ASYMMETRIC STABILITY ROBUSTNESS BOUNDS FOR UNCERTAIN LINEAR SYSTEMS WITH FIRST-ORDER LYAPUNOV METHOD

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NOMENCLATURE

A_n: discharging area of the nozzle

A_N: nominal system matrix

A_p: cross sectional area of the spool valve

B_a: viscous damping constant of torque motor and flapper

B_s: viscous damping constant of spool valve

C_{do}, C_{df}, C_{dd}: discharge coefficient of upstream, nozzles, and drain orifices

d_o, d_n, d_d: diameter of upstream, nozzles, and drain orifices

E: perturbation matrix

i, i_m: electrical current differential and maximum electrical current differental respectively

I: identity matrix

J_a: inertia of armature and attached load

K_a, K_s: spring constants of torque motor and spool valve respectively

K_L: internal leakage coefficient

K_t: torque constant of the torque motor

k_i: uncertain perturbation parameters

k_i^-, k_i^+: lower and upper bounds of uncertain parameter k_i

$k^{(j)}$: j-th vertex of perturbation parameter space's hypercube

M_s: mass of moving spool

m: number of uncertain perturbation parameters

n: dimension of a nominal matrix

P, P_o: Lyapunov matrix and nominal Lyapunov matrix respectively

ΔP: perturbation of Lyapunov matrix

P_1, P_2, P_e: hydraulic pressures

$\overline{P}_{1o}, \overline{P}_{2o}, \overline{P}_{eo}$: dimensionless pressure values at null

$\Delta\overline{P}_{1o}, \Delta\overline{P}_{2o}, \Delta\overline{P}_{eo}$: small variations of dimensionless null pressures

P_s: supplying hydraulic pressure

Q: arbitrary positive definite matrix

$Q_{o1}, Q_{o2}, Q_{n1}, Q_{n2}, Q_e, Q_{sv}, Q_L$: flow rates

QF: a quadratic function for optimal Lyapunov function

r: equivalent length of flapper

t, t_o: time and initial time respectively

u: control input vector

V: Lyapunov function

V_t, V_e: internal oil volumes

x, x_o: state vector and initial state vector respectively

x_{fo}: equilibrium flapper position

x_f, x_p: displacements of flapper and spool respectively

x_{pm}: maximum displacement of spool

α: angle of attack for an aircraft movement

β: effective bulk modulus

$\overline{\sigma}(.)$: maximum singular value of matrix (.)

ε: small constant value

$\mu(.)$: matrix measure of matrix (.)

μ_p, μ_{PU}, μ_{YS}: robustness measures

λ: performance measure of a Lyapunov function candidate

$\lambda(.)$: eigen values of the matrix (.)

$\lambda_{max}(.)$,$\lambda_{min}(.)$: largest and smallest eigenvalues of matrix (.) respectively

ρ: oil mass density

Ω, Φ: symbol of a convex hull (polytope)

Γ: a set of vertex matrices

Π: a hypercube of perturbation parameter space

Λ: performance measure of a Lyapunov function candidate

Ψ: perturbation matrix set

$|.|$: modulus matrix of (.)

$||.||$: Euclidean norm of a vector (.)

CHAPTER I

INTRODUCTION

Overview

The design and analysis of a control system is based on the mathematical model of the physical plant. One of the fundamental challenges facing a control engineer is to account for and accommodate the inaccuracies in the mathematical models of physical systems used for controller design. The presence of inaccuracies in the mathematical model results from simplifications such as lumped parameter approximations, simplified relations, ignored high-order dynamics, linearizations about operating points, neglected instrumentation uncertainties, and changes in the system component properties due to time and environmental effects.

The nominal model of a physical plant often has the form of an autonomous linear system with uncertain parameters. Uncertain perturbation parameters can exist in the form of structured or unstructured perturbations. A structured uncertainty represents those uncertainties whose sources can be explicitly identified in a parameter model. Meanwhile an unstructured uncertainty is a lumped uncertainty that may represent several uncertainties that cannot be explicitly expressed in a parameter model. In unstructured uncertainty, only the bound on the norm of the perturbation matrix is given. This approach leads to overly conservative results in many instances, since the robustness criteria do not identify the perturbation structure of uncertain parameters. This research addresses structured uncertainties.

In controller design, these uncertain perturbations can be accommodated by the use of either adaptive or robust controllers. If the bounds of the perturbations are known, robust controllers are often utilized, and this has motivated the design of robust controllers for multi-variable linear systems ([Dorato 87], [Dorato 93]). The fundamental requirement for the design of robust control is the ability to analyze system stability and robustness. Stability analysis is concerned with the state trajectories for perturbations of an initial condition from its equilibrium point or reference trajectory. In the analysis and design of robust control systems, it is essential to determine to what extent a nominal system remains stable when subject to a certain class of perturbation. These bounds for perturbation parameters in which the system remains stable are referred to as stability robustness bounds.

The approaches to estimating the stability robustness bounds of linear time-invariant systems can be viewed from two perspectives: one is the time domain approach based upon state space equations, and the other is the frequency domain approach primarily based upon system transfer function. The frequency domain approach has been extensively studied in the past ([Safonov 77], [Barrett 80], [Doyle 81], [Lehtomaki 81], [IEEE 81]). The main approach in frequency domain analysis is to extend the classical single-input, single-output stability margin to multi-input, multi-output systems by use of the singular-value decomposition method. The nonsingularity of a matrix is the criterion used to determine stability robustness bounds. Barett [Barett 80] presented a useful summary and a comparison of different robustness tests available with respect to their conservatism. One of the most important developments in robust stability analysis and control, in the frequency-domain, has been achieved in H_2 and H_∞ theories (see [Francis 87] for

H_2 and H_∞ theories).

Kharitonov's theorem [Kharitonov 78] is the most celebrated work on the stability of a family of characteristic polynomials in the frequency domain. Kharitonov proved that the stability of dynamic systems whose parameter uncertainty is restricted to a hyperrectangular domain is guaranteed by the stability of four extreme polynomials whose parameters take values at the vertices of the hyperrectangle. Kharitonov's result for the interval polynomials dramatically alleviates the excessive computational demands of a stability test which would simply invoke repeated root or eigenvalue computations over a "sufficiently fine" grid of points within the family. However, there are several restrictions in the application of Kharitonov's theorem: first, if a polynomial family has mutually dependent coefficients, which is almost always the case in control applications, Kharitonov's result can not be applied directly but should be modified to provide sufficient conditions [Bartlett 88]; second, since there are no specific guides for choosing the vertex polynomials, one may try the epsilon-iteration algorithm [Barmish 87] to enlarge the polytope if the size of the initial polytope is not satisfactory; third, Kharitonov's theorem is only applicable to the interval polynomials and only considers the strict Hurwitz property of polynomials. Hence many researchers have made efforts to generalize Kharitonov's theorem ([Bartlett 88],[Barmish 89], [Chapellat 89], [Peterson 90], [Foo 91], [Cavallo 91], [Xu 93]). The well-known Edge Theorem ([Lin 87], [Bartlett 88]) extends Kharitonov's result to polytopes of polynomials. The stability of the exposed edges of a polytope of polynomials is both necessary and sufficient for the stability of the entire polytope. For a survey and further insight into Kharitonov's type of results, see [Barmish 87], [Chapellat 88], [Jury 90] .

The natural conjecture for Kharitonov's type of results would be that the family of interval system matrices (matrices whose elements vary independently in given intervals) in the time domain is stable if and only if all the vertex matrices are stable. Bialas [Bialas 83] claimed that it is true; however, it turned out to be false as shown by Barmish and Hollot [Barmish 84] who constructed counterexamples. Barmish et al. [Barmish 88] also provided counterexamples for three plausible conjectures which are directly motivated by the results in the polynomial cases: the first conjecture is checking edges of a polytope of matrices (i.e., convex combinations of a number of matrices) in view of the Edge Theorem for polynomials; the second conjecture is checking edges of a hyperrectangle rather than a polytope; the third conjecture is mapping interval matrices onto the set of characteristic polynomials. All three conjectures failed. Similarly, Jiang [Jiang 88] attempted to prove the above conjectures for the discrete time case, which is also false as pointed out by Soh [Soh 89]. Various sufficient conditions for stability of a polytope of matrices ([Barmish 86], [Shi 86], [Cobb 89], [Kokame 90], [Qian 92], [Fang 94b]) have been proposed, while some necessary and sufficient conditions ([Kokame 91], [Wang 91b], [Qian 92]) were obtained for special vertices in the parameter domain. Barmish and Kang [Barmish 93] provided an extensive literatures survey of extreme point results for robust stability of systems with structured parametric uncertainty. For a polytope of matrices in the time domain, the stability problem is far from completely resolved. The stability problem of a polytope of matrices is treated in this research by using the convexity property of matrix measure ([Desoer 75], [Wang 91a], [Fang 94]).

Although many of the stability robustness criteria developed in the frequency domain are significant, it is also useful to analyze stability robustness in the time domain, especially when a broader class of parameter perturbations have to be considered. The time domain approach is more amenable to the consideration of structured perturbations in the form of parameter variations and nonlinearities [Siljak 89]; and the time domain approach generally involves checking only a finite number of inequalities, often just one, while the frequency domain methodology requires all criteria over the whole range of frequencies to be satisfied [Petkovski 89].

The time domain approach has been primarily based on Lyapunov theory, while root locus-based techniques [Qiu 86, Juang 87, Yedavalli 88] were sometimes utilized. The advantage of the Lyapunov-based approach is the capability to deal with nonlinear time-varying perturbations. The drawback in the application of the Lyapunov method is that estimates of the stability bounds are often too conservative to be used in high performance controller designs [Juang 87] and dependent upon the particular Lyapunov function used. Therefore, it is highly desirable to develop a better method for estimation of stability bounds. Research on the Lyapunov-based stability robustness is conducted in this study.

There is an extensive amount of literature on the use of the Lyapunov direct method in robust stability and control problems. Early efforts include [Bellman 69], [Barnett 70], [Gutman 75], [Davison 76], [Desoer 77], [Ackermann 80], and [Eslami 80]. Despite the considerable results of the time-domain stability conditions in these references, explicit stability bounds on the uncertain parameter perturbation of a linear system were first proposed by Siljak [Siljak 78] and Patel and Toda [Patel 80]. Since then, stability robustness conditions based on the Lyapunov direct method have been widely investigated ([Yedavalli 85b], [Barmish 86], [Zhou 87], [Siljak 89], [Latchman 91], [Olas 92], [Chen 93], [Olas 94a], [Olas 94b]).

Patel and Toda [Patel 80] studied the stability of linear systems with unstructured perturbation and obtained the upper Euclidean bound of the perturbation. Yedavalli [Yedavalli 85b], Yedavalli and Liang [Yedavalli 86], and Zhou and Khargonekar [Zhou 87] further improved the stability robustness bounds of the perturbation parameters. Yedavalli [Yedavalli 85b] obtained an upper bound for the interval perturbation for robust control. Yedavalli and Liang [Yedavalli 86] used a state space transformation before applying Yedavalli's results [Yedavalli 85b] to reduce the conservatism in stability bounds estimation. Barmish and DeMarco [Barmish 86] proposed a technique for the parameterization of the Lyapunov matrix to obtain less conservative stability bounds. Zhou and Khargonekar [Zhou 87] studied the robust stability of systems with perturbations that are linear combinations of a finite number of matrices and improved Yedavalli's results [Yedavalli 85b]. Siljak [Siljak 89] demonstrated that the estimation of the stability bounds of perturbation is strongly dependent upon the selection of the system state space.

Latchman and Letra [Latchman 91] proposed an optimization procedure to systematically choose the best Lyapunov matrix so that the conservatism of stability bounds is reduced. Chen and Han [Chen 93] proposed a modified Lyapunov-based method where iterative interpolations of quadratic Lyapunov functions are considered. Olas [Olas 92] proposed another approach to the problem of stability robustness based on the construction of optimal Lyapunov functions. In a series of works, Olas ([Olas 92], [Olas 94a], [Olas 94b]) proposed a recursive algorithm for the design of optimal Lyapunov functions which resulted in better estimates of stability bounds, approaching the maximum volume of the hypercube. Some of these methods ([Latchman 91], [Olas 92], [Chen 93]) provided less conservative stability estimates with recursive algorithms. However, these methods demand much computational efforts, and sometimes the procedure is empirical and subjective. For example, Olas [Olas 94b] showed that the stability bounds resulting from the recursive procedure are highly dependent on the way the estimated hypercube is enlarged; and no explicit method for the enlargement was proposed in Olas' works. Therefore the development of a better (i.e. less conservative) and easily applicable method for estimation of stability bounds is still highly desirable.

Analysis of robust stability based upon the Lyapunov theorems consists of two principal steps: first, the generation of a Lyapunov function ([Schultz 65] and [Mohler 89]); second, the determination of the robustness bounds based upon the generated Lyapunov function.

It should be noted that stability estimates were conservative because the preceding methods fail to consider the structured features of the uncertainties when generating a Lyapunov function, and because the directional property of the parameter variation was neglected. Most past works have used the zero-order Lyapunov robustness method. In the zero-order method, the nominal matrix Lyapunov equation is used to generate a nominal "P" matrix, which is then used to compute stability bounds for the perturbed system. This conventional method is referred to as the zero-order method in the sense of Taylor series expansion applying the perturbed dynamic model into the Lyapunov equation. Many early works on stability robustness bounds were restricted to bounds on the absolute values of the uncertain parameters, i.e., symmetric parameter variation with respect to the origin. These restrictions sometimes result in very conservative estimates of stability bounds.

Significant progress ([Bernstein 89], [Leal 90], [Gao 93]) has been made recently in obtaining less conservative stability bounds by removing the restrictions already listed. In [Leal 90], the Lyapunov function varies with perturbation in the system matrix so that the method is called a first-order Lyapunov robustness method. In the first-order method, the perturbed dynamic model is used to generate a perturbed P matrix. This perturbed P matrix is then used to obtain modified stability bounds. In the sense of Taylor series expansion, this method is referred to as the first-order method. Meanwhile, the bounds in [Bernstein 89, Gao 93] are not necessarily symmetric. In particular, the estimate bounds obtained in [Gao 93] are expressed in terms of the uncertain parameters, rather than a convex hull over intervals as in [Bernstein 89].

However the bounds developed in [Leal 90] were symmetric bounds and the zero-order Lyapunov method was utilized in [Gao 93]. And the asymmetric stability bound estimates obtained in [Gao 93] is needed to expand to the entire region of the hypercube with maximum and minimum values of uncertain parameters. Hence, in this research, the first-order Lyapunov robustness method is applied to the method proposed in [Gao 93] to estimate asymmetric stability bounds and to further reduce the conservatism of the stability bounds.

For the current investigation, the structure of system equations with uncertain parameters, the convexity property of matrix measure, the asymmetric structure of perturbations, and the first-order terms for the parameter perturbation with Taylor series expansion are used to prove the new lemmas and theorems which have enabled the development of a procedure for better estimation of stability robustness bounds. The stability robustness bounds obtained by the development of these theorems are revealed to be less conservative than results of the early works of ([Zhou 87], [Gao 93]). It is also proved that the new proposed method can inherit the property of the optimal Lyapunov function in Olas' works ([Olas 92], [Olas 94a], [Olas 94b]). In other words, the new method can be completed by using the Olas' results. Theorems are derived to prove that the stability bounds estimated by the asymmetric first-order method are always less conservative than those of the zero-order method under certain conditions.

The method proposed in this research provides three distinct advantages: first, ease of the application to system matrices with structured uncertain parameters; second, improved means to estimate less conservative stability bounds; third, the ability to extend to the properties of the optimal Lyapunov function for the systematic enlargement of hypercube of perturbation parameter space; fourth, the extensibility to better estimate measure of robustness bounds for eigenvalue distribution of uncertain linear systems.

Several simple examples and practical design problems are considered to demonstrate the practicality and the advantages of the proposed method for the estimation of stability bounds with respect to the previously reported methods. Examples demonstrate superiority of the stability bounds estimated by the proposed method over those obtained by other methods [Zhou 87, Gao 93]. Two practical design problems for an electrohydraulic servovalve and a fighter aircraft control show that the new method significantly improves the "conventional" bound estimates providing less conservative estimates, so that the new method can be effectively applied to practical design problems for control systems.

Contributions of This Research

This research demonstrates a new approach to the stability robustness problem of a linear time-invariant system with structured uncertain parameters. There are six principal contributions made by this research.

First, a theorem referred to as the Expansion Theorem is derived. It is to verify whether the basic asymmetric robustness bounds estimated by the Gao's method can be expanded in the full hypercube whose vertices are a linear combination of maximum and minimum values of uncertain perturbation parameters. The Expansion Theorem enables the stability test to be simply applied at a vertex to find if the stability boundaries can be expanded to the full region of the hyper-quadrant to which each vertex belongs.

Second, a new method for the analysis of stability robustness, referred to as the asymmetric first-order Lyapunov method, is developed by combining Gao's asymmetric stability bounds with the first-order Lyapunov method. The new method considers the structured features of uncertainties, when generating a quadratic Lyapunov function, and the directional property of perturbation parameters, so that it resolves the problem of conservative stability bound estimates that past methods often gave rise to.

Third, using the properties of the optimal Lyapunov function and convexity of polytope of matrices, it is mathematically proved that, under certain condition, the asymmetric first-order Lyapunov method always provides better (less conservative) estimates of stability bounds than those of Gao's asymmetric stability bounds.

Fourth, a theorem is developed to determine the optimal vertex for the application of the asymmetric first-order method. It is proved that applying the asymmetric first-order method to the vertex with better performance measure of the Lyapunov function can provide better estimates of stability bounds.

Fifth, new measures for robust eigenvalue-assignment of uncertain linear systems are presented. These robustness measures for eigenvalue-assignment are generalization of the results for the estimate of stability robustness bounds.

Sixth, a new design for a two-stage electrohydraulic servovalve with a variant drain orifice is proposed. The effect of a variant drain orifice on a first-stage flapper-nozzle is analyzed to show that the variant drain orifice can enhance the valve performance across the null position and the overall stability of the servovalve simultaneously. It is also proved that the new asymmetric first-order method is easily applicable and provides a substantially large estimation of stability bounds for the design of a two-stage electrohydraulic servovalve.

Organization of Contents

Chapter II presents a discussion of the issues of stability of Lyapunov method, and Chapter III describes principal stability robustness methods proposed so far by other authors. Theorems for Gao's asymmetric stability bound estimation and the first-order method in the Lyapunov theorem and its physical meaning in the sense of Taylor series expansion are described in Chapter IV. Based on the results of the Chapter IV, the lemmas and theorems for the expansion of Gao's result to the full hypercube and the new asymmetric stability robustness bounds with the first-order method are established in Chapter V. Two numerical examples to validate the new method are demonstrated in Chapter VI. The examples compare the results of new method with those of the early methods to establish the effectiveness of new method. Application of the new method to electrohydraulic servovalve design and to the sensor degradation problem of a fighter aircraft is also included in Chapter VII. Chapter VIII deals with robustness measures for uncertain linear systems where the eigenvalues of the perturbed systems are guaranteed to stay in a prescribed region. Similar to the results for the estimate of stability robustness bounds in Chapter V, new techniques to estimate allowable perturbation parameter bounds for the robust eigenvalue distribution are derived. Chapter IX describes the conclusion of this research and suggests the areas for future researches. Appendix A summarizes the preliminaries of uncertainties, matrix norms and matrix measure. A review of Lyapunov stability analysis is described in Appendix B. It includes the fundamental Lyapunov stability theorems, the Lyapunov equation for linear systems and the generation of the Lyapunov function. Appendix C summarizes the lemmas and theorems proposed by Gao [Gao 93]. Appendix D illustrates the design of a new two-stage electrohydraulic servovalve with a variant drain orifice to enhance both the servovalve performance and the stability of the servovalve.

CHAPTER II

A REVIEW OF ROBUST STABILITY ANALYSIS

Introduction

In the analysis and synthesis of control systems, a fundamental problem is that the mathematical description of a physical plant is always characterized by uncertainty or modeling error. In addition, parameter variations are often present in the system dynamics. The figures and characteristics of the uncertainties are described in Appendix A. In recent years, the robustness of control systems, i.e., the ability to maintain performance in face of uncertainties has received much attention, and much research effort has been devoted to the analysis and synthesis of such systems.

More attention has been given to the robustness analysis of multivariable feedback systems, however stability robustness evaluation is still a fundamental problem in control theory that has yet to be completely resolved. These problems have been investigated in both the frequency domain and the time domain. In the case of linear time-invariant systems, there are two basic common models: the state space representation and the transfer function or transfer matrix representation. For continuous systems, the former is a time domain description of the system using first-order linear differential equations, while the latter is a frequency domain description that maps the system from the time domain to the frequency domain using the Laplace transformation technique.

The most productive developments, in robust control in the frequency domain description for the system model, may be the H_2 and H_∞ theories, developed from the "small gain principle" introduced by Zames [Zames 63]. Levine and Reichert [Levin 90] provided an introduction to the H_∞ system control design, and Francis [Francis 87] also contributed an excellent introduction to H_∞ theory. Most of the investigations of this subject have been based on transfer function representation. When system uncertainties can be translated into the uncertainties of the parameters in the characteristic polynomial, Kharitonov-type approaches ([Barlett 88], [Barmish 87], [Barmish 88], [Barmish 89], [Chapellat 89]) give a good stability test that considers all possible values of the uncertain parameters. In general, it is not a trivial task to determine the characteristic polynomial from the state space model when some of the parameters are given at intervals, especially when the system order is high, so alternative approaches are desirable.

From the discussions above, it seems appropriate to focus on time domain approaches. This is because the uncertainties of a system are defined as the uncertainties of parameters in the state space model, which have specific physical meanings. For the time domain, the Lyapunov direct method has been widely used for the investigation of system robustness; this is because this method provides ready accommodation for both nonlinear and time-variant systems. It should be noted that the small gain theorem is concerned only with nominally linear systems. This is an important factor since the solution of nonlinear differential equations can be difficult or impossible.

For robustness, the application of the Lyapunov direct method consists of two principal steps: first, the generation of the Lyapunov function; second, the determination of the robustness bounds based upon the generated Lyapunov function. Patel and Toda [Patel 80] considered linear autonomous systems with nonlinear, time-varying unstructured vector perturbations and unstructured perturbations, and formulated estimates for the robustness bounds. Yedavalli [Yedavalli 85] improved the accuracy of these estimates when considering structured perturbations. The bounds obtained by the application of these methods were not directly dependent on the structure of the nominal matrix. Yedavalli and Liang [Yedavalli 86] improved estimation of the bounds by the transformation of states. For the case of structured perturbations, Zhou and Khargonekar [Zhou 87] improved the robustness bounds by separating independent perturbation elements within the perturbation matrix. Siljak [Siljak 89] suggested the use of a vector Lyapunov function introduced by Matrosov [Matrosov 82] and Bellman [Bellman 78], to reduce the conservatism of the estimates. Juang [Juang 91] considered robustness for linear time-invariant systems, including linear autonomous systems with time-varying perturbations as a special case.

In the case of structured perturbations, the conservatism of the estimates was principally caused by two factors: first, the failure to consider the structured features of the perturbations when generating a Lyapunov function; second, neglecting the directional sign of the uncertain parameters. The problem studied in the research of ([Yeda 86], [Zhou 87], [Martine 87], [Mansour 89]) was quite general. It applies to all linear time-invariant systems which have parameter variations with the additive perturbations residing at intervals symmetric to zero. It is, therefore, not surprising that the stability bounds obtained were very conservative. They are usually given as small, symmetric intervals around the origin in the parameter space. However, in many cases it is reasonable to assume that the signs of the uncertain parameters are known. For example, when an elevator on an aircraft is stuck, one can usually tell whether it is stuck upward or downward by the movement of the plane and this information can be translated into the signs of certain parameters of the system. Gao [Gao 93] first found that better (less conservative) robustness bounds can be obtained if the signs of the uncertain parameters are known. However, Gao's method was not sufficient to estimate the full stability robustness bounds allowable for the given parameters. Although this research is based on the Gao's results, it overcomes the drawback of Gao's method.

Application of the Lyapunov Method to the Stability Robustness

As mentioned in Appendix B, the Lyapunov stability theorems have been established for the perturbations of initial conditions near an equilibrium point. These theorems were subsequently extended for application to perturbations of the system parameters [Leipholz 87]. The conclusions derived from the Lyapunov direct method are: if for a system, there exists a single Lyapunov function for all choices of the perturbation parameters within a compact bounded set, the system stability of equilibrium for the nonlinear and time-variant perturbations is insured.

The uncertainties, the matrix norm and matrix measure are described in Appendix A. The remainder of this chapter reviews selected research on the robust stability analysis of nominally autonomous linear systems. Three perspectives for the approach to the problem of stability robustness are considered: first, the analytical method with matrix norm, eigenvalues and singular values; second, the stability of a convex hull (polytope) of matrices with their vertex matrices; third, numerical recursive algorithms to obtain less conservative stability bounds.

Analytical Methods

Patel and Toda [Patel 80], in an extension of their paper on robustness analysis for linear state feedback design [Patel 77], considered nonlinear unstructured vector perturbation and unstructured perturbations for nominally autonomous linear systems.

Consider the following system

$$\dot{x} = A_N x + f(x,t) \tag{2.1}$$

where A_N is a time-invariant asymptotically stable matrix and f(x,t) is a time-varying nonlinear vector function of x(t), representing the nonlinear unstructured vector perturbations within the system and f(0,t) = 0 for all times. The fundamental stability robustness problem is to determine the magnitude of perturbation f(x,t) such that the system (2.8) remains stable. The answer to this problem is obtained in the form of the following theorem.

Theorem 2.1 [Patel 80]

The system on Equation (2.1) is stable if

$$\frac{\|f(x,t)\|}{\|x\|} \leq \mu_P := \frac{\min \lambda(Q)}{\max \lambda(P)} \tag{2.2}$$

where $\|.\|$ *is a Euclidean norm of a vector (.),* $\lambda(.)$ *is an eigenvalue of matrix (.) and P is the unique positive definite solution of the Lyapunov equation*

$$A_N^T P + P A_N = -2Q \tag{2.3}$$

where Q *is a positive definite matrix.* □

Further, it was proved that for unstructured perturbations, the robustness bound μ_P on the condition (2.2) is maximal for Q = I, where I is an identity matrix. In a particularly important case of linear perturbations where

$$f(x,t) = E(t)\, x(t) \tag{2.4}$$

it was proposed that the system

$$\dot{x}(t) = (A_N + E(t))\, x(t) \tag{2.5}$$

is stable if

$$|e_{ij}(t)| \le \mu_{PU} := \frac{1}{n\,\bar{\sigma}(P)} = \frac{1}{n\,\lambda_{max}(P)} \tag{2.6}$$

where e_{ij} is the $(i,j)^{th}$ element of perturbation matrix E, n is the dimension of the nominal matrix A_N and $\bar{\sigma}(.)$ represents a maximal singular value of matrix (.).

The results of [Patel 80] did not consider the structure of perturbation. This was accomplished in the work of Yedavalli [Yedavalli 85a], where the stability robustness bound estimate obtained on (2.6) is improved in the following theorem.

Theorem 2.2 [Yedavalli 85a]

The time invariant version of the system (2.5) is stable if

$$\varepsilon := \max |e_{ij}| \le \mu_{YS} := \frac{1}{\bar{\sigma}[\,\| P \|\, U_E]_s} \tag{2.7}$$

where $\| . \|$ *is the modulus matrix which means that all elements of matrix (.) are replaced by their absolute values,* $[.]_s := [(.) + (.)^T]^{1/2}$ *is the symmetric part of the corresponding matrix, and* U_E *is a nXn matrix with all elements equal to 1.* □

It has been also shown in [Yedavalli 85a] that

$$\bar{\sigma}[\,\| P \|\, U_E]_s \le n\,\bar{\sigma}(P) \tag{2.8}$$

which analytically proves that the bound on (2.7) is better than the one on (2.6).

The follow-up paper [Yedavalli 85b] has shown that the matrix U_E can be chosen such that

$$U_{E_{ij}} = \frac{e_{ij}}{\varepsilon} \tag{2.9}$$

Stability robustness results obtained in ([Yedavalli 85a], [Yedavalli 85b]) were applied to the design of linear regulators [Yedavalli 85c] and to the stability analysis of interval matrices [Yedavalli 86a]. In [Yedavalli 86b] it was indicated that the stability robustness bounds can be improved even more by using state transformation and solving the corresponding algebraic Lyapunov equation in the new coordinates.

Based upon the fact that the stability of a system is invariant with respect to nonsingular linear transformation, Yedavalli and Liang [Yedavalli 86b] transformed the state vector through M, $x = M\hat{x}$ for the new system

$$\dot{\hat{x}} = \hat{A}(t)\,\hat{x}(t) \tag{2.10}$$

where $\hat{A}(t) = M^{-1}A(t)\,M$.

By changing the system matrix $\hat{A}$ in the Lyapunov equation while maintaining $Q = I$, $\bar{\sigma}(P)$ is reduced which results an improvement of robustness bounds (see conditions on (2.6) and (2.7)). Examples were presented to demonstrate improvement of the bounds, with respect to those achieved by Patel and Toda [Patel 80], for the structured as well as unstructured perturbations. However, with the exception of a special case limited to the diagonal transformation matrix, the question of generating the matrix M remained unsolved.

Estimates for the upper bounds of perturbation elements presented by Patel and Toda [Patel 80], Yedavalli ([Yedavalli 85a], [Yedavalli 85b]) and Yedavalli and Liang [Yedavalli 86b] were not directly related to the structure of the nominal system matrix, rather they were indirectly influenced through the matrix P. Generalization of the results of ([Yedavalli 85a], [Yedavalli 85b]) to a class of structured perturbations appearing in the feedback control systems is given by Zhou and Khargonekar [Zhou 87].

Zhou and Khargonekar [Zhou 87] considered structured perturbations. In their work it is assumed that the perturbation matrix has the form

$$E = \sum_{i=1}^{m} k_i E_i \tag{2.11}$$

where $k_i \in \left[-k_i^-, k_i^+\right]$; $-k_i^-$ and k_i^+ are negative lower and positive upper bounds of uncertain parameters respectively varying in the symmetric interval around zero. The main result of [Zhou 87] is given in the following theorem.

Theorem 2.3

The time invariant version of the system (2.5) is stable under structured perturbation defined on (2.11) if one of the following conditions is satisfied:

$$\sum_{i=1}^{m} k_i^2 < \frac{1}{\bar{\sigma}^2(P_e)}$$

$$\sum_{i=1}^{m} |k_i| \, \bar{\sigma}(P_i) < 1 \tag{2.12}$$

$$|k_j| < \frac{1}{\bar{\sigma}\left(\sum_{i=1}^{m} |P_i|\right)}, \quad j=1,\ldots,m$$

where $P_i := \tfrac{1}{2}\left(E_i^T P + P E_i\right)$, $i = 1, 2, \ldots, m$, *and* $P_e := \left[P_1, \ldots, P_m\right]$. □

These bounds were less conservative than those derived prior to this formulation. Yedavalli [Yedavalli 85a] proved that the bound obtained on Theorem 2.2 is a special case of the third condition of (2.12).

Siljak [Siljak 89] performed an extensive review of the parameter space methods for analysis and design of robust control systems, and demonstrated that estimation of the stability bounds of perturbation is strongly dependent upon the selection of the system state space.

Siljak ([Siljak 89], [Siljak 90]) also used the property of the vector Lyapunov function to develop a method, so-called "connectivity stability" ([Siljak 72], [Siljak 78]), to determine the stability of a large scale system which is composed of several decoupled subsystems. The concept of the vector Lyapunov function was introduced by Matrosov [Matrosov 82] and Bellman [Bellman 78]. This concept associates several scalar functions with a given dynamic system in such a way that each function determines a desirable stability property in a part of the state where others do not. These scalar functions are considered as components of a vector Lyapunov function. The results of "connective stability" [Siljak 90] are described as follows:

Considered the system

$$S_E : \dot{x}_i = A_i\, x_i + \sum_{j=1}^{N} k_{ij}\, A_{ij}\, x_j, \quad i = 1,\ldots,N \tag{2.13}$$

to be an interconnection of N subsystem

$$S_i : \dot{x}_i = A_i\, x_i, \quad i = 1,\ldots,N \tag{2.14}$$

where A_i are negative definite matrices and $|k_{ij}| < k_{ij}^{+}$ are perturbation elements. The system S_E can be rewritten in a compact form

$$S_E : \dot{x} = A_D\, x + A_C\, x \tag{2.15}$$

where $A_D = \{A_1,\ldots,A_N\}$ and $A_C = (k_{ij}\, A_{ij})$ are matrices of appropriate dimensions. Equation (2.14) implies that the couplings between the subsystems consist only of the perturbation elements, and the perturbation within each subsystem is unstructured.

Let $\mathcal{V}_i(x)$ represent the Lyapunov function for the i^{th} subsystem S_i such that

$$\mathcal{V}_i(x_i) = \left(x_i^T P_i x_i\right)^{1/2} \tag{2.16}$$

where P_i is the symmetric positive definite solution for the Lyapunov equation.

$$A_i^T P_i + P_i A_i = -Q_i \tag{2.17}$$

The Lyapunov function for the overall system S_E is selected as

$$V(x) = d^T \mathcal{V}(x) \tag{2.18}$$

where $\mathcal{V} \in \mathbf{R}^N$ is a vector Lyapunov function with components defined on Equation (2.16) and $d \in \mathbf{R}^N$ is a positive vector. It was then demonstrated that the overall system, S_E, is connectively stable if the following matrix is an M matrix [Siljak 78], i.e.

$$\begin{bmatrix} w_{11} & \cdots & w_{1j} \\ \vdots & \ddots & \vdots \\ w_{j1} & \cdots & w_{jj} \end{bmatrix} > 0, \quad j = 1,\ldots,N \tag{2.19}$$

where

$$w_{ij} = \begin{cases} \dfrac{1}{2}\dfrac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)} - k_{ij}^{+}\,\xi_i, & i = j \\[2ex] -k_{ij}^{+}\,\xi_{ij}, & i \neq j \end{cases} \tag{2.20}$$

and $\xi_{ij} = \lambda_{max}\left(A_{ij}^T A_{ij}\right)$.

As shown on Equation (2.20), Q_i can be set as the solution to the following problem to maximize chances of proving stability

$$\text{find: } \max_{Q_i} \left\{ \frac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)} \right\} \tag{2.21}$$

subject to the Lyapunov equation (2.17). Based upon the assumption that A_i has all distinctive eigenvalues, the maximum value of the ratio is found to be

$$\max_{Q_i} \left\{ \frac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)} \right\} = \bar{\sigma}(A_i) = \left| Re \ \ \lambda_{max}(A_i) \right| \tag{2.22}$$

For the special case, S_E is reduced to a single subsystem

$$S_E: \dot{x} = A_N \, x + E \, x \tag{2.23}$$

the system is stable if

$$\sigma(E) < \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} \tag{2.24}$$

This result is identical to that achieved by Patel and Toda [Patel 80] for the case of unstructured perturbations.

