline{H}(t,s)$ matrix reveals that $h_{ij}(t,s)$ is the response of the ith energy storage element to an impulse applied to the jth energy storage element.

Returning to the electrical network which has been chosen as an example, it has been shown that the state space

formulation of the system's operation was given by the linear vector differential equation:

$$\begin{bmatrix} \dot{i}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/RC \end{bmatrix} \times \begin{bmatrix} i(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} 0 \\ i_s(t)/C \end{bmatrix} , \begin{bmatrix} I_0 \\ E_0 \end{bmatrix}$$

The solution to the homogeneous form of the equation is easily accomplished by Laplace transformation techniques as follows:

$$\dot{Y}(t) = AY(t) \qquad Y(0) = Z$$

Making the Laplace transformation:

$$pUY(p) - Z = AY(p)$$

Note that U is understood to be the unit matrix. Solving this expression for the column vector, Z:

$$(pU - A)Y(p) = Z$$

Multiplication from the left by the inverse of the (pU - A) matrix gives:

$$Y(p) = (pU - A)^{-1} Z$$

or, making the inverse Laplace transformation:

$$Y(t) = \mathcal{L}^{-1} \left[(pU - A)^{-1}\right] Z$$

Since the A matrix is:

$$A = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/RC \end{bmatrix}$$

the (pU - A) matrix is:

$$(pU - A) = \begin{array}{|c|c|} \hline p & -1/L \\ \hline 1/C & p+(1/RC) \\ \hline \end{array}$$

and the inverse of the (pU - A) matrix is:

$$(pU - A)^{-1} = \frac{\begin{array}{|c|c|} \hline p+(1/RC) & 1/L \\ \hline -1/C & p \\ \hline \end{array}}{p^2 + pRC + 1/LC}$$

or, if the values: $R = 1/3$, $L = 1/2$, $C = 1$ are chosen:

$$(pU - A)^{-1} = \begin{array}{|c|c|} \hline (p+3)/(p+1)(p+2) & 2/(P+1)(p+2) \\ \hline -1/(p+1)(p+2) & p/(p+1)(p+2) \\ \hline \end{array}$$

Making the inverse Laplace transformations:

$$\mathcal{L}^{-1}\left((p+3)/(p+1)(p+2)\right) = 2e^{-t} - e^{-2t}$$

$$\mathcal{L}^{-1}\left(2/(p+1)(p+2)\right) = 2e^{-t} - 2e^{-2t}$$

$$\mathcal{L}^{-1}\left(-1/(p+1)(p+2)\right) = -e^{-t} + e^{-2t}$$

$$\mathcal{L}^{-1}\left(p/(p+1)(p+2)\right) = -e^{-t} + 2e^{-2t}$$

gives for the transition matrix, $\underline{T} = \mathcal{L}^{-1}\left((pU - A)^{-1}\right)$:

$$\underline{T} = \begin{array}{|c|c|} \hline 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ \hline -e^{-t} + e^{-2t} & -e^{-t} + 2e^{-2t} \\ \hline \end{array}$$

and the solution to the differential equation is:

$$Y(t) = \underline{T}(t)Z$$

It is of interest to note, from the similarity of the above expression and the definition of $\phi_j(t)$ in the integral expression for the sytem output, that the transition matrix in this solution is precisely the weighting function for the system. At time zero both the transition matrix and the product of the fundamental matrix and its inverse reduce to the unit matrix; at time equals infinity both functions reduce to zero.

In order to completely establish the relationship between the $\underline{H}(t,s)$ function, the impulse response, and the transfer function (applicable since the system is time invariant) one may consider the response of the inductor (current as a function of time) to the variation of the current source as an input. The differential equation which describes this behavior is a second order linear, constant coefficient, ordinary differential equation of the form:

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = x(t)$$

where $y(t) = i(t)$, $x(t) = i_s(t)/LC$, $a_1 = 1/RC$, $a_0 = 1/LC$. Making the Laplace transformation, with the numerical values chosen previously for R, L, and C, and solving for the transfer function:

$$H(p) = \frac{Y(p)}{X(p)} = \frac{2}{p^2 + 3p + 2}$$

or:

$$H(p) = (2/(p+1) - 2/(p+2))$$

Making the inverse Laplace transformation, the impulse response becomes:

$$H(t) = 2e^{-t} - 2e^{-2t}$$

This expression is seen to be identical to element $t_{12}(t)$ in the transition matrix, $\underline{T}$, or as mentioned above, it is identical to element $h_{12}(t)$ in the weighting function matrix, $\underline{H}(t,s)$. The failure of the dummy variable, s, to appear is explained by the fact that for time invariant systems, the independent variable in the weighting function is the difference between the values of t and s; since the "impulse" is assumed to have occurred at t = s = 0, the difference, t-s, is simply t. Note that a more careful solution of the differential equation would have taken cognizance of the fact that the impulse might not have occurred at time zero, but rather at some time, s. Had this been the case, the argument for the weighting function would have been t-s.

The fact that the element $h_{12}(t,s)$ was selected is consistent as $h_{12}(t,s)$ describes the response of the first energy storage element (the inductor) to an impulse applied to the second energy storage element (the capacitor).

Experimental Determination of the Weighting Function: From consideration of Equation (1) one recognizes that a knowledge of the weighting function, $\underline{H}(t,s)$, for the system and

its state at some point in time, T, is sufficient information from which to calculate the state of the system at any time, t, greater than T. One has only to solve the integral equation with the system input of interest substituted for X(t).

Further, one recognizes that a knowledge of the weighting function, $\underline{H}(t,s)$, consists of possessing a fundamental set of solutions to the homogeneous differential equation which describes the system's transient behavior. Assuming, as must be done, that the differential equation, the solution of which is desired, is a valid model for the behavior of the system, then the autonomous gyrations of the elements of the output vector, as the system returns to equilibrium after being appropriately displaced, constitute measures of the columns of a fundamental matrix, $\underline{\overline{\phi}}(t)$, applicable for t greater than T.

If the system, the model of which is being sought, could be manipulated such that at time t = T, the state of the system could be arbitrarily set such that:

$$\phi(T) = \begin{bmatrix} \phi_{1j}(T) \\ \vdots \\ \phi_{jj}(T) \\ \vdots \\ \phi_{Nj}(T) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad j = 1,2,\ldots,N$$

then the measured gyrations of the elements of the matrix created by adjoining the N $\phi_j(t)$ would give $\underline{H}(t,s)$ directly. In general, however, a physical system cannot be so manipulated; therefore, the weighting function must be obtained by

manipulation of the measured fundamental matrix $\underline{\overline{\phi}}(t)$ as indicated by the definition of the weighting function:

$$\underline{H}(t,s) = \underline{\overline{\phi}}(t)\underline{\overline{\phi}}^{-1}(s) \qquad t \geq s$$

It is clear, from the definition, that the weighting function for an Nth order physical system is determinable from information obtained from a set of N linearly independent autonomous responses of the system.

It must be noted that unless the variation of the system parameters is periodic, either within or between operations, the likelihood of obtaining linearly independent responses under identical parametric conditions is very doubtful. Fortunately, however, this limitation is not serious as the vast majority of systems of interest are operated according to some set pattern.

Since a set of linearly independent autonomous system responses are required for the determination of the system weighting function, it is of interest to consider techniques for generating them. Since, in general, there exists no facility for the establishment of an arbitrary distribution of "energies" among the energy storage elements of the physical system and because time "zero" is a rather arbitrary designation of some particular time during the operation of the system, the following technique is valid.

Beginning with the system in equilibrium at some time, T_0, impress upon the system some non-zero input, $X_1(t)$; the

response of the system is then:

$$Y_1(t) = \int_{T_0}^{t} \underline{H}(t,s)X_1(s)ds$$

At some time T*, greater than T_0, the distribution of "energies" among the elements is:

$$Y_1(T^*) = \int_{T_0}^{T^*} \underline{H}(T^*,s)X_1(s)ds = Z_1$$

If at time T*, the input vector is removed, $X_1(t) = 0$, then the system output for t greater than T* is:

$$Y_1(t) = \emptyset(t)\overline{\emptyset}^{-1}(T^*)Z_1 = \emptyset_1(t) \qquad t \geq T^*$$

and one column of a fundamental matrix for the system has been generated.

Repetition of the process given above employing a different input vector, $X_2(t)$ would give rise to a second autonomous response. In fact N-1 repetitions would provide sufficient information from which to create a fundamental matrix provided certain conditions have been met.

The first of these conditions has been touched upon previously, namely, that if the parameters of the system are time-varying in nature, then the variation must be identical in each of the N repetitions. The second condition is, of

course, that the N repetitions must give rise to N linearly independent response vectors.

As shown earlier, the linear independence of the $\phi_i(t)$ depends directly on that of the conditions at T*, or the Z_i. Since the N Z_i are numbers which correspond to the N forced responses evaluated at time T*, the question of the linear independence of the $\phi_i(t)$ may be answered by examination of the proposition: does there exist a set of constants, c_i, not all zero, such that the following equation holds

$$\sum_i^N c_i \int_{T_0}^{T^*} \underline{H}(T^*,s)X_i(s)ds \stackrel{?}{=} 0$$

Since the c_i are constants they may be taken inside the integral; in fact, the integrand may be written as $\underline{H}(T^*,s)X_i(s)c_i ds$. It is also apparent that the order of summation and integration may be reversed. Further, it is clear that since $\underline{H}(T^*,s)$ is a member of every term in the summation the proposition may be rephrased as follows:

$$\int_{T_0}^{T^*} \underline{H}(T^*,s) \left(\sum_i^N X_i(s)c_i\right)ds \stackrel{?}{=} 0 \quad c_i \neq 0 \quad i=1,2,\ldots,N$$

The only way in which the integral might be zero is the case where:

$$\sum_i^N X_i(s)c_i = 0$$

Clearly then, if the N input vectors used to drive the system away from equilibrium are linearly independent, then the resulting energy distributions at time T*, and hence the resulting $\emptyset_i(t)$ are linearly independent.

Consider the case wherein a system of N energy storage elements is displaced from equilibrium in N+1 linearly independent ways. Clearly, this condition is possible since there exist an infinite number of linearly independent functions which are potential forcing functions, $X_i(t)$. The result would be N+1 forced responses of the system, which if evaluated at time T*, would give rise to N+1 N-dimensional vectors, Z_i. Since each vector at T* acts as an initial vector to define the trajectory followed by the system in the subsequent autonomous return to equilibrium, N+1 autonomous trajectories are generated. It is of interest to inquire as to whether or not these trajectories are linearly independent. Each response may be written:

$$\emptyset_i(t) = \underline{\overline{\emptyset}}(t)\underline{\overline{\emptyset}}^{-1}(T^*)Z_i$$

and since by the existance and uniqueness theorems on the solutions of differential equations one may state that if two solutions are linearly independent at one point in the time domain, then they are linearly independent at every point, it is sufficient to inquire whether or not the:

$$\emptyset_i(T^*) = \overline{\emptyset}(T^*)\overline{\emptyset}^{-1}(T^*)Z_i = Z_i$$

are linearly independent.

Each of Z_i is an N-dimensional column vector and it was shown previously that there exist exactly N linearly independent N-dimensional vectors which span an N-dimensional vector space. Therefore, even though linearly independent inputs are applied to the system, the N-dimensionality of the vector space required to describe the state of the system precludes more than N of these inputs from generating linearly independent autonomous responses.

Determination of Differential Model from System Weighting Function: As mentioned earlier, the most useful form of a mathematical model for control purposes is the differential equation form. It was noted that this statement is particularly true if the system is time-varying. It is clear, from the discussion of the state space formulation of the differential equation form of a mathematical model for a physical system that all of the characteristics of the system are described in the structure of the A(t) matrix; therefore, assuming that one has available a fundamental matrix of solutions, the matrix form of the differential equation

$$\dot{\underline{\overline{\phi}}}(t) = A(t)\underline{\overline{\phi}}(t)$$

serves as a defining equation for the parameter matrix.

Since the weighting function $\underline{H}(t,s)$ has been shown to be a fundamental matrix:

$$\underline{H}(t,s) = \underline{\overline{\phi}}(t)\underline{\overline{\phi}}^{-1}(s) \qquad t \geq s$$

one may write the differential equation in the form:

$$\dot{\underline{H}}(t,s) = A(t)\underline{H}(t,s) \qquad t \geq s$$

or, upon multiplication from the right by the inverse of $\underline{H}(t,s)$:

$$\dot{\underline{H}}(t,s)\underline{H}^{-1}(t,s) = A(t)\underline{H}(t,s)\underline{H}^{-1}(t,s) = A(t)U = A(t) \qquad t \geq s$$

The implication of the above expression for the A(t) matrix in terms of the experimentally determinable weighting function is that the problem of the requirement for an infinite number of weighting function models, one of each possible parametric variation, mentioned earlier may not be as serious as first imagined. Consider a situation wherein the weighting function has been determined under some measurable parametric variation:

$$a_{ij}(t) = f_{ij}(t), \qquad f_{ij} \text{ measurable}$$

In this situation, one could obtain a differential equation representation of the system's operation by implementation of the experimental techniques given earlier and implementation of the expression for the A(t) matrix:

$$A(t) = \dot{\underline{H}}(t,s)\underline{H}^{-1}(t,s) \qquad t \geq s$$

This knowledge of the A(t) matrix might then be combined with the measured variation in the system parameters which existed during the experimental determination to define functional expressions for the elements of the A(t) matrix. Given such a

model together with the measurablity of the variation in system parameters during the operation of interest, there appears to be no fundamental barrier to the acquisition and implementation of time-varying mathematical models.

Theoretical Conclusions: In summary of the material presented in this chapter, the following statements may be made.

If a physical process or system can be adequately described in terms of the linear vector differential equation:

$$\dot{Y}(t) = A(t)Y(t) + X(t) \qquad Y(T) = z \quad T \in t$$

then the behavior of the system in response to any input, X(t), can be calculated explicitly from the linear integral equation:

$$Y(t) = \underline{H}(t,T)Y(T) + \int_{T}^{t} \underline{H}(t,s)X(s)ds$$

where:

$$\underline{H}(t,s) = \underline{\overline{\phi}}(t)\underline{\overline{\phi}}^{-1}(s) \qquad t \geq s$$

A knowledge of a fundamental matrix, $\underline{\overline{\phi}}(t)$ is therefore sufficient information from which to define a mathematical model for the physical system in the form of a weighting function.

The fundamental matrix of autonomous responses of the system (solutions to the matrix differential equation

$\dot{\underline{\Phi}}(t) = A(t)\underline{\Phi}(t)$, $\underline{\Phi}(T) = \underline{\Phi}(T)$ may be obtained from experimental measurements of the autonomous return to equilibrium by the system after it has been displaced in N (where N is the number of elements in the state vector Y(t)) linearly independent ways by forcing with N linearly independent input vectors $X_i(t)$ prior to time T. Stated mathematically:

$$\underline{\Phi}(T) = \underline{H}(T,T_0)\underline{\Phi}(T_0) + \int_{T_0}^{T} \underline{H}(T,s)\underline{X}(s)ds$$

Alternatively, if the model is desired in the differential equation form one may employ the experimentally determined weighting function together with information on the time variation of the system parameters and the relationship:

$$A(t) = \dot{\underline{H}}(t,s)\underline{H}^{-1}(t,s) \qquad t \geq s$$

to define a mathematical model for the system's operation in the differential equation formulation:

$$\dot{Y}(t) = A(t)\ Y(t) + X(t) \qquad Y(T) = Z$$

Important constraints on the technique are that:

1. all elements of the response vector Y(t) are measurable.
2. it is possible to make N experiments (for an Nth order system) under the same conditions of parametric variation.

3. certain aspects of the parameter variation are measurable.

CHAPTER III

APPLICATIONS OF THE GENERALIZED PULSE TESTING TECHNIQUE

This chapter presents three relatively simple examples of the application of the generalized pulse testing technique for the identification of linear system models. The purpose of these examples is to familiarize the reader with the weighting function form for a mathematical model, its use, the computations required for its definition, and to introduce some of the practical difficulties encountered in carrying out these computations using experimentally measured response data.

The first example employs a first order time-varying system, or rather the differential equation which describes its transient behavior, to introduce the weighting function and its use in the calculation of the response of the system to some arbitrary input. The introduction of the weighting functions consists of an analysis of the solution to the differential equation in the two regions of interest in the time domain and their use in the generation of the weighting function. The geometric interpretation of the weighting function, as a surface, is employed for the purpose of its description.

The composition calculation to determine the response of the system to some arbitrary input demonstrates the use of the weighting function and points up the role of the arguments of the weighting function as either a running variable or a parameter.

The second example employs an electrical circuit, programmed on an analog computer, as a time invariant physical system. The practicalities of system testing, data reduction, and manipulation to define a model in the form of weighting function are discussed. The results of making the required manipulations are presented in graphical form. These results provide an introduction to the difficulties encountered in the naive attempt to carry out the processes of matrix inversion and multiplication using data which are numerically imprecise. A more complete discussion of these calculations using data which, for various reasons, is imprecise, is presented in Chapter IV. Since it is possible to choose arbitrarily a specific distribution of the energy among the elements of this physical system, the autonomous responses which correspond to the weighting function may be measured. These are presented as a check of the numerical calculations.

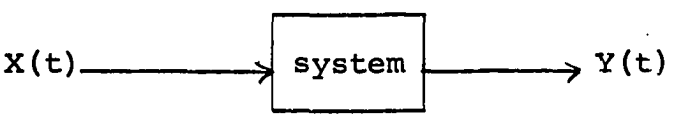
The third example also makes use of the analog computer as a physical system. In this case the electrical circuit simulates the behavior of a system which is describable in terms of a third order constant coefficient differential equation. In addition to pointing out the deleterious

effect of an increase in the size of the data matricies manipulated on the quality of the calculated weighting function, the Cannonical form for a system coefficient matrix is introduced. Finally, and perhaps most significantly, this system with its naturally occurring Cannonical coefficient matrix is used to demonstrate a technique for the definition of a system model in the form of a differential equation which utilizes the same experimental measurements used for the determination of the weighting function model.

Example 1: First Order Time-Varying System (Analytic)

A first order system, the parameter of which undergoes a single step change in magnitude represents the most elementary time-varying system of interest. Its consideration will, despite the simplicity, serve to demonstrate the generation of a weighting function and introduce the composition calculation used to compute future outputs of the system resulting from arbitrary inputs.

Consider the system shown below:



which performs the following linear operation:

$$\dot{Y}(t) = A(t)Y(t) + X(t) \qquad Y(0) = 0$$

where:

$$A = \begin{cases} -a_1 & \text{for the range } -\infty \leq t \leq T_1 \\ -a_2 & \text{for the range } T_1 \leq t \leq \infty \end{cases}$$

on the input to the system, X(t), to define the output of the system, Y(t). One may study the behavior of the weighting function, $\underline{H}(t,s)$, on the t, s plane by recalling that the weighting function was defined, in Chapter II, to be:

$$\underline{H}(t,s) = \underline{\Phi}(t)\underline{\Phi}^{-1}(s) \qquad t \geq s$$

Consideration of the t, s plane in the light of the differential equation and the definition of the weighting function indicates that there are four distinct regions of the weighting function to be considered. These regions are identified, located, and characterized as to their time dependence in Table III-1. These regions are indicated schematically below:

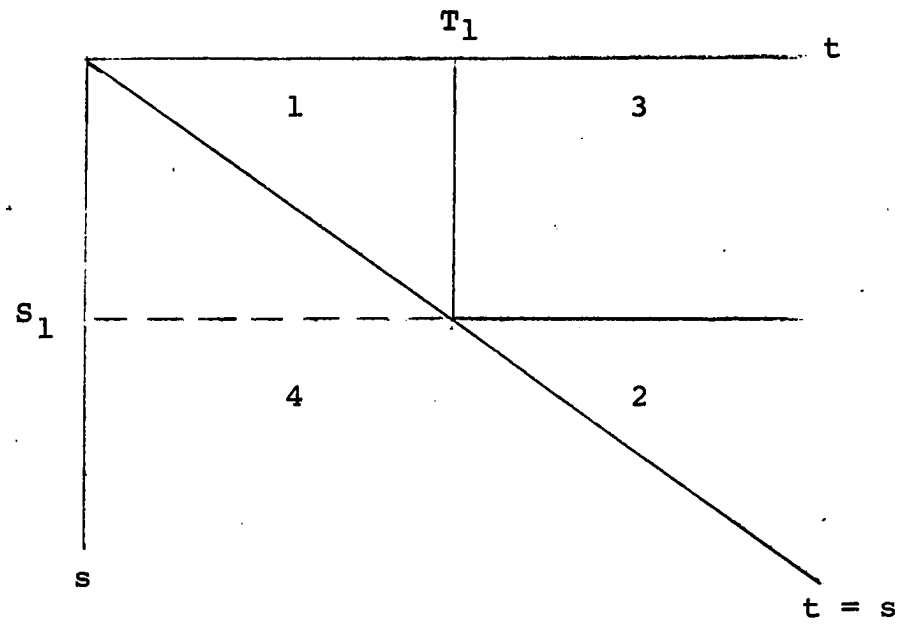


TABLE III-1

Areas of Interest of First Order Weighting Function

Region	Location	Definition of $\underline{H}(t,s)$	Time Dependence
1	$0 \le t \le T_1$ $0 \le s \le t$	$\overline{\underline{\Phi}}_1(t)\overline{\underline{\Phi}}_1^{-1}(s) = \underline{H}_1(t,s)$	Time Invariant
2	$T_1 \le t \le \infty$ $S_1 \le s \le t$	$\overline{\underline{\Phi}}_2(t)\overline{\underline{\Phi}}_2^{-1}(s) = \underline{H}_2(t,s)$	Time Invariant
3	$T_1 \le t \le \infty$ $0 \le s \le S_1$	$\overline{\underline{\Phi}}_2(t)\overline{\underline{\Phi}}_1^{-1}(s) = \underline{H}_3(t,s)$	Time-Varying
4	$0 \le t \le s$ $0 \le s \le \infty$	0	Time Invariant

Note: $\overline{\underline{\Phi}}_1(t)$ is the fundamental matrix of solution which applies prior to T_1; $\overline{\underline{\Phi}}_2(t)$ is the fundamental matrix of solution which applies after T_1.

Calculation of the weighting function for this simple system is quite straight forward as for a first order system:

$$\underline{H}(t,s) = h(t,s)$$

$$\overline{\underline{\Phi}}(t) = \Phi(t)$$

$$\overline{\underline{\Phi}}^{-1}(s) = 1/\Phi(s)$$

$$\dot{\phi}(t) = a\phi(t) \qquad \phi(0) = z_1$$

$$a = a_1 \text{ for } 0 \leq t \leq T_1$$

$$a = a_2 \text{ for } T_1 \leq t \leq \infty$$

Therefore:

$$\underline{H}_1(t,s) = e^{-a_1(t-s)}$$

$$\underline{H}_2(t,s) = e^{-a_2(t-s)}$$

$$\underline{H}_3(t,s) = e^{-a_2(t-T_1)} \; e^{a_1(s-S_1)}$$

$$\underline{H}_4(t,s) = 0$$

Notice that along the line $s = S_1$, $\underline{H}_2(t,s) = \underline{H}_3(t,s)$; along the line $t = T_1$, $\underline{H}_1(t,s) = \underline{H}_3(t,s)$; and along the line $t = s$, $\underline{H}_1(t,s) = \underline{H}_2(t,s) = \underline{H}_3(t,s) = 1.0$.

In order to develop a further appreciation for the nature of the weighting function for this simple system let its parameters be specialized as follows:

$$a_1 = 0.50$$

$$a_2 = 0.20$$

$$T_1 = 8.00$$

and to permit the calculation of $\overline{\underline{\phi}}(t)$ let $\phi(0) = 10.0$. The solution of the homogeneous form of the differential equation is:

$$\phi(t) = 10e^{-0.05t} \qquad 0 \leq t \leq T_1$$

$$\phi(t) = 10e^{0.15T} \; 1e^{-0.20t} \qquad T_1 \leq t \leq \infty$$

This solution is equivalent to the autonomous gyrations which the system would undergo if it had been forced, prior to time zero, such that at time zero the output took on the value Y(0) = 10.0. This response is plotted in Figure III-1.

The weighting function has been evaluated for values of t and s from zero to thirty (t equal to or greater than s) and is presented in Table III-2, and in Figures III-2 and III-3. Table III-2 consists of an array of values of the weighting function in the interval (for both t and s) zero to fifteen. The fact that the weighting function is symmetric in t and s for time invariant systems is reflected by the data in the two sub-arrays $0 \leq s \leq 8$, $t \geq s$ and $8 \leq s \leq 15$, $t \geq s$. The fact that symmetry, about the line $t = s/4$ is reflected in the data which appear in the sub-array $8 \leq t \leq 0$, $0 \leq s \leq 8$ is the result of the fact that the system parameter undergoes a simple step change, its magnitude being a factor of four. Figure III-2 is an "iso-weight" plot of the weighting function. Lines of constant $\underline{H}(t,s)$ are projected onto the t,s plane. The relatively different spacing of the "iso-weights" in each of the three non-trivial regions of the surface reflects the differences in the slope as a function of t and s. The isoweight plot reveals that the lines $t = 8$ and $s = 8$ are respectively a ridge and a crease in the function surface. Finally, Figure III-3 is a projection of the $\underline{H}(t,s)$ surface as viewed from the point $t = 30$, $s = 0$, along a line $t = -s$. The purpose of including a third representation of the

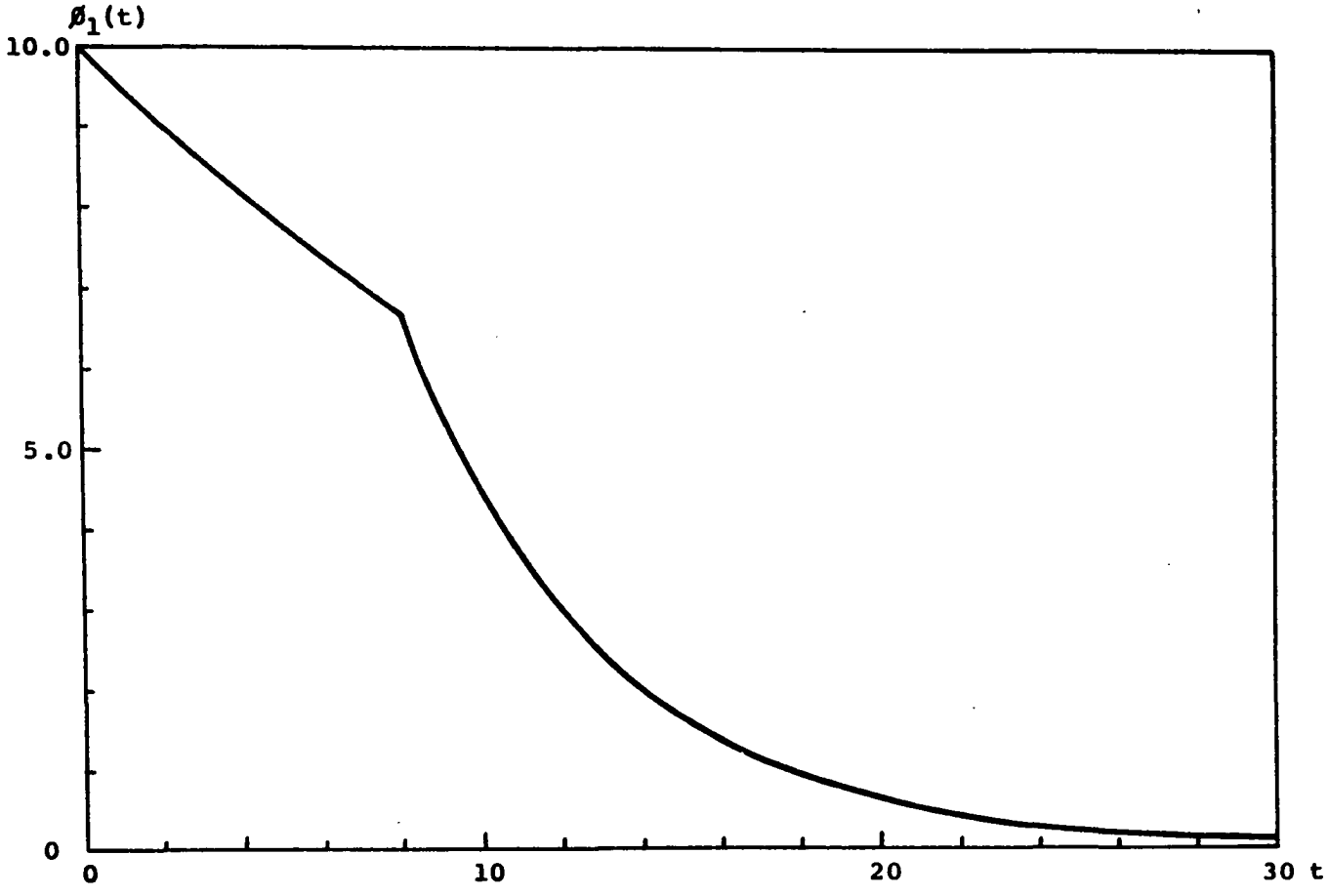


Figure III-1

Autonomous Response of First Order Time-Varying System

T S	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1.000	.951	.905	.861	.819	.779	.741	.705	.670	.549	.449	.368	.301	.247	.201	.165
1		1.000	.951	.905	.861	.819	.779	.741	.705	.577	.472	.387	.317	.259	.212	.174
2			1.000	.951	.905	.861	.819	.779	.741	.606	.497	.407	.333	.272	.223	.182
3				1.000	.951	.905	.861	.819	.779	.638	.522	.427	.350	.286	.235	.192
4					1.000	.951	.905	.861	.819	.670	.549	.449	.368	.301	.247	.202
5						1.000	.951	.905	.861	.705	.577	.472	.387	.317	.259	.212
6							1.000	.951	.905	.741	.606	.497	.407	.333	.272	.223
7								1.000	.951	.779	.638	.522	.427	.350	.286	.235
8									1.000	.819	.670	.549	.449	.368	.301	.247
9										1.000	.819	.670	.549	.449	.368	.301
10											1.000	.819	.670	.549	.449	.368
11												1.000	.819	.670	.549	.449
12													1.000	.819	.670	.549
13														1.000	.819	.670
14															1.000	.819
15																1.000

TABLE III-2

TABULATION OF FIRST ORDER WEIGHTING FUNCTION

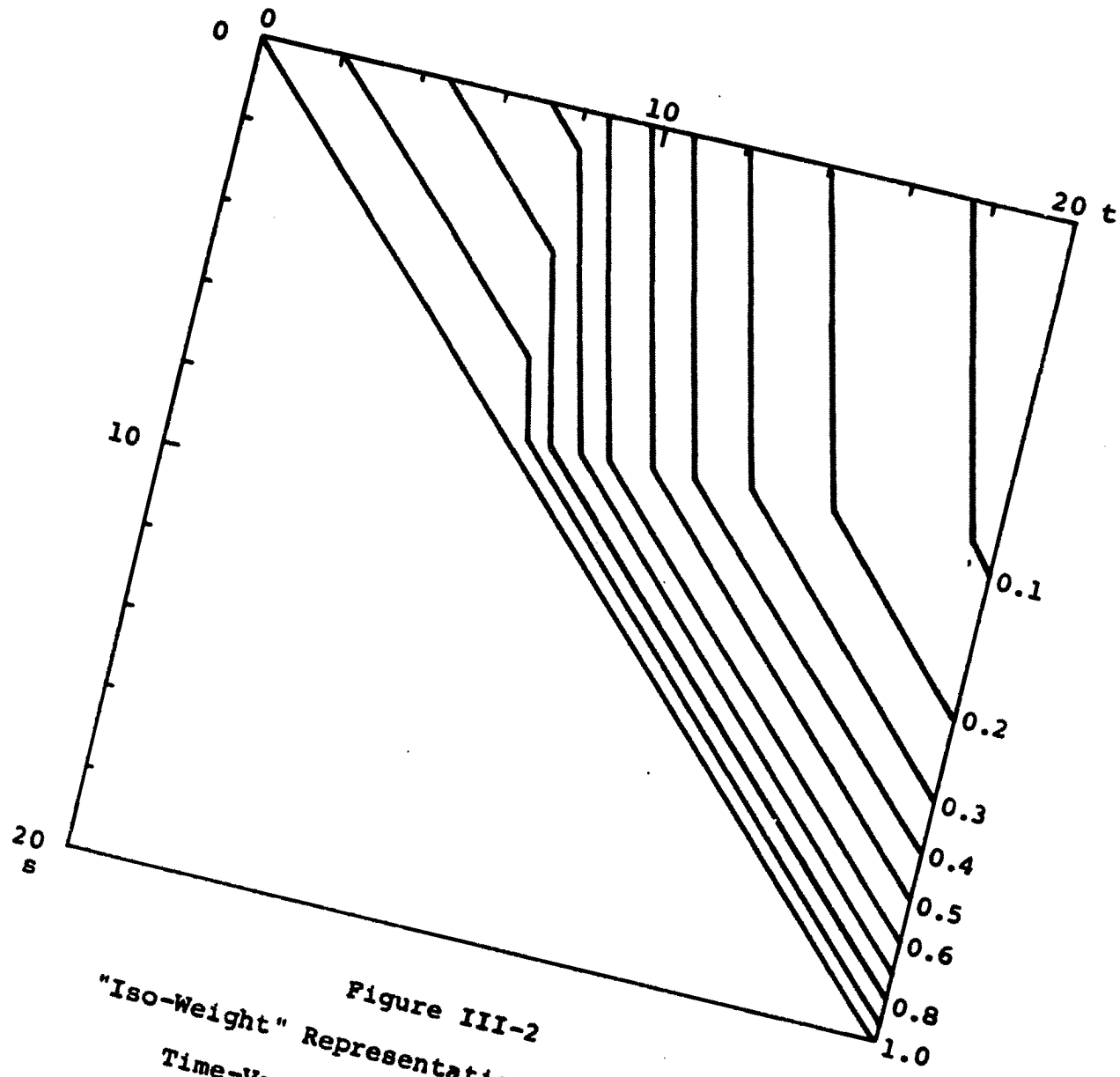


Figure III-2

"Iso-Weight" Representation of First Order Time-Varying Weighting Function

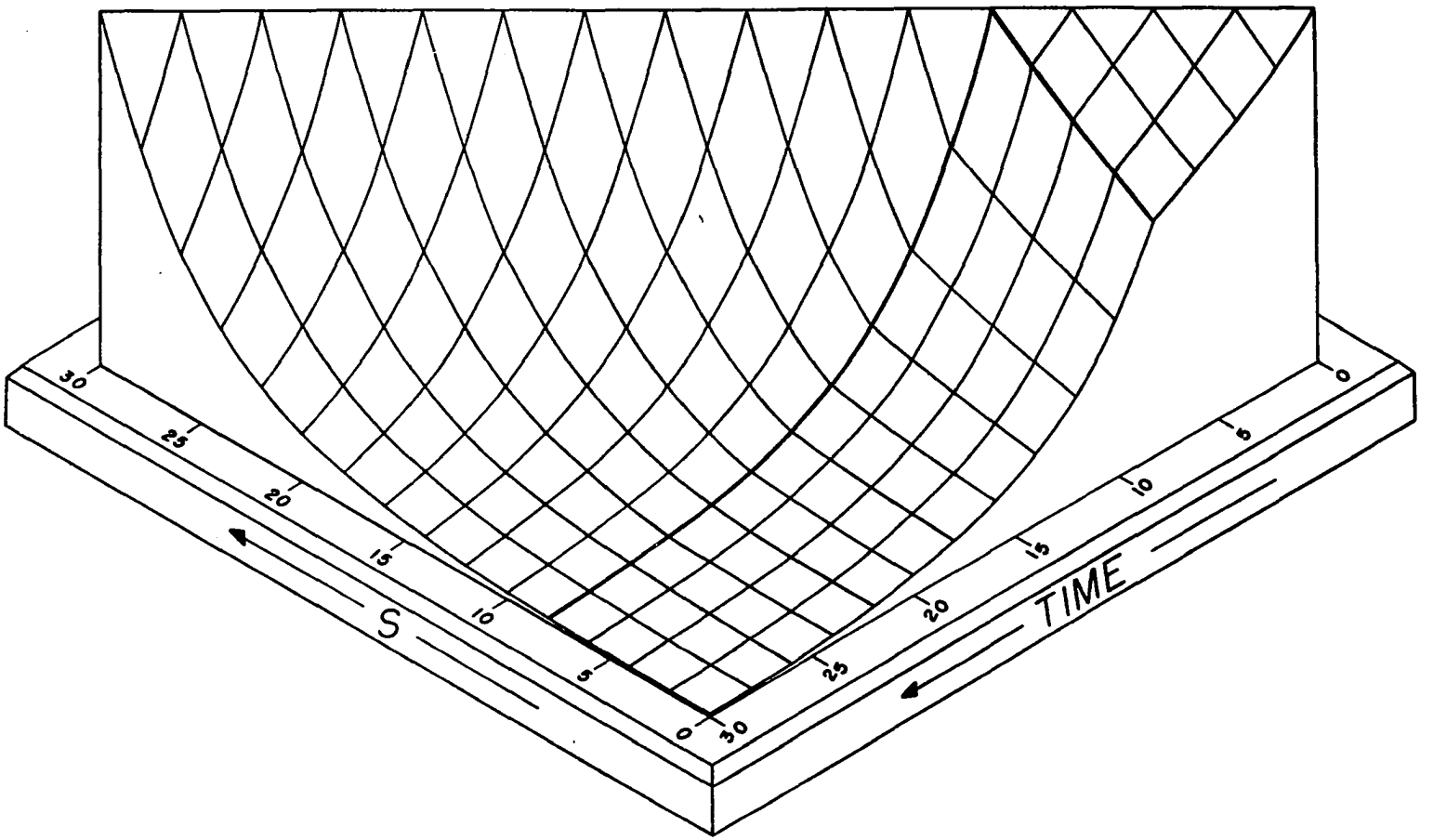


FIGURE III-3
PERSPECTIVE REPRESENTATION OF FIRST ORDER
TIME-VARYING WEIGHTING FUNCTION

weighting function is to provide an appreciation for the shape of the surface which is not entirely obvious from the preceding two representations.

In order to demonstrate the composition calculation for which the weighting function form of the system model is most useful, suppose that it is of interest to compute the value of the system output at time T*, greater than T_1 in response to an input of the form X(t) = sin(wt). For simplicity, it is assumed that the system is in equilibrium at time zero.

The composition calculation is described as follows:

$$Y(T^*) = \int_0^{T^*} \underline{H}(T^*,s)X(s)\,ds$$

As mentioned earlier, the implication of this calculation is that one forms the product of the system input X(s) and a constant t (T*) cut in the weighting function, $\underline{H}(T^*,s)$ and integrates from s = 0 to s = T*. For the case of interest the cut $\underline{H}(T^*,s)$ crosses two regions of the weighting function, namely, $\underline{H}_3(T^*,s)$ and $\underline{H}_2(T^*,s)$ necessitating breaking the integration into two parts:

$$Y(T^*) = \int_0^{T_1} \underline{H}_3(T^*,s)X(s)\,ds + \int_{T_1}^{T^*} \underline{H}_2(T^*,s)X(s)\,ds$$

or on substitution for $\underline{H}_3(T^*,s)$ and $\underline{H}_2(T^*,s)$:

$$Y(T^*) = e^{-a_2(T^*-T_1)} e^{-a_1 S_1} \int_0^{T_1} e^{a_1 s} \sin(ws)\,ds + e^{-a_2 T^*} \int_{T_1}^{T^*} e^{a_2 s} \sin(ws)\,ds$$

Integration by parts gives:

$$Y(T^*) = (a_2 \sin(wT^*) - w\cos(wT^*))/(w^2+a_2^2) + [(a_1/(w^2 + a_1^2) - a_2/(w^2+a_1^2)] \sin (wT_1) - (w/(w^2+a_1^2) - w/(w^2+a_2^2)\cos (wT_1) + we^{-a_1 T_1}/(w^2+a_1^2))e^{-a_2(T^*-T_1)}$$

which is precisely the solution to the nonhomogeneous differential equations:

$$\dot{Y}(t) = -a_1 Y(t) + \sin(wt) \qquad Y(0) = 0 \qquad 0 \le t \le T_1$$

$$\dot{Y}(t) = -a_2 Y(t) + \sin(wt) \qquad T_1 \le t \le \infty$$

evaluated at $t = T^*$.

Example 2: Second Order Time Invariant System

A linear time invariant system containing two energy storage elements was chosen to begin the investigation of practical difficulties associated with the application of the generalized pulse testing technique. A pair of simultaneous

linear constant coefficient first order differential equations was programed on the University of Oklahoma Process Dynamics Laboratory's analog computer; this electrical circuit was then used to simulate the operation of some arbitrary physical system. For the purposes of logical discussion, the experiment may be broken down into three phases: the System Testing phase, the Computational phase, and the Weighting Function Generation phase.

System Testing: The system testing phase of the experiment consisted of forcing the system, initially at equilibrium, prior to time zero, with two linearly independent input signals and measuring the autonomous gyrations of the system variables as it returned to equilibrium.

One response vector was generated by forcing the system prior to time zero with the input signal being a negative ramp function. The behavior of the system variables, in response to the negative ramp, for a few increments of time prior to zero and approximately one hundred twenty increments after zero are reproduced in Figure III-4A. A second linearly independent response vector was generated by forcing the system prior to time zero with the input signal being a sine wave. The system's response to this forcing function is reproduced, in the same range, in Figure III-4B. In each case, the responses were recorded on a six channel Sanborn recorder, each channel being fifty millimeters in width; Figure III-4 is reproduced on an equivalent grid system.

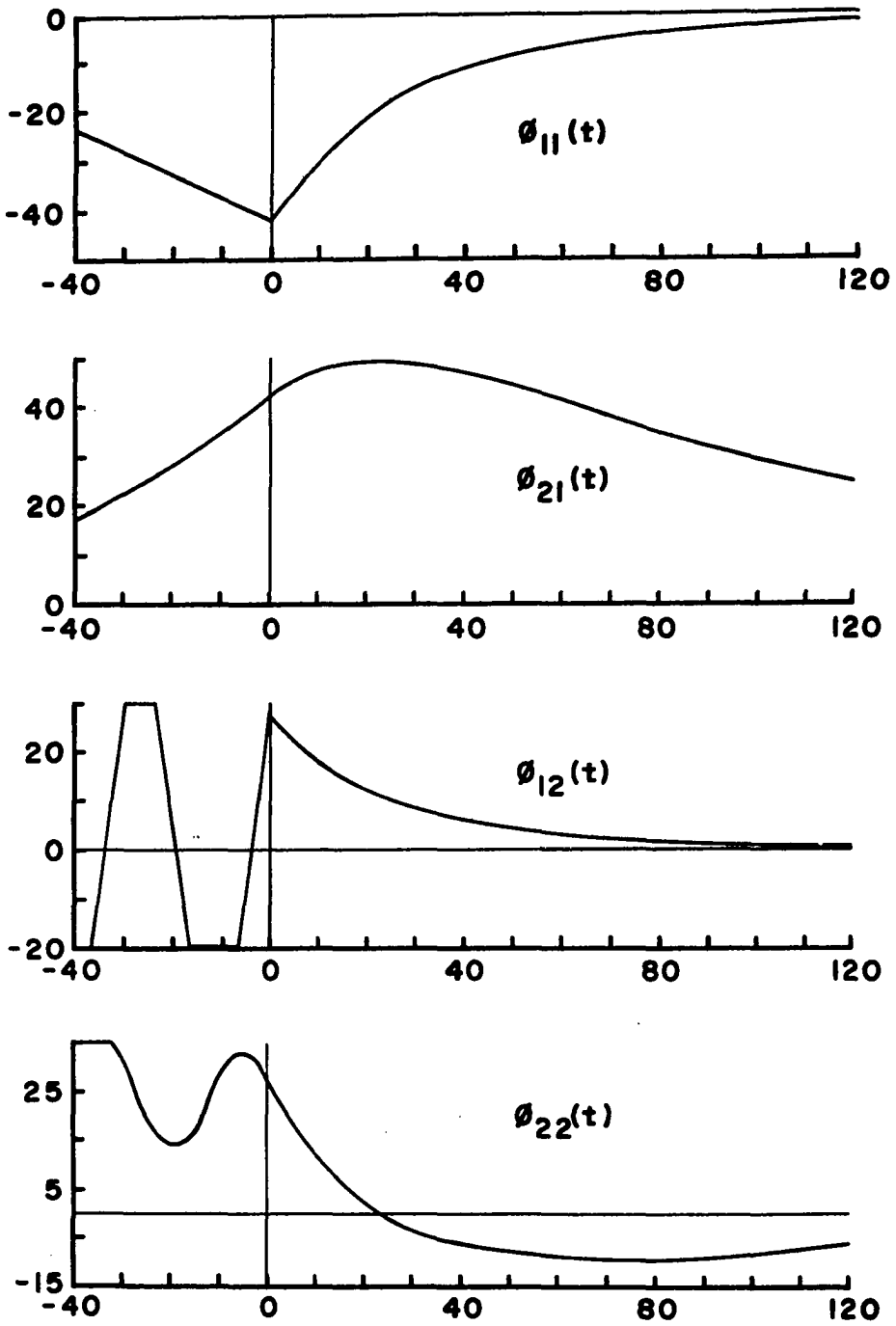


FIGURE III-4

RESPONSE TO TESTS OF SECOND ORDER TIME INVARIANT SYSTEM

The autonomous response data recorded from the two experiments described constitutes a fundamental matrix of the system responses. This data was "read," tabulated (to form four arrays of one hundred twenty elements each), and adjoined appropriately to form the $\underline{\overline{\phi}}(t)$ matrix in the range $0 \leq t \leq 120$.

Computations: A digital program was written to make the computations stated in the definition of the weighting function:

$$\underline{H}(t,s) = \underline{\overline{\phi}}(t)\underline{\overline{\phi}}^{-1}(s) \qquad t \geq s$$

The actual calculations involved are: inversion of the fundamental matrix at time $t = s$ and matrix multiplication of the fundamental matrix at each value of $t \geq s$; this procedure is repeated over the entire range $0 \leq s \leq 120$. The results of these calculations are presented in Figure III-5.

Figure III-5 presents the calculated values of the weighting function $\underline{H}(t,s)$ calculated for four values of s, namely, 0, 5, 10, and 20. The system is known, a priori, to be time invariant, therefore $h_{ij}(t,s)$ may be expressed as $h_{ij}(t-s)$ and plots for various values of s versus the argument (t-s) should lie one on top of another. It is clear, from Figure III-5 that the calculated elements $h_{11}(t-s)$, $h_{21}(t-s)$, and $h_{22}(t-s)$ represent a successful application of the technique. However, $h_{12}(t-s)$ represents a considerably less satisfactory calculation. While the general nature of the weighting function $h_{12}(t-s)$ is apparent, one would have

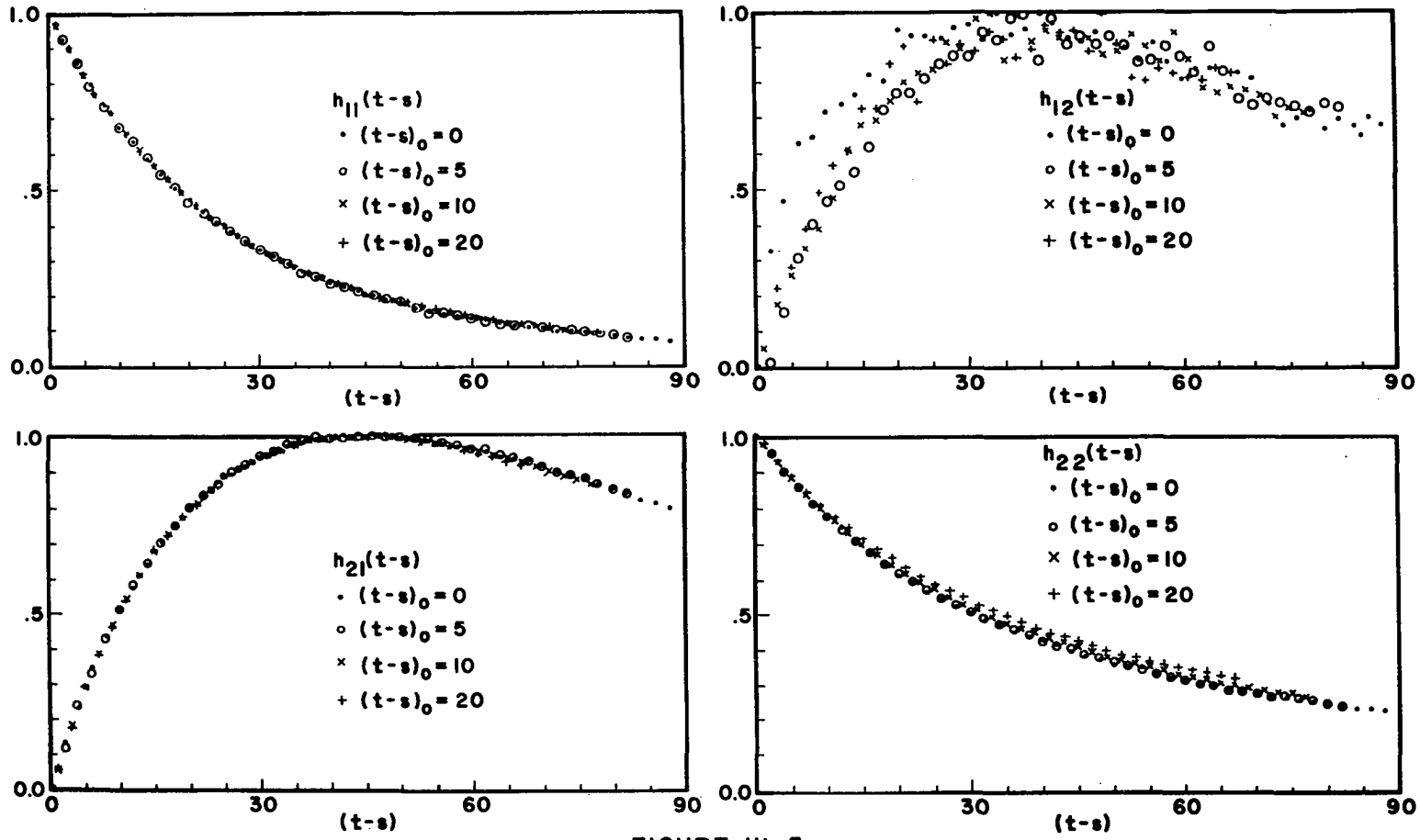


FIGURE III-5
NORMALIZED TIME INVARIANT WEIGHTING FUNCTION
CALCULATED AT S=0,5,10,20

great difficulty stating precisely what value of the weighting function to use at any given value of (t-s).

The reasons for the apparent failure of the technique to generate a weighting function of equally high quality for all four elements in the weighting function is investigated in Chapter IV; suffice it to say, at this point, that the fault lies in the accuracy and precision available in the experimental response data and a happenstance unique to the particular pair of vectors chosen, namely, the relative polarities and magnitudes of elements in the two matricies which are multiplied together.

Weighting Function Generation: The calculation of the weighting function for this particular physical system may be checked because of its nature; that is, one may solve the differential equations in their homogeneous form from appropriate initial conditions on an analog computer. In order to check the calculated weighting function two more runs were made. The first of these was made to check the elements $h_{11}(t-s)$ and $h_{21}(t-s)$ and involved solving the homogeneous form of the differential equation with the initial conditions:

$$h_1(0) = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}$$

The second run was made to check the calculation of the two elements $h_{12}(t-s)$ and $h_{22}(t-s)$, the initial conditions being:

$$h_2(0) = \begin{array}{|c|}\hline 0.0 \\ \hline 1.0 \\ \hline \end{array}$$

The results of these analog simulations are presented, in normalized form, in Figure III-6.

Example 3: Third Order Time Invariant System

A third order linear constant coefficient differential equation programed on the University of Oklahoma Process Dynamics Laboratory analog computer was chosen as an arbitrary physical system for investigation. This choice was made to permit the demonstration of three aspects of the generalized pulse testing technique. The first aspect to be demonstrated is the relationship of the elements of the weighting function for the case in which the coefficient matrix, A, is of the cannonical form. The second aspect demonstrated is the determination of a mathematical model in the differential equation formulation from experimental measurements. The third aspect is consideration of the effect of erroneous measurements in the computational phase of the generalized pulse testing technique when applied to systems of order higher than two.

An electrical circuit, programmed on an analog computer the behavior of which is described by the differential equation:

$$\dddot{y}(t) + a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = x(t)$$

where:

$$a_0 = 1.00$$

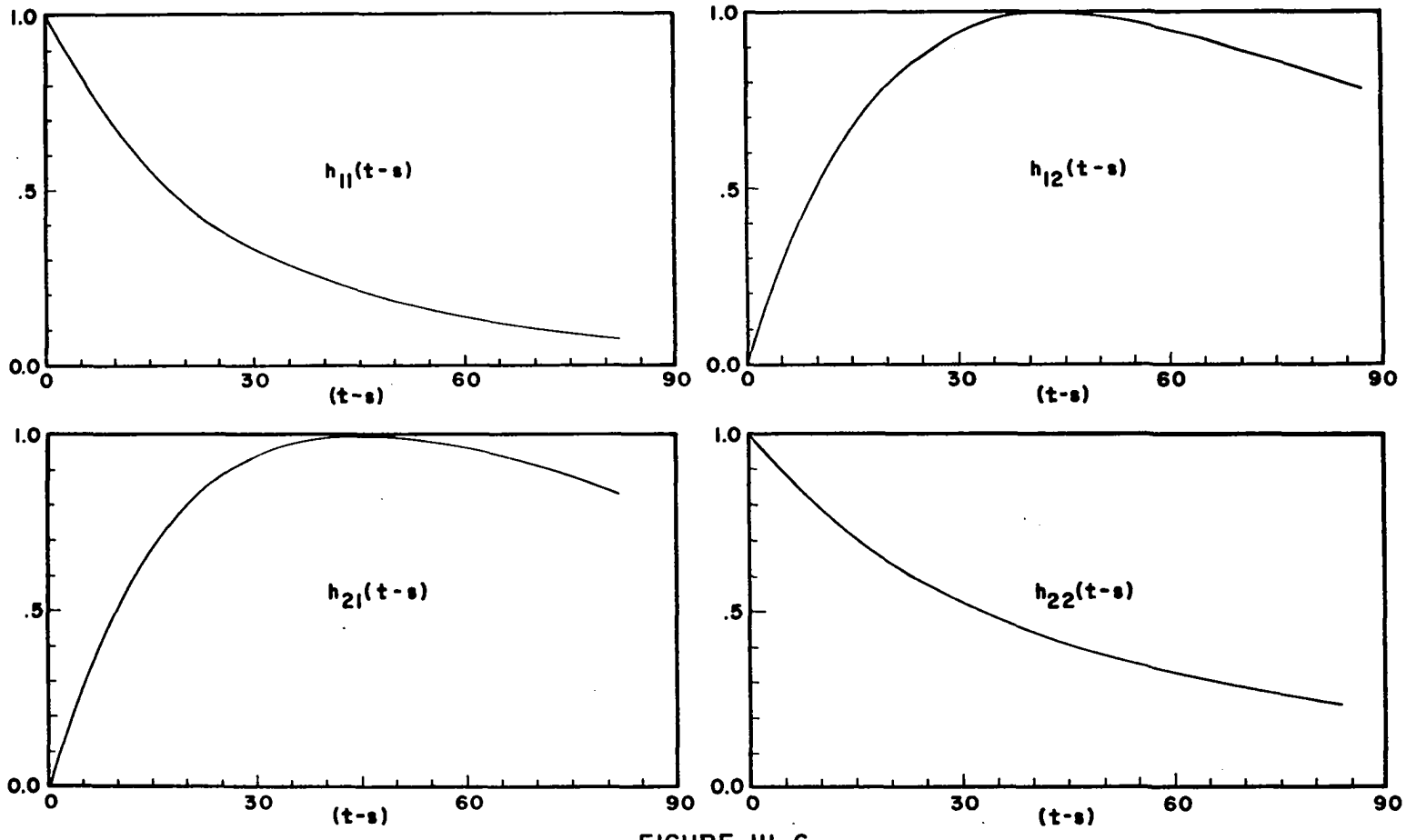


FIGURE III-6
NORMALIZED TIME INVARIANT WEIGHTING FUNCTION

$$a_1 = 0.75$$

$$a_2 = 0.98$$

$$\dddot{y}(0) = \ddot{y}(0) = \dot{y}(0) = y(0) = 0$$

was chosen as a physcial system for investigation. A schematic diagram of the analog computer program used in this example is given in Figure III-7.

