Technical Note

Generalized bending of a plate with a circular inclusion of arbitrary rigidity

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Abstract: The paper presents an exact closed-form solution for a thin isotropic plate containing a circular elastic inclusion and subjected to arbitrary bending moments at large distances from the hole. Quantitative results are presented for the bending stress concentration factors for the plate and for the inclusion.

Keywords: inclusion, plate bending, stress concentration

NOTATION

a	radius of the inclusion (see Fig. 1)
$a_1, b_1, A_i, B_i, C_i, F_i$	constants of integration
B	biaxial bending-moment ratio $\equiv M_y/M_x$
D_i	flexural rigidity $= E_i h^3/[12(1-\nu_i^2)]$
E_i	modulus of elasticity
k	$= D_2/D_1$
K_i	bending stress concentration factor
$(M_r)_i$, $(M_\theta)_i$	bending stress couples in r, θ coordinates
$(M_{r\theta})_i$	twisting stress couple in r, θ coordinates
M_x, M_y	bending stress couples in x, y coordinates
$(Q_r)_i$	radial-edge transverse shear stress resultant
r	radial position coordinate (see Fig. 1)
$(V_r)_i$	effective radial-edge shear stress resultant
w_i	normal deflection
x, y	rectangular Cartesian coordinates (see Fig. 1)
θ	angular position coordinate (see Fig. 1)
ν_i	Poisson's ratio
ρ	$= r/a$
∇^2	Laplacian operator

Subscripts

$i = 1$	plate
$i = 2$	inclusion

1 INTRODUCTION

The problem of a large thin isotropic plate containing a stress-free circular hole and subjected to either uniaxial bending moment or cylindrical bending was solved by Bickley [**1**] in 1924. Recently, this work was generalized by the present author [**2**] to the case of generalized bending. The problem of a large plate containing a rigid or elastic inclusion and subjected to three bending loading states and transverse shear was solved by Goland [**3**] in 1943.

The objective of the present work is to generalize the work in references [**2**] and [**3**] to the case of a large plate with an isotropic elastic inclusion subjected to a generalized bending moment field.

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2 ANALYSIS

The geometry of the problem is a thin isotropic plate containing a circular inclusion at the origin. Cartesian (x, y) and polar (r, θ) coordinates are measured in the mid-plane of the plate from the centre-line of the inclusion (Fig. 1). A far-field bending moment state having principal bending moments M_x and M_y per unit length is considered. The biaxial bending-moment ratio is defined as

$$B \equiv \frac{M_y}{M_x} \tag{1}$$

The values of B for various special cases are listed in Table 1, in which ν_1 denotes Poisson's ratio of the plate.

Let subscripts 1 and 2 refer to the plate and the inclusion respectively. Then, the general expression for the dimensionless deflection of the plate can be expressed in plane polar coordinates (r, θ) as

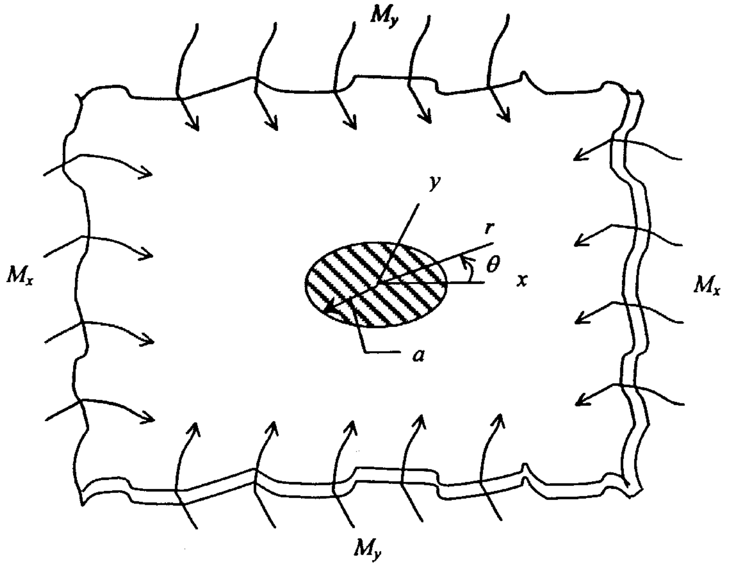

Fig. 1 A thin plate containing a circular inclusion and subjected to bending stress couples M_x and M_y at great distances from the centre of the inclusion