A general case for Zhou and Khargonekar [Zhou 87] is also considered by Juang [Juang 91] as follows:

Consider the system

$$\dot{x} = \sum_{i=1}^{m} k_i(t) \, A_i \, x \tag{2.25}$$

where $k_i \in \left[-k_i^-, \, k_i^+\right]$, $\sum_{i=1}^{m} |k_i(t)| \neq 0$ as a special case for the robustness analysis of autonomous linear systems with structured perturbations. Note that for $k_1(t) = 1$, the system is identical to that represented by Zhou and Khargonekar [Zhou 87].

First, for structured perturbations. Defining

$$\nu_j := \sum_{i=1}^{m} k_i(t)\, A_i \Big|_{k_i(t) = k_i^+ \text{ or } k_i^-}, \quad j = 1,\ldots,2^m \tag{2.26}$$

it was demonstrated that the system is stable if an invertible matrix P existed such that $\mu_2\left(P\, \nu_j\, P^{-1}\right) < 0$ for all $j = 1,\ldots,2^m$, where $\mu_2(.)$ denotes the matrix measure corresponding to a 2-norm (see Appendix A). As before, this approach left the issue of the generation of a Lyapunov function an open question, where the Lyapunov function is $V(x) = x^T P^* P x$ and P^* is the complex conjugate of P.

Polytope of Matrices

The application of Kharitonov's theorem to the robust stability of characteristic polynomials has motivated many studies for the polytope of matrices using a similar approach for the polynomial cases. Even though the plausible conjectures were proved to be false and the stability problem of a polytope of matrices is unresolved, there were some considerable achievements.

Barmish and DeMarco [Barmish 86] first considered the linear perturbation problem on Equation (2.5) with the perturbation matrices (2.11) as equivalent to the problem of determining the stability of the convex hull (polytope) of a finite set of stable matrices. Given the system and perturbation matrices of equations (2.5) and (2.11) one can generate a finite set of "extreme" matrices $\{A_1, \ldots, A_k\}$ having the following property: the system on (2.5) with perturbation matrices (2.11) is stable for all uncertain parameters k_i if and only if all matrices in the convex hull (polytope)

$$\Omega := \mathrm{conv}\{A_1, ..., A_k\} \tag{2.27}$$

are stable.

Denote the unit simplex in R^k by

$$\Gamma := \left\{ \alpha = (\alpha_1, ..., \alpha_k) \middle| \alpha_i \geq 0 \ \forall i, \sum_{i=1}^{k} \alpha_i = 1 \right\} \tag{2.28}$$

The main result in [Barmish 86] is given in the following theorem:

Theorem 2.4

The set of matrices Ω *is stable if*

$$M_{ij} := A_i^T P_j + P_j A_i < 0, \quad \forall i, j \tag{2.29}$$

where P_i *is the unique positive definite symmetric matrix satisfying the Lyapunov equation*

$$A_i^T P_i + P_i A_i = -I \tag{2.30}$$

□

Barmish and DeMarco [Barmish 86] also found that the stability robustness bounds in [Yedavalli 85b] are a special case of their works.

Theorem 2.4 verifies that a polytope of matrices is stable if there exists a positive definite quadratic function that is a Lyapunov function common to all vertex members. For normal vertex matrices, Wang [Wang 91b] derived a necessary and sufficient condition.

Theorem 2.5 [Wang 91b]

A polytope of matrices with (2.27) and (2.28) given by

$$\Omega := \left\{ A = \sum_{i=1}^{k} \alpha_i A_i \middle| \alpha_i \geq 0 \ \forall i, \sum_{i=1}^{k} \alpha_i = 1 \right\} \tag{2.31}$$

is Hurwitz stable if and only if all the vertex matrices, A_i, *are Hurwitz stable and normal,* *that is,* $A_i^* A_i = A_i A_i^*$, *where* A_i^* *is the conjugate transpose of* A_i. □

Fang and et al. provided a general sufficient condition for the stability of a family using matrix measure such that stability of the vertices guarantees stability of the convex hull of the matrix family.

Theorem 2.6 [Fang 94b]

The polytope of matrices (2.31) is stable if there exists a norm $\mu(A)$ *such that*

$$\mu(A_i) < 0, \ \forall i = 1,\ldots,k \tag{2.32}$$

□

Theorem 2.7 [Fang 94b]

Let V *be the set of vertex matrices for a polytope* Ω, *and* V *be *-closed which means* $V^* \in \Omega$. *Then* Ω *is stable if and only if there exists a matrix measure* μ *such that* $\mu(A) < 0$ *for any* $A \in V$. □

Fang et. al also proved that theorem 2.5 [Wang 91b] is another representation of their preceding theorems.

Recursive Numerical Method

Petkovski [Petkovski 89] used a time-domain stability robustness methodology in [Yedavalli 85c] to develop an iterative algorithm to determine the largest positive number e, such that the perturbed system

$$\dot{x} = (A_N + e\,E)\,x \tag{2.33}$$

where A_N is a time-invariant asymptotically stable matrix, E is given, and $e > 0$ is unknown, remains asymptotically stable. The criterion for the stability Petkovski used is in the following theorem:

Theorem 2.8

The system of Equation (2.33) is stable if

$$e < \frac{1}{\bar{\sigma}(\|P\| \, \|E\|)_s} \tag{2.34}$$

where P *is the solution of the Lyapunov matrix equation (2.3) when* Q = I. □

Latchman and Letra [Latchman 91] proposed an optimization algorithm to systematically choose the Lyapunov matrix Q so that the conservatism of stability bounds is reduced. The criterion for the stability that Latchman and Letra used is in the following theorem:

Theorem 2.9

The structurally perturbed system of equations (2.5) and (2.11) is stable if

$$\sum_{i=1}^{m} k_i^2 < \frac{1}{\left[\bar{\sigma}(M_Q)\right]^2} \tag{2.35}$$

where

$$M_Q := \begin{bmatrix} Q^{-1/2} F_1 Q^{-1/2} \\ \vdots \\ Q^{-1/2} F_m Q^{-1/2} \end{bmatrix} \tag{2.36}$$

$$F_i := \left(E_i^T P + P E_i\right), \quad i = 1, \ldots, m \tag{2.37}$$

□

The optimal choice of Q is the one which minimizes the norm of M_Q, thus providing a less conservative assessment of the upper-bound on the uncertain parameters.

Analysis and synthesis of control systems using linear matrix inequalities (LMI) have received much attention recently ([Boyd 94a], [Boyd 94b], [Boyd 95]) motivated by the advent of useful algorithms for convex optimization (interior-point methods by Nesterov and Nemirovsky [Nesterov 93] for example). As Horisberger and Belanger remarked in [Horisberger 76], stability robustness problem can be considered as a convex problem involving linear matrix inequalities.

A linear matrix inequality is a matrix inequality of the form

$$F(x) := F_o + \sum_{i=1}^{m} x_i F_i > 0 \tag{2.38}$$

where $x \in \mathbb{R}^m$ is a variable vector, and $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $i = 0, \ldots, m$ are given matrices. Thus, a linear matrix inequality is a constraint on the variable x whose set $\{x \mid F(x) > 0\}$ is convex.

For a system $\dot{x} = A\,x$ where matrix A is a polytope of matrices on (2.31), the sufficient condition for stability is the existence of a positive definite matrix P such that

$$A_i^T P + P A_i < 0, \quad i = 1, \ldots, k \tag{2.39}$$

Boyd [Boyd 94a] showed that the inequality (2.39) can be transformed to the form (2.38) in P.

Olas [Olas 92] proposed an approach to the problem of stability robustness based on the construction of the optimal Lyapunov function. The concept of the optimal Lyapunov function [Olas 94a] is summarized.

Determine the derivative of the Lyapunov function candidate V, along solutions of Equation (2.5) and introduce a function

$$\Lambda(t,x) := \frac{\dot{V}(t,x)}{V(x)}, \quad \|x\| \neq 0 \tag{2.40}$$

Define the performance measure λ of the function of the function V as an upper bound of the function $\Lambda(t,x)$

$$\lambda := \sup_{x \in R^n, x \neq 0} \ \sup_{t \in [o,\infty)} \Lambda(t,x) \tag{2.41}$$

It is said that a function V is better than $\widetilde{V}$ if the performance measure λ of the function V is smaller than the performance measure $\widetilde{\lambda}$ of $\widetilde{V}$. Denote the distance *d* between two functions $V_1 = x^T S_1 x$ and $V_2 = x^T S_2 x$ by a norm of a difference between matrices S_1 and S_2, i.e., $d = ||S_1 - S_2||$. The function V is called optimal if all the neighboring functions at less than some distance *d* from V are not better than V. When searching for the solution of the robust stability problem of Equation (2.5) with a Lyapunov matrix equation such that

$$P_o A_N + A_N^T P_o = -2I \tag{2.42}$$

in a form of a hypercube $\Pi \in \mathsf{R}^m$, one has 2^m quadratic forms

$$QF_j := x^T R_j x, \quad j = 1, \ldots, 2^m \tag{2.43}$$

where

$$R_j := \left[A_N^T P_o + P_o A_N + E^T P_o + P_o E\right]_j \tag{2.44}$$

The subscript "j" denotes j-th vertex of 2^m vertices, $k^{(1)}$, $k^{(2)}$, ..., $k^{(2^m)}$, on the hypercube Π.

Consider the arbitrary form of perturbed Lyapunov function $\Delta V = x^T \Delta P x$, and let ΔQF_j denote 2^m forms resulting from entering the vertices $k^{(j)}$ into the derivative of ΔV along the solutions of Equation (1). Then, the theorem addressed in [Olas 94a] follows.

Theorem 2.10 (Olas' Theorem)

If there is a matrix ΔP *such that the corresponding forms* ΔQF_j, $j = 1, \dots, q$ *satisfy*

$$\Delta QF_j(\zeta_j) := \zeta_j^T \, \Delta R_j \, \zeta_j < 0, \quad j = 1, \dots, q \tag{2.45}$$

where

$$\Delta R_j := \left[A_N^T \, \Delta P + \Delta P \, A_N + E^T \, \Delta P + \Delta P \, E\right]_j \tag{2.46}$$

then for sufficiently small ε *the function* $V + \varepsilon \, \Delta V$ *is better than the function* V where ζ *is a root of the form* QF_j. □

A recursive algorithm for the problem of the stability robustness was proposed by Olas ([Olas 92], [Olas 94a], [Olas 94b]). However, no systematic method to enlarge the parallelepiped Π was proposed. It was found that the stability robustness bounds resulting from the procedure of the algorithm are highly dependent on the way the parallelepiped Π is enlarged [Olas 94b].

As reviewed so far, most methods for stability robustness have used the zero-order Lyapunov method and assumed symmetric parameter variation with respect to the origin. These restrictions often result in extremely conservative estimates of stability bounds. Leal and Gibson [Leal 90] proposed a first-order Lyapunov robustness method where the Lyapunov function varies with perturbation in the system matrix. Gao and Antsaklis [Gao 93] derived an unique stability criterion for an asymmetric stability bounds. However, the stability bounds in [Leal 90] were still symmetric bounds and the zero-order Lyapunov method was utilized in [Gao 90]. Hence, as a natural next step, this research establishes an asymmetric first-order Lyapunov method using the results in [Leal 90] and [Gao 93].

Chapter III first reviews the works in [Gao 93] and [Leal 90], and then describes the drawbacks of their results.

CHAPTER III

GAO'S THEOREM AND BASIC FIRST–ORDER LYAPUNOV METHOD

Gao's Theorem for Asymmetric Stability Robustness Bounds

As reviewed in the preceding chapters, parameter variations with structured perturbations were often assumed to be intervals symmetric to zero. Therefore, the stability bounds could be very conservative. They are usually given as small, symmetric intervals around the origin in the parameter space. Gao [Gao 93] first considered the directional property of parameter variations and derived a theorem for an asymmetric stability bounds of perturbation parameters in the following manner:

Review of Gao's Theorem

Consider the linear time–invariant system represented by the state space model with perturbation E as shown below:

$$\dot{x} = (A_N + E)\,x \tag{3.1}$$

where A_N is a nxn real Hurwitz matrix.

Assume that the parameter perturbation matrix takes the form

$$E = \sum_{i=1}^{m} k_i E_i \tag{3.2}$$

where E_i are real, constant matrices and k_i are real, uncertain parameters. This form of the parameter perturbation matrix of Equation (3.2) has many useful features. First, notice that each uncertain parameter k_i can have multiple entries in the E_i matrices, each with its own scale factor. This allows an uncertain parameter that is one term of a product to be used in the E matrix. Also there is nothing to prevent a parameter from having multiple entries at the same row column address of the E_i matrix. Lastly, note that the total E matrix is linear in the parameter k_i. Because there are m parameters k_i and each term can be positive or negative, there are 2^m perturbations on E or 2^{m-1} perturbations on $\pm E$. The parameters k_i form a hypercube in the m-dimensional parameter space with a vertex of the hypercube being one of the perturbations with all maximum and minimum values of k_i.

The analysis of robustness for the system Equation (3.1) is concerned with the determination of the bounds for the perturbation elements E in which the system stays stable. Gao [Gao 93] developed following theorem for the stability robustness of the system described by equations (3.1) and (3.2). See Appendix C for the proof and related lemmas.

Theorem 3.1 (Gao's Theorem)

The system on Equation (3.1) is asymptotically stable if

$$\sum_{i=1}^{m} k_i \lambda_i < 1 \tag{3.3}$$

where

$$\lambda_i = \begin{cases} \lambda_{max}(P_i) & \text{for } k_i \geq 0 \\ \lambda_{min}(P_i) & \text{for } k_i < 0 \end{cases} \qquad i = 1,\ldots,m \tag{3.4}$$

in which

$$P_i = \tfrac{1}{2}\left(E_i^T P + P E_i\right), \qquad i = 1, 2,\ldots, m \tag{3.5}$$

□

The significance of this theorem is that it takes into consideration the directional information which is often available in practice. Consequently, it can be shown that the stability bound obtained here is always less conservative than that or equal to the bound proposed by Zhou et al. [Zhou 87] which is described

$$\sum_{i=1}^{m} |k_i|\, \bar{\sigma}(P_i) < 1 \tag{3.6}$$

where $\bar{\sigma}(.)$ denotes the largest singular value of a matrix (.).

The Shortcoming of Gao's Method

In order to show the shortcoming of the Gao's method, consider a two-dimensional perturbation problem on Equation (3.1).

In general, the stability robustness bounds for two-dimensional perturbed parameters are obtained by using Theorem 3.1 as follows:

$$\begin{cases} k_1\,\lambda_{max}(P_1) + k_2\,\lambda_{max}(P_2) < 1, & \text{for } k_1 \geq 0 \text{ and } k_2 \geq 0 \\ k_1\,\lambda_{max}(P_1) + k_2\,\lambda_{min}(P_2) < 1, & \text{for } k_1 \geq 0 \text{ and } k_2 < 0 \\ k_1\,\lambda_{min}(P_1) + k_2\,\lambda_{max}(P_2) < 1, & \text{for } k_1 < 0 \text{ and } k_2 \geq 0 \\ k_1\,\lambda_{min}(P_1) + k_2\,\lambda_{min}(P_2) < 1, & \text{for } k_1 < 0 \text{ and } k_2 < 0 \end{cases} \tag{3.7}$$

The condition (3.7) basically determines four boundary lines defining the stability bounds in four quadrant respectively. Graphically, the region of the stability bounds on the condition (3.7) is shown on Figure 3.1.

Figure 3.1 clearly illustrates that Gao's method cannot estimate the maximum robustness bounds allowable for the range of given perturbed parameters,

$$k_1 \in \left[k_1^+, k_1^-\right] = \left[\frac{1}{\lambda_{max}(P_1)}, \frac{1}{\lambda_{min}(P_1)}\right] \text{ and } k_2 \in \left[k_2^+, k_2^-\right] = \left[\frac{1}{\lambda_{max}(P_2)}, \frac{1}{\lambda_{min}(P_2)}\right].$$

This is the shortcoming of the Gao's method. The conventional methods reviewed in Chapter II provide robustness bounds expressed by the full range of $k_i \in \left[k_i^+, k_i^-\right]$. Also, notice the hypercube obtained by the Gao's method in m-dimensional perturbed parameter space has 2m vertices, while the number of vertices for a hypercube using the conventional methods is generally 2^m.

In Chapter IV, several lemmas and a theorem called the "Expansion Theorem" is developed to improve the shortcoming of Gao's method. The theorem allows one to verify whether the basic asymmetric stability bounds estimated by Gao's method can be expanded in the full hypercube in the perturbation parameter space.

First-Order Lyapunov Robustness Method

As reviewed in Chapter I and Chapter II, most previous works on stability robustness on time domain, based upon the Lyapunov stability theory, have used the zero-order method for applying the perturbed dynamic model in the Lyapunov equation. This often resulted in conservative stability bounds. Leal [Leal 90] first introduced a basic first-order method which uses a Lyapunov function varying linearly with perturbations in the system matrix. However, much of Leal's work was to optimize the Q matrix in the Lyapunov equation with the assumption of symmetric bound of stability region. This chapter shows the development of the first-order method, compares it with the zero-order method, and shows its advantage in estimating less conservative robustness bounds.

The Goal of Stability Robustness with Lyapunov Method

The fundamental Lyapunov stability theorems are described in Appendix B. Any function possessing Lyapunov stability properties is termed a Lyapunov function, and the associated system is then known to be asymptotically stable (see detail on theorems in Appendix B). There are no general rules for finding such functions; however, for a time-invariant linear system on Equation (3.1) the following function V(x(t)), is a Lyapunov function candidate:

$$V(x(t)) = x^T(t)\, P_o\, x(t) \tag{3.8}$$

where P_o is a positive definite, symmetric (Hermitian) matrix which satisfies following

Lyapunov equation

$$A^T P_o + P_o A = -Q \tag{3.9}$$

where the system matrix is given by $A = A_N + E$, and Q is a positive definite, symmetric matrix. Then one has

$$\dot{V}(x(t)) = -x^T(t)\, Q\, x(t) \tag{3.10}$$

The Lyapunov stability relationship is now used on the perturbed system defined on (3.1). Assume that the nominal system is stable but that it is unknown whether or not the perturbed system is stable. Certainly, if the perturbation "E" is sufficiently small, then the perturbed system should also be stable. The goal here is to determine the range of perturbations of matrix E for which the function on Equation (3.8) remains a Lyapunov function.

Fundamental Zero-Order Stability Condition

The zero-order method uses nominal system dynamics to compute a P_o matrix for the Lyapunov function, V(x(t)). For a given positive definite, symmetric matrix Q, P_o satisfies Equation (3.9) and V(x(t)) is given by Equation (3.8). For the perturbed system on Equation (3.1), one has

$$\frac{dV(x(t))}{dt} = -x^T(t)\left(Q - E^T P_o - P_o E\right) x(t) \tag{3.11}$$

Then V(x(t)) remains a Lyapunov function if:

$$x^T(t)\, W\, x(t) \le x^T(t)\, Q\, x(t), \ \ \forall\, x(t) \tag{3.12}$$

where

$$W := E^T P_o + P_o E \tag{3.13}$$

Condition (3.12) is the fundamental condition upon which various sufficient conditions have been developed, as reviewed in Chapter II.

Overview of the Basic First-Order Lyapunov Method

In the zero-order method, the nominal matrix Lyapunov equation is used to generate a nominal P matrix, which is then used to compute a stability bound for the perturbed system. The distinctive feature of the first-order method lies in the use of the perturbed dynamic model to generate a perturbed P matrix. This perturbed P matrix is then used to obtain a modified stability bound. In a majority of problems, the first-order method is a large improvement for estimating stability robustness bounds [Leal 90]. However, Leal and Gibson [Leal 90] did not investigate the condition under which the first-order method provides less conservative estimates of stability bounds than the zero-order method.

The procedure for computing first-order stability bounds is similar to the one for estimating zero-order stability bounds. The perturbed system matrix, A, is given by

$$A = A_N + E \tag{3.14}$$

Now define the perturbed P matrix with

$$P := P_o + \Delta P \tag{3.15}$$

As before P_o satisfies the nominal Lyapunov equation

$$A_N^T P_o + P_o A_N = -Q_o \tag{3.16}$$

The perturbed Lyapunov equation still satisfies

$$A^T P + P A = -Q \tag{3.17}$$

When this equation is expanded with Equation (3.14) and Equation (3.15) the result is

$$(A_N + E)^T \left(P_o + \Delta P\right) + \left(P_o + \Delta P\right)(A_N + E) = -Q \tag{3.18}$$

If one chooses ΔP so that

$$A_N^T \Delta P + \Delta P A_N = -\left(E^T P_o + P_o E\right) \tag{3.19}$$

then

$$Q = Q_o - \left(E^T \Delta P + \Delta P E\right) \tag{3.20}$$

Now, in order for the perturbed system Equation (3.1) to be stable, V(x(t)) should be a Lyapunov function with $\frac{dV(x(t))}{dt}$ negative definite; i.e., both the matrix P on Equation (3.15) and the matrix Q on Equation (3.20) should be positive definite.

Hence,

$$\frac{V(x(t))}{dt} = \dot{x}^T P x + x^T P \dot{x} \tag{3.21}$$

$$= \dot{x}^T (A_N + E)^T \left(P_o + \Delta P\right) x + x^T \left(P_o + \Delta P\right)(A_N + E)\, \dot{x}$$

$$= \dot{x}^T \left(A_N^T P_o + A_N^T \Delta P + E^T P_o + E^T \Delta P + P_o A_N + P_o E + \Delta P A_N + \Delta P E\right) x$$

$$= x^T \left(-Q_o + E^T \Delta P + \Delta P E\right) x$$

Therefore, the stability conditions for the perturbed system on Equation (3.1) are

$$\begin{gathered} x^T \Delta P x \le x^T P_o x, \;\; \forall x \\ x^T \left(E^T \Delta P + \Delta P E\right) x \le x^T Q_o x, \;\; \forall x \end{gathered} \tag{3.22}$$

The first and second conditions on Equation (3.22) are for the matrix P and matrix Q be positive definite, respectively.

In Chapter V, a new method for the asymmetric robustness bounds will be developed using the results of first order method. Chapter V also establishes a sufficient condition upon which the new asymmetric first-order method provides better (less conservative) estimates of stability bounds than those of Gao's zero-order method. The examples in Chapter VI show that this new method generates better stability bounds than the methods reviewed in Chapter II.

A First-Order Method Using a Taylor Series Expansion

The motivation for the first-order nomenclature will be made clear in the following development, in which a Taylor series expansion is used for the matrix P of the perturbed dynamic system.

Consider a dynamic system represented by

$$\dot{x} = A\,x = (A_N + \varepsilon\,E)\,x \tag{3.23}$$

with the perturbation matrix E is scaled by a constant ε. The Lyapunov equation associated with matrix A is given by

$$A^T P(\varepsilon) + P(\varepsilon)\,A = -Q \tag{3.24}$$

The solution P(ε) exists and is unique if the perturbed dynamic system is stable. Now assume that the solution P(ε) is represented by a Taylor series expansion with center of the nominal point, i.e. ε = 0:

$$P(\varepsilon) = P(0) + \varepsilon \left.\frac{dP(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} + \ldots + H.O.T. \tag{3.25}$$

When ε = 0 the matrix $A(\varepsilon) = A_o$ from Equation (3.23) and P(0) is given by

$$A^T P(0) + P(0)\,A = -Q \tag{3.26}$$

If $P_o := P(0)$, then Equation (3.26) is identical to Equation (3.9).

Now differentiate each side of Equation (3.24) with respect to the parameter ε to obtain

$$\frac{d}{d\varepsilon}\left(A^T P(\varepsilon) + P(\varepsilon) A\right) = -\frac{d}{d\varepsilon} Q = 0 \tag{3.27}$$

Taking the derivative of the left hand side of Equation (3.27) gives

$$\frac{dA^T}{d\varepsilon} P(\varepsilon) + A^T \frac{dP(\varepsilon)}{d\varepsilon} + \frac{dP(\varepsilon)}{d\varepsilon} A + P(\varepsilon) \frac{dA}{d\varepsilon} = 0 \tag{3.28}$$

Using $\frac{dA}{d\varepsilon} = E$ from Equation (3.23), Equation (3.28) gives

$$E^T P(\varepsilon) + P(\varepsilon) E + A^T \frac{dP(\varepsilon)}{d\varepsilon} + \frac{dP(\varepsilon)}{d\varepsilon} A = 0 \tag{3.29}$$

Since $P(0) = P_o$ and $A(0) = A_o$ when $\varepsilon = 0$, Equation (3.29) yields

$$E^T P_o + P_o E + A_o^T \left.\frac{dP(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} + \left.\frac{dP(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} A_o = 0 \tag{3.30}$$

The perturbation matrix ΔP of Equation (3.15), resulting from the parameter variation, can be considered as a first -order term of Taylor series expansion on Equation (3.25) such that

$$\Delta P := \left.\frac{dP(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \tag{3.31}$$

Then, substituting Equation (3.31) into Equation (3.30) yields

$$A_o^T \Delta P + \Delta P A_o = -\left(E^T P_o + P_o E\right) \tag{3.31}$$

Note that Equation (3.31) is identical to Equation (3.19) and this completes the development of the Taylor series expansion of $P(\varepsilon)$. This result reveals that the nomenclatures zero-order or first-order methods imply, in the sense of the Taylor series expansion, the number of higher-order terms used to represent the parameter perturbation property.

The basic first-order Lyapunov method is combined with Gao's method to obtain the asymmetric first-order stability robustness method as described in Chapter V.

CHAPTER IV

EXPANSION OF GAO'S ASYMMETRIC STABILITY BOUNDS

General Properties of Convexity

Convexity is an important property in the estimation of stability bounds because its use can greatly reduce the amount of searching in the parameter space. Hence, the definition and the lemmas for convexity are introduced for future use before the lemmas and a theorem for the expansion of Gao's basic asymmetric stability bounds is developed.

Definition 4.1

A real valued function f *defined on a convex subset of a linear space is convex if*

$$f\big((1-\theta)\, u_1 + \theta\, u_2\big) \leq (1-\theta)\, f(u_1) + \theta\, f(u_2) \tag{4.1}$$

holds for all θ *such that* $0 \leq \theta \leq 1.$ □

Lemma 4.1

Let f *be a convex function on a convex subset S of a linear space, and let* $\{\mu_1, \mu_2, \ldots, \mu_m\}$ *be a finite collection of points in S. If*

$$\theta_i \geq 0 \;\forall i \textit{ and } \sum_{i=1}^{m} \theta_i = 1 \tag{4.2}$$

then

$$f\left(\sum_{i=1}^{m} \theta_i \, \mu_i\right) \le \sum_{i=1}^{m} \theta_i \, f(\mu_i) \tag{4.3}$$

<u>Proof</u>

Assume that Lemma 4.1 is true for some positive integer n. By the definition of convexity, it is true for m = 2. Now suppose

$$\sum_{i=1}^{m+1} \beta_i = 1, \; \beta_i \ge 0 \;\; \forall \; i, \; 0 < \beta_{m+1} < 1 \tag{4.4}$$

and write

$$\sum_{i=1}^{m+1} \beta_i \, \mu_i = \beta_{m+1} \, \mu_{m+1} + (1-\beta_{m+1}) \sum_{i=1}^{m} \theta_i \, \mu_i \tag{4.5}$$

where

$$\theta_i := \frac{\beta_i}{\left(1 - \beta_{m+1}\right)} \text{ for } 0 \le i \le m \;\text{ and } \sum_{i=1}^{m} \theta_i = 1 \tag{4.6}$$

Since the function f is convex,

$$f\left(\sum_{i=1}^{m+1} \beta_i \, \mu_i\right) \le \beta_{m+1} \, f(\mu_{m+1}) + \left(1-\beta_{m+1}\right) f\left(\sum_{i=1}^{m} \theta_i \, \mu_i\right) \tag{4.7}$$

$$\le \beta_{m+1} \, f(\mu_{m+1}) + \left(1-\beta_{m+1}\right) \sum_{i=1}^{m} \theta_i \, f(\mu_i) = \sum_{i=1}^{m+1} \beta_i \, f(\mu_i)$$

□

Lemma 4.2

Let $\{\mu_1, \mu_2, \ldots, \mu_m\}$ *be a finite collection of points in* $\mathbf{R}^m$, *let S be the convex hull of* $\{\mu_1, \mu_2, \ldots, \mu_m\}$, *and let* f *be a convex function defined in S. Then*

$$\max_{\mu \in S} f(\mu) = f(\mu_k) \quad \textit{for some } k \tag{4.8}$$

Proof

Each μ in S can be written as

$$\mu = \sum_{i=1}^{m} \mu_i \, \theta_i \tag{4.9}$$

where

$$\sum_{i=1}^{m} \theta_i = 1, \quad \theta_i \geq 0 \quad \forall\, i \tag{4.10}$$

It follows from Lemma 4.1 that

$$f\left(\sum_{i=1}^{m} \theta_i \, \mu_i\right) \leq \sum_{i=1}^{m} \theta_i \, f(\mu_i) \tag{4.11}$$

$$\leq \left(\sum_{i=1}^{m} \theta_i\right) \max_i f(\mu_j) = \max_i f(\mu_j) \qquad \square$$

Lemma 4.2 is a generalization of the fact that a convex function of a single variable defined over a closed interval will achieve its maximum at one end of the interval.

Lemmas and Theorem for Expansion

Several lemmas and a theorem are derived in this chapter for the expansion of

Gao's asymmetric stability bounds. They allow one to verify whether the basic asymmetric robustness bounds estimated by Gao's method can be expanded in the full hypercube Π in the perturbation parameter space.

Let $\mu(A)$ denote the matrix measure of matrix A. Matrix measure have different values corresponding to the different induced matrix norms (see Appendix A). In 2-norm case, i.e., for the induced matrix norm $\|A\|_2 = \sqrt{\lambda_{max}(A^* A)}$ given the vector norm $|x|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$, the induced matrix measure is obtained [Desoer 75] by

$$\mu_2(A) = \lambda_{max}\left(A + A^*\right)/2 \tag{4.12}$$

where A^* is a conjugate matrix of A. For the sake of simplicity $\mu(.)$ is used for $\mu_2(.)$ from now on.

Define the admissible perturbation matrix set

$$\Psi = \left\{ E \mid \lambda(A_N + E) \in C^- \right\} \tag{4.13}$$

where A_N is a stable nominal matrix and C^- is the set of all complex numbers with negative real parts.

Lemma 4.3

$$E \in \Psi \text{ if } \mu(P_o E) < 1 \tag{4.14}$$

where P_o *is symmetric, positive definite, and satisfies the Lyapunov equation*

$$P_o A_N + A_N^T P_o = -2 I \tag{4.15}$$

Proof

Asymptotic stability is guaranteed if

$$(A_N + E)^T P_o + P_o (A_N + E) = -2 I + E^T P_o + P_o E < 0 \tag{4.16}$$

Then

$$\mu(P_o\, E) = \lambda_{max}\left(\frac{E^T\, P_o + P_o\, E}{2}\right) < 1 \tag{4.17}$$

□

Define

$$\Psi_1 = \{E \mid \mu(P_o\, E) < 1\} \tag{4.18}$$

so that Ψ_1 is a subset of Ψ.

Lemma 4.4

Ψ_1 is convex.

<u>Proof</u>

For any E_a and $E_b \in \Psi_1$, let $E_t = (1 - \theta)\, E_a + \theta\, E_b$ where $0 \le \theta \le 1$, then

$$\mu(P_o\, E_t) \le (1 - \theta)\, \mu(P_o\, E_a) + \theta\, \mu(P_o\, E_b) < (1 - \theta) + \theta = 1 \tag{4.19}$$

Hence $E_t \in \Psi_1$ which means Ψ_1 is convex. □

Lemma 4.5

Assume Φ is a polytope in a parameter space. Then $\Phi \in \Psi_1$ if all of its vertices are in Ψ_1.

<u>Proof</u>

Let v_i denote the vertices of Φ, i = 1,...,k. Then a point in Φ, ψ, is represented by a convex combination of these vertices such that

$$\psi = \sum_{i=1}^{k} \theta_i\, v_i, \text{ for } \forall\, \theta_i \ge 0 \text{ and } \sum_{i=1}^{k} \theta_i = 1 \tag{4.20}$$

The triangle inequality, $\mu(A+B) \le \mu(A) + \mu(B)$, and the property $\mu(\alpha A) = \alpha\, \mu(A)$, $\forall\alpha\ge 0$ of matrix measure yield

$$\mu\left(\sum_{i=1}^{k} \theta_i\, v_i\right) \le \sum_{i=1}^{k} \theta_i\, \mu(v_i), \quad \text{for } \forall\,\theta_i \ge 0 \text{ and } \sum_{i=1}^{k} \theta_i = 1 \tag{4.21}$$

The condition (4.21) is generally known as a convexity property of matrix measure. Hence one has

$$\mu(P\psi) = \mu\left(P \sum_{i=1}^{k} \theta_i\, v_i\right) \le \sum_{i=1}^{k} \theta_i\, \mu(P\, v_i) \le \max\left[\mu(P\, v_i)\right] < 1 \tag{4.22}$$

$\square$

It is shown that the Gao's method is a corollary of three lemmas as follows:

For a parameter perturbation, $E = \sum_{i=1}^{m} k_i\, E_i$, one has

$$\mu(P_oE) = \lambda_{max}\left(\frac{E^TP_o + P_o\,E}{2}\right) = \lambda_{max}\left(\frac{\sum_{i=1}^{m} k_i\left(E_i^TP_o + P_o\,E_i\right)}{2}\right) = \lambda_{max}\left(\sum_{i=1}^{m} k_iP_i\right) < 1 \tag{4.23}$$

where P_i are defined in Equation (3.5).

Condition (4.23) leads to Gao's results (see lemmas C.1 to C3 in Appendix C for proofs) such that

$$\lambda_{max}\left(\sum_{i=1}^{m} k_iP_i\right) \le \sum_{i=1}^{m} \lambda_{max}(k_iP_i) \le \sum_{i=1}^{m} k_i\lambda_i < 1 \tag{4.24}$$

Now, define

$$\Pi_o = \left\{ k \in R^m \,\middle|\, \lambda_{max}\left(\sum_{i=1}^{m} k_i P_i \right) < 1 \right\}$$

$$\Pi_1 = \left\{ k \in R^m \,\middle|\, \sum_{i=1}^{m} k_i \lambda_i < 1 \right\} \tag{4.25}$$

Lemma 4.6

Π_o *and* Π_1 *are convex.*

<u>Proof</u>

Similar to the proof of Lemma 4.4, Π_o and Π_1 can be easily proved to possess the convexity property. □

The fully expanded stability robustness bounds in Π_1 would be denoted by

$$\Pi = \left\{ k \in R^m \,\middle|\, \left(\sum_{i=1}^{m} k_i \lambda_i < 1 \right) \cup \left(k^- \le k \le k^+ \right) \right\} \tag{4.26}$$

Lemma 4.7

Π *is convex.*

<u>Proof</u>

The vertices of Π_1, if they exist, are always on the hyper-axes in the parameter space. The vertices of Π are linear unit combination of those of Π_1 which is convex. Hence Π is convex. □

Lemma 4.8 (Fang, Loparo and Feng [Fang 94])

If there exists a positive definite matrix P_o *such that* $A^T P_o + P_o A$ *is stable for any*

$A \in \Gamma$, *then the polytope of matrix* A *is stable where* Γ *is the set of vertex matrices.*

$\square$

Theorem 4.1 (Expansion Theorem)

Gao's stability robustness bounds, Π_1, *are fully expanded in the hypercube* Π *if* $A_N + E(k^{(j)}),\ \forall j = 1,\ldots,2^m$ *are negative definite, where* $k^{(j)}$ *is the j-th vertex of* Π.