In order to recast the differential equation into the state space formulation, the following variables are introduced:

$$y_1(t) = y(t)$$

$$y_2(t) = \dot{y}(t)$$

$$y_3(t) = \ddot{y}(t)$$

The definitions of these new variables may be combined with the third order differential equation to give the first order vector differential equation:

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x(t) \end{bmatrix} \qquad Y(0) = 0$$

The coefficient matrix possessing the properties that the elements in the diagonal one to the right of the major diagonal are unity and that all non-zero elements exclusive of these and those in a single row of the matrix are zero is

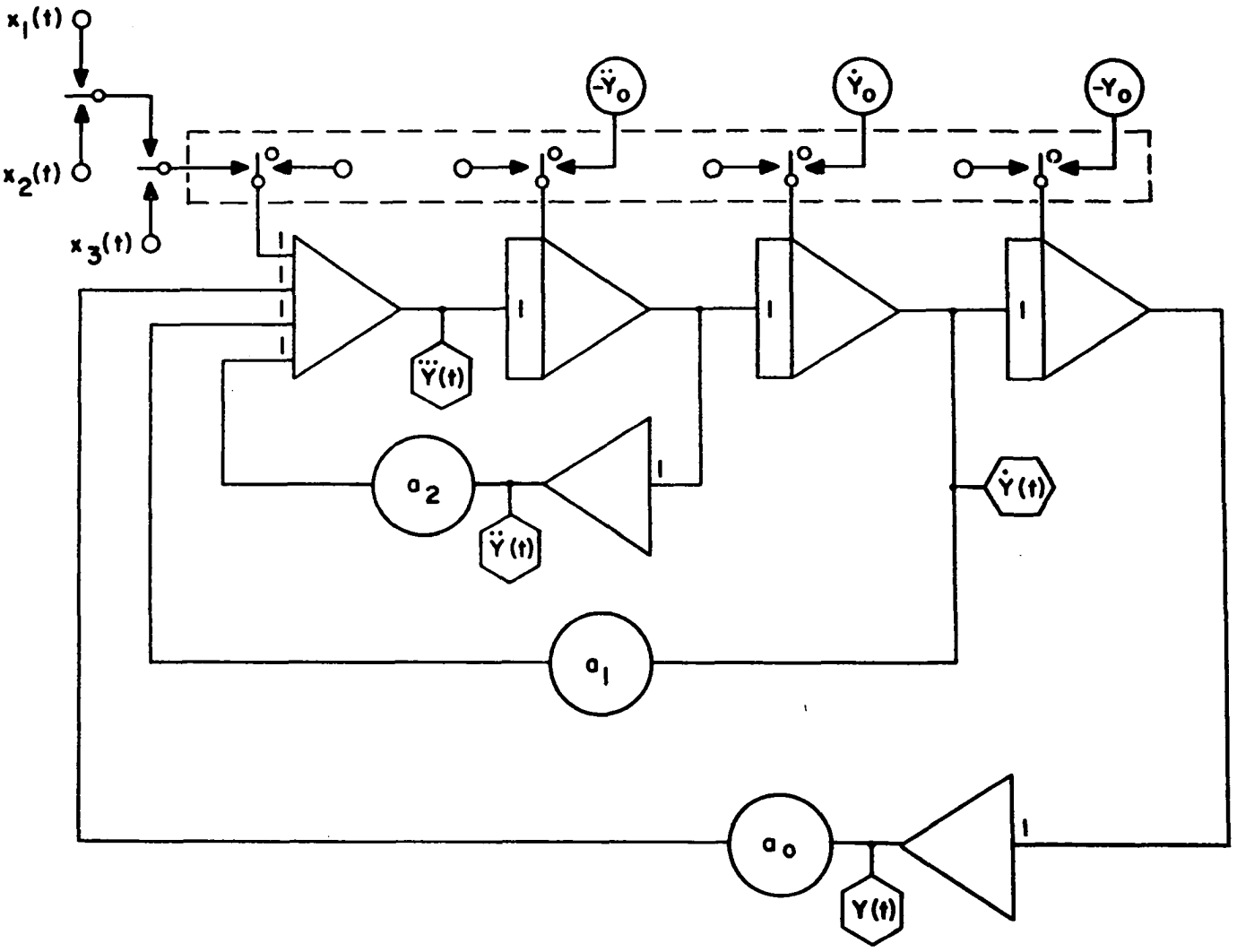


FIGURE III-7, A

SYSTEM SCHEMATIC FOR TESTS OF THIRD ORDER TIME INVARIANT SYSTEM

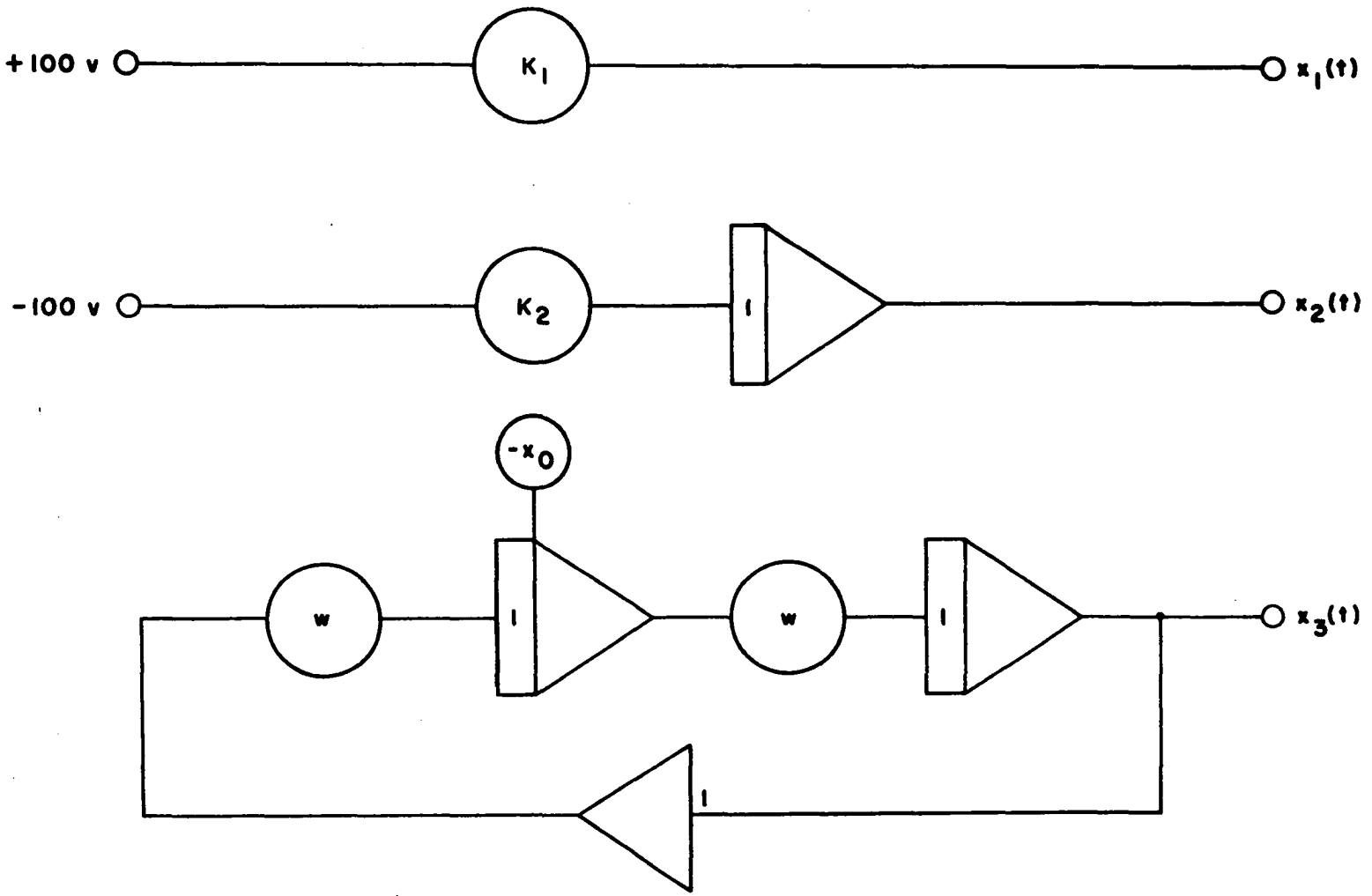


FIGURE III-7, B

INPUT SIGNAL GENERATION SCHEMATIC FOR TESTS OF THIRD ORDER TIME INVARIANT SYSTEM

said to be of the cannonical form. Note that if the non-zero elements appeared in any row other than that corresponding to the non-zero element in the input vector, then the Nth order differential equation would necessarily be of the form:

$$[L]\ y(t) = [M]\ x(t) \qquad y(0) = 0$$

where:

$$[L] = \sum_{i=0}^{N} a_i d^i/dt^i$$

$$[M] = \sum_{i=0}^{N-1} b_i d^i/dt^i$$

Since the weighting function, $\underline{H}(t,s)$ is expressible in terms of a fundamental set of solutions to the homogeneous form of the differential equation:

$$\underline{H}(t,s) = \overline{\underline{\phi}}(t)\overline{\underline{\phi}}^{-1}(s) \qquad t \geq s$$

where:

$$\dot{\overline{\underline{\phi}}}(t) = A\overline{\underline{\phi}}(t) \qquad \overline{\underline{\phi}}(0) = Z$$

and since the ith element of a solution vector has been defined to be the time derivative of the i-1st element, it is clear that $h_{ij}(t,s)$, (the response of the ith energy storage element to an input on the jth energy storage element) is the time derivative of the element $h_{i-1,j}(t,s)$.

As shown in the previous chapter, a knowledge of the weighting function and its time derivative $\partial\underline{H}(t,s)/\partial t$ is

sufficient information from which to determine the coefficient matrix, A. Recognition of the fact that the weighting function is nothing more than a particular fundamental matrix for the system makes clear the fact that the coefficient matrix, A, is determinable from the relation:

$$A = \dot{\underline{\phi}}(t)\underline{\phi}^{-1}(t)$$

Noting the relation mentioned above between the elements in the response vector for the particular type of system chosen (cannonical A matrix) it is clear that for this case a great savings of labor can be effected over the general case in which the fundamental matrix must be differentiated with respect to time. It is only necessary to measure one signal (the sum of signals entering the first integrator) in addition to those which must be measured to calculate the weighting function. Of course the fact that the required signal is measurable is not generally true but rather is a consequence of the fact that any voltage existing in the analog circuit is measurable. In general one would necessarily become involved with the problems associated with differentiation of experimental data.

$$\underline{\phi}(t) = \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) & \phi_{13}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \phi_{23}(t) \\ \phi_{31}(t) & \phi_{32}(t) & \phi_{33}(t) \end{vmatrix} = \begin{vmatrix} \phi_1(t) & \phi_2(t) & \phi_3(t) \\ \dot{\phi}_1(t) & \dot{\phi}_2(t) & \dot{\phi}_3(t) \\ \ddot{\phi}_1(t) & \ddot{\phi}_2(t) & \ddot{\phi}_3(t) \end{vmatrix}$$

$$\dot{\underline{\phi}}(t) = \begin{vmatrix} \dot{\phi}_{11}(t) & \dot{\phi}_{12}(t) & \dot{\phi}_{13}(t) \\ \dot{\phi}_{21}(t) & \dot{\phi}_{22}(t) & \dot{\phi}_{23}(t) \\ \dot{\phi}_{31}(t) & \dot{\phi}_{32}(t) & \dot{\phi}_{33}(t) \end{vmatrix} = \begin{vmatrix} \dot{\phi}_1(t) & \dot{\phi}_2(t) & \dot{\phi}_3(t) \\ \ddot{\phi}_1(t) & \ddot{\phi}_2(t) & \ddot{\phi}_3(t) \\ \dddot{\phi}_1(t) & \dddot{\phi}_2(t) & \dddot{\phi}_3(t) \end{vmatrix}$$

For simplicity the analog runs which were made to check the calculated values for the weighting function ($\underline{H}(0) = U$) were used to provide the required fundamental matrix and the additional vector element, $\dddot{\phi}_i(t)$. The results of making the digital calculation of the elements of the coefficient matrix are presented in Table III-3.

The system testing phase of the investigation of the third order system was accomplished in a manner similar to that discussed in connection with the second order time invariant system (Example 2). Since this system is of third order it was necessary to perform three experimental runs. The linearly independent forcing functions chosen were $x_1(t) = K_1$, $x_2(t) = K_2 t$, and $x_3(t) = x_0 \sin(wt)$. As before, the response of the system for t greater than zero (unforced) was recorded, read, tabulated, and adjoined appropriately to form nine arrays, each one corresponding to an element in a fundamental matrix.

As before, the calculation of the inverse matrix and the weighting function as the product of the fundamental matrix and its inverse was accomplished using a digital computer. The analog computer was used to generate a weighting function

TABLE III-3

DIGITAL COMPUTATION OF THIRD ORDER COEFFICIENT MATRIX

Time	Third Order Coefficient Matrix		
0	-0.0000	1.0000	-0.0000
	0.0000	-0.0000	1.0000
	-0.0980	-0.7519	-1.0000
2	-0.0000	1.0000	0.0000
	0.0000	0.0000	1.0000
	-0.0980	-0.7520	-1.0000
5	0.0000	1.0000	-0.0000
	0.0000	0.0000	1.0000
	-0.0980	-0.7519	-1.0000
10	0.0000	1.0000	-0.0000
	0.0000	-0.0000	0.0000
	-0.0980	-0.7519	-1.0000
20	-0.0000	1.0000	-0.0000
	-0.0000	-0.0000	1.0000
	-0.0980	-0.7519	-1.0000
50	-0.0000	1.0000	-0.0000
	0.0000	-0.0000	1.0000
	-0.0981	-0.7520	-1.0000
100	-0.0000	1.0000	0.0000
	0.0000	-0.0000	1.0000
	-0.0980	-0.7518	-1.0000

against which the digitally computed weighting function was checked. As in Example 2, the "check" weighting function was generated by solving the matrix differential equation:

$$\dot{\underline{\Phi}}(t) = A\underline{\Phi}(t) \qquad \underline{\Phi}(0) = U$$

The results of the calculation of the normalized weighting function, $\underline{H}(t,s)$, for $s = 0$ and normalized data from the solution for the "check" weighting function are presented in Figure III-8. As one would reasonably expect, the effect of larger matricies on the computational phase of the technique (inversion of matricies and matrix product formation) is deleterious. While the general criterion for high quality computational results presented in Chapter IV is applicable, its implementation for systems of order greater than two has arbitrarily been placed beyond the scope of this project.

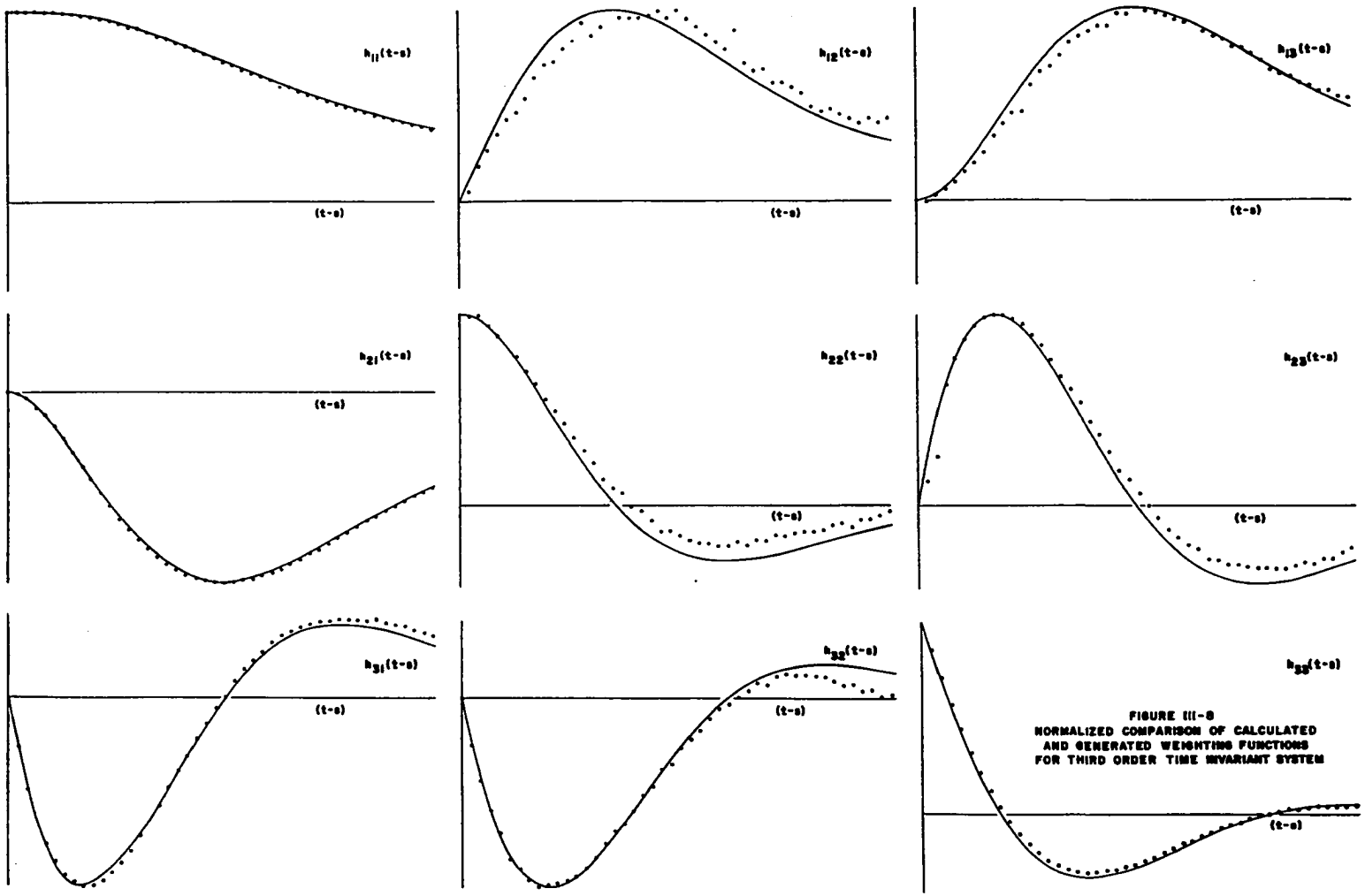


FIGURE III-8
NORMALIZED COMPARISON OF CALCULATED AND GENERATED WEIGHTING FUNCTIONS FOR THIRD ORDER TIME INVARIANT SYSTEM

CHAPTER IV

THE EFFECT OF SMALL ERRORS IN EXPERIMENTAL DATA

Examples 2 and 3 in Chapter III provided an introduction to the deleterious effect of imprecisely known fundamental matricies on the quality of the weighting function definable by the generalized pulse testing technique. In Example 2 the manipulations of the experimental data (read from continuous recordings of analog voltages) generated very satisfactory estimates of three of the four elements of the weighting function as indicated by a comparison of Figures III-5 and III-6. This same comparison, on the fourth element, $h_{12}(t-s)$, reveals an unexpected and quite dramatic failure of the technique to provide a sound estimate of the weighting function for the response of the first energy storage element to inputs applied to the second energy storage element. Example 3, on the other hand, reveals a general degradation of the quality of the estimate which is of the magnitude and nature one might expect upon increasing the size of the matricies being manipulated. This chapter is devoted to three topics generally concerned with the problem of manipulating erroneous data: a formal error criterion for the calculation, the derivation of maximum error bounds

for the calculation specifically for two by two matricies, and a discussion of the sources and effects of these errors. The final section consists of an application of the bounding estimates to the unsatisfactory element in the weighting function calculated in Example 2.

Formal Error Criterion: A formal error criterion for the manipulation of combined matrix inversion and multiplication which is the basis for the definition of weighting function models in the generalized pulse testing technique is sufficiently complex so as to make a heuristic approach profitable. This particular derivation is based on consideration of the determination of the weighting function for a second order system; its result, however, is applicable to N by N matricies.

Consider the fundamental matrix of homogeneous responses:

$$\underline{\underline{\phi}}(t) = \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{vmatrix}$$

and its inverse at time t = s:

$$\underline{\underline{\phi}}^{-1}(s) = I(s) = \begin{vmatrix} i_{11}(s) & i_{12}(s) \\ i_{21}(s) & i_{22}(s) \end{vmatrix}$$

The manipulation required to generate the weighting function at a particular point on the t, s surface is:

$$\underline{H} = \overline{\underline{\phi}}\, I$$

Note: Neglecting the arguments in the above equation and those following, except where confusion could arise, will simplify the presentation.

Writing out the previous equation in detail:

$$\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = \begin{vmatrix} \phi_{11} i_{11} + \phi_{12} i_{21} & \phi_{11} i_{12} + \phi_{12} i_{22} \\ \phi_{21} i_{11} + \phi_{22} i_{21} & \phi_{21} i_{12} + \phi_{22} i_{22} \end{vmatrix}$$

emphasizes the functional relationship:

$$\underline{H} = f(\phi_{11}, \phi_{21}, \phi_{12}, \phi_{22}, i_{11}, i_{21}, i_{12}, i_{22})$$

the derivative of which is:

$$\begin{aligned} \dot{\underline{H}} = {} & (\partial f/\partial \phi_{11})\dot{\phi}_{11} + (\partial f/\partial \phi_{21})\dot{\phi}_{21} + \\ & (\partial f/\partial \phi_{12})\dot{\phi}_{12} + (\partial f/\partial \phi_{22})\dot{\phi}_{22} + \\ & (\partial f/\partial i_{11})\dot{i}_{11} + (\partial f/\partial i_{21})\dot{i}_{21} + \\ & (\partial f/\partial i_{12})\dot{i}_{12} + (\partial f/\partial i_{22})\dot{i}_{22} \end{aligned}$$

or, if very small finite differences are used to replace differentials:

$$\Delta\underline{H} = (\partial f/\partial \phi_{11})\Delta\phi_{11} + (\partial f/\partial \phi_{21})\Delta\phi_{21} +$$

$$(\partial f/\partial \phi_{12})\Delta\phi_{12} + (\partial f/\partial \phi_{22})\Delta\phi_{22} +$$

$$(\partial f/\partial i_{11})\Delta i_{11} + (\partial f/\partial i_{21})\Delta i_{21} +$$

$$(\partial f/\partial i_{12})\Delta i_{12} + (\partial f/\partial i_{22})\Delta i_{22}$$

Alternately, the expression for the weighting function may be differentiated to give:

$$\dot{\underline{H}} = \overline{\underline{\phi}}\,\dot{I} + \dot{\overline{\underline{\phi}}}\,I$$

or by replacing differentials with small differences:

$$\Delta\underline{H} = \overline{\underline{\phi}}\Delta I + \Delta\overline{\underline{\phi}}I$$

where, of course:

$$\Delta\overline{\underline{\phi}} = \begin{vmatrix} \Delta\phi_{11} & \Delta\phi_{12} \\ \Delta\phi_{21} & \Delta\phi_{22} \end{vmatrix}$$

and

$$\Delta I = \begin{vmatrix} \Delta i_{11} & \Delta i_{12} \\ \Delta i_{21} & \Delta i_{22} \end{vmatrix}$$

The elements of the $\Delta\overline{\underline{\phi}}$ and ΔI matricies are to be regarded as the errors occurring in the measurement of the elements of

the fundamental matrix and these errors as compounded by the inversion of the fundamental matrix.

The partial derivatives appearing in the expression for the weighting function error matrix, $\Delta\underline{H}$, are thus seen to be replaceable in terms of values of $\overline{\underline{\phi}}$ and I as follows:

$$\Delta h_{11} = \Delta\phi_{11} i_{11} + \Delta\phi_{12} i_{21} + \phi_{11}\Delta i_{11} + \phi_{12}\Delta i_{21}$$

$$\Delta h_{21} = \Delta\phi_{21} i_{11} + \Delta\phi_{22} i_{21} + \phi_{21}\Delta i_{11} + \phi_{22}\Delta i_{21}$$

$$\Delta h_{12} = \Delta\phi_{11} i_{12} + \Delta\phi_{12} i_{22} + \phi_{11}\Delta i_{12} + \phi_{12}\Delta i_{22}$$

$$\Delta h_{22} = \Delta\phi_{21} i_{12} + \Delta\phi_{22} i_{22} + \phi_{21}\Delta i_{12} + \phi_{22}\Delta i_{22}$$

In order to express the elements of the inversion error matrix in terms of measurable quantities, one may again write a functional form, differentiate it appropriately, replace differentials with small differences, and substitute into the differentiated expression. For example:

$$i_{11} = \phi_{22}/(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})$$

where the argument of ϕ_{ij} is understood to be (s). Differentiation of the functional relation:

$$i_{11} = f(\phi_{11}, \phi_{21}, \phi_{12}, \phi_{22})$$

gives:

$$\dot{i}_{11} = (\partial f/\partial\phi_{11})\dot{\phi}_{11} + (\partial f/\partial\phi_{21})\dot{\phi}_{21} +$$

$$(\partial f/\partial\phi_{12})\dot{\phi}_{12} + (\partial f/\partial\phi_{22})\dot{\phi}_{22}$$

or by replacing differentials with small differences:

$$\Delta i_{11} = (\partial f/\partial \phi_{11})\Delta\phi_{11} + (\partial f/\partial \phi_{21})\Delta\phi_{21} + (\partial f/\partial \phi_{12})\Delta\phi_{12} + (\partial f/\partial \phi_{22})\Delta\phi_{22}$$

Performing the indicated differentiations gives:

$$\partial i_{11}/\partial \phi_{11} = -\phi_{22}\phi_{22}/(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})^2$$

$$\partial i_{11}/\partial \phi_{21} = \phi_{12}\phi_{22}/(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})^2$$

$$\partial i_{11}/\partial \phi_{12} = \phi_{21}\phi_{22}/(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})^2$$

$$\partial i_{11}/\partial \phi_{22} = -\phi_{21}\phi_{12}/(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})^2$$

which, upon substitution, gives:

$$\Delta i_{11} = \frac{(\phi_{12}\Delta\phi_{21}\ \phi_{22}\Delta\phi_{11})\phi_{22} + (\phi_{22}\Delta\phi_{12} - \phi_{12}\Delta\phi_{22})\phi_{21}}{(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})^2}$$

Similar manipulation of the remaining three elements of the inverse matrix gives:

$$\Delta i_{21} = \frac{(\phi_{21}\Delta\phi_{11} - \phi_{11}\Delta\phi_{21})\phi_{22} + (\phi_{11}\Delta\phi_{22} - \phi_{21}\Delta\phi_{12})\phi_{21}}{(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})^2}$$

$$\Delta i_{12} = \frac{(\phi_{22}\Delta\phi_{11} - \phi_{12}\Delta\phi_{21})\phi_{12} + (\phi_{12}\Delta\phi_{22} - \phi_{22}\Delta\phi_{12})\phi_{11}}{(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})^2}$$

$$\Delta i_{22} = \frac{(\phi_{11}\Delta\phi_{21} - \phi_{21}\Delta\phi_{11})\phi_{12} + (\phi_{21}\Delta\phi_{12} - \phi_{11}\Delta\phi_{22})\phi_{11}}{(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})^2}$$

Examination of the assembled ΔI matrix reveals that one may write it in the shorter matrix form:

$$\Delta I = -I\Delta\underline{\overline{\phi}}I$$

where the matrix $\Delta\underline{\overline{\phi}}$ is understood to be evaluated at t = s.

The final result then is an expression for the error in the calculated weighting function in terms of the fundamental matrix at time t, the inverse of the fundamental matrix at time s, and the errors in the measured values of these matricies:

$$\Delta\underline{H}(t,s) = \Delta\underline{\overline{\phi}}(t)I(s) - \underline{\overline{\phi}}(t)I(s)\Delta\underline{\overline{\phi}}(s)I(s)$$

If the maximum error in a $\underline{\overline{\phi}}$ matrix is considered, then the error criterion may be expressed in terms of this maximum error estimate and the values of the fundamental matrix and its inverse at times t and t = s respectively. The result is an estimate of the maximum error in the weighting function:

$$\Delta\underline{H}_{max}(t,s) = (U - \underline{\overline{\phi}}(t)I(s))\ \Delta M_{max} I(s)$$

or, since $I(s) = \underline{\overline{\phi}}^{-1}(s)$ and $\underline{H}(t,s) = \underline{\overline{\phi}}(t)\underline{\overline{\phi}}^{-1}(s)$:

$$\Delta\underline{H}_{max}(t,s) = (U - \underline{H}(t,s)\ \Delta M_{max}\ \underline{\overline{\phi}}^{-1}(s)$$

where ΔM_{max} is a matrix of numbers which correspond to the maximum error existing in each of the elements of a measured fundamental matrix. If one desires to obtain a percentage estimate of the error, then the initial formulation which considers the errors at time t and time t = s individually must be used. This percentage formulation would probably be more useful for the situation wherein the errors at the various points in the time domain were available and important, i.e., the case in which the responses contained more than one zero crossing, therefore, introducing regions of very poor quality calculations on a percentage basis within one generally high quality estimate.

Specialization of Error Criterion for a Two by Two Matrix: While the general criterion for the errors in a weighting function calculation given above is valid and asthetically pleasing, its practical usefulness is somewhat obscure. One simply does not have available the "true" values of the elements of the fundamental matrix and its inverse. In order to arrive at a more useful version of the criterion for maximum error in the calculated values of the weighting function for a second order system a slightly different approach was made.

Consider a matrix of measured values of the fundamental set of responses of a physical system:

$$M = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix}$$

Clearly, if $m_{ij} = \phi_{ij} \pm e_{ij}$, then $\phi_{ij} = m_{ij} \mp e_{ij}$ and the fundamental matrix becomes:

$$\underline{\overline{\phi}} = \begin{vmatrix} m_{11} \mp e_{11} & m_{12} \mp e_{12} \\ m_{21} \mp e_{21} & m_{22} \mp e_{22} \end{vmatrix}$$

and the inverse of the fundamental matrix becomes:

$$\underline{\overline{\phi}}^{-1} = \frac{\begin{vmatrix} (m_{22} \mp e_{22}) & -(m_{12} \mp e_{12}) \\ -(m_{21} \mp e_{21}) & (m_{11} \mp e_{11}) \end{vmatrix}}{((m_{11} \mp e_{11})(m_{22} \mp e_{22}) - (m_{12} \mp e_{12})(m_{21} \mp e_{21})}$$

The expression for the inverse of the fundamental matrix indicates that the error growth in the inversion operation depends on the magnitude and polarity of both the measurements and their associated errors and the ratio of magnitudes among both the measured elements and their associated errors, as well as the ratio of the magnitudes and polarities of the measured elements to the magnitudes and polarities of the associated errors. Further, the question of what constitutes the most erroneous inversion of the true matrix has no clear cut answer.

The criterion chosen to define the most erroneous inverse may be stated as follows: That particular configuration

of error polarities which gives rise to a determinant of minimum absolute magnitude generates the most erroneous inverse matrix. Clearly, it may be possible, by some arbitrarily chosen arrangement of error polarities, to find a single element in an inverse matrix which is more in error than that predicted by concentrating attention exclusively on the determinant. The savings in computational labor affected by this choice of error criterion and by the assumption that all errors are equal in absolute magnitude to the largest error reduces the problem of estimating the error generation in the inversion operation to a tractable level. Since the effect, if any, of the assumption concerning the magnitude of the determinant is to underestimate the error generation and the effect of the assumption of equal measurement error in all elements is certainly to overestimate the error generation, the combination of these assumptions may reasonably be expected to be valid. This expectation has been borne out by much experience.

Focusing attention on the determinant of the fundamental matrix, one writes:

$$\nabla = (m_{11} \mp e)(m_{22} \mp e) - (m_{21} \mp e)(m_{12} \mp e)$$

or:

$$\nabla = m_{11}m_{22} - m_{21}m_{12} - e(\pm m_{11} \pm m_{22} \pm m_{12} \pm m_{21} \pm e \pm e)$$

where e is the absolute magnitude of largest measurement error.

Clearly only the terms appearing within the parentheses need be considered in the determination of the minimum determinant. The four sign choices in the original determinant give rise to sixteen combinations of possible error. The parentheses is evaluated for each of these choices and that one giving the maximum value is chosen as the error polarity configuration which generates the worst error in the inversion process. The maximum error generated in the inversion process is then calculated by inverting both the measured fundamental matrix and the measured fundamental matrix as modified by the predicted "worst" error configuration; the difference between corresponding elements of these matricies is an estimate of the maximum error introduced by the inversion operation. Written in symbolic form, the matrix of maximum inversion errors is defined as:

$$\Delta I_{max} = M^{*-1} - M^{-1}$$

where:

$$M^{*} = M + E$$

and E is a matrix of errors of the polarity configuration chosen to minimize the determinant of M^{*}.

The manipulations required for the definition of the weighting function in terms of the fundamental matrix and its inverse are, as given previously:

$$\underline{H}(t,s) = \overline{\underline{\Phi}}(t)\overline{\underline{\Phi}}^{-1}(s) \qquad t \geq s$$

However, the calculation actually performed is:

$$\underline{H}'(t,s) = M(t)I(s) \qquad t \geq s$$

where M(t) is the matrix of adjoined measured system responses, existing at time t and I(s) is the matrix which is the result of inverting M(t) which existed at time t = s; of course, $\underline{H}'(t,s)$ is an estimate of the true weighting function $\underline{H}(t,s)$. As noted above, the matrix of measured values may be expressed as a combination of the fundamental matrix and a matrix, the elements of which are the errors made in measuring the fundamental matrix:

$$M(t) = \overline{\underline{\Phi}}(t) + \Delta M(t)$$

Similarly, the calculated inverse matrix, I, may be expressed in terms of the inverse of the fundamental matrix and a matrix the elements of which are the difference between the calculated and true values of the inverse:

$$I(s) = \overline{\underline{\Phi}}^{-1}(s) + \Delta I(s)$$

Thus the expression for the calculated weighting function may be expanded as follows:

$$\underline{H}'(t,s) = \overline{\underline{\Phi}}(t)\overline{\underline{\Phi}}^{-1}(s) + (M(t) - \Delta M(t))\,\Delta I(s) +$$

$$\Delta M(I(s) - \Delta I(s)) + \Delta M(t)\,\Delta I(s)$$

which becomes, upon simplification and rearrangement:

$$\underline{H}'(t,s) - \underline{H}(t,s) = M(t)\Delta I(s) + \Delta M(t)I(s) - \Delta M(t)\Delta I(s)$$

The left hand side of the above equation is clearly the error in the calculated value of the weighting function. Substitution of $\Delta I_{max}(s)$ for $\Delta I(s)$ and E for $\Delta M(t)$ renders this expression an estimate of the maximum error in the calculation of the weighting function:

$$\Delta \underline{H}_{max}(t,s) = M(t)\Delta I_{max}(s) + EI(s) + E\Delta I_{max}(s)$$

It should be noted that through this equation, the absolute magnitude of the maximum error in the calculation, as a function only of the estimated maximum error in any element of the measured set of system responses is defined; further, these error limits are not to be applied to the true weighting function, but rather to the calculated value. It is clear, however, that if a calculated value lies within given error limits of the true value, say $\underline{H}(t,s) \pm \epsilon$, then the true value must lie within these same error limits applied to the calculated value. These designations being equivalent, the values of the weighting functions determined by the generalized pulse test technique are reported as being the calculated value plus or minus the maximum error predicted for its calculation.

Possible Sources of Errors in the Data: There are five distinct sources of errors inherent in the acquisition and reduction procedures followed in the process of providing experimental data for the computational phase of the identification technique. Needless to say, the error in any one piece of data may be, and probably is, due to a combination

of two or more of these types. In order to discuss faulty data on a logical basis it is necessary to treat the error sources individually. The five sources of errors may be expressed as follows:

1. Imprecision due to limits of recording equipment.
2. Imprecision due to random reading error.
3. Inaccuracy due to erroneous equilibrium state designation.
4. Faults due to erroneous designation of input removal time.
5. Inaccuracy due to erroneous designation of applicable signal attenuation or amplification.

The first two sources of error are important in all of the experiments performed, regardless of whether the source of the data is the analog computer or the physical process chosen for the experiment. The remaining three sources are particularly important in considerations of the data acquired from the physical process, although if the total variable, rather than its transient portion, is recorded in the analog type of experiment, then the designation of the equilibrium state of the system becomes of prime importance.

As background for the discussion of the five sources of error it is profitable to examine the data acquisition and reduction procedures. Consider first the problem of acquisition of data from the analog computer experiments.

This discussion is equally valid for those experiments involving the experimental apparatus as the procedure followed was to telemeter data from the apparatus to the analog computer, amplify it further, and record computer voltages which were proportional to the state variables of the process carried out in the apparatus.

As described in Example 2 of Chapter III, the system testing phase of the determination consisted of making a series of continuous recordings of the voltages, existing in an electrical network, which were analogous to the state variables of the process being simulated. These recordings were made on a Sanborn six channel hot wire recording oscillograph. Each recording channel has a span of fifty millimeters width with gradations each millimeter. Assuming that, at best, one is able to estimate the position of the hot wire trace (approximately 0.15 millimeters wide) to the nearest 0.25 millimeter, one could expect to report data with a maximum resolution of 0.5 per cent. While for most uses this resolution is adequate, it represents two significant figures with up to fifty per cent uncertainty in a third.

Experience has shown that the maximum error bound predictions made in the computation phase may be uncomfortably large for calculations based on two significant figure data. Further, and far more important, it has shown that no guarantee can be made that the calculated values will lie in a sufficiently close packed pattern so as to permit a sound estimate of the true value.

The result of these considerations is an indication that there exists a limit on the precision of the data acquired which is purely an equipment constraint, namely, that each recorder channel has a finite span. Since the resolution analysis includes the implication that the attenuation controls on the recorder are set to utilize the full recorder span, circumvention of the difficulty requires either alternate or additional auxiliary recording equipment. The second choice was made and implemented by the development of switching equipment to provide eleven adjustable finite steps of recorder reference voltage zero suppression, thus effectively telescoping 550 millimeters of channel span onto the original fifty millimeter span. With this addition, the equivalent resolution became three significant figures with up to fifty per cent uncertainty in a fourth.

Regardless of the increase in precision introduced by the development of the zero suppression switch arrangement, any number read and taken to be the value of the recorded variable is merely an estimate of the true value. In an effort to make this estimate as nearly correct as possible, there is a great tendency to "over-read" the chart. This instinctive reaction introduces a small random variation in the data which may be noted by examination of the second differences of the data. While "over-reading" may be condemned, it, together with analyses of second difference information is more logically viewed as an attempt to extract the maximum amount of information contained in the recording.

Consideration of the additional complications of reducing experimental information generated in the laboratory reveals the remaining three sources of imprecision. The actual data reduction procedures vary between types of signals but the amplification of signal level is common to all types. In general, the procedures involve transducing the physical variable to a voltage, preamplification, transmission to the analog computer, further amplification, and recording this amplified voltage as discussed previously. The operations of signal transducing and amplification involve both zero suppression and amplification factors which must be determined correctly if one is to extract the signal from the recordings. Quite obviously, since one cannot ordinarily measure the transient portion of physical variables to the complete exclusion of their level, the adjustments to be made and calibrated on the reduction equipment may introduce errors in both the attenuation factor and steady state or equilibrium level correction to be applied to the recorded voltages.

Finally, the question of the designation of the time at which the recorded responses truly become autonomous responses of the physical system must be considered. It is quite reasonable to expect to be able to switch an electrical input to zero at an arbitrary instant as is the case in the analog computer experiments. Unfortunately it is impossible to exercise such complete control over inputs, such as temperature or flow rate, to a physical process. It must be

understood that the concept of setting the input to zero should be more precisely stated when dealing with most physical processes. Since all of the theory has been developed using only the transient portion of the responses, setting an input to zero is the equivalent of instantaneously setting the total input variable to its equilibrium level. In general, it is impossible to make an instantaneous change in a physical variable, therefore the "time at which the input is removed" must be taken, for experimental purposes, to mean the time at which the input variable settles to its equilibrium level. Since this point on a recording of experimental data is quite arbitrary, a reasonable amount of care must be exercised in its designation. The computational phase of the identification process treats all data as if it were bonafide autonomous response data. Therefore, an error in the designation of the initial piece of data for which the computations are applicable may be the source of a vastly misleading estimate of the weighting function.

Effects of Erroneous Data: Three of the five types of data inaccuracies mentioned in this chapter involve some sort of systematic error. It is not unreasonable to expect that the effect of this type of error on the quality of the calculated weighting function should follow some sort of pattern. If such an error pattern could be recognized, then steps could be taken to prevent its recurrence, if not to correct for it. Therefore, it is of interest to investigate

the nature of the effects of the various types of error on the quality of the estimate of a weighting function. The complexity of these effects is such that a complete quantitative treatment has arbitrarily been placed beyond the scope of this work. Instead, a qualitative presentation of their nature is offered together with evidence in support of the validity of the prediction technique, given in this chapter, as a practical estimate of the maximum errors associated with the calculated values of the weighting function.

The investigation of the effects of the various types of error was accomplished by simulation of the responses of a physical system, the transient behavior of which is described by the vector differential equation:

$$\dot{Y}(t) = AY(t) + X(t) \qquad Y(T) = Z$$

where:

$$A = \begin{array}{|c|c|} \hline & 1 \\ \hline -0.2 & -0.3 \\ \hline \end{array}$$

by calculations carried out to five significant figure precision on a digital computer. This particular system was chosen because the underdamped nature of its response leads to multiple zero crossings by its weighting function and because the duration of its autonomous response is relatively short.

The "experimental" data were generated each 0.5 units of the independent variable from calculation of the numerical solution of the differential equation (Runge-Kutta) made each 0.01 units of the independent variable. The relatively high frequency of calculation was required to assure the specified precision because of the oscillatory nature of the solution. The actual calculations consisted of solving the non-homogeneous matrix differential equation:

$$\begin{bmatrix} \dot{y}_{11}(t) & \dot{y}_{12}(t) \\ \dot{y}_{21}(t) & \dot{y}_{22}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.2 & -0.3 \end{bmatrix} \times \begin{bmatrix} y_{11}(t) & y_{12}(t) \\ y_{21}(t) & y_{22}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x_{21}(t) & x_{22}(t) \end{bmatrix}$$

where:

$$x_{21}(t) = 50$$
$$x_{22}(t) = 5 \sin (0.0628t)$$

in the range $0 \leq t \leq T$ and the matrix differential equation:

$$\begin{bmatrix} \dot{\phi}_{11}(t) & \dot{\phi}_{12}(t) \\ \dot{\phi}_{21}(t) & \dot{\phi}_{22}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.2 & -0.3 \end{bmatrix} \times \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix}$$

where:

$$\underline{\bar{\Phi}}(T) = \begin{array}{|c|c|} \hline y_{11}(T) & y_{12}(T) \\ \hline y_{21}(T) & y_{22}(T) \\ \hline \end{array}$$

in the range $T \leq t \leq (T+50)$.

The spirit of the investigation was to compute the weighting function for the system, based on five significant figure data and to compare the results of this "ideal" calculation to the results of calculations of the weighting function made with data which were controlled corruptions of the "ideal" data. The types of error studied in this manner are inaccuracies due to erroneous attenuation factor determination, and faults due to erroneous designation of the initial instant of autonomous response. By regarding a piece of data corrupted by errors in the steady state determination as "ideal" data either misread or containing random measurement error, the investigation of steady state errors may be used to shed light on these sources of imprecision.

Figure IV-1 is a presentation of the weighting function for this system calculated from the "ideal" data for two values of s, zero and twenty. The quality of the convergence of calculated values to a smooth curve is an indication both that the system is time invariant and that five significant figure "experimental" data are truly "ideal."

Consideration of Figure IV-2, which is a plot of the absolute magnitude of the determinant of the fundamental set

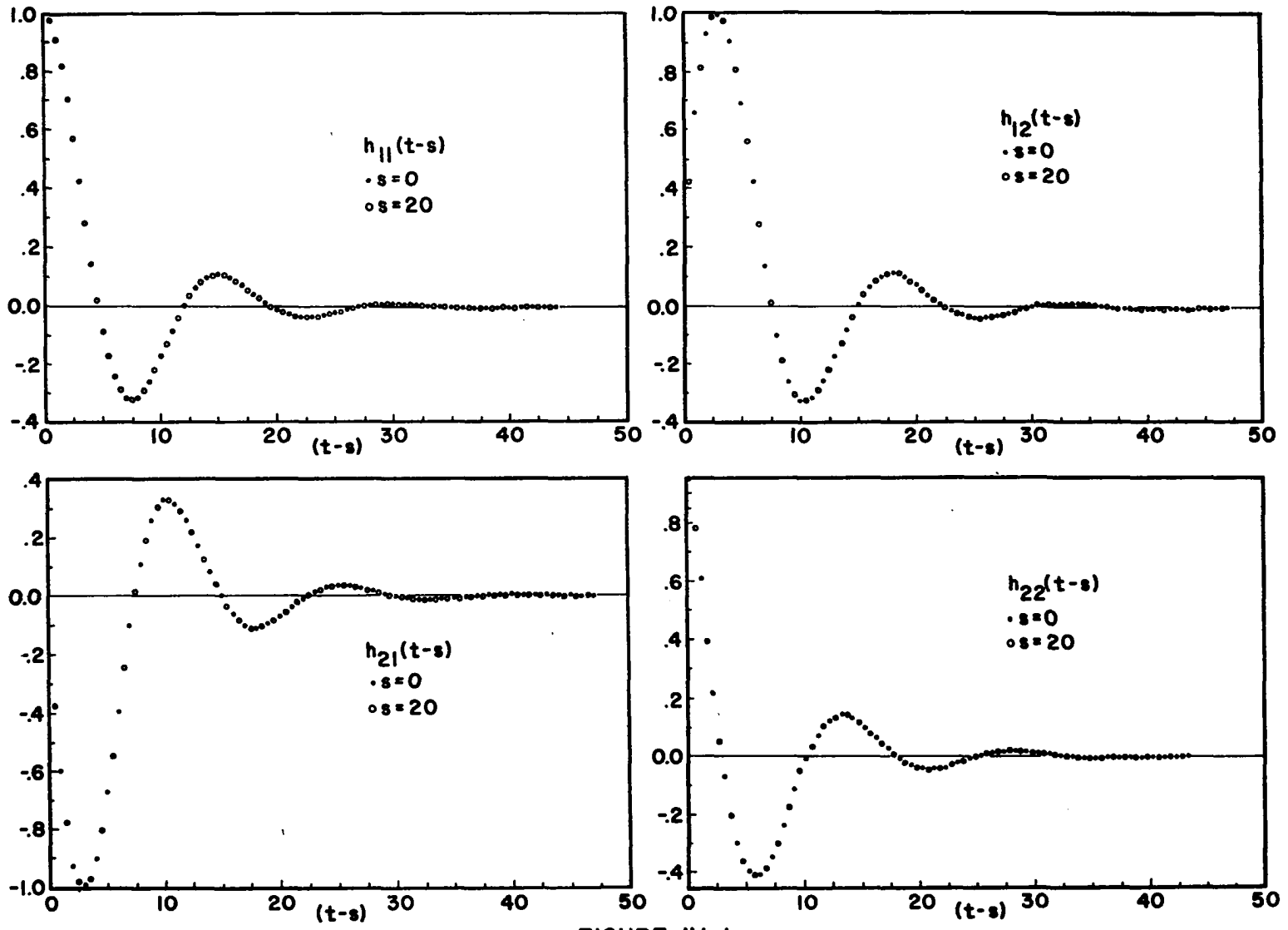


FIGURE IV-1
NORMALIZED WEIGHTING FUNCTION FOR SECOND ORDER TIME INVARIANT SYSTEM CALCULATED FROM IDEAL DATA

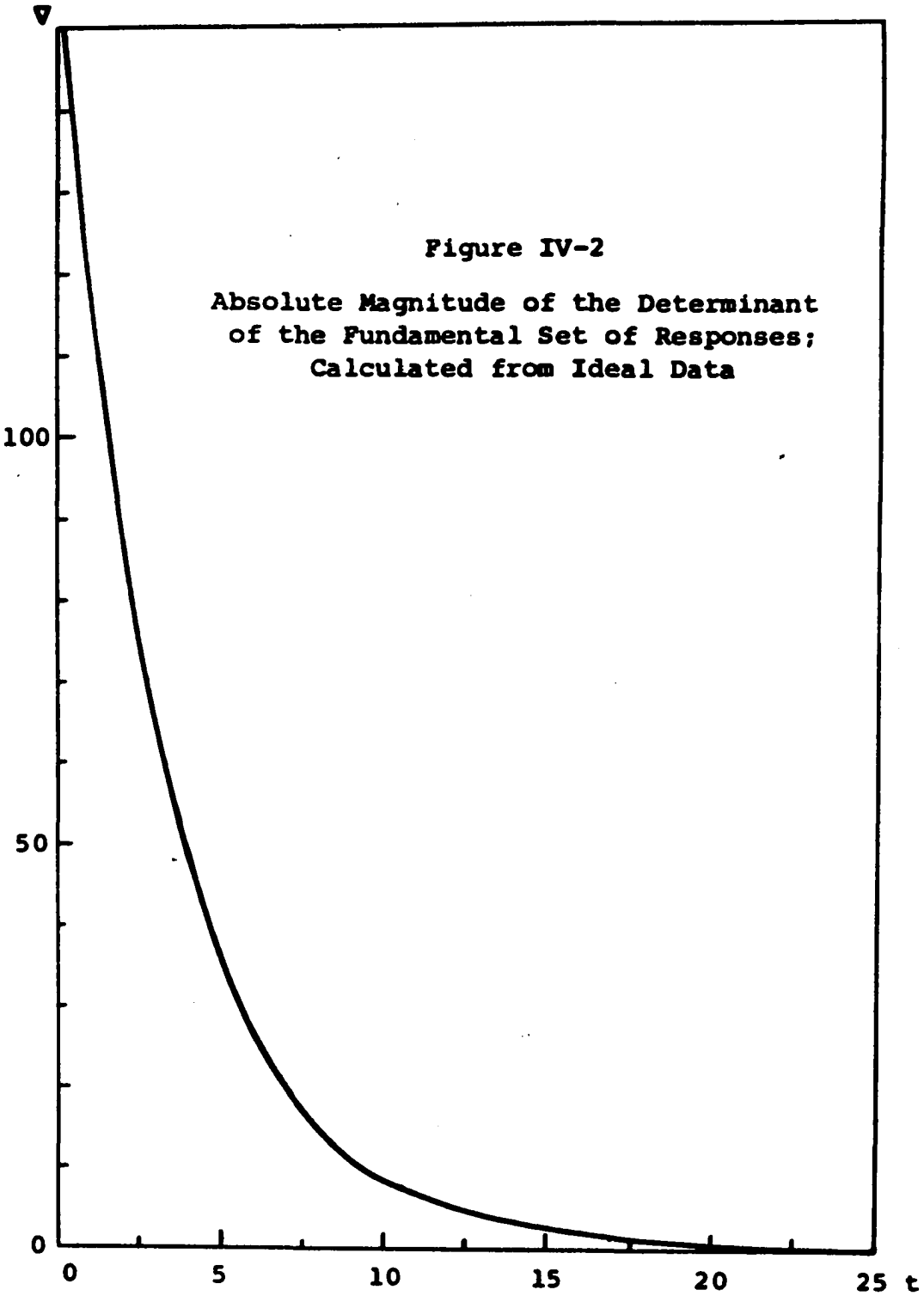


Figure IV-2

Absolute Magnitude of the Determinant of the Fundamental Set of Responses; Calculated from Ideal Data

of autonomous response, emphasizes the quality of the "ideal" data. Recall that the weighting function is calculated from the expression:

$$\underline{H}(t,s) = \underline{\Phi}(t)\underline{\Phi}^{-1}(s) \qquad t \geq s$$

and that the elements of $\underline{\Phi}^{-1}(s)$ are inversely proportional to the determinant of $\underline{\Phi}(s)$. It is apparent that as s increases, any error in the elements of $\underline{\Phi}(s)$, hence in its determinant, may generate very large errors in the elements of $\underline{\Phi}^{-1}(s)$, hence in the elements of the weighting function. Clearly, if one is able to see that the maximum error predicted for the inversion operation was large enough to include the possibility that the matrix being inverted was a singular matrix, then there would be no point in attempting to calculate the weighting function at that value of s.

Figure IV-3 presents the calculated values of the determinant of the fundamental matrix modified by the addition of steady state errors equal in magnitude to one per cent of its maximum value. The vertical bars in the Figure indicate the predicted maximum errors associated with the inversion of the erroneous fundamental matrix. The fact that the calculated values of the determinant are much closer to the "ideal" determinant than required by the error bounds is simply a statement that this particular corruption was not the worst possible choice. The sense of the maximum error bounds predicted is that for s greater than ten units, there

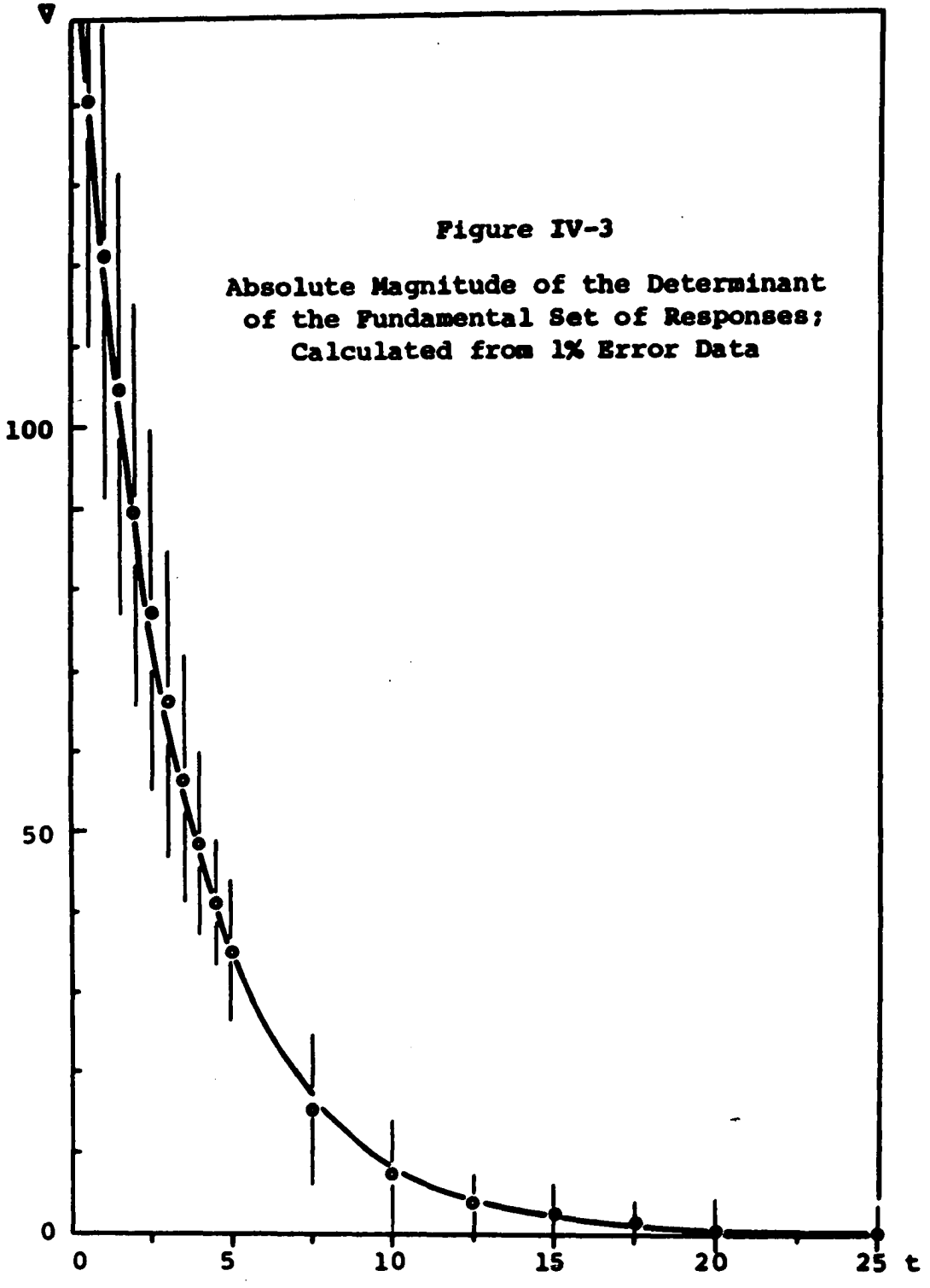


Figure IV-3

Absolute Magnitude of the Determinant of the Fundamental Set of Responses; Calculated from 1% Error Data

can be no guarantee that the matrix, with its estimated inaccuracy was not singular; hence there is no guarantee that the value of the calculated weighting function is valid.

Figure IV-4 shows the normalized comparison of the "ideal" weighting function to that calculated using one per cent steady state error data. The value of the parameter s in the calculation was zero. It should be noted that while the "ideal" results lie within the predicted maximum error bounds associated with the calculated values, these bounds are not overly generous. The variation in the size of the error bounds associated with specific elements differs because of the relative sizes of multiplicative factors and of the normalization constants.

Figure IV-5 also shows the normalized comparison of the "ideal" weighting function to that calculated using one per cent error data. The difference is that the value of the parameter s in this calculation was five. Consideration of the relative sizes of the ratio of predicted maximum error in the determinant to its magnitude (see Figure IV-3) indicates that the values calculated at s equals five are less reliable than those calculated at s equals zero. This indication is borne out in Figure IV-5 by the increased predicted error bounds and a slight worsening of the estimate of the "ideal" weighting function.

Figure IV-6 again shows the normalized comparison of the "ideal" weighting function to that calculated using one

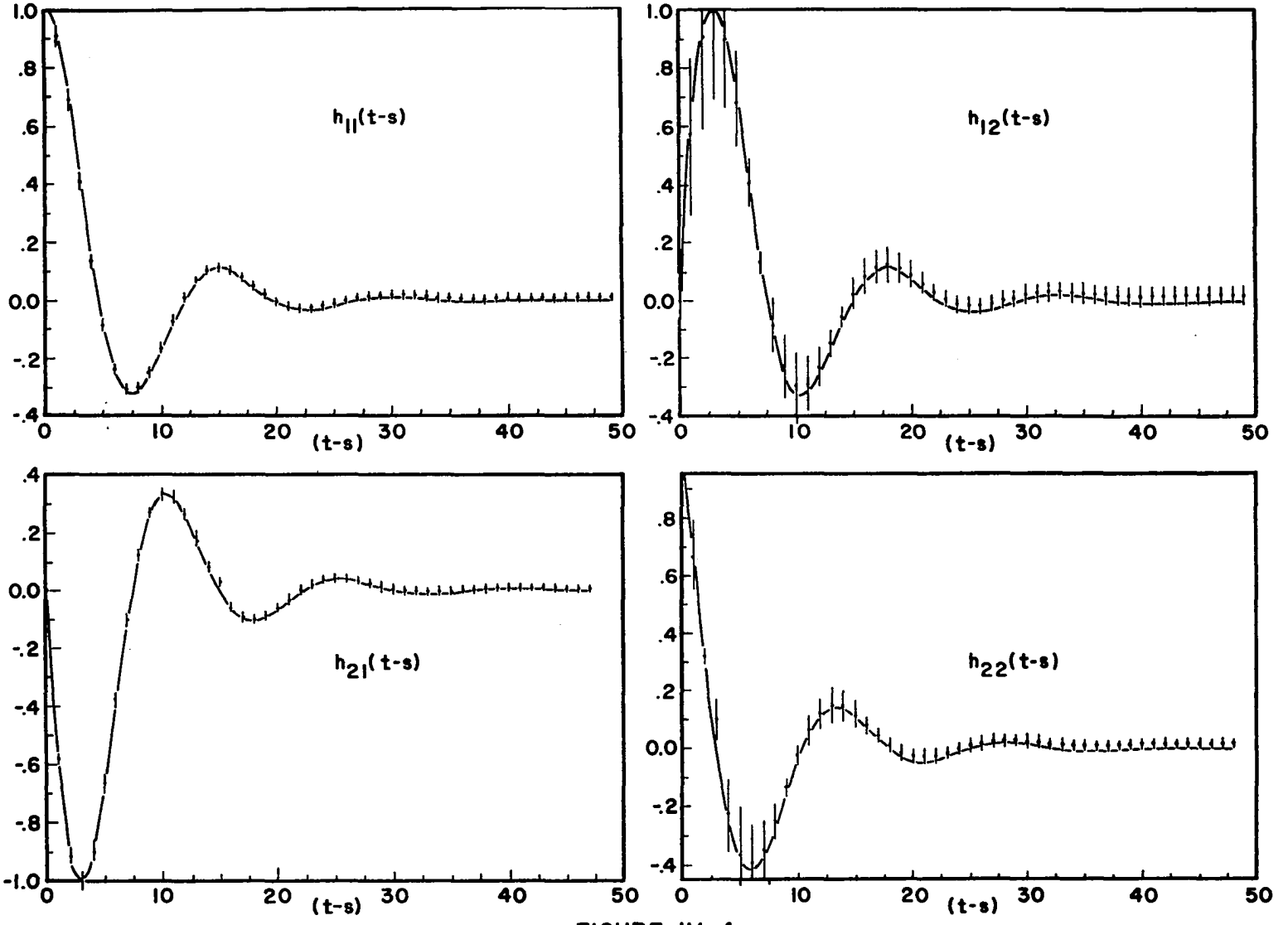


FIGURE IV-4
NORMALIZED WEIGHTING FUNCTION FOR SECOND ORDER TIME INVARIANT SYSTEM
CALCULATED FROM 1% ERROR DATA AT S=0

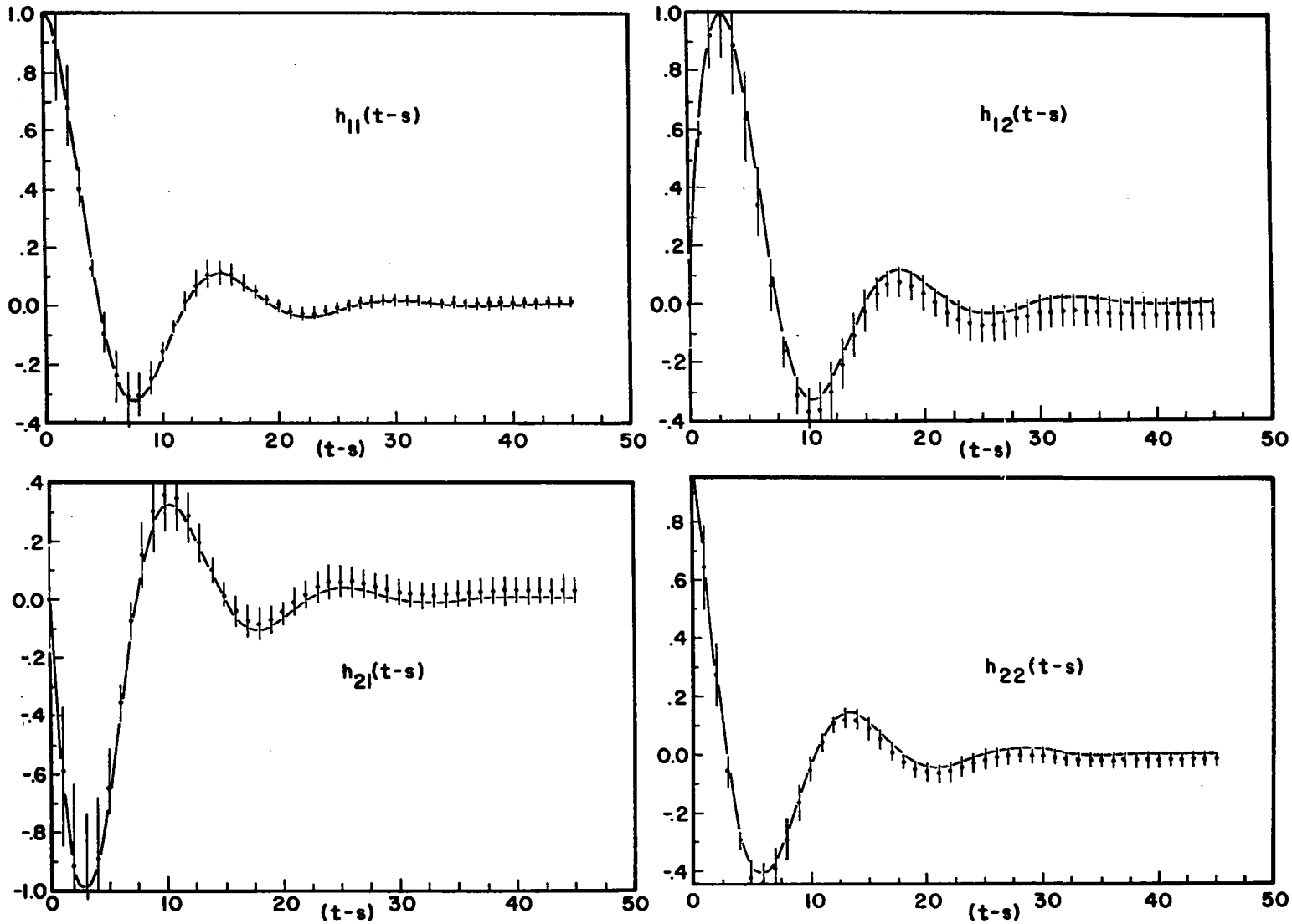


FIGURE IV-5
NORMALIZED WEIGHTING FUNCTION FOR SECOND ORDER TIME INVARIANT SYSTEM
CALCULATED FROM S=5,1% ERROR DATA

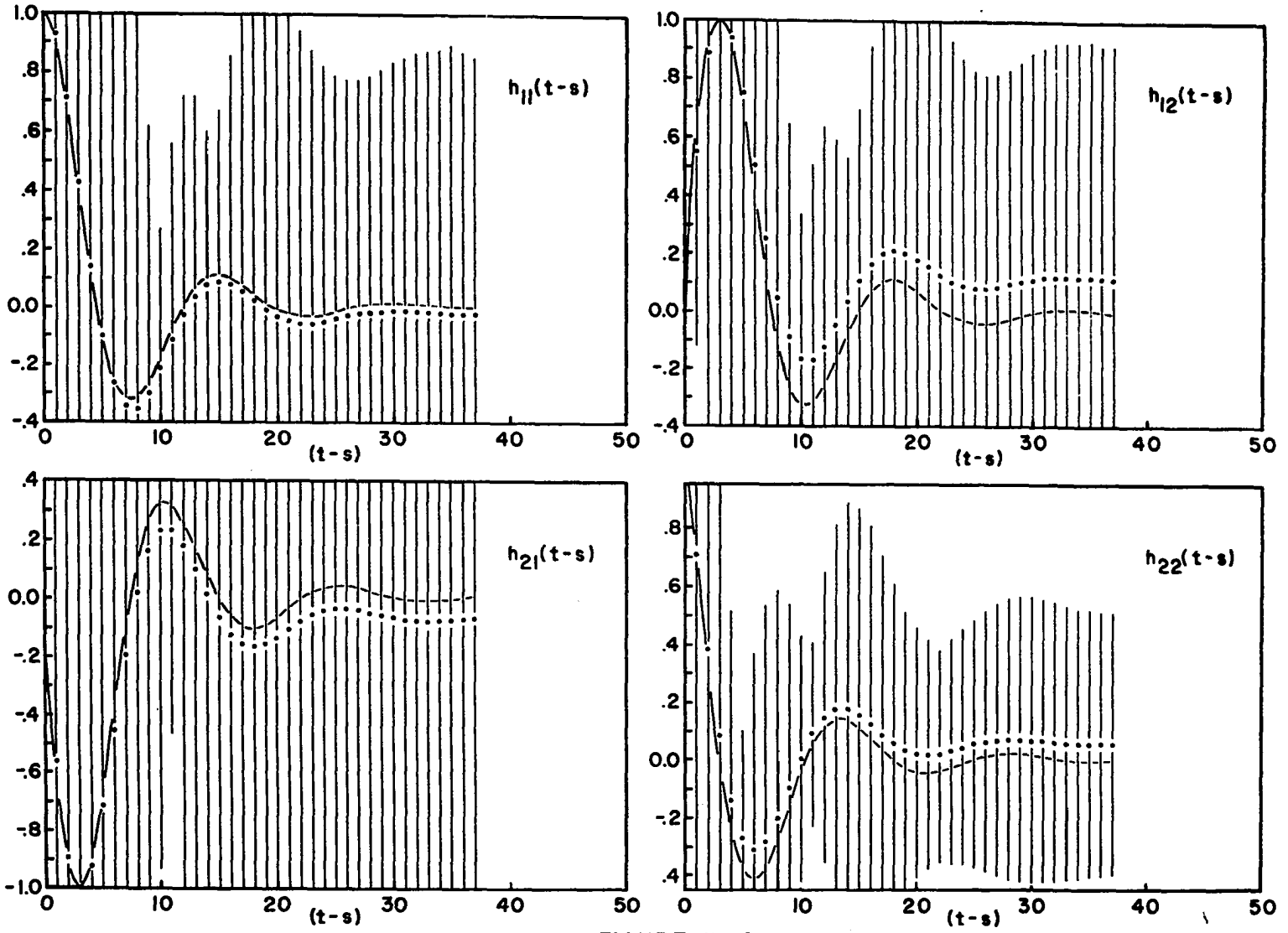


FIGURE IV-6
NORMALIZED WEIGHTING FUNCTION FOR SECOND ORDER TIME INVARIANT SYSTEM
CALCULATED FROM S=12.5, 1% ERROR DATA

per cent error data. The value of s at which these calculations were made is twelve and one half. Consideration of Figure IV-3 indicates that very little reliability may be attached to these calculations. The extent of the predicted maximum error bounds in Figure IV-6 also reflect this indication. It should be noted that while the calculated values represent a much better estimate of the weighting function than the maximum error bounds predict, the polarity of the deviation from the "ideal" calculation has changed and the estimate itself is considerably worse.