Table 1 Values of biaxial bending-moment ratio for various loading cases

Loading case	B
Uniaxial bending stress	0
Cylindrical bending (uniaxial bending strain)	ν_1
Balanced biaxial bending (axisymmetric)	1
Pure twisting	−1

$$\frac{w_1}{a} = -\frac{M_x a}{2D_1}[a_1\rho^2 + b_1\rho^2\cos(2\theta) + A_1 \ln\rho + B_1\cos(2\theta) + C_1\rho^{-2}\cos(2\theta) + F_1] \quad (2)$$

where a is the radius of the inclusion, a_1, b_1, A_1, B_1, C_1 and F_1 are constants to be determined, D_1 is the plate flexural rigidity and $\rho \equiv r/a$.

For the inclusion, which is assumed to be solid, i.e. not to contain a hole, the dimensionless deflection is

$$\frac{w_2}{a} = -\frac{M_x a}{2D_2}[A_2\rho^2 + B_2\rho^2\cos(2\theta) + C_2\rho^4\cos(2\theta)] \quad (3)$$

where A_2, B_2 and C_2 are constants to be determined and D_2 is the inclusion flexural rigidity.

The boundary conditions at great distances from the inclusion (ρ increasing without bound) are that M_x must be finite, that the transverse shear force Q_x must vanish at $\theta = 0$ rad, that $M_y = BM_x$ must be finite and that Q_y must vanish at $\theta = \pi/2$. Thus,

$$M_r = \frac{M_x}{2}[(1+B) + (1-B)\cos(2\theta)]$$
$$M_{r\theta} = -\frac{M_x}{2}(1-B)\sin(2\theta), \qquad Q_r = 0 \quad (4)$$

To relate the plate displacement w_1 to the bending and twisting moments, the constitutive relations

$$M_r = -\frac{D_1}{a^2}(w_{,\rho\rho} + \nu_1\rho^{-1}w_{,\rho} + \nu_1\rho^{-2}w_{,\theta\theta})$$
$$M_{r\theta} = -(1-\nu_1)\frac{D_1}{a^2}\left(\frac{w}{\rho}\right)_{,\rho\theta} \quad (5)$$

are used. Combining equations (2), (4), and (5), it is found that

$$a_1 = \frac{1+B}{2(1+\nu_1)}, \quad b_1 = \frac{1-B}{2(1-\nu_1)} \quad (6)$$

The interfacial continuity conditions which must be satisfied at the interface ($\rho = 1$) are matching deflection, radial slope, radial bending moment and radial effective shear force:

$$w_1(1, \theta) = w_2(1, \theta) \quad (7)$$

$$\frac{\partial w_1}{\partial r}(1, \theta) = \frac{\partial w_2}{\partial r}(1, \theta) \quad (8)$$

$$(M_r)_1(1, \theta) = (M_r)_2(1, \theta) \tag{9}$$

$$(V_r)_1(1, \theta) = (V_r)_2(1, \theta) \tag{10}$$

Satisfaction of equation (7) leads to the following two relations:

$$A_2 = \frac{D_2}{D_1}(a_1 + F_1) \tag{11}$$

$$B_2 + C_2 = \frac{D_2}{D_1}(b_1 + B_1 + C_1) \tag{12}$$

Solution of equation (8) yields the two relations

$$A_2 = \frac{D_2}{D_1}\left(a_1 + \frac{A_1}{2}\right) \tag{13}$$

$$B_2 + 2C_2 = \frac{D_2}{D_1}(b_1 - C_1) \tag{14}$$

Since the radial bending stress couple is

$$(M_r)_i = -D_i\left[\frac{\partial^2 w_i}{\partial r^2} + \nu_i\left(\frac{1}{r}\frac{\partial w_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_i}{\partial \theta^2}\right)\right]$$