Proof

It is known that the Lyapunov equation $A^T P_o + P_o A = -I$ yields a positive definite solution if $\lambda(A) \in C^-$ (see Theorem B.6 in Appendix B). Since Π is convex, the set of matrices $A = A_N + \sum_{i=1}^{m} k_i E_i$ is a polytope (convex hull). Hence using Lemma 4.8, if $A = A_N + E(k^{(j)})$ is negative definite at all vertices $j = 1,\ldots,2^m$, then the polytope of matrix A is stable in the hypercube Π. $\square$

Remark 4.1

If $A_N + E(k^{(j)}),\ j = 1,\ldots,2^m$ *is negative definite, Gao's stability robustness bounds in j-th hyper-quadrant are fully expanded respectively, where* $k^{(j)}$ *is the j-th vertex of* Π.

$\square$

Graphical Illustration for a Three-Dimensional Example

The preceding lemmas and theorem imply that the expanded zero-order stability robustness bounds can be tested only on 2^m vertices points of the hypercube parameter space for k_i's to expand Gao's stability robustness bounds to the full parallelepiped

$\Pi = \{k \in R^m: k^- \le k \le k^+\}$. Figure 4.1 illustrates the comparison of the stability bounds estimated by Gao's method and the expanded stability bound estimate by Theorem 4.1 for a three-dimensional perturbation parameter system. As shown in Figure 4.1, Gao's method yields stability bounds whose vertices, if they exist, are always located on the hyperspace axes. Each vertex of Gao's stability bounds is a maximum or a minimum value of each perturbation parameter, k_1^+, k_1^-, ..., k_m^+, k_m^-, and the number of all vertices would be 2m. Vertices of the expanded volume of the hypercube with these maximum and minimum perturbation parameters are linear combination of these extreme parameter values, and the number of vertices is 2^m. For example, in Figure 4.1, $k^{(1)} = (k_1^+, k_2^+, k_3^+)$ and $k^{(2)} = (k_1^+, k_2^+, k_3^-)$. The stability test using Theorem 4.1 and Corollary 4.1 at vertices $k^{(1)}$ and $k^{(2)}$ respectively proves whether the stability boundaries can be expanded the full region of the hyper-quadrant to which each vertex belongs.

A Discussion of Expansion Theorem

As mentioned in Chapter I and Chapter II, the natural conjectures of Kharitonov's type of results to the family of interval system matrices have failed and the stability robustness problem for a polytope of matrices is not yet resolved. The significance of the Expansion Theorem is that it enables a Kharitonov's like stability test at the vertices of the hypercube to expand Gao's stability bound estimates to the full hypercube of perturbation parameter space. The versatility of the Expansion Theorem is that it can be applied at each vertex independently. In other words, the stability boundaries at each hyper-quadrant can be expanded by satisfying the sufficient condition independently regardless of the results of stability tests at the other vertices.

CHAPTER V

ASYMMETRIC FIRST-ORDER STABILITY ROBUSTNESS BOUNDS

Theorem for an Asymmetric First-Order Lyapunov Method

In most earlier work on stability robustness as well as Gao's method, the zero-order method has been used to generate the Lyapunov function. This often resulted in conservative robustness bounds. In this research, the first-order method is combined with the Gao's method so that the perturbed dynamic model is used to generate a perturbed P matrix. This perturbed P matrix is then used to obtain modified better stability bounds.

Define the perturbed P matrix with

$$P = P_o + \Delta P \tag{5.1}$$

in which matrix P_o satisfies the nominal Lyapunov equation

$$A_N^T P_o + P_o A_N = -2I \tag{5.2}$$

Define

$$\Delta P_i := \frac{E_i^T \Delta P + \Delta P E_i}{2}, \quad i = 1,...,m \tag{5.3}$$

and

$$\Delta\lambda_i := \begin{cases} \lambda_{max}(\Delta P_i) & \text{for } k_i \geq 0 \\ \lambda_{min}(\Delta P_i) & \text{for } k_i < 0 \end{cases} \tag{5.4}$$

The perturbed Lyapunov equation still satisfies

$$A^T P + P A = -Q \tag{5.5}$$

Theorem 5.1 (Asymmetric First-Order Lyapunov Method)

The linear system described by Equations (3.1) and (3.2) is asymptotically stable if

$$\sum_{i=1}^{m} k_i \, \Delta\lambda_i < 1 \tag{5.6}$$

and if P *is a positive definite matrix where* ΔP *satisfies*

$$A_N^T \Delta P + \Delta P \, A_N = -(E^T P_o + P_o E) \tag{5.7}$$

<u>Proof</u>

Let the candidate Lyapunov function $V(x,t) = x(t)^T P \, x(t)$, then

$$\begin{aligned} \frac{dV(x,t)}{dt} &= \dot{x}^T P \, x + x^T P \, \dot{x} \\ &= [(A_N + E) \, x]^T (P_o + \Delta P) \, x + x^T (P_o + \Delta P) (A_N + E) \, x \\ &= x^T (A_N^T P_o + A_N^T \Delta P + E^T P_o + E^T \Delta P + P_o A_N + P_o E + \Delta P \, A_N + \Delta P \, E) \, x \\ &= x^T (E^T \Delta P + \Delta P \, E - 2I) \, x + x^T (A_N^T \Delta P + \Delta P \, A_N + E^T P_o + P_o E) \, x \\ &= x^T \left(\sum_{i=1}^{m} k_i \, E_i^T \Delta P + \sum_{i=1}^{m} k_i \, \Delta P \, E_i - 2I \right) x \end{aligned} \tag{5.8}$$

$$=x^T\left[\sum_{i=1}^{m} k_i \left(E_i^T\, \Delta P + \Delta P\, E_i\right) - 2I\right] x$$

$$= 2\, x^T \left(\sum_{i=1}^{m} k_i\, \Delta P_i - I\right) x$$

For the system on Equation (3.1) to be asymptotically stable

$$\lambda\left(\sum_{i=1}^{m} k_i\, \Delta P_i\right) < 1 \tag{5.9}$$

It was proved by Olas (see lemmas C.1 to C.3 in Appendix C for proofs) that

$$\lambda\left(\sum_{i=1}^{m} k_i\, \Delta P_i\right) \le \sum_{i=1}^{m} \lambda_{max}\left(k_i\, \Delta P_i\right) \le \sum_{i=1}^{m} k_i\, \Delta\lambda_i < 1 \tag{5.10}$$

□

Note that the positive definite matrix Q for the first-order method is obtained using the equations (5.2), (5.5) and (5.7).

$$Q = 2\, I - \left(E^T\, \Delta P + \Delta P\, E\right) \tag{5.11}$$

The matrices P and Q should be positive definite so that a system equation satisfies the Lyapunov matrix equation to be asymptotically stable. Equation (5.7) provides ΔP which results in positive definite matrix Q on Equation (5.11). One approach for the problem of stability robustness is to search for a matrix Q that maximizes the size of the hypercube of stability estimates. Nonlinear optimization techniques ([Leal 90], [Latchman 91]) were used to directly find optimal matrix Q. It can be said that the Olas' works ([Olas 92], [Olas 94a], [Olas 94b]) to find the optimal Lyapunov function is also a second approach to find the optimal matrix Q.

The asymmetric first-order method proposed in Theorem 5.1 overcomes the shortcoming of Olas' algorithm. It is shown, in Theorem 5.1, that the perturbed Lyapunov matrix ΔP is obtained using Equation (5.7), and the increment of parallelepiped Π, of uncertain parameters, is not arbitrary but explicitly enlarged by the inequality condition (5.6). Following theorems show how the optimal Lyapunov function property in Olas' Theorem completes the first-order method in Theorem 5.1.

Theorem 5.2

The stability bounds estimated by the asymmetric first-order method are always better (less conservative) than those of zero-order method if the Asymmetric First-Order Method proposed in Theorem 5.1 satisfies the condition for an optimal Lyapunov function in Theorem 2.10, i.e., $\Delta QF_k < 0$*(*R_k *is negative definite) on the vertex k where the first-order method is applied.*

<u>Proof</u>

Define $P = P_o + \Delta P$, $V = V_o + \Delta V = x^T \left(P_o + \Delta P\right) x$, then the variation ΔQF_k is determined by

$$\Delta QF_k = x^T R_k\, x = x^T \left[A_N^T \Delta P + \Delta P\, A_N + E^T \Delta P + \Delta P\, E\right]_k x \qquad (5.12)$$

Substituting Equation (5.7) into Equation (5.12) yields

$$\Delta QF_k = x^T \left[E^T (\Delta P - P_o) + (\Delta P - P_o)\, E\right]_k x < 0,\ \forall x,\ x \neq 0 \qquad (5.13)$$

Using Equation (3.5) and Equation (5.3) one has

$$\left[E^T (\Delta P - P_o) + (\Delta P - P_o) E\right]_k = \left[\sum_{i=1}^{m} k_i E_i^T(\Delta P - P_o) + \sum_{i=1}^{m} k_i(\Delta P - P_o)E_i\right]_k$$

$$= \sum_{i=1}^{m} k_i\left[\left(E_i^T \Delta P + \Delta P E_i\right) - \left(E_i^T P_o + P_o E_i\right)\right]_k \tag{5.14}$$

$$= 2 \sum_{i=1}^{m} k_i\left[\Delta P_i - P_i\right]_k$$

Hence, Equation (5.13) yields

$$\Delta QF_k = 2\, x^T \left[\sum_{i=1}^{m} k_i\left(\Delta P_i - P_i\right)\right]_k x < 0, \quad \forall x,\ x \neq 0 \tag{5.15}$$

Equation (5.15) implies that

$$\lambda\left[\sum_{i=1}^{m} k_i\left(\Delta P_i - P_i\right)\right]_k \leq \sum_{i=1}^{m} \lambda_{max}\left[k_i\left(\Delta P_i - P_i\right)\right]_k \leq \sum_{i=1}^{m} \left[k_i\left(\Delta\lambda_i - \lambda_i\right)\right]_k < 0 \tag{5.16}$$

Hence

$$\left|\Delta\lambda_i\right|_k < \left|\lambda_i\right| \tag{5.17}$$

Consider that the vertex on i-th axis of the stability bound's hypercube in the perturbation parameter space is given by $1/\lambda_i$ for the zero-order method and $1/\Delta\lambda_i$ for the first-order method as shown in Figure 1. Therefore one can conclude, using condition (5.17), that the estimate of stability bounds by the first-order method is better than those of the zero-order method. □

There may be more than one vertex of the zero-order stability bounds which satisfy the conditions described in Theorem 5.1 and Theorem 5.2. In such a case, using the performance measure of the Lyapunov function defined in Equation (2.39), one can determine the vertex where the first-order method is applied as follows:

Let $V := x^T P_o x$, $\Delta V := x^T \Delta\widetilde{P} x$ and $P := P_o + \Delta P := P_o + \varepsilon \Delta\widetilde{P}$. Determine matrix perturbation, ΔP, by Equation (5.7) at each vertex, $k^{(1)},\ldots,k^{(2^m)}$. Consider two performance measures λ_a and λ_b of the functions Λ_a and Λ_b respectively, where subscript "a" or "b" denotes a vertex among the vertices of the zero-order stability bound estimates which satisfies the conditions in Theorem 5.1 and Theorem 5.2. Assume the Lyapunov function V_a is better than V_b, i.e., λ_a is smaller than λ_b, then one has the following Theorem.

Theorem 5.3

Stability bound estimates by the first-order method applied on the vertex "a" are better than those applied on the vertex "b" if the matrix $\lambda_a \Delta P_a - \lambda_b \Delta P_b$ *is positive definite.*

Proof

Using the results in Equation (5.8), one has

$$\Lambda_{a \text{ or } b} = \left.\frac{\dot{x}^T P x + x^T P \dot{x}}{x^T P x}\right|_{a \text{ or } b} = \left.\frac{\dot{V} + \varepsilon \Delta\dot{V}}{V + \varepsilon \Delta V}\right|_{a \text{ or } b} \tag{5.18}$$

For small ε, applying the Taylor series expansion to Equation (5.18) yields

$$\Lambda_{a \text{ or } b} = \left[\frac{\dot{V}}{V} + \varepsilon \left(\frac{\Delta\dot{V}}{V} - \frac{\dot{V}}{V}\frac{\Delta V}{V}\right)\right]_{a \text{ or } b} \tag{5.19}$$

Meanwhile as Olas [Olas 94a] suggested

$$\left.\frac{\dot{V}}{V}\right|_j = \frac{x^T R_j x}{x^T P_o x} = \frac{y^T P_o^{-0.5} R_j P_o^{-0.5} y}{y^T y} \tag{5.20}$$

where $y := P_o^{0.5} x$ and R_j is defined in Equation (2.42). The maximum value of $\dot{V}/V|_j$ is equal to the largest eigenvalue of the matrix $P_o^{-0.5} R_j P_o^{-0.5}$. Let $\bar{\lambda}_j$ denote the largest eigenvalue of the matrix $P_o^{-0.5} R_j P_o^{-0.5}$. Then one has $\bar{\lambda}_a < \bar{\lambda}_b$ since $\lambda_a < \lambda_b$.

$$\lambda_a - \lambda_b = \sup\Lambda_a - \sup\Lambda_b = \max_{x \in R^n}\left[\frac{\dot{V}}{V} + \varepsilon\left(\frac{\Delta\dot{V}}{V} - \frac{\dot{V}\Delta V}{V\ V}\right)\right]_a - \max_{x \in R^n}\left[\frac{\dot{V}}{V} + \varepsilon\left(\frac{\Delta\dot{V}}{V} - \frac{\dot{V}\Delta V}{V\ V}\right)\right]_b < 0 \quad (5.21)$$

Then condition (5.21) yields

$$\left(\left[\frac{\Delta\dot{V}}{V}\right]_a - \bar{\lambda}_a\left[\frac{\Delta V}{V}\right]_a\right) - \left(\left[\frac{\Delta\dot{V}}{V}\right]_b - \bar{\lambda}_b\left[\frac{\Delta V}{V}\right]_b\right) < 0 \quad (5.22)$$

which leads to

$$\left([\Delta\dot{V}]_a - [\Delta\dot{V}]_b\right) - \left(\bar{\lambda}_a[\Delta V]_a - \bar{\lambda}_b[\Delta V]_b\right) < 0 \quad (5.23)$$

Similar to the proof for Theorem 5.2, condition (5.23) derives

$$\left(\Delta QF_a - \Delta QF_b\right) - x^T\left(\bar{\lambda}_a \Delta P_a - \bar{\lambda}_b \Delta P_b\right) x < 0 \quad (5.24)$$

Hence, if $\bar{\lambda}_a \Delta P_a - \bar{\lambda}_b \Delta P_b$ is positive definite, then

$$\sum_{i=1}^{m}\left[k_i(\Delta\lambda_i - \lambda_i)\right]_a - \sum_{i=1}^{m}\left[k_i(\Delta\lambda_i - \lambda_i)\right]_b = \sum_{i=1}^{m} k_i\left[(\Delta\lambda_i)_a - (\Delta\lambda_i)_b\right] < 0 \quad (5.25)$$

which means

$$\left|(\Delta\lambda_i)_a\right| < \left|(\Delta\lambda_i)_b\right| \quad (5.26)$$

Thus the stability bound estimates at vertex "a" are better (less conservative) than that at the vertex "b". □

Application Procedure for the Asymmetric First-Order Lyapunov Method to Estimate the Stability Robustness Bounds

The whole procedure to estimate the stability robustness bounds using the asymmetric first-order Lyapunov method is proposed in the following manner:

First, estimate the stability bounds using the Gao's zero-order method described in Theorem 3.1.

Second, apply the Expansion Theorem to the vertices of stability boundaries estimated by Gao's method to verify whether Gao's stability bounds can be expanded in the full hypercube.

Third, using Theorem 5.3, find the optimal vertex to apply the first-order method among the vertices of the stability bound hypercube obtained in second stage.

Fourth, estimate the stability bounds by applying the asymmetric first-order method of Theorem 5.1 to the optimal vertex identified in third stage, then expand the stability bounds using the Expansion Theorem.

Discussions of the Asymmetric First-Order Lyapunov Method

First, the significance of the asymmetric first-order Lyapunov method is that it resolves the conservatism problem of stability bound estimates that conventional methods often produce. Two major causes of conservative stability bound estimate are the failure to consider structured features of the uncertainties when generating a Lyapunov function and the neglecting directional property of the perturbation parameters. These causes are resolved by considering a perturbed dynamic model of the Lyapunov matrix equation to generate a Lyapunov function and the directional property of the perturbation parameters.

Without mathematical proof, Olas and Ahmadkhanlou [Olas 94b] asserted that the optimal Lyapunov function ensures larger stability bound estimation than the other quadratic functions in its neighborhood could provide. Theorem 5.3 shows a condition under which Olas and Ahmadkhanlou's assertion can be proved. The significance of the asymmetric first-order Lyapunov method from a standpoint of Olas' optimal Lyapunov function is that the enlargement of the stability bounds for uncertain parameters can be obtained systematically as shown in Equation (5.7). Even though the stability bounds with Olas' algorithm [Olas 92] are highly dependent on the way the estimated stability bounds are enlarged [Olas 94b], no explicit method for the enlargement was proposed in Olas' works.

Theorem 5.2 shows that the condition for an optimal Lyapunov function in Olas' Theorem 2.10 can be used to complete the asymmetric first-order Lyapunov method. Theorem 5.2 proves that if the perturbed matrix $P = P_0 \pm \Delta P$ is positive definite, then the asymmetric first-order method always provides larger stability bounds than those of Gao's method. In other words, Theorem 5.2 determines the absolute amount of perturbation $|\Delta P|$ of nominal Lyapunov matrix P_0 to obtain better estimates of stability bounds by applying the asymmetric first-order method.

CHAPTER VI

EXAMPLES

Example 1: A Second-Order System

Consider the two-dimensional, second-order the system in [Zhou 87].

$$\dot{x} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} x + k_1 \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} x + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x \tag{6.1}$$

in which the stability bound obtained by Zhou and Khargonekar [Zhou 87] is

$$|k_1| + |k_2| < 1, \quad \text{for any } k_1, k_2 \tag{6.2}$$

The exact stability bound for the system on Equation (6.1) is

$$-k_1 + k_2 < 2 \tag{6.3}$$

The eigenvalues of P_i defined in Equation (3.5) are

$$\lambda(P_1) = \{-1, 0\} \text{ and } \lambda(P_2) = \{1, 0\} \tag{6.4}$$

Hence, the zero-order stability bounds given by Theorem 3.1 are determined by

$$\begin{array}{ll} k_2 < 1 & \text{for } k_1 \geq 0, k_2 \geq 0 \\ \forall\, k_1, k_2 & \text{for } k_1 \geq 0, k_2 < 0 \\ k_1 + k_2 < 1 & \text{for } k_1 < 0, k_2 \geq 0 \\ k_1 > -1 & \text{for } k_1 < 0, k_2 < 0 \end{array} \tag{6.5}$$

From the results of the fundamental zero-order robustness bounds on condition (6.5), one has the vertices of the 2-dimensional k_1-k_2 perturbed parameters plane: $(-1, 0)$, $(0, 1)$, $(-1, -\infty)$, and $(\infty, 1)$. The expansion Theorem 4.1 is satisfied at the vertex $(-1, 1)$ on the k_1-k_2 domain. Hence the expanded Gao's stability bounds are determined as follows:

$$k_1 > -1 \text{ and } k_2 < 1 \tag{6.6}$$

The eigenvalues of ΔP_i on the vertex $(-1, 1)$ are

$$\lambda(\Delta P_1) = \{-0.5, 0\} \text{ and } \lambda(\Delta P_2) = \{0.5, 0\} \tag{6.7}$$

Hence the vertices of the 2-dimensional k_1-k_2 perturbed parameters plane are: $(-2, 0)$, $(0, 2)$, $(-2, -\infty)$, and $(\infty, 2)$. Using Theorem 5.1, the asymmetric first-order stability robustness bounds are determined by following four divisions:

$$\begin{array}{ll} 0.5\, k_2 < 1 & \text{for } k_1 \geq 0,\ k_2 \geq 0 \\ \forall\, k_1, k_2 & \text{for } k_1 \geq 0,\ k_2 < 0 \\ 0.5\, k_1 + 0.5\, k_2 < 1 & \text{for } k_1 < 0,\ k_2 \geq 0 \\ 0.5\, k_2 > -1 & \text{for } k_1 < 0,\ k_2 < 0 \end{array} \tag{6.8}$$

The expansion Theorem 4.1 is not satisfied at the vertex (-2, 2). Therefore the robustness bounds cannot be expanded further. The comparison between various robustness bounds for this example is shown on Figure 6.1. As shown on Figure 6.1, the stability region using the new first-order asymmetric method is less conservative, and approaches the exact stability region.

Example 2: A Third-Order System

Consider the two-dimensional, third-order system in [Zhou 87] given by

$$\dot{x} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix} x + k_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} x + k_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \tag{6.9}$$

Zhou and Khargonekar [Zhou 87] proved that the system is stable within the following robustness bounds:

$$0.6052\,|k_1| + 0.3512\,|k_2| < 1, \quad \text{for any } k_1, k_2 \tag{6.10}$$

The exact stability bound for the system on Equation (6.9) is

$$k_1 < 1.75 \ \text{ and } \ k_2 < 3 \tag{6.11}$$

The eigenvalues of P_1 and P_2 are

$$\lambda(P_1) = \{0.6052, 0, -0.0338\} \text{ and } \lambda(P_2) = \{0.3512, 0, -0.0487\} \tag{6.12}$$

and the stability bounds derived using Theorem 3.1 are

$$\begin{aligned} &0.6052\,k_1 + 0.3512\,k_2 < 1 && \text{for } k_1 \geq 0, k_2 \geq 0 \\ &0.0338\,k_1 + 0.3512\,k_2 < 1 && \text{for } k_1 < 0, k_2 \geq 0 \\ &0.6052\,k_1 + 0.0487\,k_2 < 1 && \text{for } k_1 \geq 0, k_2 < 0 \\ &0.0338\,k_1 + 0.0487\,k_2 < 1 && \text{for } k_1 < 0, k_2 < 0 \end{aligned} \tag{6.13}$$

From the conditions of zero-order robustness bounds on (6.13), one has the vertices of the 2-dimensional k_1-k_2 perturbed parameters plane: (–29.586, 0), (0, 2.847), (1.652, 0), and (0, –20.534). The Expansion Theorem 4.1 is satisfied at the four vertices of the k_1-k_2 domain: (–29.586, –20.534), (1.652, –20.534), (–29.586, 2.847), and (1.652, 2.847). Hence the robustness bounds are fully expanded on the ranges of the k_1 and k_2 parameters. At the vertices (–29.586, –20.534), (1.652, –20.534), (–29.586, 2.847), the perturbed matrices $P = P_o + \Delta P$, where ΔP is obtained by Equation (5.7), are not positive definite. On the other hand, the matrix P at the vertex (1.652, 2.847) is positive definite since their eigenvalues are all positive. The eigenvalues of ΔP_i at the vertex (1.652, 2.847) are

$$\begin{cases} \lambda(\Delta P_1) = \{0.5714, 0, -0.0318\} \\ \lambda(\Delta P_2) = \{0.3705, 0, -0.0131\} \end{cases} \tag{6.14}$$

Using Theorem 5.1, the asymmetric first-order stability robustness bounds are determined by the four divisions:

$$\begin{aligned} &0.5714\ k_1 + 0.3705\ k_2 < 1 && \text{for } k_1 \geq 0, k_2 \geq 0 \\ &0.0318\ k_1 + 0.3705\ k_2 < 1 && \text{for } k_1 < 0, k_2 \geq 0 \\ &0.5714\ k_1 + 0.0131\ k_2 < 1 && \text{for } k_1 \geq 0, k_2 < 0 \\ &0.0318\ k_1 + 0.0131\ k_2 < 1 && \text{for } k_1 < 0, k_2 < 0 \end{aligned} \tag{6.15}$$

The vertices of the bounds defined on the condition (7.14) are: (1.7501, 0), (0, 2.6991), (–31.4465, 0), and (0, –76.3358). At these vertices, the expansion Theorem 4.1 is satisfied. The comparison between the various robustness bounds for this example is shown on Figure 6.2.

A Discussion of Examples

As the numerical examples demonstrated, the proposed asymmetric first-order method estimated less conservative stability bounds for perturbation parameters than those obtained by conventional methods. In particular, Example 1 shows that new method can easily estimate the infinite bound of the stability region while the recursive iteration method demands much computational effort to identify the infinite boundaries of stability estimates [Olas 94b].

CHAPTER VII

CASE STUDIES

Case 1: A Two-Stage Electrohydraulic Servovalve Design

Problem Statement

A two-stage electrohydraulic servovalve with a drain orifice is considered as a practical plant for the application of stability robustness. The stability of a two-stage electrohydraulic servovalve has received much attention for many years and is not completely resolved (see [Merritt 67], [Martin 76], [Watton 87], [Akers 90]). A new design for a two-stage electrohydraulic servovalve is introduced in Appendix D. A feature of the new servovalve is a variant drain orifice damping on the first-stage flapper-nozzle in order to increase the system performance as well as the system stability.

The dynamics of an electrohydraulic servovalve are complex and highly nonlinear; many physical properties and characteristics of the electrohydraulic servovalve are hard to measure or affected by conditions such as temperature, wear, and oil contamination. Empirical or experimental data have been used in the design stage to determine the stability robustness of an electrohydraulic servovalve, however a few analytical methods for determining the stability robustness of an electrohydraulic servovalve have been published (see for example [Merritt 67], [Nikiforuk 69], [de Pennington 74], [Watton 87], [Akers 90]).

Recently, Watton [Watton 87] addressed a stability criterion for the design of an electrohydraulic servovalve using the linearization method and some assumptions for simplification. However, Watton's stability criterion is too conservative to be used for the practical design of a two-stage electrohydraulic servovalve. Watton also neglected the effect of the viscous damping constant of the second-stage spool which is not negligible in practice. Watton's criterion provides narrow tolerance limits of the uncertain parameters resulting in high manufacturing cost. Therefore, a new stability robustness method for the design of the two-stage electrohydraulic servovalve is desirable.

Modeling of Servovalve Dynamics

The new method for the stability robustness bounds in preceding chapters could be applied to the design of a servovalve. Here, in order to illustrate effectiveness of the new stability robustness method when applied to practical design problems, a conventional two-stage electrohydraulic servovalve with a fixed drain orifice is considered [see Watton 87 for details]. The results of the stability analysis using the new method are compared to Watton's stability criterion.

The simplified configuration of the flapper-nozzle valve with a fixed drain orifice is shown on Figure D.2, Appendix D. From the continuity of flow it is seen that

$$
\begin{aligned}
\frac{V_t}{\beta}\frac{dP_1}{dt} &= Q_{o1} - Q_{n1} - Q_{sv} - Q_L \\
\frac{V_t}{\beta}\frac{dP_2}{dt} &= Q_{o2} - Q_{n2} + Q_{sv} + Q_L \\
\frac{V_e}{\beta}\frac{dP_e}{dt} &= Q_{n1} + Q_{n2} - Q_e
\end{aligned}
\tag{7.1}
$$

where

$$
\begin{aligned}
Q_{o1} &= C_{do}\frac{\pi\, d_o^2}{4}\sqrt{\frac{2}{\rho}(P_s - P_1)} \\
Q_{o2} &= C_{do}\frac{\pi\, d_o^2}{4}\sqrt{\frac{2}{\rho}(P_s - P_2)} \\
Q_{n1} &= C_{df}\,\pi\, d_n\,(x_{fo} - x_f)\sqrt{\frac{2}{\rho}(P_1 - P_e)} \\
Q_{n2} &= C_{df}\,\pi\, d_n\,(x_{fo} + x_f)\sqrt{\frac{2}{\rho}(P_2 - P_e)} \\
Q_{sv} &= A_p\frac{dx_p}{dt} \\
Q_L &= K_L\,(P_1 - P_2) \\
Q_e &= C_{dd}\frac{\pi\, d_d^2}{4}\sqrt{\frac{2}{\rho}P_e}
\end{aligned}
\tag{7.2}
$$

The dynamics of the torque motor with a flapper are given by

$$r\, K_t\, i = J_a \frac{d^2 x_f}{dt^2} + B_a \frac{dx_f}{dt} + K_a\, x_f + A_n\, r^2 (P_1 - P_2) \tag{7.3}$$

And the dynamics of the second-stage spool valve are given by

$$A_p (P_1 - P_2) = M_s \frac{d^2 x_p}{dt^2} + B_s \frac{dx_p}{dt} + K_s\, x_p \tag{7.4}$$

Define following variables for the non-dimensionalization such that

$$k_o = C_{do} \frac{\pi\, d_o^2}{4} \sqrt{\frac{2\, P_s}{\rho}}, \quad k_n = C_{df}\, \pi\, d_n\, x_{fo} \sqrt{\frac{2\, P_s}{\rho}} \tag{7.5}$$

and

$$\lambda = \frac{C_{dd}\, d_d^2}{C_{do}\, d_o^2} \tag{7.6}$$

$$\overline{P}_1 = \frac{P_1}{P_s}, \quad \overline{P}_2 = \frac{P_2}{P_s}, \quad \overline{P}_e = \frac{P_e}{P_s}, \quad \overline{x}_f = \frac{x_f}{x_{fo}}, \quad \overline{x}_p = \frac{x_p}{x_{pm}}, \quad \overline{i} = \frac{i}{i_m} \tag{7.7}$$

$$C = \frac{P_s\, V_t}{\beta}, \quad C_e = \frac{P_s\, V_e}{\beta} \tag{7.8}$$

Then, Equation (7.1) and Equation (7.2) yield

$$C \frac{d\overline{P}_1}{dt} = k_o \sqrt{1 - \overline{P}_1} - k_n (1 - \overline{x}_f) \sqrt{\overline{P}_1 - \overline{P}_e} - A_p \, x_{pm} \frac{d\overline{x}_p}{dt} - K_L \, P_s (\overline{P}_1 - \overline{P}_2) \tag{7.9}$$

$$C \frac{d\overline{P}_2}{dt} = k_o \sqrt{1 - \overline{P}_2} - k_n (1 + \overline{x}_f) \sqrt{\overline{P}_2 - \overline{P}_e} + A_p \, x_{pm} \frac{d\overline{x}_p}{dt} + K_L \, P_s (\overline{P}_1 - \overline{P}_2) \tag{7.10}$$

$$C_e \frac{d\overline{P}_e}{dt} = k_n (1 - \overline{x}_f) \sqrt{\overline{P}_1 - \overline{P}_e} + k_n (1 + \overline{x}_f) \sqrt{\overline{P}_2 - \overline{P}_e} - \lambda \, k_o \sqrt{\overline{P}_e} \tag{7.11}$$

Using the Taylor series expansion at the null position of flapper, one has

$$\sqrt{1-\bar{P}_1} \approx -\frac{1}{2}\frac{1}{\sqrt{1-\bar{P}_{1o}}}\Delta\bar{P}_1 \tag{7.12}$$

$$\sqrt{1-\bar{P}_2} \approx -\frac{1}{2}\frac{1}{\sqrt{1-\bar{P}_{2o}}}\Delta\bar{P}_2 \tag{7.13}$$

$$\begin{aligned}(1-\bar{x}_f)\sqrt{\bar{P}_1-\bar{P}_e} \approx -\sqrt{\bar{P}_{1o}-\bar{P}_{eo}}\,\Delta\bar{x}_f + \frac{1}{2}\frac{1}{\sqrt{\bar{P}_{1o}-\bar{P}_{eo}}}\Delta\bar{P}_1 - \\ \frac{1}{2}\frac{1}{\sqrt{\bar{P}_{1o}-\bar{P}_{eo}}}\Delta\bar{P}_e\end{aligned} \tag{7.14}$$

$$\begin{aligned}(1+\bar{x}_f)\sqrt{\bar{P}_2-\bar{P}_e} \approx \sqrt{\bar{P}_{2o}-\bar{P}_{eo}}\,\Delta\bar{x}_f + \frac{1}{2}\frac{1}{\sqrt{\bar{P}_{2o}-\bar{P}_{eo}}}\Delta\bar{P}_2 - \\ \frac{1}{2}\frac{1}{\sqrt{\bar{P}_{2o}-\bar{P}_{eo}}}\Delta\bar{P}_e\end{aligned} \tag{7.15}$$

$$\sqrt{\bar{P}_e} \approx \frac{1}{2}\frac{1}{\sqrt{\bar{P}_{eo}}}\Delta\bar{P}_e \tag{7.16}$$

Substituting the equations (7.12) to (7.16) into the equations (7.9) to (7.11) and defining the state variables such that

$$x^T = [x_1,\ \ldots,\ x_7] = \left[\Delta\bar{x}_f,\ \Delta\dot{\bar{x}}_f,\ \Delta\bar{x}_p,\ \Delta\dot{\bar{x}}_p,\ \Delta P_1,\ \Delta P_2,\ \Delta P_e\right],\quad u = \bar{i} \tag{7.17}$$

one has the following state-space form of servovalve dynamics:

$$\dot{x} = A\,x + B\,u \tag{7.18}$$

where

$$\text{(i)}: -\frac{k_o}{2C\phi_1}-\frac{k_n}{2C\phi_{1e}}-\frac{K_L P_s}{C}$$

$$\text{(ii)}: -\frac{k_o}{2C\phi_2}-\frac{k_n}{2C\phi_{2e}}-\frac{K_L P_s}{C}$$

$$\text{(iii)}: -\frac{k_n\left(\phi_{1e}-\phi_{2e}\right)}{2C_e}$$

$$\text{(iv)}: -\frac{k_n}{2C_e}\left(\frac{1}{\phi_{1e}}+\frac{1}{\phi_{2e}}\right)-\frac{\lambda k_o}{2C_e\phi_e}$$

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-K_a}{J_a} & \frac{-B_a}{J_a} & 0 & 0 & \frac{-A_n r^2 P_s}{J_a x_{fo}} & \frac{A_n r^2 P_s}{J_a x_{fo}} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{-K_s}{M_s} & \frac{-B_s}{M_s} & \frac{A_p P_s}{M_s x_{pm}} & \frac{-A_p P_s}{M_s x_{pm}} & 0 \\ \frac{k_n\phi_{1e}}{C} & 0 & 0 & \frac{-A_p x_{pm}}{C} & \text{(i)} & \frac{K_L P_s}{C} & \frac{k_n}{2C\phi_{1e}} \\ \frac{-k_n\phi_{2e}}{C} & 0 & 0 & \frac{A_p x_{pm}}{C} & \frac{K_L P_s}{C} & \text{(ii)} & \frac{k_n}{2C\phi_{2e}} \\ \text{(iii)} & 0 & 0 & 0 & \frac{k_n}{2C_e\phi_{1e}} & \frac{k_n}{2C_e\phi_{2e}} & \text{(iv)} \end{bmatrix}$$

$$B^T=\begin{bmatrix} 0 & \frac{rK_t i_m}{J_a x_{fo}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

in which the variables defined for simplification are

$$\phi_1 := \sqrt{1-\overline{P}_{1o}},\ \phi_2 := \sqrt{1-\overline{P}_{2o}},\ \phi_{1e} := \sqrt{\overline{P}_{1o}-\overline{P}_{eo}},\ \phi_{2e} := \sqrt{\overline{P}_{2o}-\overline{P}_{eo}},\ \phi_e := \sqrt{\overline{P}_{eo}}.$$

Stability Consideration

Since the servovalve system equation (7.18) is not a regulator form for Equation (3.1), one needs to consider the relationship between BIBO (Bounded Input-Bounded Output) stability and Lyapunov stability. It is said that a dynamic system described by a vector differential equation $\dot{x} = f(x,u,t)$ is BIBO stable (any bounded input u produces a bounded output x) if for all x_o and t_o, and for all bounded inputs $u(t)$, $t_o \le t < \infty$, which satisfy the inequality $||u|| < M$ for all $t \ge t_o$ where $M < \infty$ is a constant. The output motion $x(t,x_o,t_o)$ is bounded.