Figure IV-7 presents the values of the determinant of the fundamental matrix estimated from data modified by the addition of errors, equal in magnitude to two per cent of its maximum value, to each element. The vertical bars which represent the maximum error bound predicted for the inversion operation are larger than those appearing in Figure IV-3, thus reflecting the increased data error. The calculated values lie further from the "ideal" calculation than do those of Figure IV-3 but much closer to it than the predicted error bounds require. This behavior is the result of the fact that the error configuration chosen has a less deleterious effect on the calculation than at least one other choice. It should be noted that, based on the predicted maximum error, determinants evaluated at values of the parameter s greater than seven and one half must be used cautiously as the possibility of a singular matrix exists.

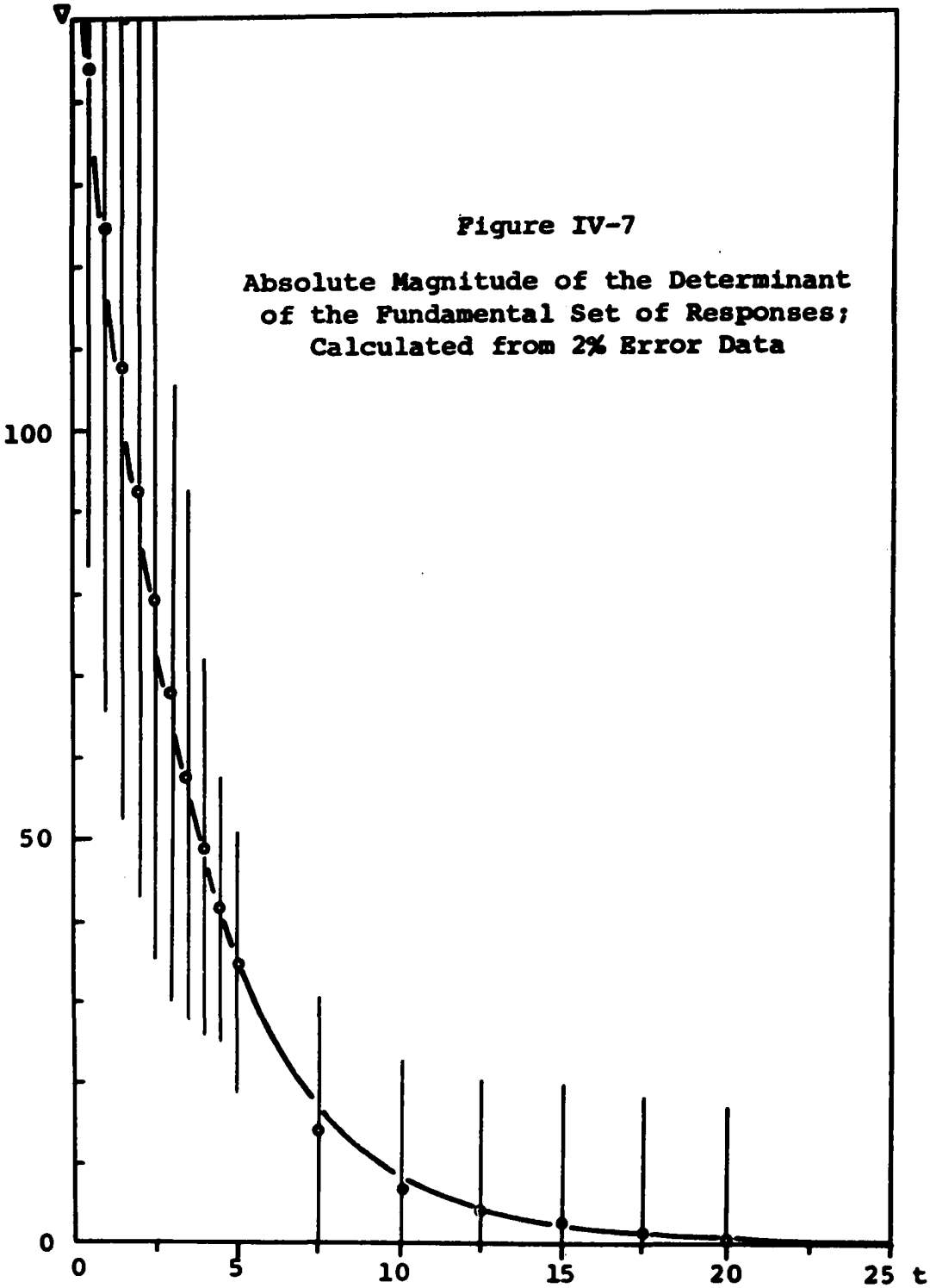


Figure IV-7

Absolute Magnitude of the Determinant of the Fundamental Set of Responses; Calculated from 2% Error Data

Figure IV-8 shows the normalized comparison of the "ideal" weighting function to that calculated using the two per cent error data. The value of the parameter s in the calculation is zero. It should be noted that the calculated values are displaced further from the "ideal" weighting function than are those of Figures IV-4, 5, 6 and that the predicted maximum errors are larger than those of these figures. The calculated values of elements h_{11} and h_{21} deviate from the "ideal" by an amount nearly equal to the limit predicted by the error bounds, thus indicating that these elements are very sensitive to the particular choice of error configuration.

Figure IV-9 again shows the comparison of the "ideal" weighting function to that calculated using the two per cent error data. The value of the parameter s for this calculation was five. Comparison of Figure IV-8 and IV-9 shows, as did a similar comparison of Figures IV-4, 5, 6, that values calculated for different values of s deviate from one another. In general, the magnitude of the deviation from the "ideal" becomes larger as the value of the parameter s is increased. The source of this quality degradation is the quality of the inverse matrix used in the calculation. Note, however, that the maximum error predicted has grown sufficiently so as to include the "ideal" calculation within its bounds.

The result of making the calculation at steadily larger values of the parameter s is the prediction of steadily larger error bounds until the situation is reached wherein

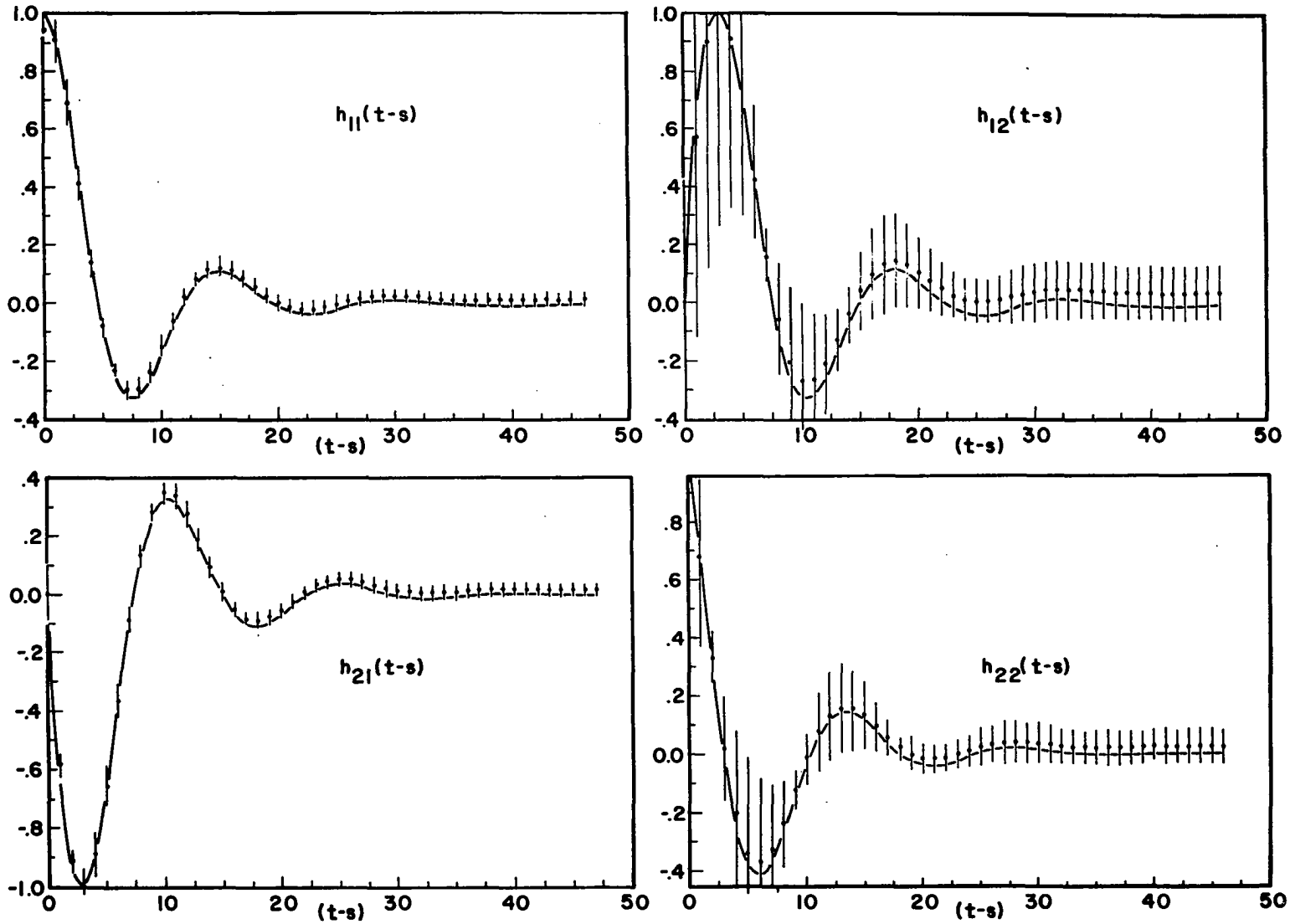


FIGURE IV-8
NORMALIZED WEIGHTING FUNCTION FOR SECOND ORDER TIME INVARIANT SYSTEM
CALCULATED FROM 2% ERROR DATA AT S=0

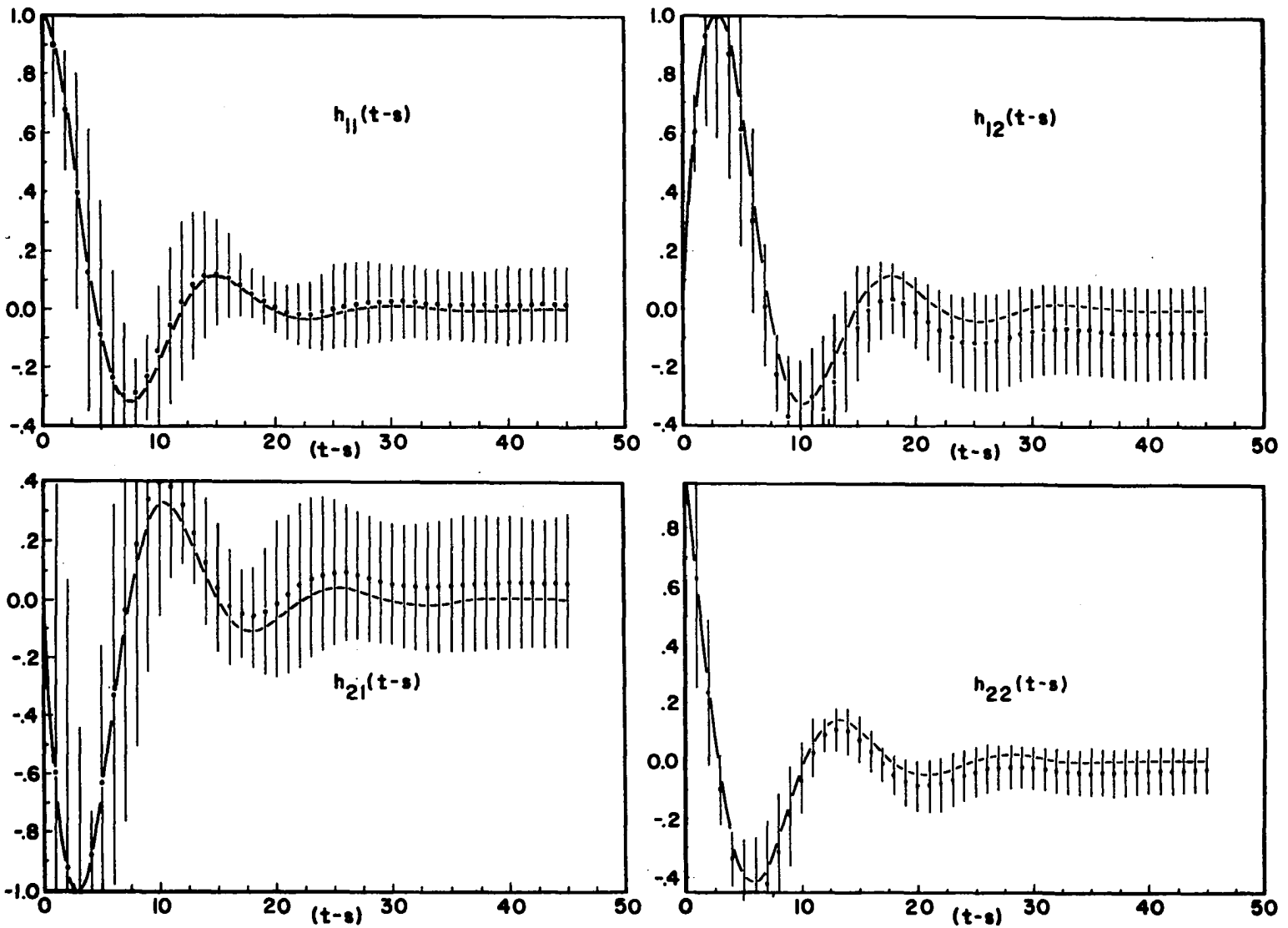


FIGURE IV-9
NORMALIZED WEIGHTING FUNCTION FOR SECOND ORDER TIME INVARIANT SYSTEM
CALCULATED FROM S=5, 2% ERROR DATA

virtually any weighting function is covered by the field defined by the error-bounded calculation. This senseless waste of computational effort may be avoided by the refusal to make a calculation at a value of s from which a plot of the error-bounded determinant, i.e., Figures IV-3 and IV-7, indicates the possibility of a singular matrix. Since Figure IV-7 indicates this possibility beginning at s equals seven and one half, no calculation of the weighting function for s equals ten was made for the two per cent error data.

To conclude the qualitative study of the effect of steady state errors on the calculated weighting function, the following statements are appropriate.

1. The effect of a constant error in the data used is an error in the final value approached by the calculated function.
2. In general, the magnitude of the error in the calculated function is proportional to that of the data.
3. No guarantees regarding the polarity of the error in the calculated function can be given for a specific polarity of the data error. The dependence on the relative magnitude of the data elements appears to control polarity.

These qualitative results may be used to gain an appreciation of the effect of random measurement error on the quality of the calculated weighting functions. Consider

one specific calculated point on any of the Figures IV-4, 5, 6, 8, 9. It is the result of making the weighting function calculation using erroneous data. More specifically it is the result of using data which quite randomly happened to be in error by a positive measurement error equal in magnitude to some percentage of the maximum value of each element. The deviation of this calculated point from the "ideal" is an estimate of the effect of that size random error on the calculation. It is difficult to imagine measurement errors in excess of one per cent, so Figures IV-4, 5, and 6 probably reflect the approximate magnitude of the random measurement error effect in a more realistic manner than do Figures IV-8 and 9.

If one considers truly random errors in the data, then it is to be expected that some of the calculated values would be greater than the true weighting function and some would be smaller. The effect of these errors on a plot of the calculated weighting function would be to cause it to have the appearance of a braod saw-tooth trace. Since the true weighting function is a continuous, smooth curve, its location necessarily is between the "valleys" of the saw-tooth; thus the region of possible location of the true function tends to be constrained to be somewhat smaller than that predicted by the maximum error bounds.

Figures IV-10, 11, and 12 show the effect of two, five, and ten per cent attenuation errors on the major diagonal elements of the calculated weighting function. Calculations made

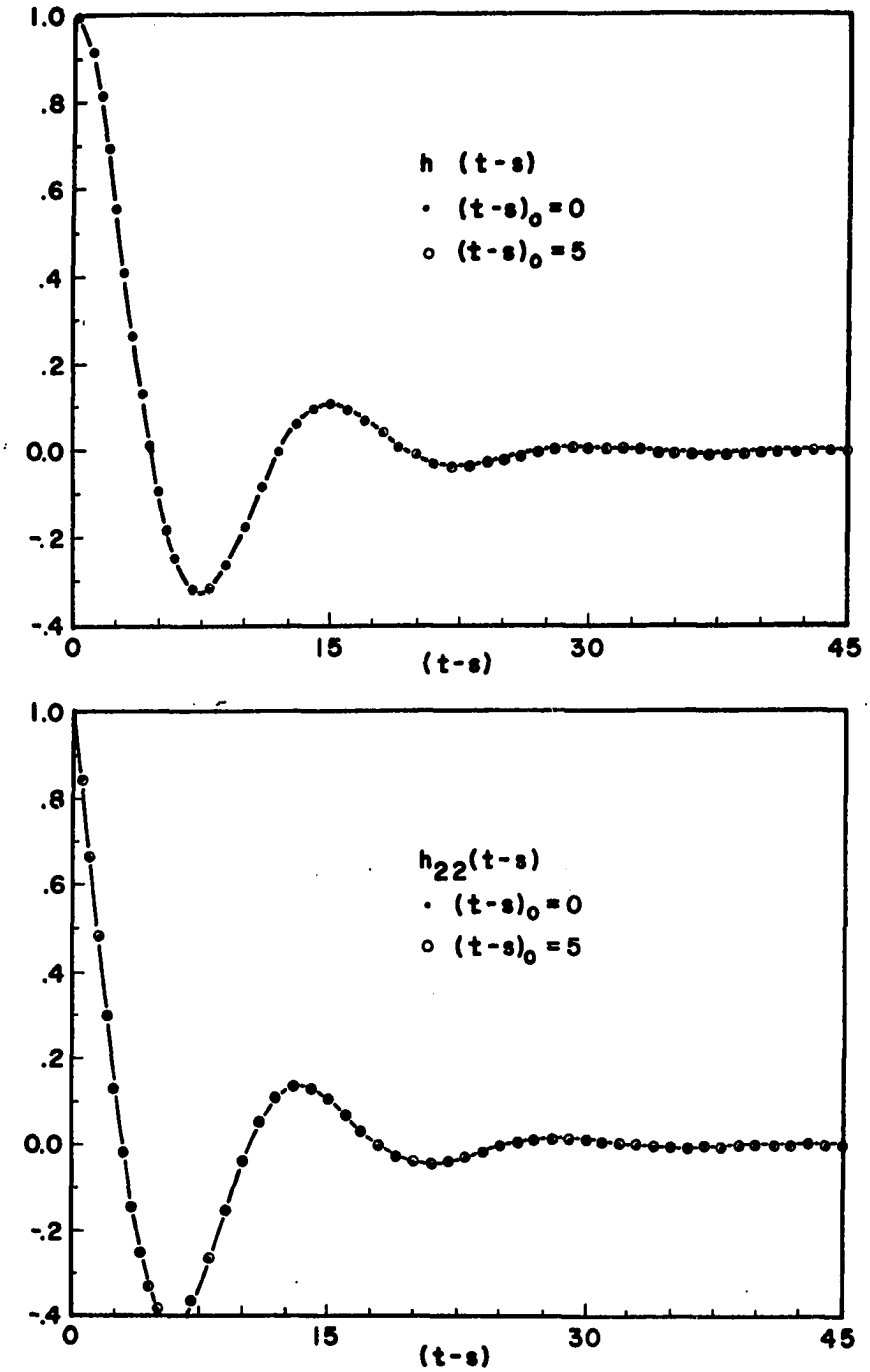
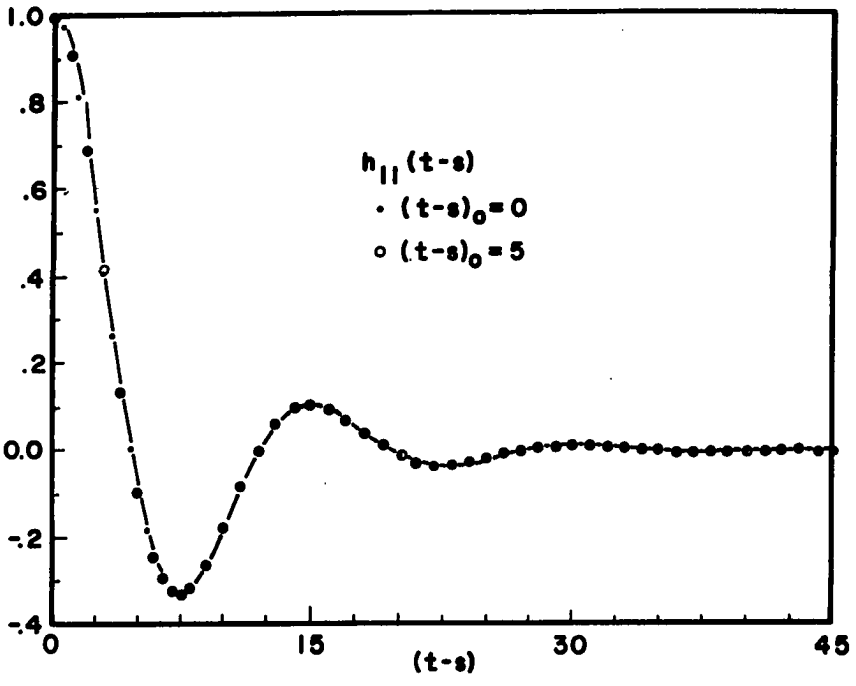


FIGURE IV-10
THE EFFECT OF 2% ATTENUATION ERROR ON THE MAJOR DIAGONAL ELEMENTS OF A SECOND ORDER TIME INVARIANT WEIGHTING FUNCTION



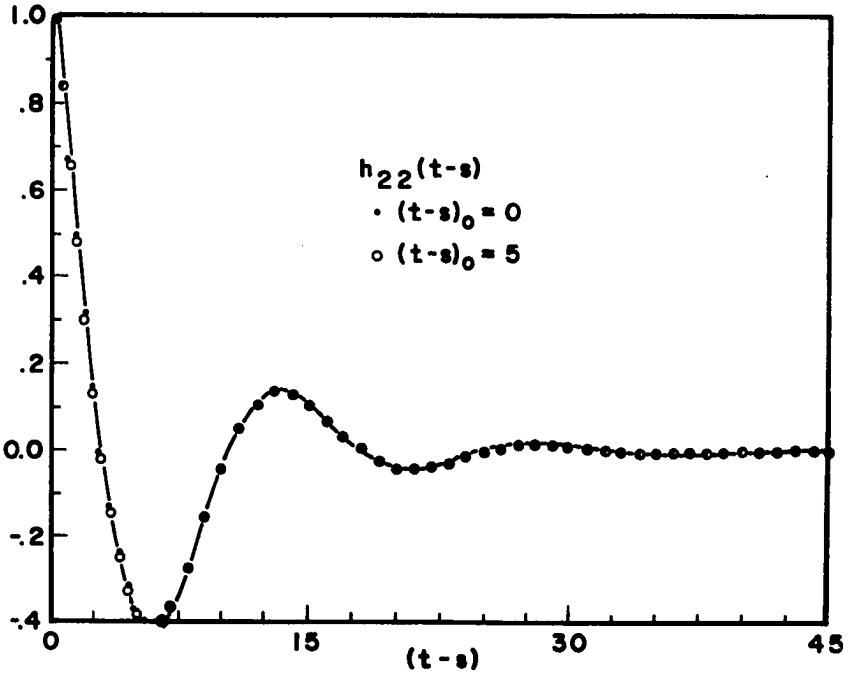


FIGURE IV-11
THE EFFECT OF 5% ATTENUATION ERROR ON THE MAJOR DIAGONAL ELEMENTS OF A SECOND ORDER TIME INVARIANT WEIGHTING FUNCTION

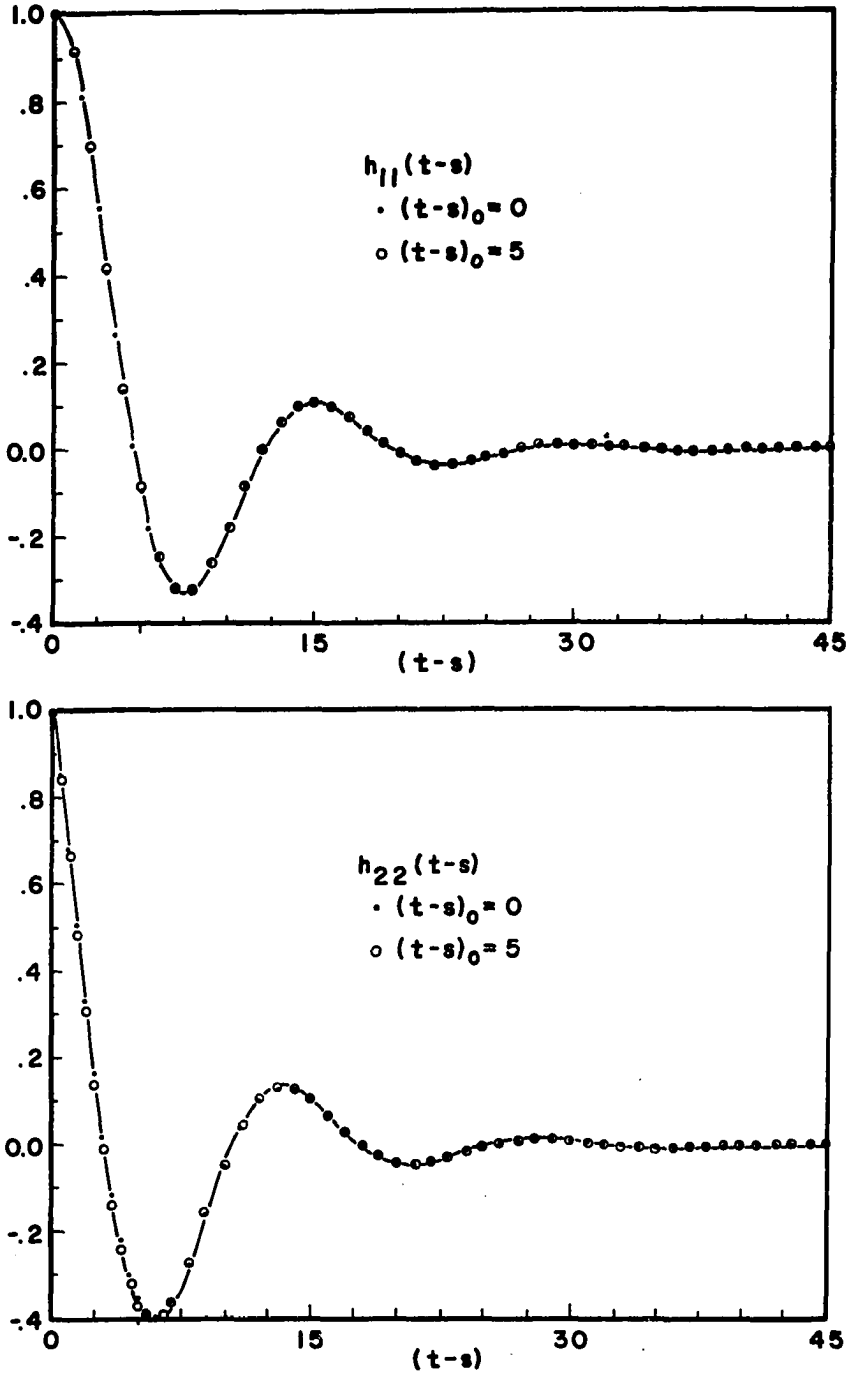


FIGURE IV-12
THE EFFECT OF 10% ATTENUATION ERROR ON THE MAJOR DIAGONAL ELEMENTS OF A SECOND ORDER TIME INVARIANT WEIGHTING FUNCTION

with the value of the parameter s equal to zero and five are compared to the "ideal" weighting function in each of these figures. The weighting function calculation automatically normalizes the major diagonal elements at time t equals s; hence in a normalized calculation only the minor diagonal elements are considered for normalization. The process of arbitrarily normalizing the minor diagonal elements nullifies the effect of proportional type errors, of which attenuation imprecisions are an example. Therefore only the major diagonal elements are presented in these figures.

Figure IV-10 reveals that the element which contained the erroneous data has a negligible effect on element $h_{11}(t-s)$ and a relatively small effect on element $h_{22}(t-s)$. Further, for values of the argument (t-s) greater than fifteen, the values of the weighting function calculated for s equals five are indistinguishable from those for s equals zero.

Figure IV-11 reveals a behavior pattern very similar to that of Figure IV-10, the only difference being in the magnitude of the effect. Figure IV-12 likewise shows nothing new in terms of the nature of the effect, with the exception of larger deviations from the "ideal" calculation at larger values of the argument (t-s). It would appear from the study presented here that any effect of small errors in the attenuation determined for the elements of the fundamental set of responses may be for most purposes neglected if normalized calculations are employed in the identification process.

The effect of gross errors in attenuation on the weighting function calculation has not been considered here for two reasons. The first of these is that the intent of this section was to study the effect of errors of the magnitude to be expected from considerations of amplifier drift and minor calibration adjustments rather than gross errors attributable to careless data reduction. The second reason is that it should be quite obvious that gross error in reporting attenuation factors would lead to catastrophic errors in the calculation. Further, these effects are very closely tied to the relative magnitudes and polarities of an individual set of autonomous responses; hence no general conclusions regarding their effect may be reached.

Figure IV-13 presents the results of a study of the effect of erroneous designation of the time at which the measured data actually become autonomous responses of the system. Figure IV-13 shows values of the weighting function calculated from data for values of the parameter s equal to minus ten, minus five, and zero. The data for "negative" time, required for these calculations, was obtained from forced system response data generated prior to the removal of the linearly independent input signals.

As mentioned previously, the computational phase of the identification process treats the initial data it receives as if it were autonomous response data, generating the unit matrix at time t equals s. Examination of Figure IV-13

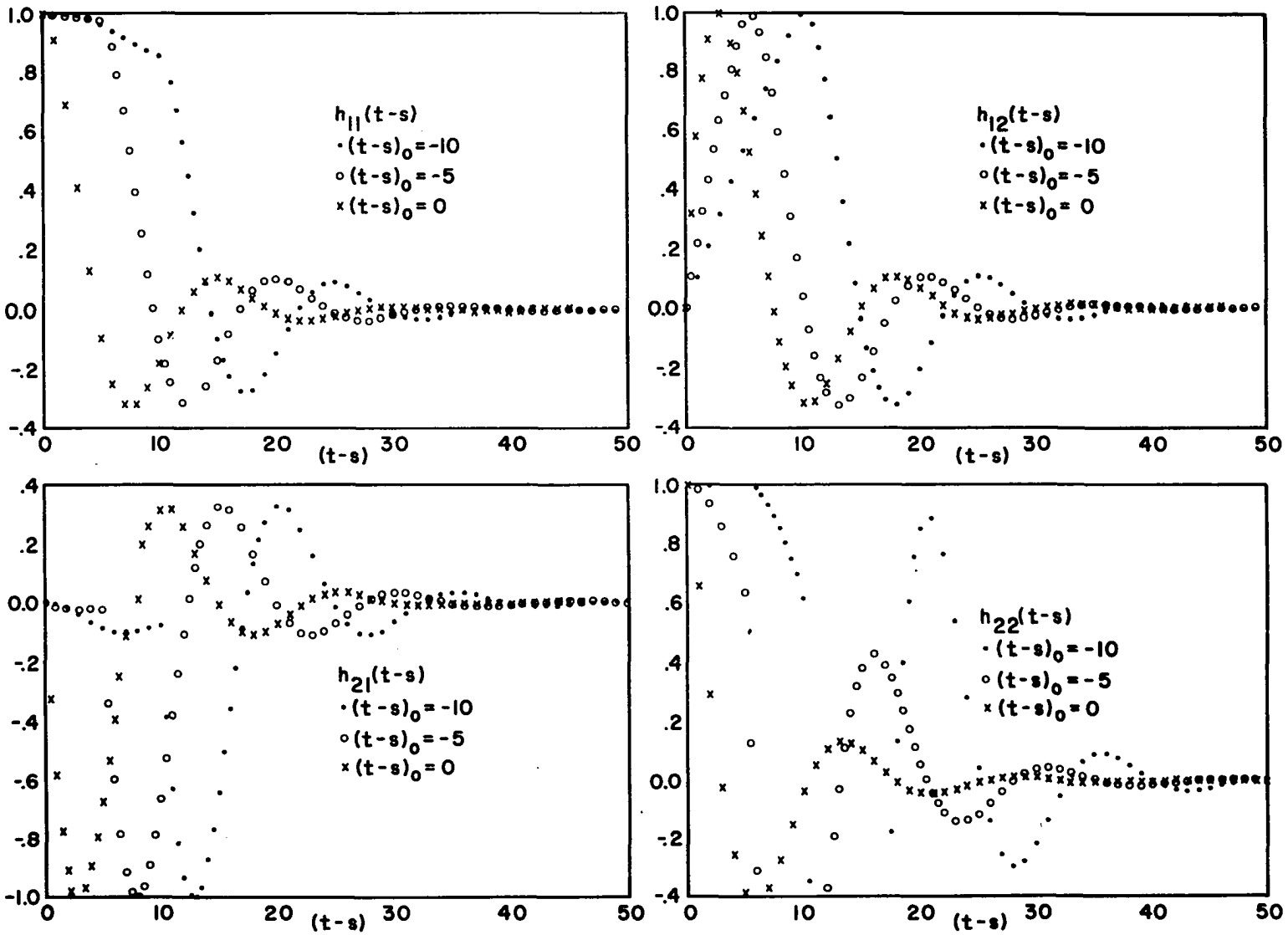


FIGURE IV-13
THE EFFECT OF ERRORS IN DESIGNATION OF THE INITIAL INSTANT OF AUTONOMOUS RESPONSE

indicates that in the range of the argument (t-s) greater than zero but less than that at which the response data is truly autonomous, the weighting function calculation proceeds smoothly but generates some strange function. Promptly at the value of the argument (t-s) at which the data becomes representative of the autonomous responses of the system, the calculations begin to generate a function which is very similar to, although distinctly different from, the true weighting function. The difference is that the "initial" value of the similar behavior is definitely incorrect. Therefore a simple shift of the function along the (t-s) axis cannot successfully correct the erroneous function.

Two important characteristics of the weighting function calculated from data containing this type of error are:

1. The calculated values of the major diagonal elements tend to be larger than those of the true function. Often these values exceed the theoretical maximum value of one.
2. The calculated values of the minor diagonal elements tend to be smaller than those of the true function. The derivative of a trace of these calculated points generally can be expected to undergo a violent discontinuity at the point at which the measured response data becomes truly autonomous.

In general, fault of this type which may occur in the measured response data may be spotted readily because of their

catastrophic effect on the calculated weighting function. The remedy for this type of fault is quite simple; one simply discards response data until the effect disappears.

There is one possibility of this type of error which represents a very serious and fundamentally intolerable situation. It is the case wherein there exists an unrecognized cause of non-autonomous system behavior such as an unrecognized and uncontrolled input to the system. The weighting function which would be calculated in this very plausible situation would certainly be erroneous and would bear little, if any, resemblance to the function being sought. The treatment of this situation would involve non-homogeneous, and possibly non-linear, weighting functions; therefore it has arbitrarily been placed beyond the scope of this investigation. The solution of this problem would represent a very important extension of this investigation.

Application of Maximum Error Estimates: At this point it is of interest to return to the consideration of Example 2 of Chapter III. Figure III-5 is a graphical representation of the second order time invariant weighting function calculated from the autonomous response data generated using an analog computer circuit as a physical system. Three of the four elements of the weighting function are very satisfactory estimates of the true function as shown by a comparison with Figure III-6. The fourth element, $h_{12}(t-s)$ is such poor quality that a sound estimate of the true function is not possible.

The nature of the physical system tested was such that the actual instant of autonomous response as well as the actual steady state and attenuation values could be precisely determined. Therefore the only meaningful criticism of the experiment falls on the precision of the measurements. Consideration of Figure III-4A and B reveals deviations from steady state of forty-two, forty-nine, twenty-eight, and twenty-seven millimeters respectively for the elements ϕ_{11}, ϕ_{21}, ϕ_{12}, and ϕ_{22}. Taking the maximum reading precision to be 0.25 millimeters, elements ϕ_{11} and ϕ_{21} possess a precision of approximately 0.5 per cent while elements ϕ_{12} and ϕ_{22} possess a precision of approximately one per cent.

The discussion presented earlier in this chapter with regard to the precision requirements on experimental data to guarantee high quality estimates of the true weighting function indicates that one might expect marginal results in this case. This limitation has been seen to be the situation. Therefore it is of interest to apply the maximum error bound estimates to the calculated values of the h_{12} element and to inquire as to whether or not its true value is included within these predicted bounds. Figure IV-14, 15, 16, and 17 show the results of the calculation of the $h_{12}(t-s)$ element for values of the parameter s equal to zero, five, ten and twenty, using a maximum measurement error estimate of plus or minus 0.125 millimeters.

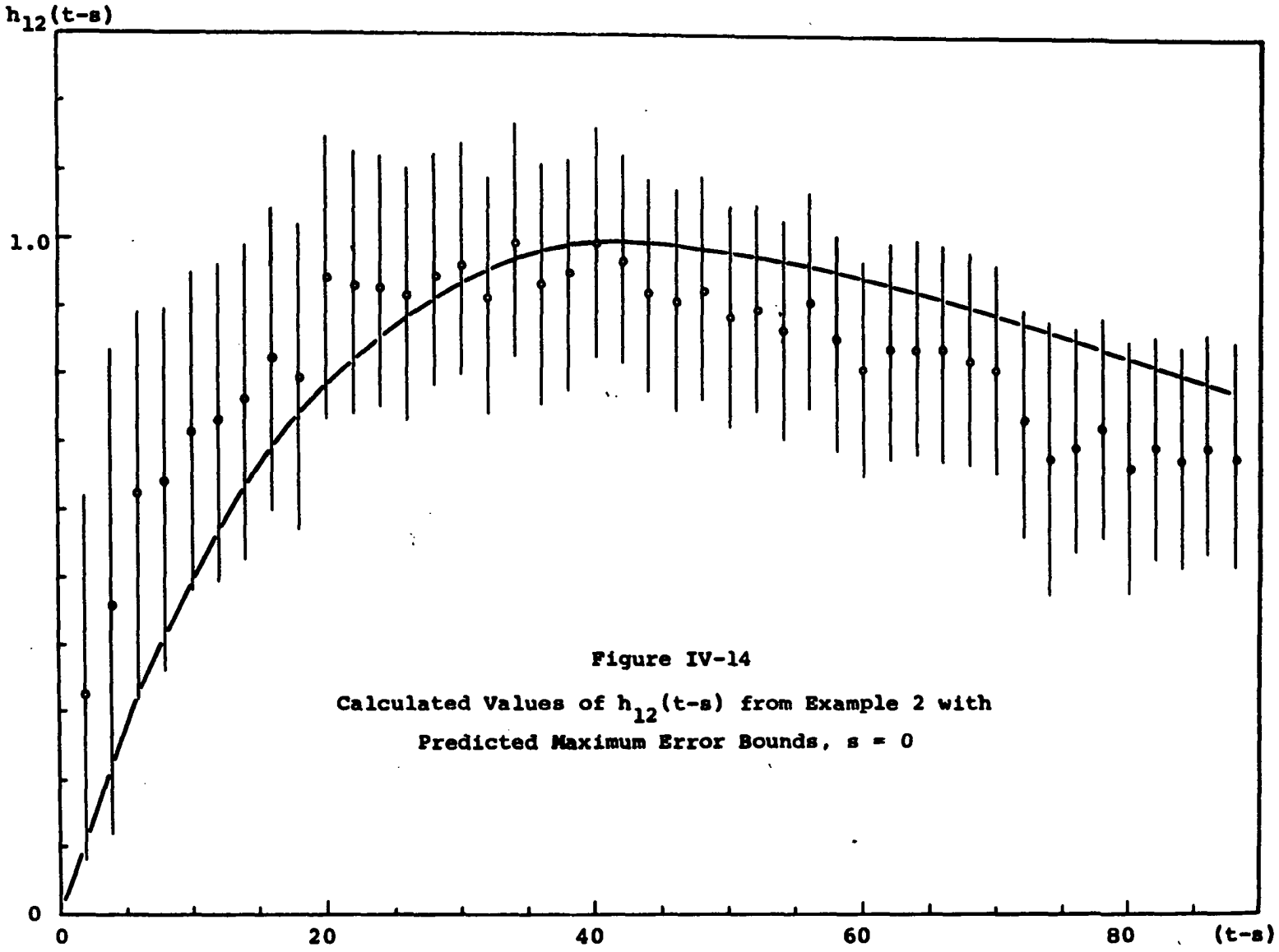


Figure IV-14

Calculated Values of $h_{12}(t-s)$ from Example 2 with Predicted Maximum Error Bounds, s = 0

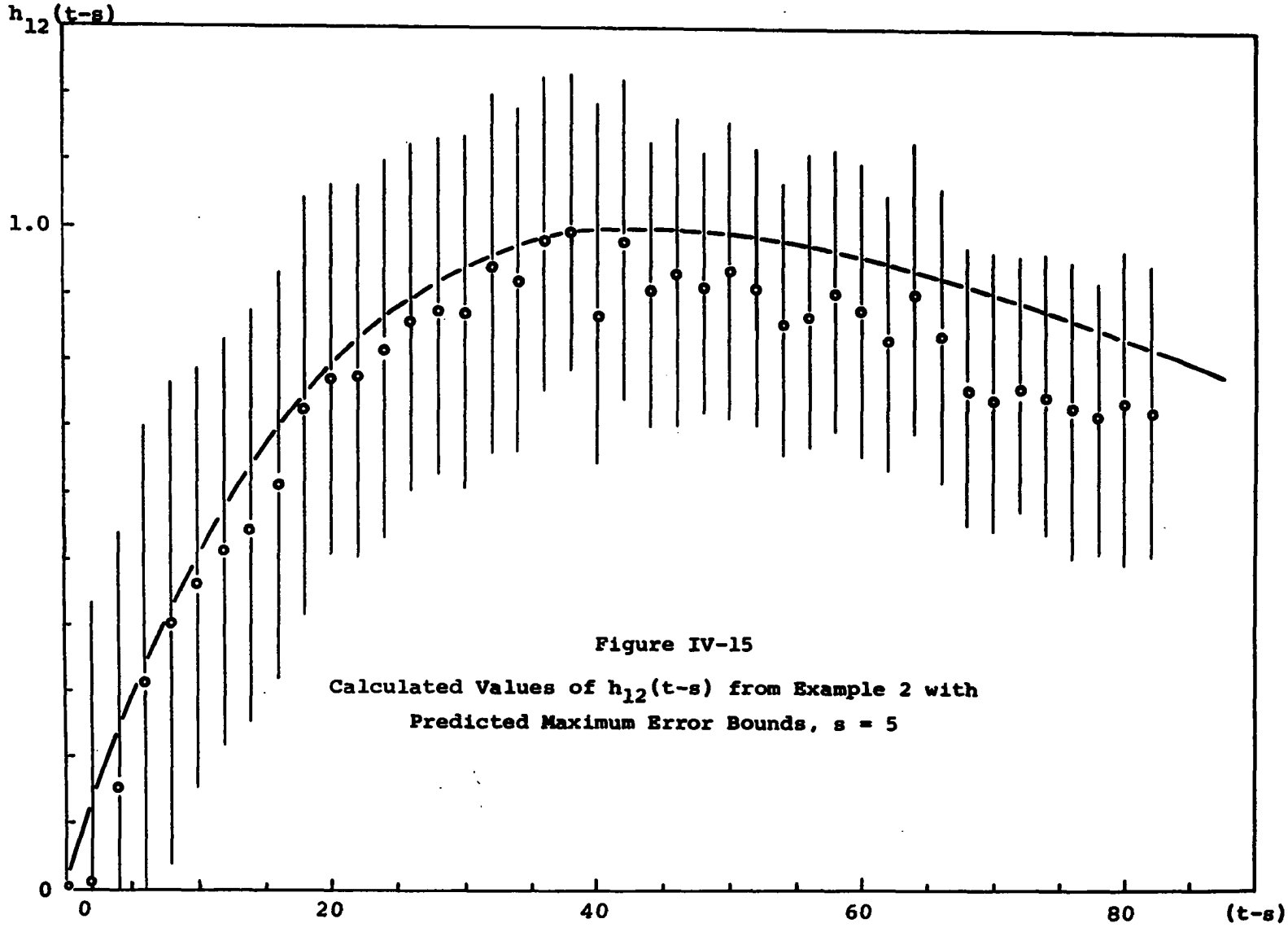


Figure IV-15

Calculated Values of $h_{12}(t-s)$ from Example 2 with Predicted Maximum Error Bounds, s = 5

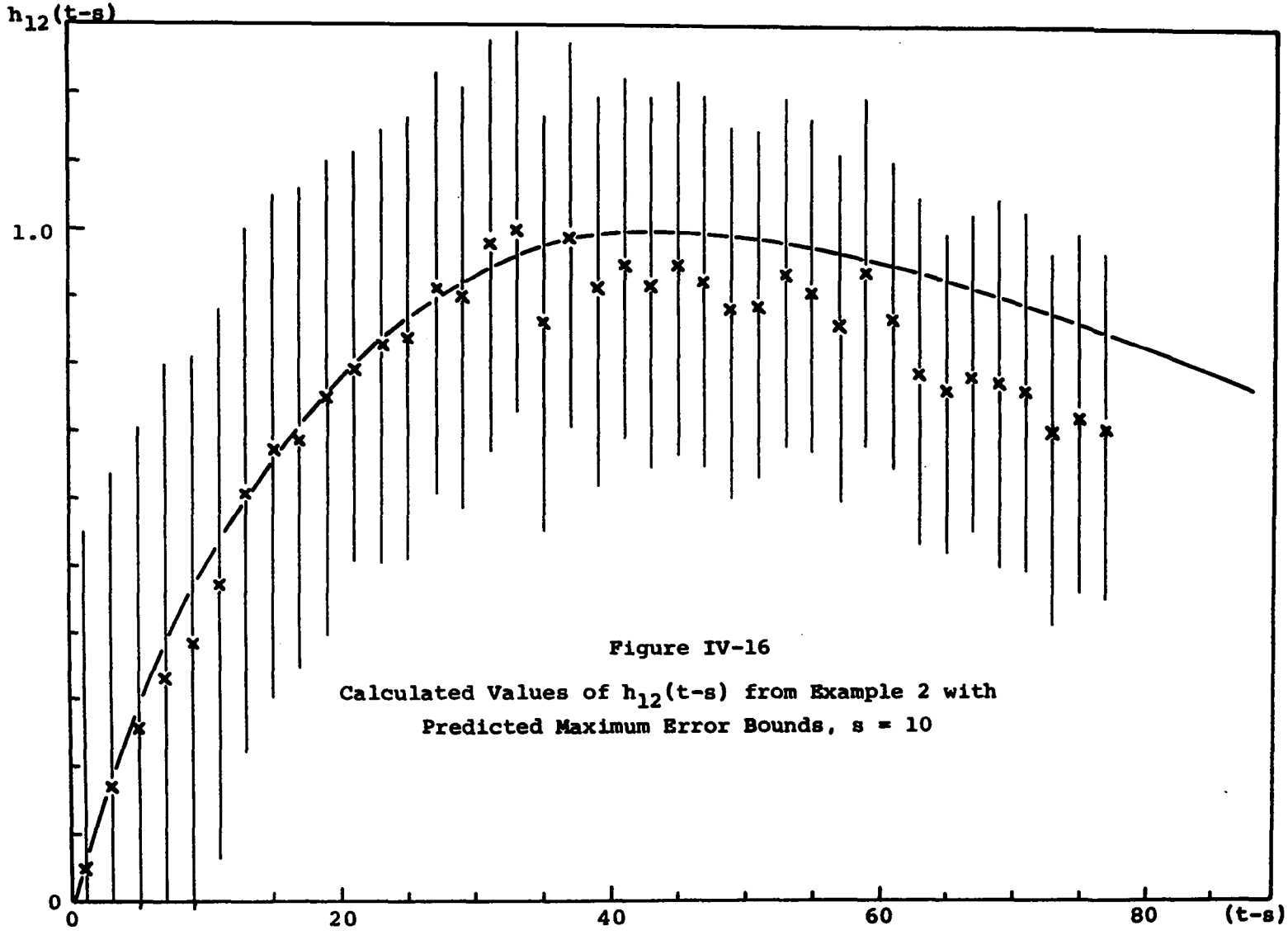


Figure IV-16

Calculated Values of $h_{12}(t-s)$ from Example 2 with Predicted Maximum Error Bounds, s = 10

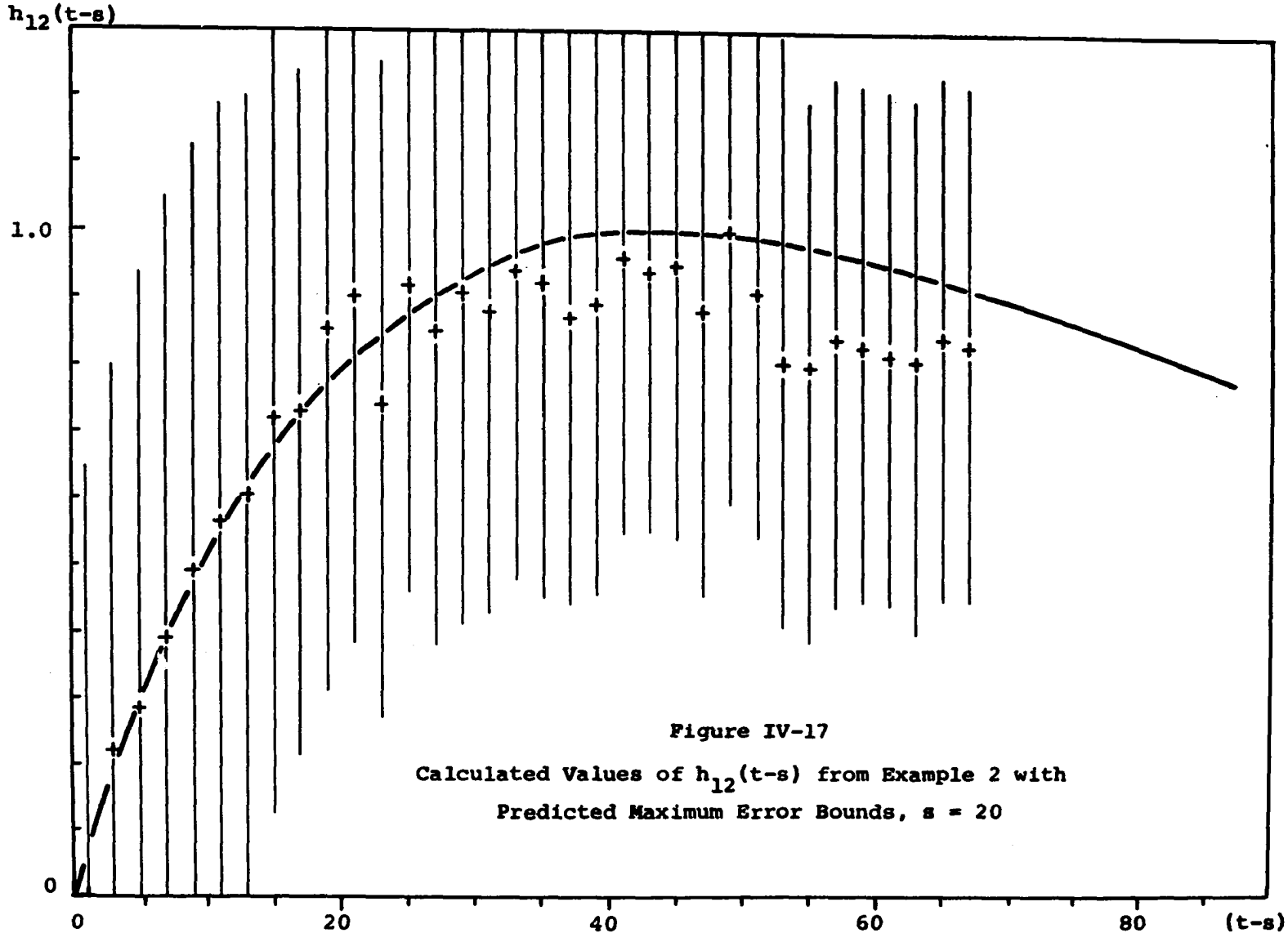


Figure IV-17

Calculated Values of $h_{12}(t-s)$ from Example 2 with Predicted Maximum Error Bounds, s = 20

Figure IV-14 reveals that the true weighting function is included in the range of values defined by the calculated values with their associated maximum error bounds. The extremes of the band of maximum errors follow the saw-tooth pattern suggested earlier for the effect of random errors in the response data. Further, the fact that the true function lies very close to the limit of the maximum error band at several values of the argument (t-s) implies that the error estimates are of a realistic magnitude.

Figure IV-15 again shows that the true weighting function is included in the range of values defined by the calculated values with their associated maximum error bounds. It should be noted that these maximum error bounds are larger than those of Figure IV-14. This enlargement is a result of the decreased reliability which is predicted for the matrix inversion operation at larger values of the parameter s. Because of the fact that the true weighting function is more easily accommodated within the maximum error bounds, there exists the implication that the predicted worsening of the quality of the inverse matrix may not actually have been experienced.

Figure IV-16 follows the pattern established in the Figures IV-14 and 15. The difference in this figure is one of magnitude of the effect rather than its nature. The predicted maximum errors have become large enough so that one might question the advisability of using the calculated values as an estimate of the true weighting function.

Figure IV-17 also shows a continuation of the pattern established in Figures IV-14 and 15. The magnitude of the predicted maximum calculational errors has become so large that virtually any reasonable weighting function could conceivably be represented by the calculated values.

It should be noted that the error predictions are for maximum and not probable error in the calculation. It is apparent that the random errors made in the measurement of the system's autonomous responses have not had the deleterious effect on the quality of the calculated weighting function that they might have had. However, since this type of information generally can not be obtained, the quality estimates as expressed in this chapter are very useful and appear to be quite valid.

CHAPTER V

ADDITIONAL APPLICATIONS OF THE GENERALIZED PULSE TESTING TECHNIQUE

This chapter is devoted to the presentation of studies of two additional applications of the generalized pulse testing technique for the determination of the dynamic characteristics of linear systems.

The first study is concerned with the application of the technique to a backmix chemical reactor in an effort to determine its dynamic heat transfer characteristics when operated as a time invariant system. This study is of great interest as it represents the initial attempt to apply the technique to an actual physical process (excluding computers) and as such introduces the questions of model applicability and data quality. The study is also of interest because through it one is able to assess the "cost", in terms of both experimental and calculational effort, of employing the technique to a system of the type so successfully modeled by frequency response, statistical, and impulse response techniques. Of course, once this "cost" has been assessed, one is in a position to compare it to that of the other techniques.