where $i = 1, 2$, satisfaction of equation (9) gives the following two equations:

$$2(1 + \nu_2)A_2 = 2(1 + \nu_1)a_1 - (1 - \nu_1)A_1 \tag{15}$$

$$(1 - \nu_2)B_2 + 6C_2 = (1 - \nu_1)b_1 - 2\nu_1 B_1 + 3(1 - \nu_1)C_1 \tag{16}$$

The effective radial-edge shear force is

$$(V_r)_i = (Q_r)_i + \frac{1}{r}\frac{\partial (M_{r\theta})_i}{\partial \theta}$$

where

$$(Q_r)_i = -D_i \frac{\partial}{\partial r}(\nabla^2 w_i)$$

$$(M_{r\theta})_i = -(1 - \nu_i)D_i\left(\frac{1}{r}\frac{\partial^2 w_i}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial w_i}{\partial \theta}\right)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Thus, satisfaction of equation (10) requires that

$$2(1 - \nu_2)B_2 - 6(1 + \nu_2)C_2 = 2(1 - \nu_1)b_1 - 2(3 - \nu_1)B_1 - 6(1 - \nu_1)C_1 \tag{17}$$

Equations (11) to (17) constitute seven equations in the seven unknown coefficients. Their solutions are as follows:

$$A_1 = \frac{1 + B}{1 + \nu_1}\frac{1 + \nu_1 - (1 + \nu_2)k}{1 - \nu_1 + (1 + \nu_2)k}$$

$$A_2 = \frac{1 + B}{1 + \nu_1}\frac{1}{1 - \nu_1 + (1 + \nu_2)k}$$

$$B_1 = \frac{1 - B}{1 - \nu_1}\frac{1 - \nu_1 - (1 - \nu_2)k}{3 + \nu_1 + (1 - \nu_2)k}$$

$$B_2 = \frac{1 - B}{1 - \nu_1}\frac{2}{3 + \nu_1 + (1 - \nu_2)k}$$

$$C_1 = -\frac{B_1}{2}, \qquad C_2 = 0, \qquad F_1 = \frac{A_1}{2} \tag{18}$$

where

$$k \equiv \frac{D_2}{D_1} \tag{19}$$

Since

$$(M_\theta)_i = -D_i\left(\frac{1}{r}\frac{\partial w_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_i}{\partial \theta^2} + \nu_i\frac{\partial^2 w_i}{\partial r^2}\right)$$

it can be shown that

$$\frac{(M_\theta)_1}{M_x} = (1 + \nu_1)a_1 + \tfrac{1}{2}(1 - \nu_1)A_1\rho^{-2} + [(-1 + \nu_1)b_1 - 2B_1\rho^{-2} - 3(1 - \nu_1)C_1\rho^{-4}]\cos(2\theta) \tag{20}$$

and

$$\frac{(M_\theta)_2}{M_x} = (1 + \nu_2)A_2 + [-(1 - \nu_2)B_2 + 6\nu_2 C_2\rho^2]\cos(2\theta) \tag{21}$$

The bending moment stress concentration factors for the plate ($i = 1$) and inclusion ($i = 2$) are defined as

$$K_i = \frac{(M_{\max})_i}{M_x} \tag{22}$$

where $(M_{\max})_i$ is the maximum value of $(M_r)_i$, $(M_\theta)_i$ and $(M_{r\theta})_i$.