It may be concluded that the Lyapunov stability concept is concerned with the internal dynamics of the system, whereas the BIBO stability reflects their external behavior. In general, Lyapunov stability does not guarantee BIBO stability, and vice versa. However, if a linear, time-invariant, differential system is asymptotically stable (in the large) in the Lyapunov sense, it is also BIBO stable; that is, any bounded input will produce a bounded output [Zadeh 63]. The converse is also true, provided the system is completely observable and controllable. Hence the new method for stability robustness bounds can be applied to the stability of the servovalve system on Equation (7.18).

A Typical Numerical Example

Table 7.I shows typical servovalve parameters. From the data on Table 7.I and by [Watton 87], the values of some parameters on Equation (7.18) at the null position are estimated such that

$$\lambda = \frac{C_{dd}\, d_d^2}{C_{do}\, d_o^2} = 4.2, \qquad Z = \left(\frac{k_n}{k_o}\right)^2 = 16\left(\frac{C_{df}\, d_n\, x_{fo}}{C_{do}\, d_o^2}\right)^2 = 1.01 \tag{7.19}$$

and

$$\overline{P}_{eo} = \frac{4\,Z}{4\,Z + (1+Z)\,\lambda^2} = 0.1023 \tag{7.20}$$

$$\overline{P}_{1o} = \overline{P}_{2o} = \frac{4\,Z + \lambda^2}{4\,Z + (1+Z)\,\lambda^2} = 0.5489 \tag{7.21}$$

Hence

$$\phi_1 = \phi_2 = 0.9475, \quad \phi_{e1} = \phi_{e2} = 0.6683, \quad \phi_e = 0.3198 \tag{7.22}$$

Substituting the values of the parameters on Table 7.I into the matrix A on Equation (7.18), and finding the eigenvalues of matrix A yields

$$10^5 \times \left[-1.5601 \pm 1.1152i,\ -0.0116,\ -0.1939,\ -0.0005,\ -5.6693,\ -0.1326\right]$$

Since all eigenvalues have negative real values, the system on Equation (7.18) is asymptotically stable.

Suppose that the uncertain parameters are the viscous damping constants of the flapper-nozzle and that of the second-stage spool: i.e., $k_1 = B_a$, $k_2 = B_s$. Applying Gao's zero-order stability bounds to the system matrix, one obtains the following robustness bounds condition.

$$5.1302 \times 10^5\, k_1 + 1.8505 \times 10^4\, k_2 < 1, \quad k_1 > 0, \quad k_2 > 0 \tag{7.23}$$

The Expansion Theorem 4.1 is satisfied at the vertices of the stability region defined on condition (7.23). Then applying the first-order method in Chapter V at the vertex $\left(1.9492 \times 10^{-6}, 5.4039 \times 10^{-5}\right)$, one also obtains the following stability robustness bounds

$$9.1408\, k_1 + 0.04644\, k_2 < 1, \quad k_1 > 0, \quad k_2 > 0 \tag{7.24}$$

Since the Expansion Theorem 4.1 is satisfied at the vertices obtained on condition (7.24), the stability robustness bounds are fully expanded. Figure 7.1 graphically shows the results of the stability robustness bounds resulting from the conditions of (7.23) and (7.24). This figure reveals that the first-order method estimates significantly less conservative stability bounds than the zero-order method.

Analysis and Discussion

Watton [Watton 87] proposed a sufficient condition for a two-stage electrohydraulic servovalve with a drain orifice as follows:

$$\frac{B_a^2\, x_{fo}}{J_a\, P_s\, A_n\, r^2} > f_g \tag{7.25}$$

where f_g is a gain function defined in [Watton 87]. The gain function f_g is mainly determined by the parameters of the servovalve configuration and flow characteristics.

In the stability criterion of the condition (7.25), Watton neglected the effect of the viscous damping of the second-stage spool valve. Even neglecting the effect of the viscous damping of the spool on the Watton's criterion, the maximum value of the damping constant of the flapper-nozzle in this example would be estimated less than 3.9775×10^{-4} lb_f in s / rad. This value is too small to be compared with a practical one. The supply pressure should be also decreased for stability if the viscous damping constant of the flapper decreases. Hence precise estimation of the viscous damping is important for the economic operation and the design of the electrohydraulic servovalve.

The purpose of Watton's study was to assess the effect of the drain orifice damping on the performance and the stability of an electrohydraulic servovalve. Since a drain orifice is attached to the flapper-nozzle, a small back pressure increases the stability of the servovalve, but decreases performance characteristics such as the null pressure sensitivity. However one cannot easily use the sufficient condition (7.25) to determine the size of the drain orifice for stability, because parameters of the drain orifice such as d_d and V_e are not explicitly included in the condition (7.25). This drawback is resolved if the proposed method for the stability robustness is utilized.

Hence, one can find that the proposed method for stability robustness provides superior stability bound estimates compared to those of conventional methods, and that the new method can be complementary to the Watton's criterion and applicable to the practical design of a two-stage electrohydraulic servovalve with a drain orifice.

Case 2: Sensor Degrading Accommodation with Incomplete Information

Problem Statement

This is the type of control system failure where some of the sensors lose accuracy, and it is called a "sensor degrading" failure. When failure information is incomplete, the major concern is system stability. If the system can be stabilized quickly, immediate catastrophic consequences can be avoided and time is available to obtain more accurate information on the failure. The stability analysis, to check if the nominal system still remains stable for any sensor degradation, should occur before the stabilization action. If result of the stability analysis is negative, then one can take the linear quadratic (LQ) approach to desensitize the system with respect to the uncertain sensor degradation. By utilizing the results of the preceding chapters one is able to analyze, in an aircraft flight control system for example, how much sensor failure the system can tolerate. See ([Etkin 72], [Etkin 82]) for the sensors utilized in an aircraft.

Modeling of Aircraft Control Motion

A typical formulation of the longitudinal motion in a fighter aircraft is given by [Sparks 93]

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha & Z_q \\ M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_{\delta_{PV}} \\ M_{\delta_{PV}} \end{bmatrix} \delta_{PV} \tag{7.26}$$

where

α, q: angle of attack and pitch rate

Z_0, M_0: longitudinal dimensional stability derivatives

δ_{PV}: pitch vectoring nozzle deflection

The gyroscope sensor produces measurement of pitch rate in the body axis, and angle of attack signal is constructed by augmenting the vane measurement from the air data unit with inertial data.

Assume that the pitch rate sensor and the angle of attack sensing apparatus are degrading. Define the sensor degradation

$$\bar{\alpha} := r_1\, \alpha, \quad \bar{q} := r_2\, q \tag{7.27}$$

where r_1 and r_2 are the degrading constants for the angle of attack sensing apparatus and the pitch rate sensor, respectively.

Assuming the LQ controller where a constant state feedback, u = K x, is used to minimize the quadratic cost function

$$J = \int_0^{\infty} \left(x^T Q x + u^T R u \right) dt \tag{7.28}$$

where Q and R are positive definite symmetric weighting matrices for the states and inputs, respectively. The control gain K is obtained by solving the Riccati equation.

Substituting Equation (7.27) into Equation (7.26) and using the constant state feedback control yields the perturbation matrix

$$E = r_1 \begin{bmatrix} Z_{\delta_{PV}} k_{11} & 0 \\ M_{\delta_{PV}} k_{11} & 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 & Z_{\delta_{PV}} k_{12} \\ 0 & M_{\delta_{PV}} k_{12} \end{bmatrix} \tag{7.29}$$

where k_{ij} is ij_th component of the gain matrix K.

A Typical Numerical Example

Typical data for a fighter aircraft in [Sparks 93] are used for the numerical calculation.

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0.0264 & 0.9905 \\ -0.8810 & -0.2079 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} -0.0520 \\ -4.3434 \end{bmatrix} \delta_{PV} \tag{7.30}$$

Using this model, the stability robustness bounds analysis for the perturbed parameters r_1 and r_2 is performed, i.e., $k_1 = r_1$, $k_2 = r_2$. When Q = 10 I and R = I where I is an identity matrix, the feedback gain K = [3.0923, 3.2918] is obtained in Equation

(7.28). Then one has

$$E_1 = \begin{bmatrix} -0.1608 & 0 \\ -13.4310 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & -0.1712 \\ 0 & -14.2977 \end{bmatrix} \tag{7.31}$$

The eigenvalues of P_1 and P_2 estimated by Gao's Theorem are

$$\lambda(P_1) = \{69.0815,\ -90.3626\},\quad \lambda(P_2) = \{0.7593,\ -168.9734\} \tag{7.32}$$

and the stability bounds derived are

$$\begin{aligned} &69.0815\ k_1 + 0.7593\ k_2 < 1 \quad \text{for } k_1 \geq 0,\ k_2 \geq 0 \\ &90.3626\ k_1 + 0.7593\ k_2 < 1 \quad \text{for } k_1 < 0,\ k_2 \geq 0 \\ &69.0815\ k_1 + 168.9734\ k_2 < 1 \quad \text{for } k_1 \geq 0,\ k_2 < 0 \\ &90.3626\ k_1 + 168.9734\ k_2 < 1 \quad \text{for } k_1 < 0,\ k_2 < 0 \end{aligned} \tag{7.33}$$

The Expansion Theorem is satisfied at the four vertices of the k_1-k_2 domain: (0.0145, 1.3170), (–0.0111, 1.3170), (–0.0111, –0.0059), and (0.0145, –0.0059). Hence the stability bounds are expanded into the full ranges of Gao's zero-order estimates. The perturbed matrices P obtained by Equation (5.7) at the vertices (0.0145, 1.3170) and (–0.0111, 1.3170) are not positive definite, while the matrices P obtained at the vertices (–0.0111, –0.0059) and (0.0145, –0.0059) are positive definite. Denote v_1 and v_2 for the vertices (–0.0111, –0.0059) and (0.0145, –0.0059) respectively. Then, using Theorem 5.3, one can find that the vertex v_2 is better than the vertex v_1 to apply the first-order method. The eigenvalues of ΔP_i at the vertex v_1 are

$$\lambda(\Delta P_1) = \{-47.6941,\ 41.8053\},\quad \lambda(\Delta P_2) = \{0.1032,\ -95.1714\} \tag{7.34}$$

And, the eigenvalues of ΔP_i at the vertex v_2 are

$$\lambda(\Delta P_1) = \{27.1281,\ -25.6049\},\quad \lambda(\Delta P_2) = \{0.0117,\ -56.1240\} \tag{7.35}$$

which yield following first-order stability robustness bounds

$$
\begin{aligned}
&27.1281\ k_1 + 0.0117\ k_2 < 1 \quad \text{for } k_1 \geq 0,\ k_2 \geq 0 \\
&25.6049\ k_1 + 0.0117\ k_2 < 1 \quad \text{for } k_1 < 0,\ k_2 \geq 0 \\
&27.1281\ k_1 + 56.1240\ k_2 < 1 \quad \text{for } k_1 \geq 0,\ k_2 < 0 \\
&25.6049\ k_1 + 56.1240\ k_2 < 1 \quad \text{for } k_1 < 0,\ k_2 < 0
\end{aligned}
\tag{7.36}
$$

Since the Expansion Theorem is satisfied at the vertices (0.0369, 85.4701) and (–0.03906, 85.4701), the stability bounds can be expanded in the quadrants where these vertices belong to. Figure 7.2 illustrates several stability bound estimates for the perturbed parameters k_1 and k_2. This figure shows that the degradation of pitch rate sensor is more tolerable than the failure of angle of attack sensing apparatus for the aircraft to maintain stability.

A Discussion of Case Studies

Many earlier works for stability robustness often used simple numerical examples as shown in Chapter VI to verify their results. However, as demonstrated in the preceding two practical examples, the stability bounds estimated by conventional stability robustness methods are often too conservative to be used for the analysis of practical systems. Case 1, for the design of a two-stage electrohydraulic servovalve, shows that conventional methods have estimated extremely conservative stability bounds for uncertain parameters. Even Gao's method for asymmetric stability bounds failed to provide practical, useful estimates. On the other hand, the proposed asymmetric first-order method, which was easily applicable, provided a substantially large estimation of stability bounds for practical usage.

CHAPTER VIII

APPLICATION TO ASYMMETRIC ROBUSTNESS MEASURE OF EIGENVALUE DISTRIBUTION

Introduction

For a linear time-invariant control system, it is known that the performance specifications for the system can be satisfied by suitably assigning the poles of the system. However, due to the presence of uncertainty or variation of parameters for a system with an approximated model, the poles of the designed system shift away from their prescribed locations so that the performance of the system may be seriously degraded. Therefore, it is desirable to estimate the allowable bounds of uncertain perturbation parameters under which the eigenvalues of the perturbed system stay in the prescribed region.

Recently, robust eigenvalue-assignment problems, to maintain the stability and meet additional performance requirements of a perturbed system, received much attention ([Juang 89], [Juang 90], [Shieh 90], [Chou 91], [Juang 93], [Horng 93], [Alt 93], [Chouaib 94]). However, similar to the stability robustness problem, most estimation techniques developed assume symmetric bounds of perturbation parameters around the origin to provide conservative estimates of robustness bounds. This chapter deals with asymmetric robustness measures for linear systems with structured uncertainties where the eigenvalues of the perturbed systems are guaranteed to stay in a prescribed region. Based upon the Lyapunov approach as shown in preceding chapters, new techniques to estimate allowable perturbation parameter bounds are derived. Examples are given to illustrate proposed methods.

Preliminary Results

Consider a line L which separates the complex plane into two open half-planes, H and $\overline{H}$, as shown in Figure 8.1. The line L intersects the real axis at $(\alpha, 0)$, the imaginary axis at $(0, j\beta)$, and makes an angle θ with respect to the imaginary axis, where θ is assumed positive in a counterclockwise sense and $-\pi \le \theta \le \pi$.

Juang et. al [Juang 90] proposed the following lemmas:

Lemma 8.1

All the eigenvalues of the constant matrix A *lie in the region* H *if and only if matrix* $e^{-j\theta}(A - \alpha I)$ *or matrix* $e^{-j\theta}(A - j\beta I)$ *is stable.* □

Lemma 8.2

All the eigenvalues of a constant matrix A *lie in the region* H *if and only if*

$$\left[e^{-j\theta}(A - \alpha I)\right]^* P + P\left[e^{-j\theta}(A - \alpha I)\right] = -2\,I \tag{8.1}$$

or

$$\left[e^{-j\theta}(A - j\beta I)\right]^* P + P\left[e^{-j\theta}(A - j\beta I)\right] = -2\,I \tag{8.2}$$

has a unique positive definite Hermitian solution P, *where* * *denotes the conjugate transpose.* □

Based upon the Lyapunov approach, Juang et. al [Juang 90] addressed a technique for calculating the robustness bounds for eigenvalue-assignment in any prescribed region taking account not only stability robustness but also certain types of performance robustness.

Consider the perturbed system described by

$$\dot{x} = (A_N + E)\,x \tag{8.3}$$

where A_N is a nominally stable matrix and E represents a perturbation matrix which has a form

$$E = \sum_{i=1}^{m} k_i\, E_i \tag{8.4}$$

where k_i are uncertain perturbation parameters and E_i are constant matrices which denote the structure of the perturbations.

Then the criterion for the analysis of eigenvalue-assignment robustness in [Juang 90] is shown in the following theorem:

Theorem 8.1

If all the eigenvalues of a matrix A_N *lie in the region* H, *then the eigenvalues of the perturbed matrix* $A_N + E$ *will remain in the same region if*

$$\sum_{i=1}^{m} |k_i| \leq \mu := \frac{1}{\sum_{i=1}^{m} \| |M_i| \|} \tag{8.5}$$

where

$$M_i = \frac{e^{j\theta}\, E_i^{*}\, P + P\, E_i\, e^{-j\theta}}{2} \tag{8.6}$$

$\|.\|$ *denotes the spectral norm and* P *is the unique Hermitian matrix obtained by Equation (8.1) or Equation (8.2).* □

Zero-Order Asymmetric Robustness Measure

As shown in Theorem 8.1, the criterion for eigenvalue-assignment robustness by Juang [Juang 90] provides symmetric intervals around the origin for the uncertain parameter perturbations which resulting in conservative estimates of robustness bounds. Hence, an asymmetric robustness measure to resolve the conservatism is proposed.

Lemma 8.3

All eigenvalues of matrix M_i *defined on Equation (8.6) are real values.*

Proof

The Lyapunov equation on Equation (8.1) has a unique Hermitian positive definite matrix, P. Alt and Jabbari [Alt 93] addressed that matrix P is represented by

$$P = P_{Re} + j\, P_{Im}, \quad P_{Re} = P_{Re}^T, \quad P_{Im} = -P_{Im}^T \tag{8.7}$$

where P_{Re} is a real term and P_{Im} is an imaginary term of a complex matrix P respectively. Hence Equation (8.6) yields

$$\begin{aligned} e^{j\theta}E_i^TP + P\,E_ie^{-j\theta} &= \left(\cos\theta + j\,\sin\theta\right) E_i^T\,(P_{Re}+jP_{Im}) + (P_{Re}+jP_{Im})\,E_i\left(\cos\theta - j\,\sin\theta\right) \\ &= \cos\theta\;E_i^TP_{Re} - \sin\theta\;E_i^TP_{Im} + j\,\sin\theta\;E_i^TP_{Re} + j\,\cos\theta\;E^TP_{Im} + \\ &\quad \cos\theta\;P_{Re}\,E_i + \sin\theta\;P_{Im}\,E_i - j\,\sin\theta\;P_{Re}\,E_i + j\,\cos\theta\;P_{Im}\,E_i \end{aligned} \tag{8.8}$$

Since the real terms of the right-hand side of Equation (8.8) are

$$\begin{aligned} &\cos\theta\left(E_i^TP_{Re}+P_{Re}E_i\right) + \sin\theta\left(-E_i^TP_{Im}+P_{Im}E_i\right) = \\ &\qquad \cos\theta\left[(P_{Re}E_i)^T + P_{Re}E_i\right] + \sin\theta\left[(P_{Im}\,E_i)^T + P_{Im}E_i\right] \end{aligned} \tag{8.9}$$

and the imaginary terms are

$$\sin\theta\left(E_i^T P_{Re} - P_{Re} E_i\right) + \cos\theta\left(E_i^T P_{Im} + P_{Im} E_i\right) = \tag{8.10}$$

$$\sin\theta\left[(P_{Re}E_i)^T - P_{Re}E_i\right] + \cos\theta\left[-(P_{Im}\,E_i)^T + P_{Im}E_i\right]$$

Equation (8.9) and Equation (8.10) show that the matrix M_i on Equation (8.6) is a complex matrix whose real terms are symmetric, imaginary terms are skew-symmetric, and diagonal terms are real values. Hence all eigenvalues of matrix M_i are real. □

Theorem 8.2

If all the eigenvalues of a matrix A_N *lie in the region* H, *then the eigenvalues of the perturbed matrix* $A_N + E$ *will remain in the same region if*

$$\sum_{i=1}^{m} k_i\,\lambda_{Mi} < 1 \tag{8.11}$$

where

$$\lambda_{Mi} = \begin{cases} \lambda_{max}(M_i) & \text{for } k_i \geq 0 \\ \lambda_{min}(M_i) & \text{for } k_i < 0 \end{cases} \qquad i = 1, \ldots, m \tag{8.12}$$

in which $\lambda_{max}(.)$ *and* $\lambda_{min}(.)$ *are largest and smallest real eigenvalues of matrix (.) respectively.*

Proof

Since all the eigenvalues of A_N lie in the region H, according to Lemma 8.2 there must exist a Hermitian solution P on Equation (8.1) or Equation (8.2), so for the system

$$\dot{x} = \left[e^{-j\theta}(A_N + E - \alpha I)\right] x \tag{8.13}$$

or

$$\dot{x} = \left[e^{-j\theta}\left(A_N + E - j\beta I\right)\right] x \tag{8.14}$$

the Lyapunov function candidate is chosen as

$$V = x^* P x \tag{8.15}$$

Differentiating Equation (8.15), and using equations (8.1), (8.2), (8.13) and (8.14) yields

$$\dot{V} = x^* \left(e^{j\theta} E P + P E e^{-j\theta} - 2 I \right) x \tag{8.16}$$

Substituting Equation (8.4) and Equation (8.6) into Equation (8.16), one has

$$\dot{V} = 2 x^* \left(\sum_{i=1}^{m} k_i M_i - I \right) x \tag{8.17}$$

Hence following condition is necessary to satisfy $\dot{V} < 0$

$$\lambda\left(\sum_{i=1}^{m} k_i M_i \right) < 1 \tag{8.18}$$

Since Lemma 3 proves that all eigenvalues of matrix M_i are real, one can apply Gao's [Gao 93] result

$$\lambda\left(\sum_{i=1}^{m} k_i M_i \right) \leq \sum_{i=1}^{m} \lambda_{max}(k_i M_i) \leq \sum_{i=1}^{m} k_i \lambda_{Mi} < 1 \tag{8.19}$$

□

Gao's method is a special case when $\theta = 0$ and $\alpha = 0$ in Theorem 2.

First-Order Asymmetric Robustness Measure

In order to improve the conservatism of robustness bounds estimates, the first-order Lyapunov method [Leal 90] can be combined with the zero-order asymmetric robustness method proposed in Theorem 8.2.

Define the perturbed P matrix

$$P := P_o + \Delta P \tag{8.20}$$

in which matrix P_o satisfies the nominal Lyapunov equation on Equation (8.1) or Equation (8.2), i.e.,

$$\left[e^{-j\theta}(A_N - \alpha I)\right]^* P_o + P_o\left[e^{-j\theta}(A_N - \alpha I)\right] = -2\, I \tag{8.21}$$

or

$$\left[e^{-j\theta}(A_N - j\beta I)\right]^* P_o + P_o\left[e^{-j\theta}(A_N - j\beta I)\right] = -2\, I \tag{8.22}$$

And define

$$\Delta M_i := \frac{e^{j\theta}\, E_i^T\, \Delta P + \Delta P\, E_i\, e^{-j\theta}}{2}, \quad i = 1, \ldots, m \tag{8.23}$$

and

$$\Delta\lambda_{Mi} := \begin{cases} \lambda_{max}(\Delta M_i) & \text{for } k_i \geq 0 \\ \lambda_{min}(\Delta M_i) & \text{for } k_i < 0 \end{cases} \quad i = 1, \ldots, m \tag{8.24}$$

The first-order asymmetric robustness measure is obtained in the following theorem.

Theorem 8.3

For a linear system on Equation (8.3) where all the eigenvalues of a nominal matrix A_N *lie in the region* H *on Figure 8.1, the eigenvalues of the perturbed matrix* $A_N + E$ *remain in the region* H *if*

$$\sum_{i=1}^{m} k_i\, \Delta\lambda_{Mi} < 1 \tag{8.25}$$

and if perturbed matrix P *is a positive definite matrix in which* ΔP *satisfies*

$$A_H^* \Delta P + \Delta P A_H = -\left(e^{j\theta} E^T P_o + P_o E e^{-j\theta}\right) \tag{8.26}$$

where

$$A_H := e^{-j\theta} (A_N - \alpha I) \tag{8.27}$$

<u>Proof</u>

As proof for theorem 8.2, for the system described on Equation (8.13) or Equation (8.14), the Lyapunov function candidate is chosen as

$$V = x^* P x = x^* \left(P_o + \Delta P\right) x \tag{8.28}$$

Differentiating Equation (8.28) yields

$$\begin{aligned} \dot{V} = {} & x^* \left(e^{j\theta} E^T \Delta P + \Delta P E e^{-j\theta} - 2I\right) x + \\ & x^* \left(A_H^* \Delta P + \Delta P A_H + e^{j\theta} E^T P_o + P_o E e^{-j\theta}\right) x \end{aligned} \tag{8.29}$$

Using Equation (8.23) and Equation (8.26), Equation (8.29) yields

$$\dot{V} = 2 x^* \left(\sum_{i=1}^{m} k_i \Delta M_i - 1 \right) x < 0 \tag{8.30}$$

Similarly as for M_i in Lemma 8.3, one can prove that all eigenvalues of matrix ΔM_i are real values. Hence, using the results addressed in [Gao 93], it is shown that

$$\lambda\left(\sum_{i=1}^{m} k_i \Delta M_i \right) \le \sum_{i=1}^{m} \lambda_{max}\left(k_i \Delta M_i\right) \le \sum_{i=1}^{m} k_i \Delta\lambda_{Mi} < 1 \tag{8.31}$$

□

Application Procedure for the Asymmetric First-Order Lyapunov Method to the Robustness Measure of Eigenvalue-Location Assignment

A procedure is proposed as follows, to measure the robustness bounds of uncertain perturbation parameters for the eigenvalue-location assignment using the asymmetric first-order Lyapunov method.

First, estimate the robustness bounds for uncertain perturbation parameters using the asymmetric zero-order method described in Theorem 8.2.

Second, find the vertices to apply the first-order Lyapunov method among the vertices of the robustness bounds hypercube obtained at the first stage. The first-order method can be applied at the vertices where the estimated Lyapunov matrices "P" are positive definite.

Third, estimate the robustness bounds by applying the asymmetric first-order Lyapunov method of Theorem 8.3 to the vertices identified at the second stage. The common boundaries of the robustness bounds are final measure when the first-order method is applied at more than one vertex.

Example 1

Consider a two-dimensional, third-order system in [Zhou 87, Gao 93] given by

$$\dot{x} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix} x + k_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} x + k_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \tag{8.32}$$

Assume $\theta = \pi/6$, $\alpha = 1$ for the location of eigenvalues as shown in Figure 8.1 for the system on Equation (8.20). Juang's criterion described in Theorem 8.1 yields the following robustness bounds:

$$|k_1| + |k_2| \leq 1.3946 \tag{8.33}$$

For the asymmetric zero-order robustness measure described in Theorem 8.2, the eigenvalues of M_1 and M_2 are

$$\lambda(M_1) = \{0.4442, -0.0156, 0\}, \ \lambda(M_2) = \{0.2729, 0, -0.0480\} \tag{8.34}$$

Hence, the robustness bounds obtained in Theorem 8.2 are as follows:

$$\begin{aligned} &0.4442\ k_1 + 0.2729\ k_2 < 1 && \text{for } k_1 \geq 0, k_2 \geq 0 \\ &0.0156\ k_1 + 0.2729\ k_2 < 1 && \text{for } k_1 < 0, k_2 \geq 0 \\ &0.4442\ k_1 + 0.0480\ k_2 < 1 && \text{for } k_1 \geq 0, k_2 < 0 \\ &0.0156\ k_1 + 0.0480\ k_2 < 1 && \text{for } k_1 < 0, k_2 < 0 \end{aligned} \tag{8.35}$$

The comparison between various stability and robustness bounds is shown on Figure 8.2. In this example, the bounds for the specification $\theta = \pi/6$, $\alpha = 1$, obtained by Theorem 8.2

are wider than the stability bounds estimated by Gao's method, since the specification $\theta = {}^{\pi}/_{6}$, $\alpha = 1$ includes certain unstable region.

Consider the same system described by Equation (8.32), as far as the performance specifications are considered, all eigenvalues of the system even under parameter perturbations are required to lie within the shaded region H_1 as shown on Figure 8.3; The eigenvalues $A_N + E$ should stay inside the specified region and the allowable bounds for the uncertain perturbation parameters k_1 and k_2 are of interest.

Assume, for eigenvalue locations, that the specifications of $\theta = {}^{\pi}/_{6}$, $\alpha_1 = -1$ and $\alpha_2 = -10$ on Figure 8.3 are required. Then the desired region H_1 is bounded by four lines L_1 to L_4. The specifications of θ and α for each line L_1 to L_4, corresponding to the eigenvalues of M_1 and M_2 are respectively listed on Table 8.I. The bound values for $\mu := \dfrac{1}{\sum_{i=1}^{m} \|M_i\|}$ in Theorem 8.1 for Juang's criterion are also included on Table 8.I.

Figure 8.4 illustrates the robustness bounds of uncertain perturbation parameters in order that the eigenvalues of the system matrix stay in the prescribed region. The bounds obtained by Juang's criterion are:

$$|k_1| + |k_2| \leq \mu = \min\{\mu_1, \mu_2, \mu_3, \mu_4\} = 0.5758 \tag{8.36}$$

The shaded common region, $A_R = A_{L_1} \cap A_{L_2} \cap A_{L_3} \cap A_{L_4}$, where A_{L_i}, i = 1,..., 4 are asymmetric robustness bounds corresponding to the boundary lines L_1 to L_4 respectively, shows the robustness bounds obtained by the proposed zero-order asymmetric robustness measure in Theorem 8.2. This figure reveals that the asymmetric robustness bounds for uncertain perturbation parameters are less conservative than symmetric bounds obtained by

Juang's method. Given the values of uncertain parameters k_1 and k_2 at four vertices of the figure that defines the asymmetric robustness bounds illustrated on Figure 8.4, transient responses on time-domain are obtained as shown on Figure 8.5. The initial values for state variables were $(x_1, x_2, x_3) = (-10, 10, -20)$. This figure shows that the system responses are overdamped and satisfy the specifications of eigenvalue locations.

Example 2

Consider a system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} x + k_1 \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} x + k_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x \tag{8.37}$$

Suppose that the eigenvalues of the system on Equation (8.37) are required to be located as shown on Figure 8.6. From Figure 8.6, the damping constant ζ and the natural frequency ω_n are specified by

$$1 \geq \zeta \geq \sin(\pi/6) = 0.5, \quad 20 \geq \omega_n \geq 1 \tag{8.38}$$

Using Equation (8.37) and Equation (8.38), one can obtain the exact bounds of uncertain parameters k_1 and k_2 such that

$$0.9 \geq k_1 - k_2 \geq -39, \quad 35 \geq k_1 \geq -4 \tag{8.39}$$

Similar to example 1, the bounds of uncertain perturbation parameters k_1 and k_2 to satisfy the eigenvalues-assignment are estimated by criterion on Theorem 8.2 as well as Juang's criterion on Theorem 8.1. Table 8.II shows the results of the numerical estimation, and Figure 8.7 shows the results graphically. These results show that the bounds estimated by the zero-order asymmetric robustness method in Theorem 8.2 are less conservative than those obtained by Juang's method.

The system on Equation (8.37) is also taken to validate the first-order asymmetric robustness measure in Theorem 8.3. Using the results on Table 8.II, one can apply the first-order asymmetric robustness method at each vertex of the robustness bounds estimated by the zero-order asymmetric robustness method. The results are summarized on Table 8.III. Figure 8.8 shows that the robustness bounds estimated first-order asymmetric method is considerably less conservative than those obtained by Juang's method. Figure 8.9 demonstrates the transient responses of the system at the extreme vertices of perturbation parameter bound estimates to show that the performance requirements are satisfied in the estimated bounds.

A Discussion of Asymmetric Measure for Robust Eigenvalue-Assignment

New measures for robust eigenvalue-assignment of uncertain systems are generalization of the asymmetric zero-order and first-order method for stability robustness bounds described in preceding chapters. The proposed methods for estimating asymmetric robustness bounds are less conservative than those obtained conventional Juang's method. In numerical example, the first-order asymmetric robustness measure provided better (less conservative) estimates of perturbation bounds than those of the zero-order method.

CHAPTER IX

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

Conclusions

The subject of the current investigation has been the stability robustness of nominally linear time-invariant system with structured uncertainties. Gao's asymmetric stability bounds and the first-order Lyapunov method were combined to develop a new method for the estimation of stability bounds.

Lemmas with the properties of matrix measure and convexity have formulated the Expansion Theorem in Chapter IV. The Expansion Theorem provides a sufficient condition under which a simple test at a vertex guarantees the expansion of the stability boundaries to the full region of the hyper-quadrant to which each vertex belongs.

The Theorem 5.1 establishes a new method to estimate asymmetric first-order stability bounds. The new approach, with the asymmetric first-order robustness method and the Expansion Theorem, provides three distinct advantages: first, ease of the application given by system matrices with structured perturbation parameters; second, improved means to estimate less conservative stability bounds; third, using the properties of the optimal Lyapunov function for systematically enlarging the perturbation parameter space hypercube. As proved in Chapter V, the new method has the properties of the optimal Lyapunov function in Olas' work. Theorem 5.2 with the properties of the optimal Lyapunov function proves that stability bounds estimated by the asymmetric first-order method are always less conservative than those of Gao's zero-order method under certain condition. Also, Theorem 5.3 shows that application of the asymmetric first-order method to the vertex which possesses a better performance measure of the Lyapunov function results in less conservative estimates of stability bounds.

It was demonstrated in Chapter VI that the results obtained through application of the proposed approach are superior to those obtained from the application of conventional methods. In practical cases, the proposed technique effectively estimated the practical bounds of uncertain parameters for the design of a two-stage electrohydraulic servovalve and the sensor degradation problem for a fighter aircraft stability. Especially for the design of a two-stage electrohydraulic servovalve, the practicality of the proposed method was evident, demonstrating that the new method is easily applicable and provides a substantially large estimation of stability bounds for practical usage. It was further demonstrated that the conventional stability criterion for a two-stage electrohydraulic servovalve and past methods for stability robustness provide extremely conservative estimations of stability bounds for uncertain parameters. It was shown in Chapter VIII that the proposed approach for the estimates of stability robustness bounds can be generalized to estimate asymmetric measure for robust eigenvalue distribution.

As described in Appendix D, a variant drain orifice on the first-stage flapper-nozzle enhances valve performance across the null position and overall stability of the servovalve simultaneously. The size and the features of a drain orifice with the uncertain parameters of an electrohydraulic servovalve could be determined by using the proposed technique of asymmetric first-order stability robustness.