Although application of the generalized pulse testing technique to the backmix reactor, operated as a time-varying system, was planned at the outset of the investigation, practical considerations of time and equipment forced cancellation of the study. Therefore the second study presented is that of a hypothetical second order time-varying system, programmed on a digital computer. There are a number of reasons for the inclusion of this study in addition to those presented in Chapter III.

Aside from the consideration of completeness, this study has been included to demonstrate the technique further, for only in its application to time-varying systems does the technique realize its full potential.

Inclusion of this study does provide the mechanism by which at least a partial assessment of the "cost" of employing the technique of time-varying systems may be made. Since the other applicable techniques (statistical and "impulse" response) have been discussed in the literature, this study is of value.

The final reason for inclusion of this study in the Chapter concerned with practical considerations of applying the technique is that through it the very practical benefits of redundant testing of time-varying systems may be introduced.

Time Invariant System

The familiar backmix chemical reactor was chosen as the system to be used in the investigation of the experimental difficulties associated with the application of the generalized

pulse testing technique to an actual physical process. As no chemical reaction was carried out in the experiments, the dynamic behavior studied was exclusively that of heat transfer from the contents of the reactor, through its wall, to the coolant flowing in the reactor jacket. This seemingly mundane choice of physical process was made for several reasons, the most important being that previous investigations (15, 16, 32) have shown that its heat transfer dynamics may be successfully characterized in terms of a linear second order mathematical model.

Two of the cited investigations obtained experimental information on the process' response by frequency response techniques (15, 32); the third employed a statistical determination technique (16). The success of these investigations implies either that the actual nonlinearities of the process had a negligible effect on its response, or that the data acquisition techniques and subsequent operations on that data are sufficiently insensitive to small variations so as to permit description of the process with a relatively simple mathematical model. In either case the important point is that these techniques have been successful.

The material presented in Chapters III and IV suggests that the calculations involved in the determination of the process weighting functions by the generalized pulse testing technique may be very sensitive to small deviations from linear behavior (considered previously as data error). Therefore it becomes important to determine whether or not actual

process data can be used successfully in the matrix calculations required for the determination of the weighting functions. Clearly, success in this application is prerequisite to further consideration of the technique as a practical tool for the determination of process models.

Other considerations entering the choice of the backmix reactor as the physical system to be studied were primarily matters of convenience. Notably, temperatures and liquid flow rates may be conveniently, if not easily, measured and controlled. Since no chemical reaction was carried out in the reactor, the problems of material acquisition and disposal as well as those of product analysis were nonexistent; in fact closed process fluid and coolant cycles were used.

The details of the physical system are probably best presented through comparison with the hypothetical reactor considered in the derivation of a linear second order mathematical model of the actual reactor dynamics. The procedure will be to derive the mathematical model, then discuss the differences between the hypothetical and actual reactors, thus pointing up the liberties taken with reality in order to generate a model of reasonable complexity.

Linear Process Model: Figure V-1 is a schematic representation of the hypothetical reactor. Process fluid enters the reactor at temperature T_{in} and flow rate F. As it is tacitly assumed that the contents of the reactor are perfectly mixed, the outlet temperature is identical to the bulk temperature T_b. The contents of the reactor give up heat to

the reactor wall, which is assumed to present an appreciable resistance to heat transfer. The wall is assumed to operate at a temperature T_w which is not a function of position. The reactor wall, in turn, gives up heat to the coolant flowing in the jacket. In order to limit the model of the reactor at second order it is necessary to assume that the coolant operates at a temperature T_c and possesses an infinite capacity to absorb heat.

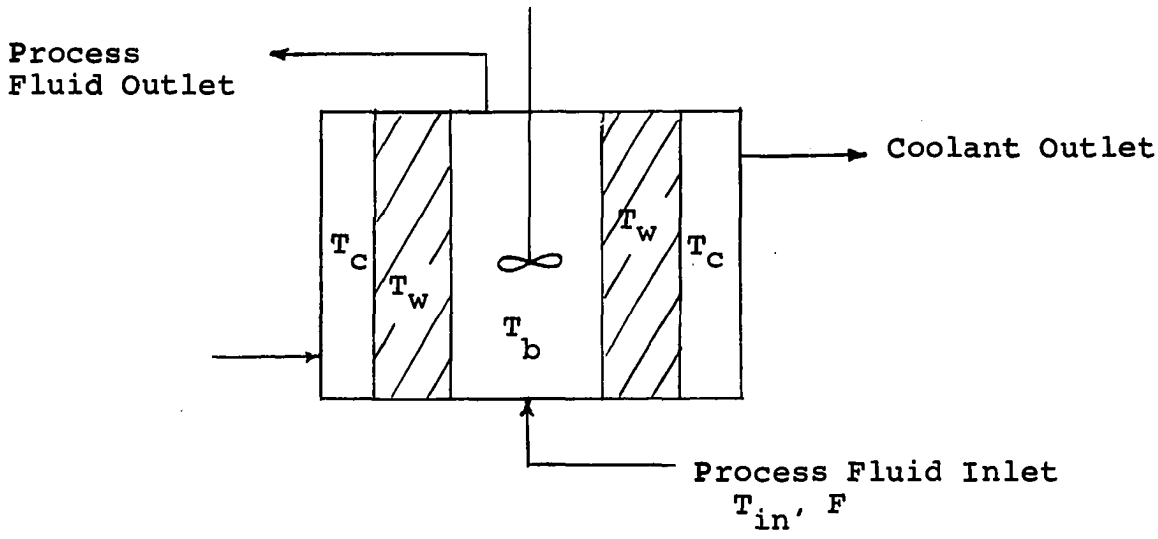


Figure V-1. Schematic Diagram of the Hypothetical Reactor

An energy balance on the contents of the reactor gives rise to the following differential equation:

$$\dot{T}_b = -\left[(FC_{pb} + h_iA_i)/V_b\rho_bC_{pb}\right]T_b + (h_iA_i/V_b\rho_bC_{pb})T_w + (FC_{pb}/V_b\rho_bC_{pb})T_{in}$$

Similarly an energy balance on the reactor wall gives rise to the differential equation:

$$\dot{T}_w = -[(h_oA_o + h_iA_i)/V_w\rho_wC_{pw}]T_w + (h_iA_i/V_w\rho_wC_{pw})T_b + (h_oA_o/V_w\rho_wC_{pw})T_c$$

where the notation is as follows:

T = Temperature

F = Flow rate (mass)

V = Volume

A = Area for heat transfer

ρ = Density

Cp = Heat Capacity

h = Convective heat transfer coefficient

with the subscript notation:

in = reactor inlet

b = bulk reactor contents (process fluid)

w = reactor wall

c = coolant

i = inside wall

o = outside wall

and $\dot{T}$ indicates the derivative of temperature with respect to time.

Assuming that the process fluid flow rate F is to be maintained at some constant value, it is apparent that there are two time invariant energy storage elements: the contents of the reactor and the reactor wall. The state variables are the temperatures associated with these energy storage elements. Further, there are two possible input variables: the inlet and coolant temperatures. The assumption that the densities

and heat capacities of the process fluid and reactor wall are negligible functions of temperature completes the derivation of the linear total variable model.

Extraction of the steady state portion of the total variables from these equations ($T(t) = T_{ss}(t) + T^*(t)$) leads to the vector differential equation which describes the transient behavior of the system's state

$$\dot{Y}(t) = AY(t) + X(t)$$

where the state vectors and coefficient matrix have the definitions given below:

$$\dot{Y}(t) = \begin{bmatrix} T_b^*(t) \\ T_w^*(t) \end{bmatrix}$$

$$Y(t) = \begin{bmatrix} T_b^*(t) \\ T_w^*(t) \end{bmatrix}$$

$$X(t) = \begin{bmatrix} (FC_{pb}/V_b\rho_bC_{pb})T_{in}^*(t) \\ (h_oA_o/V_w\rho_wC_{pw})T_c^*(t) \end{bmatrix}$$

$$A = \begin{bmatrix} -((FC_{pb}+h_iA_i)/V_b\rho_bC_{pb}) & (h_iA_i/V_b\rho_bC_{pb}) \\ (h_iA_i/V_w\rho_wC_{pw}) & -((h_oA_o+h_iA_i)/V_w\rho_wC_{pw}) \end{bmatrix}$$

Physical System: The reactor used as a source of dynamic heat transfer data in this investigation is visible in the upper center of Figure V-2. Also visible in the photograph are portions of the signal conditioning system (upper left), the reactor coolant cycle (lower left and center), and the flow splitter (right center).

Figure V-3 is a sketch of the reactor used as the source of dynamic heat transfer data for this investigation. The sketch has not been drawn to scale nor has any detail not germane to the discussion to follow been included. The reactor jacket, support and "head" secions, as well as the agitator are parts of the reactor used by Stewart (32). The original reaction vessel was replaced by a thick walled tube specially cast of printer's type metal.

The derivation of the mathematical model assumed that the wall temperature was not a function of position. It was also assumed that the wall presented an appreciable resistance to heat transfer. Clearly these two assumptions are not consistent, for if the resistance of the wall is appreciable, then its temperature will be given by the solution to the partial differential equation:

$$\dot{T}_w(t) = D\nabla^2 T_w(t)$$

subject to appropriate boundary conditions. However, if the thermal diffusivity of the wall material is high, then the transient portion of the wall response (exclusive of that due

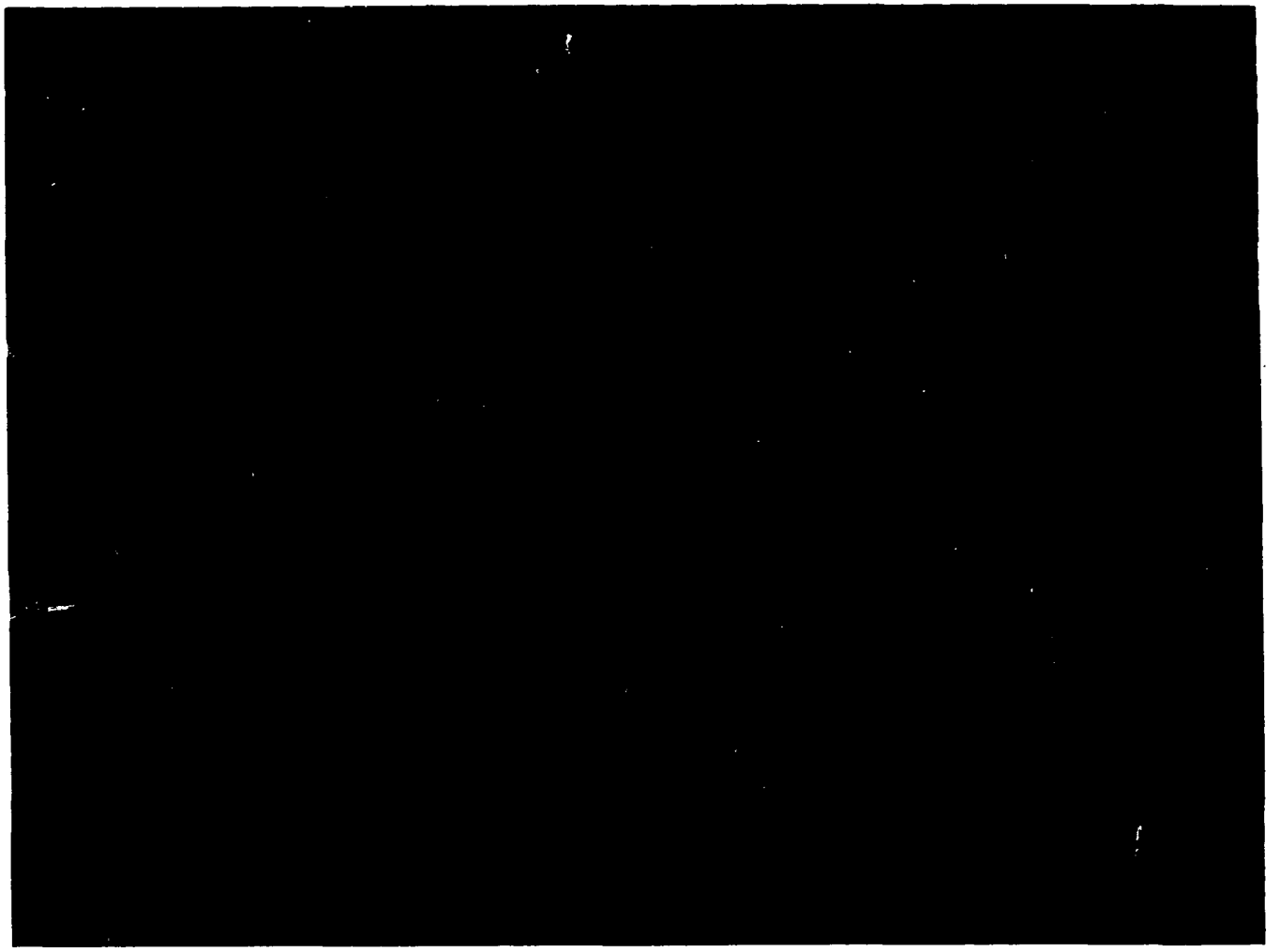


Figure V-2

View of the Continuous, Backmix Reactor

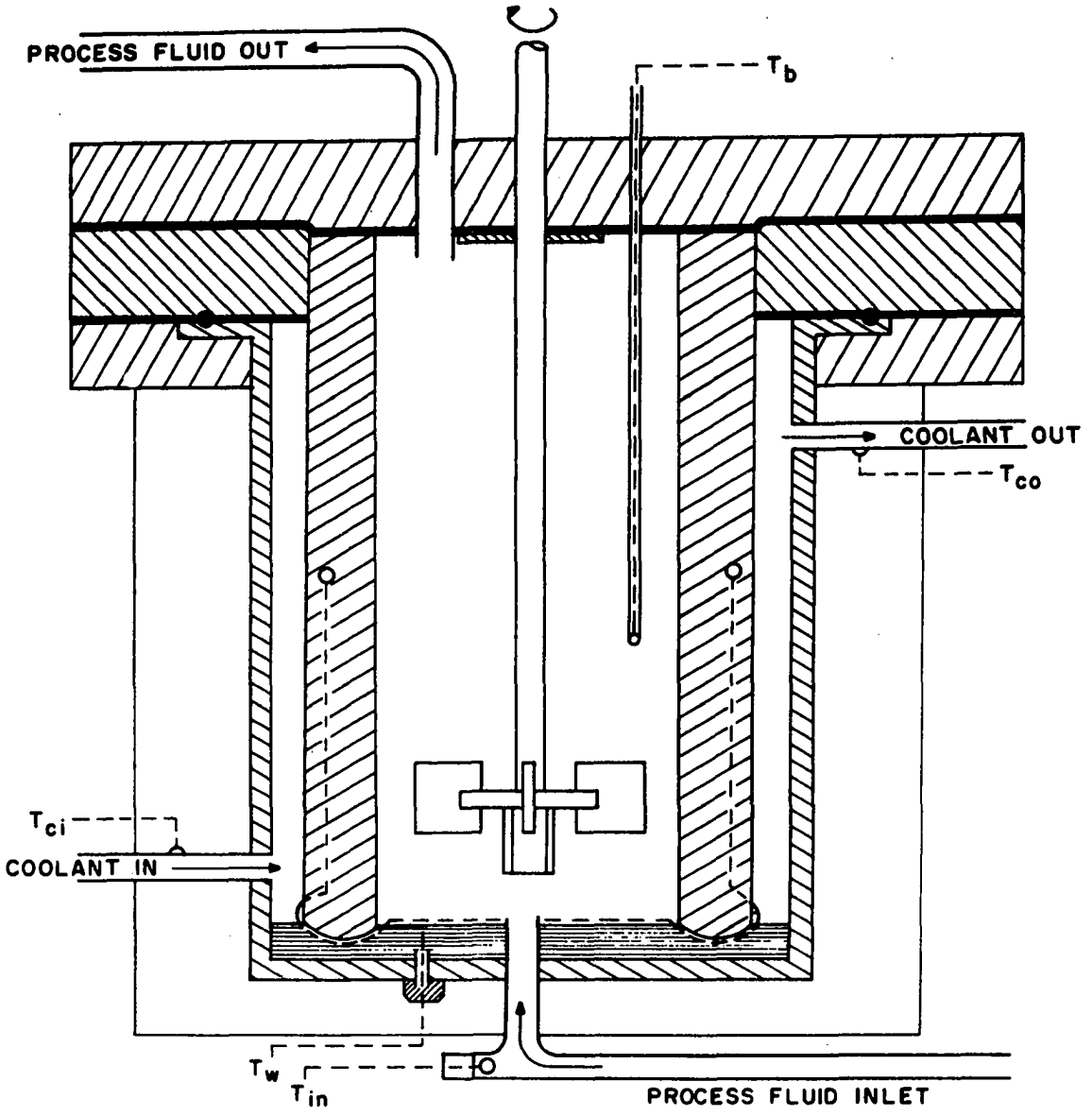


FIGURE V-3

SKETCH OF THE BACKMIX REACTOR

to forcing by T_b or T_c) would be very short-lived and the temperature profile would approach a logarithmic function of the radius and at worst a monotonic function of the reactor's axial dimension. Thus measurement of the wall temperature at the log mean radius and the median axial position can reasonably be expected to represent the "average" wall temperature.

On the basis of this consideration, a thick walled tube was cast of printer's type metal and installed in the reactor. This material was chosen over other high thermal diffusivity materials because of its low melting point. The low melting point material was desirable for two reasons. The first, and most obvious, was the simplicity of the casting technique which could be employed. The second reason was that, at the time of casting, four thermocouples could be positioned at approximately the chosen position with reasonable assurance that their integrity would be maintained. The average of these four measurements was taken to be the wall temperature, T_w.

The installation of the wall in the reactor involved press fitting the tube into the lower section of the reactor "head", effectively making it part of the reactor wall. Since this additional "wall volume" does not operate under the same conditions of heat flux and temperature as the tube section, one would expect deviations from the "average" wall temperature to be largest in this region. In fact these deviations might be large enough to affect materially the measured temperature.

In an effort to minimize this expected deviation, a low thermal conductivity gasket material was used to prevent intimate contact of the lower "head" section with the upper "head" and jacket support sections. The bottom of the tube was sealed from the jacket chamber and thermally insulated from the jacket wall with the same material.

The second state variable, namely the process fluid temperature, was measured at a point approximately half the distance between the reactor inlet and outlet. The assumption of perfect mixing seems to be justified by the results obtained by Stewart (32) with essentially the same reactor system.

Recall that in the derivation of the mathematical model the constraint that the model be of second order was introduced by the assumption that the coolant possesses an infinite capacity to absorb heat. Not only was this assumption somewhat unrealistic on the grounds that such a coolant is nonexistent but it also implied that the coolant operates at a temperature T_c which is physically not measurable. This deviation from reality was minimized in the experimental program by the use of very high coolant flow rates. The temperature, T_c, was taken to be the average of the temperatures measured at the inlet and outlet of the reactor jacket. These temperatures differed by less than three degrees Fahrenheit for the range of operating conditions used.

The process fluid inlet temperature was measured immediately prior to the fluid's entry into the reactor. The

physical arrangement of the reactor inlet was such that the conduit connections were in intimate contact with the jacket wall, thus providing a small heat transfer area which was not accounted for in the model. The effect of this arrangement on the model, if any, would be the indication of a falsely large value for both the reactor volume and inside convection heat transfer coefficient.

One final criticism of the model might be the fact that no accounting has been made of heat losses. Even though the actual reactor is partially insulated, it would be unreasonable to assume that the losses are negligible. The reason for neglect of the heat losses in the model is that it would be impossible to account for them correctly--should they appear in the wall equation, the process fluid equation, or both? If the answer is both, the question is how the total heat loss (which is measurable) should be split between the two equations. Notice that while the neglect of the heat loss is indeed a poor assumption, it is <u>not</u> unreasonable to assume that the heat losses for the total variable model would be the same as the heat losses at steady state. Therefore since the transient variable model is obtained by subtracting the steady state expressions from those of the total variable model, its heat losses are probably negligible.

<u>Support Systems</u>: Thus far the discussion of the physical system has dealt strictly with the reactor itself, ignoring the necessary support systems. This approach has been made intentionally in an effort to direct the reader's attention

to the experimental investigation rather than to the experimental apparatus.

The discussion of the support systems is conveniently separable into four sections and hence is presented in Appendices D, E, F, and G. These Appendices should be consulted for details concerned with the support systems.

Appendix D contains a description of the Data Acquisition and Signal Conditioning System. This system consists of the electronic circuitry required to amplify, condition, and record voltages which are proportional to the temperatures and flow rate measured in the experiments.

Appendix E gives a description of the closed Process Fluid Cycle, excluding the path through the reactor itself. This cycle is made up of the constant temperature feed tanks, pumps, flow controllers, flow splitter, surge and storage tanks. The hydraulic function generator forms part of this cycle but is described separately.

Appendix F describes the Hydraulic Function Generation System. This system consists of the electronic circuitry for the electronic function generation, transducing equipment for conversion of the voltage function to a position function, and the hydraulic cylinder which transduces this position function to a temperature function at constant process fluid flow.

Appendix G contains a description of the closed Reactor Coolant Cycle. Included here are the Freon refrigeration system, coolant feed tank, pump and temperature control circuitry.

The reactor and the associated support systems are shown pictorially in Figures V-2, V-4, V-5, V-6, and V-7. Figure V-4 shows an overall view of the apparatus including portions of the Data Acquisition and Signal Conditioning System (left center), the Reactor Coolant Cycle (lower center), the Process Fluid Cycle (right center), and the laboratory instrument panel (right center).

Figure V-5 is a view of the laboratory instrument panel (left) and the analog computer and Sanborn recorder (right) which formed a portion of the Data Acquisition and Signal Conditioning System.

Figure V-6 is a view of the rear of the laboratory instrument panel showing transducers, controllers, and switching circuitry used in the various support systems.

Figure V-7 shows that portion of the experimental apparatus housed in the analog computer laboratory (adjacent to the process laboratory). The signal voltages proportional to the process variables are transmitted from that portion of the Data Acquisition and Signal Conditioning System located in the process laboratory to that portion of the system programmed on the computer, further conditioned, and recorded on the Sanborn recorder which is located at the left of the picture.

Experimental Procedure: With the implicit assumption that the model actually represents the dynamic behavior of the physical system, consider the A matrix of the model:

Figure V-4

Overall View of the Experimental Apparatus

Figure V-5

View of the Process Laboratory Instrument Panel

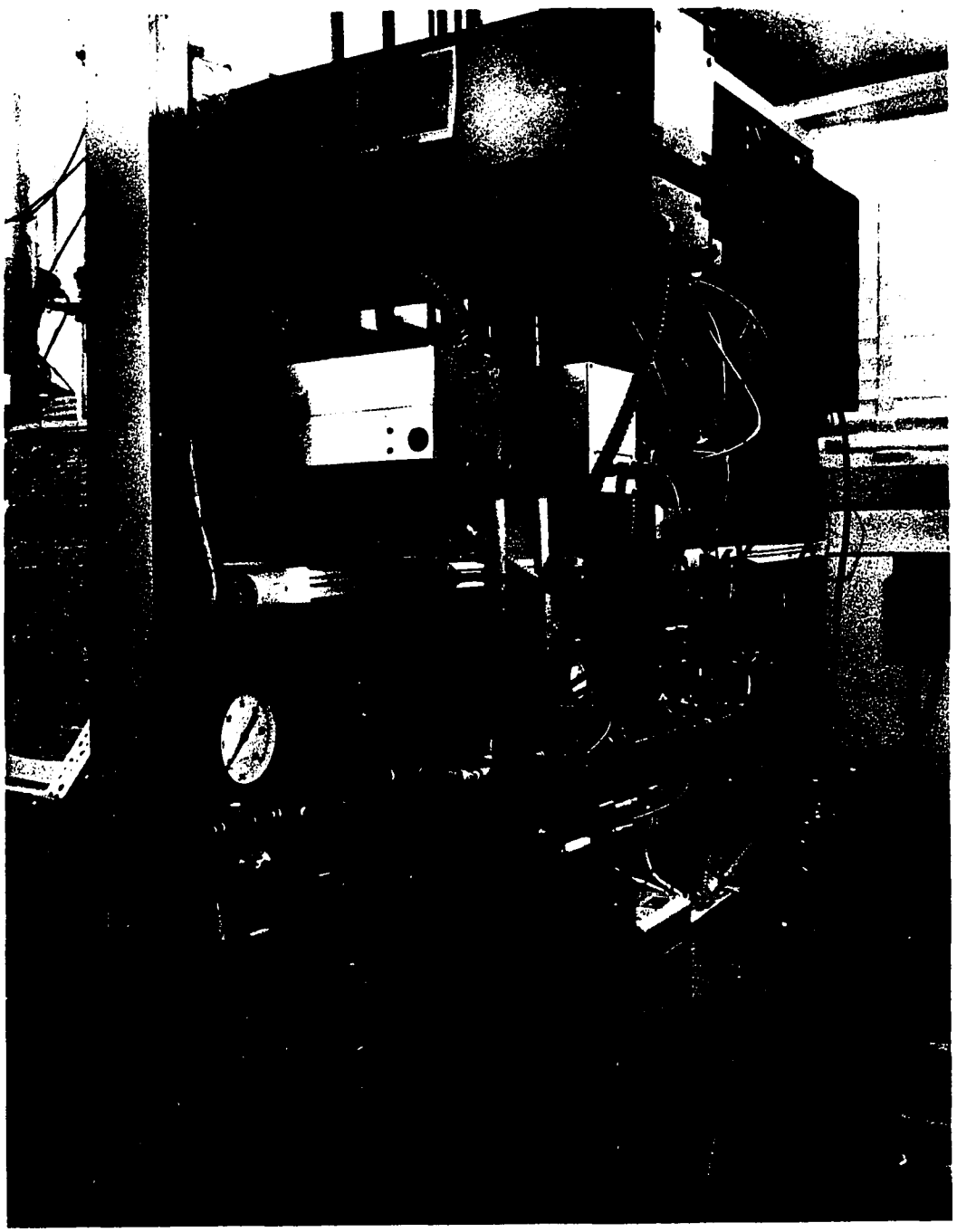


Figure V-6

Rear View of the Instrument Panel

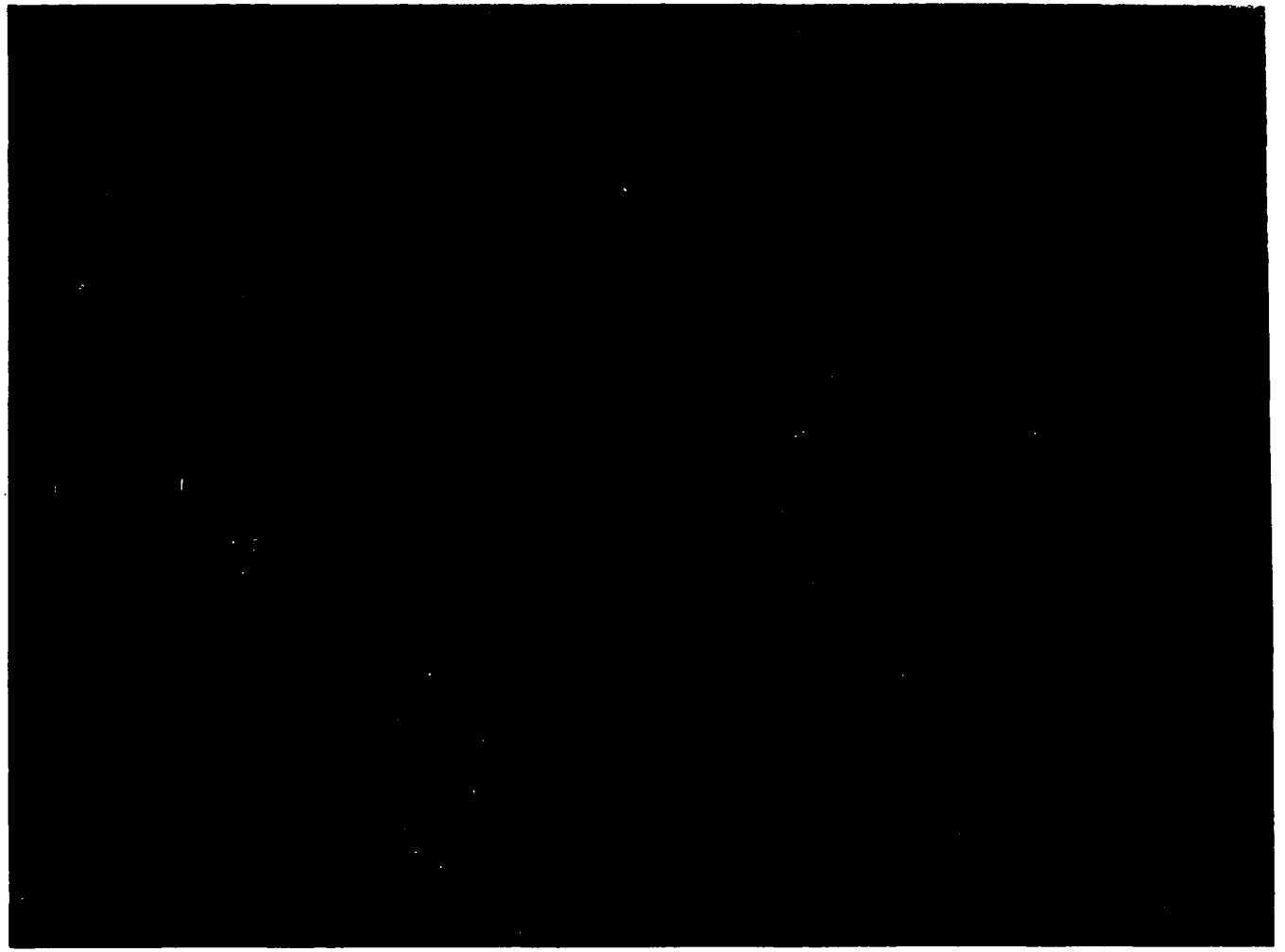


Figure V-7

Donner Model 3100 D Analog Computer and Auxiliary Equipment

$$A = \begin{vmatrix} -[(FC_{pb}+h_iA_i)/V_b\rho_bC_{pb}] & (h_iA_i/V_b\rho_bC_{pb}) \\ (h_iA_i/V_w\rho_wC_{pw}) & -[(h_oA_o+h_iA_i)/V_w\rho_wC_{pw}] \end{vmatrix}$$

Provided that the process fluid flow rate is held constant, the elements of the matrix are seen to be time invariant within the model assumptions. Therefore one requirement on the experimental procedure is that the flow rate must be held constant.

Consider the system forcing vector X(t):

$$X(t) = \begin{vmatrix} (FC_{pb}/V_b\rho_bC_{pb})T^*_{in}(t) \\ (h_oA_o/V_w\rho_wC_{pw})T^*_c(t) \end{vmatrix}$$

As it has already been pointed out, the flow rate, F, is to be held constant. The requirements of the generalized pulse testing theory are that during the period of data acquisition (homogeneous response) both $T^*_{in}(t)$ and $T^*_c(t)$ must be zero. In terms of the real temperatures, the requirement is that during the period of data acquisition, the reactor inlet and coolant temperatures must be at their steady state values.

The state variables, $T^*_b(t)$ and $T^*_w(t)$, are the transient portions of the measurable temperatures $T_b(t)$ and $T_w(t)$. Therefore in the measurement of these temperatures the steady state values were suppressed by using the zero suppression capability of the signal conditioning equipment.

In summary, the data taken in each experiment consisted of:

1. The zero suppressed measurement of $T_w(t)$.
2. The zero suppressed measurement of $T_b(t)$.
3. The zero suppressed measurement of $T_{in}(t)$.
4. The zero suppressed measurement of $(t_{ci}(t) + T_{co}(t))/2$.
5. The measurement of the flow rate, $F(t)$.

Voltages, proportional to these five process signals together with an indication of the arbitrary t_o and of chronological time were recorded on a six channel Sanborn oscillographic strip chart recorder. Figure V-8 is a photograph of a portion of the recorded process data from a typical experiment.

The theoretical aspects of the testing technique discussed in Chapter II suggest that the procedure followed in the acquisition of homogeneous response data from the actual reactor involves driving the system away from its equilibrium state (steady state for a particular T_{in}, T_c, and F) and by forcing either or both T_{in} and T_c prior to time t_o. At the arbitrary time t_o, the forced variable is to be returned to its equilibrium value; the subsequent gyrations of the state variables [$T_w(t)$ and $T_b(t)$] as the system returns to its equilibrium state constitute the dynamic information acquired in the experiment.

Reflection, on the constraints of the model, suggests that T_c <u>not</u> be employed as a forcing variable. However, due to the nature of the physical system being studied, one could employ the process fluid flow rate, F, to force the system into some, possibly very desirable, non-equilibrium state

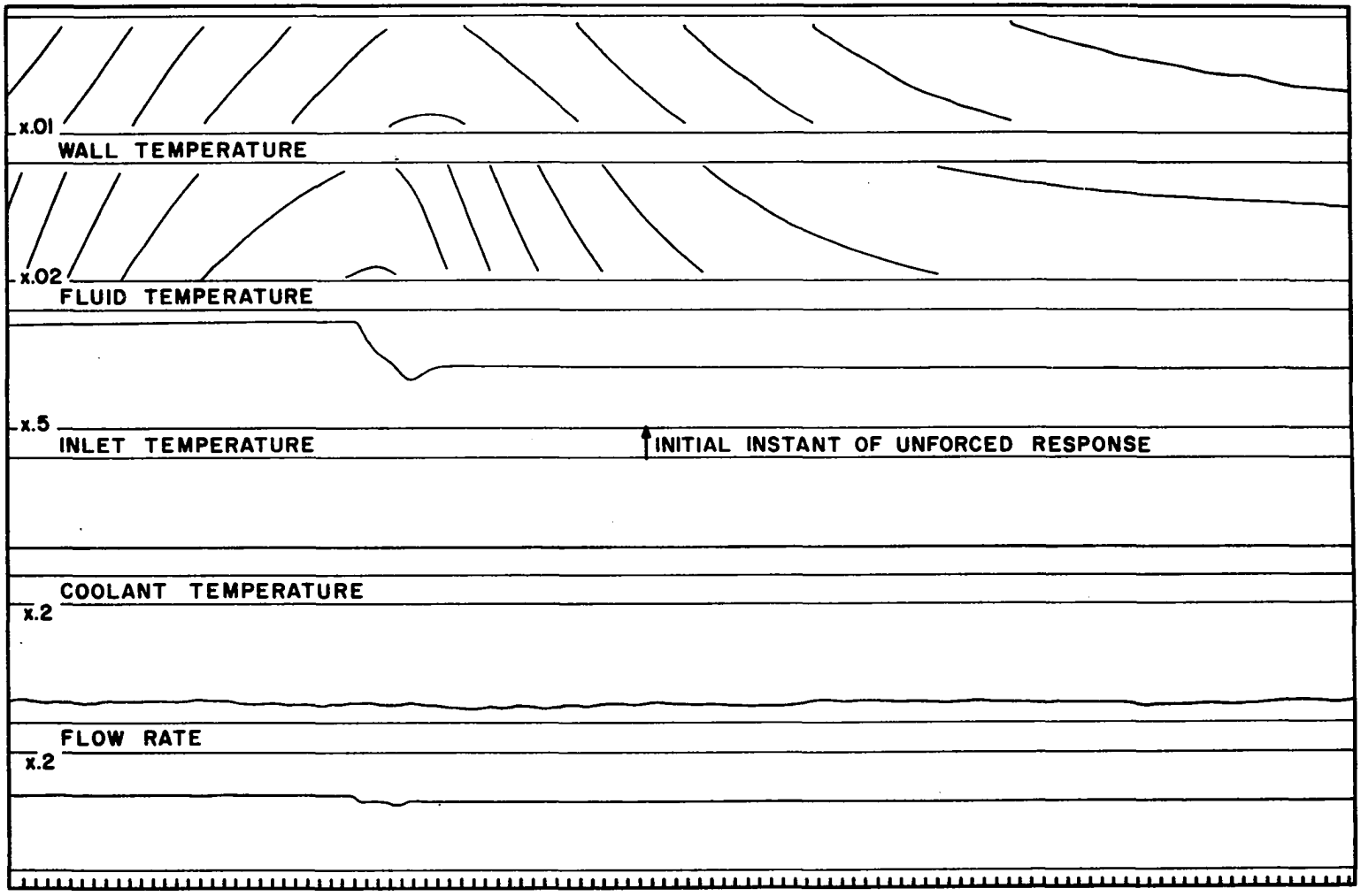


FIGURE V-8
A PORTION OF THE RECORDED DATA FROM A TYPICAL EXPERIMENT

provided that F assumed its equilibrium value at time t_o. Unfortunately, experimentation showed that the hysteresis effect of the valves used to control the flow rate precluded its use as a forcing variable. Therefore all of the experiments reported herein were conducted using the inlet process fluid temperature as the forcing variable.

This temperature forcing, at constant flow rate, was accomplished through the use of a hydraulic function generator, which was designed and constructed for this investigation. The details of the function generator appear in Appendix F.

The procedure followed in making a dynamic data acquisition experiment is summarized below:

1. Establish equilibrium conditions in the reactor at some particular level of T_{in}, T_c, and F.
2. Make final adjustments (zero suppression) to data acquisition and signal conditioning system.
3. Drive system into appropriate non-equilibrium state through use of hydraulic function generator operating on the process fluid inlet temperature.
4. Follow system trajectory away from equilibrium until some desirable distribution of energies is achieved.
5. Choosing this instant as the arbitrary time t_o, return the forcing variable $[T_{in}(t)]$ to its original level.
6. Follow the system's trajectory as it returns to its equilibrium state.

7. Establish that the system has returned to its equilibrium state.

Data Reduction Procedure: For various reasons, such as the difficulty mentioned with the flow control valve, and what appeared to be drift in the signal conditioning system, it was found that completion of procedure step seven was not always possible. Therefore the experimentally measured homogeneous system trajectories used in the calculation phase of this investigation were chosen on the basis of experimental assurance that the system did in fact return to its initial equilibrium state. Of course further selection was made on the basis of the linear independence of the forcing functions employed in the various experiments. On these bases, the experiments chosen for use in the calculation phase of the investigation consist of a set of four response trajectories generated by forcing functions which include positive and negative steps as well as sine waves for which the state variables were on both positive and negative excursions at the time of withdrawal of the forcing function. These experimental response data are reproduced in tabular form in Appendix B.

Given a pair of response vectors which satisfy the criteria given above, the first step in data reduction is the choice of time t_o. While this time was selected during the experiment, one is faced with the fact that physical systems do not respond instantaneously. Thus, in order to be assured

that the response vector is truly homogeneous, it is necessary to discard the data prior to the time at which the recorded value of the input variable returns to its equilibrium level. As a consequence of the physical imperfections of the hydraulic function generator, it was also necessary to establish that the process inlet flow rate, F, had returned to its equilibrium level.

Granting these requirements, consider the recordings of the state variables, $T_w^*(t)$ and $T_b^*(t)$. For the purpose of discussion, consider Figure V-9. This figure shows idealized recordings of raw data corresponding to the experimental measurements of the zero suppressed wall and bulk temperatures. Recall that in Chapter IV the requirement for precision in the experimental measurements was discussed. A technique for obtaining the necessary precision was suggested which involved telescoping eleven chart spans onto one such span through incremental suppression of the recording's zero level. The second step in the reduction of the experimental data involves the re-assembly of the homogeneous response vectors. This procedure will be described in terms of the segmented vectors of Figure V-9.

This figure has been constructed such that two small divisions on the horizontal run correspond to one unit of chronological time; further, that each "channel" is forty small divisions in width with zero suppression increments occurring each time the vector reaches a point five small divisions from the "channel" boundary. Thus the figure shows,

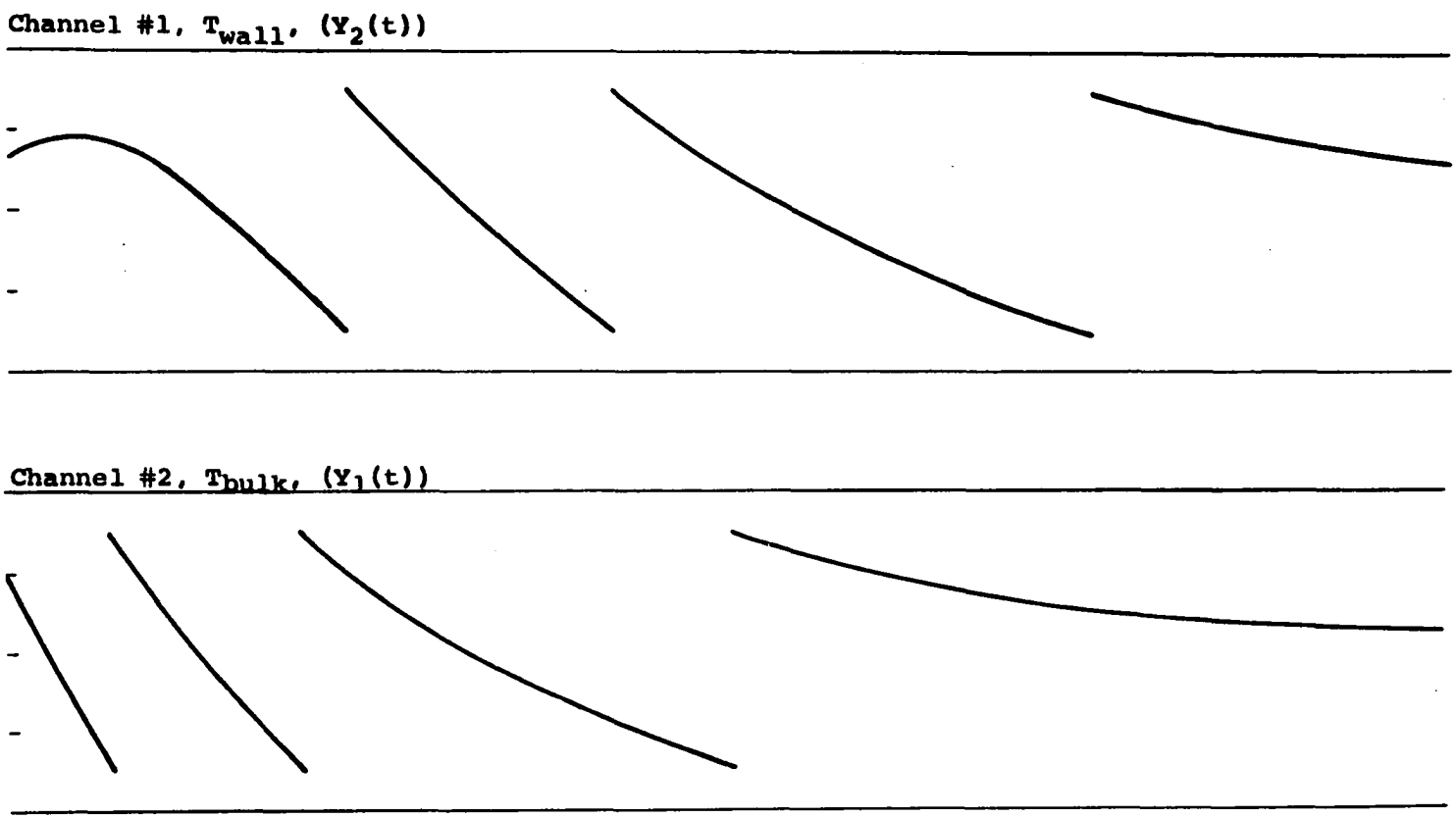


Figure V-9

Raw Data, Homogeneous Response Vector

for each of the vectors, four "windows", each thirty small divisions wide, telescoped onto a single chart with a forty small division span. In the ninety units of time shown, the true zero of the recording is incrementally suppressed three times.

The problem of reconstruction of the vector is, of course, solved quite easily as follows:

Assume that the steady state value is at twenty small divisions above the lower "channel" boundary with no incremental zero suppression. In order to determine the value of the assembled homogeneous response vector from the raw data segments, the following formulae are used:

$$AHRV(t) = RD(t) + 3x30 - 20 \qquad t \text{ on segment } 1$$

$$AHRV(t) = RD(t) + 2x30 - 20 \qquad t \text{ on segment } 2$$

$$AHRV(t) = RD(t) + 30 - 20 \qquad t \text{ on segment } 3$$

$$AHRV(t) = RD(t) + 0 - 20 \qquad t \text{ on segment } 4$$

This procedure has been followed and a plot of the assembled homogeneous response vector appears as Figure V-10.

The task of adjusting the incremental zero suppression equipment to a standard increment proved quite difficult even though the equipment was infinitely adjustable. As a result, the assembly procedure was modified slightly from that given above; a summation of the zero suppression increments applicable was used in place of the product of a standard increment with the appropriate number of increments. The details of this procedure appear in Appendix B.

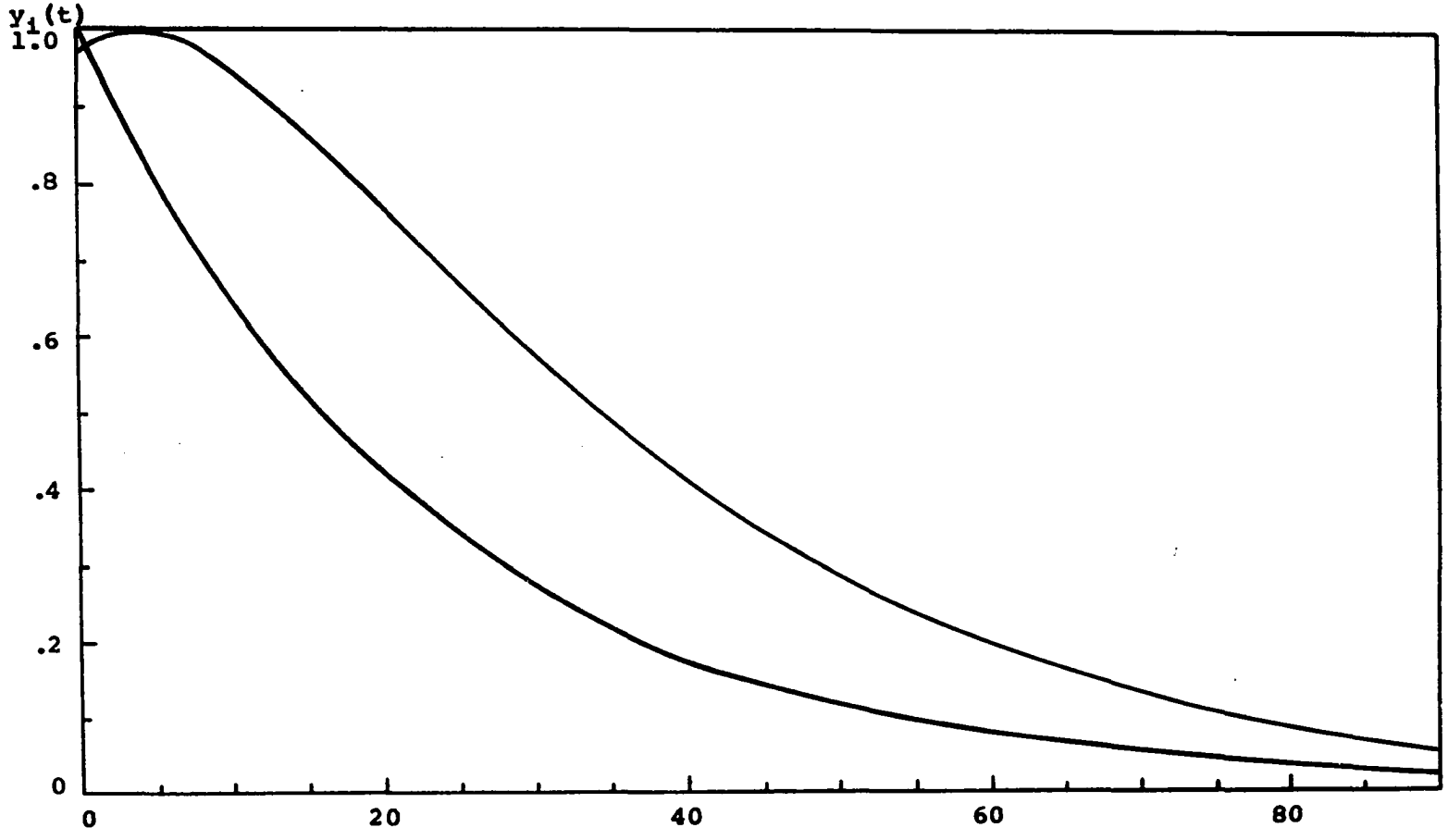


Figure V-10

Assembled Homogeneous Response Vector

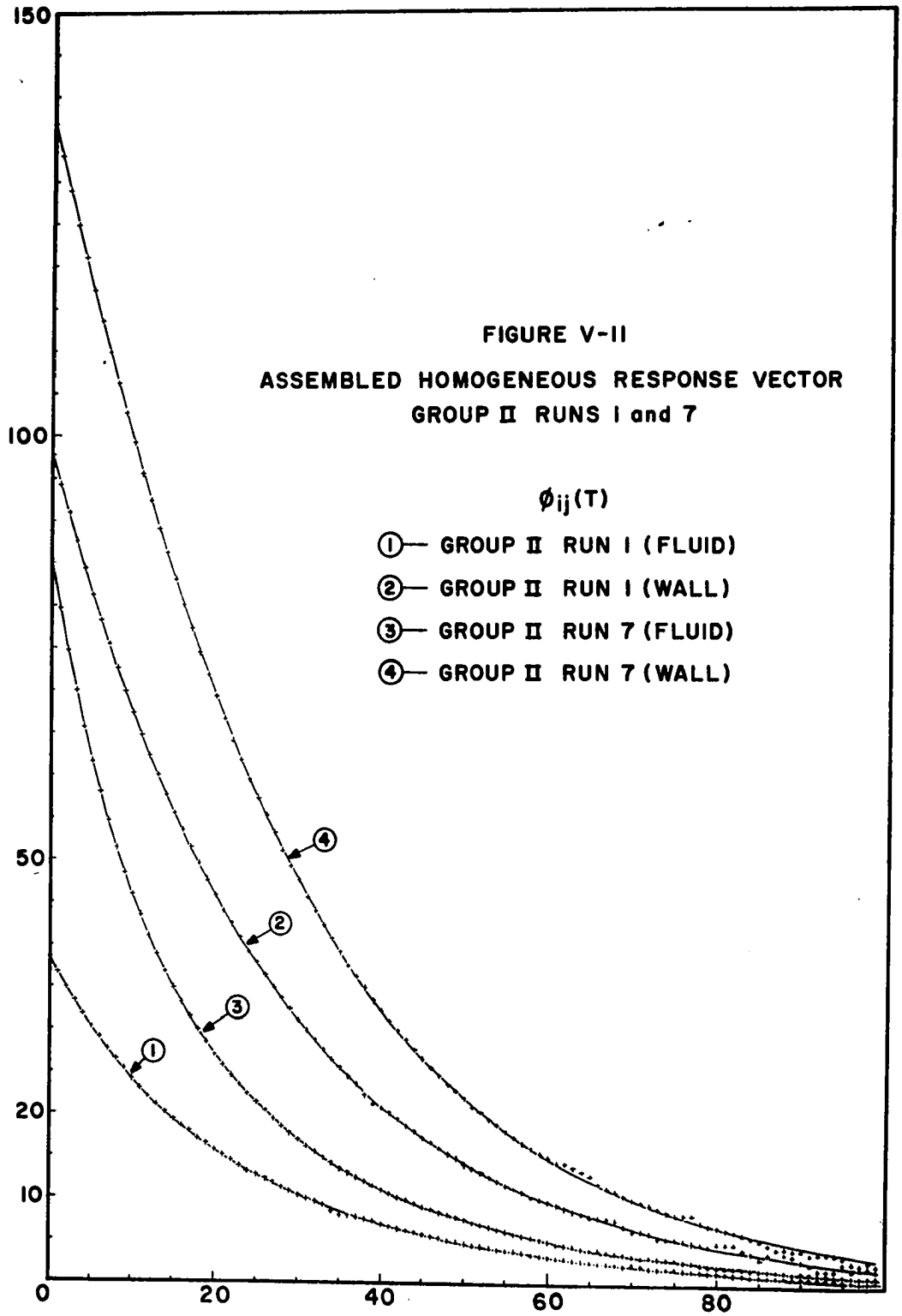
Figures V-11 and V-12 present, graphically, the elements of the four experimentally measured homogeneous response vectors used in the calculational phase of the investigation. Figure V-11 shows the elements of the homogeneous response vectors assembled from experiments Group II, Runs 1 and 7. Figure V-12 shows the absolute value of the elements of the homogeneous response vectors assembled from experiments Group II, Runs 6 and 8. The Table below describes the forcing functions employed in these experiments.

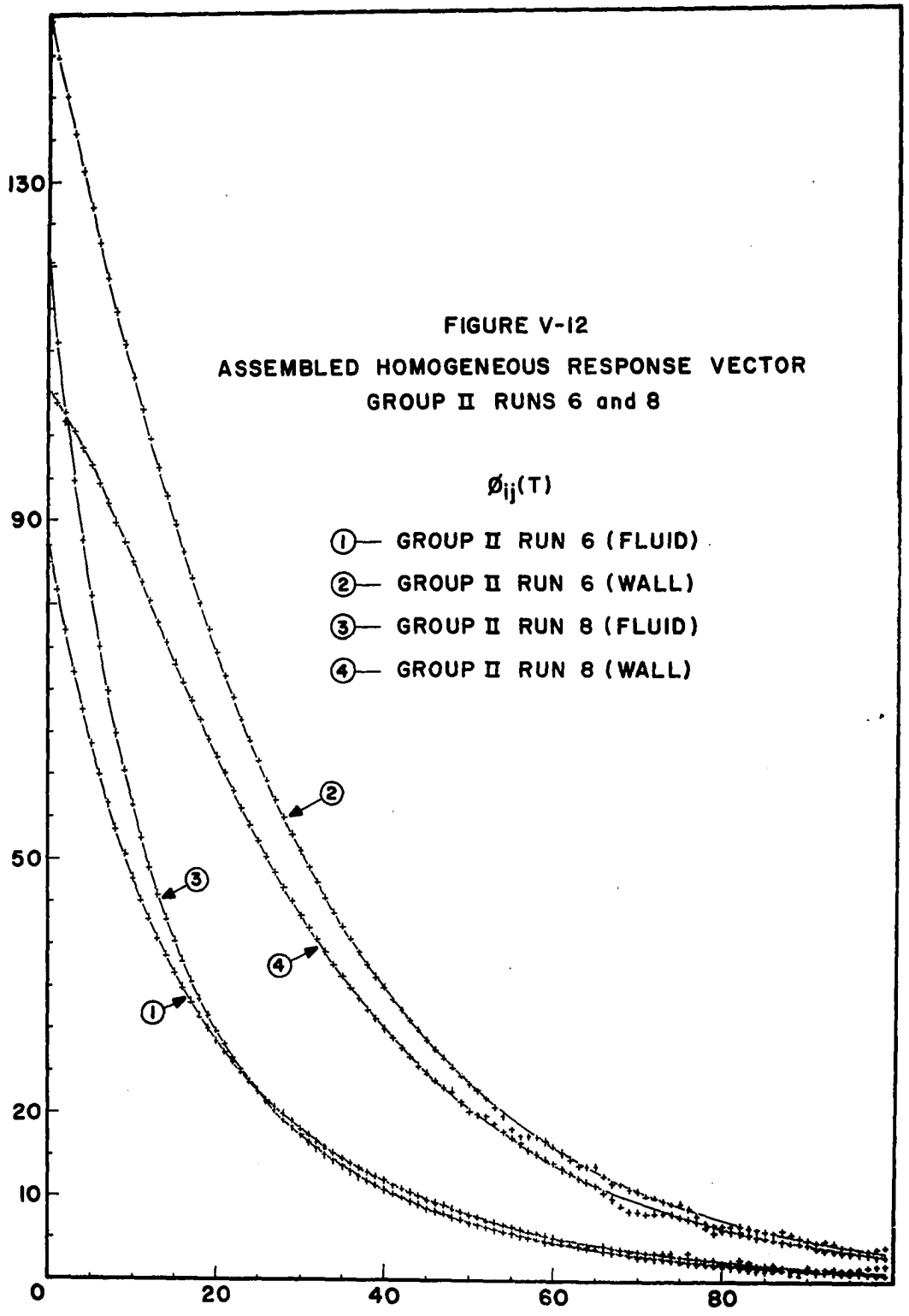
TABLE V-1

SUMMARY OF EXPERIMENTAL CONDITIONS

Experiment	Forcing Function	State Polarity at t_o
GII,R1	Positive Step	Positive
GII,R6	Sine Wave	Negative
GII,R7	Sine Wave	Positive
GII,R8	Negative Step	Negative

It should be noted that the data in Figures V-11 and V-12, indicated by the plotting symbols, show random, low amplitude, relatively high frequency deviations from the smooth response one would expect from a system describable by the simple process model proposed above. These deviations are attributed to four possible causes: the type of integral filtering used in the signal conditioning system (see Appendix D), small uncontrolled variations in the inlet and coolant


150
100
50
20
10
0
20
40
60
80
FIGURE V-II
ASSEMBLED HOMOGENEOUS RESPONSE VECTOR
GROUP II RUNS I and 7
$\phi_{ij}(T)$
① GROUP II RUN I (FLUID)
② GROUP II RUN I (WALL)
③ GROUP II RUN 7 (FLUID)
④ GROUP II RUN 7 (WALL)


130
90
50
20
10
0
20
40
60
80
FIGURE V-12
ASSEMBLED HOMOGENEOUS RESPONSE VECTOR
GROUP II RUNS 6 and 8
$\phi_{ij}(T)$
① GROUP II RUN 6 (FLUID)
② GROUP II RUN 6 (WALL)
③ GROUP II RUN 8 (FLUID)
④ GROUP II RUN 8 (WALL)

temperatures, momentary failures (instability) of components in the data acquisition and signal conditioning system, and, of course, the lack of sophistication of the process model. While the magnitude of the variations of the temperatures is not greater as the system approaches equilibrium, the fraction of the total "input" to the individual "energy storage elements" from these "sources" is indeed greater. Any trend in the experimental response away from that typical of the model is easily explained in terms of the failure of the constraints on the model to correspond with reality.

As a result of the presence of these small deviations, smooth curves were drawn through the reliable portion of the elements and extrapolated to determine the equilibrium level of the experimental data. These smooth curves have been included in Figures V-11 and V-12. In order to prevent degeneration of the normalized calculations, certain data in the range of $65 \leq t \leq 75$ seconds have been replaced with values read from these smooth curves.

Calculation Procedure: A digital computer was employed to carry out the calculations indicated by the theory of the generalized pulse testing technique.

The information required as input for the calculation consisted of a pair of linearly independent homogeneous response vectors from the experimental apparatus. This information, together with an estimate of the maximum "reading" error, was used to compute an estimate of the elements of the weighting

function representation of the mathematical model. As discussed in Chapter IV, calculations were made to determine the maximum error to be expected in these estimates.

The computer program contained several options, the selection of which permitted some freedom concerning the aspects of a particular calculation. Examples of these options include choices as to whether or not:

1. The calculated weighting functions are normalized.
2. Possibly singular matrices (by the maximum error criteria) are used in the calculation.
3. A graphical display, in addition to the tabular form, is made of the calculated weighting functions.
4. A deck of cards containing the calculated weighting functions is punched.

The theory implies that the procedure for determining the non-zero regions of the weighting function ($0 \leq t \leq \infty$, $0 \leq s \leq t$) is as follows: for successive points in the s variable the weighting function is calculated for all points in the range $s \leq t \leq \infty$. Even though the maximum value of t for which experimental data are available is substituted for the upper limit on the range of t, a huge number of calculations are being implied. While these calculations are clearly necessary for complete characterization of a time-varying system, the requirement for the definition of a time invariant weighting function is considerably less severe. Because of the symmetrical nature of the weighting function for a time

invariant system, the calculations need only be made at enough values of s to insure that the function is truly symmetrical. Therefore the computer program was written to make the calculation at increments of s chosen for a particular calculation.

Table V-1 presented a summary of the forcing functions and state variable polarities for the measured homogeneous response vectors which were considered suitable for use in the calculation phase of the investigation. Three sets of calculations were made to determine the weighting function representation of the model for the backmix reactor. These calculations were made using the vector pairs Group II-Runs 1 and 7, Group II-Runs 6 and 8, and Group II-Runs 7 and 8. These pairs are seen, by examination of Table V-1, to be linearly independent.