For biaxial ratios in the range $-1 \leqslant B \leqslant 1$, the maximum values of moment are the circumferential values at the interfacial radius ($\rho = 1$) at an orientation $\theta = \pi/2$. Thus,

$$K_i = \frac{(M_\theta)_i(1, \pi/2)}{M_x} \tag{23}$$

Finally,

$$K_1 = 1 + \tfrac{1}{2}(1 - \nu_1)A_1 + 2B_1 + 3(1 - \nu_1)C_1 \tag{24}$$

and

$$K_2 = (1 + \nu_2)A_2 + (1 - \nu_2)B_2 \tag{25}$$

3 NUMERICAL RESULTS AND DISCUSSION

Numerical values of K_1 and K_2 are given in Figs 2 and 3 respectively for various values of the dimensionless material parameters ν_1, ν_2 and k and bending moment ratio B.

It can be seen that both the K_1 and the K_2 curves have a value of unity when the stiffness ratio $D_2/D_1 = 1$. This makes sense because then the entire plane is homogeneous, i.e. there is no discontinuity.

When the stiffness ratio $D_2/D_1 = 0$, the inclusion vanishes and the plate contains simply a hole. Then the plate stress concentration K_1 depends upon the biaxial ratio B and the equation coincides exactly with that given in reference [**2**], which reduces to those given by Bickley [**1**] for the cases that he considered. The inclusion stress concentration factor vanishes for D_2/D_1, because in fact there is no inclusion present in this case.

As the inclusion stiffness is increased, the stress concentration K_1 in the plate decreases (Fig. 2) while that (K_2) in the inclusion increases. It is interesting to note that the various families of curves for K_2 cross at $D_2/D_1 = 1$. Thus, the effects of the biaxial ratio B reverse at this point. For instance, for $D_2/D_1 = 0.5$, the largest value of K_2 corresponds to $B = 1$ while, for $D_2/D_1 = 2$, the largest value of K_2 corresponds to $B = -1$. In contrast, the various curves for K_1 converge at $D_2/D_1 = 1$ but they do not cross, i.e. they do not reverse.

As a practical example, suppose that a plate has a hole. Now determine the effects of inserting an inclusion and of changing the biaxial ratio. It can be seen from Fig. 2 that the stiffer the inclusion (the higher the D_2/D_1 ratio), the lower is the bending stress ratio K_1; for instance, for $B = 0$ and $D_2/D_1 = 3$, K_1 is only 0.5. This means that the presence of a stiff inclusion reduces the maximum bending moment from approximately to $1.7M_x$ to $0.5M_x$. The most beneficial biaxial ratio B for a D_2/D_1 ratio of 3 is $B = -1$, which reduces the maximum bending moment to 0.3 approximately.

This question may arise: what is the effect of purely Poisson ratio mismatch, i.e. $D_2/D_1 = 1$ but $\nu_2 \neq \nu_1$? As a