Future Research Areas

There can be many possible avenues for future research. Prime candidates for such research efforts are addressed in the following paragraphs.

1. Transformation of the theorems for discrete-time systems. The results in this research are derived for continuous-time linear systems. A similar approach can be used to derive the first-order Lyapunov method for discrete-time linear systems as Gao derived corresponding results for the zero-order method in [Gao 93]. Thus, the use of the first-order Lyapunov method for the stability robustness in discrete-time domain with time delays can be a challenging research area.

2. Generalization of the results of stability robustness bounds for the robust eigenvalue-location assignment. As shown in Chapter VIII, the stability robustness problem is a special case of general robust eigenvalue-location assignment. In Chapter VIII, by using the asymmetric first-order Lyapunov method, theorems for the robust eigenvalue-location are derived to estimate the bounds of uncertain parameters similarly to the theorems for the stability robustness bound estimates. However, there remains a large research area concerning how the results obtained for the stability robustness bounds, such as the Expansion Theorem, the optimal condition of the first-order Lyapunov method, and optimal vertex condition, can be applied to the general robust eigenvalue-distribution problem.

3. Development of a recursive numerical algorithm. Since the stability bounds estimated by the proposed methods depend on the Q matrix of the Lyapunov equation, it is necessary to find a trend to obtain the optimum Q matrix. The proposed asymmetric first-order Lyapunov method in this research is obtained under the initial condition, Q = 2I, and could be recursively iterated. As mentioned in earlier Chapters, the proposed approach

can inherit the results of Olas' optimal Lyapunov function. The development of a recursive numerical algorithm with the asymmetric first-order Lyapunov method, in conjunction with Olas' optimal Lyapunov function, might also prove to be a fruitful avenue of research.

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APPENDIX A

PRELIMINARIES

Uncertainties

The system uncertainties may come from different sources. Due to the limitations in measurements, some parameters in either the time domain or frequency domain model may not be exactly known. Instead, there is a range of possible values these parameters can assume. For example, the resistance of a resistor is normally given by a nominal value and a percentage that represents the possible variation. This type of uncertainty in a system is called a parameter uncertainty because it can be expressed as variations of certain parameters in the system model. In contrast to parameter uncertainty, the unmodeled dynamics in a system constitute the non-parametric uncertainty because they cannot be directly described by variations of the system parameters. The unmodeled dynamics are the system dynamics that are not, or cannot be, incorporated in the mathematical model of the system and they are usually associated with the system behavior in response to relatively high-frequency inputs. In general, modeling techniques always leave ambiguities in the system model, either parametric or non-parametric, or both. Therefore, this is a problem all control engineers may face.

Sources of Uncertainties

The designing stage of a dynamic system is preceded by the modeling stage. In general, this model is not a true representation of the real system, i.e., there are uncertainties in the plant model. These uncertainties can be attributed to two major sources: the one source is external disturbance; the other source is model uncertainty.

External disturbances are often modeled statistically because they are not known, a priori. Disturbance signals can not be controlled and they are not dependent on the plant model.

Model uncertainties are caused by the following two reasons:

First, imprecise knowledge of the plant. Although the structure of the system equations can be obtained from basic laws of physics and engineering, or by experimental means, the numerical values of the parameters are only known within certain tolerances;

Second, simplifications and linearizations. Although the dynamics of the physical system may be known accurately, the designer may choose to simplify the model, e.g., one may reduce the order of the model or linearize the nonlinear components in order to simplify the calculations.

Types of Uncertainties

In most practical design cases, uncertainties can be categorized into two groups: one group is called unstructured uncertainty ; the other group is called structured uncertainty. Their existence is dependent upon the physics of the physical plant under consideration.

Structured uncertainty represents those uncertainties whose sources can be explicitly identified in a parametric model. The source of this type of uncertainty is an imprecise knowledge of the model parameters. Consider a state matrix for the plant perturbation represented by the summation of a nominal fixed matrix and perturbation matrix, that is,

$$A(t) = A_N + E(t) \tag{A.1}$$

Most engineering plants such as an aircraft or a robot can be described with known dynamical equations. The existing design uncertainties are with regard to the values of specific physical system parameters. Examples of structured perturbations in aircraft models include the parameter values for the spring constant, mass, inertia, aerodynamic coefficients, and changes in air pressure. These values cannot be considered as constant values, but they affect only specific system parameters [Bhattachargga 87].

An unstructured uncertainty is a lumped uncertainty that may represent several uncertainties that cannot explicitly be accomplished in a parametric model. This includes those that occur due to modeling approximation, e.g., linearization and unmodeled dynamics by neglecting high frequency components. Modeling continuous systems as finite lumped masses is one of the examples of unmodeled dynamics. In the unstructured perturbations, only the norm of the perturbation matrix E is specified. When possible, perturbations elements should be modeled as structured perturbations, since less conservative stability robustness bounds may then be obtained. These are the basic facts which have motivated growing interest in the robust control of systems with structured perturbation.

Models of Uncertainties

There are three prominent ways to model unstructured uncertainties. For a nominal transfer function matrix $G_0(s)$, which represents the best model of the true plant behavior, the true plant transfer function G(s) can be represented as one of following three forms:

$$G(s) = G_o(s) + \Delta_a(s)$$

$$G(s) = G_o(s)\left[I + \Delta_{im}(s)\right] \tag{A.2}$$

$$G(s) = \left[I + \Delta_{om}(s)\right] G_o(s)$$

where Δ_a is an additive unstructured uncertainty, Δ_{im} is an input multiplicative unstructured uncertainty, and Δ_{om} is an output multiplicative unstructured uncertainty. Since matrix multiplication does not commute in general, the location of a multiplicative uncertainty is critical in MIMO systems. There is no structural limitation imposed on Δ_a, Δ_{im} and Δ_{om}. The only limitation imposed on these uncertainties are the "size" of the matrix measured in an appropriate matrix norm. On the other hand, these uncertainty matrices will inherit a certain structural form that is determined by the knowledge of parameter uncertainties when they represent structured uncertainties.

In the case of structured perturbations, the system matrix is usually written in the following form:

$$\dot{x} = \left(A_N + \sum_{i=1}^{m} k_i E_i\right) x \tag{A.3}$$

where k_i is a perturbation element, also called parameter perturbation and E_i is a constant matrix, called i^{th} perturbation matrix. The advantage of this form is that it separates each of the independent perturbation parameters from the others.

Matrix Norms

The robustness of a system is measured by the "size" of a certain transfer function in many current design methods. For MIMO systems, this transfer function is a matrix itself. The "size" of a matrix is measured by its norm. Following three matrix norms have been most commonly found in the current robust control literature.

Singular Values

The singular values of an mxn matrix A, denoted $\sigma_i(A)$, are defined to be non-negative square-roots of the eigenvalues of A^HA, i.e., $\sigma_i(A) = \sqrt{\lambda_i(A^HA)}$ where A^H denotes the conjugate transpose of A. By convention, the singular values of A are ordered as follows:

$$\overline{\sigma} = \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r = \underline{\sigma}, \;\; \sigma_{r+1} = \ldots = \sigma_n \tag{A.4}$$

where r is the rank of matrix A, $\overline{\sigma}$ is the largest singular value and $\underline{\sigma}$ the smallest non-zero singular value. The largest singular value is defined to be the Hilbert or spectral norm of A, i.e.,

$$||A||_s = \overline{\sigma} \tag{A.5}$$

H_2 Norm

Let G(s) be a matrix whose entries are analytic functions of the complex variables in the open right-half s-plane. The H_2 norm of G(s) is denoted by $||\, G(s)\, ||_2$ and is defined to be

$$||\, G(s)\, ||_2 = \frac{1}{2\pi} \sqrt{\int_{-\infty}^{\infty} \mathrm{trace}\left[G^H(j\omega)\, G(j\omega)\right] d\omega} \tag{A.6}$$

$$= \frac{1}{2\pi} \sqrt{\int_{-\infty}^{\infty} \sum_{i=1}^{n} [\sigma_i\, G(j\omega)]^2\, d\omega}$$

H_∞ Norm

Let G(s) be defined as stated above. The H_∞ norm of G(s) is denoted by $||\, G(s)\, ||_\infty$ and is defined to be

$$||\, G(s)\, ||_\infty = \sup_\omega \bar{\sigma}(G(j\omega)),\ \omega \in \mathbf{R} \tag{A.7}$$

where sup(.) denotes the least upper bound operator. Therefore, the H_∞ norm is the largest singular value evaluated along $j\omega$ axis. For SISO systems, the H_∞ norm is the largest distance from the origin to the Nyquist plot.

Matrix measure has been used in stability analysis probably since the 1950's (see [Coppel 65], [Desoer 75]). It is not a norm, however it is induced from matrix norms and has some interesting relationships with norms.

The matrix measure of matrix $A \in C^{n\times n}$ is defined as follows:

$$\mu(A) := \lim_{h\to\infty^+} \frac{\|I + h\,A\| - 1}{h} \tag{A.8}$$

where $\|.\|$ is any matrix norm induced by a vector norm (see for example [Coppel 65], [Desoer 75], [Vidyasagar 78]). It can be explained as the directional derivative on I_n in the direction of A. The matrix measure has different values corresponding to the different induced matrix norms. For example:

$$\mu_1(A) := \max_j \left\{ \mathrm{Re}(a_{jj}) + \sum_{i=1, i\neq j}^{n} |a_{ij}| \right\}, \text{ when } \|A\|_1 = \max_j \left(\sum_{i=1}^{n} |a_{ij}| \right) \tag{A.9}$$

$$\mu_2(A) := \max_i \left\{ \lambda_i\left(\frac{A^* + A}{2} \right) \right\}, \text{ when } \|A\|_2 = \max_i \left(\sqrt{\lambda_i(A^*A)} \right) \tag{A.10}$$

$$\mu_\infty(A) := \max_i \left\{ \mathrm{Re}(a_{ii}) + \sum_{i=1, i\neq j}^{n} |a_{ij}| \right\}, \text{ when } \|A\|_\infty = \max_i \left(\sum_{j=1}^{n} |a_{ij}| \right) \tag{A.11}$$

From the definitions of matrix measure it is clear that $\mu_1(A) \le \|A\|_1$ and $\mu_\infty(A) \le \|A\|_\infty$. These equalities hold when matrix A is real and has nonnegative diagonal entries, whereas $\mu_2(A) \le \|A\|_2$ and the equality holds when A is symmetric and positive definite. The main properties of matrix measure are as follows:

(i) $\mu(I_n) = 1,\ \mu(-I_n) = -1$

(ii) $\mu(\alpha\, A) = \alpha\, \mu(A),\ \forall \alpha \geq 0$

(iii) $-\|A\| \leq -\mu(-A) \leq \mu(A) \leq \|A\|$

(iv) $\|Ax\| = -\|Ax\| \geq -\mu(A)\|A\|$, for all vector x with proper dimension

(v) $\mu(A) \geq \mathrm{Re}(\lambda_i)$ and $-\mu(-A) \leq \mathrm{Re}(\lambda_i),\ \forall i$

(vi) $\mu(A+B) \leq \mu(A) + \mu(B)$

(vii) $\mu(A+B) \geq \max\{\mu(A)-\mu(-B),\ -\mu(A)+\mu(B)\}$

(viii) $|\mu(A)-\mu(B)| \leq \max\{|\mu(A-B)|,\ |\mu(B-A)|\}$

Remarks:

(a) From property (i) one can see that a matrix measure is different from a norm, a matrix measure could be a negative real value.

(b) Matrix measure can also be zero, however $\mu(A) = 0$ doesn't mean $A = 0$. This is another difference between a norm and a matrix measure.

(c) Properties (iii) and (iv) show the relationships between a norm and a matrix measure.

(d) Property (iii) tells the relation with eigenvalues. In 2-norm case,

$\mu(A) = \max_i \lambda_i(A)$ and $-\mu(-A) = \min_i \lambda_i(A)$ where A is symmetric.

APPENDIX B

A REVIEW OF LYAPUNOV STABILITY ANALYSIS

Introduction

Stability of the dynamic system is the fundamental requirement in design of control systems. In general, issues of stability are concerned with the state trajectory. This occurs when the system is perturbed from the equilibrium point or a reference trajectory. There are a number of different definitions of stability, and the underlying concept which is common to each is described as follows: Employ some measure called the norm, which characterizes the state at any desired time; let the state whose stability is under investigation be perturbed, then define measures for perturbation as well as for the norm. From this concept, it follows that stability is defined as follows: If the perturbation does not exceed the defined measure, then the perturbed state is stable when the change in the norm caused by the perturbation does not exceed its established measure. From the engineering point of view, these analyses are important because of the state perturbations caused by the existence of such external disturbances as noise and environmental changes around the equilibrium points [Leipholz 87]. The specific definition of Lyapunov stability for an equilibrium point is given in the following manner.

Stability in the Sense of Lyapunov

If the solutions for the state equations are available, it is easy to determine stability for a particular case. However, solving the nonlinear differential equations is frequently a difficult or impossible task. The objective of Lyapunov stability theorems is to analyze system stability in the absence of knowledge of solutions to the system differential equations. In theory, an isolated (i.e., zero-input) system remains in the equilibrium state if that is where it initially started. In this sense, Lyapunov stability is concerned with the behavior of the system trajectories when the initial state is near the equilibrium point. As mentioned earlier, the results of this analysis are important because of the existence of external disturbances such as noise and environmental influences. Initially, Lyapunov stability theorems have been established for perturbations of initial condition near an equilibrium point. However, as explained in Chapter II on issues of robustness, these theorems can be extended and thus applied to the case of system parameter perturbations.

The underlying concept for the Lyapunov theorems is as follows: consider a system with no external forces acting upon it. If "0" denotes one of the system equilibrium points, it can be assumed that it is possible to define a function which represents the total energy of the system, such that it is equal to zero at the point of origin and positive elsewhere. And if the system dynamics are such that the energy of the system is nonincreasing over time, dependent upon the nature of the energy function, the stability of equilibrium point "0" is implied. The virtue of the Lyapunov theorem has been to employ this concept in a mathematical form.

Basic Definitions (Refer to [Hahn 63], [Barnett 70], [Vidyasagar 78] in detail)

Consider the vector differential equation

$$\dot{x} = f(x,t) \tag{B.1}$$

Then, assume that "0" denotes an equilibrium point of the system equation (B.1), which is done since the equilibrium point can always be transferred by a simple transformation of the states. As described by many authors, the basic definitions of stability for the equilibrium points are as follows:

Definition B.1

The equilibrium point 0 at time t_o is said to be stable if, for $\varepsilon > 0$, there exists a $\delta(t_o,\varepsilon) > 0$, such that $\|x(t_o)\| < \delta(t_o,\varepsilon) \Rightarrow \|x(t)\| < \varepsilon,\ \forall\ t \geq t_o$. It is said to be uniformly stable over $[t_o,\infty)$ if, for each $\varepsilon > 0$, there exists $\delta(t_o,\varepsilon) > 0$ such that

$\|x(t_1)\| < \delta(\varepsilon),\ t_1 \geq t_o \Rightarrow \|x(t)\| < \varepsilon,\ \forall\ t \geq t_1$. □

Definition B.2

The equilibrium point 0 at time t_o is unstable if it is not stable at t_o. □

Definition B.3

The equilibrium point 0 at time t_o is said to be asymptotically stable at t_o if first, it is stable at time t_o, and second, there exists a number $\delta_1(t_o) > 0$ such that

$\|x(t_o)\| < \delta_1(t_o) \Rightarrow \|x(t)\| \rightarrow 0$, as $t \rightarrow \infty$.

It is uniformly asymptotically stable over $[t_o,\infty)$ if first, it is uniformly stable over $[t_o,\infty)$, and second, there exists a number $\delta_1 > 0$ such that $\|x(t_o)\| < \delta_1,\ t_1 \geq t_o \Rightarrow \|x(t)\| \rightarrow 0$, as $t \rightarrow \infty$. □

Definition B.4

The equilibrium point 0 at time t_o is said to be globally asymptotically stable if it is asymptotically stable for all initial states (i.e., $x(t) \to 0$ as $t \to \infty$, regardless of $x(t_o)$); thus, if 0 is a globally asymptotically stable equilibrium point at time t_o for a given system, then it should be the only equilibrium point at time t_o. □

Lyapunov Stability Theorems (Refer to [La Salle 61], [Hahn 63], [Zubov 64], [Lehnigk 66], [Lyapunov 66] in detail proofs)

In order to investigate the stability of a system of differential equations without having to solve them, Lyapunov proposed some methods in his doctoral dissertation in 1892 [Lyapunov 66]. Although Lyapunov's theory was introduced at the end of the nineteenth century, it was not recognized for its vast applications until the 1960s. Since then it has become a major part in controls, system theory, and other fields. According to the sense of Lyapunov, the stability of dynamic systems can be determined in terms of certain scalar functions known as Lyapunov functions [Halanay 93]. The basic stability theorems for the Lyapunov direct method are as follows:

Let $\dot{x} = f(x,t)$, where $f(0,t) = 0 \ \forall \ t$, describe a given system equation. It follows:

Theorem B.1

The equilibrium point 0 *at time* t_o *is stable if there exists a continuously differentiable local positive definite function (l.p.d.f.)* $V(x,t)$ *such that*

$\dot{V}(x,t) \leq 0, \ \forall \ t \geq t_o, \ \forall \ x \in B_r$ *for some ball* B_r.

If V(x,t) is a decrescent locally positive definite function in Theorem B.1, the *equilibrium point 0 at time* t_o *is said to be uniformly stable over* $[t_o,\infty)$. $\square$

Theorem B.2

The equilibrium point 0 *at time* t_o *for the system is asymptotically stable over the interval* $[t_o,\infty)$ *if there exists a continuously differentiable l.p.d.f.* $V(x,t)$ *such that*

$\frac{dV(x,t)}{dt}$ *is a l.p.d.f.* $\square$

Theorem B.3

The equilibrium point 0 *at time* t_o *is globally asymptotically stable if there exists a continuously differentiable decrescent p.d.f.* $V(x,t)$ *such that* $\dot{V}(x,t) \le -G(|x|)\ \forall\ t \ge t_o$, $\forall\ x \in \mathbf{R}^n$, *where* G *is a function belonging to class K.* $\square$

Theorem B.4

The equilibrium point 0 *at time* t_o *is unstable if there exists a continuously differentiable decrescent function* $V(x,t)$ *such that first,* $\frac{dV(x,t)}{dt}$ *is a l.p.d.f., and second,* $V(0,t) = 0$, *and there exists points* x *arbitrary close to* 0 *such that* $V(x,t_o > 0)$. $\square$

Clearly, the advantage of the Lyapunov stability theorems is that they do not require a solution of the state equations; in contrast, they are disadvantaged in that only sufficient conditions are provided. If a particular function fails to satisfy all of the conditions, then no conclusions can be drawn and another function candidate should be attempted. For this reason, a function is referred to as a Lyapunov candidate when subject to the testing under the conditions described above. If all the conditions for one of the theorems can be satisfied, then it can be termed a Lyapunov function. However, it is difficult to find a Lyapunov function for a given system. The choice of a Lyapunov function is relatively easy for the case of linear or weakly nonlinear systems.

Lyapunov Method for Linear Autonomous Systems

Basic Theorems

Consider the following linear autonomous system

$$\dot{x} = A\,x \tag{B.2}$$

Theorem B.5 ([Hahn 63] or [Willems 70])

For the system on Equation (B.2) the origin is asymptotically stable if and only if all the characteristic roots of matrix A *have negative real parts.* □

The second method of Lyapunov takes a particularly simple form when Lyapunov function is selected in the quadratic form

$$V(x,t) = x^T P\,x \tag{B.3}$$

where P is a positive definite symmetric (Hermitian) matrix. The derivative of V(x,t) along the solution of the system Equation (B.2) is obtained by

$$\frac{dV(x,t)}{dt} = \dot{x}^T P\,x + x^T P\,\dot{x} = (A\,x)^T P\,x + x^T P\,A\,x = x^T\left(A^T P + P\,A\right)x \tag{B.4}$$

Consider the equation which is referred to as the Lyapunov matrix equation:

$$A^T P + P\,A = -\,Q \tag{B.5}$$

Then the following theorems are fundamental in the study of linear autonomous systems:

Theorem B.6 ([Hahn 63], [Barnett 70])

The system (B.2) is asymptotically stable if and only if there exists a symmetric positive definite matrix P *which is the unique solution of the Lyapunov matrix equation*

(B.5) for any given symmetric positive definite matrix Q. ☐

Theorem B.7 ([Hahn 63], [Barnett 70])

The null solution of (B.2) is unstable if at least one of the characteristic roots of matrix A *has positive real part.* ☐

Theorem B.8 [Khalil 92]

An equilibrium point of a time invariant dynamical system is stable if there exists a continuously differentiable scalar function $V(x)$ *such that along the system trajectories* $V(x) > 0$, $V(0) = 0$, *and* $\dot{V}(x) \leq 0$ *of Equation (B.4) are satisfied. And if* $\dot{V}(x) < 0$ *is satisfied then the system is asymptotically stable.* ☐

The Solution of Lyapunov Equation

Due to broad applications, the solution of Lyapunov matrix equation has been subject of very active research for the past thirty years (see [MacFarlane 63], [Barnett 66a], [Barnett 66b], [Bingulac 70], [Chen 84], [Lancaster 85], [Mori 86a], [Mori 86b], [Mori 87]). Especially, in the 1970s growing use of digital computers, which resulted in celebrated algorithms for a numerical solution of the continuous-time algebraic Lyapunov equation (see [Davison 68], [Bartels 72], [Golub 79], [Hammarling 82], [Subrahmanyam 86] for examples).

For a time invariant linear system, the condition for existence of an unique solution of a Lyapunov matrix equation (B.5) is given by following theorem.

Theorem B.9 ([Chen 71], [Lancaster 85])

Equation (B.5) provides a unique solution for P *corresponding to every* $Q \in \mathbf{R}^{n\times n}$ *if and only if* $\lambda_i + \lambda_j \neq 0$, $\forall$ i, j, *where* $\lambda_1,\ldots,\lambda_n$ *are the eigenvalues of* A. ☐

Lyapunov Function Generation

Of the different techniques for the generation of a Lyapunov function, the most important factor is to determine a function which provides the least conservative results. In the case of stability analysis, conservatism of results is referred to as the estimated size of regions of stability for state perturbations around equilibrium points or reference trajectories. However, for the analysis of robust stability, the conservatism refers to the estimated size of the robustness bounds. To determine less conservative estimates, the nominal part of the system as well as the structure of the perturbation elements must be considered when generating the Lyapunov functions.

Apart from these two cases, there are a quadratic Lyapunov function for linear systems quoted precedingly and the so-called "Lure problem" (see details in [Lefschetz 65] and [Aizerman 64]). There is no certain method of finding a Lyapunov function for a general nonlinear problem. A number of suggestions have been made for such construction in the general case: refer to [Krasovskii 57], [Ingwerson 61], [Zubov 62], [Schultz 62] for a variety of techniques for the generation of a Lyapunov function. For comprehensive study of the generation of a Lyapunov function, refer to Mohler [Mohler 89] and Schultz[Schultz 65].

APPENDIX C

GAO'S LEMMAS AND THEOREM

Since a new method of stability robustness is developed in this research by using the Gao's lemmas and a theorem [Gao 93], the results of Gao's work are briefly summarized as follows:

Consider the linear time–invariant system represented by the state space model with perturbation E as shown below:

$$\dot{x} = (A_N + E)\,x \tag{C.1}$$

where A_N is nxn real Hurwitz matrix. Assume that the parameter perturbation matrix, E, takes the form

$$E = \sum_{i=1}^{m} k_i E_i \tag{C.2}$$

where E_i are real constant matrices and k_i are real uncertain parameters.

Lemma C.1

Let $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$, and $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$, *be eigenvalues of the Hermitian matrices* A, B *and* C = A + B, *then*

$$\alpha_i + \beta_1 \leq \gamma_i \leq \alpha_i + \beta_n \qquad i = 1,\ldots,n \tag{C.3}$$

□

Lemma C.2

For any Hermitian matrices P_i, $i = 1,\ldots,m$,

$$\lambda\left(\sum_{i=1}^{m} k_i P_i\right) \leq \sum_{i=1}^{m} \lambda_{max}(k_i P_i) \tag{C.4}$$

where λ_i *are defined by*

$$\lambda_i = \begin{cases} \lambda_{max}(P_i) & \text{for } k_i \geq 0 \\ \lambda_{min}(P_i) & \text{for } k_i < 0 \end{cases} \qquad i = 1,\ldots,m \tag{C.5}$$

in which $\lambda(.)$ *denote all possible eigenvalues of the matrix* $(.)$, *and* $\lambda_{max}(.)$, $\lambda_{min}(.)$ *are the largest and smallest eigenvalues, respectively.*

<u>Proof</u>

This lemma can be proved by using Lemma C.1 and mathematical induction. For m = 2, Equation (C.4) reduces Equation (C.3) where $A = k_1P_1$, $B = k_2P_2$. Assume that Equation (C.4) is valid for m = k, that is

$$\lambda\left(\sum_{i=1}^{k} k_i P_i\right) \leq \sum_{i=1}^{k} \lambda_{max}(k_i P_i) \tag{C.6}$$

then one only needs to prove that it is also valid for m = k+1. Note that

$$\lambda\left(\sum_{i=1}^{k+1} k_i P_i\right) = \lambda\left(\sum_{i=1}^{k} k_i P_i + k_{k+1} P_{k+1}\right) \tag{C.7}$$

Let $A = \sum_{i=1}^{k} k_i P_i$, $B = k_{k+1} P_{k+1}$, then $\lambda\left(\sum_{i=1}^{k+1} k_i P_i\right) = \lambda(A + B)$. By Lemma C.1 and Equation (C.6), one has

$$\lambda\left(\sum_{i=1}^{k+1} k_i P_i\right) = \lambda(A + B)$$

$$\leq \lambda_{max}(A) + \lambda_{max}(B)$$

$$= \sum_{i=1}^{k} \lambda_{max}(k_i P_i) + \lambda_{max}(k_{k+1} P_{k+1})$$

$$= \sum_{i=1}^{k+1} \lambda_{max}(k_i P_i)$$

$\square$

Lemma C.3

$$\lambda_{max}(k_i P_i) = \begin{cases} k_i \lambda_{max}(P_i) & \text{for } k_i \geq 0 \\ k_i \lambda_{min}(P_i) & \text{for } k_i < 0 \end{cases} \qquad i = 1,\ldots,m \tag{C.8}$$

<u>Proof</u>

If $\lambda(P_i) = \{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in}\}$, then $\lambda(k_i P_i) = \{k_i\lambda_{i1}, k_i\lambda_{i2}, \ldots, k_i\lambda_{in}\}$, and $\lambda_{max}(k_i P_i) = \max\{k_i\lambda_{i1}, k_i\lambda_{i2}, \ldots, k_i\lambda_{in}\}$. Here one only proves the case where n = 2. For n > 2, it can be proved in the same way. Since for any k, $\lambda_1, \lambda_2 \in \mathbf{R}$, clearly

$$\max\{k\lambda_1, k\lambda_2\} = \begin{cases} k \max\{\lambda_1, \lambda_2\} & \text{for } k \geq 0 \\ k \min\{\lambda_1, \lambda_2\} & \text{for } k < 0 \end{cases} \tag{C.9}$$

This is illustrated as follows: for λ_1, λ_2 with the same sign on Equation (C.9) is obvious; otherwise one will have $\max\{\lambda_1, \lambda_2\} \geq 0$ and $\min\{\lambda_1, \lambda_2\} < 0$, therefore, for $k \geq 0$, $\max\{k\lambda_1, k\lambda_2\} = k \max\{\lambda_1, \lambda_2\}$ and for $k < 0$, $\max\{k\lambda_1, k\lambda_2\} = k \min\{\lambda_1, \lambda_2\}$.

□

Theorem C.1

The system on Equation (C.1) is asymptotically stable if

$$\sum_{i=1}^{m} k_i \lambda_i < 1 \tag{C.10}$$

where λ_i are defined on Equation (C.5).

Proof

Since it is assumed that the matrix A_N on Equation (C.1) is Hurwitz, there exists a symmetric positive definite matrix P which is the unique solution of the Lyapunov equation (see Theorem B.6 in Appendix B for explanation) represented by

$$P\,A_N + A_N^T\,P + 2\,I = 0 \tag{C.11}$$

Let the candidate Lyapunov function $V(x) = x^T P x$. And define

$$P_i = \frac{E_i^T P + P E_i}{2}, \qquad i = 1, 2, \ldots, m \tag{C.12}$$

where E_i are real constant matrices using on Equation (C.4). Note that P_i are real and symmetric, and therefore they are Hermitian matrices.

Then

$$\frac{dV}{dt} = \dot{x}^T P x + x^T P \dot{x}$$

$$= [(A_N + E)\,x]^T\, P\, x + x^T\, P\,[(A_N + E)\,x]$$

$$= x^T \left(A_N^T\, P + P\, A_N + E^T\, P + P\, E\right) x$$

$$= x^T \left(E^T\, P + P\, E - 2\, I\right) x$$

$$= x^T \left(\sum_{i=1}^{m} k_i E_i^T P + \sum_{i=1}^{m} k_i P E_i - 2\, I \right) x$$

$$= x^T \left(\sum_{i=1}^{m} k_i \left(E_i^T P + P E_i\right) - 2\, I \right) x$$

$$= 2\, x^T \left(\sum_{i=1}^{m} k_i P_i - I \right) x$$

Define the matrix M such that

$$M = \sum_{i=1}^{m} k_i P_i - I \tag{C.13}$$

Note that M is an nxn Hermitian matrix. For the system on Equation (C.1) to be asymptotically stable, one needs $\frac{dV}{dt} < 0$, or equivalently, one needs matrix M to be negative definite. Since a Hermitian matrix is negative definite if and only if all its eigenvalues are negative, the following condition is necessary:

$$\lambda\left(\sum_{i=1}^{m} k_i P_i \right) < 1 \tag{C.14}$$

From Lemma C.1 to Lemma C.3 it is shown that

$$\lambda\left(\sum_{i=1}^{m} k_i P_i\right) \leq \sum_{i=1}^{m} \lambda_{max}(k_i P_i) \leq \sum_{i=1}^{m} k_i \lambda_i \tag{C.15}$$

Hence the system on Equation (C.1) is asymptotically stable if the condition (C.10) is satisfied. $\square$

APPENDIX D

ON THE DESIGN OF AN ELECTROHYDRAULIC SERVOVALVE WITH A VARIANT DRAIN ORIFICE DAMPING

Introduction

Since Moog [Moog 53] developed the first two-stage electrohydraulic servovalve with a flapper-nozzle valve in 1950, many researches have identified the important factors affecting steady-state behavior and dynamic response ([Zaborszky 58], [Feng 59], [Merritt 67], [Nikiforuk 69], [Pennington 74], [Martin 76], [Arafa 87a], [Arafa 87b], [Liaw 90]). The flapper-nozzle valve [Maskrey 78] has been extensively used as a first-stage valve, because it is comparatively simple to construct and relatively reliable to operate. As a first-stage, the flapper-nozzle valve appreciably reduces the valve threshold and provides a high dynamic response because of its lower mass. Most recently, the effects of a damper attached to the outlet of the flapper-nozzle valve have been studied ([Watton 87], [Lin 89], [Akers 90]). Watton [Watton 87] placed a drain orifice in the flapper-nozzle return line (see Figure D.1), thus creating a small back pressure which, in turn, improved servovalve performance. He showed that the drain orifice damper reduces the power loss and may eliminate the high frequency valve whistle associated with servovalve instability. By directly attaching a squeeze film damper to the flapper, Lin [Lin 89], and Akers [Akers 90] caused the damping force to increase exponentially as the flapper distance increased so that the first stage was a stable-alone dynamic system. However, as Watton [Watton 87] noted, the system gain of the servovalve at null condition of the flapper-nozzle is reduced as the drain orifice is attached. Also, the drain orifice deteriorates the uniform linearity of the gain function over the flapper displacement range. Even though the squeeze film damper provides the flapper-nozzle valve-alone stability, it destabilizes the servovalve when the feedback flow force increases, causing the high frequency valve whistle [Watton 87].

A new damper design is proposed to overcome the shortcomings of Watton's design for a fixed drain orifice. Basically, the damper on Figure D.2 has the similar configuration to the drain orifice on Figure D.1. However, contrary to the drain orifice on Figure D.1, the flapper divides the drain chamber into two parts so that the return pressures in each side of the drain chamber depend on flapper movement. As the flapper moves away from the null position, the returning pressure difference on the flapper increases and, in turn, the damping force increases. It is expected that the newly designed damper on Figure D.2 will improve both performance and stability, i.e., increase null pressure sensitivity and enhance servovalve stability.