The choice of these pairings permitted the investigation of the possibility that the response of the system to positive inputs might differ from that to negative inputs due to the variation of the physical properties of the process fluid with temperature.

Results of the Investigation: The results of the three sets of calculations are presented in Figures V-13, V-14, V-15, and V-16.

Figure V-13 is a photograph of the tabular form of the computed weighting function. The first page of the computer output is devoted to identification of the particular calculation and a summary of its procedural aspects. The statement "the vector choice for this calculation is 14" is

GENERALIZED PULSE TEST CALCULATION (2X2)
R. A. BISHOP RUN 46

CALCULATION PROCEDURE
T= LOWER LIMIT TO UPPER LIMIT WITH INCREMENT

75 DATA POINTS PER VECTOR. CALCULATIONS MADE EVERY 1 POINTS. T = 0 INITIALLY.

CALCULATION USES DATA FROM EXPERIMENTAL APPARATUS (GROUP II, RUNS 7 AND 8).
GROUP II RUN 7 FORCING FUNCTION.. SINE WAVE ON TINLET. (POSITIVE AT RELEASE)
GROUP II RUN 8 FORCING FUNCTION.. STEP ON TINLET. (NEGATIVE)
DATA IDENTIFICATION.. 11=TOUTR7 21=TWALL7 12=TOUTR8 22=TWALL8

RUN PARAMETERS

ELEMENTS	11	21	12	22
S. S. CORRECTION	0.000	0.000	0.000	0.000
ATTENUATION	0.100	0.050	0.100	0.050
EST. MEAS. ERROR	0.250	0.250	0.250	0.250

OPTION ONE NOT DESIRED .... POSSIBLY SINGULAR MATRICES ARE NOT CALCULATED
OPTION TWO DESIRED .... RESULTS ARE NORMALIZED
OPTION THREE NOT DESIRED .... A DATA DECK IS NOT PUNCHED
OPTION FOUR DESIRED .... A GRAPH DISPLAY IS DRAWN
THE VECTOR CHOICE FOR THIS CALCULATION IS 14

FIGURE V-13

TABULAR FORM OF WEIGHTING FUNCTION ESTIMATE

REDUCED DATA FOR RUN 46, S = 0

REAL TIME.....T		11	21	- ELEMENTS - 12	22
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

T-S	PHI OF T AND WORST POSSIBLE ERRORS				DETERMINANT	PHI INVERSE OF S AND MAXIMUM ERROR				DET ERROR
0	[illegible]	[illegible]	[illegible]	[illegible]	37.527	[illegible]	[illegible]	[illegible]	[illegible]	-0.010

NORMALIZING FACTORS = 1.000 4.091 / 11.795 1.000

T	T-S	H(T,S) AND ASSOCIATED MAXIMUM ERROR				T	T-S	H(T,S) AND ASSOCIATED MAXIMUM ERROR			
0	0	[illegible]	[illegible]	[illegible]	[illegible]	1	1	[illegible]	[illegible]	[illegible]	[illegible]
2	2	[illegible]	[illegible]	[illegible]	[illegible]	3	3	[illegible]	[illegible]	[illegible]	[illegible]
4	4	[illegible]	[illegible]	[illegible]	[illegible]	5	5	[illegible]	[illegible]	[illegible]	[illegible]
6	6	[illegible]	[illegible]	[illegible]	[illegible]	7	7	[illegible]	[illegible]	[illegible]	[illegible]
8	8	[illegible]	[illegible]	[illegible]	[illegible]	9	9	[illegible]	[illegible]	[illegible]	[illegible]
10	10	[illegible]	[illegible]	[illegible]	[illegible]	11	11	[illegible]	[illegible]	[illegible]	[illegible]
12	12	[illegible]	[illegible]	[illegible]	[illegible]	13	13	[illegible]	[illegible]	[illegible]	[illegible]
14	14	[illegible]	[illegible]	[illegible]	[illegible]	15	15	[illegible]	[illegible]	[illegible]	[illegible]
16	16	[illegible]	[illegible]	[illegible]	[illegible]	17	17	[illegible]	[illegible]	[illegible]	[illegible]
18	18	[illegible]	[illegible]	[illegible]	[illegible]	19	19	[illegible]	[illegible]	[illegible]	[illegible]
20	20	[illegible]	[illegible]	[illegible]	[illegible]	21	21	[illegible]	[illegible]	[illegible]	[illegible]
22	22	[illegible]	[illegible]	[illegible]	[illegible]	23	23	[illegible]	[illegible]	[illegible]	[illegible]
24	24	[illegible]	[illegible]	[illegible]	[illegible]	25	25	[illegible]	[illegible]	[illegible]	[illegible]
26	26	[illegible]	[illegible]	[illegible]	[illegible]	27	27	[illegible]	[illegible]	[illegible]	[illegible]
28	28	[illegible]	[illegible]	[illegible]	[illegible]	29	29	[illegible]	[illegible]	[illegible]	[illegible]
30	30	[illegible]	[illegible]	[illegible]	[illegible]	31	31	[illegible]	[illegible]	[illegible]	[illegible]
32	32	[illegible]	[illegible]	[illegible]	[illegible]	33	33	[illegible]	[illegible]	[illegible]	[illegible]
34	34	[illegible]	[illegible]	[illegible]	[illegible]	35	35	[illegible]	[illegible]	[illegible]	[illegible]
36	36	[illegible]	[illegible]	[illegible]	[illegible]	37	37	[illegible]	[illegible]	[illegible]	[illegible]
38	38	[illegible]	[illegible]	[illegible]	[illegible]	39	39	[illegible]	[illegible]	[illegible]	[illegible]
40	40	[illegible]	[illegible]	[illegible]	[illegible]	41	41	[illegible]	[illegible]	[illegible]	[illegible]
42	42	[illegible]	[illegible]	[illegible]	[illegible]	43	43	[illegible]	[illegible]	[illegible]	[illegible]
44	44	[illegible]	[illegible]	[illegible]	[illegible]	45	45	[illegible]	[illegible]	[illegible]	[illegible]
46	46	[illegible]	[illegible]	[illegible]	[illegible]	47	47	[illegible]	[illegible]	[illegible]	[illegible]
48	48	[illegible]	[illegible]	[illegible]	[illegible]	49	49	[illegible]	[illegible]	[illegible]	[illegible]
50	50	[illegible]	[illegible]	[illegible]	[illegible]	51	51	[illegible]	[illegible]	[illegible]	[illegible]
52	52	[illegible]	[illegible]	[illegible]	[illegible]	53	53	[illegible]	[illegible]	[illegible]	[illegible]
54	54	[illegible]	[illegible]	[illegible]	[illegible]	55	55	[illegible]	[illegible]	[illegible]	[illegible]
56	56	[illegible]	[illegible]	[illegible]	[illegible]	57	57	[illegible]	[illegible]	[illegible]	[illegible]
58	58	[illegible]	[illegible]	[illegible]	[illegible]	59	59	[illegible]	[illegible]	[illegible]	[illegible]
60	60	[illegible]	[illegible]	[illegible]	[illegible]	61	61	[illegible]	[illegible]	[illegible]	[illegible]
62	62	[illegible]	[illegible]	[illegible]	[illegible]	63	63	[illegible]	[illegible]	[illegible]	[illegible]

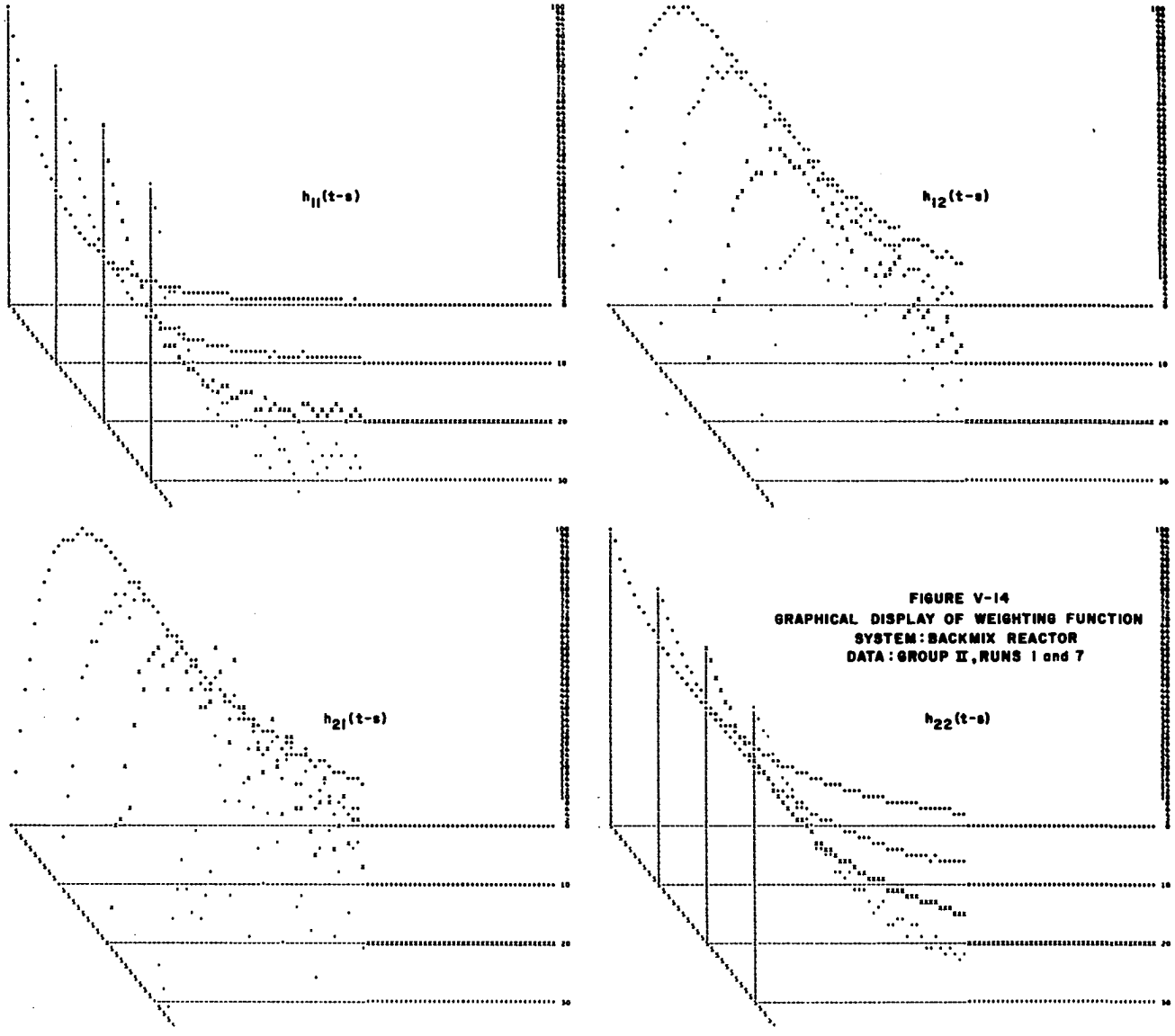


FIGURE V-14
GRAPHICAL DISPLAY OF WEIGHTING FUNCTION
SYSTEM: BACKMIX REACTOR
DATA: GROUP II, RUNS 1 and 7

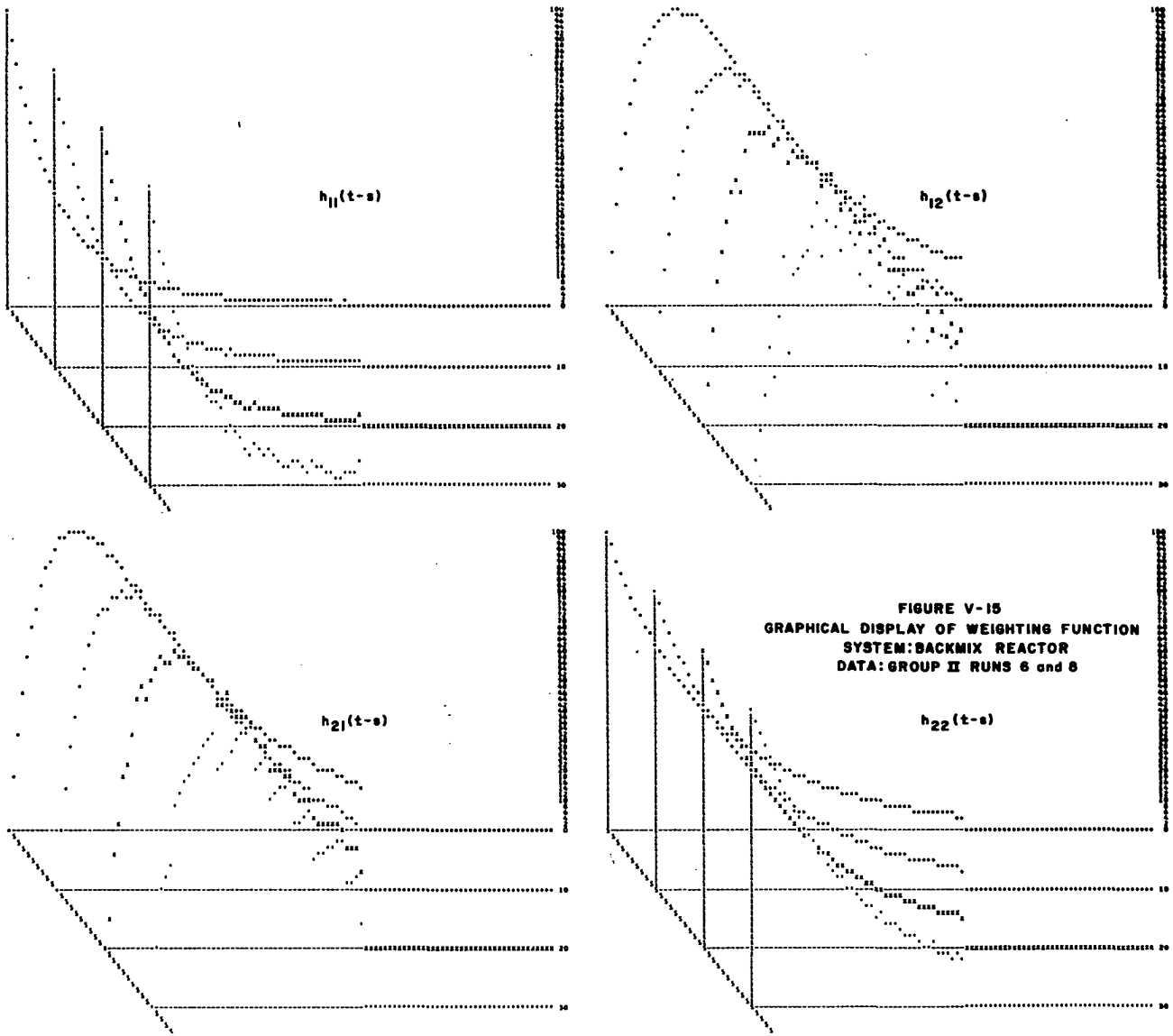


FIGURE V-15
GRAPHICAL DISPLAY OF WEIGHTING FUNCTION
SYSTEM: BACKMIX REACTOR
DATA: GROUP II RUNS 6 and 8

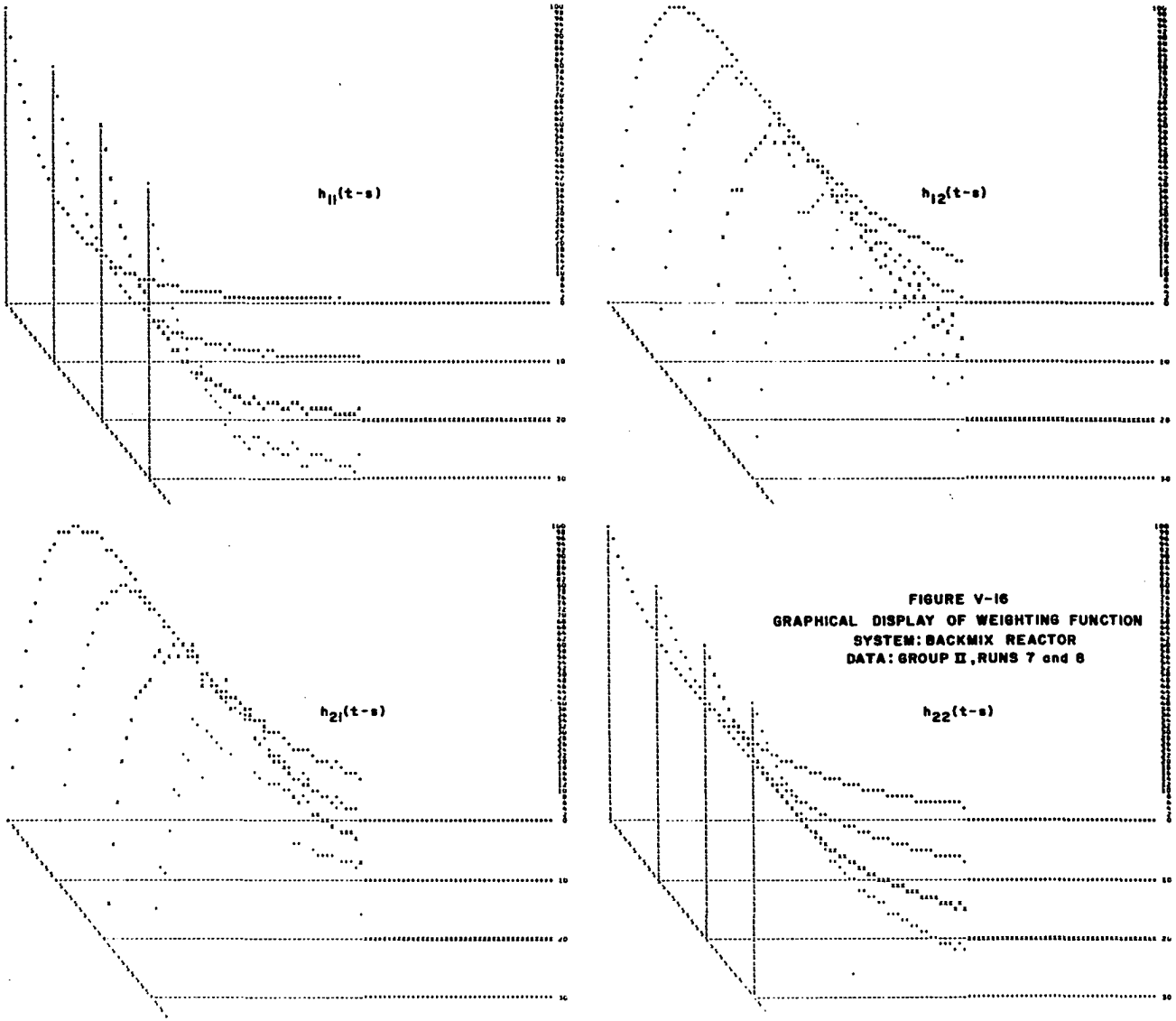


FIGURE V-16
GRAPHICAL DISPLAY OF WEIGHTING FUNCTION
SYSTEM: BACKMIX REACTOR
DATA: GROUP II, RUNS 7 and 8

an indication of the location, on each card of a combined input deck, where the computer is to find the data to be used in the calculation. In this case the first vector on the card is Group II-Run 7; the fourth vector on the data card is Group II-Run 8. The second and third pages of computer output list the input data contained on the cards and that data as modified by considerations of steady state correction and attenuation. The remaining pages of computer output list the calculated values of the weighting function and the associated maximum error prediction. The format may not be completely obvious, therefore the following diagram is given:

t	t-s	$h_{11}(t,s)$	$e_{11}(t,s)$	$h_{12}(t,s)$	$e_{12}(t,s)$
		$h_{21}(t,s)$	$e_{21}(t,s)$	$h_{22}(t,s)$	$e_{22}(t,s)$

Figure V-14 is a photograph of the assembled graphical display of the weighting functions calculated from the homogeneous response vectors Group II-Runs 1 and 7. The necessity for such an output format is clear after perusal of Figure V-13. This calculation used two positive energy distributions at time t_o and is therefore taken to represent the response of the system to positive input signals.

Figure V-15 is a photograph of the assembled graphical display of the weighting functions calculated from the homogeneous response vectors Group II-Runs 6 and 8. This calculation used two negative energy distributions at time t_o and is therefore taken to represent the response of the system to negative input signals.

Figure V-16 is a photograph of the assembled graphical display of the weighting functions calculated from the homogeneous response vectors Group II-Runs 7 and 8. This calculation used one positive and one negative energy distribution at time t_o and is therefore taken to be the response of the system to a mixture of positive and negative input signals.

These three figures are seen to be essentially identical, thus indicating that the temperature variation of the physical properties of the process fluid is not of sufficient magnitude to cause difficulty in the definition of a process model of sophistication comparable to those obtained from frequency response and statistical techniques.

The calculated values of the weighting functions exhibit characteristics similar to those noted in the discussion of weighting function estimates in Chapters III and IV. As seen previously, the quality of the estimate degenerates rapidly with decreasing magnitude of the determinant of the fundamental matrix. The choice of the option not to calculate weighting functions at values of the parameter s for which the fundamental matrix is possibly singular resulted in the termination of useful calculations after those made at s = 30. As is quite apparent upon examination of the quality of the estimate obtained from the calculation at s = 30, further calculation would have been a waste of time.

The second characteristic noted earlier, namely the apparent greater sensitivity of the quality of the estimate of

second order weighting functions to data imperfections, is also seen in these calculations. It must be noted that the reduced magnitude of these weighting functions themselves rather than a larger error is responsible for the proportionally greater scatter indicated in a normalized presentation of the weighting functions.

TIME-VARYING SYSTEM

The inclusion, in this chapter, of an application of the generalized pulse testing technique to a system which contains a parameter that varies significantly with time is desirable for three reasons. The most obvious is that only in such an application is the full potential of the technique realized. The determination of a mathematical model of such a process from experimental measurements is beyond the capability of frequency response methods and represents an exceedingly lengthy and complex calculation if statistical techniques are employed. Therefore the relative simplicity of the required calculations and the versatility of this method come into excellent focus.

The second reason for the inclusion of this application is that through it some appreciation for the "cost", in terms of calculational effort, may be gained. Admittedly an application which involved the difficulties associated with the acquisition of dynamic response data from an actual time-varying system as well as those associated with manipulation of such data would constitute a much more valid assessment of the true

cost of using the technique. However, since one of the primary concerns of this investigation is with the ramifications of the technique itself (as stated in Chapter II), the following material is very useful.

The third reason that inclusion of the following material is desirable is that it serves as a vehicle for the natural introduction of the practical concept of redundant testing of time-varying systems. The process of redundant testing represents a mechanism for obtaining improved quality of the weighting function estimates for time-varying systems at large values of the variable t = s.

Description of the System: The time-varying system chosen as the source of dynamic data for this study was the purely hypothetical one described by the vector differential equation:

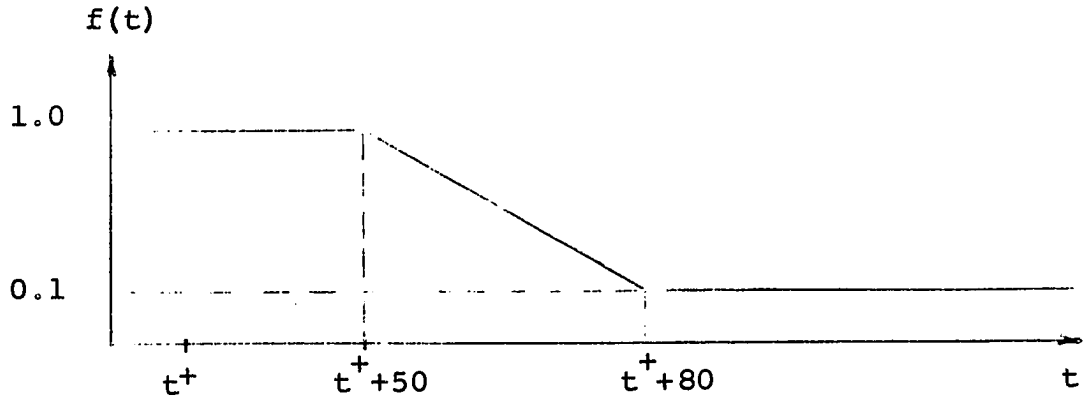
$$\dot{Y}(t) = A(t)Y(t) + X(t) \qquad Y(0) = 0$$

where the time variation of the parameter matrix, A(t) is given by:

$$A(t) = \left|\begin{array}{c|c} 0.0 & 1.0 \\ \hline -0.2f(t) & -0.3 \end{array}\right|$$

$$f(t) = \begin{cases} 1.0 & 0 \le t < t^+ + 50 \\ 2.5 - 0.03(t-t^+) & t^+ + 50 \le t \le t^+ + 80 \\ 0.1 & t^+ + 80 \le t \le \infty \end{cases}$$

The inclusion of the parameter, t^+, provides a reference time for the purpose of locating the region of the t domain for which a mathematical model is obtained. The variation of the parameter matrix element, a_{21}, is as shown diagramatically below:



Consider the analytic solution to the differential equation:

$$\ddot{y}(t) + .3\dot{y}(t) + .2y(t) = 0 \qquad y(0) = k,\ \dot{y}(0) = k'$$

which is just the system definition (provided $t < t^+ + 50$) written in the form of a second order ordinary differential equation. The solution is seen to a degenerate oscillation given by:

$$y(t) = \exp(-.15t)(k\cos.422t + (.15kk'/.422)\sin.422t)$$

The amplitude of the solution at the point $t = t^+ + 50$ is seen to be no greater than $k\exp(-(7.5 + .15t^+))$. Assuming that $t^+ \geq 0$, the maximum value of the solution is 0.00056k; hence for all practical purposes, the response is complete.

Consider now the analytic solution to the differential equation:

$$\ddot{y}(t) + .3\dot{y}(t) + .02y(t) = 0 \qquad y(0) = k,\ \dot{y}(0) = k'$$

which is the system definition (if t^+ were chosen to be -80) written in the form of a second order ordinary differential equation. The solution is seen to be composed of the sum of negative exponentials:

$$y(t) = (10k' + 2k)\exp(-.1t) - (10k' + k)\exp(-.2t)$$

Evaluation of the slowest decaying exponential indicates that at $t = t^+ + 130$, the amplitude of the solution is 0.00675k. Clearly the response is complete.

It is obvious that measurements of the homogeneous response of this system may be terminated after fifty units of time without loss of measurable information. This fact makes the computation of the numbers to be considered as "experimental data" convenient but introduces an interesting problem of system identification.

Procedure Equivalent to Data Acquisition Experiments: Since this system is hypothetical, it was simulated on a digital computer, the experiments on a real system being replaced by the calculation of the numerical solution (Runge-Kutta-Gill method) to the system describing equations. The non-homogeneous form of the vector differential equation was solved in the range of $0 \leq t \leq t^+ + s$ $(0 \leq s \leq t^+ + 80)$, and the homogeneous

form of the equation was solved in the range $t^+ + s \leq t \leq t^+ + s + 50$. These solutions correspond to the forced responses of the system terminating at $t = t^+ + s$ and the subsequent free responses. Calculations were made every 0.1 units of t with the result reported every 0.5 units of t. This calculation frequency provided more than 148 evaluations of the solution per cycle at the natural frequency of the system. Doubling the calculation frequency made no change in the numbers calculated, therefore the conclusion that numerical approximation was valid seems justified.

As has been seen previously, two experiments are required to obtain the information necessary for the characterization of a second order system. Therefore, the calculations outlined above were made twice, once with the non-homogeneous portion of the equation chosen to be a sine wave of amplitude, M, and frequency w, and once with the forcing function chosen to be a step of amplitude M.

Suppose that the value of t^+ is chosen to be a large positive number; then the solution to the differential equation:

$$\ddot{y}(t) + .3\dot{y}(t) + .2y(t) = x(t) \qquad \dot{y}(0) = y(0) = 0$$

is given by the sum of the homogeneous solution (form displayed above) and a particular solution. For $t > 50$ the homogeneous portion has been shown to be negligible, therefore at $t = t^+$:

$$y(t^+) = -M/((w^2-.2)^2+.09w^2)(.3w\cos wt^+ + (w^2-.2)\sin wt^+)$$

$$\dot{y}(t^+) = -wM/((w^2-.2)^2+.09w^2)(w^2-.2)\cos wt^+ - .3w\sin wt^+)$$

for the case in which $x(t) = M\sin wt$. Notice that if, rather than choosing t^+ very large and solving the non-homogeneous form of the equation with zero initial conditions, one chose $t^+ = 0$ and solved the non-homogeneous differential equation with the initial conditions:

$$y(t^+) = -.3M/((w^2-.2)^2+.09w^2)$$

$$\dot{y}(t^+) = -(w^2-.2)wM/((w^2-.2)^2+.09w^2)$$

then the solutions for $t \geq t^+$ would be identical. This idea was incorporated into the computer program for both sets of calculations (initial conditions for response to step were $y(0) = 5M$, $\dot{y}(0) = 0$) in order to minimize the computer time required to generate the "experimental data".

Five sets of calculations were made at values of the parameter, s, equal to 0, 20, 40, 60, and 80. The reason for these repeated calculations is explained in the section on redundant testing.

Redundant System Testing: As mentioned in the section on the description of the system, the fact that the homogeneous response of this system is complete within fifty time increments after it begins, poses an interesting problem in system identification. This problem is rooted in the fact that the onset of the parameter variation does not occur until $t = t^+ + 50$, a point at which the response begun at $t = t^+$ retains insufficient amplitude for recognition of the variation.

This situation leads to one of two types of difficulty depending on the measurability and controllability of the parameter variation.

If one assumes that the parameter variation is not controllable but is measurable, then the ultimate success or failure of efforts to obtain a model for the system depends on whether or not the variation is repetitive.

Consider the case in which the variation is repetitive. Without a priori knowledge that the system is time-varying, one would proceed to drive the system from equilibrium using an appropriate forcing function, remove this forcing function upon achievement of some desirable energy distribution, and measure the subsequent homogeneous response. Since two linearly independent response vectors are required for the determination of a second order system's characteristics, this process would be repeated using a second appropriate forcing function.

Assume that the desirable energy distribution was achieved at time $t = t^+$ in the first experiment, then at time $t = t^+ + 50$ the response would be complete. Assume further that the desirable energy distribution for the second experiment was achieved at the time $t = t^+ + 50 + T$, where $T \geq 30$. The response of the system in the second experiment would be complete by time $t = t^+ + T + 100$.

Examination of the equations which describe the system's behavior indicates that these two measured responses would not represent the same system, therefore subsequent calculation

based on the fundamental matrix formed by adjoining these measured responses is doomed to failure.

It is clear that unless the parameter variation is repetitive (it was assumed not controllable) the generalized pulse testing technique is not applicable. Similarly, if the parameter variation is not measurable then the technique cannot be employed. These situations are examples of the basis for the constraint on the measurability of the parameter variation given at the end of Chapter II.

Alternatively, if the parameter variations are weakly controllable (in the sense that they may be "reset"), then the outlook is much brighter. Consider the case wherein the parameter variation is "resettable". Essentially what is being suggested is that a means exists for returning to the time $t = t^+$. In this situation the first data acquisition experiment proceeds as above; the second proceeds in a similar manner, namely a second homogeneous response vector which is representative of the system in the range $t^+ \leq t \leq t^+ + 50$ is obtained. While subsequent calculation of the weighting function based on these responses determines the model for the system in the specified range of time, the model contains no clue that the nature of the system is time-varying.

Figure V-17 shows a diagram of the t,s plane which illustrates the situation. The homogeneous responses obtained experimentally serve as the basis for the calculation of an estimate of the weighting function which applies in the area

based on the fundamental matrix formed by adjoining these measured responses is doomed to failure.

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Figure V-17 shows a diagram of the t,s plane which illustrates the situation. The homogeneous responses obtained experimentally serve as the basis for the calculation of an estimate of the weighting function which applies in the area

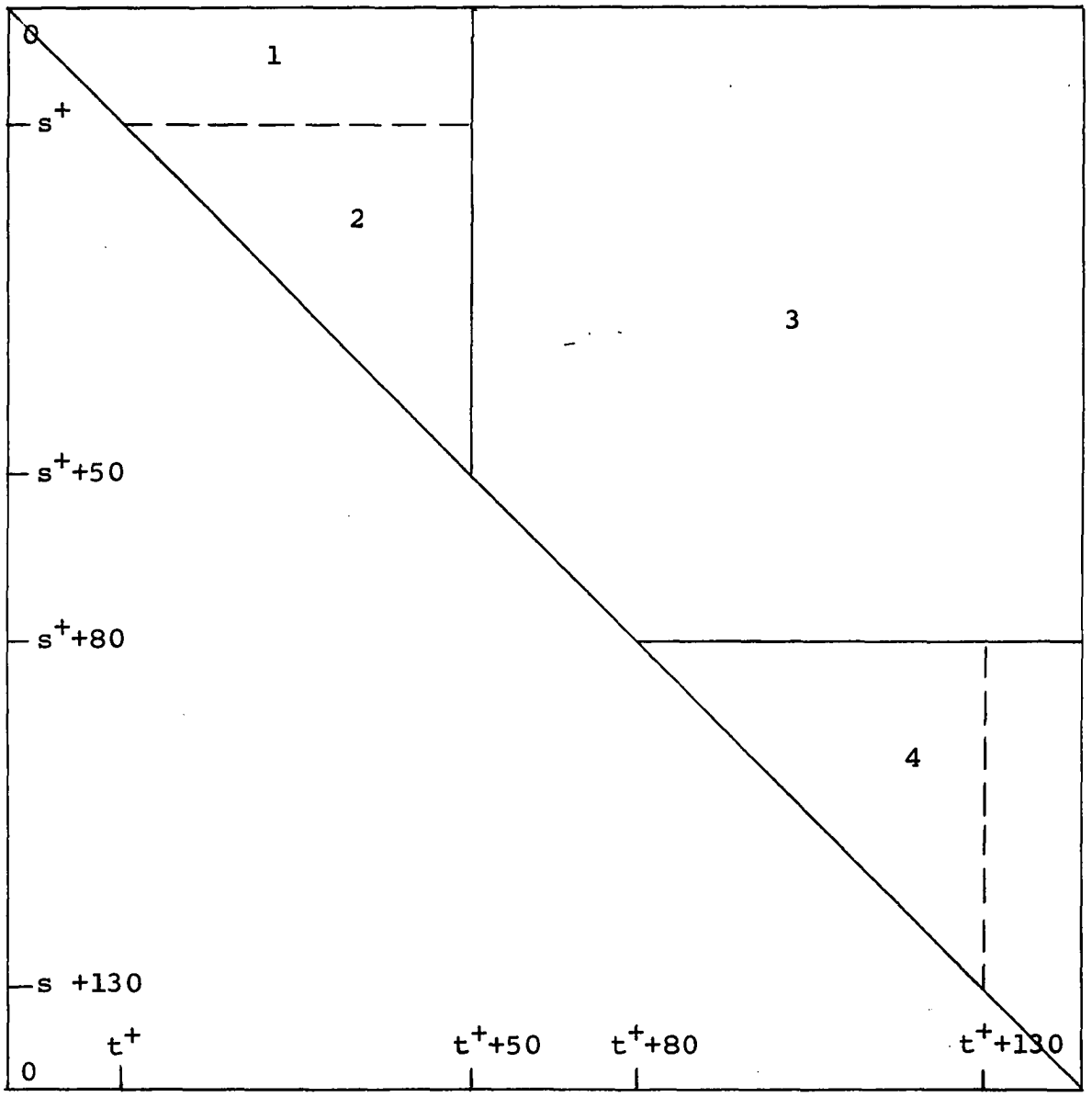


Figure V-17

Regions of Interest on T,S Plane

for Time-Varying System

designated by the number 2. It is clear that the areas designated 1 and 2 represent the region in which the system is time invariant with $a_{21} = 1.0$. The area designated by the number 3 defines the region in which the system is time-varying. The area designated by the number 4 represents the region in which the system is again time invariant ($a_{21} = 0.1$). Since area 2 is completely contained within the first time invariant region of the t,s plane, the subsequent time-varying nature of the system goes unrecognized in the weighting function determination.

The clue to the means by which this difficulty may be handled is contained in the discussion of the case in which the parameter variation was not controllable. Recall that in that instance the system represented by the second homogeneous response vector differed from that represented by the first due exclusively to the passage of time. Therefore in order to obtain homogeneous response information on which to base calculations in the region $t \geq s \geq 50$, one has only to make repeated experiments which involve forcing the system in the range $t > t^+$. Notice that in the case of a time invariant system, these tests provide no new information and hence are redundant.

Figure V-18 shows the coverage on the t,s plane made possible through redundant tests at values of $t = s = t^+ + 0$, 20, 40, 60, and 80 units of time respectively.

Examination of Figure V-18 reveals that certain areas of the t,s plane are covered by the calculations made possible

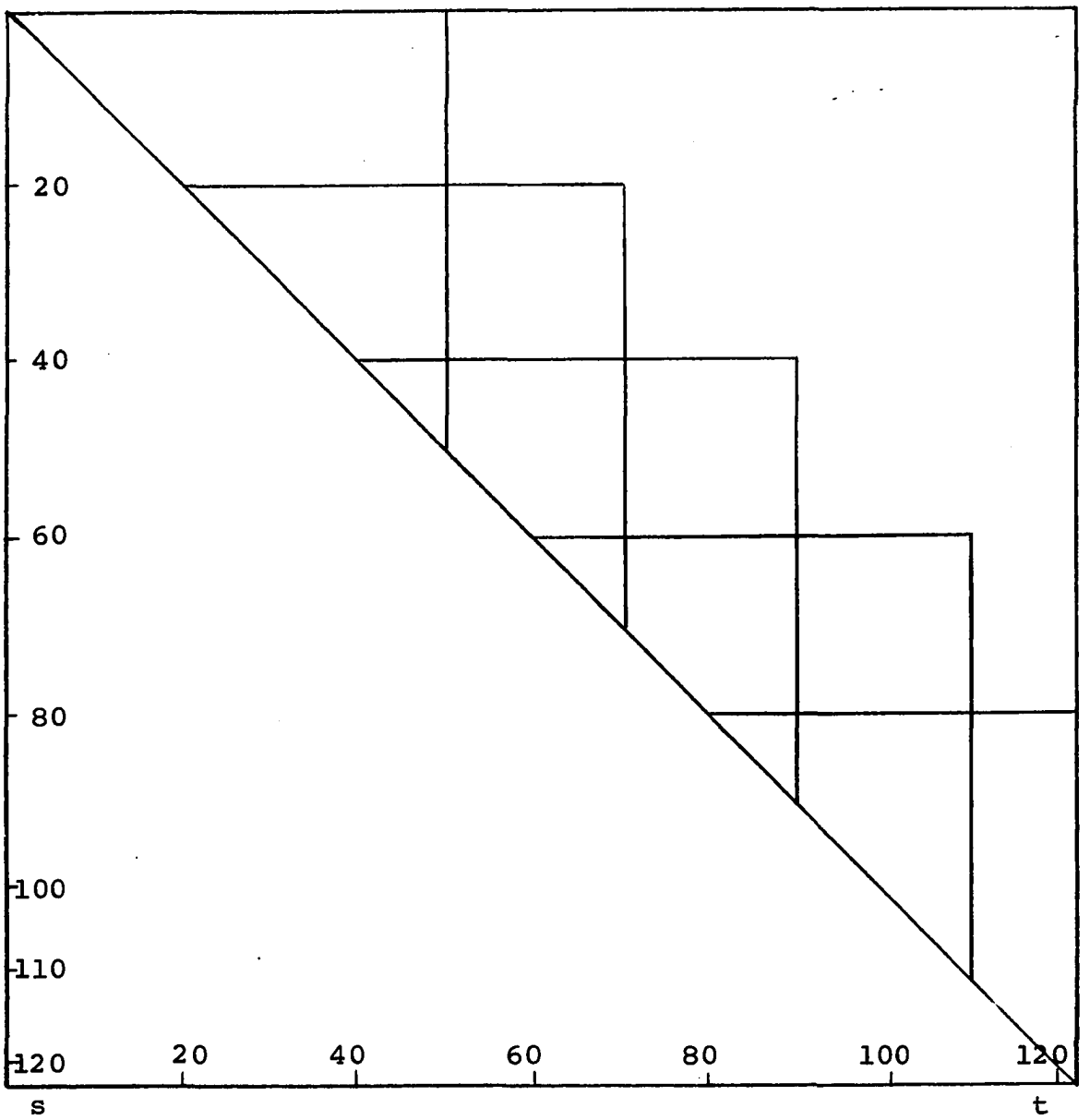


Figure V-18

T,S Plane Coverage by Redundant Testing

on the basis of more than one set of tests. It has been recognized that the magnitude of the determinant of the fundamental matrix decreases rapidly with increasing time. Further, it has been recognized that the quality of the estimates of the weighting functions is a strong function of this magnitude. Therefore it is clear that an improved estimate of the weighting function may be obtained from the calculations based on the redundant set of tests.

For example, consider the calculation of the weighting function at $s = 25$, $25 \leq t \leq 50$ from the homogeneous response data acquired by removing the forcing function at $t = s = 0$. It is clear that at $t = s = 25$, the magnitude of the determinant of the fundamental matrix will be very small, thus implying that the errors in the estimate of the weighting function are likely to be very large. Since an experiment has been carried out in which the forcing function was removed at $t = s = 20$, the data for calculating the weighting function in the range $s = 25$, $25 \leq t \leq 70$ is available. Not only may the calculations be made, but the magnitude of the determinant of the fundamental matrix which is formed from the second set of experiments must be much larger at $t = s = 25$ than that formed from the first set of experiments. Therefore one may expect that magnitude of the errors in the estimate of the weighting function would be much smaller.

Calculation Procedure: The data reduction process for the "experimental data" used in this study was trivial

since the homogeneous response vectors were available in digital form from the numerical solution of the system describing equations. These data were computed to three decimal precision; hence an estimate of 0.001 was used as the maximum error contained therein. The origin of the 1.0 attenuation factors and the 0.0 steady state corrections is obvious. The "data" were "measured" at increments of 0.5 Δ t.

The estimates of the weighting functions were calculated using a slightly modified version of the digital computer program listed in Appendix C. A time scale change of $\bar{t} = 0.5t$ was made for convenience in the calculations which were made at intervals of $\Delta\bar{t}$ for parameters of $s = \bar{t} + 0, 10, 20, 30, 40, 50$, increments of $\bar{t}$ respectively for the five sets of "experimental data" corresponding to removal of the forcing function at $t = s = 0, 20, 40, 60$, and 80 increments of t respectively. These results were punched on cards which were subsequently used as input information for a slightly modified version of the plotting routine which appears in Appendix C.

It is of interest to note that calculations were not made for parameters of $s = \bar{t}$ at which the maximum error criteria indicated the possibility of a singular fundamental matrix. Further, the estimates of the weighting functions were not normalized in the calculation as normalization at this point would destroy one of the more interesting characteristics of the time-varying weighting functions.

Results of the Study: The results of the calculation of estimates of the time-varying weighting function

characterization of the system under consideration appear graphically in Figures V-19, V-20, V-21, and V-22. A typical section of the tabular form of these results has been photographed and appears as Figure V-23.

Advantage has been taken of the benefits associated with redundant testing in order to present the best estimates of the weighting functions over the range of the t,s plane which is shown in these figures. Table V-2 summarizes the procedure followed in the selection of the estimates for plotting.

Figure V-19 is a photograph of the normalized graphic display of the computed estimate of the time-varying weighting function element $h_{11}(t,s)$. The two areas of time invariant parametric behavior ($0 \leq t \leq 50$, $s \leq t$ and $80 \leq t \leq \infty$, $s \leq t$) are reflected by the identical nature of the cuts at s = 0, 20, and 40 and at s = 80 and 100. The time variation of the parameters ($50 \leq t \leq 80$) is reflected in the cut at s = 60. Notice that the maximum value (1.0) exists along the line t = s; this behavior is a consequence of the definition of the weighting function as: $\underline{H}(t,s) = \underline{\overline{\phi}}(t)\underline{\overline{\phi}}^{-1}(s)$.

Figure V-20 is a photograph of the normalized graphic display of the computed estimate of the time-varying weighting function element $h_{12}(t,s)$. As noted in connection with Figure V-19, the two areas of time invariant behavior are reflected by the cuts at s = 0, 20, 40 and at s = 80, 100. The cut at s = 60 is different due to the time varying nature of the

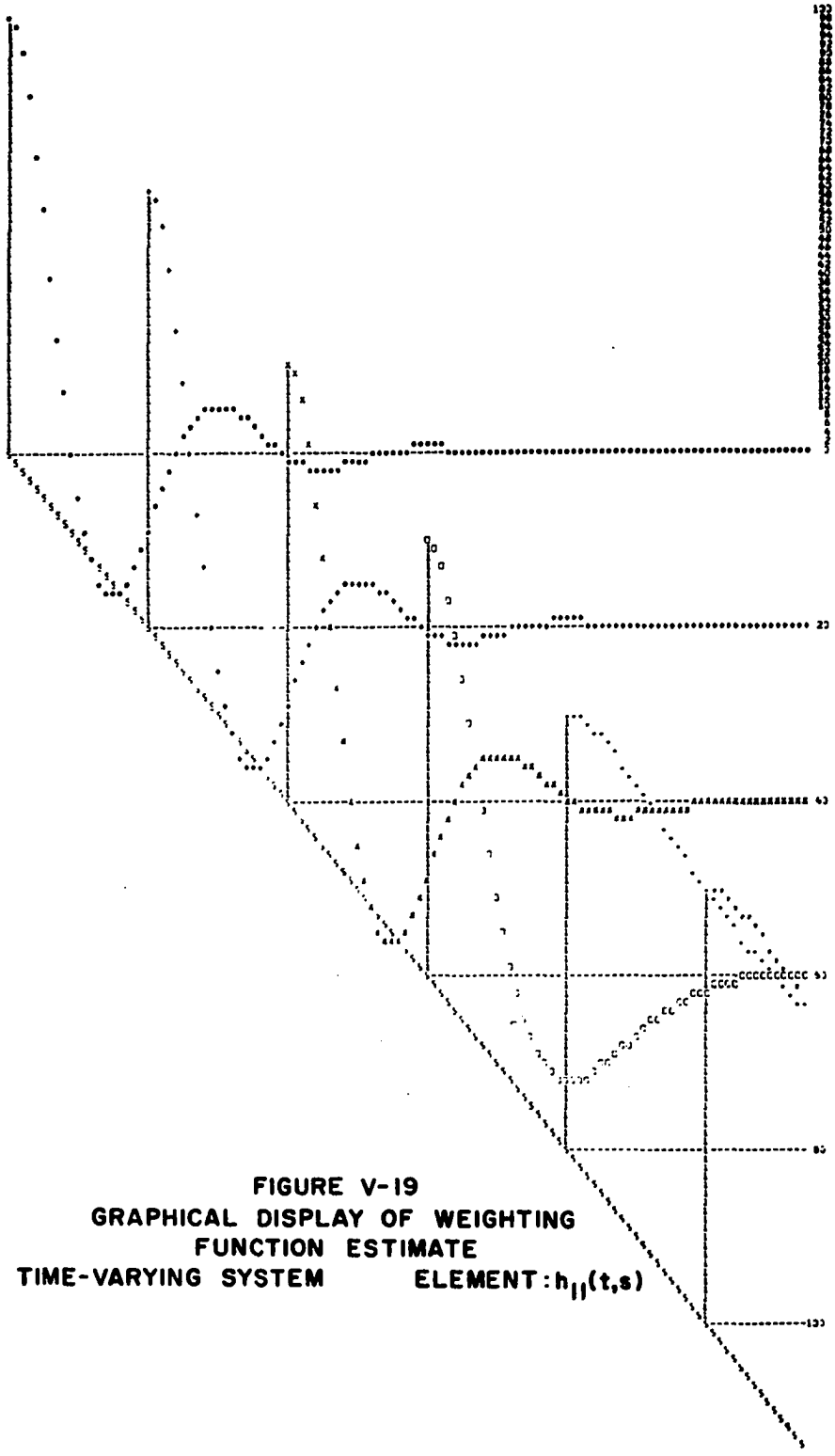


FIGURE V-19
GRAPHICAL DISPLAY OF WEIGHTING
FUNCTION ESTIMATE
TIME-VARYING SYSTEM ELEMENT: $h_{11}(t,s)$

FIGURE V-20
GRAPHICAL DISPLAY OF WEIGHTING
FUNCTION ESTIMATE
TIME-VARYING SYSTEM ELEMENT: $h_{21}(t,s)$

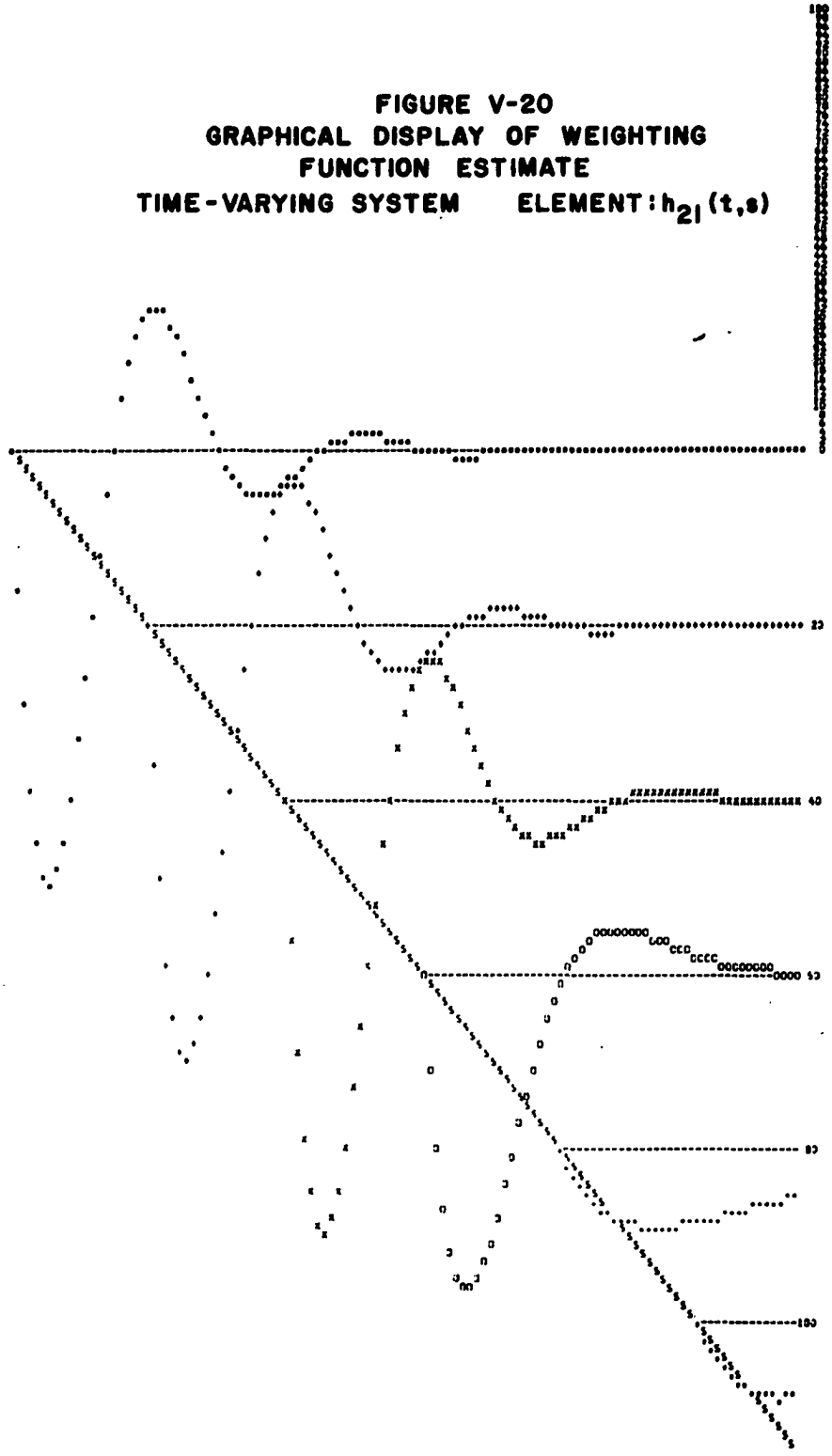
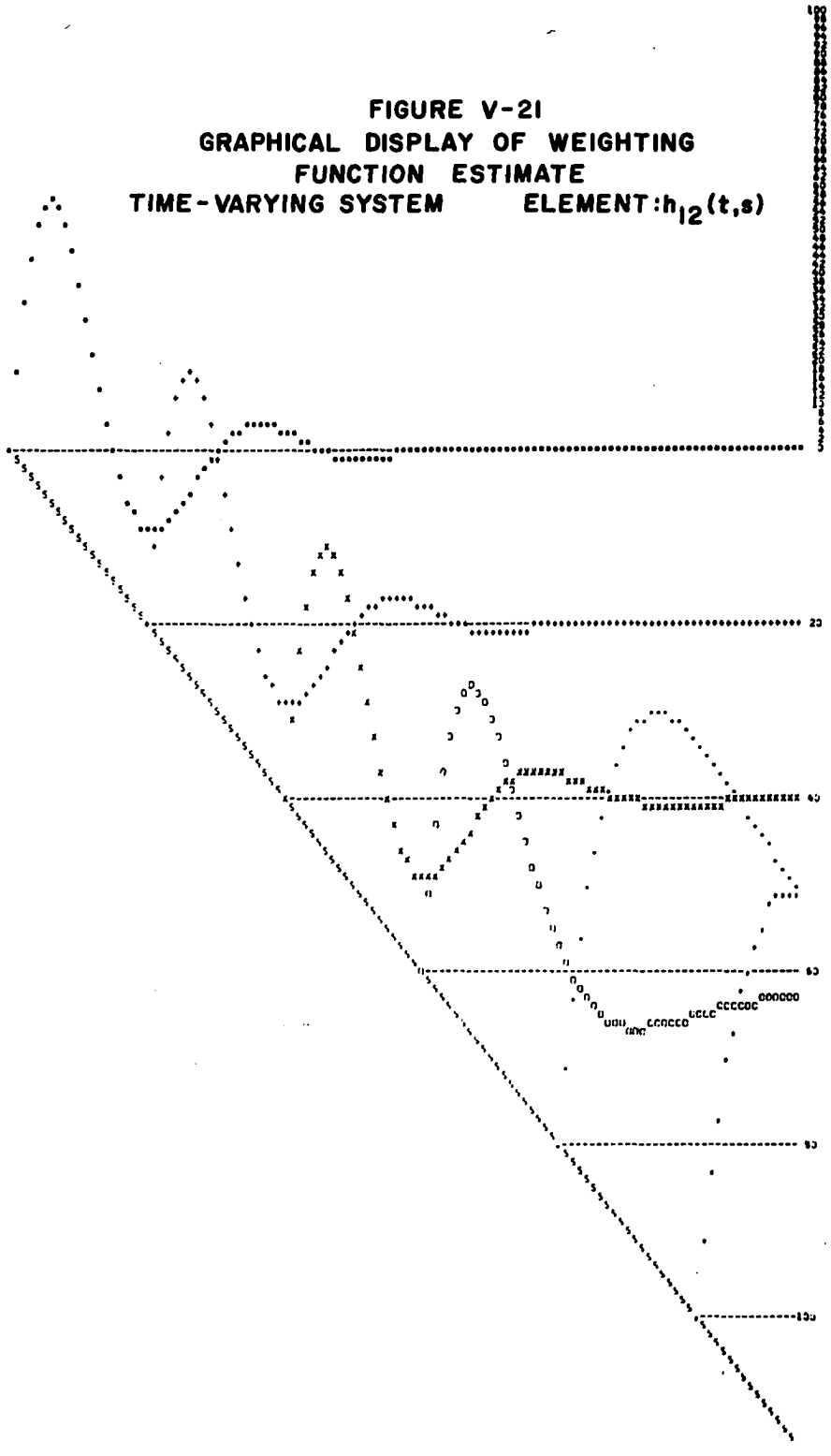


FIGURE V-21
GRAPHICAL DISPLAY OF WEIGHTING
FUNCTION ESTIMATE
TIME-VARYING SYSTEM ELEMENT: $h_{12}(t,s)$



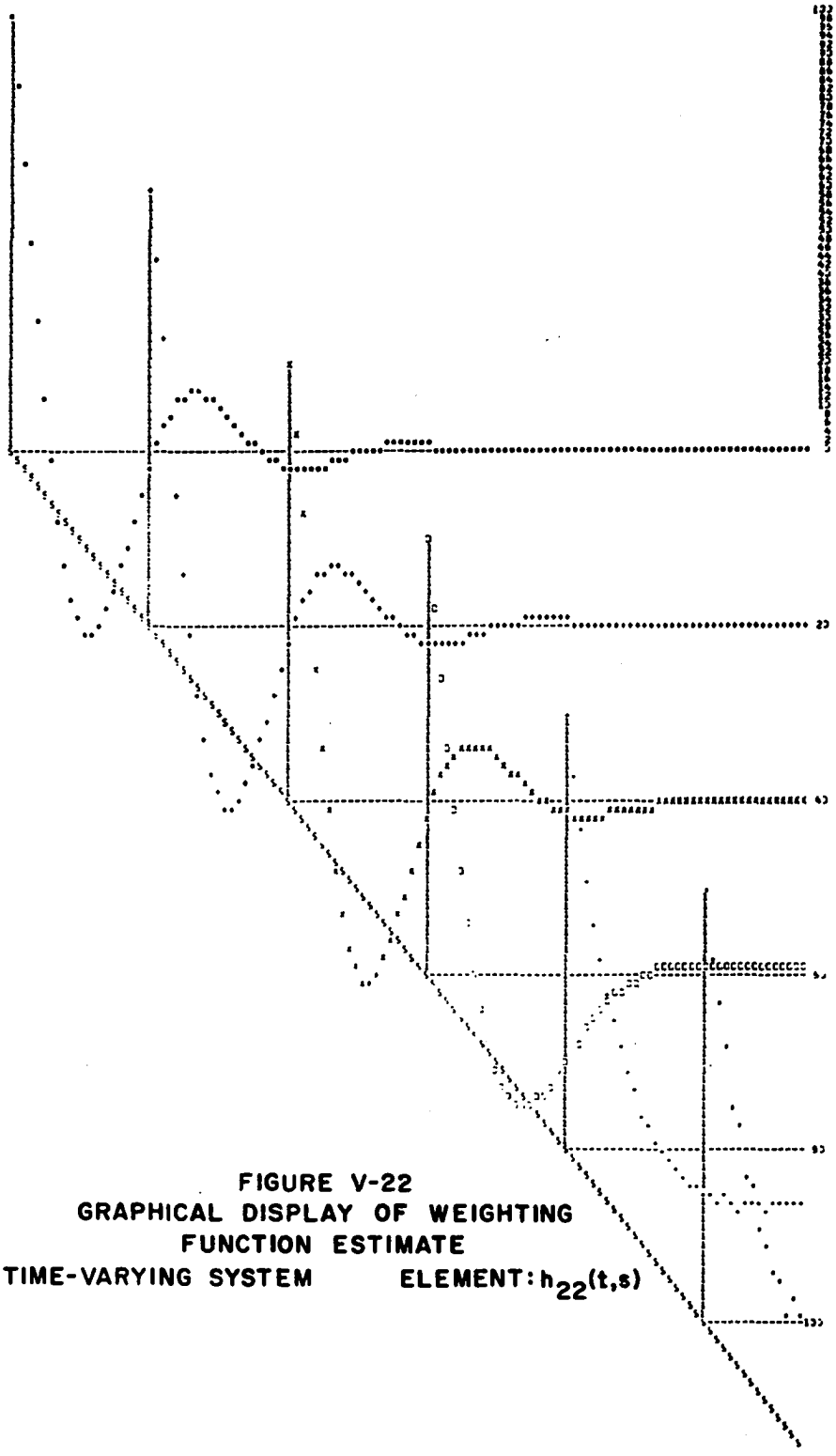


FIGURE V-22
GRAPHICAL DISPLAY OF WEIGHTING
FUNCTION ESTIMATE
TIME-VARYING SYSTEM ELEMENT: $h_{22}(t,s)$

GENERALIZED PULSE TEST CALCULATION (2X2)
R. A. BISHOP RUN 33

CALCULATION PROCEDURE

T= LOWER LIMIT	TO UPPER LIMIT	WITH INCREMENT
3	40	10
53	200	50
203	200	1

101 DATA POINTS PER VECTOR. CALCULATIONS MADE EVERY 1 POINTS.

RUN PARAMETERS

ELEMENTS	11	21	12	22
S. S. CORRECTION	0.000	0.000	0.000	0.000
ATTENUATION	1.000	1.000	1.000	1.000
EST. READ. ERROR	0.001	0.001	0.001	0.001

OPTION ONE NOT DESIRED .... POSSIBLY SINGULAR MATRICES ARE NOT CALCULATED
OPTION TWO NOT DESIRED .... RESULTS ARE NOT NORMALIZED
OPTION THREE DESIRED .... A DATA DECK IS PUNCHED
THE VECTOR CHOICE FOR THIS CALCULATION IS 12

DATA SOURCE — TIME VARYING SYSTEM, DY/DT = AY + U, WHERE
A11 = 0.0, A12 = 1.0, A21 = -3.2F, A22 = -.03
F = 1.0 FOR T = 0 TO 50, F = 12.5 - .23 T) FOR T = 50 TO 80
F = 0.1 FOR T GREATER THAN 80.
AUTONOMOUS BEHAVIOR BEGINS AT T = 23.3

RAW DATA FOR RUN 33

POINT	11	21	- ELEMENTS - 12	22
0	38.062	3.967	41.966	-0.297
1	38.346	-2.768	40.443	-4.044
2	36.281	-5.899	37.642	-7.115
3	32.640	-8.241	32.474	-9.430
4	28.879	-9.914	28.338	-10.987
5	22.049	-13.096	22.606	-11.818
6	17.290	-11.236	16.638	-11.968
7	11.707	-11.818	10.717	-11.979
8	6.360	-13.380	5.131	-10.692
9	1.663	-9.229	0.086	-9.434
10	-2.019	-7.066	-4.200	-7.983
11	-6.376	-6.334	-7.884	-6.242
12	-9.119	-4.722	-10.493	-4.513
13	-11.096	-3.116	-12.322	-2.067
14	-12.260	-1.590	-13.321	-1.226
15	-12.710	-2.203	-13.976	0.200
16	-12.502	1.030	-13.162	1.410
17	-11.745	1.990	-12.198	2.400
18	-10.558	2.749	-10.883	3.175
19	-9.034	3.276	-9.104	3.623
20	-7.312	3.570	-7.220	3.879
21	-5.491	3.672	-5.265	3.913
22	-3.670	3.584	-3.339	3.764
23	-1.933	3.342	-1.526	3.463
24	-0.349	2.979	0.104	3.063
25	1.031	2.529	1.983	2.561
26	2.171	2.025	2.637	1.996
27	3.052	1.497	3.491	1.426
28	3.669	3.976	0.865	0.876
29	4.030	3.478	4.371	0.358
30	4.155	3.039	4.432	-0.102
31	4.070	-3.350	4.200	-0.494
32	[illegible]	[illegible]	[illegible]	[illegible]
33	[illegible]	[illegible]	[illegible]	[illegible]
34	[illegible]	[illegible]	[illegible]	[illegible]
35	[illegible]	[illegible]	[illegible]	[illegible]
36	[illegible]	[illegible]	[illegible]	[illegible]
37	[illegible]	[illegible]	[illegible]	[illegible]
38	[illegible]	[illegible]	[illegible]	[illegible]
39	[illegible]	[illegible]	[illegible]	[illegible]
40	[illegible]	[illegible]	[illegible]	[illegible]
41	[illegible]	[illegible]	[illegible]	[illegible]
42	[illegible]	[illegible]	[illegible]	[illegible]
43	[illegible]	[illegible]	[illegible]	[illegible]
44	[illegible]	[illegible]	[illegible]	[illegible]
45	[illegible]	[illegible]	[illegible]	[illegible]
46	[illegible]	[illegible]	[illegible]	[illegible]
47	[illegible]	[illegible]	[illegible]	[illegible]
48	[illegible]	[illegible]	[illegible]	[illegible]
49	[illegible]	[illegible]	[illegible]	[illegible]
50	[illegible]	[illegible]	[illegible]	[illegible]
51	[illegible]	[illegible]	[illegible]	[illegible]
52	[illegible]	[illegible]	[illegible]	[illegible]

FIGURE V-23
TABULAR FORM OF WEIGHTING FUNCTION ESTIMATE

T=S	PHI OF T AND WORST POSSIBLE ERRORS				DETERMINANT	PHI INVERSE OF S AND MAXIMUM ERROR				DET ERROR
20	38.692 0.987	-0.001 -0.001	41.566 -0.257	-0.031 0.031	-40.182	0.009 0.319	-0.000 0.000	0.620 -0.776	0.001 0.001	-0.002

NORMALIZING FACTORS

1.222 1.000
1.333 1.000

T	T-S	H(T,S) AND ASSOCIATED MAXIMUM ERROR				T	T-S	H(T,S) AND ASSOCIATED MAXIMUM ERROR			
20	0	1.000 -0.000	0.000 0.000	0.000 1.000	0.136 0.031	21	1	0.976 -0.092	0.000 0.000	0.666 0.839	0.181 0.009
22	2	0.911 -0.187	0.000 0.000	0.835 0.640	0.085 0.317	23	3	0.812 -0.224	0.000 0.000	1.120 0.477	0.009 0.023
24	4	0.650 -0.262	0.000 0.000	1.312 0.297	0.073 0.027	25	5	0.551 -0.204	0.000 0.000	1.418 0.123	0.009 0.029
26	6	0.409 -0.289	0.000 0.000	1.443 -0.023	0.045 0.033	27	7	0.266 -0.280	0.000 0.000	1.307 -0.153	0.029 0.029
28	8	0.131 -0.275	0.000 0.000	1.294 -0.296	0.019 0.027	29	9	0.009 -0.229	0.000 0.000	1.149 -0.334	0.002 0.024
30	10	-0.097 -0.193	0.000 0.000	0.944 -0.384	0.037 0.025	31	11	-0.183 -0.153	0.000 0.000	0.704 -0.413	0.010 0.016
32	12	-0.249 [illegible]	0.000 [illegible]	0.516 [illegible]	0.025 [illegible]	33	13	-0.294 [illegible]	0.000 [illegible]	0.352 [illegible]	0.003 [illegible]
34	14	[illegible]	[illegible]	[illegible]	[illegible]	35	15	[illegible]	[illegible]	[illegible]	[illegible]
36	16	[illegible]	[illegible]	[illegible]	[illegible]	37	17	[illegible]	[illegible]	[illegible]	[illegible]
38	18	[illegible]	[illegible]	[illegible]	[illegible]	39	19	[illegible]	[illegible]	[illegible]	[illegible]
40	20	[illegible]	[illegible]	[illegible]	[illegible]	41	21	[illegible]	[illegible]	[illegible]	[illegible]
42	22	[illegible]	[illegible]	[illegible]	[illegible]	43	23	[illegible]	[illegible]	[illegible]	[illegible]
44	24	[illegible]	[illegible]	[illegible]	[illegible]	45	25	[illegible]	[illegible]	[illegible]	[illegible]
46	26	[illegible]	[illegible]	[illegible]	[illegible]	47	27	[illegible]	[illegible]	[illegible]	[illegible]
48	28	[illegible]	[illegible]	[illegible]	[illegible]	49	29	[illegible]	[illegible]	[illegible]	[illegible]
50	30	[illegible]	[illegible]	[illegible]	[illegible]	51	31	[illegible]	[illegible]	[illegible]	[illegible]
52	32	[illegible]	[illegible]	[illegible]	[illegible]	53	33	[illegible]	[illegible]	[illegible]	[illegible]
54	34	[illegible]	[illegible]	[illegible]	[illegible]	55	35	[illegible]	[illegible]	[illegible]	[illegible]
56	36	[illegible]	[illegible]	[illegible]	[illegible]	57	37	[illegible]	[illegible]	[illegible]	[illegible]
58	38	[illegible]	[illegible]	[illegible]	[illegible]	59	39	[illegible]	[illegible]	[illegible]	[illegible]
60	40	[illegible]	[illegible]	[illegible]	[illegible]	61	41	[illegible]	[illegible]	[illegible]	[illegible]

TABLE V-2

PLAN USED FOR SELECTION OF WEIGHTING FUNCTION BASED ON REDUNDANT TESTS

Region of t,s plane for which $\underline{H}(t,s)$ was calculated	Initial value t for which homogeneous version of the differential equation was solved
$0 \leq s \leq 20$ $s \leq t \leq 50$	0
$20 \leq s \leq 40$ $s \leq t \leq 70$	20
$40 \leq s \leq 60$ $s \leq t \leq 90$	40
$60 \leq s \leq 80$ $s \leq t \leq 110$	60
$80 \leq s \leq 130$ $s \leq t \leq 130$	80

system parameters. It should be noted that the magnitude of the response as well as its general shape change as a function of time. Since this element is the weighting function of the system when it is looked upon as a system described by the ordinary second order differential equation:

$$\ddot{y}(t) + .3\dot{y}(t) + .2y(t) = x(t)$$

where x(t) is the input and y(t) is the output, one infers that magnitude of the response of an overdamped system to a specific input is larger than that of an underdamped system, which is true.