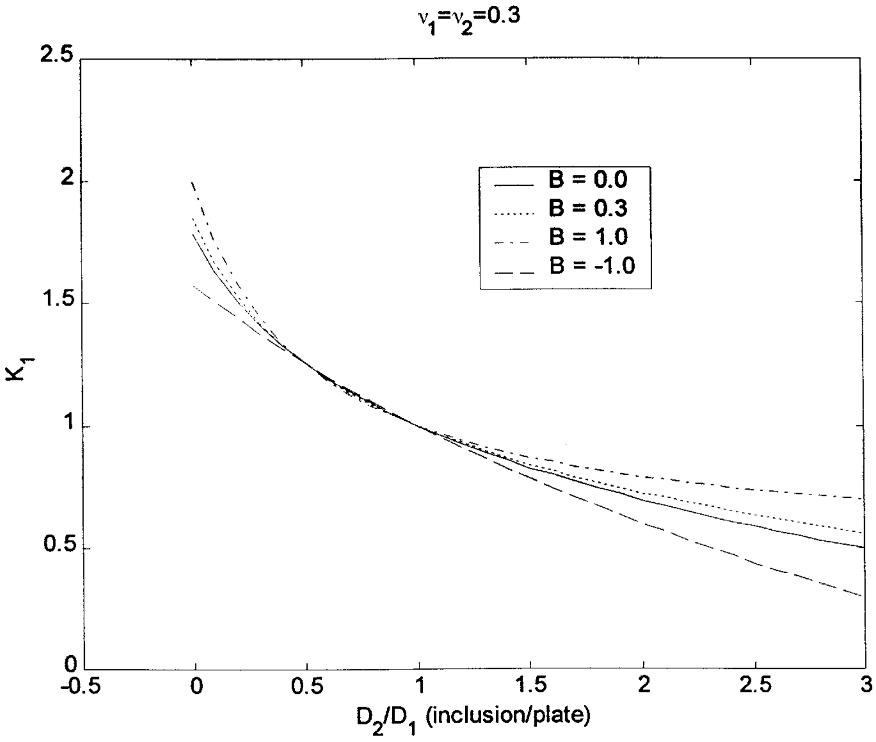

Fig. 2 Dimensionless maximum circumferential bending moment *in the plate* as a function of the ratio of the flexural rigidity of the inclusion to that of the plate

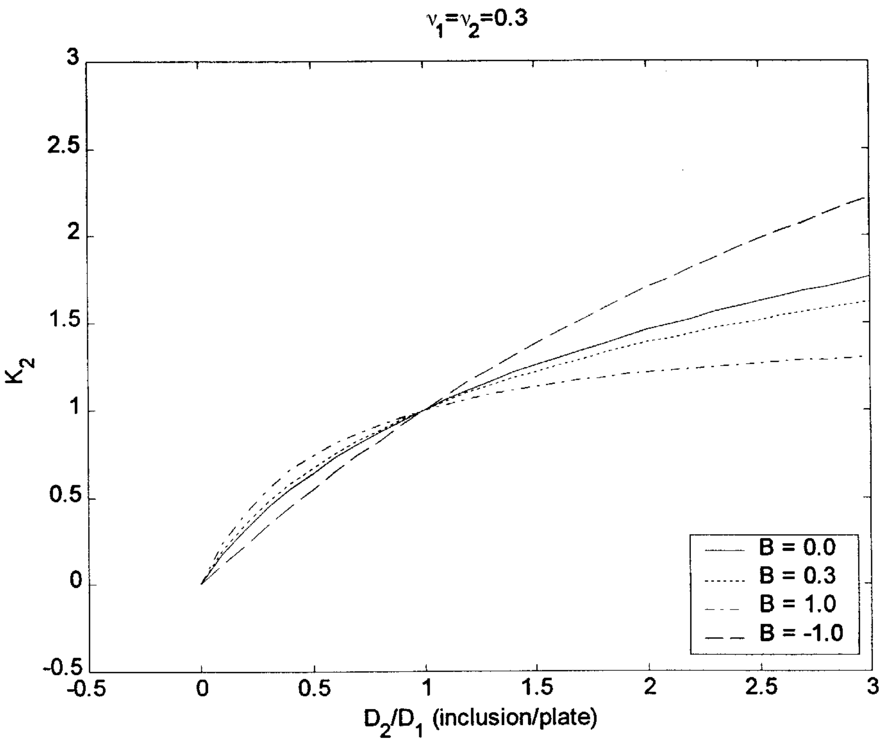

Fig. 3 Dimensionless maximum circumferential bending moment *in the inclusion* as a function of the ratio of the flexural rigidity of the inclusion to that of the plate

numerical example, this case is calculated: $\nu_1 = 0.3$, $\nu_2 = 0.1$ and $B = 0$. The results are $K_1 = 0.9653$ and $K_2 = 1.0823$. Thus, the stress in the plate is slightly relieved while the stress in the inclusion is somewhat elevated. In contrast, if Poisson's ratios are reversed, i.e. $\nu_1 = 0.1$, $\nu_2 = 0.3$ and $B = 0$, the resulting values are $K_1 = 1.0008$ and $K_2 = 0.9466$, i.e. K_1 is larger but K_2 is somewhat lower.

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