Steady-State Characteristics of the Flapper-Nozzle Valve

Consider the flapper-nozzle stage as shown on Figure D.2. If we assume a blocked-load so that $Q_s = 0$ and no leakage,

$$Q_{o1} = Q_{n1} = Q_{e1},\ Q_{o2} = Q_{n2} = Q_{e2} \tag{D.1}$$

where

$$Q_{o1} = C_{do}\, A_o \sqrt{\frac{2}{\rho}(P_s - P_1)}$$

$$Q_{o2} = C_{do}\, A_o \sqrt{\frac{2}{\rho}(P_s - P_2)}$$

$$Q_{n1} = C_{df}\, \pi\, d_n\, (x_{fo} - x_f) \sqrt{\frac{2}{\rho}(P_1 - P_{e1})}$$

$$Q_{n2} = C_{df}\, \pi\, d_n\, (x_{fo} + x_f) \sqrt{\frac{2}{\rho}(P_2 - P_{e2})}$$

$$Q_{e1} = C_{dd}\, w_d\, (x_{fo} - x_f) \sqrt{\frac{2}{\rho}P_{e1}}$$

$$Q_{e2} = C_{dd}\, w_d\, (x_{fo} + x_f) \sqrt{\frac{2}{\rho}P_{e2}}$$

where C_{df}, C_{do}, C_{dd} = the unitless discharge coefficients for curtain, nozzle, and drain orifice, and where

A_o = orifice area $\left(A_o = \frac{\pi\, d_o^2}{4}\right)$, in^2

d_n, d_o = diameters of nozzle area and orifice area, in

w_d = area gradient of drain orifice, in

x_{fo}, x_f = equilibrium flapper position and flapper displacement, in

Equation (D.1) is simplified by

$$\begin{aligned} k_o \sqrt{1 - \overline{P}_1} &= k_n\,(1 - \overline{x}_f) \sqrt{\overline{P}_1 - \overline{P}_{e1}} \\ k_o \sqrt{1 - \overline{P}_2} &= k_n\,(1 + \overline{x}_f) \sqrt{\overline{P}_2 - \overline{P}_{e2}} \end{aligned} \tag{D.2}$$

and

$$k_n(1-\bar{x}_f)\sqrt{\bar{P}_1-\bar{P}_{e1}} = \gamma\, k_n(1-\bar{x}_f)\sqrt{\bar{P}_{e1}}$$
$$k_n(1+\bar{x}_f)\sqrt{\bar{P}_2-\bar{P}_{e2}} = \gamma\, k_n(1+\bar{x}_f)\sqrt{\bar{P}_{e2}} \tag{D.3}$$

where

$$k_o = C_{do}\, A_o \sqrt{\frac{2P_s}{\rho}},\ k_n = C_{df}\, \pi\, d_n\, x_{fo} \sqrt{\frac{2P_s}{\rho}}$$

$$\bar{P}_1 = \frac{P_1}{P_s},\ \bar{P}_2 = \frac{P_2}{P_s},\ \bar{P}_{e1} = \frac{P_{e1}}{P_s},\ \bar{P}_{e2} = \frac{P_{e2}}{P_s}$$

$$\bar{x}_f = \frac{x_f}{x_{fo}},\ \gamma = \frac{C_{dd}\, w_d}{C_{df}\, \pi\, d_n}$$

Then, Equation (D.2) yields

$$\bar{P}_1 = \frac{1+\bar{P}_{e1}\, Z(1-\bar{x}_f)^2}{1+Z(1-\bar{x}_f)^2}$$
$$\bar{P}_2 = \frac{1+\bar{P}_{e2}\, Z(1+\bar{x}_f)^2}{1+Z(1+\bar{x}_f)^2} \tag{D.4}$$

where

$$Z = \left(\frac{k_n}{k_o}\right)^2 = 16\left(\frac{C_{df}\, d_n\, x_{fo}}{C_{do}\, d_o^2}\right)^2$$

Equation (D.3) yields

$$\bar{P}_{e1} = \frac{\bar{P}_1}{1+\gamma^2},\qquad \bar{P}_{e2} = \frac{\bar{P}_2}{1+\gamma^2} \tag{D.5}$$

Then, equations (D.4) and (D.5) can be combined to determine the steady-state characteristics of the flapper-nozzle valve, i.e.,

$$\overline{P}_1 = \frac{1}{1 + Z\,(1 - \overline{x}_f)^2 - \dfrac{Z\,(1 - \overline{x}_f)^2}{1 + \gamma^2}}$$

$$\overline{P}_2 = \frac{1}{1 + Z\,(1 + \overline{x}_f)^2 - \dfrac{Z\,(1 + \overline{x}_f)^2}{1 + \gamma^2}} \tag{D.6}$$

The nondimensional flow loss, $\overline{Q}_e$, and power loss, $\overline{W}_e$, are written as follows:

$$\overline{Q}_e = \frac{Q_{e1} + Q_{e2}}{k_n}$$

$$= (1 - \overline{x}_f)\sqrt{\overline{P}_1 - \overline{P}_{e1}} + (1 + \overline{x}_f)\sqrt{\overline{P}_2 - \overline{P}_{e2}} \tag{D.7}$$

$$= \frac{\gamma}{\sqrt{1 + \gamma^2}}\left[(1 - \overline{x}_f)\sqrt{\overline{P}_1} + (1 + \overline{x}_f)\sqrt{\overline{P}_2}\right]$$

$$\overline{W}_e = \frac{W_e}{P_s\,k_n} = \overline{Q}_e \tag{D.8}$$

The nondimensional characteristics, $\overline{P}_1 - \overline{P}_2$, $\overline{P}_{e1} - \overline{P}_{e2}$, and $\overline{Q}_e = \overline{W}_e$ are shown on the figures D.3 through D.5 for values of the parameter Z = 1, 2, 3, 4. No drain orifice is represented by the configuration as $\gamma \to \infty$.

At the null condition, i.e., $\overline{x}_f = 0$, equations (D.5) through (D.8) yield

$$\overline{P}_1 = \overline{P}_2 = \frac{1+\gamma^2}{(1+Z)\left(1+\gamma^2\right) - Z} \tag{D.9}$$

$$\overline{P}_{e1} = \overline{P}_{e2} = \frac{1}{(1+Z)\left(1+\gamma^2\right) - Z} \tag{D.10}$$

$$\overline{W}_e = \overline{Q}_e = \frac{2\gamma}{\sqrt{(1+Z)\left(1+\gamma^2\right) - Z}} \tag{D.11}$$

For servovalve with force feedback, it is common to have a value of $Z = 1$ so that, with no drain orifice, the null pressures on either side of the nozzle are at half supply pressure. Servovalves with direct feedback tend to have higher values of Z, typically up to 4 [Watton 87]. Figure D.3 shows that the flapper-nozzle stage without a drain orifice has a maximum null gain (i.e., a maximum null pressure sensitivity) when $Z = 1$.

Consequently, the figures D.3 through D.5 reveal the following steady-state characteristics of the flapper-nozzle valve with drain orifice. First, when γ is finite and $Z \geq 2$, the gain around null increases as γ decreases. Second, the pressure differential versus the flapper displacement, i.e., $(P_1 - P_2) / x_f$, generally becomes more linear as γ decreases. Third, a decrease in γ or an increase in Z is accompanied by an increase in back pressure differential, $|P_{e1} - P_{e2}|$, but with a decrease in flow and power loss. Figure D.6 illustrates the preferred null range of operation where the back pressure is selected to be less than 10 percent of the supply pressure.

Linearized Transfer Function of the Flapper-Nozzle Connected to a Spool Valve

The load flow into the spool valve is represented by

$$Q_s = Q_{o1} - Q_{n1} = Q_{n2} - Q_{o2} \tag{D.12}$$

Equation (D.12) is

$$\begin{aligned} Q_s &= k_o\sqrt{1-\overline{P}_1} - k_n(1-\overline{x}_f)\sqrt{\overline{P}_1-\overline{P}_{e1}} \\ Q_s &= k_n(1+\overline{x}_f)\sqrt{\overline{P}_2-\overline{P}_{e2}} - k_o\sqrt{1-\overline{P}_2} \end{aligned} \tag{D.13}$$

Considering the small-signal dynamic response, then the steady-state flow characteristics of Equation (D.13) may be linearized about an operating point $\overline{x}_{fo}$, $\overline{P}_{1o}$, $\overline{P}_{2o}$, $\overline{P}_{e1o}$, $\overline{P}_{e2o}$ as in equations (D.14) and (D.15)

$$\begin{aligned} \Delta Q_s = {} & k_n\sqrt{\overline{P}_{1o}-\overline{P}_{e1o}}\,\Delta\overline{x}_f - \frac{k_o}{2\sqrt{1-\overline{P}_{1o}}}\Delta\overline{P}_1 \\ & - \frac{k_n(1-\overline{x}_{fo})}{2\sqrt{\overline{P}_{1o}-\overline{P}_{e1o}}}\Delta\overline{P}_1 + \frac{k_n(1-\overline{x}_{fo})}{2\sqrt{\overline{P}_{1o}-\overline{P}_{e1o}}}\Delta\overline{P}_{e1} \end{aligned} \tag{D.14}$$

$$\begin{aligned} \Delta Q_s = {} & k_n\sqrt{\overline{P}_{2o}-\overline{P}_{e2o}}\,\Delta\overline{x}_f + \frac{k_o}{2\sqrt{1-\overline{P}_{2o}}}\Delta\overline{P}_2 \\ & + \frac{k_n(1+\overline{x}_{fo})}{2\sqrt{\overline{P}_{2o}-\overline{P}_{e2o}}}\Delta\overline{P}_2 - \frac{k_n(1+\overline{x}_{fo})}{2\sqrt{\overline{P}_{2o}-\overline{P}_{e2o}}}\Delta\overline{P}_{e2} \end{aligned}$$

Combining equations (D.5) and (D.14) yields

$$\Delta Q_s = k_{x1}\,\Delta\bar{x}_f + (k_{a1} - k_{a2} - k_{a3})\,\Delta\bar{P}_1$$
$$\Delta Q_s = k_{x2}\,\Delta\bar{x}_f + (k_{b2} - k_{b1} + k_{b3})\,\Delta\bar{P}_2 \tag{D.15}$$

where

$$k_{x1} = k_n\sqrt{\bar{P}_{1o} - \bar{P}_{e1o}},\quad k_{x2} = k_n\sqrt{\bar{P}_{2o} - \bar{P}_{e2o}}$$

$$k_{a1} = -\frac{k_o}{\sqrt{1 - \bar{P}_{1o}}},\quad k_{b1} = -\frac{k_o}{\sqrt{1 - \bar{P}_{2o}}}$$

$$k_{a2} = \frac{k_n\,(1 - \bar{x}_{fo})}{\sqrt{\bar{P}_{1o} - \bar{P}_{e1o}}},\quad k_{b2} = \frac{k_n\,(1 + \bar{x}_{fo})}{\sqrt{\bar{P}_{2o} - \bar{P}_{e2o}}}$$

$$k_{a3} = -\frac{k_{a2}}{1 + \gamma^2},\quad k_{b3} = -\frac{k_{b2}}{1 + \gamma^2}$$

It is frequently assumed that the dynamic flow contribution is dominated by the spool valve velocity component and that the oil compressibility and leakage effects are negligible. Thus, the flow equation of the spool is simply represented by

$$Q_s = a_p\,\frac{dx_p}{dt} \tag{D.16}$$

where a_p = cross sectional area of the spool valve, in^2

x_p = spool displacement from null position, in.

Spool inertia is also assumed to dominate the dynamics of momentum equation and is combined with a resisting spring force which exists for a servovalve with direct feedback. Thus, the dynamic equation for spool movement is

$$a_p \left(\Delta P_1 - \Delta P_2\right) = \left(m\, s^2 + k\right) x_p \tag{D.17}$$

where m = spool mass, slug

k = restraining spring stiffness, lb/in.

Combining equations (D.15) and (D.16) yields

$$\Delta\overline{P}_1 = \frac{x_{pm}\, a_p}{k_{a1} - k_{a2} - k_{a3}}\, s\, \Delta\overline{x}_p - \frac{k_{x1}}{k_{a1} - k_{a2} - k_{a3}}\, \Delta\overline{x}_f$$

$$\Delta\overline{P}_2 = \frac{x_{pm}\, a_p}{k_{b2} - k_{b1} + k_{b3}}\, s\, \Delta\overline{x}_p - \frac{k_{x2}}{k_{b2} - k_{b1} + k_{b3}}\, \Delta\overline{x}_f \tag{D.18}$$

where x_{pm} = maximum displacement of the spool, in

$\Delta\overline{x}_p = \dfrac{\Delta x_p}{x_{pm}}$, unitless.

Hence, one has

$$\Delta\overline{P}_1 - \Delta\overline{P}_2 = -\, x_{pm}\, a_p\, B\, s\, \Delta\overline{x}_p + A\, \Delta\overline{x}_f \tag{D.19}$$

where

$$A = \frac{k_{x2}}{k_{b2} - k_{b1} + k_{b3}} - \frac{k_{x1}}{k_{a1} - k_{a2} - k_{a3}} \tag{D.20}$$

$$B = \frac{1}{k_{b2} - k_{b1} + k_{b3}} - \frac{1}{k_{a1} - k_{a2} - k_{a3}} \tag{D.21}$$

Combining equations (D.17) and (D.19) yields the transfer function relating nondimensional spool movement to nondimensional flapper movement as follows:

$$\frac{\Delta \bar{x}_p}{\Delta \bar{x}_f} = \frac{A \dfrac{a_p P_s}{x_{pm}}}{m s^2 + x_{pm} \dfrac{a_p^2 P_s}{x_{pm}} B s + k}$$

$$= \frac{A C}{\dfrac{m}{\omega_n^2} s^2 + \dfrac{2 \alpha}{\omega_n} k_b B s + 1} \qquad \text{(D.22)}$$

where

$$\omega_n = \sqrt{\frac{k}{m}}, \ \alpha = \frac{a_p^2 P_s}{2 \sqrt{m k}\, k_n}, \ C = \frac{a_p P_s}{k\, x_{pm}}$$

It is now appropriate to define the gain function, f_g, and the damping function, f_d, such that

$$f_g := A \qquad \text{(D.23)}$$

$$f_d := k_n B \qquad \text{(D.24)}$$

Consequently, it follows from equations (D.22) and (D.24) that the damping ratio of the second order transfer function is given by

$$\zeta := \alpha f_d \qquad \text{(D.25)}$$

Before pursuing the generalized transfer function of Equation (D.22) and its variation with operating conditions, it is worthwhile considering its nature at the null condition. At the null condition of a flapper-nozzle valve, i.e., $x_f = 0$, equations (D.5) and

(D.6) yield

$$\overline{P}_1 = \overline{P}_2 = \frac{1 + \gamma^2}{Y}$$
$$\overline{P}_{e1} = \overline{P}_{e2} = \frac{1}{Y} \tag{D.26}$$

where $Y = 1 + (1 + Z)\,\gamma^2$.

Hence, one has

$$k_{x1} = k_{x2} = k_n \frac{\gamma}{\sqrt{Y}}$$
$$k_{a1} = k_{b1} = -\,k_o \frac{\sqrt{Y}}{2\sqrt{Z}\,\gamma}$$
$$k_{a2} = k_{b2} = k_n \frac{\sqrt{Y}}{2\,\gamma} \tag{D.27}$$
$$k_{a3} = k_{b3} = -\,k_n \frac{\sqrt{Y}}{2\,\gamma\left(1 + \gamma^2\right)}$$

Then, using the equations (D.21), (D.24), and (D.27), the null damping function is

$$f_d = \frac{4\,Z\,\gamma\left(1 + \gamma^2\right)}{Y^{3/2}} \tag{D.28}$$

And, for a servovalve with no drain orifice, i.e., as $\gamma \rightarrow \infty$

$$(f_d)_\infty = \frac{4\,Z}{(1 + Z)^{3/2}} \tag{D.29}$$

Therefore the damping constant ratio at null can be written as

$$\frac{\zeta \text{ with drain orifice}}{\zeta \text{ without drain orifice}} = \frac{\gamma\left(1+\gamma^2\right)(1+Z)^{3/2}}{\left[1+(1+Z)\,\gamma^2\right]^{3/2}} \tag{D.30}$$

$$= \left(\frac{1+\gamma^2}{\gamma^2}\right)\left(1-\overline{P}_{e1,2}\right)^{3/2}$$

Also using the equations (D.20), (D.23), and (D.27), null gain function is represented by

$$f_g = \frac{4\,Z\,\gamma^2\left(1+\gamma^2\right)}{Y^2} \tag{D.31}$$

And, as $\gamma \rightarrow \infty$

$$\left(f_g\right)_\infty = \frac{4\,Z}{(1+Z)^2} \tag{D.32}$$

Therefore the gain ratio at null can be written as

$$\frac{\text{gain with drain orifice}}{\text{gain without drain orifice}} = \frac{\gamma^2\left(1+\gamma^2\right)(1+Z)^2}{\left[1+(1+Z)\,\gamma^2\right]^2} \tag{D.33}$$

$$= \left(\frac{1+\gamma^2}{\gamma^2}\right)\left(1-\overline{P}_{e1,2}\right)^2$$

Figures D.7 and D.8 illustrate the nondimensional plots of the damping constant ratio of Equation (D.30) and the gain ratio of Equation (D.33), respectively. These figures reveal that both the damping constant and the gain at null always increase as Z increases for any value of γ and that, in general, the damping constant and the gain at null reach peak values in the range of $0 \langle \gamma \langle 2$. Consequently, figures D.7 and D.8 show the typical range of parameters for practical implementation of the drain orifice. For all cases of $\gamma \geq 2$ and Z ≥ 2, both the damping constant ratio and the gain ratio at null increase as the area gradient of the drain orifice decreases.

Figures D.9 and D.10 show that the damping function, f_d, and the gain function, f_g, for a particular parameter, Z, may increase and reach the maximum at operating points away from null. Highly resistive drain orifices will also tend to produce flatter characteristics away from null and would certainly prevent the low gain and damping that occur at extreme flapper movements. With a drain orifice, an increase in the gain function occurs when $Z \geq 2$, and in all cases, an increase in the damping function always occurs. Generally the damping function and the gain function become more linear over the range of flapper displacement as the drain orifice is more resistive, that is γ decreases. It would appear that a high value of Z is required with γ values less than 2.

Dynamics of Torque Motor/Flapper-Nozzle First-Stage Assembly

On the first stage of an electrohydraulic servovalve, the torque which is produced by the electromotive force is proportional to the armature current. It is opposed by retarding torques due to the flapper restraining spring force, the static and dynamic fluid forces, the viscous friction force and the flapper acceleration force ([Merritt 67], [Nikiforuk 69]). This is expressed by

$$K_t \, \Delta i = \left(J_a \, s^2 + B_a \, s + K_{at}\right) \Delta\theta + \Delta F_f \, r \tag{D.34}$$

where i = electrical current differential, amp

K_t = torque constant of the torque motor, in-lb/amp

J_a = inertia of armature and any attached load, in-lb-sec^2

B_a = viscous damping coefficient of mechanical armature mounting and load, in-lb-sec

K_{at} = total spring constant of torque motor and armature, $K_{at} = K_a - K_m$, in-lb/rad

K_a = mechanical torsion spring constant of armature pivot, in-lb/rad

K_m = magnetic spring constant of torque motor, in-lb/rad

F_f = static and dynamic fluid flow forces on the flapper, in-lb

r = equivalent length of flapper, in.

Nikiforuk, et. al [Nikiforuk 69] showed that the fluid force on the flapper with no drain orifice is dominated by the static fluid force

$$F_f = a_n (P_1 - P_2) \tag{D.35}$$

Hence, the fluid force on the flapper with a drain orifice is similarly given by

$$\begin{aligned} F_f &= a_n (P_1 - P_2) + a_d (P_{e1} - P_{e2}) \\ &= a_n \left(1 + \frac{n}{1+\gamma^2}\right)(P_1 - P_2) \end{aligned} \tag{D.36}$$

where a_d = equivalent drain pressure sensing area of the flapper except nozzle area, $a_d = w_d\, h - a_n$, in^2

h = height of the drain chamber, in

n = ratio of $\frac{a_d}{a_n}$

For the blocked load condition, the dynamics of the first-stage, torque motor/flapper-nozzle valve assembly can be represented using equations (D.19), (D.34) and (D.36). The block diagram on Figure D.11 shows the dynamics of the first-stage assembly. The loop transfer function is

$$G(s)H(s) = \frac{K}{\dfrac{s^2}{\omega_n^2} + \dfrac{2\,\zeta\, s}{\omega_n} + 1} \tag{D.37}$$

where θ_m = maximum angular displacement of flapper, rad

$$K = \frac{P_s\, a_n\, r}{K_{at}\, \theta_m} A \left(1 + \frac{n}{\gamma^2}\right)$$

$$\omega_n^2 = K_{at} \,/\, J_a$$

$$\zeta = B_a \,/\, 2\sqrt{K_{at}\, J_a}$$

The transfer function of the first stage assembly is

$$\frac{\Delta\left(\bar{P}_1 - \bar{P}_2\right)}{\Delta\bar{i}} = \frac{K_t \dfrac{i_m}{\theta_m} A}{J_a\, s^2 + B_a\, s + K_{at} + \dfrac{r\, A\, P_s\, a_n}{\theta_m}\left(1 + \dfrac{n}{1 + \gamma^2}\right)} \tag{D.38}$$

where i_m = maximum current differential, amp

$$\bar{i} = \frac{i}{i_m}$$

The dynamics of the first-stage assembly are always stable since the contours of the polar plots of the transfer function G(s)H(s) do not encircle the point $-1 + j0$. However, an increase of the gain function K makes the contour closely approach to the point $-1 + j0$. Hence, in terms of stability, the parameter γ must be large and the area ratio n must be small. A highly resistive drain orifice causes the first-stage system to have a low damping constant and a high natural frequency. With a high null gain for the small value of γ, the practical value of γ must be determined.

Interactions Between the Torque Motor and the Hydraulic Stage

By combining equations (D.19), (D.22), (D.34), and (D.36), the dynamics of a two-stage electrohydraulic servovalve are represented as shown on Figure D.12. It is then possible to show that the response of the servovalve will contain oscillatory components using the Routh stability criterion. By adding a fixed drain orifice and deriving the sufficient condition, Watton [Watton 87] showed that these oscillatory components will not exist. By using Routh stability criterion, the sufficient condition for the stability of a servovalve with a variant drain orifice is obtained by

$$\frac{B_a^2 \, x_{pm}}{J_a \, P_s \, a_n \left(1 + \dfrac{n}{1 + \gamma^2}\right) r} > f_g \tag{D.39}$$

As previously mentioned, the parameter γ and the area ratio n must be balanced, i.e., sufficiently large and small, respectively, to increase overall servovalve stability. However, as shown on Figure D.10, the gain function, f_g, is a function of γ, and its peak value on a stroke of the flapper increases as γ increases. Therefore the value of γ need to be determined to design an actual variant drain orifice.

For example, consider a commercially available servovalve having the following data:

$B_a = 0.24$ lb in sec	$x_{fo} = 2.6 \times 10^{-3}$ in
$J_a = 6.84 \times 10^{-4}$ lb in^2	$P_s = 1500$ psi
$a_n = 7.13 \times 10^{-2}$ in^2	$r = 0.5$ in

It is assumed that the servovalve is a direct feedback type of which typical value of Z is 4, and that $\gamma = 1$ and $n = 3$ (i.e., $a_d = 4\ a_n$). When $\gamma = 1$, the back pressure in the drain chamber at null is 25% of the supply pressure. Then, using the condition (D.39), the two sides of the inequality can be plotted using the information on Figure D.9. Figure D.13 shows several plots of inequalities of (D.39).

High frequency oscillations are identified when the horizontal straight lines on Figure D.13 intersect the gain function f_g. The undesirable range of operation increases significantly when the variant drain orifice is attached to the flapper. For high supply pressures, the developed theory predicts an increase in audible noise resulting from instability over most of the operating range of the flapper displacement.

Findings and a Discussion

Table D.I summarizes the effects of a variant drain orifice compared to a fixed drain orifice. The specific advantages acquired by using a variant drain orifice on the first-stage flapper-nozzle of a two-stage electrohydraulic servovalve are as follows:

1. A reduction of the flow and the power loss through the flapper-nozzle stage.

2. An increase in the null pressure sensitivity of the flapper-nozzle valve.

3. An increase in the linearity of the gain function of the flapper-nozzle valve.

4. An increase in the null gain of the overall servovalve dynamics.

5. An increase in the uniformity of the gain function for each flapper stroke.

However, a highly resistive drain orifice may deteriorate the stability of servovalve dynamics. Hence, the features and the size of the variant drain orifice must be determined considering stability.

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TABLE 7.I

Numerical Values for The Parameters of A Two-Satge Electrohydraulic Servovalve

parameter	value	parameter	value
J_a	1.75×10^{-6} lb_f in s^2 / rad	x_{fo}	0.0015 in
B_a	0.036 lb_f in s / rad	C_{do}, C_{df}	0.68
K_a	32 lb_f in / rad	C_{dd}	0.63
d_n	0.0145 in	ρ	8.12×10^{-5} lb_f s^2 / in^4
r	0.6882 in	A_p	0.0515 in^2
K_t	18.5 lb_f in / amp	M_s	3.02×10^{-5} lb_f s^2 / in
β	2.16×10^5 lb_f / in^2	B_s	5.15 lb_f s / in
V_t	0.003 in^3	K_s	340 lb_f / in
V_e	0.0005 in^3	x_{pm}	0.05 in
K_L	8.79×10^{-4} in^5 / lb_f s	i_m	1 amp
P_s	2000 lb_f / in^2	d_o	0.0093 in
d_d	0.0198 in		

TABLE 8.I

Various Values Depending on the Specifications of θ and α

Line	θ	α	$\lambda(M_1)$	$\lambda(M_2)$	Juang's Criterion
L_1	$\pi/6$	1	0.4442 −0.0156 0	0.2729 0 −0.0480	$\mu_1 = 1.3946$
L_2	0	−1	1.2089 −0.2089 0	0.5277 0 −0.0991	$\mu_2 = 0.5758$
L_3	$-\pi/6$	1	0.4442 −0.0156 0	0.2729 0 −0.0480	$\mu_3 = 1.3946$
L_4	π	−10	0.0017 −0.3421 0	0.0426 0 −0.2007	$\mu_4 = 1.8428$

TABLE 8.II

Various Values for Example 1

Line	θ	α	$\lambda(M_1)$	$\lambda(M_2)$	Juang's Criterion
L_1	$\pi/6$	0	2.1659 −3.3659	2.8581 −2.0248	$\mu_1 = 0.1607$
L_2	0	−1	3.2158 −4.4380	3.8000 −3.0222	$\mu_2 = 0.1214$
L_3	$-\pi/6$	0	2.1659 −3.3659	2.8581 −2.0248	$\mu_3 = 0.1607$
L_4	π	−10	0.3530 −0.0685	0.0628 −0.1683	$\mu_4 = 1.9182$

TABLE 8.III

Various Values for Example 2

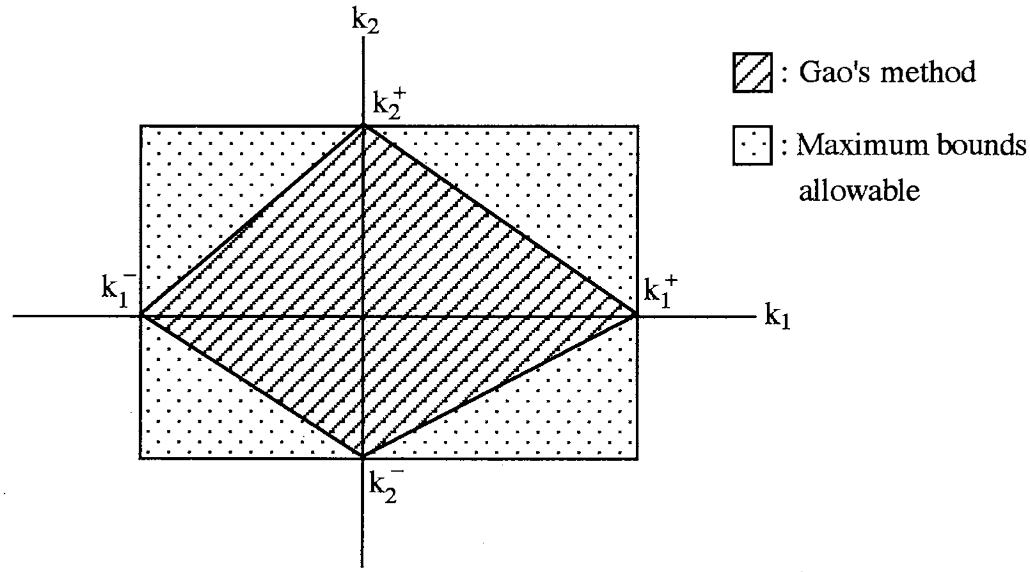
Line	vertex		$\lambda(\Delta M_1)$		$\lambda(\Delta M_2)$		P: positive definite?
	k_1	k_2					
L_1 or L_3	0.4617	0	0.8099	−0.1204	0.0072	−0.4894	yes
	−0.2971	0	0.0775	−0.5211	0.3149	−0.0046	yes
	0	0.3499	0.1390	−0.4305	0.3941	−0.1997	yes
	0	−0.4939	0.6077	−0.1961	0.2819	−0.5563	yes
L_2	0.3110	0	1.5698	−1.8634	1.7181	−1.4071	yes
	−0.2253	0	1.3503	−1.1375	1.0196	−1.2449	yes
	0	0.2632	1.7130	−1.4108	1.2790	−1.5616	yes
	0	−0.3309	1.7739	−2.1538	1.9635	−1.6081	yes
L_4	2.8329	0	0.3301	−0.0581	0.0287	−0.1672	yes
	−14.5985	0	-	-	-	-	no
	0	15.9236	-	-	-	-	no
	0	−5.9418	0.2988	−0.0633	0.0285	−0.1666	yes

TABLE D.I

Comparison of Effects of Drain Dampers

Characteristics	Fixed Drain Orifice	Variant Drain Orifice
Flow and power loss	↓	↓
Null pressure sensitivity of flapper-nozzle valve	↓	↑
Linearity of gain function of flapper-nozzle valve	↓	↑
Null gain of servovalve	↓	↑
Null damping constant	↓	↑
Uniformity of gain function	↔	↑
Stability of first-stage	↑	↕
Overall Stability	↑	↕