Figure V-21 is a photograph of the normalized graphic display of the computed estimate of the time-varying weighting function element $h_{21}(t,s)$. Again the time invariant behavior of the system in certain ranges is reflected by the cuts at s = 0, 20, 40, and 80, 100. Examination of the results plotted in this figure shows that as was the case with element $h_{12}(t,s)$ the maximum magnitude of the response, as well as its general shape, is a function of time. In this case, however, the magnitude is a maximum when the system is underdamped.

Figure V-22 is a photograph of the normalized graphic display of the computed estimate of the time-varying weighting function element $h_{22}(t,s)$. As seen previously, the time invariant portions of the system response are reflected in the cuts at s = 0, 20, 40, and 80, 100; the time-varying portions are reflected in the cut at s = 60. Again, as in element

$h_{11}(t,s)$ the magnitude of the maximum response is constrained to be unity by the definition of the weighting function.

Figure V-23 is a photograph of a typical section of the tabular form of the computed estimate of the time-varying weighting function, $\underline{H}(t,s)$. With the exception of the comment that t as listed in the photograph is actually $\overline{t}$ in the notation of this report, there is little to be said about the figure. It has been included only for the sake of completeness.

CHAPTER VI

RESULTS, CONCLUSIONS, AND RECOMMENDATIONS

The results of and conclusions drawn from this investigation are presented in this chapter. The discussion has been sectioned along lines corresponding to the specific areas of interest which developed in the course of the investigation. These areas may be summarized as follows:

1. Theoretical Aspects of the Generalized Pulse Testing Technique.
2. Calculation and Presentation of the Weighting Function Estimates.
3. Acquisition of the Required Dynamic Responses.
4. Application of the Technique to an Actual Physical Process.
5. Application of the Technique to a hypothetical Time-Varying System.
6. Comparison with Other Applicable Modeling Techniques to Assess the Relative Costs.

Recommendations for future work concerned with the development of the generalized pulse testing technique appear as the final section of this chapter as they represent questions which have been left unanswered by this investigation.

Also, these areas tend to represent the current interests of the author.

Theoretical Aspects of the Generalized Pulse Testing Technique: No new theories, either mathematical or in terms of systems work, have been developed in the course of this investigation. Instead, certain results from the theory of ordinary differential equations have been combined with the concepts of that area of systems work concerned with the determination of mathematical models for physical processes. The result is a coherent structure that defines a novel method by which the characterization of linear systems, regardless of their particular time dependence, may be achieved.

The relationship between the weighting function, determined through application of this technique, and the impulse response (alternatively the transfer function), determined by the standard techniques, has been thoroughly elucidated in Chapter II. This relationship may be summarized by noting that for systems which may be described adequately by the linear differential equation:

$$\dot{Y}(t) = A(t)Y(t) + X(t) \qquad Y(T) = Z \quad T \in t$$

the behavior of the system in response to any input, $X(t)$, may be calculated explicitly from the linear integral equation:

$$Y(t) = \underline{H}(t,T)Y(T) + \int_{T}^{t} \underline{H}(t,s)X(s)\,ds \qquad t \geq s$$

where $\underline{H}(t,s)$ is that N by N fundamental matrix of solution to the homogeneous form of the differential equation which when evaluated at t = T is the unit matrix. If the A matrix is of the canonical form then the $h_{1N}(t,s)$ element of the weighting function matrix is identically the impulse response of the system for which the descriptive equation is the N^{th} order ordinary differential equation:

$$[L(t)]\ y(t) = x(t) \qquad y^{N-1}(0) = \ldots\ldots = y(0) = 0$$

In the case of time invariant systems, the transfer function is the ratio of the Laplace transforms of the output and input functions.

$$H(p) = \mathcal{L}\ \{h_{1N}(t-s)\} = y(p)/x(p)$$

Through comparison of the solution of the non-homogeneous form of the differential equation by the method of "variation of parameters" with the integral representation of the model, one is able to show that:

$$\underline{H}(t,s) = \underline{\overline{\phi}}(t)\underline{\overline{\phi}}^{-1}(s) \qquad s \leq t$$

This expression suggests that measurement of any fundamental matrix of homogeneous responses, $\underline{\overline{\phi}}(t)$, will permit characterization of the system in terms of its weighting functions. Further, since the weighting function matrix evaluated at t = s is seen to be the unit matrix, the measurement of a set of unforced system responses is recognized as a

generalization of the measurement of the impulse response of the system; hence the name: generalized pulse testing technique.

The method by which a fundamental matrix of unforced system responses in the range $t \geq T$ may be measured is stated mathematically as follows:

$$\dot{\underline{\emptyset}}(t) = A(t)\underline{\emptyset}(t) \qquad t \geq T$$

$$\underline{\emptyset}(T) = \underline{H}(T,T_o)\underline{\emptyset}(T_o) + \int_{T_o}^{T} \underline{H}(t,s)\underline{X}(s)\,ds$$

where $\underline{X}(t)$ is a non-singular matrix of system forcing functions.

The final step in the theoretical development is concerned with the relation of the weighting function to the linear differential operator, $A(t)$. The transition between the integral (weighting function) and the differential representations of the model may be defined in terms of the matrix equivalent of the homogeneous differential equation:

$$\dot{\underline{H}}(t,s) = A(t)\underline{H}(t,s) \qquad \underline{H}(t,t) = U$$

Multiplication of this equation, from the right, by the inverse of the weighting function matrix yields:

$$\dot{\underline{H}}(t,s)\underline{H}^{-1}(t,s) = A(t)$$

which simplifies to the following form provided the evaluation is $s = t$.

$$A(t) = \dot{\underline{H}}(t,t)$$

There are two constraints on the class of system to which the technique is applicable which are implied by the mathematical statements but which deserve special notation.

1. Since the differential equation which describes the system behavior is vectorial in nature, all elements of the state vector must be measurable.
2. Since the fundamental set of unforced system responses is a matrix formed by adjoining response vectors, the time dependent nature of the system must be either repetitive and measurable or controllable (and measurable).

Granting the constraints on the nature of the physical process mentioned above, it is clear that time variation of the system parameters represents no fundamental difficulty in the determination of its mathematical model by application to the technique. Further, the state space formulation used in the elucidation of the technique points up the extreme ease with which it may be applied to multiple input/output systems; the only single input/output systems actually recognized being those which are characterizable in a one dimensional state space.

Three potential advantages to the use of the technique as a practical method for the determination of mathematical models are implied by the mathematical statement of the way in which the fundamental matrix of unforced responses is to be generated.

Since the forcing function, applied prior to the arbitrary time, T, is a vector quantity, any one of its elements may be chosen for actual system manipulation. Of course the choice is made on the basis of the relative convenience and controllability of the actual signals which correspond to the elements of the vector.

The mathematical statement of the way in which a set of unforced responses is to be obtained indicates the advantage gained by the generalization of the impulse testing technique. Rather than attempting to impress an impulse of sufficient strength to permit measurement of the response, one simply drives the system away from the equilibrium state and releases it. This feature represents a significant practical advantage in the study of physical processes which characteristically display sluggish response.

A third potential advantage of the technique is that all possible input-output relationships are determined for the same amount of experimental and computational effort required for the determination of one such relationship. This advantage is particularly important in the study of multiple input/output systems.

Calculation and Presentation of Weighting Function Estimates: While the calculation of the weighting function matrix for a system is quite simple as a theoretical proposition, its application to data obtained from an actual physical process is not a trivial matter. Due to the combined

effects of nonlinearities which are unaccounted for in the model and imperfections in the measured data, the calculations are necessarily based on an estimate of the true homogeneous response of a linear system. The error contained in the measured data is severely compounded by the nature of the matrix operations required; therefore one must regard the calculated result as an estimate of the true weighting function. If this estimate is to be used with any confidence one must have an appreciation for its quality. Examples 2 and 3 in Chapter III demonstrate the validity of this statement.

The material in Chapter IV is devoted to the consideration of the effect of small errors in the data on the quality of the estimate. Recognizing that instead of calculation the weighting function as the product of the fundamental matrix, and time t, and its inverse, at time s, the estimate:

$$\underline{H}'(t,s) = (\overline{\underline{\Phi}}(t) + \Delta\overline{\underline{\Phi}}(t))(\overline{\underline{\Phi}}(s) + \Delta\overline{\underline{\Phi}}(s))^{-1}$$

is actually calculated, the error in the estimate ($\underline{H}'(t-s)$ - $\underline{H}(t,s)$) may be expressed as follows:

$$\Delta\underline{H}(t,s) = (\Delta\overline{\underline{\Phi}}(t) - \overline{\underline{\Phi}}(t)I(s)\Delta\overline{\underline{\Phi}}(s))I(s)$$

$$I(s) = \overline{\underline{\Phi}}^{-1}(s)$$

Alternatively, if a maximum error matrix (for all values of t and s), ΔM_{max}, is defined one may write:

$$\Delta \underline{H}_{max}(t,s) = (U - \underline{H}(t,s))\Delta M_{max}\overline{\emptyset}^{-1}(s)$$

These expressions for the error in the estimate suffer, from the practical point of view, in that they involve knowledge which is normally not available, namely the values of the actual weighting function and its inverse.

The error criterion has been specialized for case of the calculation of a second order weighting function matrix. This was accomplished by consideration of the calculation in two phases: matrix inversion and matrix multiplication. Defining a matrix ΔI_{max} as:

$$\Delta I_{max} = (M + E)^{-1} - M^{-1}$$

where M is the matrix of measured unforced system responses and E is that matrix of maximum errors with their polarities chosen so as to minimize the determinant of (M + E), the maximum error in the calculation of the weighting function estimate as:

$$\underline{H}'(t,s) = M(t)I(s) \qquad t \geq s$$

$$I(s) = M^{-1}(s)$$

is shown to be:

$$\Delta \underline{H}_{max}(t,s) = M(t)\Delta I_{max}(s) + EI(s) + E\Delta I_{max}(s)$$

These three equations form the basis for the digital computer program used in the investigation. A listing of the Fortran statements in the program appears in Appendix C. The procedure followed in the calculations was to consider the variable s as a parameter and the variable t as the running variable; hence the calculations correspond to cuts in the weighting function hypersurface along lines of constant s. Examples of the tabular form of the computed weighting function estimates with associated maximum error appear in Figures V-13 and V-23.

While the tabular form of presentation of the weighting function estimate is desirable for subsequent utilization, it is not desirable from the point of view of assessing the characteristics and general quality of the estimate. This statement is particularly true in the case of a time-varying system. Therefore a digital computer program was written which plots the weighting function estimates in a manner which approaches a perspective view of the weighting function surface. Examples of this form of the presentation of the computed estimates of the weighting function appear as Figures V-14, V-15, V-16, V-19, V-20, V-21, and V-22. The listing of Fortran statements for the plotting program is included in Appendix C.

Acquisition of the Required Dynamic Responses: The theory of the generalized pulse testing technique is quite specific regarding the method by which the fundamental set

of homogeneous system responses is to be obtained. The state variables are measured as the system returns to equilibrium along N (the order of the system) linearly independent trajectories. These trajectories are the system's unforced responses to a set of non-equilibrium states existing at an arbitrary time zero which, in turn, are the result of forcing the system, prior to time zero, in N linearly independent ways.

Since the measured responses of any real systems are actually estimates of true linear responses, the requirements on the set of forcing functions used are more complicated than the linear independence requirement of the theory. The question of what constitutes a desirable set of forcing functions must be answered in terms of the constitution of a desirable set of energy distributions at time zero. While this investigation produced no quantitative answers to the question, the following discussion (qualitative in nature and based mainly on experience) may be helpful.

It has been noted that the quality of the estimate of the weighting function calculated at a specific value of the dummy time variable, s, is proportional to the magnitude of the determinant of the fundamental matrix evaluated at time t = s. Further, it has been noted that as time increases (and the system approaches equilibrium), the errors contained in the response data become proportionally much larger. Finally, at some large value of time, t, there is no assurance that the measured responses are linearly independent of one another.

The obvious conclusion is that the set of energy distributions which gives rise to the largest value of the determinant when evaluated at time zero is most desirable. Consider the determinant of the fundamental matrix of a second order system for the purpose of discussion:

$$\nabla = \phi_{11}\phi_{22} - \phi_{21}\phi_{12}$$

It is clear, from this equation, that the easiest (and probably best) way to maximize the determinant is to generate a mismatch of polarities among the elements, either three positive and one negative or vice versa. This scheme works nicely for systems which are underdamped but is virtually impossible to implement if the system's response is sluggish.

A second way to maximize the determinant is to achieve a mismatch among the magnitudes of the elements, either large main diagonal elements combined with small minor diagonal elements or vice versa. This situation may be achieved easily provided that one is free to force both energy storage elements alternately. However, care must be exercised to avoid difficulties caused by the ratio of the magnitude of the small element to the maximum measurement error estimate becoming uncomfortably small.

A less obvious conclusion regarding the distribution of energies at time zero involves consideration of the "shape" of the trajectories. In certain cases it is possible to achieve distributions which give rise to convexo-concave

trajectories. This type of response, when combined with the more usual concave trajectory, may give rise to a determinant which retains a greater magnitude for larger values of time, t. In this case the rate of degeneration in the quality of the weighting function is somewhat smaller.

The comment regarding the ratio of the magnitudes of the responses to those of the error estimates points up the second important feature of a desirable set of initial energy distributions. The magnitudes of the elements must differ from their equilibrium values by an amount which is large compared to the estimated error in the measurements if high quality in the estimate of the weighting function is to be insured.

Immediately one recognizes the dilemma. From the point of view of maximizing the determinant, it is desirable to use a vector which possesses one small element; however, this policy severely limits the quality of the estimate possible at large values of the dummy time variable, s.

A reasonable policy for the choice of initial energy distributions for the study of the backmix reactor was found to include a combination of the "shape" concept with a modification of the polarity mismatch concept (two positive and two negative). The magnitudes of the vector elements at time zero were all greater than 100 times that of the maximum estimated error.

The table given below is based on information gained from the study of the heat transfer dynamics of the backmix

TABLE VI-1

COMPARISON OF DETERMINANT: ERROR RATIO WITH WEIGHTING FUNCTION ESTIMATE QUALITY BASED ON BACKMIX CHEMICAL REACTOR STUDY

Ratio: ∇ / |error|	Quality
1750 1500	Superior
1000	
620 395 334 138	Good
100	
87 72 31 11 10	Poor
10	
7	Worthless

reactor. It compares the ratio of the magnitudes of the "initial" determinants and attenuation corrected maximum errors with qualitative assessments of the quality of the calculated weighting functions.

The process of acquiring the unforced system trajectories from a real system necessarily involves conditioning of the measured signal as well as its amplification. The process of suppressing the equilibrium level of the various state variables in order to transform the total variable (which is measurable) to the transient form (which is required in the model) is quite well known and understood. A second type of signal conditioning, used in the study of the backmix reactor, is equally well known and understood but deserves mention.

It is well known that measurements of temperature in liquid streams which are in turbulent flow are characteristically noisy. This characteristic may be explained in terms of the fact that the velocity measured, at some point in space, as a function of time shows chaotic variations. Since heat transfer in this situation proceeds principally by the mechanism of convection, one must expect similar chaotic variation in temperature.

In order to describe heat transfer under these conditions one normally writes an energy balance in terms of a temperature variable which is the sum of the "time smoothed" temperature and the fluctuation about this temperature. The

definition of "time smoothed" temperature is, of course:

$$\overline{T}(t) = (1/t^*) \int_0^{t^*} T(t)\,dt = (1/t^*) \int_0^{t^*} (\overline{T}+T')\,dt$$

which implies that $\overline{T} = \overline{T}$ and $\overline{T}' = 0$. The resulting energy balance then includes a term which describes additional convective heat transfer due to the turbulent nature of the flow.

Implicit in the definition of the "time smoothed" temperature is the requirement that the period of integration, t^*, be large compared to the period of the chaotic fluctuations but small compared to variations in temperature caused by dynamic behavior.

The temperatures which were measured in the experimental phase of the study of the backmix reactor displayed this characteristic low amplitude, very high frequency fluctuation. Therefore the measurements were "time smoothed" in the signal conditioning system as elucidated in Appendix D. It should be mentioned that fluctuations which appeared in the measurement of the wall temperature (which were attributed to electronic noise) were also eliminated by the "time smoothing" process implemented in the signal conditioning system.

Since the study of the backmix reactor involved operation of real experimental apparatus, which cannot respond

instantaneously, that point of the recorded system trajectories which corresponds to the initial instant of unforced response cannot be selected with precision. The procedure followed was to choose a point well separated from "time zero" at which the inlet temperature had clearly returned to its equilibrium value. This choice was made on the basis of the catastrophic effect a selection of a premature time has on the calculated estimate of the weighting function (see Figure IV-13).

While this procedure had no particularly adverse effect on the study of the time invariant system, the situation would have been more serious had the system under study been time-varying. In that case the largest value of time at which unforced response begins would define the smallest value of $t = s$ for which the technique would be applicable.

The final comment regarding the acquisition of the unforced response data is concerned with the problem of reducing the recordings of the conditioned system response data to a form suitable for use in the computational phase of a given study. The procedure followed is described in Chapter V and in Appendix D. Briefly, it consisted of transcribing the measured temperatures from stripchart recordings to tabular form, reassembling the incrementally zero suppressed response trajectories, and generating a deck of punched cards for use as input to the computer program. This task is not trivial.

However, it is the opinion of the author that it would be unrealistic to count this task as a major portion of the cost of employing the generalized pulse technique to the study of a physical system, as subsequent studies would surely employ more sophisticated instrumentation which would avoid the necessity for the transcription and reassembly portions of the task.

Application of the Technique to an Actual Physical Process: As stated in Chapter V, the purpose of the study of the heat transfer dynamics of the backmix chemical reactor was twofold. 1) The study was used to determine whether or not the nature of the required matrix operations would preclude use of the technique as a practical means for obtaining simple mathematical models for physical processes. 2) Assuming that reasonable estimates of the weighting function may be calculated, the study provided a means by which the cost, in terms of both experimental and computational effort, could be assessed.

The results of the three sets of weighting function estimate calculations made in this study appear in Figures V-13, V-14, V-15, and V-16, the latter three figures being graphical displays of the computed estimates. These figures indicate that smooth estimates of the weighting function, consistent within the maximum error bounds, were produced in the computation phase of the study.

Since the three sets of weighting function estimates are superposable (within the error bounds), one must conclude

that the variation of the system parameters with temperature, in the range studied, has a negligible effect on the system's response.

The graphical displays of the weighting function estimates, do not provide information on the maximum error associated therewith. To evaluate the quality further, consider the plot of a typical estimate with its associated maximum error bounds presented in Figure VI-1. This Figure depicts the cut in the normalized weighting function surface along the line s = 0 as computed from data acquired from the experiments designated Group II, Runs 7 and 8.

Figure VI-1 reveals that the first order weighting function estimates are quite smooth and are characterized by a very narrow band of predicted maximum error. The second order weighting functions are also quite smooth but have a much wider maximum error band associated with them. The width of the error bands reflects the fact that, while the magnitudes of the first and second order weighting functions are quite different, the magnitudes of the maximum error associated with their computation by matrix operations are similar. The normalization process presents the error on what is essentially a percentage basis:

$$h'_{11}(t-s) = h_{11}(t-s) \pm 0.015h'_{11}(t-s)_{max}$$

$$h'_{21}(t-s) = h_{21}(t-s) \pm 0.143h'_{21}(t-s)_{max}$$

$$h'_{12}(t-s) = h_{12}(t-s) \pm 0.133h'_{12}(t-s)_{max}$$

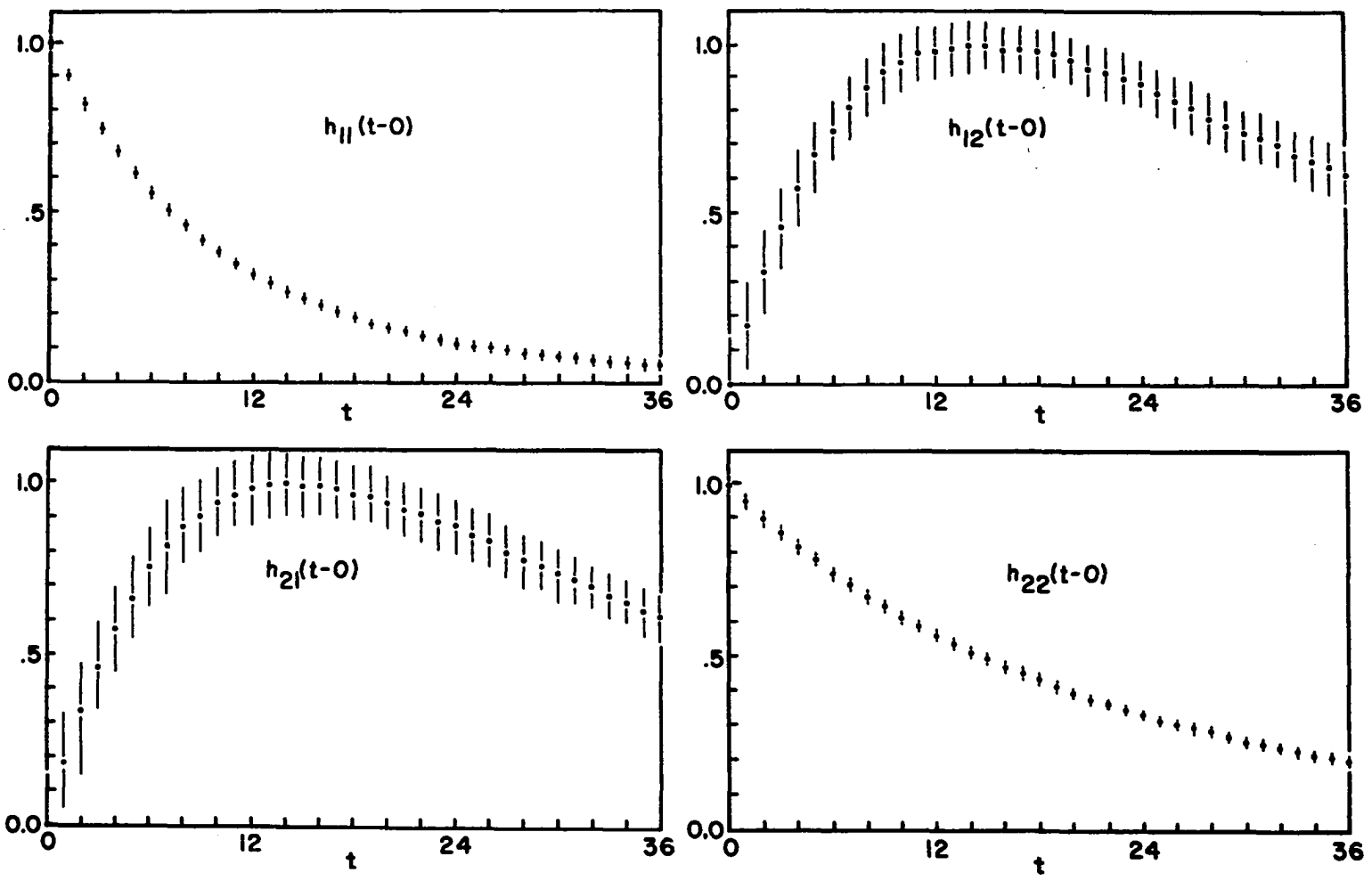


FIGURE VI-1
BACKMIX REACTOR WEIGHTING FUNCTION ESTIMATE, H(t-0)
DATA : GROUP II RUNS 7 and 8

$$h'_{22}(t-s) = h_{22}(t-s) \pm 0.020 h'_{22}(t-s)_{max}$$

It is necessary to bear in mind, however, that these percentages represent the maximum error in the estimates. The smoothness of the estimates implies that they are much better than required by the maximum error predictions.

In order to assess the validity of the model consider the roughestimate of the A matrix obtainable by extrapolating the first differences in the calculated weighting function to time, t = 0. This procedure is an approximation of the relationship:

$$A = \dot{\underline{H}}(t,t) \qquad t = 0$$

As indicated by Figure VI-2, the approximate A matrix is:

$$A = \begin{array}{|c|c|} \hline -0.098 & 0.038 \\ \hline 0.018 & -0.051 \\ \hline \end{array} \quad [=] \ sec^{-1}$$

This information, combined with the measured flow rate (F = 0.6 gal./min.) and physical properties of the process fluid and reactor wall material given by Haskins (21) as:

$C_{p\ oil} = 0.538$ BTU/lb. °F.

$\rho_{oil} = 52.3$ lbs/ft.^{3}

$C_{p\ wall} = 0.042$ BTU/lb. °F.

$\rho_{wall} = 603.2$ lbs/ft.^{3}

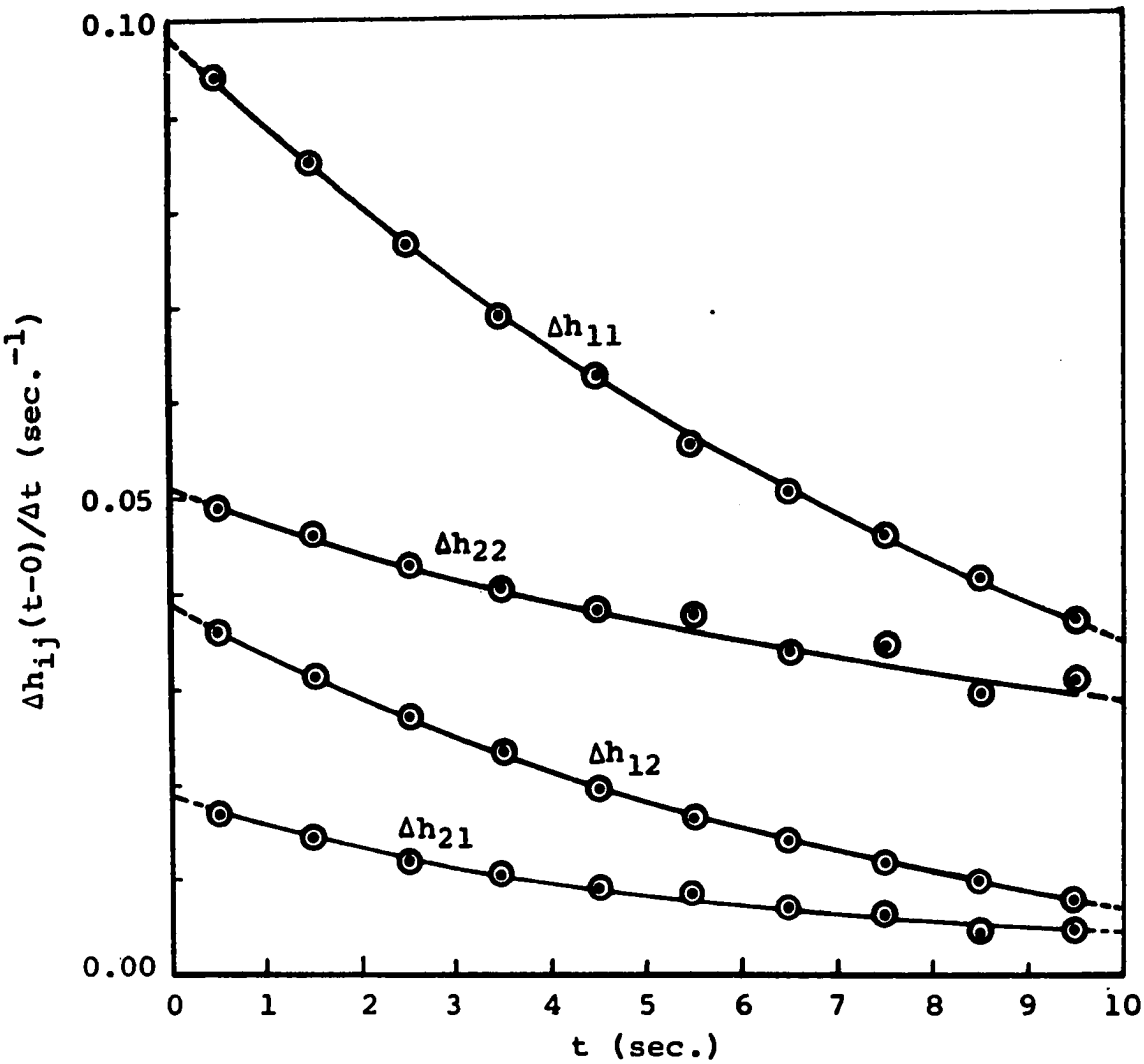


Figure VI-2

Graphical Determination of the Elements

of the Coefficient Matrix

permit calculation of certain system parameters as follows:

$$V_b = -F(vol)/(a_{11} + a_{12})$$

$$h_iA_i = a_{12}V_b\rho_{oil}\ C_{p\ oil}$$

$$V_w = (a_{12}V_b\rho_{oil}\ C_{p\ oil})/(a_{21}\rho_{wall}\ C_{p\ wall})$$

$$h_oA_o = -(a_{22} + a_{21})V_w\ \rho_{wall}\ C_{p\ wall}$$

Haskins estimated V_b and V_w from measurements made on the printer's type metal tube which was cast for use as a reactor wall (see Chapter V) and estimated h_iA_i and h_oA_o from an energy balance for steady state operation of the reactor. A comparison of the values for these parameters as determined in the two investigations appears in Table VI-2.

It is obvious that there are significant differences in the estimates taken from the two investigations and it is of interest to attempt to rationalize them.

TABLE VI-2

COMPARISON OF SYSTEM PARAMETERS

Parameter	This Investigation	Haskins (21)
V_b = ft.^{3}	0.023	0.0287
V_w = ft.^{3}	0.053	0.0246
h_iA_i = BTU/hr.°F.	85.9	25.6
h_oA_o = BTU/hr.°F.	157.0	26.3

The volume of the reactor, V_b, calculated from measurements of the cast tube dimensions neglects the volume of the agitator and the thermocouple well (estimated as 0.0028 ft^3). Therefore the fact that the volume determined in this investigation is smaller than that estimated by Haskins is not surprising.

The difference in the magnitude of the estimates of the reactor wall volume, V_w, may be attributed to the fact that the estimate calculated in this investigation is an apparent value. As was pointed out in the section of Chapter V which dealt with the assumptions necessary to obtain a simple model, the effect of the "additional" wall volume due to intimate contact between the lower section of the reactor head and the thick walled tube is neglected in the model. Therefore one might expect the effective volume of the wall to be larger than that of the tube.

The steady state experiments from which Haskins estimated the heat transfer parameters were conducted with the reactor operating at lower process fluid temperature and flow rates than those chosen as the equilibrium state in this investigation. This fact accounts for the direction, but probably not the magnitude, of the difference in the estimates of these parameters. It is interesting to note that since the heat flux out of the reactor wall is inversely proportional to the wall volume (see energy balance on reactor wall), the large apparent wall volume determined in this investigation is partially responsible for the large value of $h_o A_o$.

Finally, one must recognize that the estimation of these parameters involved calculations based on handbook data and experimental measurements which probably are not perfectly accurate and precise.

Based on the smoothness of the estimates of the weighting functions, the relatively small maximum error predictions, and the qualitative consistency with Haskins' estimates of the reactor parameters, it is the conclusion of the author that mathematical model of the backmix reactor's heat transfer dynamics is valid. As a consequence of the success obtained in this study one must conclude that the generalized pulse testing technique may be employed with data of the quality obtainable experimentally.

The cost of employing the generalized pulse testing technique for the determination of the mathematical model of a physical system's dynamic behavior may be separated into three areas: data acquisition, data reduction, and estimate computation. Since this cost obviously depends largely on the nature of the physical system, only the cost of the techniques application to the backmix reactor can be discussed.

The cost of acquiring the raw data in this study involved the design and construction of a special signal transducer to be used in the manipulation of the temperature of the process fluid as it entered the reactor, development of an appropriate data acquisition and signal conditioning system from laboratory instruments and analog computer components, and the expenditure of the man hours required to

conduct the experiments. These costs are nominally the same as those associated with the acquisition of data for model determination by either frequency response or impulse response techniques.

The cost, in terms of man-hours expended, of reducing the raw data to a form suitable for use with the digital computer program was very high. This cost is that associated with "reading" strip chart recordings of the elements of the unforced responses as segmented by the incremental zero suppression equipment, assembling this information into tabular representations of the response vectors, and generating the deck of cards which served as input to the program. While this cost was high, it must be pointed out that it represents a cost which could be eliminated from subsequent studies by the purchase of appropriate recording equipment, thus trading dollars spent once for generally useful equipment against numberless tedious hours.

The cost of computing the estimate of the weighting function may be specialized in terms of the rental rates for time on a specific computer. One hour of IBM 7072 time at the rate of $200/hr. was required for the calculation and presentation of the three estimates of the weighting functions of the backmix reactor presented in Chapter V as Figures V-13, V-14, V-15, and V-16. It should be noted that this time does not include the IBM 1410 time required to print the computed estimates (twenty-one minutes).

Application of the Technique to a Hypothetical Time-Varying System: As stated in Chapter V, the purpose of the study of the dynamic behavior of the hypothetical time-varying system was threefold. 1) The study provided a means by which the power, relative simplicity, and versatility of the technique could be demonstrated. 2) It provided a partial estimate of the cost of applying the technique to a time-varying system. 3) It served as a vehicle for the introduction and discussion of the concept of redundant testing.

Graphical displays of the computed estimates of the elements of the system weighting functions were presented in Figures V-19, V-20, V-21, and V-22. These displays contain the best estimates of the weighting functions (selected on the basis of the benefits of redundant testing) available from five sets of calculations.

The nature of the input data (digital solution of a time-varying differential equation) was such that very small maximum computational errors were predicted (see Figure V-23), therefore the presentation is made mainly for tutorial purposes.

The characteristic behavior of the weighting functions for this particular system involves variation in the maximum amplitude of the second order weighting functions as the nature of the system changes from underdamped to overdamped.

The parameters of the hypothetical system were chosen such that its homogeneous response was essentially complete

after fifty units of time had elapsed. A second feature of the system was that the parametric variation did not begin until time zero plus fifty units; therefore this variation could not be measured in the homogeneous responses beginning at time zero.

The concept of redundant testing was introduced to obtain the information necessary to the elucidation of the time-varying nature of the system. Implementation of this concept involves repetition of the system tests at choices of the initial instant of homogeneous response greater than time zero (obviously redundant for time invariant systems). The result, for time-varying systems, is homogeneous response information at larger values of t and s (arguments of the weighting function) which permit estimation of the model. Since this procedure permits duplication of the computed estimates at certain points on the $\underline{H}(t,s)$ surface, the more reliable estimate may be selected for subsequent use (or presentation).

The cost which may be assessed from this study is purely that of computing the weighting function estimates from information of the homogeneous response of the system. As in the case of the backmix reactor study the cost may be expressed in terms of the rental rate for computer time and the required time for calculation. The calculations from which the information plotted in the graphical displays was selected required one hour and eighteen minutes at $200/hr.

or $260. The time required to plot the selected results was approximately twelve minutes at $200/hr. ($40) plus approximately six minutes of IBM 1410 time.

It must be concluded that the technique has shown great promise as a tool for the investigation of time-varying physical systems provided the required response information can be obtained (a problem not specifically considered in this investigation).

Assessment of Comparative Costs: The assessment of the costs associated with applying any particular technique to the determination of the mathematical model for a physical process is quite difficult. Clearly the cost involves consideration of the nature of the particular process, the computational facilities available, and so forth. Further, any attempt to compare the relative costs of the various methods involves a value judgment on the part of the assessor which involves personal preferences. Therefore, the standard techniques which have applicability to general types of systems are compared on an "order of preference by the author" basis. Experience in the application of each of the techniques to single input-output time invariant physical processes and at least computational experience in the application of impulse response and statistical techniques to multiple input/output physical systems should insure some degree of validity to the comparison.

Designating the various techniques as follows:

Generalized Pulse = 1

Impulse = 2

Frequency Response = 3

Statistical = 4

the comparison is given in Table VI-3.

TABLE VI-3

COMPARISON OF ORDER OF PREFERENCE AMONG VARIOUS MODELING TECHNIQUES FOR VARIOUS TYPES OF SYSTEMS

(Order: most . . . . . least)

System Types	Time Dependence of Parameters	
	Time Invariant	Time-Varying
Single Input/Output	2,3,..(1/4)	(2/1),..4
Multiple Input/Output	1,2,4,3	1,2,....4

Recommendations for Future Work: There are two fundamental constraints on the generalized pulse testing technique which limit its usefulness as a tool for the determination of mathematical models of physical processes. The first of these is the requirement that each element of the system's unforced response vector must be measurable. The second constraint is the requirement that the system's dynamic response be described in terms of linear differential equations.

It is usually necessary to simplify the mathematical description of a physical process somewhat in order to apply any of the standard data acquisition techniques on a reasonable basis. In the case of the generalized pulse testing technique, as elucidated herein, this procedure often involves generation of "imaginary" state variables which cannot be physically measured.

This situation arose in the study of the backmix reactor's heat transfer dynamics at the point in the model derivation at which the reactor wall temperature was introduced. It was assumed that the wall temperature is not a function of any space variable. The question of measuring this non-existant variable was answered in this case by the measurement of the temperature at a geometric location thought to be roughly characteristic of the entire wall.

Clearly this approach is not usually reasonable. Therefore, an investigation of the possibilities of making a linear transformation (change of base of the state vector space) which would permit description of the system behavior in terms of all physically measurable quantities would be valuable.

The constraint of linear behavior by the physical system to which the generalized pulse testing technique is applicable is, of course, the source of a large area for future work. As is obvious, from consideration of the simplifying assumptions made in order to cast the energy balances

on the contents of the reactor and its wall into the class of equations required, the backmix chemical reactor was actually a slightly non-linear system.

None the less, a linear mathematical model which is valid in the range of temperature covered was obtained. Answers to the usual questions of model applicability and range of validity are complicated in the case of the technique used herein because of the sensitive nature of the matrix calculations involved. Since the computations do not differentiate between deviations from linear behavior which are due to measurement error and those due to system non-linearities, it is important to define the extent of non-linearity which is tolerable.

The necessity of finding an alternative method for making the transition from the integral to the differential representation of the model is emphasized by consideration of the propagation of error associated with the weighting function estimates.

The author is currently investigating one such possibility. It involves optimization of the coefficients of the matrix in question by minimizing the mean square error between computed weighting function estimates and the analytic weighting functions obtained by solution of the vector differential equation. Of course this approach is an iterative procedure which uses the A matrix obtained by numerical differentiation as an initial differential representation of the system's behavior.

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APPENDIX A

NOMENCLATURE

Upper Case Letters

AHRV	=	Assembled homogeneous response vector
A_i	=	Heat transfer area (inside reactor wall)
A_0	=	Heat transfer area (outside reactor wall)
A(t)	=	N by N coefficient matrix of differential form of the mathematical model
C	=	Capacitance
C_{pb}	=	Heat capacity of process fluid
C_{pw}	=	Heat capacity of reactor wall (printer's typemetal)
E	=	N by N matrix of errors in measurement of a fundamental matrix with polarities chosen to minimize its determinant
F	=	Process fluid flow rate
H(p)	=	System transfer function
$\underline{H}(t,s)$	=	N by N matrix of system weighting functions
$\underline{H}_i(t,s)$	=	Weighting function in the ith region of t,s plane
$\underline{H}'(t,s)$	=	Computed estimate of the weighting function
I(s)	=	The inverse of a fundamental matrix; the sum of this inverse and $\Delta I(s)$
K_i	=	A specific constant

L	=	Inductance; linear differential operator (in square brackets)
M	=	N by N matrix of measured values of the fundamental matrix; amplitude of a sine wave; linear differential operator (in square brackets)
M*	=	The matrix sum of M and E
P(t)	=	N vector of weighting functions
Q	=	Any quantity
R	=	Resistance
RD	=	Reading (from strip chart recording)
S	=	A specific value of s (dummy time)
S_1	=	A specific value of s
T	=	A specific value of t (time)
T_b	=	Temperature of process fluid in reactor
T_c	=	Temperature of coolant (imaginary)
T_{ci}	=	Temperature of coolant (inlet)
T_{co}	=	Temperature of coolant (outlet)
T_{in}	=	Temperature of process fluid at reactor inlet
T_w	=	Temperature of reactor wall
$\underline{T}$	=	N by N state transition matrix
T*	=	A specific value of t; steady state temperature
T_0	=	A specific value of t
T_1	=	A specific value of t
U	=	Identity matrix
V_b	=	Reactor volume
V_w	=	Reactor wall volume
X(p)	=	Laplace transform of system input

$X(t)$ = N vector representing system input

$Y(p)$ = Laplace transform of system output

$Y(t)$ = N vector representing system output

Z = N vector representing system output at specific t

ZSI = Zero Suppression Increment

Lower Case Letters

a_i = A coefficient of an Nth order differential equation

$a_{ij}(t)$ = An element of the coefficient matrix A(t)

c_i = An arbitrary constant

dt = A differential period of time

$e(t)$ = Voltage

$f_{ij}(t)$ = A measurable element of the coefficient matrix A(t)

$h_i(t)$ = Convective heat transfer coefficient (inside reactor wall)

$h_o(t)$ = Convective heat transfer coefficient (outside reactor wall)

$h_i(t)$ = Solution to differential equation under constraints $h(0) = 1$, $\dot{h}(0) = 0$ or $h(0) = 0$, $\dot{h}(0) = 1$

$h_{ij}(t,s)$ = An element of the weighting function matrix $\underline{H}(t,s)$

$i(t)$ = Current

$i_{ij}(s)$ = An element of the inverse of the fundamental matrix I(s)

k = A specific constraint on the solution to a differential equation

k' = A specific constraint on the derivative of the solution to a differential equation

l	=	Any signal
m_{ij}	=	An element of the matrix of measured values M
p	=	The argument of the Laplace transformation
s	=	A dummy variable for t (time)
ssc	=	Steady state correction
t	=	Time
t^+	=	A specific value of t
$\overline{t}$	=	Scaled time
t^*	=	An increment of time
$t_{ij}(t)$	=	An element of the state transition matrix $\underline{T}$
w	=	Frequency of a sine wave
$x(t)$	=	An element of the N vector X(t)
$y(t)$	=	An element of the N vector Y(t)
z	=	An element of the N vector Z

Greek and Special Symbols

Δ	=	A small amount
$\Delta\underline{H}(t,s)$	=	The difference between $\underline{H}'(t,s)$ and $\underline{H}(t,s)$
$\Delta\underline{H}_{max}(t,s)$	=	Maximum error in a system weighting function estimate
$\Delta I(s)$	=	Error in the inverse of the fundamental matrix
ΔM_{max}	=	Maximum error in measured fundamental matrix
$\Delta\overline{\underline{\emptyset}}(t)$	=	Error in fundamental matrix
$\overline{\underline{\emptyset}}(t)$	=	Fundamental matrix
δ	=	Dirac delta function (unit impulse)
ρ_b	=	Process fluid density
ρ_w	=	Printer's typemetal density

$\phi(t)$ = A fundamental response of the system (N vector)

$\phi_{ij}(t)$ = An element of the fundamental matrix

$\cdot$ = Time derivative

∂ = Partial derivative

∇ = Determinant of a matrix

∇^2 = Laplacian differential operator

$\mathcal{L}$ = Laplace transformation operator

APPENDIX B

EXPERIMENTAL DATA

This appendix is devoted to presentation of the raw experimental data from Group II runs 1, 6, 7, 8, the corresponding reduced data (homogeneous system response vectors), and the combination of reduced data and values from the smooth curves of Figures V-11 and V-12 which were used as input to the digital computer program for calculation of the weighting function estimates.

Table A-1 contains tabulated values of pen position, in m.m., at one second intervals as recorded from the strip chart recordings of the state variables (T_b and T_w). The numbers which appear in the columns immediately to the right of the data are to be added to it in order to assemble the response vectors. They represent appropriately summed zero suppression increments and steady state corrections.

While the steady state correction is obtained from the strip chart recording, the contribution to the sum of incremental zero suppression is obtained from a knowledge of the initial increment and the suppression policy. The individual suppression increments are listed in Figure A-1.