↓ : decrease ↑ : increase ↔ : little effect ↕ : adjustable variance



$$k_1^+ = 1/\lambda_{max}(P_1), \quad k_1^- = 1/\lambda_{min}(P_1)$$

$$k_2^+ = 1/\lambda_{max}(P_2), \quad k_2^- = 1/\lambda_{min}(P_2)$$

Figure 3.1 Stability bounds for a 2-dimensional perturbation system

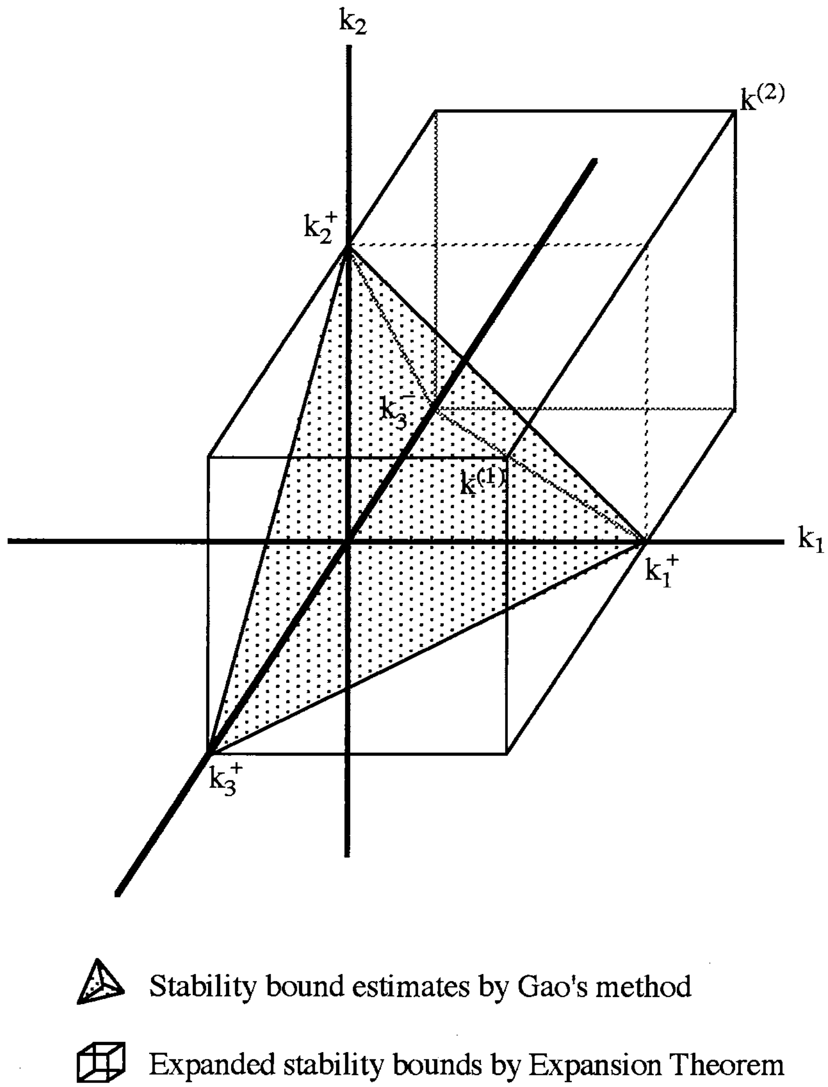


Figure 4.1 Part view of stability bounds for a 3-dimensional perturbation system

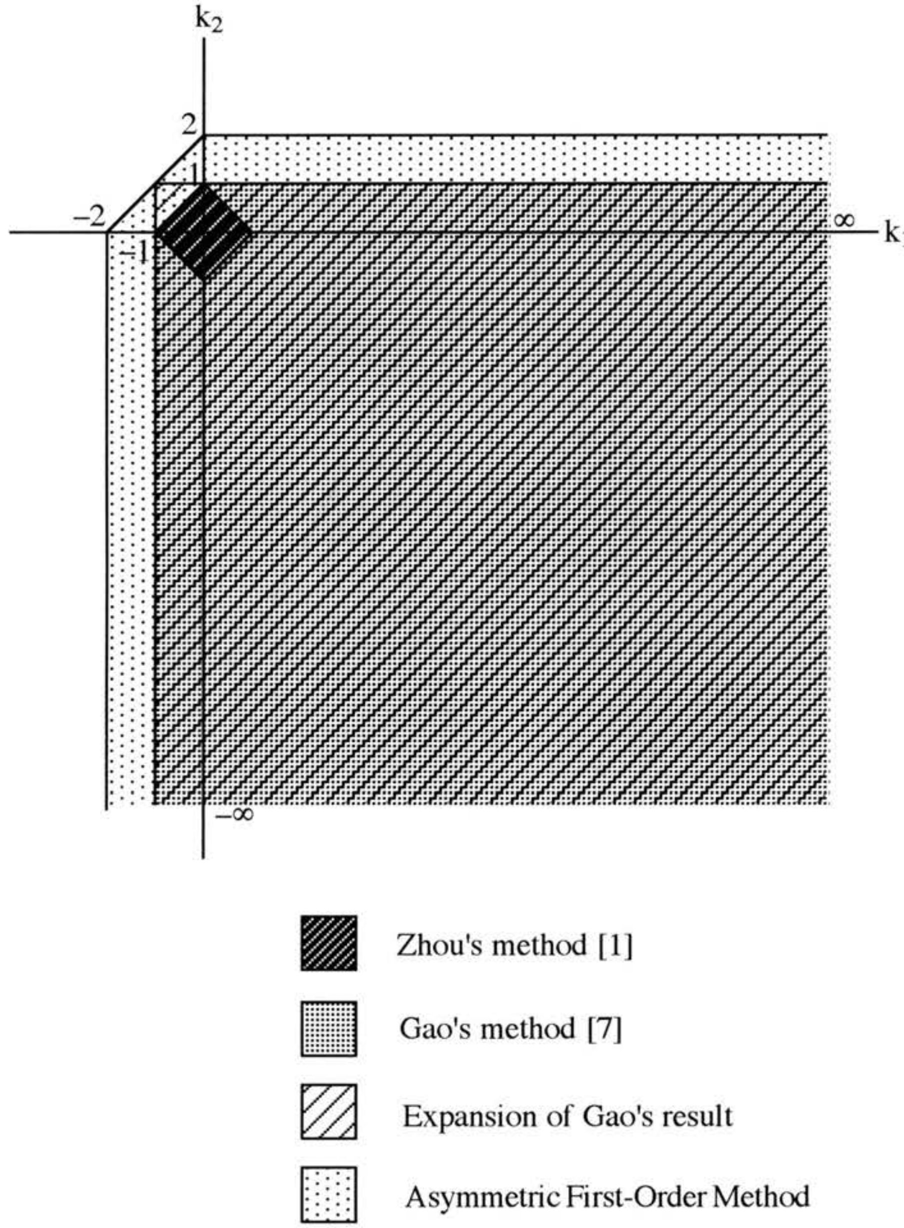


Figure 6.1 Comparison between the stability bound estimates for example 1

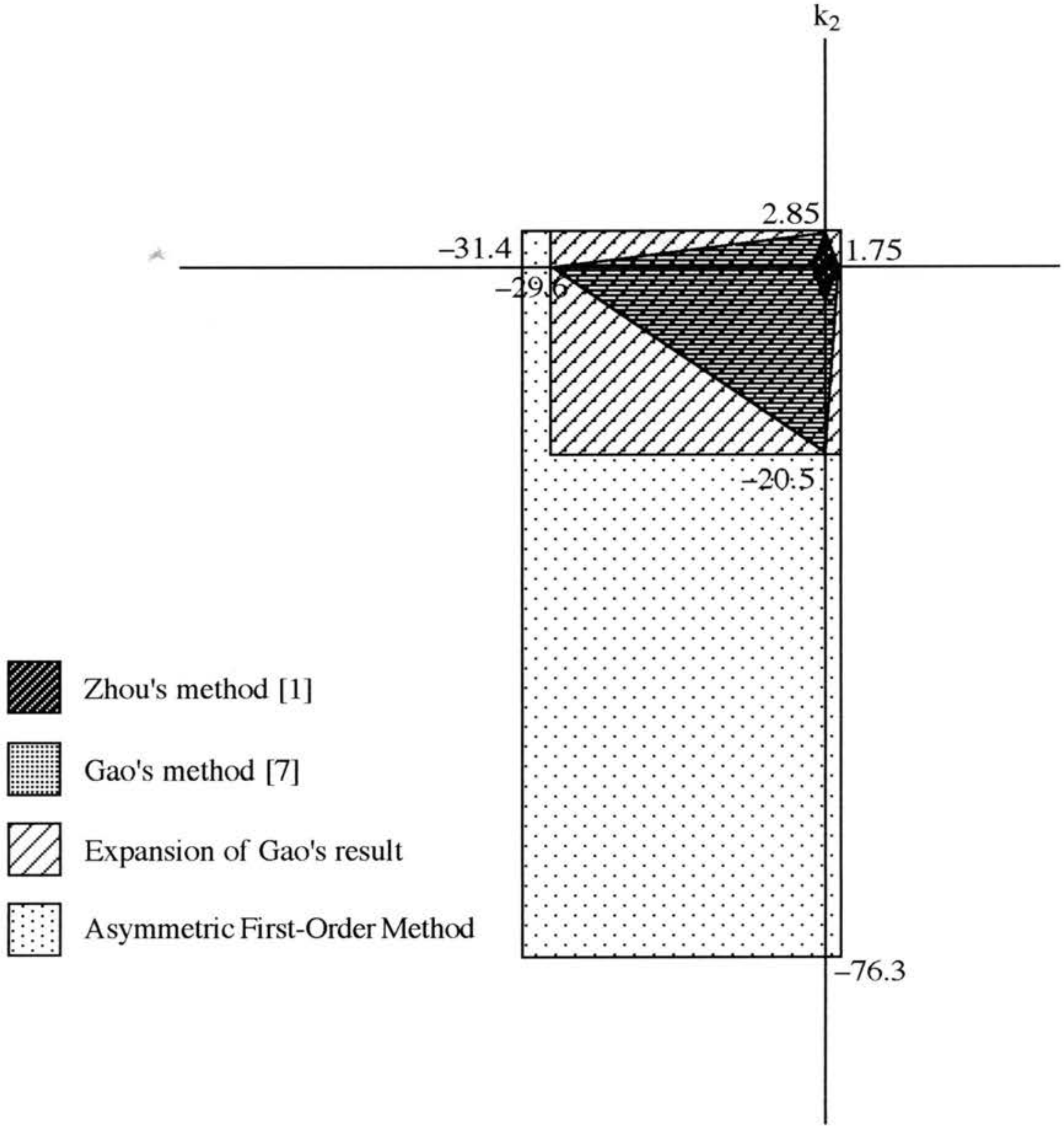


Figure 6.2 Comparison between the stability bound estimates for example 2

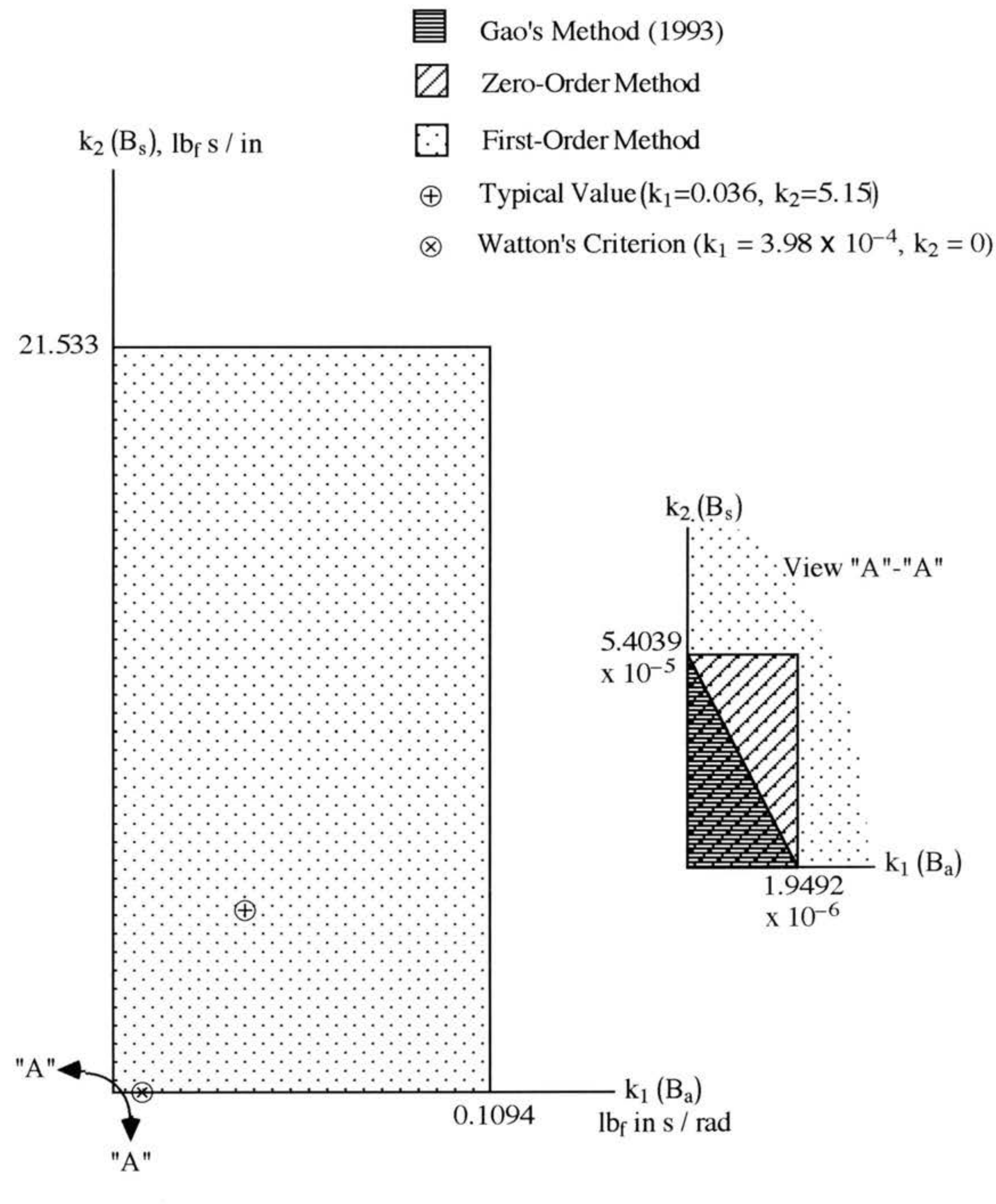


Figure 7.1 Stability bound estimates for a two-stage electrohydraulic servovalve example

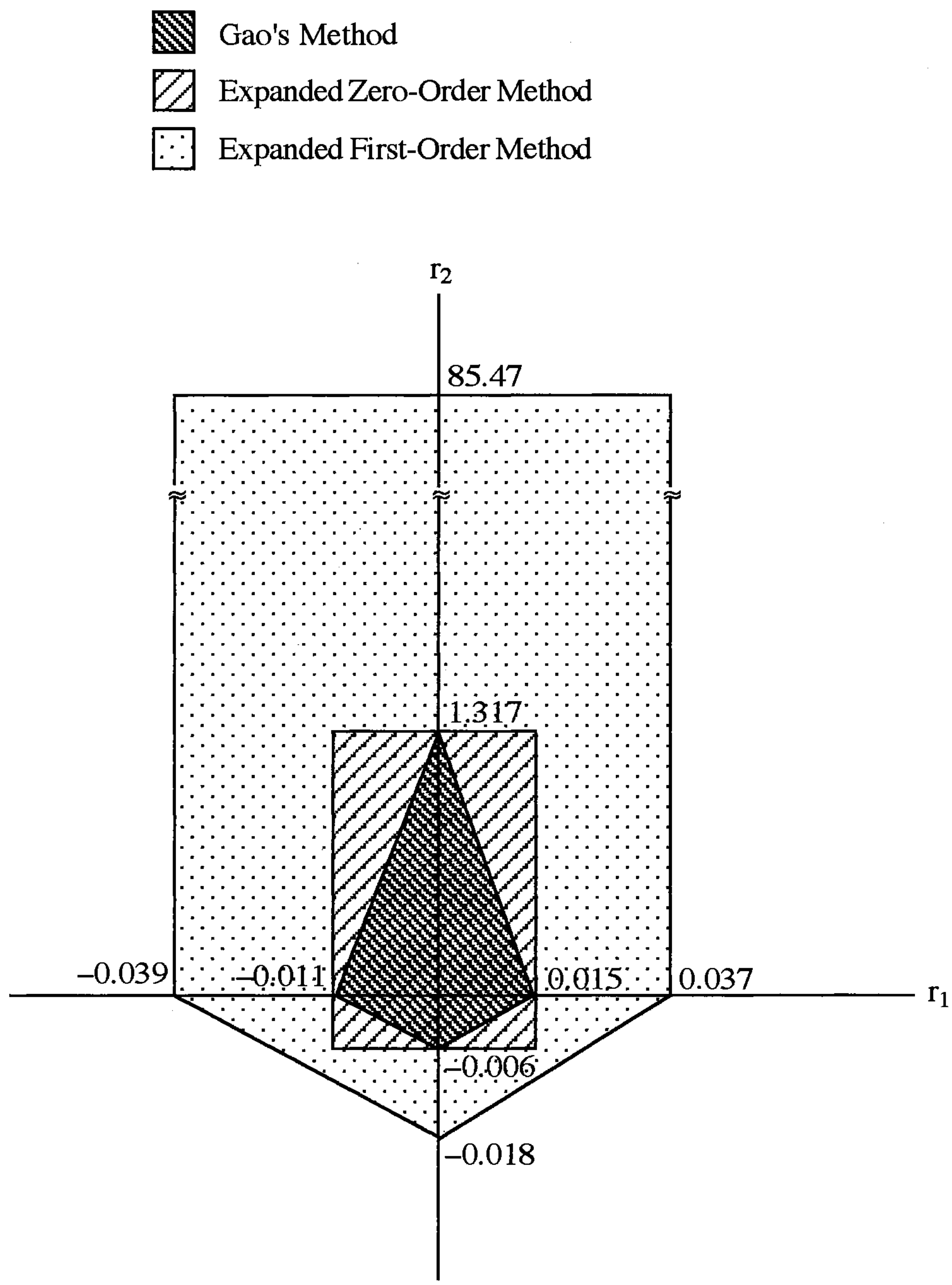


Figure 7.2 Stability bounds for a fighter aircraft example

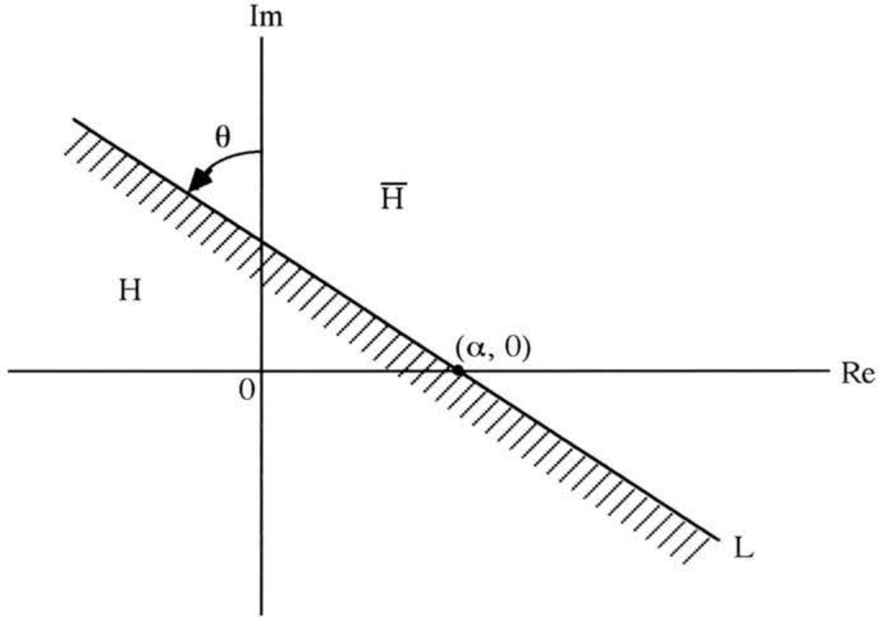


Figure 8.1 Two open half-planes seperated by a line L

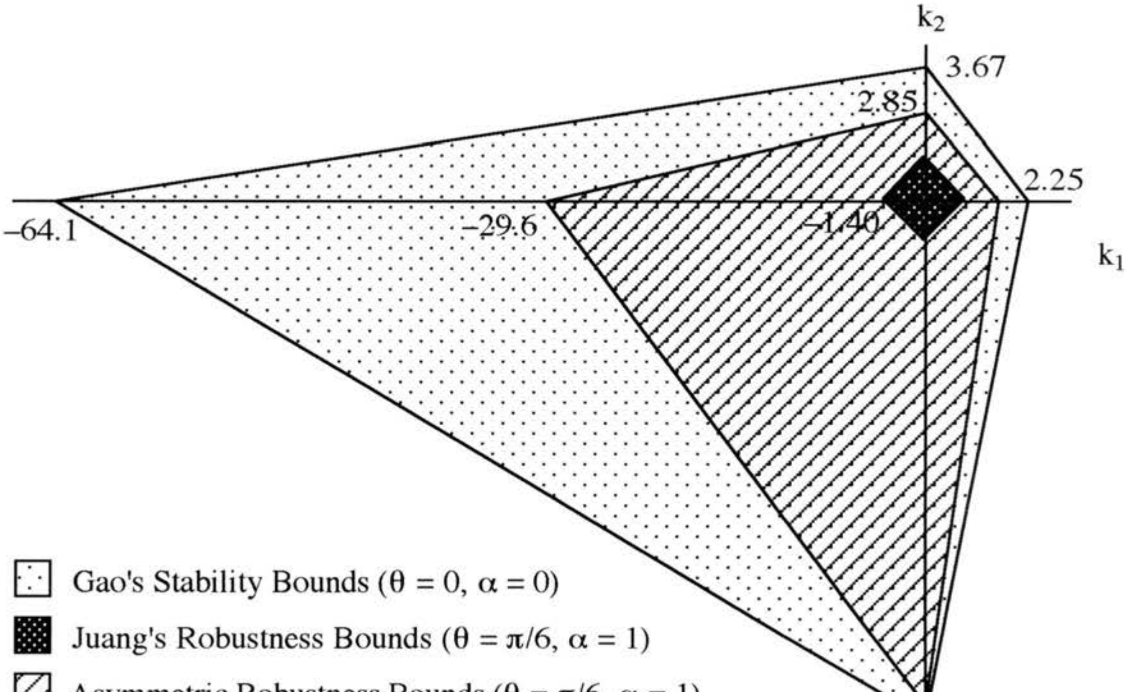


Figure 8.2 Various stability and robustness bound estimates

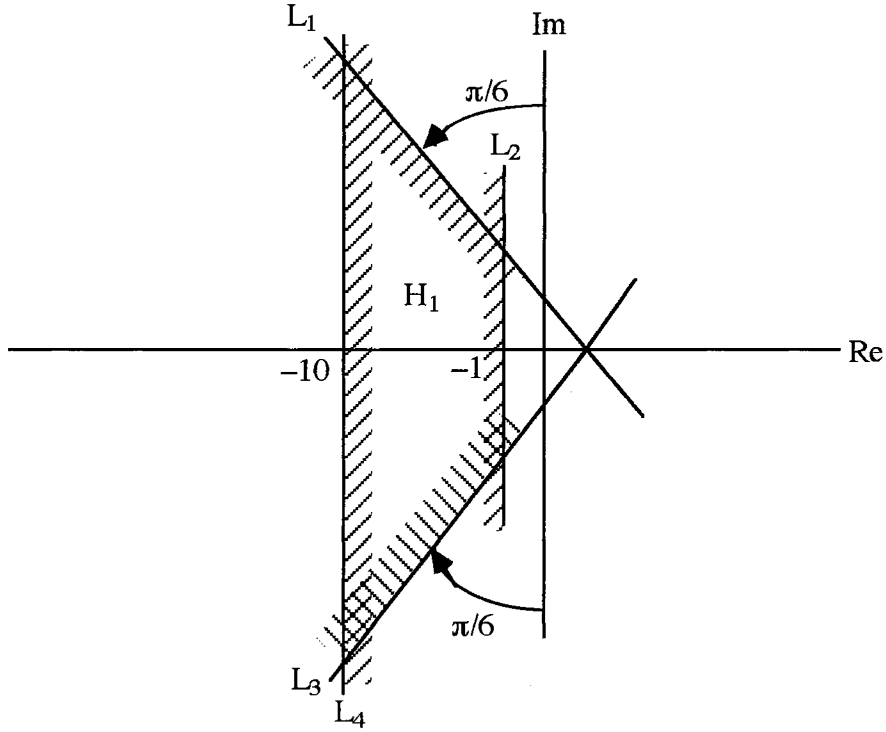


Figure 8.3 Region H_1 for eigenvalue assignment

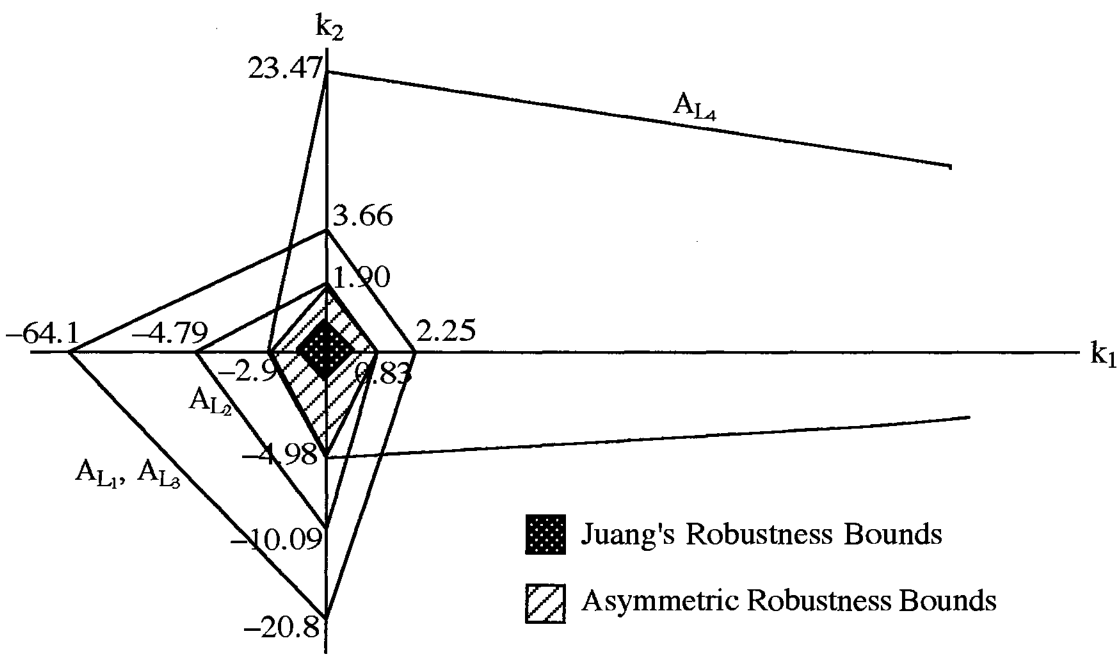
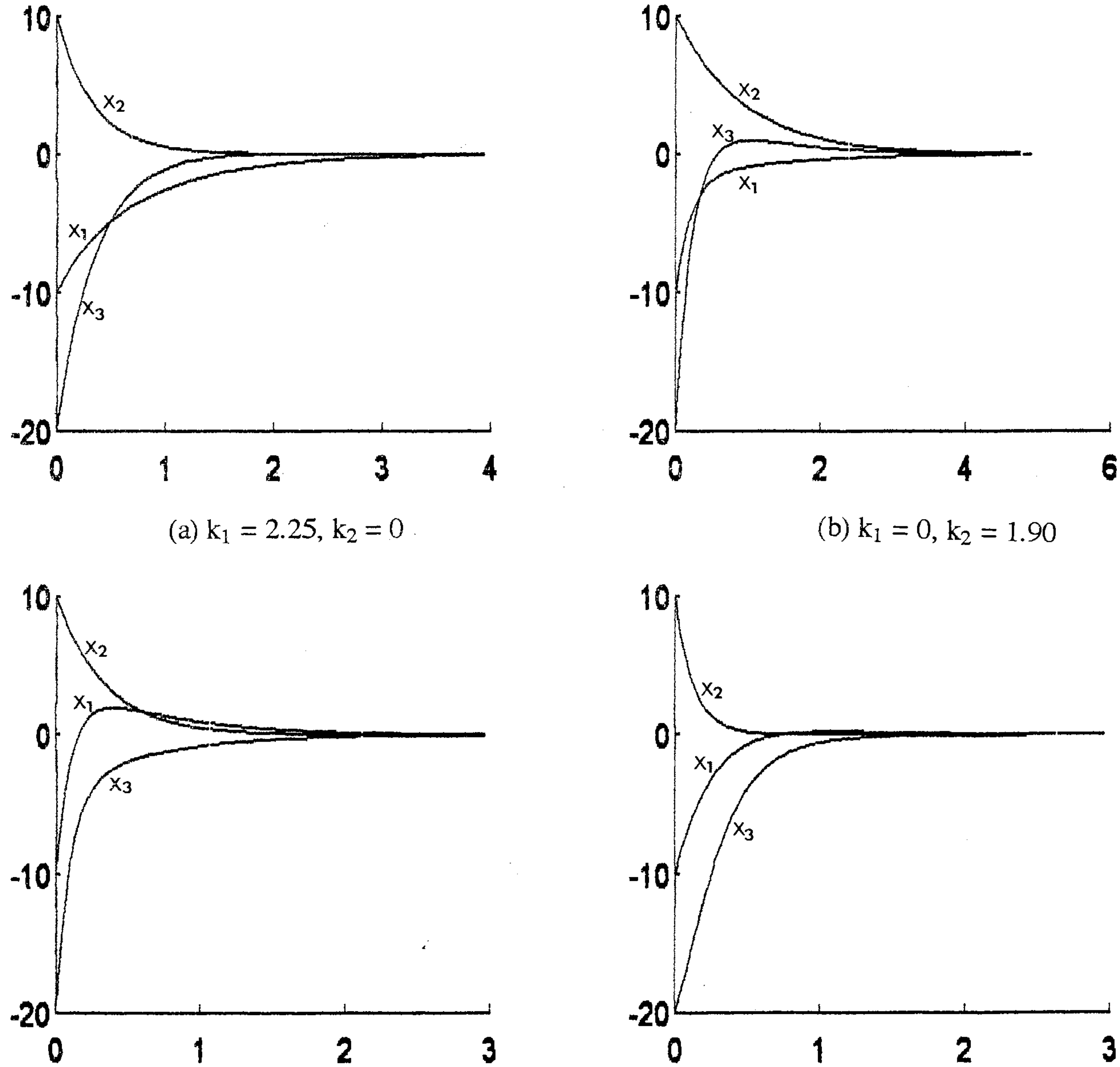


Figure 8.4 Comparison of robustness bound estimates for example 1



(a) $k_1 = 2.25$, $k_2 = 0$

(b) $k_1 = 0$, $k_2 = 1.90$

(c) $k_1 = -2.90$, $k_2 = 0$

(d) $k_1 = 0$, $k_2 = -4.98$

Figure 8.5 Transient responses at the vertices for example 1

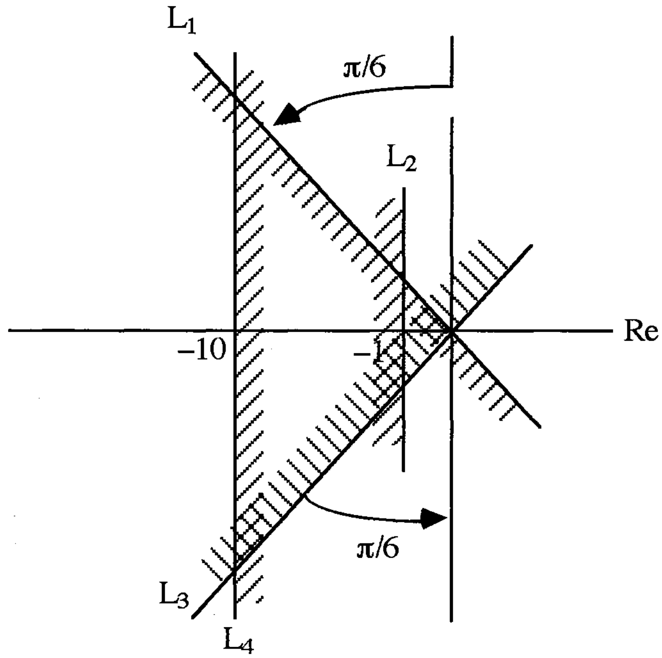


Figure 8.6 Eigenvalue assignment for example 2

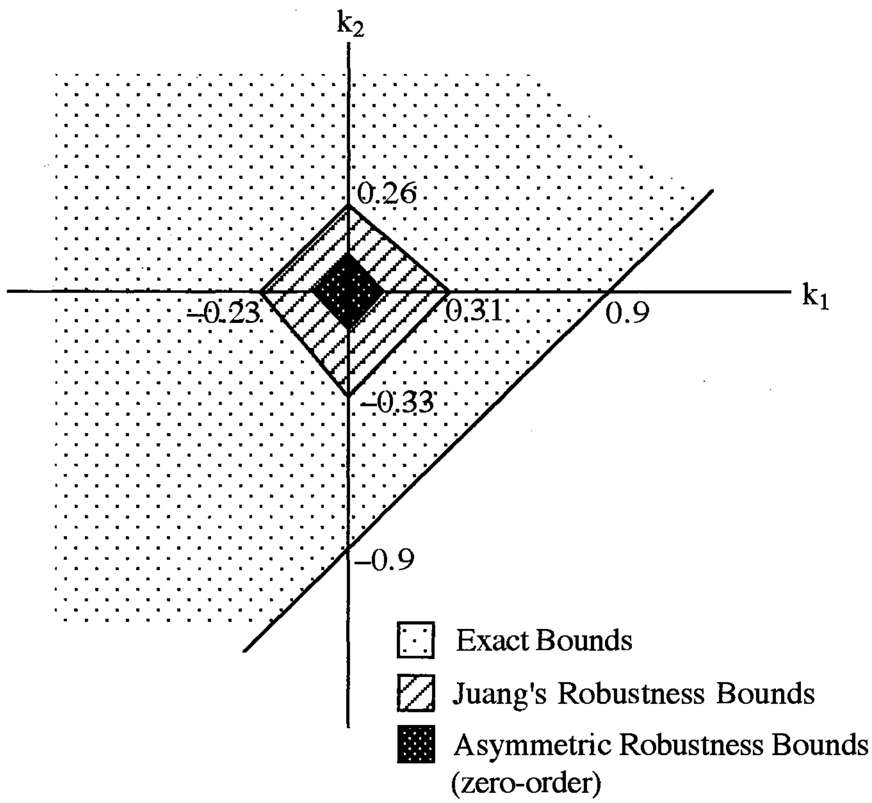


Figure 8.7 Comparison of robustness bound estimates for example 2 (I)

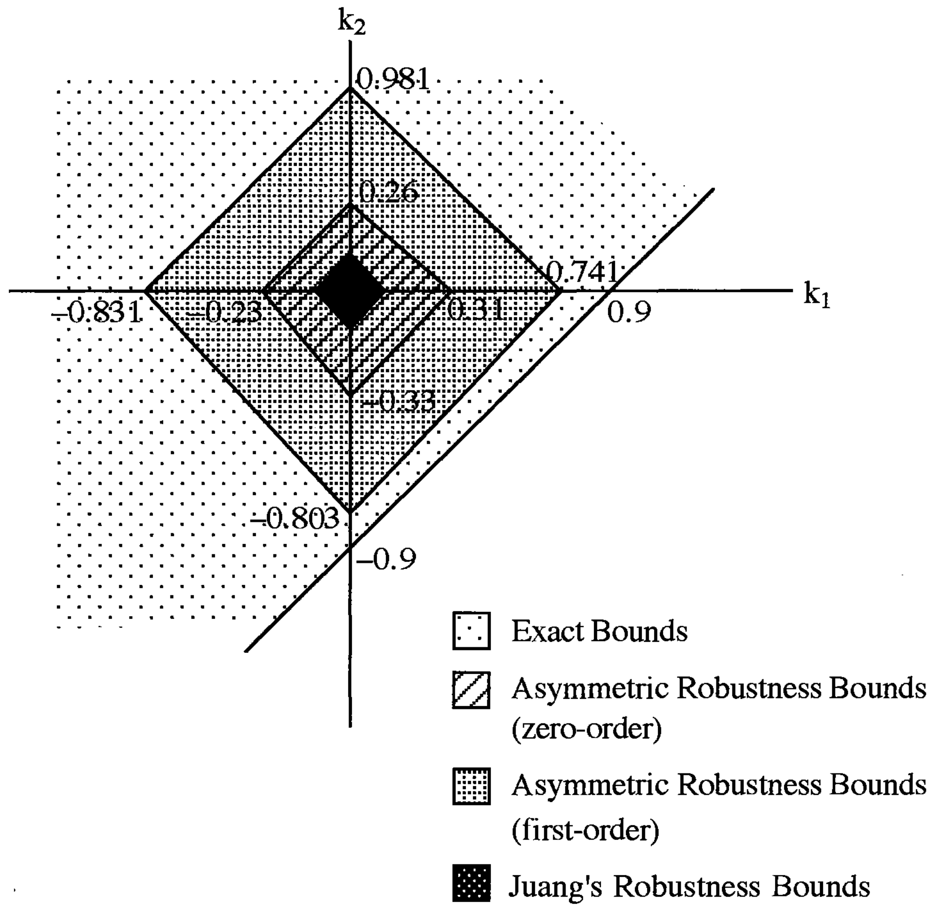
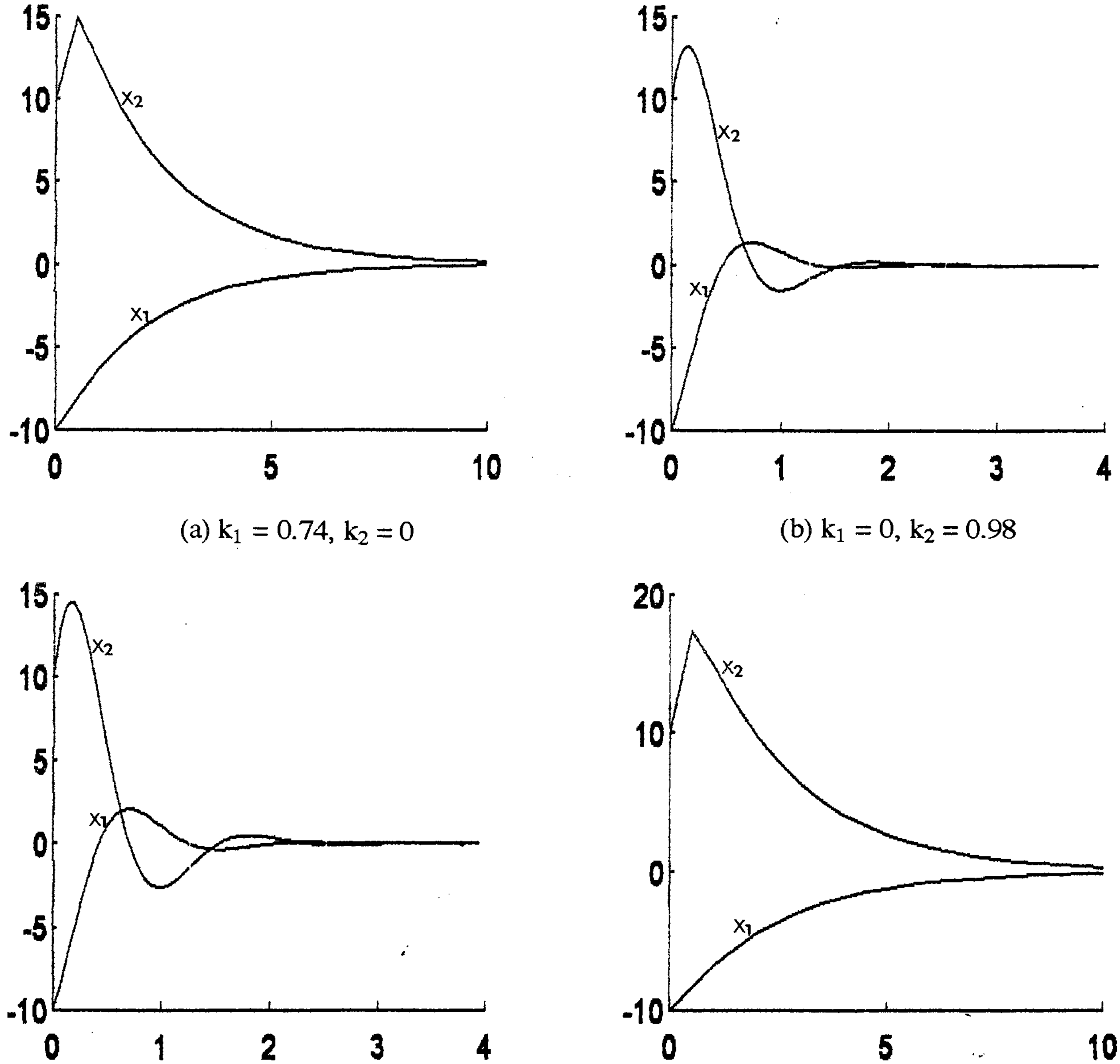


Figure 8.8 Comparison of robustness bound estimates for example 2 (II)