TABLE A-1

RAW DATA WORKSHEET

SOURCE Group II, Run I SYSTEM Backmix Reactor SIGMA Time Invariant

COMPONENT Bulk FORCED T inlet FUNCTION Pos. Step

ATTENUATION 0.10°F/mm STEADY STATE COR. -10.0 mm READING ERROR 0.25 mm

DATES: RUN 10/12/63 READ 3/30/64 REDUCED 5/25/64 RUN/page II/1/1

No.	Data	No.	Data	No.	Data	No.	Data
1	48.5 -10.0	26	22.8 -10.0	51	14.7 -10.0	76	11.8
2	46.7	27	22.3	52	14.5	77	11.7
3	44.9	28	21.8	53	14.4	78	11.6
4	43.3	29	21.3	54	14.2	79	11.6
5	41.8	30	20.9	55	14.0	80	11.5
6	40.3	31	20.4	56	13.9	81	11.5
7	39.0	32	20.0	57	13.7	82	11.4
8	37.7	33	19.6	58	13.6	83	11.4
9	36.4	34	19.3	59	13.4	84	11.3
10	35.4	35	18.5	60	13.3	85	11.1
11	34.2	36	18.0	61	13.2	86	11.1
12	33.1	37	18.0	62	13.1	87	11.0
13	32.1	38	17.8	63	12.9	88	11.0
14	31.2	39	17.6	64	12.8	89	11.0
15	30.3	40	17.3	65	12.7	90	11.0
16	29.4	41	17.0	66	12.6	91	11.0
17	28.6	42	16.7	67	12.5	92	11.0
18	28.0	43	16.5	68	12.4	93	11.0
19	27.1	44	16.2	69	12.5	94	10.9
20	26.5	45	16.0	70	12.2	95	10.8
21	25.7	46	15.7	71	12.5	96	10.8
22	25.1	47	15.5	72	12.3	97	10.7
23	24.5	48	15.3	73	12.0	98	10.5
24	23.9	49	15.1	74	11.9	99	10.5
25	23.2	50	14.9	75	11.8	100	10.6

RAW DATA WORKSHEET

SOURCE Group II, Run I SYSTEM Backmix Reactor SIGMA Time Invariant

COMPONENT Wall FORCED T inlet FUNCTION Pos. Step

ATTENUATION 0.05°F/mm STEADY STATE COR. -19.2 mm READING ERROR 0.25 mm

DATES: RUN 10/12/63 READ 3/30/64 REDUCED 5/25/64 RUN/page II/1/2

No.	Data		No.	Data		No.	Data	No.	Data
1	31.4	+66.5	26	12.0		51	33.5	76	24.6
2	27.8		27	10.5		52	32.9	77	24.4
3	24.4		28	9.1		53	32.4	78	24.2
4	21.1		29	7.8		54	31.9	79	24.0
5	17.9		30	6.5		55	31.4	80	23.8
6	14.7		31	5.3		56	30.9	81	24.2
7	11.7		32	4.1		57	30.5	82	24.2
8	49.5	+25.8	33	48.0	-19.2	58	30.1	83	24.2
9	46.7		34	46.9		59	29.4	84	23.5
10	44.0		35	45.8		60	29.2	85	23.0
11	41.4		36	44.8		61	28.9	86	22.5
12	38.9		37	43.8		62	28.5	87	23.0
13	36.5		38	42.9		63	28.1	88	23.6
14	34.2		39	41.5		64	27.8	89	23.1
15	31.9		40	40.5		65	27.5	90	22.8
16	29.7		41	40.0		66	27.1	91	22.2
17	27.6		42	39.5		67	26.9	92	21.8
18	25.7		43	38.7		68	26.9	93	21.2
19	23.8		44	37.9		69	26.7	94	21.2
20	21.8		45	37.2		70	26.0	95	21.2
21	20.0		46	36.5		71	25.2	96	21.9
22	18.3		47	35.9		72	25.5	97	21.2
23	16.6		48	35.2		73	25.2	98	21.3
24	15.0		49	34.6		74	24.7	99	21.5
25	13.4		50	34.0		75	24.6	100	21.2

RAW DATA WORKSHEET

SOURCE Group II, Run 6 SYSTEM Backmix Reactor SIGMA Time Invariant

COMPONENT Bulk FORCED T inlet FUNCTION Sine Wave (-)

ATTENUATION 0.10°F/mm STEADY STATE COR. -26.8 mm READING ERROR 0.25 mm

DATES: RUN 10/12/63 READ 3/31/64 REDUCED 5/25/64 RUN/page I/6/1

No.	Data		No.	Data		No.	Data	No.	Data
1	26.2	-113.3	26	4.2	-26.8	51	18.9	76	23.9
2	31.7		27	5.2		52	19.2	77	23.3
3	36.8		28	6.1		53	19.5	78	24.1
4	41.4		29	7.0		54	19.8	79	24.2
5	45.7		30	7.9		55	20.1	80	24.3
6	5.6	-69.2	31	8.7		56	20.4	81	24.4
7	9.3		32	9.5		57	20.6	82	24.5
8	12.7		33	10.2		58	20.8	83	24.0
9	15.8		34	10.9		59	21.1	84	24.3
10	18.8		35	11.6		60	21.3	85	24.7
11	21.5		36	12.2		61	21.5	86	24.8
12	24.0		37	12.8		62	21.9	87	24.7
13	26.4		38	13.4		63	22.3	88	25.0
14	28.6		39	13.9		64	22.5	89	26.2
15	30.7		40	14.5		65	22.6	90	26.1
16	32.6		41	15.0		66	22.8	91	24.8
17	34.4		42	15.4		67	22.6	92	25.2
18	36.1		43	15.9		68	22.8	93	25.3
19	37.7		44	16.3		69	22.9	94	25.0
20	39.2		45	16.8		70	23.1	95	25.3
21	40.6		46	17.2		71	23.2	96	25.2
22	42.0		47	17.5		72	23.4	97	25.2
23	43.2		48	17.9		73	23.5	98	24.9
24	44.4		49	18.3		74	23.3	99	24.8
25	45.5		50	18.6		75	23.3	100	24.8

RAW DATA WORKSHEET

SOURCE Group II,Run 6 SYSTEM Backmix Reactor SIGMA Time Invariant

COMPONENT Wall FORCED T inlet FUNCTION Sine Wave (-)

ATTENUATION 0.05°F/mm STEADY STATE COR.-21.6 mm READING ERROR 0.25 mm

DATES: RUN 10/12/63 READ 3/31/64 REDUCED 5/26/64 RUN/page II/6/2

No.	Data		No.	Data	No.	Data		No.	Data
1	46.2	-195.2	26	6.2	51	44.4		76	12.1
2	5.7	-150.2	27	8.5	52	45.3		77	12.9
3	10.2		28	10.7	53	46.2		78	13.7
4	14.6		29	12.8	54	47.3		79	14.6
5	19.0		30	14.9	55	2.1	-21.6	80	14.8
6	23.3		31	16.9	56	3.6		81	14.8
7	27.5		32	18.8	57	4.3		82	14.8
8	31.6		33	20.6	58	4.3		83	14.8
9	35.6		34	22.4	59	4.3		84	15.1
10	39.5		35	24.1	60	5.0		85	15.4
11	43.3		36	25.7	61	5.7		86	15.6
12	6.3	-109.5	37	27.3	62	6.3		87	15.8
13	9.9		38	28.9	63	7.1		88	15.3
14	13.4		39	30.3	64	7.8		89	15.8
15	16.8		40	31.8	65	7.8		90	16.5
16	20.1		41	33.1	66	7.9		91	16.7
17	23.3		42	34.4	67	9.0		92	16.9
18	26.4		43	35.7	68	9.8		93	16.7
19	29.5		44	36.9	69	9.9		94	16.8
20	32.4		45	38.1	70	10.6		95	17.4
21	35.2		46	39.2	71	10.8		96	17.5
22	37.9		47	40.3	72	11.2		97	18.2
23	40.5		48	41.4	73	11.6		98	18.3
24	1.4	-67.8	49	42.4	74	12.0		99	18.5
25	3.8		50	43.3	75	12.4		100	18.6

RAW DATA WORKSHEET

SOURCE Group II, Run 7 SYSTEM Backmix Reactor SIGMA Time Invariant

COMPONENT Bulk FORCED T inlet FUNCTION Sine Wave (+)

ATTENUATION 0.10°F/mm STEADY STATE COR. -19.8 mm READING ERROR 0.25 mm

DATES: RUN 10/12/63 READ 3/30/64 REDUCED 5/27/64 RUN/page II/7/1

No.	Data		No.	Data	No.	Data	No.	Data
1	18.7	+66.7	26	41.2	51	27.2	76	22.8
2	13.1		27	40.2	52	26.9	77	22.6
3	8.0		28	39.3	53	26.6	78	22.4
4	3.3		29	38.4	54	26.3	79	22.3
5	43.1	+22.6	30	37.6	55	26.1	80	22.2
6	39.1		31	36.8	56	25.8	81	22.0
7	35.4		32	36.1	57	25.6	82	21.9
8	32.0		33	35.4	58	25.4	83	21.8
9	28.9		34	34.7	59	25.1	84	21.8
10	26.0		35	34.1	60	24.9	85	21.7
11	23.3		36	33.5	61	24.7	86	21.6
12	20.8		37	32.9	62	24.5	87	21.5
13	18.4		38	32.4	63	24.4	88	21.5
14	16.2		39	31.9	64	24.2	89	21.5
15	14.2		40	31.4	65	24.0	90	21.3
16	12.3		41	30.9	66	23.8	91	21.4
17	10.5		42	30.4	67	23.8	92	21.4
18	8.9		43	30.0	68	23.5	93	21.2
19	7.3		44	29.6	69	23.4	94	21.3
20	5.9		45	29.2	70	23.3	95	21.2
21	46.9	-19.8	46	28.8	71	23.1	96	21.1
22	45.6		47	28.5	72	23.0	97	21.0
23	44.4		48	28.1	73	23.0	98	21.0
24	43.3		49	27.8	74	22.9	99	20.9
25	42.2		50	27.5	75	22.8	100	21.0

RAW DATA WORKSHEET

SOURCE Group II, Run 7 SYSTEM Backmix Reactor SIGMA Time Invariant

COMPONENT Wall FORCED T inlet FUNCTION Sine Wave (+)

ATTENUATION 0.05°F/mm STEADY STATE COR.-26.8 READING ERROR 0.25 mm

DATES: RUN 10/12/63 READ 3/31/64 REDUCED 5/27/64 RUN/page II/7/2

No.	Data		No.	Data	No.	Data		No.	Data
1	35.0	+101.8	26	37.9	51	2.5		76	35.1
2	31.0		27	35.8	52	1.7		77	35.1
3	27.0		28	33.8	53	0.9		78	35.2
4	23.1		29	31.8	54	46.3	-26.8	79	34.2
5	19.2		30	29.9	55	45.6		80	33.9
6	15.4		31	28.1	56	44.9		81	33.8
7	11.6		32	26.3	57	44.2		82	33.4
8	7.9		33	24.6	58	43.5		83	33.2
9	4.2		34	22.9	59	42.9		84	32.8
10	41.4	+61.1	35	21.3	60	42.3		85	32.6
11	37.9		36	19.8	61	41.7		86	32.2
12	34.5		37	18.3	62	41.1		87	31.7
13	31.2		38	16.9	63	41.0		88	31.3
14	28.0		39	15.5	64	40.6		89	31.3
15	24.9		40	14.2	65	40.1		90	31.0
16	21.9		41	12.9	66	39.6		91	31.1
17	18.9		42	11.7	67	38.6		92	30.6
18	16.1		43	10.5	68	38.1		93	30.7
19	13.3		44	9.4	69	37.7		94	30.7
20	10.6		45	8.3	70	37.3		95	30.6
21	49.7	+19.4	46	7.2	71	36.9		96	30.3
22	47.2		47	6.2	72	36.5		97	29.6
23	44.8		48	5.3	73	36.1		98	29.4
24	42.4		49	4.3	74	35.8		99	29.8
25	40.1		50	3.4	75	35.4		100	29.3

RAW DATA WORKSHEET

SOURCE Group II, Run 8 SYSTEM Backmix Reactor SIGMA Time Invariant

COMPONENT Bulk FORCED T inlet FUNCTION Neg. Step

ATTENUATION 0.10°F/mm STEADY STATE COR. -20.8 mm READING ERROR 0.25 mm

DATES: RUN 10/12/63 READ 4/2/64 REDUCED 5/27/64 RUN/page II/8/1

No.	Data		No.	Data		No.	Data	No.	Data
1	35.5	-156.1	26	40.7		51	13.8	76	18.2
2	45.0		27	41.9		52	14.1	77	18.3
3	4.8	-107.3	28	43.0		53	14.3	78	18.4
4	12.6		29	1.6	-20.8	54	14.6	79	18.5
5	19.7		30	2.5		55	14.9	80	18.6
6	26.2		31	3.4		56	15.1	81	18.7
7	32.1		32	4.3		57	15.3	82	18.9
8	37.5		33	5.1		58	15.5	83	18.9
9	42.4		34	5.8		59	15.8	84	18.9
10	46.9		35	6.5		60	16.0	85	19.3
11	6.9	-63.2	36	7.2		61	16.1	86	19.5
12	10.7		37	7.8		62	16.3	87	19.4
13	14.1		38	8.4		63	16.5	88	19.3
14	17.3		39	8.9		64	16.7	89	19.3
15	20.2		40	9.5		65	16.8	90	19.4
16	22.9		41	10.0		66	17.0	91	19.4
17	25.3		42	10.4		67	17.1	92	19.4
18	27.6		43	10.9		68	17.3	93	19.5
19	29.7		44	11.3		69	17.4	94	19.5
20	31.7		45	11.7		70	17.6	95	19.6
21	33.5		46	12.1		71	17.8	96	19.6
22	35.1		47	12.5		72	17.8	97	19.7
23	36.7		48	12.8		73	17.9	98	19.7
24	38.1		49	13.2		74	18.0	99	19.8
25	39.5		50	13.5		75	18.1	100	19.8

RAW DATA WORKSHEET

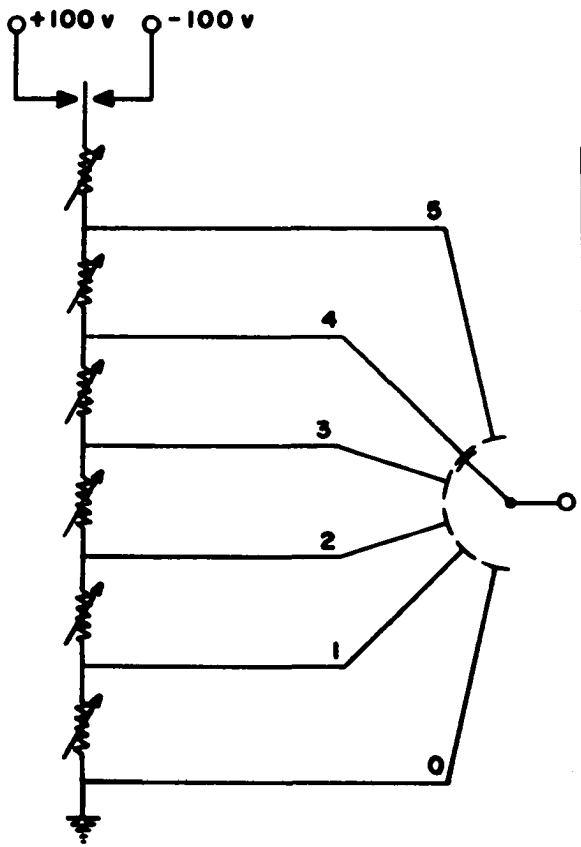
SOURCE Group II, Run 8 SYSTEM Backmix Reactor SIGMA Time Invariant

COMPONENT Wall FORCED T inlet FUNCTION Neg. Step

ATTENUATION 0.05°F/mm STEADY STATE COR. -10.4 mm READING ERROR 0.25 mm

DATES: RUN 10/12/63 READ 4/2/64 REDUCED 5/28/64 RUN/page II/8/2

No.	Data		No.	Data	No.	Data	No.	Data	
1	32.5	-137.8	26	3.3	51	35.4	76	47.7	
2	33.9		27	5.1	52	35.8	77	47.9	
3	36.2		28	6.9	53	35.8	78	48.2	
4	37.3		29	8.7	54	36.8	79	48.9	
5	39.3		30	10.3	55	37.9	80	49.5	
6	41.4		31	11.9	56	38.6	81	49.2	
7	43.5		32	13.5	57	39.2	82	49.4	
8	45.8		33	15.0	58	39.9	83	4.4	-10.4
9	6.4	-96.1	34	16.5	59	40.4	84	4.7	
10	8.8		35	17.9	60	41.0	85	4.9	
11	11.1		36	19.3	61	41.6	86	5.1	
12	13.5		37	20.6	62	42.1	87	5.4	
13	15.9		38	21.9	63	42.6	88	5.6	
14	18.3		39	23.2	64	43.1	89	5.8	
15	20.6		40	24.4	65	43.6	90	5.9	
16	23.0		41	25.5	66	44.0	91	5.9	
17	25.3		42	26.6	67	44.8	92	6.3	
18	27.5		43	27.7	68	45.5	93	6.5	
19	29.8		44	28.7	69	46.4	94	6.7	
20	31.9		45	30.0	70	46.9	95	6.9	
21	34.1		46	30.7	71	47.0	96	6.9	
22	36.2		47	31.6	72	47.2	97	6.9	
23	38.2		48	32.5	73	47.1	98	6.6	
24	40.2		49	32.9	74	47.1	99	6.3	
25	1.4	-55.4	50	34.2	75	47.4	100	6.1	



LEVEL	BULK	WALL
0-1	42.4 m.m.	46.2 m.m.
1-2	44.1	41.7
2-3	48.8	40.7
3-4	46.3	45.0
4-5	45.2	40.5

FIGURE A-1

SCHEMATIC DIAGRAM OF INCREMENTAL ZERO SUPPRESSION CIRCUITS WITH INCREMENT VALUES

The steady state correction and zero suppression information were combined with the strip chart recording readings according to the following formulae:

To obtain the assembly constant for a reading on ith segment from equilibrium:

For positive vectors:

$$AHRV_i = RD_i + ssc + \sum_{j=0}^{i-1} ZSI_{jtoj+1}$$

For negative vectors:

$$AHRV_i = RD_i + ssc - \sum_{j=0}^{i-1} ZSI_{jtoj+1}$$

where:

i = an integer number of zero suppressions

RD_i = a reading in the interval recorded i increments from equilibrium

$AHRV_i$ = a value of the assembled homogeneous response vector corresponding to RD_i

ZSI_{jtoj+1} = the increment of zero suppression between intervals j and j+1

Table A-2 and Table A-3 contain tabulations of the assembled homogeneous response vectors corresponding to the state variables in the four sets of experiments. Certain locations in these tables contain two entries. The entry on the left is the result of the assembly process; the entry on the right is the result of reading the smooth curves of

Figures V-11 and V-12. These figures are, of course, graphical displays of the information contained in the tables.

Table A-4 is a list of the deck of punched cards used as input information in the computational phase of the study of the heat transfer dynamics of the backmix chemical reactor.

TABLE A-2

REDUCED DATA WORKSHEET

SOURCE Group II, Runs 7 & 1 SYSTEM Backmix Reactor SIGMA Time Invariant

ϕ_{i1} FORCING FUNCTION Run 7 + sine T inlet ϕ_{i2} FORCING FUNCTION Run 1 + step T inlet

ATTENUATION 0.10 0.05 0.10 0.05 (^{o}F/mm

S.S. CORRECTION 0.0 0.0 0.0 0.0

READING ERROR 0.25 mm

No.	ϕ_{11}(7B)	ϕ_{21}(7W)	ϕ_{12}(1B)		ϕ_{22}(1W)	
1	85.4	136.8	38.5		97.9	
2	79.8	132.8	36.7		94.3	
3	74.7	128.8	34.9		90.9	
4	70.0	124.9	33.3		87.6	
5	65.7	121.0	31.8		84.4	
6	61.7	117.2	30.3		81.2	
7	58.0	113.4	29.0		78.2	
8	54.6	109.7	27.7		75.3	
9	51.5	106.0	26.4		72.5	
10	48.6	102.5	25.4	25.3	69.8	
11	45.9	99.0	24.2		67.2	
12	43.4	95.6	23.1		64.7	
13	41.0	92.3	22.1		62.3	
14	38.8	89.1	21.2		60.0	
15	36.8	86.0	20.3		57.7	
16	34.9	83.0	19.4		55.5	
17	33.1	80.0	18.6		53.4	
18	31.5	77.2	18.0	17.9	51.5	51.4
19	29.9	74.4	17.1		49.6	49.5
20	28.5	71.7	16.5	16.4	47.6	
21	27.1	69.1	15.7		45.8	
22	25.8	66.6	15.1		44.1	
23	24.6	64.2	14.5		42.4	
24	23.5	61.8	13.9		40.8	
25	22.4	59.5	13.2	13.3	39.2	

REDUCED DATA WORKSHEET
(CONTINUED)

RUN/page 7 & 1/2

SIGMA Time Invariant

No.	ϕ_{11}	ϕ_{21}	ϕ_{21}		ϕ_{22}	
26	21.4	57.3	12.8		37.8	
27	20.4	55.2	12.3		36.3	
28	19.5	53.2	11.8		34.9	
29	18.6	51.2	11.3		33.6	
30	17.8	49.3	10.9		32.3	
31	17.0	47.5	10.4		31.1	
32	16.3	45.7	10.0		29.9	
33	15.6	44.0	9.6		28.8	
34	14.9	42.3	9.3		27.7	
35	14.3	40.7	8.5	8.9	26.6	
36	13.7	39.2	8.0	8.5	25.6	
37	13.1	37.7	8.0	8.2	24.6	
38	12.6	36.3	7.8	7.9	23.7	
39	12.1	34.9	7.6		22.3	22.8
40	11.6	33.6	7.3		21.3	21.9
41	11.1	32.3	7.0		20.8	21.1
42	10.6	31.1	6.7		20.3	
43	10.2	29.9	6.5		19.5	
44	9.8	28.8	6.2		18.7	
45	9.4	27.7	6.0		18.0	
46	9.0	26.6	5.7		17.3	
47	8.7	25.6	5.5		16.7	
48	8.3	24.7	5.3		16.0	
49	8.0	23.7	5.1		15.4	
50	7.7	22.8	4.9		14.8	

REDUCED DATA WORKSHEET
(CONTINUED)

RUN/page 7 & 1/3

SIGMA Time Invariant

No.	ϕ_{11}		ϕ_{21}		ϕ_{12}		ϕ_{22}	
51	7.4		21.9		4.7		14.3	
52	7.1		21.1		4.5		13.7	
53	6.8		20.3		4.4		13.2	
54	6.5		19.5		4.2		12.7	
55	6.3		18.8		4.0		12.2	
56	6.0		18.1		3.9		11.7	
57	5.8		17.4		3.7		11.3	
58	5.6		16.7		3.6		10.9	
59	5.3		16.1		3.4		10.2	10.4
60	5.1		15.5		3.3		10.0	
61	4.9		14.9		3.2		9.7	
62	4.7		14.3		3.1		9.3	
63	4.6		14.2	13.8	2.9		8.9	
64	4.4		13.8	13.2	2.8		8.6	
65	4.2		13.3	12.7	2.7		8.3	
66	4.0		12.8	12.2	2.6		7.9	
67	4.0	3.9	11.8		2.5		7.7	7.6
68	3.7		11.3		2.4		7.7	7.3
69	3.6		10.9		2.3		7.5	7.1
70	3.5		10.5		2.2		6.8	
71	3.3		10.1		2.5	2.1	6.3	6.5
72	3.2		9.7		2.3	2.1	6.3	
73	3.2	3.1	9.3		2.0		6.0	
74	3.1	3.0	9.0		1.9		5.5	5.8
75	3.0	2.8	8.6		1.8		5.4	5.6

REDUCED DATA WORKSHEET
(CONTINUED)

RUN/page 7 & 1/4

SIGMA Time Invariant

No.	ϕ_{11}		ϕ_{21}		ϕ_{12}		ϕ_{22}	
76	3.0	2.7	8.3		1.8		5.4	
77	2.8	2.6	8.3	8.0	1.7		5.2	
78	2.6	2.5	8.4	7.7	1.6		5.0	
79	2.5	2.4	7.4		1.6		4.8	
80	2.4	2.3	7.1		1.5		4.6	
81	2.2		7.0	6.8	1.5		5.0	4.4
82	2.1	2.2	6.6		1.4		5.0	4.2
83	2.0	2.1	6.4	6.3	1.4	1.3	5.0	4.1
84	2.0		6.0	6.1	1.3		4.3	3.9
85	1.9		5.8		1.1	1.2	3.8	
86	1.8		5.4	5.6	1.1	1.2	3.3	3.6
87	1.7	1.8	4.9	5.4	1.0	1.1	3.8	3.5
88	1.7		4.5	5.2	1.0	1.1	4.4	3.4
89	1.7	1.6	4.5	5.0	1.0	1.1	3.9	3.2
90	1.5	1.6	4.2	4.8	1.0		3.6	3.1
91	1.6	1.5	4.3	4.6	1.0		3.0	
92	1.6	1.5	3.8	4.4	1.0	0.9	2.6	2.9
93	1.4		3.9	4.3	1.0	0.9	2.0	2.8
94	1.5	1.3	3.9	4.1	0.9		2.0	2.7
95	1.4	1.3	3.8	3.9	0.8		2.0	2.6
96	1.3	1.2	3.5	3.8	0.8		2.7	2.5
97	1.2		2.8	3.6	0.7	0.8	2.0	2.4
98	1.2		2.6	3.5	0.5	0.7	2.1	2.3
99	1.1		3.0	3.4	0.5	0.7	2.3	2.2
100	1.2	1.1	2.5	3.2	0.6	0.7	2.0	2.1

TABLE A-3

REDUCED DATA WORKSHEET

SOURCE Group II, Runs 6&8 SYSTEM Backmix Reactor SIGMA Time Invariant

ϕ_{11} FORCING FUNCTION Run 6 (-)Sine T inlet ϕ_{12} FORCING FUNCTION Run 8 - Step T inlet

ATTENUATION 0.10 0.05 0.10 0.05 (^{o}F/mm)

S.S. CORRECTION 0.0 0.0 0.0 0.0

READING ERROR 0.25 mm

No.	$-\phi_{11}$(6B)	$-\phi_{21}$(6W)	$-\phi_{12}$(8B)	$-\phi_{22}$(8W)	
1	87.1	149.0	120.6	105.3	
2	81.6	144.5	111.1	103.9	
3	76.5	140.0	102.5	101.6	102.3
4	71.9	135.6	94.7	100.5	
5	67.6	131.2	87.6	98.5	
6	63.6	126.9	81.1	96.4	
7	59.9	122.7	75.2	94.3	
8	56.5	118.6	69.8	92.0	
9	53.4	114.6	64.9	89.7	
10	50.4	110.7	60.4	87.3	
11	47.7	106.9	56.3	85.0	
12	45.2	103.2	52.5	82.6	
13	42.8	99.6	49.1	80.2	
14	40.6	96.1	45.9	77.8	
15	38.5	92.7	43.0	75.5	
16	36.6	89.4	40.3	73.1	
17	34.8	86.2	37.9	70.8	
18	33.1	83.1	35.6	68.6	
19	31.5	80.0	33.5	66.3	
20	30.0	77.1	31.5	64.2	
21	28.6	74.3	29.7	62.0	
22	27.2	71.6	28.1	59.9	
23	26.0	69.0	26.5	57.9	
24	24.8	66.4	25.1	55.9	
25	23.7	64.0	23.7	54.0	

REDUCED DATA WORKSHEET
(CONTINUED)

RUN/page 6 & 8/2

SIGMA Time Invariant

No.	$-\phi_{11}$	$-\phi_{21}$	$-\phi_{12}$	$-\phi_{22}$	
26	22.6	61.6	22.5	52.1	
27	21.6	59.3	21.3	50.3	
28	20.7	57.1	20.2	48.5	
29	19.8	55.0	19.2	46.7	
30	18.9	52.9	18.3	45.1	
31	18.1	50.9	17.4	43.5	
32	17.3	49.0	16.5	41.9	
33	16.6	47.2	15.7	40.4	
34	15.9	45.4	15.0	38.9	
35	15.2	43.7	14.3	37.5	
36	14.6	42.1	13.6	36.1	
37	14.0	40.5	13.0	34.8	
38	13.4	38.9	12.4	33.5	
39	12.9	37.5	11.9	32.2	
40	12.3	36.0	11.3	31.0	
41	11.8	34.7	10.8	29.9	
42	11.4	33.4	10.4	28.8	
43	10.9	32.1	9.9	27.7	
44	10.5	30.9	9.5	26.7	
45	10.0	29.7	9.1	25.4	25.7
46	9.6	28.6	8.7	24.7	
47	9.3	27.5	8.3	23.8	
48	8.9	26.4	8.0	22.9	
49	8.5	25.4	7.6	22.5	22.0
50	8.2	24.5	7.3	21.2	

REDUCED DATA WORKSHEET
(CONTINUED)

RUN/page 6 & 8/3

SIGMA Time Invariant

No.	$-\phi_{11}$		$-\phi_{21}$		$-\phi_{12}$		$-\phi_{22}$	
51	7.9		23.4	23.5	7.0		20.0	20.4
52	7.6		22.5	22.6	6.7		19.6	
53	7.3		21.6	21.8	6.5		19.6	18.9
54	7.0		20.5	20.9	6.2		18.6	18.1
55	6.7		19.5	20.1	5.9		17.5	
56	6.4		18.0	19.4	5.7		16.8	
57	6.2		17.3	18.6	5.5		16.2	
58	6.0		17.3	17.9	5.3		15.5	
59	5.7		17.3	17.2	5.0		15.0	
60	5.5		16.6		4.8		14.4	
61	5.3		15.9		4.7		13.8	
62	4.9	5.1	15.3		4.5		13.3	
63	4.5	4.9	14.5	14.7	4.3		12.8	
64	4.3	4.7	13.8	14.2	4.1		12.3	
65	4.2	4.5	13.8	13.6	4.0		11.8	
66	4.0	4.3	13.7	13.1	3.8		11.4	
67	4.2		12.6		3.7		10.6	11.0
68	4.0		11.8	12.1	3.5		9.9	10.5
69	3.9		11.7		3.4		9.0	10.1
70	3.4		11.0	11.2	3.2		8.5	9.8
71	3.6		10.8		3.0	3.1	8.4	9.4
72	3.4		10.4		3.0		8.2	9.0
73	3.3		10.0		2.9		8.3	8.7
74	3.5	3.2	9.6		2.8		8.3	
75	3.5	3.0	9.2		2.7		8.0	

REDUCED DATA WORKSHEET
(CONTINUED)

RUN/page 6 & 8/4

SIGMA Time Invariant

No.	$-\phi_{11}$		$-\phi_{21}$		$-\phi_{12}$		$-\phi_{22}$	
76	2.9		9.5	8.9	2.6		7.7	
77	3.5	2.8	8.7	8.5	2.5		7.5	7.4
78	2.7		7.9	8.2	2.4		7.2	7.1
79	2.6		7.0	7.9	2.3		6.5	6.9
80	2.5		6.8	7.6	2.2		5.9	6.6
81	2.4		6.8	7.3	2.1		6.2	6.3
82	2.3		6.8	7.0	1.9	2.0	6.0	6.1
83	2.8	2.2	6.8		1.9		6.0	5.9
84	2.5	2.1	6.5		1.9		5.7	5.6
85	2.1		6.2		1.5	1.8	5.5	5.4
86	2.0		6.0		1.3	1.7	5.3	5.2
87	2.1	1.9	5.6		1.4	1.7	5.0	
88	1.8		6.3	5.6	1.5	1.6	4.9	4.8
89	0.6	1.8	5.8	5.3	1.5		4.6	
90	0.7	1.7	5.1		1.4	1.5	4.5	
91	2.0	1.6	4.9		1.4		4.5	4.3
92	1.6		4.7		1.4		4.1	
93	1.5		4.9	4.6	1.3		3.9	4.0
94	1.8	1.4	4.8	4.4	1.3		3.7	3.8
95	1.5	1.4	4.2		1.2		3.5	3.7
96	1.6	1.3	4.1		1.2		3.5	
97	1.6	1.3	3.4	3.9	1.1		3.5	3.4
98	1.9	1.2	3.3	3.8	1.1		3.8	3.3
99	2.0	1.2	3.1	3.6	1.0		4.1	3.1
100	2.0	1.1	3.0	3.5	1.0		4.3	3.0

TABLE A-4

LIST OF INPUT INFORMATION FOR THE COMPUTATIONAL PHASE OF THE BACKMIX CHEMICAL REACTOR STUDY

Fluid	Wall	Fluid	Wall	Fluid	Wall	Fluid	Wall
85.4	136.8	38.5	97.9	-87.1	-149.0	-120.6	-105.3
79.8	132.8	36.7	94.3	-81.6	-144.5	-111.1	-103.9
74.7	128.8	34.9	90.9	-76.5	-140.0	-102.5	-102.3
70.0	124.9	33.3	87.6	-71.9	-135.6	-94.7	-100.5
65.7	121.0	31.8	84.4	-67.6	-131.2	-87.6	-98.5
61.7	117.2	30.3	81.2	-63.6	-126.9	-81.1	-96.4
58.0	113.4	29.0	78.2	-59.9	-122.7	-75.2	-94.3
54.6	109.7	27.7	75.3	-56.5	-118.6	-69.8	-92.0
51.5	106.0	26.4	72.5	-53.4	-114.6	-64.9	-89.7
48.6	102.5	25.3	69.8	-50.4	-110.7	-60.4	-87.3
45.9	99.0	24.2	67.2	-47.7	-106.9	-56.3	-85.0
43.4	95.6	23.1	64.7	-45.2	-103.2	-52.5	-82.6
41.0	92.3	22.1	62.3	-42.8	-99.6	-49.1	-80.2
38.8	89.1	21.2	60.0	-40.6	-96.1	-45.9	-77.8
36.8	86.0	20.3	57.7	-38.5	-92.7	-43.0	-75.5
34.9	83.0	19.4	55.5	-36.6	-89.4	-40.3	-73.1
33.1	80.0	18.6	53.4	-34.8	-86.2	-37.9	-70.8
31.5	77.2	17.9	51.4	-33.1	-83.1	-35.6	-68.6
29.9	74.4	17.1	49.5	-31.5	-80.0	-33.5	-66.3
28.5	71.7	16.4	47.6	-30.0	-77.1	-31.5	-64.2
27.1	69.1	15.7	45.8	-28.6	-74.3	-29.7	-62.0
25.8	66.6	15.1	44.1	-27.2	-71.6	-28.1	-59.9
24.6	64.2	14.5	42.4	-26.0	-69.0	-26.5	-57.9
23.5	61.8	13.9	40.8	-24.8	-66.4	-25.1	-55.9
22.4	59.5	13.3	39.2	-23.7	-64.0	-23.7	-54.0
21.4	57.3	12.8	37.8	-22.6	-61.6	-22.5	-52.1
20.4	55.2	12.3	36.3	-21.6	-59.3	-21.3	-50.3
19.5	53.2	11.8	34.9	-20.7	-57.1	-20.2	-48.5
18.6	51.2	11.3	33.6	-19.8	-55.0	-19.2	-46.7
17.8	49.3	10.9	32.3	-18.9	-52.9	-18.3	-45.1
17.0	47.5	10.4	31.1	-18.1	-50.9	-17.4	-43.5
16.3	45.7	10.0	29.9	-17.3	-49.0	-16.5	-41.9
15.6	44.0	9.6	28.8	-16.6	-47.2	-15.7	-40.4
14.9	42.3	9.3	27.7	-15.9	-45.4	-15.0	-38.9
14.3	40.7	8.9	26.6	-15.2	-43.7	-14.3	-37.5
13.7	39.2	8.5	25.6	-14.6	-42.1	-13.6	-36.1
13.1	37.7	8.2	24.6	-14.0	-40.5	-13.0	-34.8
12.6	36.3	7.9	23.7	-13.4	-38.9	-12.4	-33.5
12.1	34.9	7.6	22.8	-12.9	-37.5	-11.9	-32.2
11.6	33.6	7.3	21.9	-12.3	-36.0	-11.3	-31.0
11.1	32.3	7.0	21.1	-11.8	-34.7	-10.8	-29.9
10.6	31.1	6.7	20.3	-11.4	-33.4	-10.4	-28.8
10.2	29.9	6.5	19.5	-10.9	-32.1	-9.9	-27.7
9.8	28.8	6.2	18.7	-10.5	-30.9	-9.5	-26.7
9.4	27.7	6.0	18.0	-10.0	-29.7	-9.1	-25.7
9.0	26.6	5.7	17.3	-9.6	-28.6	-8.7	-24.7
8.7	25.6	5.5	16.7	-9.3	-27.5	-8.3	-23.8
8.3	24.7	5.3	16.0	-8.9	-26.4	-8.0	-22.9
8.0	23.7	5.1	15.4	-8.5	-25.4	-7.6	-22.0
7.7	22.8	4.9	14.8	-8.2	-24.5	-7.3	-21.2
7.4	21.9	4.7	14.3	-7.9	-23.5	-7.0	-20.4
7.1	21.1	4.5	13.7	-7.6	-22.6	-6.7	-19.6
6.8	20.3	4.4	13.2	-7.3	-21.8	-6.5	-18.9
6.5	19.5	4.2	12.7	-7.0	-20.9	-6.2	-18.1
6.3	18.8	4.0	12.2	-6.7	-20.1	-5.9	-17.5
6.0	18.1	3.9	11.7	-6.4	-19.4	-5.7	-16.8
5.8	17.4	3.7	11.3	-6.2	-18.6	-5.5	-16.2
5.6	16.7	3.6	10.9	-6.0	-17.9	-5.3	-15.5
5.3	16.1	3.4	10.4	-5.7	-17.2	-5.0	-15.0
5.1	15.5	3.3	10.0	-5.5	-16.6	-4.8	-14.4
4.9	14.9	3.2	9.7	-5.3	-15.9	-4.7	-13.8
4.7	14.3	3.1	9.3	-5.1	-15.3	-4.5	-13.3
4.6	13.8	2.9	8.9	-4.9	-14.7	-4.3	-12.8
4.4	13.2	2.8	8.6	-4.7	-14.2	-4.1	-12.3
4.2	12.7	2.7	8.3	-4.5	-13.6	-4.0	-11.8
4.0	12.2	2.6	7.9	-4.3	-13.1	-3.8	-11.4
3.9	11.8	2.5	7.6	-4.2	-12.6	-3.7	-11.0
3.7	11.3	2.4	7.3	-4.0	-12.1	-3.5	-10.5
3.6	10.9	2.3	7.1	-3.9	-11.7	-3.4	-10.1
3.5	10.5	2.2	6.8	-3.7	-11.2	-3.2	-9.8
3.3	10.1	2.1	6.5	-3.6	-10.8	-3.1	-9.4
3.2	9.7	2.1	6.3	-3.4	-10.4	-3.0	-9.0
3.1	9.3	2.0	6.0	-3.3	-10.0	-2.9	-8.7
3.0	9.0	1.9	5.8	-3.2	-9.6	-2.8	-8.3
2.8	8.6	1.8	5.6	-3.0	-9.2	-2.7	-8.0

APPENDIX C

DIGITAL COMPUTATION PROGRAMS

This appendix is devoted to presentation of the Fortran statements of the digital computer programs used to calculate and prepare graphical displays of the estimate of weighting functions. These statements appear as Table A-5.

TABLE A-5
FORTRAN STATEMENTS OF DIGITAL COMPUTATION PROGRAM

```
JOB  276,  BISHOP, TIME(25)
08/18/64
FORTRAN

      DIMENSION LIMIT(2,3),INCR(3),ATTN(2,2),SSC(2,2),REROR(2,2),
     1RAW11(200),RAW12(200),RAW21(200),RAW22(200),DAT11(200),DAT12(200),
     2DAT21(200),DAT22(200),REDER(2,2),PHI(2,2),PHINV(2,2),EINV(2,2),
     3HOFTS(2,2),EH(2,2),ER(2,2),MARK(3),T(200),TMS(200),H11(200),H12(20
     10),H21(200),H22(200),HER11(200),HER12(200),HER21(200),HER22(200),N
     2T(200),NTMS(200)
      READ(1, 499)NDECK
      DO 48 K = 1, NDECK
      READ(1, 500)MOP1,MOP2,MOP3,MOP4,INITL
      READ(1,501)LIMIT,INCR,MOSTM,INCRM,RUN
      READ(1,2)ATTN,SSC,REROR
    2 FORMAT(8(F6.2,1X),4(F5.2,1X))
      NAT = MOP4 - 13
      IF(NAT)601,602,603
  601 NET = 1
      GO TO 610
  602 NET = 2
      GO TO 610
  603 NAT = MOP4 - 23
      IF(NAT)604,605,606
  604 NET = 3
      GO TO 610
  605 NET = 4
      GO TO 610
  606 NAT = MOP4 - 34
      IF(NAT)607,608,609
  607 NET = 5
      GO TO 610
  608 NET = 6
      GO TO 610
  609 WRITE(2, 599)
      WRITE(2,598)MOP4
  598 FORMAT(10X,I5)
      GO TO 48
  610 DO 333 I = 1, MOSTM
      GO TO(611, 612, 613, 614, 615, 616),NET
  611 READ(2,650)RAW11(I),RAW21(I),RAW12(I),RAW22(I)
      GO TO 333
  612 READ(2,651)RAW11(I),RAW21(I),RAW12(I),RAW22(I)
      GO TO 333
  613 READ(2,652)RAW11(I),RAW21(I),RAW12(I),RAW22(I)
      GO TO 333
  614 READ(2,653)RAW11(I),RAW21(I),RAW12(I),RAW22(I)
      GO TO 333
  615 READ(2,654)RAW11(I),RAW21(I),RAW12(I),RAW22(I)
      GO TO 333
  616 READ(2,655)RAW11(I),RAW21(I),RAW12(I),RAW22(I)
  333 CONTINUE
  650 FORMAT(7X,4(F5.2,5X))
  651 FORMAT(7X,2(F5.2,2X),14X,2(F5.2,2X))
  652 FORMAT(7X,2(F5.2,2X),28X,2(F5.2,2X))
  653 FORMAT(21X, 4(F5.2,2X))
  654 FORMAT(21X,2(F5.2,2X),14X,2(F5.2,2X))
  655 FORMAT(35X, 4(F5.2, 2X))
  499 FORMAT(I1)
  500 FORMAT(5(I5,5X))
  501 FORMAT(9X,I3,2X,I3,2X,I3, 2X,I3,2X,I3,2X,I3,2X,I2,3X,I2,3X,I2,3X,
     1I3,2X,I2,3X,I3)
  599 FORMAT(10X,26H INCORRECT DATA SELECTION  )
      WRITE(2,4)
    4 FORMAT(1H1)
      WRITE(2,5)
    5 FORMAT(38X,42H GENERALIZED PULSE TEST CALCULATION (2X2) )
      WRITE(2,6)RUN
    6 FORMAT(38X,14H K. A. BISHOP ,19X, 5H RUN ,I3///)
      WRITE(2,7)
    7 FORMAT(48X,23H CALCLLATION PROCEDURE /)
      WRITE(2,8)
    8 FORMAT(36X,46H I= LOWER LIMIT TO UPPER LIMIT WITH INCREMENT )
      WRITE(2,9)LIMIT(1,1),LIMIT(2,1),INCR(1)
    9 FORMAT(45X,I3,12X,I3,I3X,I2)
```

```
      WRITE(2,10)LIMIT(1,2),LIMIT(2,2),INCR(2)
   10 FORMAT(45X,I3,12X,I3,13X,I2)
      WRITE(2,11)LIMIT(1,3),LIMIT(2,3),INCR(3)
   11 FORMAT(45X,I3,12X,I3,13X,I2///)
      WRITE(2,12)MOSTM,INCRM
   12 FORMAT(10X,I3,49H DATA POINTS PER VECTOR, CALCULATIONS MADE EVERY
     1,I2,9H POINTS. ///)
      WRITE(2,13)
   13 FORMAT(52X,16H RUN PARAMETERS //)
      WRITE(2,14)
   14 FORMAT(10X,10H ELEMENTS ,12X,4H 11 ,15X,4H 21 ,16X,4H 12 ,16X
     1,4H 22 /)
      WRITE(2,15)SSC
   15 FORMAT(10X,18H S. S. CORRECTION 2X,F7.3,13X,F7.3,13X,F7.3,13X,F7.3
     1/)
      WRITE(2,16)ATTN
   16 FORMAT(10X,13H ATTENUATION 8X,F6.3,14X,F6.3,14X,F6.3,14X,F6.3 /)
      WRITE(2,17)REROR
   17 FORMAT(10X,18H EST. READ. ERROR ,3X,F6.3,14X,F6.3,14X,F6.3,14X,F6.
     13///////)
      GO TO (401, 4C0),MOP1
  4C0 WRITE(2, 4C2)
  402 FORMAT(10X, 85H OPTION ONE DESIRED ..............POSSIBLY SINGULA
     1R MATRICES ARE CALCULATED                    /)
      GO TO 4C3
  4C1 WRITE(2, 4C4)
  4C4 FORMAT(10X, 85H OPTION ONE NOT DESIRED ..... POSSIBLY SINGULA
     1R MATRICES ARE NOT CALCULATED                /)
  4C3 GO TO (4C5, 4C6),MOP2
  4C5 WRITE(2, 407)
  4C7 FORMAT(10X, 85H OPTION TWO NOT DESIRED ..... RESULTS ARE NOT
     1NORMALIZED                                   /)
      GO TO 4C8
  4C6 WRITE(2, 4C9)
  4C9 FORMAT(10X, 85H OPTION TWO DESIRED ..... RESULTS ARE NORM
     1ALIZED                                       /)
  4C8 GO TO (41C, 411),MOP3
  410 WRITE(2, 412)
  412 FORMAT(10X, 85H OPTION THREE NOT DESIRED ..... A DATA DECK IS N
     1OT PUNCHED                                   /)
      GO TO 413
  411 WRITE(2, 414)
  414 FORMAT(10X, 85H OPTION THREE DESIRED ..... A DATA DECK IS P
     1UNCHED                                       /)
  413 WRITE(2,520) MOP4
  520 FORMAT(10X,43H THE VECTOR CHOICE FOR THIS CALCULATION IS ,I2//)
      WRITE (2,18)
   18 FORMAT(1H1)
      WRITE(2,19)RUN
   19 FORMAT(54X,19H RAW DATA FOR RUN  ,I3//)
      WRITE(2,20)
   20 FORMAT(16X,7H POINT ,10X,4H 11 ,15X,4H 21 ,2X,14H - ELEMENTS - ,4H
     1 12 ,16X,4H 22 /)
      NP = 0
      DO 211 I=1,MOSTM
      WRITE(2,21)NP,RAW11(I),RAW21(I),RAW12(I),RAW22(I)
   21 FORMAT(18X,I3,9X,F8.3,12X,F8.3,12X,F8.3,12X,F8.3)
      NP = NP + 1
  211 CONTINUE
      WRITE(2,22)
   22 FORMAT(1H1)
      DO 23 M=1,MOSTM
      DAT11(M)=( RAW11(M) + SSC(1,1))*ATTN(1,1)
      DAT21(M) = (RAW21(M) + SSC(2,1))*ATTN(2,1)
      DAT12(M) = (RAW12(M) + SSC(1,2))*ATTN(1,2)
   23 DAT22(M) = (RAW22(M) + SSC(2,2))*ATTN(2,2)
      DO 24 I=1,2
      DO 24 J=1,2
   24 REDER(I,J) = REROR(I,J)*ATTN(I,J)
  713 WRITE(2,25)RUN
   25 FORMAT(52X,22H REDUCED DATA FOR RUN ,I2//)
      WRITE(2,26)
   26 FORMAT(16X,7H POINT ,10X,4H 11 ,15X,4H 21 ,2X,14H - ELEMENTS - ,4H
     1 12 ,16X,4H 22 /)
      NP = 0
      DO 277 I=1,MOSTM
      WRITE(2,27)NP,DAT11(I),DAT21(I),DAT12(I),DAT22(I)
   27 FORMAT(18X,I3,9X,F8.3,12X,F8.3,12X,F8.3,12X,F8.3)
      NP = NP + 1
```

TABLE A-5 CONTINUED

TABLE A-5 CONTINUED

```
277 CONTINUE
    WRITE(2, 300)
    WRITE(2, 312)
    I=1
 30 LMTLO = LIMIT(1,I) + 1
    LMTHI = LIMIT(2,I) + 1
    INCRL = INCR(I)
    DO 90 L = LMTLO, LMTHI, INCRL
    MARKT = L - 1 + INITL
    PHI(1,1) = DAT11(L)
    PHI(1,2) = DAT12(L)
    PHI(2,1) = DAT21(L)
    PHI(2,2) = DAT22(L)
    CALL ERVRS(PHI, REDER,PHINV,EINV,ER,MARK,PHDET,ERDET)
    PHDET = PHDET
    ERDET = ERDET
    M1 = MARK(1)
    M2 = MARK(2)
    M3 = MARK(3)
    GO TO (52, 51),M1
 51 WRITE(2, 313)
    WRITE(2, 302)
    WRITE(2, 315) MARKT, PHI(1,1), ER(1,1), PHI(1,2), ER(1,2)
    WRITE(2, 316) PHI(2,1), ER(2,1), PHI(2,2), ER(2,2)
    GO TO 69
 52 GO TO(53,61),MOP1
 53 GO TO (54, 57),M2
 54 WRITE(2, 312)
    WRITE(2, 313)
    WRITE(2, 302)
    WRITE(2, 317) MARKT, PHI(1,1),ER(1,1),PHI(1,2),ER(1,2),PHDET,PHINV
   1(1,1),EINV(1,1),PHINV(1,2),EINV(1,2),ERDET
    WRITE(2, 318)PHI(2,1),ER(2,1),PHI(2,2),ER(2,2),PHINV(2,1),EINV(2,1
   1),PHINV(2,2),EINV(2,2)
    WRITE(2, 301)
    GO TO (55, 56),M3
 55 WRITE(2, 305)
    GO TO 68
 56 WRITE(2, 319)
    WRITE(2, 304)
    GO TO 68
 57 WRITE(2, 313)
    WRITE(2, 302)
    WRITE(2, 320)MARKT,PHI(1,1),ER(1,1),PHI(1,2),ER(1,2),PHDET,ERDET
    GO TO (59, 60),M3
 59 WRITE(2, 321)PHI(2,1),ER(2,1),PHI(2,2),ER(2,2)
    GO TO 69
 60 WRITE(2, 322) PHI(2,1), ER(2,1), PHI(2,2), ER(2,2)
    GO TO 69
 61 WRITE(2, 312)
    WRITE(2, 313)
    WRITE(2, 302)
    WRITE(2, 317) MARKT,PHI(1,1),ER(1,1),PHI(1,2),ER(1,2),PHDET,PHINV(
   11,1),EINV(1,1),PHINV(1,2),EINV(1,2),ERDET
    WRITE(2, 318)PHI(2,1),ER(2,1),PHI(2,2),ER(2,2),PHINV(2,1),EINV(2,1
   1),PHINV(2,2),EINV(2,2)
    WRITE(2, 301)
    GO TO (62, 63),M3
 62 WRITE(2, 301)
    GO TO 64
 63 WRITE(2, 319)
 64 GO TO (65, 66),M2
 65 WRITE(2, 301)
    GO TO 67
 66 WRITE(2, 323)
 67 WRITE(2, 303)
    GO TO 68
 69 WRITE(2, 307)
    GO TO 90
 68 DO 77 M = L, MOSTM, INCRM
    PHI(1,1) = DAT11(M)
    PHI(1,2) = DAT12(M)
    PHI(2,1) = DAT21(M)
    PHI(2,2) = DAT22(M)
    CALL MAMUL(PHI, PHINV, HOFTS)
    CALL  HRROR(PHI, PHINV, EINV, REDER, EH)
    MARKS = M - L
    MARKT = M - 1 + INITL
    T(M) = MARKT
```

TABLE A-5 CONTINUED

```
      TMS(M) = MARKS
      NT(M) = T(M)
      NTMS(M) = TMS(M)
      H11(M) = HOFTS(1,1)
      H12(M) = HOFTS(1,2)
      H21(M) = HOFTS(2,1)
      H22(M) = HOFTS(2,2)
      HER11(M) = EH(1,1)
      HER12(M) = EH(1,2)
      HER21(M) = EH(2,1)
      HER22(M) = EH(2,2)
      IF (M - L) 72, 72, 70
   70 IF(ABSF(H12(M)) - HI12)71,72,72
   71 GO TO 73
   72 HI12 = ABSF(H12(M))
   73 IF (M - L) 76, 76, 74
   74 IF(ABSF(H21(M)) - HI21)75,76,76
   75 GO TO 77
   76 HI21 = ABSF(H21(M))
   77 CONTINUE
      FAC11 = 1.0
      FAC22 = 1.0
   78 GO TO (79, 80),MOP2
   79 FAC12 = 1.0
      GO TO 81
   80 FAC12 = 1.0/HI12
      HI12 = 0.0
   81 GO TO (82, 83),MOP2
   82 FAC21 = 1.0
      GO TO 84
   83 FAC21 = 1.0/HI21
      HI21 = 0.0
   84 INCRP = 2*INCRM
      WRITE(2, 314)
      WRITE(2, 302)
      DO 92 M = L, MOSTP
      H12(M) = H12(M)*FAC12
      H21(M) = H21(M)*FAC21
      HER12(M) = HER12(M)*FAC12
      HER21(M) = HER21(M)*FAC21
   92 CONTINUE
  709 DO 87 M = L, MOSTM, INCRP
      IF(M - MOSTM)85, 86, 87
   85 WRITE(2,324)NT(M),NTMS(M),H11(M),HER11(M),H12(M),HER12(M),NT(M+1),
     1NTMS(M+1),H11(M+1),HER11(M+1),H12(M+1),HER12(M+1)
      WRITE(2, 325)H21(M),HER21(M),H22(M),HER22(M),H21(M+1),HER21(M+1),H
     122(M+1),HER22(M+1)
      WRITE(2, 301)
      GO TO 87
   86 WRITE(2,326)NT(M),NTMS(M),H11(M),HER11(M),H12(M),HER12(M)
      WRITE(2, 327)H21(M),HER21(M),H22(M),HER22(M)
      WRITE(2, 301)
   87 CONTINUE
      WRITE(2, 301)
      WRITE(2, 399)FAC11, FAC12, FAC21, FAC22
      GO TO (5C, 88),MOP3
   88 DO 89 M= L, MOSTM, INCRP
      ICARD = M+1-L
      S = T(M)- TMS(M)
   89 WRITE(2, 328) ICARD,H11(M),H12(M),H21(M),H22(M),T(M),S
   90 CONTINUE
      IF(L-LIMIT(2,3)) 40,44,44
   40 IF(L-LIMIT(2,1)) 42,41,41
   41 I=I+1
      GO TO 3C
   42 WRITE(2,43)
   43 FORMAT(10X,13H  LIMIT ERROR)
   44 CONTINUE
   48 CONTINUE
  300 FORMAT(1H1)
  301 FORMAT(1H0)
  302 FORMAT(1H0/)
  303 FORMAT(1H0//)
  304 FORMAT(1H0///)
  3C5 FORMAT(1HC////)
  307 FORMAT(1H0/////)
  312 FORMAT(1H0/////////)
  313 FORMAT(11OH      =S      PHI CF T AND WORST POSSIBLE ERRORS    DETERM
     1INANT  PHI INVERSE CF S AND MAXIMUM ERROR      DET ERROR )
```

```
314 FORMAT(47H   T    T-S    H(T,S) AND ASSOCIATED MAXIMUM ERROR,13X,48H
    1 T      T-S    H(T,S) AND ASSOCIATED MAXIMUM ERROR )
315 FORMAT(I6,5X,2(F7.3,1X,F7.3,5X),9X,17H SINGULAR MATRIX )
316 FORMAT(11X, 2(F7.3, 1X, F7.3, 5X))
317 FORMAT(I6, 2(5X,F7.3,1X, F7.3),3X,F10.3, 3X, 2(F7.3,1X,F7.3,5X), F
    19.3)
318 FORMAT(11X,2(F7.3,1X,F7.3,5X),10X,2(F7.3,1X,F7.3,5X))
319 FORMAT(18X, 36H ERROR GREATER THAN SIZE OF ELEMENT )
320 FORMAT(I6,2(5X,F7.3,1X,F7.3),3X,F9.3,2X,34H  POSSIBILITY OF A SING
    1ULAR MATRIX ,5X,F10.3)
321 FORMAT(11X, 2(F7.3, 1X, F7.3, 5X))
322 FORMAT(11X,2(F7.3,1X,F7.3,5X),10X, 35H ERROR GREATER THAN SIZE OF
    1ELEMENT )
323 FORMAT(18X, 34H POSSIBILITY OF A SINGULAR MATRIX )
324 FORMAT(I4,I5,2(2X,F7.3,1X,F7.3,3X),11X,I4,I5,2(2X,F7.3,1X,F7.3,3X)
    1)
325 FORMAT(9X, 2(2X,F7.3,1X,F7.3,3X),20X,2(2X,F7.3,1X,F7.3,3X))
326 FORMAT(I4,I5,2(2X,F7.3,1X,F7.3,3X))
327 FORMAT(9X, 2(2X,F7.3,1X,F7.3,3X))
328 FORMAT(1H ,I5,5X,4(F10.5), 2(5X,F5.1))
399 FORMAT(2(11X, F7.3, 1X, F7.3/)/)
    END
```

TABLE A-5
CONTINUED

FORTRAN

```
      SUBROUTINE MAMUL (A,B,C)
      DIMENSION A(2,2),B(2,2),C(2,2)
      C(1,1)=A(1,1)*B(1,1)+A(1,2)*B(2,1)
      C(1,2)=A(1,1)*B(1,2)+A(1,2)*B(2,2)
      C(2,1)=A(2,1)*B(1,1)+A(2,2)*B(2,1)
      C(2,2)=A(2,1)*B(1,2)+A(2,2)*B(2,2)
       RETURN
      END
```

FORTRAN

```
      SUBROUTINE INVRS (A,B,DET)
       DIMENSION A(2,2),B(2,2)
      DET=A(1,1)*A(2,2)-A(2,1)*A(1,2)
      IF(DET)1,2,1
    1 B(1,1)=A(2,2)/DET
      B(1,2)=-A(1,2)/DET
      B(2,1)=-A(2,1)/DET
      B(2,2)=A(1,1)/DET
    2 RETURN
      END
```

FORTRAN

```
      SUBROUTINE HRROR (A,B,C,D,E)
      DIMENSION A(2,2),B(2,2),C(2,2),D(2,2),E(2,2),F(2,2),G(2,2),H(2,2)
```

```
      CALLMAMUL(A,C,F)
      CALLMAMUL(D,B,G)
      CALL MAMUL(D,C,H)
      DO1 I=1,2
      DO1 J=1,2
    1 E(I,J)=ABSF(F(I,J))+ABSF(G(I,J))+ABSF(H(I,J))
      RETURN
      END
```

TABLE A-5
CONTINUED

FORTRAN

```
      SUBROUTINE ERVRS(A,B,C,D,ER,MARK,DET,EDET)
  B IS MATRIX OF ASSOCIATED ERRORS . SHOULD ALL BE EQUAL IN THIS SUBROUTINE
  D MATRIX IS DIFFERENCE IN ELEMENTS OF PHI INV. AND PHI+ERROR INV.
  EDET IS DIFFERENCE IN DETERMINANTS OF PHI AND DETERM. OF PHI + ERRORS.
      DIMENSION A(2,2),B(2,2),C(2,2),D(2,2),ER(2,2),F(2,2),G(2,2),MARK(3
     1)
      M=0
      TEMP=C.
      E=B(1,1)
      IF(E-B(1,2))82,83,83
   82 E=B(1,2)
   83 IF(B(2,1)-E)85,85,84
   84 E=B(2,1)
   85 IF(B(2,2)-E)87,87,86
   86 E=B(2,2)
   87 DO 2 I=1,2
      DO 2 J=1,2
      IF(ABSF(A(I,J))-E)28,28,29
   28 MARK(3) = 2
      GO TO 2CO
   29 MARK(3) = 1
    2 CONTINUE
  2CO CONTINUE
    3 CALL INVRS(A,C,DET)
      DET = DET
      N=0
      S=     A(1,1)+A(2,2)-A(2,1)-A(1,2)
      GO TO 2C
    4 S=-E-E-A(1,1)+A(2,2)-A(2,1)-A(1,2)
      GO TO 2C
    5 S=+E+E+A(1,1)+A(2,2)-A(2,1)+A(1,2)
      GO TO 2C
    6 S=    -A(1,1)+A(2,2)-A(2,1)+A(1,2)
      GO TO 2C
    7 S=+E+E+A(1,1)+A(2,2)+A(2,1)-A(1,2)
      GO TO 2C
    8 S=    -A(1,1)+A(2,2)+A(2,1)-A(1,2)
      GO TO 2C
    9 S=     A(1,1)+A(2,2)+A(2,1)+A(1,2)
      GO TO 2C
   10 S=-E-E-A(1,1)+A(2,2)+A(2,1)+A(1,2)
      GO TO 2C
   11 S=-E-E+A(1,1)-A(2,2)-A(2,1)-A(1,2)
      GO TO 2C
   12 S=    -A(1,1)-A(2,2)-A(2,1)-A(1,2)
      GO TO 2C
   13 S=     A(1,1)-A(2,2)-A(2,1)+A(1,2)
      GO TO 2C
   14 S= E+E-A(1,1)-A(2,2)-A(2,1)+A(1,2)
      GO TO 2C
   15 S=     A(1,1)-A(2,2)+A(2,1)-A(1,2)
      GO TO 2C
   16 S= E+E-A(1,1)-A(2,2)+A(2,1)-A(1,2)
      GO TO 2C
   17 S=-E-E+A(1,1)-A(2,2)+A(2,1)+A(1,2)
      GO TO 2C
```

TABLE A-5 CONTINUED

```
   18 S=    -A(1,1)-A(2,2)+A(2,1)+A(1,2)
   20 M=M+1
      IF(DET)23,30,21
   21 MARK(1) = 1
      IF(S-TEMP)22, 25, 25
   22 TEMP=S
      N=M
      GO TO 25
   23 MARK(1) = 1
      IF(S-TEMP)25, 25, 24
   24 TEMP =S
      N=M
   25 GO TO (4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19),M
   30 MARK(1) = 2
      GO TO 50
   19 IF(N-8)53,53,52
   52 ER(1,1)=-E
      GO TO 54
   53 ER(1,1)=E
      GO TO 45
   54 N=N-8
   45 IF(N-4)56,56,55
   55 ER(1,2)=-E
      GO TO 57
   56 ER(1,2)=E
      GO TO 75
   57 N=N-4
   75 IF(N-2)59,59,58
   58 ER(2,1)=-E
      GO TO 510
   59 ER(2,1)=E
      GO TO 105
  510 N=N-2
  105 IF(N-1)112,112,111
  111 ER(2,2)=-E
      GO TO 113
  112 ER(2,2)=E
  113 DO 114 I=1,2
      DO 114 J=1,2
  114 F(I,J)=A(I,J)+ER(I,J)
      CALL INVRS(F,G,FDET)
      FDET=FDET
      IF(DET)71,50,72
   71 IF(FDET)73,74,74
   72 IF(FDET)74,74,73
   73 EDET = ABSF(FDET) - ABSF(DET)
      MARK(2) = 1
      GO TO 76
   74 MARK(2) = 2
   76 DO 77 I=1,2
      DO 77 J=1,2
   77 D(I,J) = ABSF(G(I,J)) - ABSF(C(I,J))
   50 RETURN
      END
```

START

TABLE A-5 CONTINUED

```
JOB  276, BISHOP, TIME(20)
   CS/17/64       16.58.41.14
FORTRAN

  MOUNT LARGE SCRATCH TAPE CN DRIVE 3

  PROGRAM PLOTS 10 GRAPHS AT SELECTED VALUES OF S
      114 POINTS PLOTTED FOR FIRST GRAPH
 REPEATS FOUR TIMES FOR 4 ELEMENTS OF A 2 X2 MATRIX

 CONTROL CARDS REQUIRED IN ORDER

 CARD            VARIABLES                FORMAT
  1               S INITIAL               4F10.0
  1               S INCREMENT
  1               S RESOLUTION
  1               H RESOLUTION
  1            NORMALIZE OPTION            I5
               1 IN COLUMN 45 IF NOT,
               2 IN COLUMN 45 IF

   2 SYMBOLS USED IN PLOTTING AND FOR DRAWING COORDINATES
   2 BLANK S 1 - * + C X 8 . , = $ /

 DATA FOLLOWS (NEED NOT BE ORDERED)

 LAST CARD MUST HAVE 9999 IN AN I5 FORMAT
    DIMENSION A(115),B(115),C(115),D(115),E(115),F(115),G(115),H(115)
   1 ,O(115),P(115),STORE(4)
    WRITE(2,51)
51 FORMAT(4H1NO.,13X,2HY1,13X,2HY2,13X,2HY3,13X,2HY4,12X,1HT,13X,1HS,
   1 5X  //)
    BIGN=0.
    DO 56 J=1,4
56 STORE(J)=0.
    REWIND 3
    READ(1,1)SIPOI,SINC,SRES,HRES,NORM
 1 FORMAT(4F10.0,I5  )
    READ(1,52)BL,AK,AL,AM,AA,AB,AC,AD,AE,AF,AG,AH,AI,AJ
                  S  1  -  *  +  C  X  8  .  ,  =  $  /
52 FORMAT(14A1)
    SIP1I=SIPOI+SINC
    SIP2I=SIP1I+SINC
    SIP3I=SIP2I+SINC
    SIP4I=SIP3I+SINC
    SIP5I=SIP4I+SINC
    SIP6I=SIP5I+SINC
    SIP7I=SIP6I+SINC
    SIP8I=SIP7I+SINC
    SIP9I=SIP8I+SINC
 2 READ(1,3)N,B,X,Y,Z,T,S
 3 FORMAT(I5,5X,4F10.0,2(5X,F5.0))
    IF(1-S)2,64,64
64 WRITE(2,50)N,B,X,Y,Z,T,S
50 FORMAT(1H ,I5,6F15.4)
```

TABLE A-5 CONTINUED

```
      WRITE TAPE 3, S,T,H,X,Y,Z,N
      IF(N-9999)4,13,4
    4 IF(ABSF(H)-STORE(1))6,6,5
    5 STORE(1)=ABSF(H)
    6 IF(ABSF(X)-STORE(2))8,8,7
    7 STORE(2)=ABSF(X)
    8 IF(ABSF(Y)-STORE(3))10,10,9
    9 STORE(3)=ABSF(Y)
   10 IF(ABSF(Z)-STORE(4))12,12,11
   11 STORE(4)=ABSF(Z)
   12 GO TO 2
   13 DO 59 J=1,4
      IF(STORE(J)-BIGN)59,59,58
   58 BIGN=STORE(J)
   59 CONTINUE
      REWIND 3
      DO 47 M=1,4
      GO TO (61,60),NORM
   60 DIV=STORE(M)
      BIGN=1.
      GO TO 62
   61 DIV=1.
   62 SIZE= 2.*BIGN/HRES +(SIPOI+9.*SINC)/SRES +5.
      DO 14 I=1,115
      A(I)=-SIPCI/SRES
      B(I)=-SIP1I/SRES
      C(I)=-SIP2I/SRES
      D(I)=-SIP3I/SRES
      E(I)=-SIP4I/SRES
      F(I)=-SIP5I/SRES
      G(I)=-SIP6I/SRES
      H(I)=-SIP7I/SRES
      O(I)=-SIP8I/SRES
   14 P(I)=-SIP9I/SRES
      WRITE(2,2020)M,DIV
 2020 FORMAT(I5,F10.3 /)
   15 GO TO(16,17,18,19),M
   16 READ TAPE 3,S,T,H,X,Y,Z,N
      GO TO 20
   17 READ TAPE 3,S,T,X,H,Y,Z,N
      GO TO 20
   18 READ TAPE 3,S,T,X,Y,H,Z,N
      GO TO 20
   19 READ TAPE 3,S,T,X,Y,Z,H,N
   20 INT=T
      IF(FLOATF(INT)-T)15,55,15
   55 I=INT+1
      IF(N-9999)21,45,21
   21 IF(T-114.)210,210,15
  210 IF(T-S)22,24,24
   22 WRITE(2,23)S,T
   23 FORMAT(2CH S GREATER THAN T,S=,F10.6,2HT=,F10.6)
      GO TO 15
   24 IF(S-SIPCI)26,25,26
   25 A(I)=H/(HRES*DIV)+A(I)
      GO TO 15
   26 IF(S-SIP1I)28,27,28
   27 B(I)=H/(HRES*DIV)+B(I)
      GO TO 15
```

TABLE A-5 CONTINUED

```
   28 IF(S-SIP2I)30,29,30
   29 C(I)=b/(HRES*DIV)+C(I)
      GO TO 15
   30 IF(S-SIP3I)32,31,32
   31 D(I)=b/(HRES*DIV)+D(I)
      GO TO 15
   32 IF(S-SIP4I)34,33,34
   33 E(I)=b/(HRES*DIV)+E(I)
      GO TO 15
   34 IF(S-SIP5I)36,35,36
   35 F(I)=b/(HRES*DIV)+F(I)
      GO TO 15
   36 IF(S-SIP6I)38,37,38
   37 G(I)=b/(HRES*DIV)+G(I)
      GO TO 15
   38 IF(S-SIP7I)40,39,40
   39 H(I)=b/(HRES*DIV)+H(I)
      GO TO 15
   40 IF (S-SIP8I)42,41,42
   41 O(I)=b/(HRES*DIV)+C(I)
      GO TO 15
   42 IF(S-SIP9I)44,43,44
   43 P(I)=b/(HRES*DIV)+P(I)
   44 GO TO 15
   45 WRITE(2,46)M,SIPOI,SIP1I,SIP2I,SIP3I,SIP4I,SIP5I,SIP6I,SIP7I,
     1SIP8I,SIP9I
   46 FORMAT(1H1,////////,60X,17HGRAPH OF ELEMENT ,I1,//65X,11HS INITIAL
     1 =,F5.1,//58X,18HOTHER VALUES OF S ,F5.1 //(76X,F5.1/))
   48 WRITE(2,49)SRES,HRES
   49 FORMAT(1HC,37X,26HONE LINE IS EQUIVALENT TO ,F5.3,19H UNIT IN S, AND TO
     1ND TO ,F5.3,11H UNIT IN H.      )
      REWIND 3
      CALL GRAPH(A,B,C,D,E,F,G,H,O,P)
   47 CONTINUE
      COMMON SIPOI,SIP1I,SIP2I,SIP3I,SIP4I,SIP5I,SIP6I,SIP7I,SIP8I,SIP9I
     1 ,SINC,SIZE,SRES,HRES,BIGN,BL,AK,AL,AM,AA,AB,AC,AD,AE,AF,AG,AH,AI,
     2 AJ
      END
```

FORTRAN

```
      SUBROUTINE GRAPH(A,B,C,D,E,F,G,H,O,P)
      COMMON SIPOI,SIP1I,SIP2I,SIP3I,SIP4I,SIP5I,SIP6I,SIP7I,SIP8I,SIP9I
     1 ,SINC,SIZE,SRES,HRES,BIGN,BL,AK,AL,AM,AA,AB,AC,AD,AE,AF,AG,AH,AI,
     2 AJ
      DIMENSION A(115),B(115) ,C(115),D(115),E(115),F(115),G(115),
     1H(115),O(115),P(115),Q(115)
      WRITE(2,2)
    2 FORMAT(1H1)
    1 N2=SIZE
      IC=SIPOI+1.
      I1=SIP1I+1.
      I2=SIP2I+1.
      I3=SIP3I+1.
      I4=SIP4I+1.
```

TABLE A-5
CONTINUED

```
      I5=SIP5I+1.
      I6=SIP6I+1.
      I7=SIP7I+1.
      I8=SIP8I+1.
      I9= SIP9I+1.
      LMARK=2
      DO 3 J=1,N2
      DO 36 M=1,115
   36 Q(M)=BL
      NUMBR=BIGN/HRES+.5
      L=FLOATF(NUMBR-J)+1.
      IF(L)83,84,84
   84 ML=FLOATF(L)*HRES*100.
      LMARK=1
      GO TO 85
   83 ML=-FLOATF(L)*SRES
   85 DO 41 KL=1,10
      IF((SINC/SRES)*FLCATF(KL-1)+FLCATF(L))41,42,43
   42 LSTRT=SINC*FLCATF(KL-1)+1.
      IF(LSTRT-115)82,82,81
   82 DO 44 LINE=LSTRT,115
   44 Q(LINE)=AM
   81 LMARK=1
      GO TO 43
   41 CONTINUE
   43 DO 6 I=1,115
      T=I-1
      IF(T-SIPCI)51,50,51
   50 IF(A(IC)-FLCATF(L))69,69,71
   71 IF(SIPCI/SRES+FLOATF(L))69,69,70
   51 IF(T-SIP1I)53,52,53
   52 IF(B(I1)-FLCATF(L))69,69,72
   72 IF(SIP1I/SRES+FLOATF(L))69,69,70
   54 IF(C(I2)-FLCATF(L))69,69,73
   53 IF(T-SIP2I)55,54,55
   73 IF(SIP2I/SRES+FLOATF(L))69,69,70
   55 IF(T-SIP3I)57,56,57
   56 IF(D(I3)-FLCATF(L))69,69,74
   74 IF(SIP3I/SRES+FLOATF(L))69,69,70
   57 IF(T-SIP4I)59,58,59
   58 IF(E(I4)-FLCATF(L))69,69,75
   75 IF(SIP4I/SRES+FLOATF(L))69,69,70
   59 IF(T-SIP5I)61,60,61
   60 IF(F(I5)-FLCATF(L))69,69,76
   76 IF(SIP5I/SRES+FLOATF(L))69,69,70
   61 IF(T-SIP6I)63,62,63
   62 IF(G(I6)-FLCATF(L))69,69,77
   77 IF(SIP6I/SRES+FLOATF(L))69,69,70
   63 IF(T-SIP7I)65,64,65
   64 IF(H(I7)-FLCATF(L))69,69,78
   78 IF(SIP7I/SRES+FLOATF(L))69,69,70
   65 IF(T-SIP8I)67,66,67
   66 IF(O(I8)-FLCATF(L))69,69,79
   79 IF(SIP8I/SRES+FLOATF(L))69,69,70
   67 IF(T-SIP9I)69,68,69
   68 IF(P(I9)-FLCATF(L))69,69,80
   80 IF(SIP9I/SRES+FLOATF(L))69,69,70
   70 Q(I)=AL
   69 IF(-FLOATF(L)*SRES-FLCATF(I-1))40,35,40
```

TABLE A-5
CONTINUED

```
 35 Q(I)=AK
 40 IF(P(I))177,178,178
177 MP=P(I)-.5
    GO TO 179
178 MP=P(I)+.5
179 IF(MP-L)4,98,4
 98 IF(1-SIP9I)4,31,31
 31 Q(I)=AJ
    GO TO 6
  4 IF(O(I))174,175,175
174 MO=O(I)-.5
    GO TO 176
175 MO=O(I)+.5
176 IF(MO-L)12,97,12
 97 IF(1-SIP8I)12,29,29
 29 Q(I)=AI
    GO TO 6
 12 IF(H(I))171,172,172
171 MH=H(I)-.5
    GO TO 173
172 MH=H(I)+.5
173 IF(MH-L)16,96,16
 96 IF(1-SIP7I)16,27,27
 27 Q(I)=AH
    GO TO 6
 16 IF(G(I))168,169,169
168 MG=G(I)-.5
    GO TO 170
169 MG=G(I)+.5
170 IF(MG-L)18,95,18
 95 IF(1-SIP6I)18,25,25
 25 Q(I)=AG
    GO TO 6
 18 IF(F(I))165,166,166
165 MF=F(I)-.5
    GO TO 167
166 MF=F(I)+.5
167 IF(MF-L)20,94,20
 94 IF(1-SIP5I)20,23,23
 23 Q(I)=AF
    GO TO 6
 20 IF(E(I))162,163,163
162 ME=E(I)-.5
    GO TO 164
163 ME=E(I)+.5
164 IF(ME-L)22,93,22
 93 IF(1-SIP4I)22,21,21
 21 Q(I)=AE
    GO TO 6
 22 IF(D(I))159,160,160
159 MD=D(I)-.5
    GO TO 161
160 MD=D(I)+.5
161 IF(MD-L)24,92,24
 92 IF(1-SIP3I)24,19,19
 19 Q(I)=AD
    GO TO 6
 24 IF(C(I))156,157,157
156 MC=C(I)-.5
```

TABLE A-5
CONTINUED

```
      GO TO 158
  157 MC=C(I)+.5
  158 IF(MC-L)26,91,26
   91 IF(T-SIP2I)26,17,17
   17 Q(I)=AC
      GO TO 6
   26 IF(B(I))153,154,154
  153 MB=B(I)-.5
      GO TO 155
  154 MB=B(I)+.5
  155 IF(MB-L)28,90,28
   90 IF(T-SIP1I)28,11,11
   11 Q(I)=AB
      GO TO 6
   28 IF(A(I))150,151,151
  150 MA=A(I)-.5
      GO TO 152
  151 MA=A(I)+.5
  152 IF(MA-L)6,89,6
   89 IF(T-SIPCI)6,5,5
    5 Q(I)=AA
    6 CONTINUE
      GO TO(10,13),LMARK
   10 WRITE(2,7)C,ML
    7 FORMAT(1H*,115A1,I3)
      LMARK=2
      GO TO 3
   13 WRITE(2,14)C
   14 FORMAT (1H*,115A1)
    3 CONTINUE
      RETURN
      END
```

START

APPENDIX D

DATA ACQUISITION AND SIGNAL CONDITIONING SYSTEM

A schematic diagram of the data acquisition and signal conditioning system is shown in Figure A-2.

Wall Temperature: The temperature of the reactor wall is measured at four locations (ninety degrees apart, approximately midway along the axial dimension, and approximately at the log mean radius) by means of copper-constantan thermocouples (twenty-four gauge). The thermocouples are connected in aprallel; therefore the voltage measured (reference to an ice bath at 32°F.) is proportional to the average of the four measurements.

The thermocouple voltage is amplified, in the Process Laboratory, by means of a Sanborn, Model 350-1500, low-level dc preamplifier (Model 350-2 plug-in unit). The amplifier is operated at a gain of 2,000 (nominal)with up to ± 100 mv. of zero suppression. The output voltage is transmitted, by wire, to the instrument panel where it is connected to one of the remote lines which terminate on the analog computer's program board.

A first order filter network (time constant of 0.4 sec. (nominal) and zero frequency gain set such that the

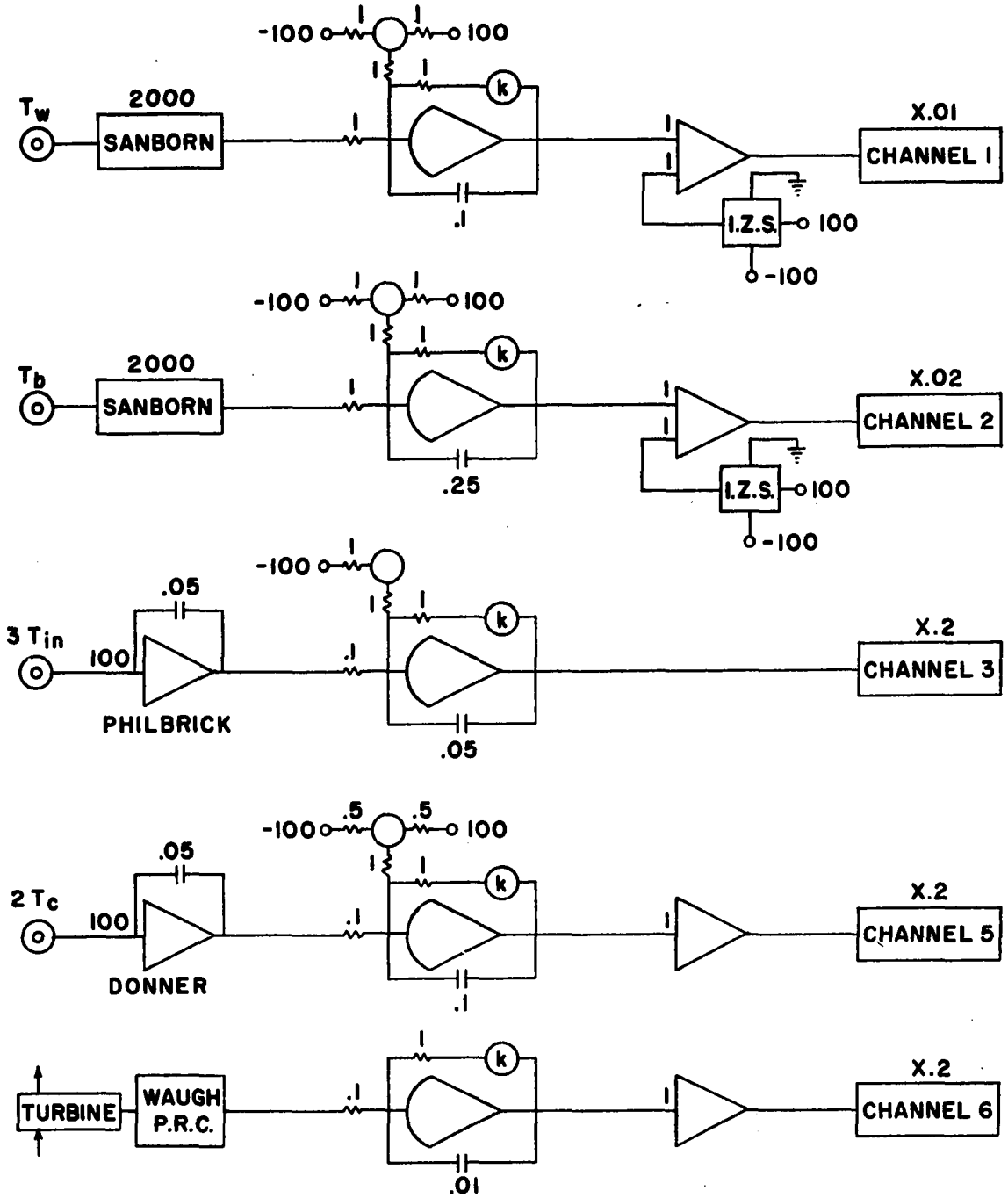


FIGURE A-2
SCHEMATIC DIAGRAM OF SIGNAL
CONDITIONING SYSTEM

system gain is 8.34×10^3) is programed on the computer program board. This filter conditions the amplified thermocouple voltage by further amplifying it, further suppressing its zero level, and time smoothing it. The filter is discussed further in a section to follow.

The output voltage from the smoothing filter is summed with the output voltage from the incremental zero suppression circuit (see Chapters IV, V, and Appendix B). This voltage, which corresponds to the transient portion of the temperature response of the reactor wall, is recorded, at an attenuation setting of 0.01 volts per millimeter, in channel 1 of the six channel Sanborn, Model 156-1100C, strip chart recorder.

These choices of system gain, recorder attenuation, and an appropriate amount of zero suppression result in the capability to record a voltage as a pen position on a fifty millimeter chart such that a 2.5°F. change in wall temperature is represented by a full scale deflection of the recording pen. Note: This statement is based upon a linear relationship between thermocouple voltage and temperature of 0.024 millivolts per degree Fahrenheit.

Bulk Fluid Temperature: The temperature of the process fluid in the reactor is measured by means of a copper-constantan thermocouple in an oil field thermowell (stainless steel) which is inserted, through the reactor head, into the reactor chamber (approximately midway along the axial dimension and somewhat off center).

This thermocouple voltage (reference to an ice bath) is amplified, in the Process Laboratory, by means of a Sanborn low-level preamplifier (similar to that used in the wall temperature measuring system) which is operated at a nominal gain of 2,000. The output voltage is transmitted, as described above, to the analog computer for conditioning.

There it is smoothed, amplified, and its zero is further suppressed in a first order filter (time constant of 1.1 sec. (nominal) and zero frequency gain set such that the system gain is 8.34×10^3). The resulting signal is summed with the output voltage from an incremental zero suppression circuit and recorded, at an attenuation setting of 0.02 volts per millimeter, in channel 2 of the six channel Sanborn recorder.

These choices of system gain, recorder attenuation, and an appropriate amount of zero suppression result in the capability to record a voltage (proportional to the transient portion of the reactor bulk temperature) as a pen position on a fifty millimeter chart. A change of five degrees in this temperature is represented by a full scale deflection of the pen. This range assumes that a change of one degree Fahrenheit in the bulk temperature causes a 0.024 millivolt change at the input to the Sanborn preamplifier.

Inlet Temperature: A thermopile, consisting of three copper-constantan thermocouples connected in series and mounted in a teflon plug, which is installed in the

reactor inlet stream, is used to acquire data on the behavior of the temperature of the process fluid entering the reactor.

The thermopile voltage (reference to an ice bath) thus generated is the input to a Philbrick operational amplifier which is operated at a gain of 100 and is equipped with a 0.05 microfarad feedback capacitor. The output of this amplifier is transmitted to the analog computer program borad as described above.

There it is conditioned by passage through a first order filter network (time constant of 1.4 sec. and zero frequency gain of 278) prior to being recorded (at an attenuation setting of 0.2 volts per millimeter) in channel 3 of the Sanborn recorder.

This choice of system gain, recorder attenuation, and an appropriate zero suppression results in the capability to record a voltage as a pen position on a fifty millimeter chart such that a fifty degree change in the inlet temperature is represented by a full scale deflection of the recording pen. Note that this recording channel is used primarily to provide a check on the fact that the system forcing has been removed. Therefore the amplification requirement is not as severe as are those of channels 1 and 2. It was the policy, however, during the experimental phase of the investigation, to employ smaller recorder attenuation settings to establish removal of system forcing more precisely.

Coolant Temperature: The temperature of the coolant, as discussed in Chapter V, is a fictitious variable and therefore cannot be measured. An approximation of the required temperature may be obtained by averaging the measurements of the temperature of the coolant as it enters and leaves the reactor cooling jacket. These temperatures are measured by means of copper-constantan thermocouples which are inserted through the wall of the Polyflow tube which carries the coolant.

The thermocouple voltages (reference to an ice bath) are summed at the input junction of a Donner, Model 3101, operational amplifier which is operated at a gain of 100 and equipped with a 0.05 microfarad feedback capacitor. The output of this preamplifier is transmitted, as described above, to the program board of the analog computer.

There it is filtered by passage through a first order filter network (time constant of 2.1 sec. and zero frequency gain of 208). The conditioned signal is then recorded, at an attenuation of 0.2 volts per millimeter, in channel 5 of the Sanborn six channel recorder.

This choice of system gain, recorder attenuation, and appropriate zero suppression results in the capability to record a voltage as a pen position of a fifty millimeter chart such that a ten degree change in the average of the inlet and outlet temperatures is represented by a full scale deflection of the recorder pen. This channel is used essentially to provide a check on the assumption (in the derivation

of the model equations) that the coolant temperature remains constant.

Reactor Flow Rate: The flow rate of process fluid through the reactor is measured by means of a Waugh, Model FL-6SB-1, turbine flow sensor, which generates electrical pulses at a rate which is proportional to the turbine rotation rate. Since the turbine is driven by passage of process fluid through the sensor, the pulse rate is proportional to the flow rate.

The train of pulses generated in the sensor is transmitted, by wire, to the Process Laboratory instrument panel where it becomes the input signal to the Waugh, Model FR-111, pulse rate converter, which generates a dc voltage, between zero and 250 millivolts, proportional to the pulse rate. This dc voltage signal is transmitted, via the remote lines, to the program board of the analog computer.

This signal is used by the feedback flow controller associated with the hydraulic function generator (see Appendix F, specifically the point designated 4 in Figure A-7) in addition to its use as a measure of the flow rate. This signal is conditioned by passage through a first order smoothing and amplification filter (time constant of 0.04 sec. and zero frequency gain of 40) prior to being recorded.

The output signal from the conditioning filter is recorded, at an attenuation of 0.2 volts per millimeter, in channel 6 of the six channel recorder.

The combination of system gain and recorder attenuation permits recording the 250 millivolt maximum output from the pulse rate converter on a fifty millimeter chart span. As is seen in Figure A-3, the calibration of the pulse rate converter for a particular range of flow sensor causes some discrepancy.

System Calibration: The calibration of the data acquisition and signal conditioning system proceeded on two levels. Firstly, the characteristics of the particular flow sensor and of a thermocouple, made from the same reel of copper-constantan wire as those used in the investigation were established (see Figures A-3 and A-4). Secondly, the gains of the systems were checked periodically during the investigation and adjusted if necessary.

The calibration of the flow sensor and reactor flow rate measurement system was accomplished as follows.

1. Using analog computing components and a Hewlett-Packard, Model 412A, vacuum tube voltmeter, the gain of the conditioning filter was set at forty.
2. With the flow sensor, pulse rate converter, and the flow rate controller (see Appendix F) in operation, steady state operation of the experimental apparatus was established.
3. Setting the flow rate at various levels (by means of the flow rate controller's set

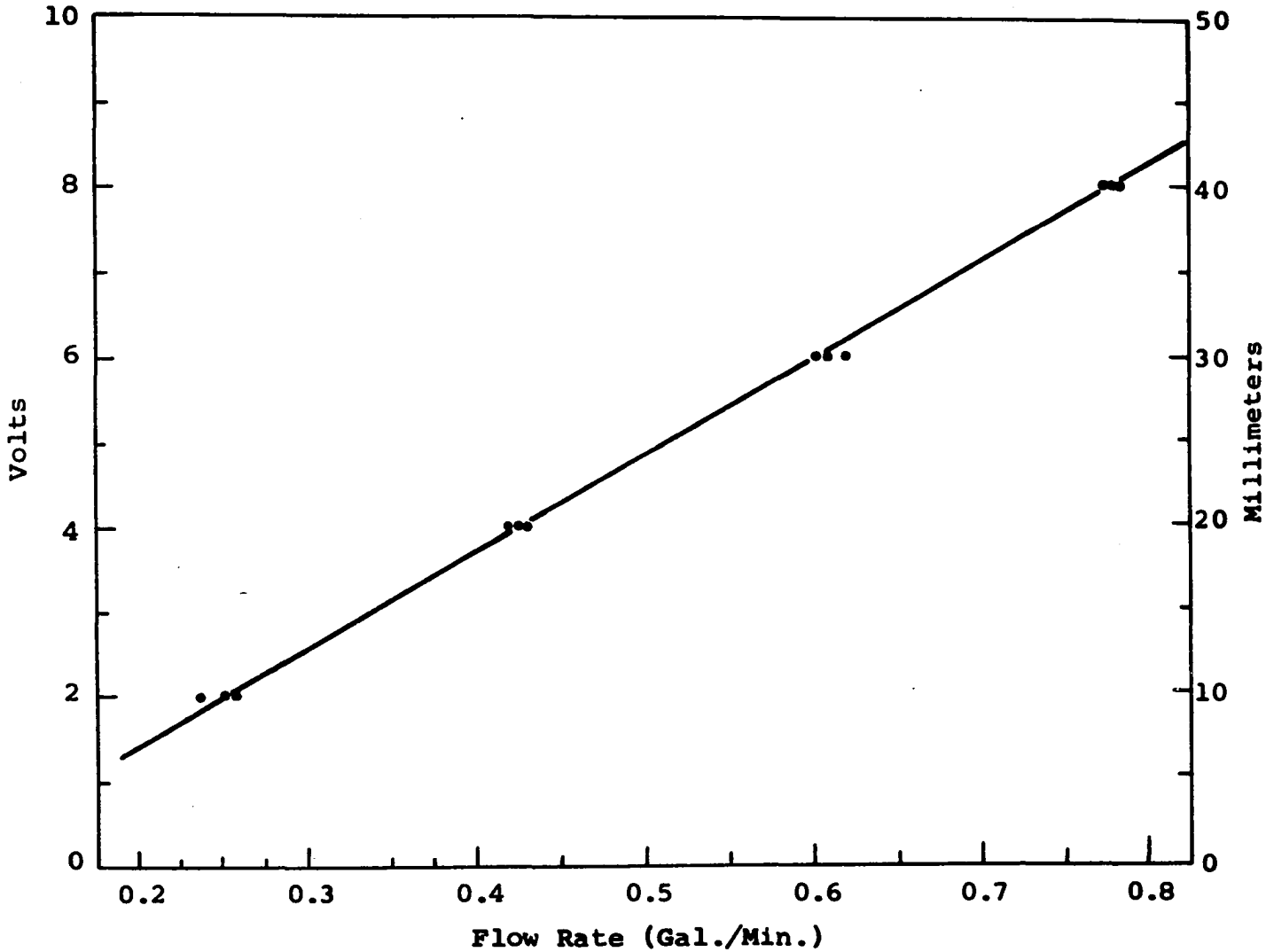


Figure A-3

Flow Meter Calibration

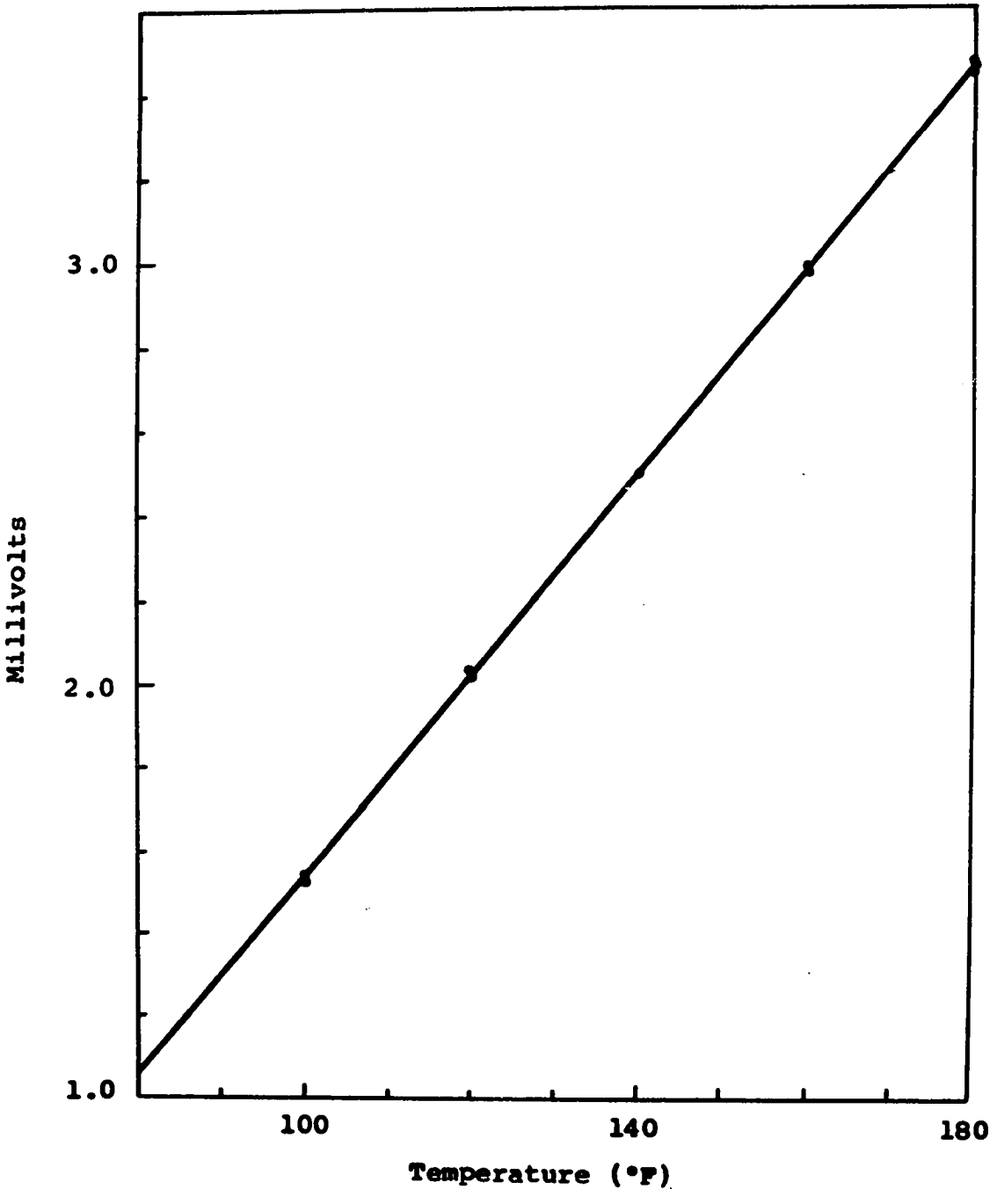


Figure A-4

Standard Thermocouple Calibration

point) in random order, the calibration experiments were made. These experiments consisted of measuring the time required to fill a standard bucket (6552 c.c.) and noting the pen position in channel 6.

The calibration curve was established at temperatures (measured at the reactor inlet, indicated by channel 3) of 124, 140, and 160°F. It was found that variation of the curve due to density changes was less than the variation in the experimental measurements of one flow rate at one temperature. Therefore, the curve determined at 140°F. was used and is presented in Figure A-3.

The method employed to calibrate the thermocouples used in the investigation is of questionable value, as it simply does not include the idiosyncracies of the individual thermocouples. However, it is the considered opinion of the author that calibration of the specific thermocouples would be pointless in the light of the difficulty of linearizing the true characteristics for recording purposes; the error which is introduced by the data reduction procedure (see Appendix B) and the fact that multiple thermocouples are used in three of the four data acquisition systems.

The actual reason for calibrating a "standard" thermocouple at all is simply to establish that its characteristic is essentially linear.

The procedure used to check the gain of the signal conditioning systems involved replacement of the sensor with

a voltage of known magnitude. The indication (at the recorder) of a precalculated pen position constituted the desired check. As an example, consider the wall temperature system:

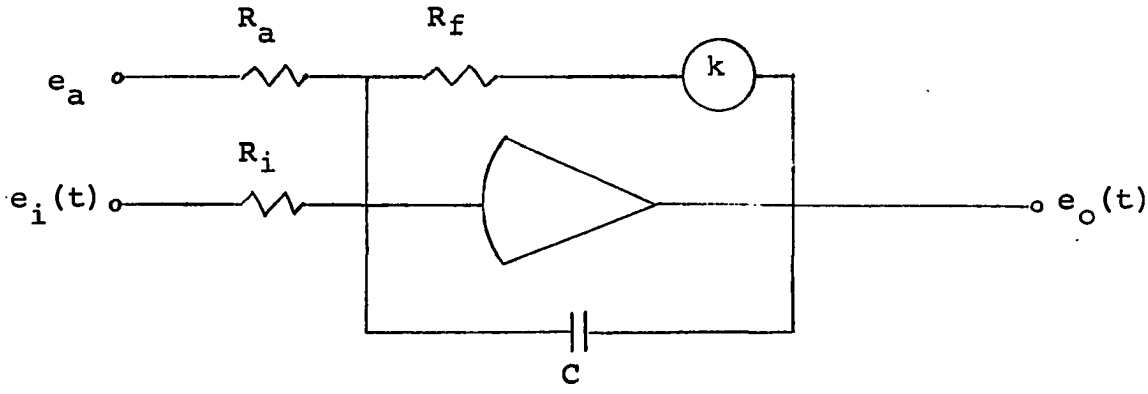
1. The four thermocouples (connected in parallel) are replaced with a one millivolt test signal.
2. With no zero suppression and an attenuation setting of 0.2 volts per millimeter (channel 1), the pen position should be:

$$\frac{1\text{mv}}{1000\text{mv}} \left| \frac{9340\text{v}}{1\text{v}} \right| \frac{1\text{ mm}}{0.2\text{v}} = 41.7\text{ mm}$$

3. Suppose that rather than 41.7, the pen position is 40.3. The setting of the coefficient potentiometer associated with the variable gain summing amplifier used in the conditioning filter would be reduced until the desired pen position is achieved.

Assuming that the data acquisition element (the thermocouple) does provide a signal change of 0.024 millivolts per degree of wall temperature change, resetting of the attenuation of channel 1 to 0.01 volts per millimeter completes the check.

Signal Conditioning Filters: With minor variations to take care of signal polarity and the lack of the necessity to suppress the zero level of a signal, the schematic diagram given below represents the filter network employed in each of the five signal conditioning systems.



The circuit consists of a variable gain summing amplifier (see Bishop and Sims (6, 7) for details) equipped with a capacitor as a feedback element. A combination of Kirchoff's first law with the common approximations for the operation of a high gain operational amplifier may be employed to derive the differential equation which describes the response of the network:

$$e_o(t) + (k/CR_f)e_o(t) = -(1/CR_i)e_i'(t)$$

$$e_i'(t) = e_i(t) + R_a e_a/R_i$$

which may be expressed, in terms of Laplace transforms, as:

$$H(p) = (E_o(p)/E_i'(p) = (R_f/R_i)/((CR_f/k)p + 1)$$

Replacement of the argument of the Laplace transform with jw and separating the resulting complex number into its real and imaginary parts gives:

$$H(jw) = x + jy$$

$$x = (R_f/kR_i)/(1 + (CR_fw/k)^2)$$

$$y = -(R_f^2C_w/k^2R_i)/[1 + (CR_fw/k)^2]$$

The normalized magnitude ratio is then:

$$\left|A_N\right| = kR_i/R_f(x^2 + y^2)^{\frac{1}{2}}$$

or:

$$\left|A_N\right| = 1/[1 + (CR_fw/k)^2]^{\frac{1}{2}}$$

The above expression reveals the reason for giving time constants, (CR_f/k) and zero frequency gains, (R_f/kR_i) as nominal values earlier.

The normalized magnitude ratio for the conditioning filters used on the wall and bulk temperatures are shown in Figure A-5. They are presented in comparison to the normalized magnitude ratios of the wall and bulk energy storage elements as determined from the computed estimates of the weighting functions. This figure is of interest as it indicates that the "noise" which was smoothed out of the measured responses was just that--noise.

An alternative way of stating the proposition is that if the reactor is viewed as a filter, its characteristic response involves much stronger attenuation of the signal at a given frequency than does the characteristic response of the conditioning filter. Therefore it is

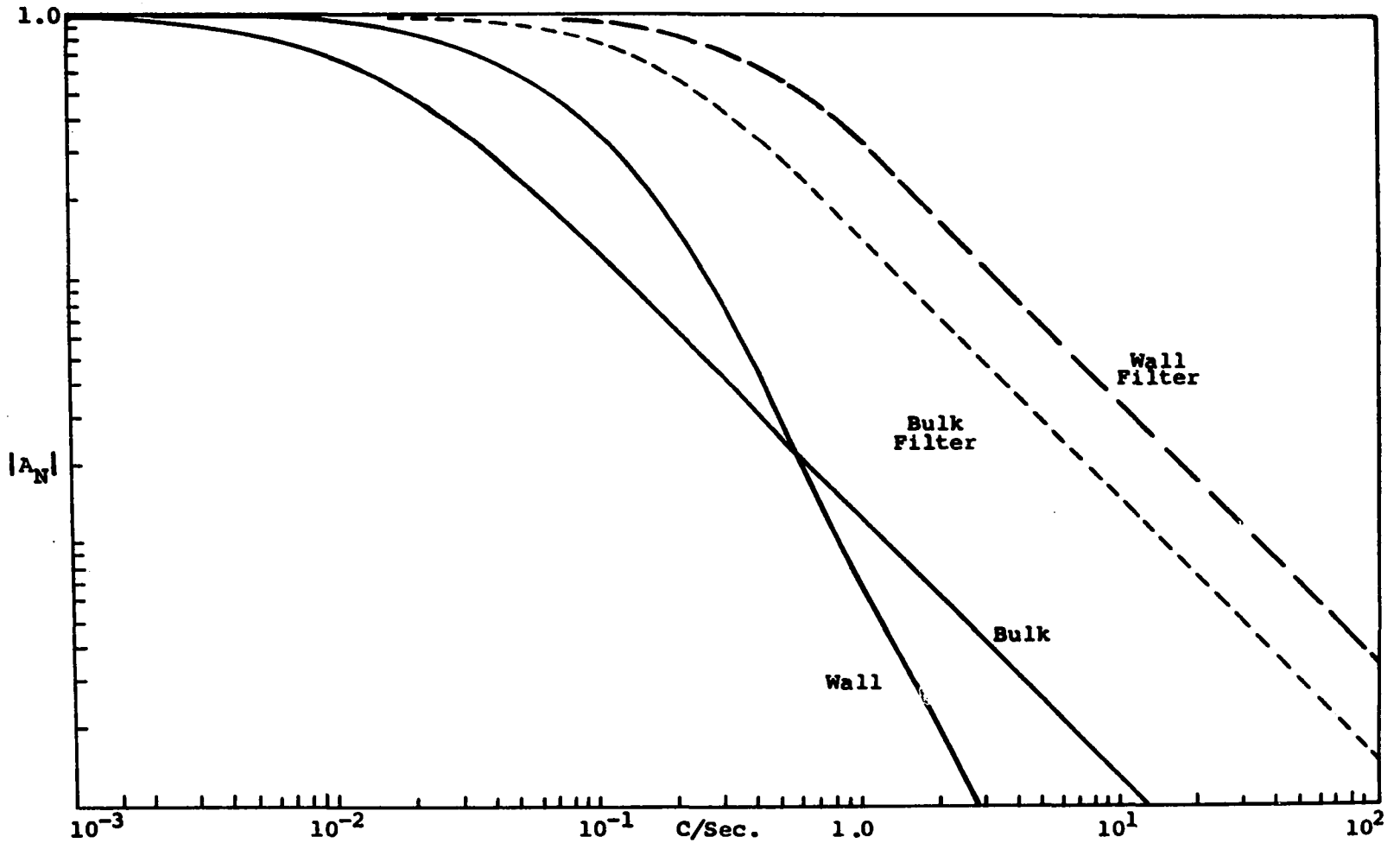


Figure A-5

Comparison of Normalized Magnitude Ratios for the Reactor System and Noise Filters

doubtful that any important level of response of the reactor has been lost because of the filters.

It is well known that the chaotic temperature variation with time seen when turbulent flow conditions prevail may be treated as a statistical variation about a time smoothed temperature, defined as:

$$\overline{T}(t) = (1/t^*) \int_0^{t^*} (\overline{T}+T')\,dt$$

where $\overline{T}$ is the time smoothed temperature, T' is the fluctuation about that temperature, and t* is an increment of time which is long compared to the period of T' but short compared to the time required for the system to respond. If this approach is made to smoothing $(\overline{T}+T')$ for successive intervals of t*, the result is a series of steps, the amplitude of which is $\overline{T}$ during a particular t*.

The filtering approach made in this investigation is recognized as essentially a series of time smoothings, where the initial instant of the second smoothing increment occurs at dt rather than t*. Of course the smoothing increment is approximately $5CR_f/k$.

APPENDIX E

CLOSED PROCESS FLUID CYCLE

A schematic diagram of the closed process fluid cycle, exclusive of the path through the hydraulic function generator (see Appendix F) and the reactor (see Chapter V), is shown in Figure A-6. The item and stream designations given therein correspond to those in the description which follows and in other appendicies which deal with the description of the experimental apparatus.

The process fluid (Super Service hydraulic lift oil) is pumped by the surge pump, P1 (1/2 hp, centrifugal), from the surge tank T3 (42 gal.), through a heat exchanger and into the constant temperature source tank, T1 (42 gal.). The purpose of cooling the process fluid with tap water in the exchanger is to insure that the temperature of the fluid entering T1 is lower than that of the contents of T1.

Skipping the equipment associated with T1 and the outlet stream which passes through the flow controller for the moment, the process fluid is removed from T1 through a standpipe (to maintain the tank level) and pumped by the crossover pump, P3 (1/10 hp, centrifugal), into the second constant temperature source tank, T2 (42 gal.). This tank

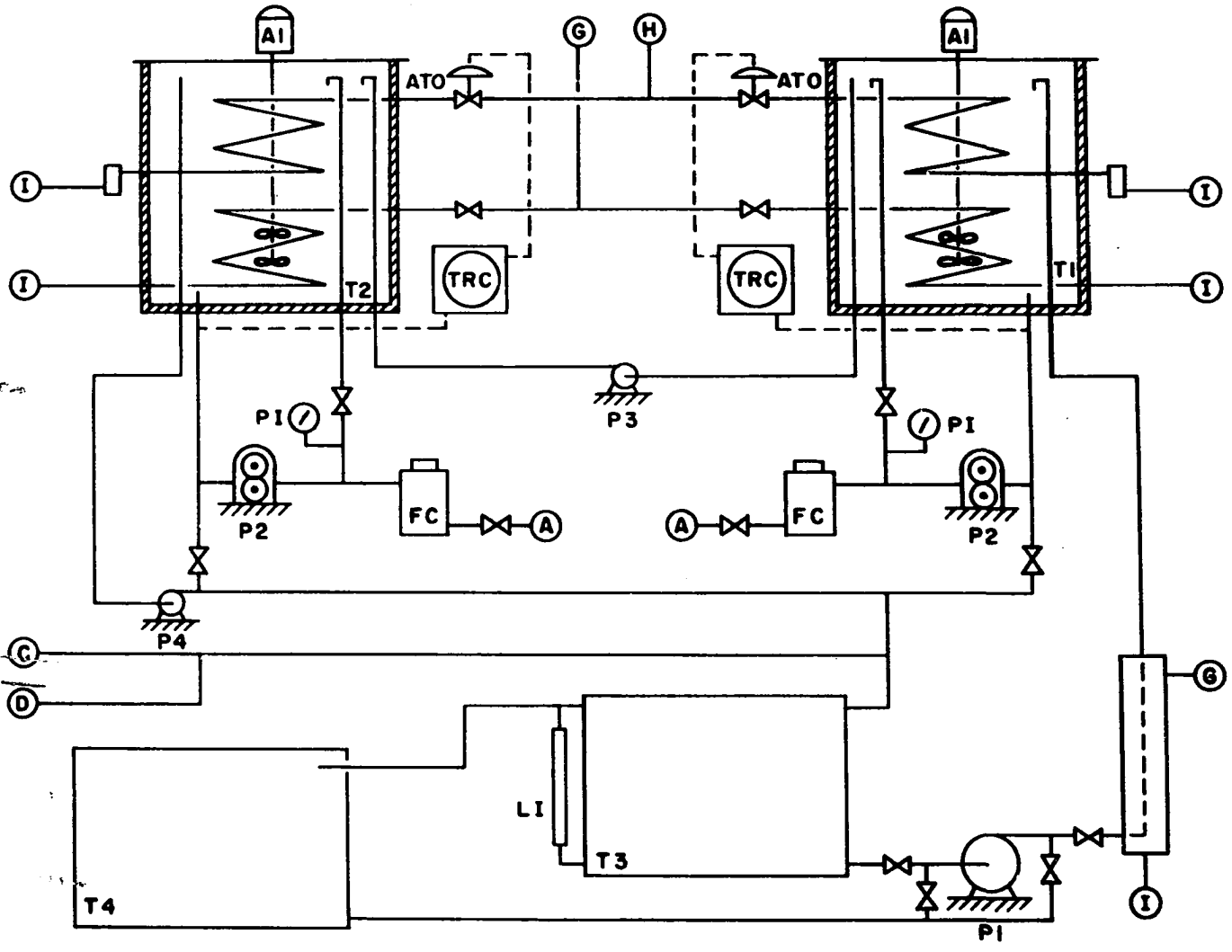


FIGURE A-6

SCHEMATIC DIAGRAM OF CLOSED PROCESS FLUID CYCLE

is maintained at a temperature which is higher than that maintained in tank T1.

The process fluid is removed from tank T2 through a standpipe (to maintain a liquid level) and pumped by the surge return pump, P4 (1/10 hp, centrifugal), back to the surge tank T3 via the surge tank manifold.

In as much as the two constant temperature source tanks, T1 and T2 are similarly constructed and equipped, only tank T1 need be described. The tank, which is well agitated by means of a Lightnin mixer, A1 (model NC2, 1/8 hp, 1725 rpm), is equipped with two sets of internal coils. One coil carries tap water, thus providing a relatively constant heat load which is large compared to the heat losses to the surroundings. The second coil carries steam for heating purposes.

The temperature of the contents of the tank is controlled by a feedback control system consisting of a copper-constantan thermocouple, installed in the tank outlet, (sensor), a Minneapolis Honeywell Brown Electronic Potentiometer Pyrometer, (two mode recording controller), and a Research Controls, Model 75S, ATO, pneumatic control valve which throttles the flow of steam to the heating coil (manipulative element).

Process fluid is pumped by the feed pump, P2 (California Bronze, 3 gal./min., gear pump driven by a Goulds #2, 3/4 hp electric motor), from the bottom of the tank to the inlet of the Kates Model MF Flowrator (range: 0.1 to 1.5

gal./min.). The needle valve in the process fluid return line (downstream of the pump) is used in conjunction with the adjacent pressure indicator to adjust the pressure upstream of the flow controller.

The points designated A in Figure A-6 represent connections to the hydraulic function generator. The systems upstream of these points perform the function of making available two streams of essentially constant temperature and flow rate.

The points designated C and D in Figure A-6 represent connections to the flow splitter section of the hydraulic function generator and the outlet of the reactor respectively. These streams are combined in the surge tank manifold and pass back to the surge tank.

The storage tank, T4 (55 gal.), together with the surge tank provide storage for the process fluid when the system is not being operated.

The points designated G on the schematic diagram represent connections to tap water facilities. The point designated H represents a connection to a source of steam. The points labeled I represent connections to drains.

A test of the ability of the system to perform the required function showed the capability to maintain temperatures of 110°F. and 180°F. within ± 0.2°F. for two hours under flow conditions of 2.0 ± .03 gal./min. for the combined outlet streams (measured with an appropriate turbine flow rate sensor at the outlet of the mixing section of the hydraulic function generator).

APPENDIX F

HYDRAULIC FUNCTION GENERATOR

A schematic diagram of the hydraulic function generator which was used to force the temperature of the process fluid at the reactor inlet is shown in Figure A-7. The item and stream designations given therein correspond to those in the description which follows and in the other appendicies which deal with the description of the experimental apparatus.

As described in Appendix E, the process fluid is available (at points A in Figures A-6 and A-7) at two distinct and essentially constant temperatures and flow rates. These streams pass through the chambers of the double ended double acting piston and cylinder arrangement (see Figure A-7). The cylinder is twenty-four inches long, four inches in diameter (inside) and is made of brass. The piston consists of double acting rubber cups mounted on a one inch diameter stainless steel shaft. The cylinder heads are aluminum and have been drilled and tapped to accept 1/2 inch pipe connections.

After passage through the piston and cylinder arrangement, the two streams are joined in the mixing section.

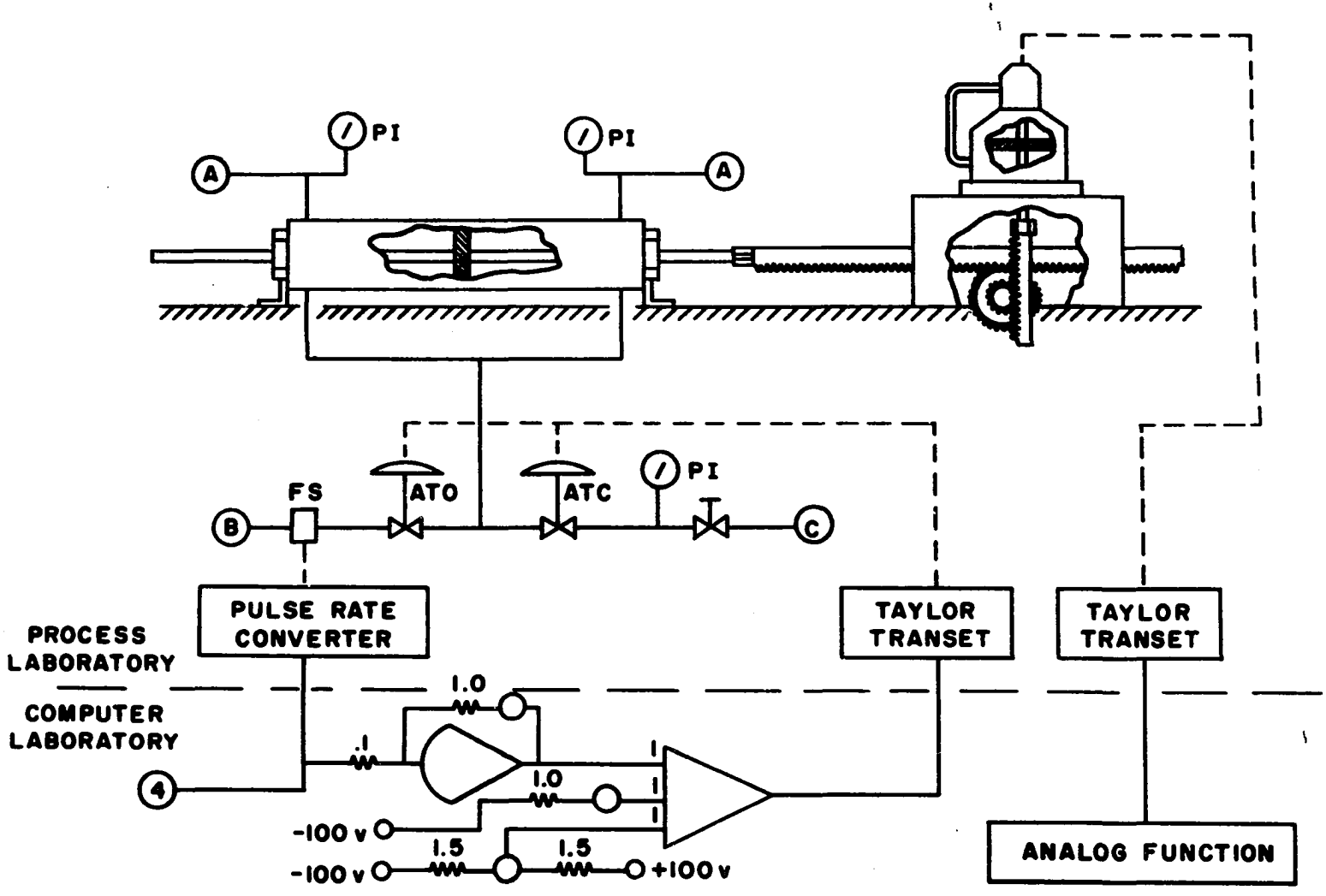


FIGURE A-7

SCHEMATIC DIAGRAM OF THE INLET TEMPERATURE FORCING SYSTEM

Subsequently the stream (of intermediate temperature) is separated into two portions in the flow splitter section. One portion of the flow passes through the flow rate sensor (Waugh Turbine Flowmeter, Model FL-6SB-1, range: 0.15 to 1.0 gal./min.) and then to the reactor inlet (point B in Figure A-7). The other portion of the stream passes through a needle valve (adjusted to approximate the reactor pressure drop) and then to the surge tank, T3, manifold (points C in Figures A-7 and A-6).

In principle, the operation performed by the piston and cylinder arrangement is that of generation of a constant flow rate stream, the temperature of which is proportional to the derivative of piston position. This operation may be explained simply as follows.

Assume that process fluid is entering the chambers of the cylinder at 120 and 140°F. respectively at identical flow rates of 0.8 gal./min.. Provided that the piston is stationary, the flow rate and temperature of the stream leaving the mixing section is 1.6 gal./min. at 130°F. Now let the piston be moved toward the low temperature end of the cylinder at a constant rate of 19.6 in./min. The flow into each chamber remains 0.8 gal./min. but the flow rate out of the low temperature chamber is 0.8 plus the rate of decrease of chamber volume (0.1 gal./min.). Similarly the flow rate out of the high temperature chamber is reduced to 0.7 gal./min. The result is 1.6 gal./min. leaving the mixing section at a temperature of 128.7°F.

If the piston is stopped, the ratio of flow rates returns to its original value and the temperature, at the exit of the mixing section, returns to its original value according to that section's time constant. The approximate relation between the temperature of the fluid leaving the mixing section and the rate of change of chamber volume, or derivative of piston position, is apparent.

Clearly there are important practical limitations to the actual operation of the function generator. For example, the rate at which the piston may be moved is limited by the flow rates through the chambers. Also, the finite length of the cylinder places a limitation of the period of time for which a monotonic function may be applied to the generator.

An advantage of the particular nature of the function generator's operation, in terms of providing the system forcing for generalized pulse testing, is that to "remove" the forcing function (see Chapter II), one has only to stop moving the piston.

In order to generate motion of the piston, a Fischer pneumatic valve operator is employed. The four inch stroke of the valve operator (in response to a 3 to 15 psi control air signal) is multiplied by a factor of five and changed, in direction, by 90° in a double rack and pinnion gear box which was designed and constructed for this investigation. The piston is, of course, connected directly to the output rack of the gear box.

The integral of the desired forcing function is generated electrically by the analog computer and transmitted to the instrument panel in the Process Laboratory. There the signal voltage is transduced by means of a Taylor Transet, Model 701T electro-pneumatic transducer to a 3 to 15 psi signal which is transmitted to the valve operator.

A requirement for successful operation of the function generator which has been mentioned only as an assumption is that the flow rates into the chambers are, in fact, constant. Even though the flow controllers, C2, (Figure A-6) are endowed with impressive specifications (with respect to the quality of flow control in the face of downstream pressure fluctuations), it was found necessary to introduce a feedback control system on the flow splitter section.

The flow rate sensor provides a train of electrical pulses, the frequency of which is proportional to the flow rate. A transducer (Waugh Pulse Rate Converter, Moder FR-111, 0 to 250 mv output) converts the frequency of pulses to a continuous voltage. This voltage is transmitted to the analog computer, where it is amplified and negatively summed with an arbitrary control signal. The resulting error signal is transmitted to the Process Laboratory instrument panel where it is transduced (Taylor Transet, Model 701T) to a 3 to 15 psi pneumatic signal.

This pneumatic signal is transmitted to a pair of linear control valves (Research Controls, Type 75, $c_v = 0.2$,

range spring: 3 to 15 psi), one of which operates air to close, the other, air to open. These valves control the flow of process fluid through the legs of the flow splitter section.

Notice that this control system would permit introduction of arbitrary variation of the flow rate simply by making the arbitrary control signal something other than a constant voltage. This technique was employed to attempt non-linear forcing of the reactor. Unfortunately, the hysteresis of the control valves was of such magnitude that following extreme manipulation of the flow rate, the return to equilibrium level was not reproducible; thus precluding the use of the technique.

APPENDIX G

CLOSED REACTOR COOLANT CYCLE

A schematic diagram of the reactor coolant cycle is shown in Figure A-8. The item and stream designations given therein correspond to those in the description which follows and in other appendicies which deal with the description of the experimental apparatus.

The coolant (a 50% by weight mixture of glycol and water) was maintained at 30°F. in the coolant reservoir, T5, by means of a freon refrigeration system (T6, the compressor, and the coil in T5).

The coolant is withdrawn from the reservoir by the coolant feed pump P5 (1/4 hp, gear pump) and fed to the reactor's cooling jacket (point E in Figure A-8). After passage through the jacket, the coolant returns to the reservoir (from point F in Figure A-8).

The coolant reservoir is well agitated by a Precision Scientific Company mixer, A2, which is rated at twenty watts.

The temperature of the coolant in the reservoir is controlled by means of a simple on-off controller which uses a Fenwal bi-metallic switch (Catalog number 17552-0, range: -100 to 600°F.) as a temperature sensor.

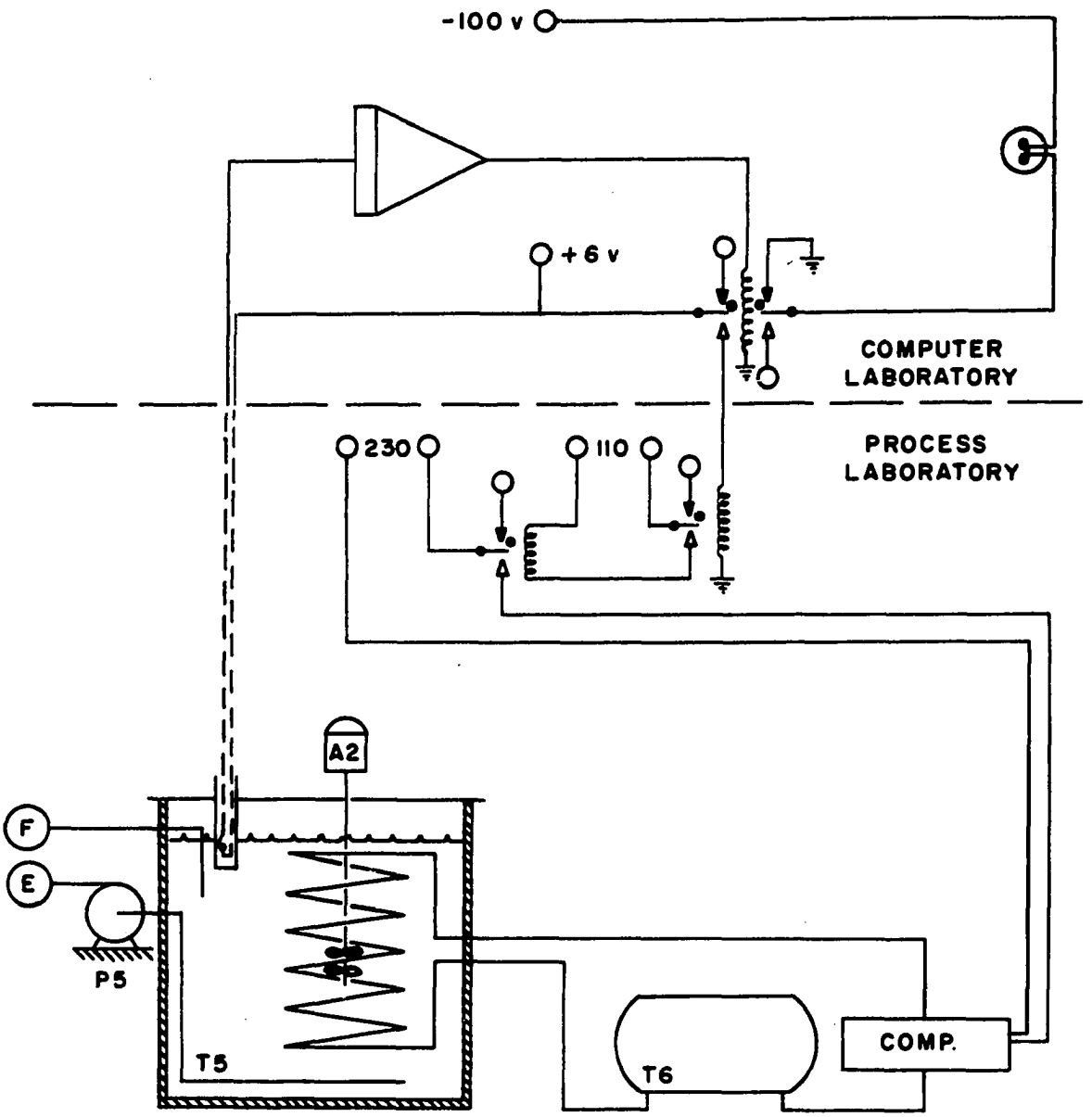


FIGURE A-8

SCHEMATIC DIAGRAM OF COOLANT CYCLE

A temperature rise closes the switch connecting a six volt (dc) signal, through a chatter smoothing circuit, across the coil of a computing relay (located in the analog computer). Closure of the computing relay contacts connects six volts (dc) across the coil of a relay located on the process laboratory instrument panel. Activation of this relay places 110 volts (AC) across the coil of a power relay on the panel which in turn completes the 230 volt (AC) power circuit to the freon refrigeration system compressor.

Operation of the compressor causes a pressure rise in the freon reservoir, T6, and subsequent passage of freon through the refrigeration coil located in the coolant reservoir.

The subsequent drop in coolant temperature causes the bi-metallic switch to open, thus deactivating all relays and disconnecting the power to the compressor.

The closure of the computing relay contacts also makes the ground connection for a circuit which lights a small neon bulb on the analog computer console, thus providing remote indication of normal operation.