(a) $k_1 = 0.74$, $k_2 = 0$

(b) $k_1 = 0$, $k_2 = 0.98$

(c) $k_1 = -0.83$, $k_2 = 0$

(d) $k_1 = 0$, $k_2 = -0.80$

Figure 8.9 Transient responses at the vertices for example 2

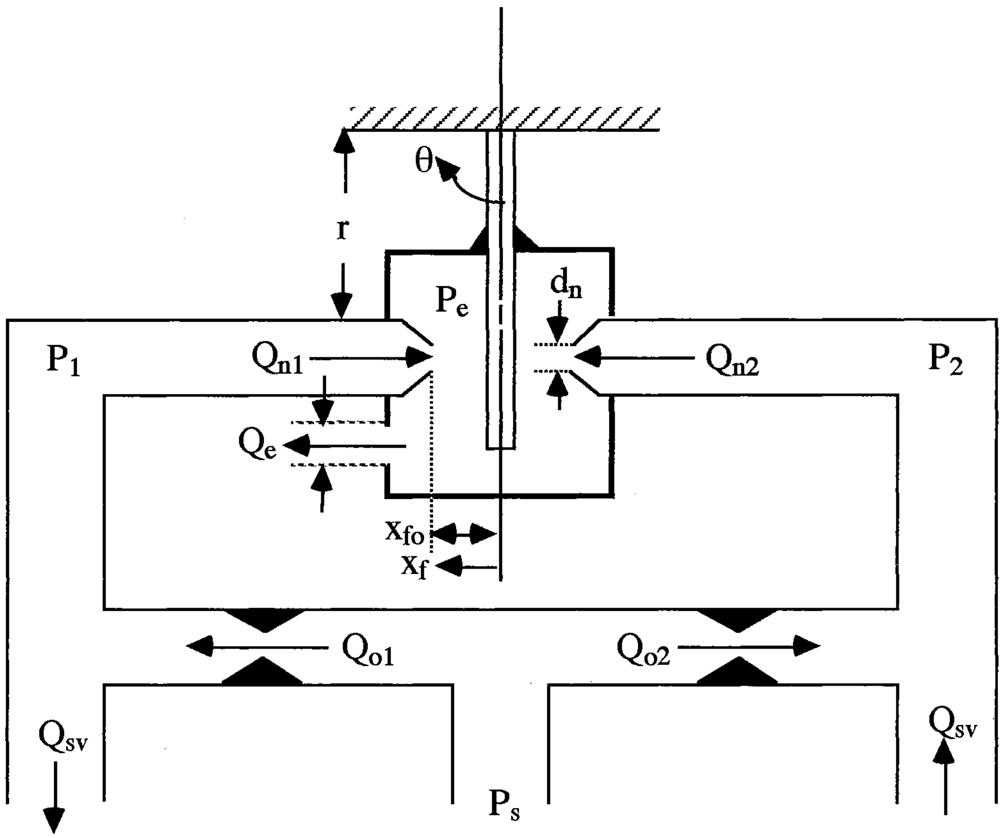


Figure D.1 Flapper-nozzle with a fixed drain orifice

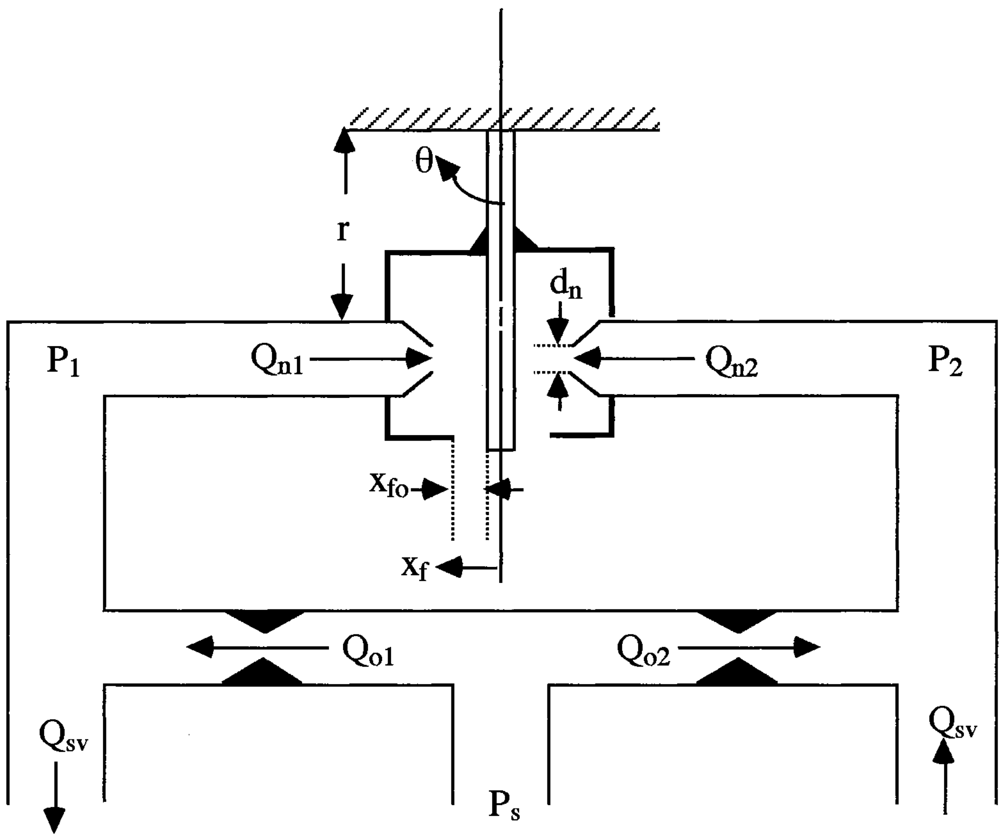


Figure D.2 Flapper-nozzle with a variant drain orifice

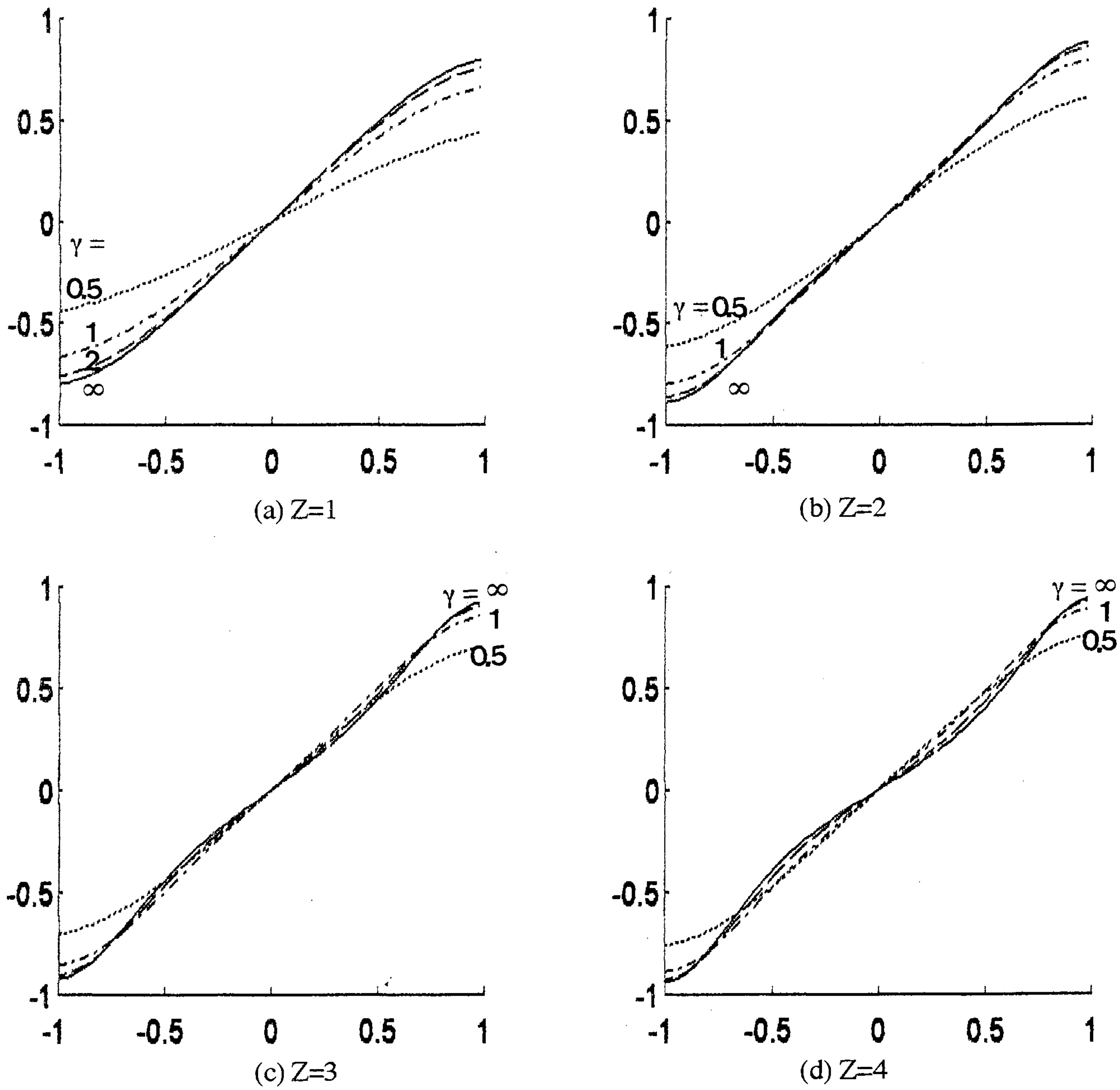


Figure D.3 Steady-state characteristics, $\overline{P}_1 - \overline{P}_2$

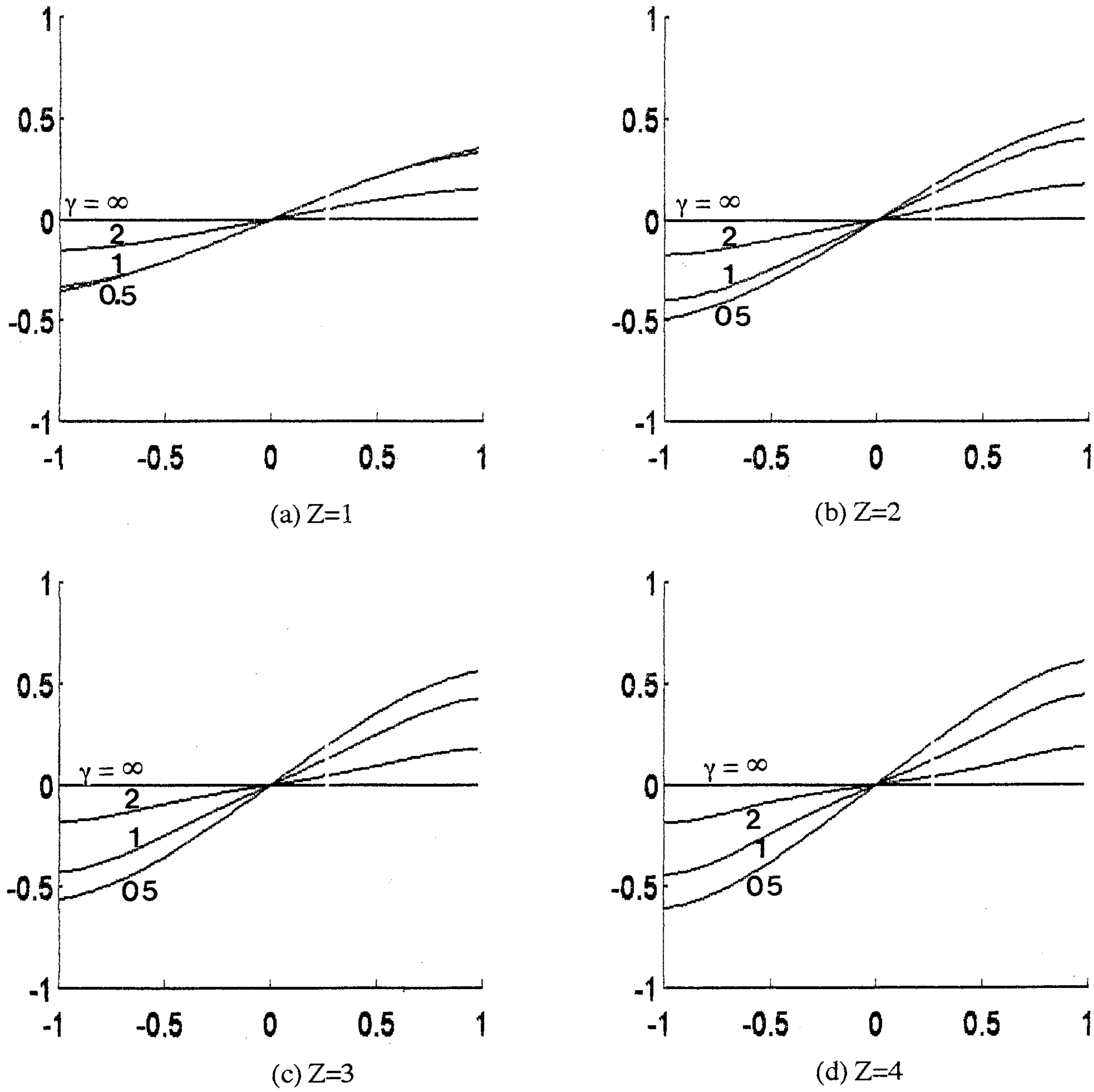
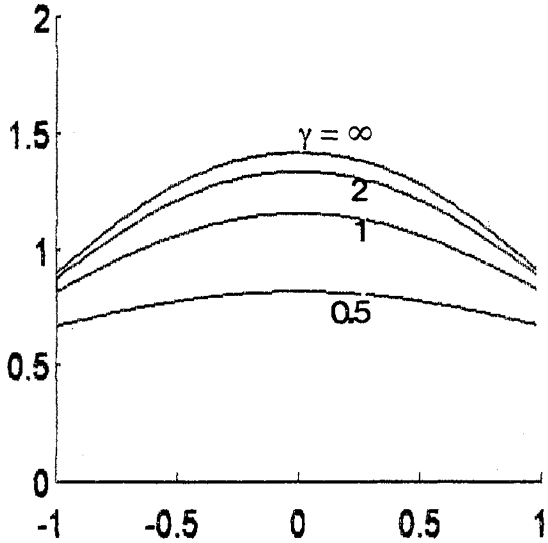
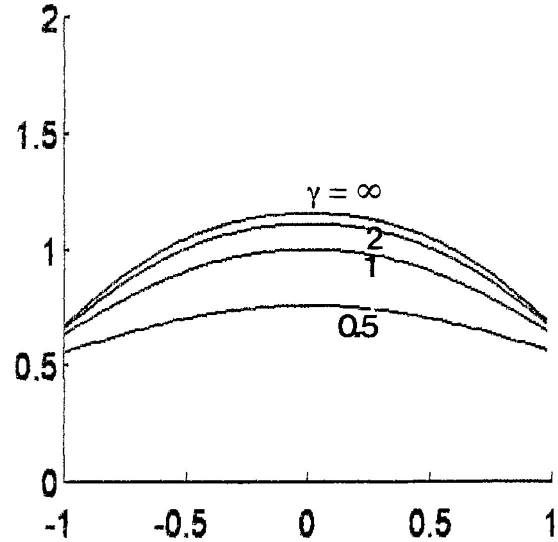


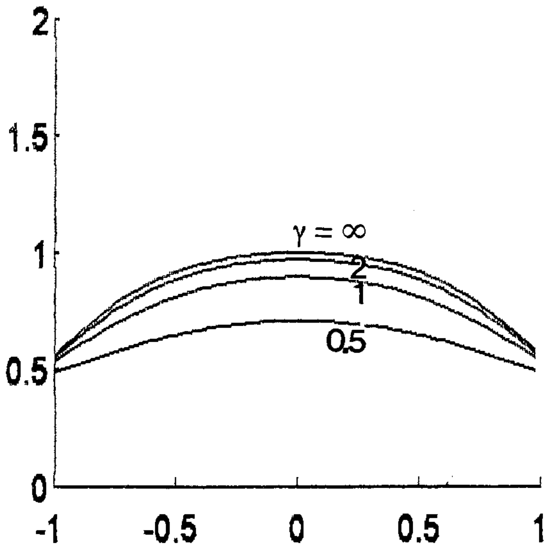
Figure D.4 Steady-state characteristics, $\overline{P}_{e1}-\overline{P}_{e2}$



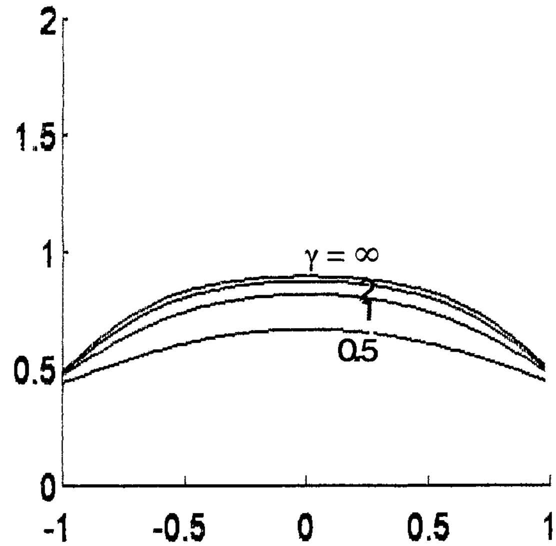
(a) Z=1



(b) Z=2



(c) Z=3



(d) Z=4

Figure D.5 Steady-state characteristics, $\overline{W}_e$ or $\overline{Q}_e$

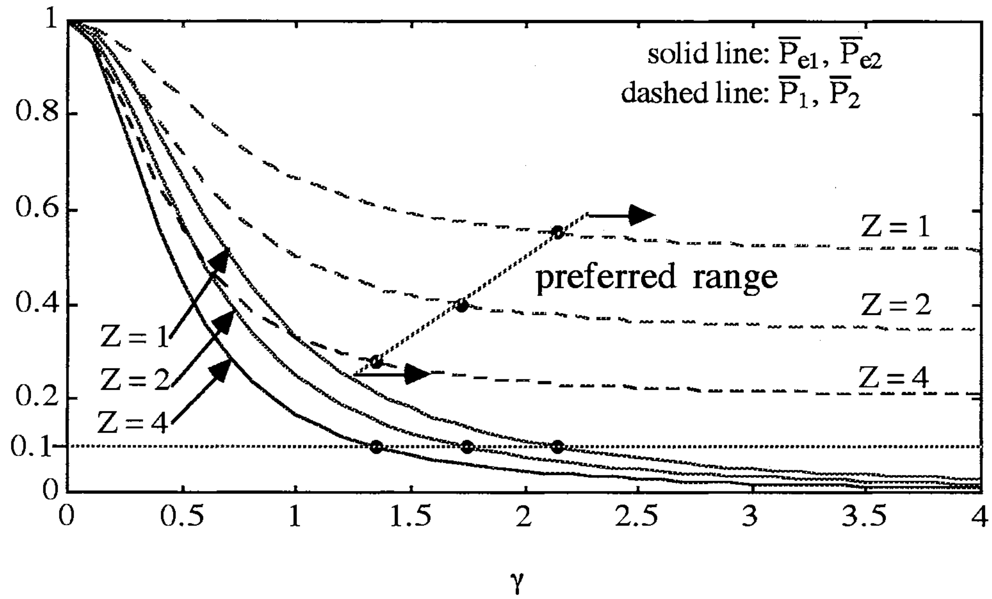


Figure D.6 Pressure characteristics at null

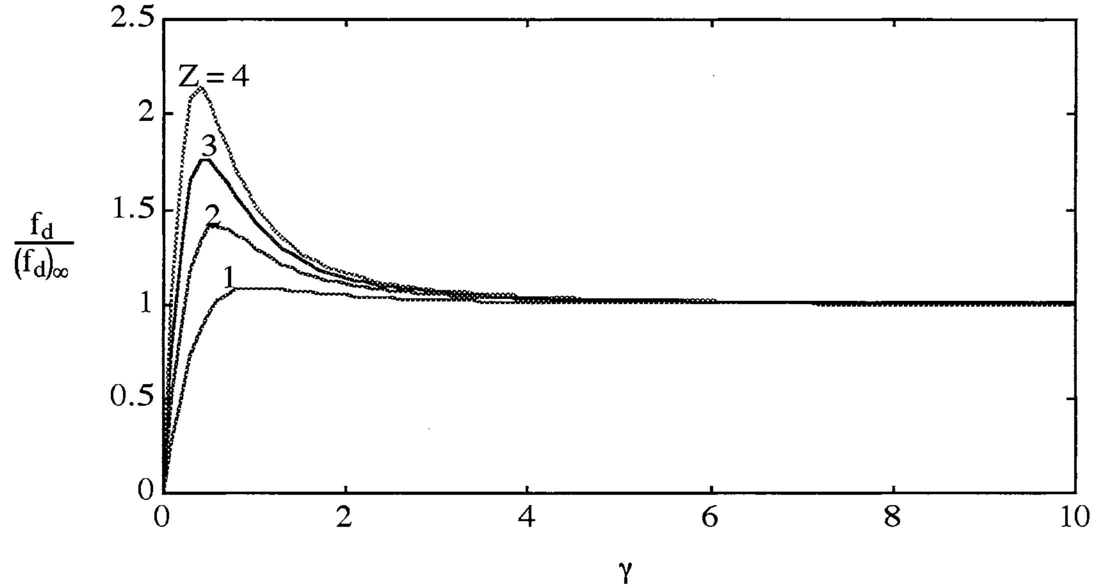


Figure D.7 Variation of null damping constant ratio

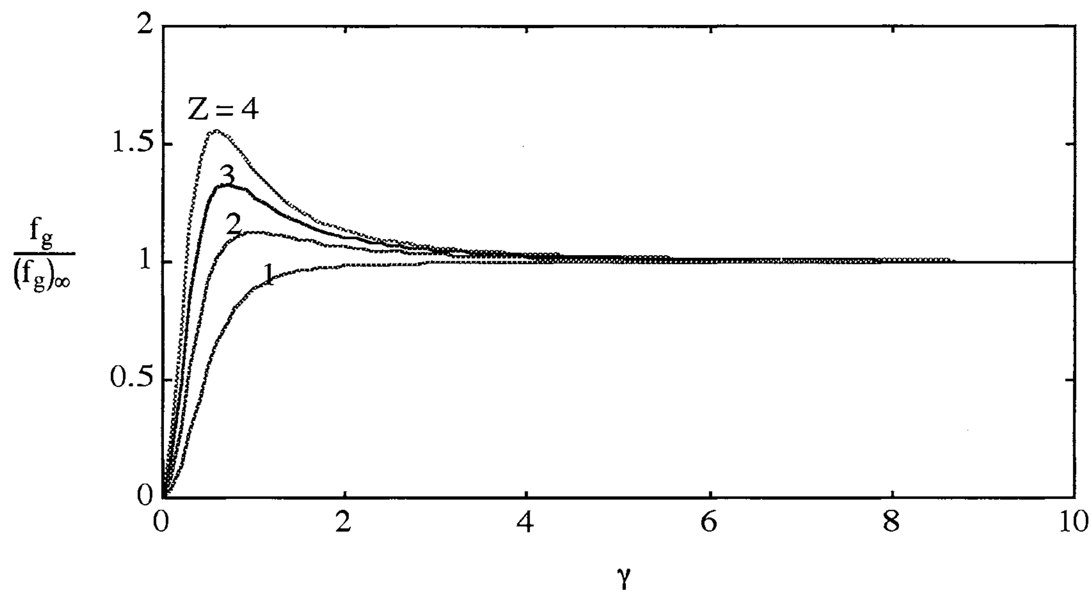


Figure D.8 Variation of null gain ratio

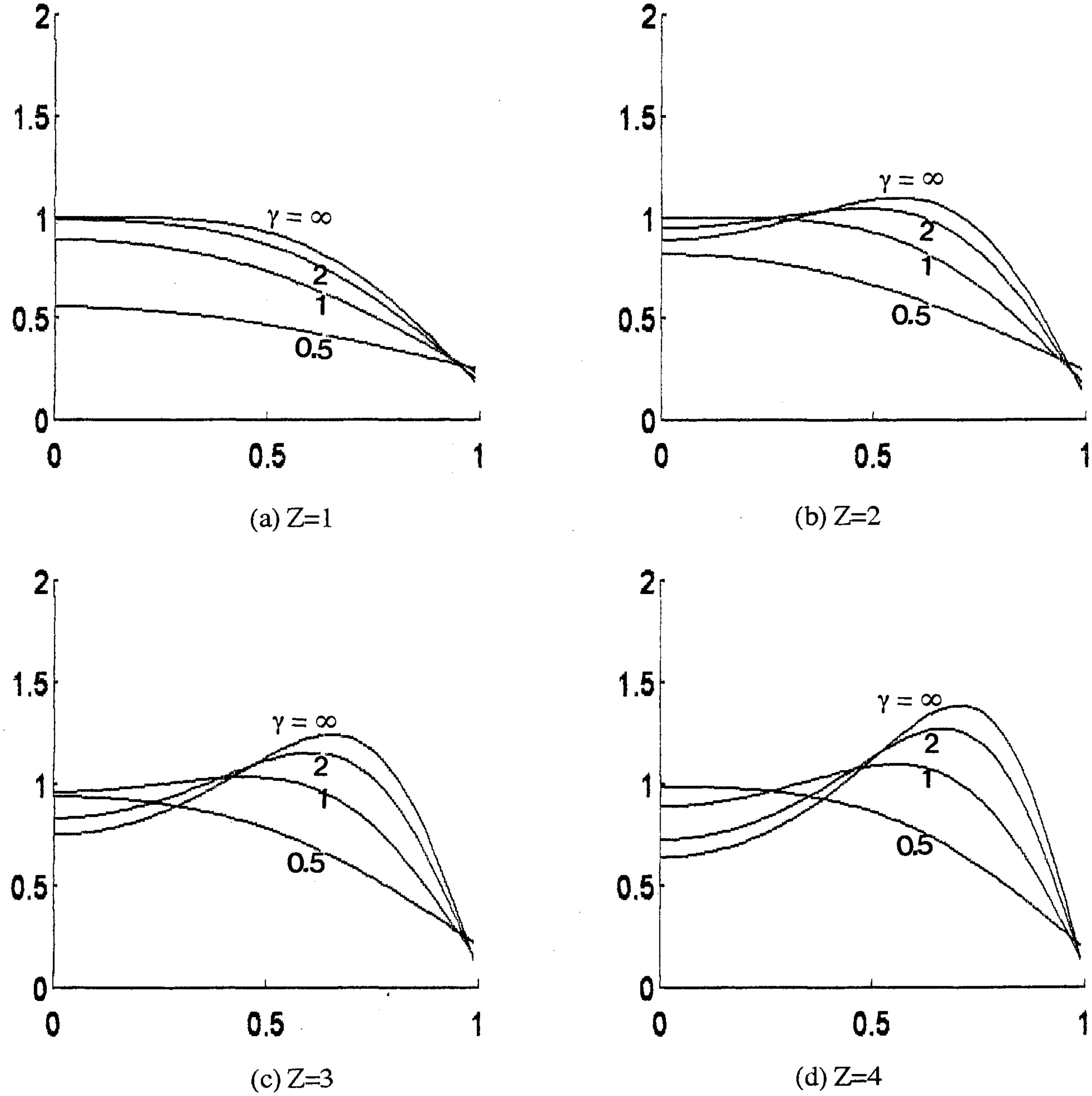


Figure D.9 Variation of gain function

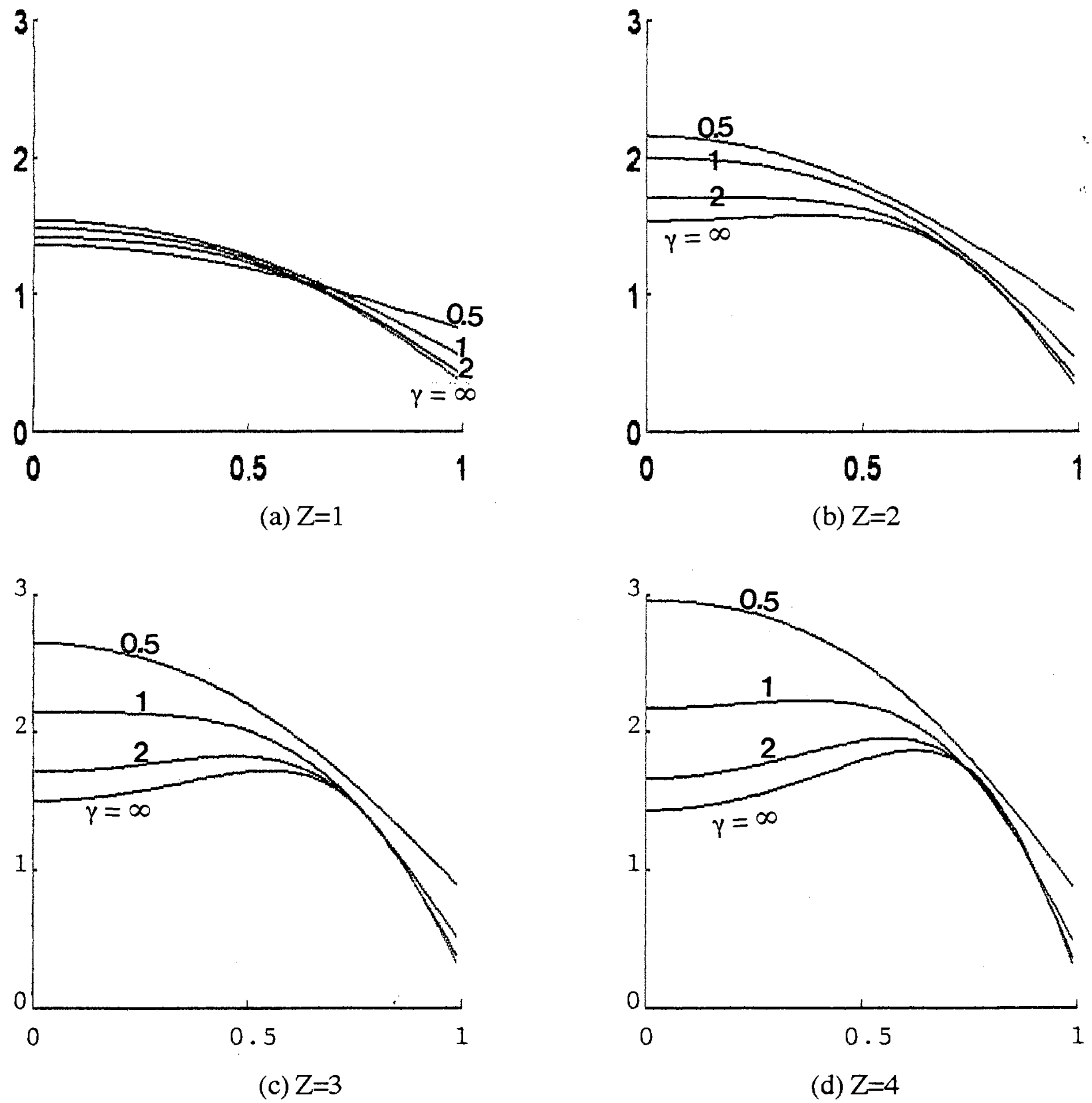


Figure D.10 Variation of damping function

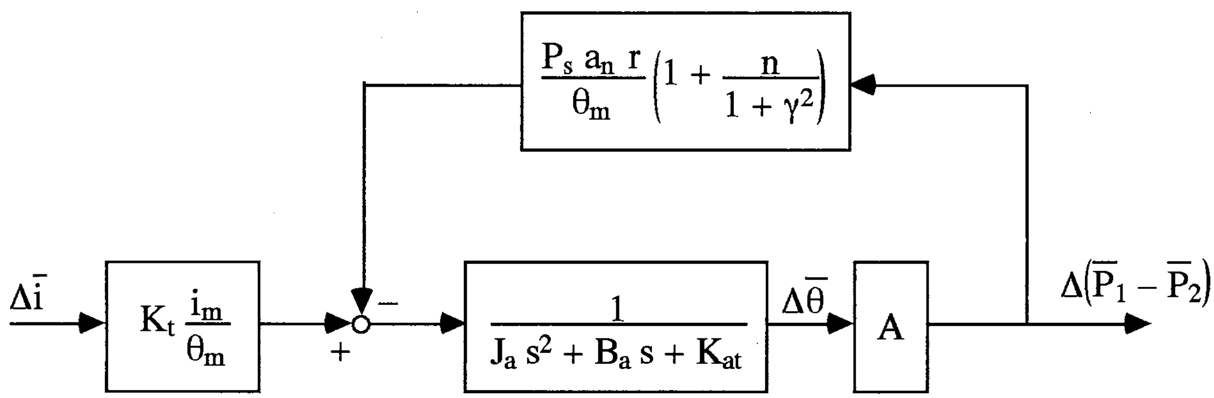
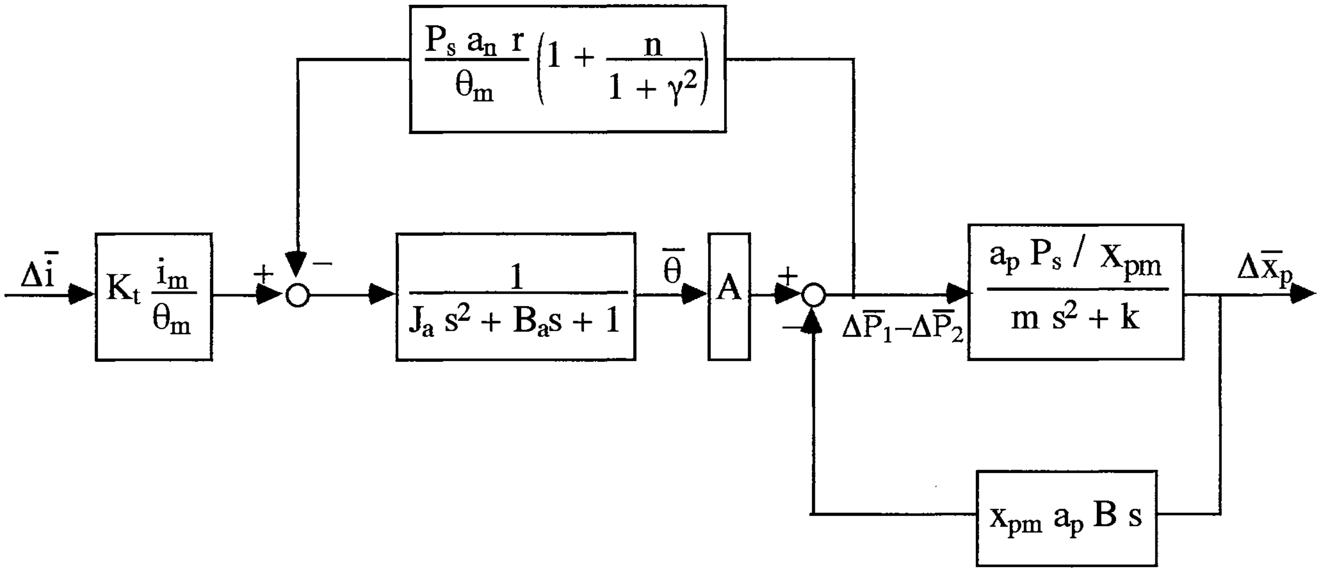


Figure D.11 Block diagram for the dynamics of first-stage assembly



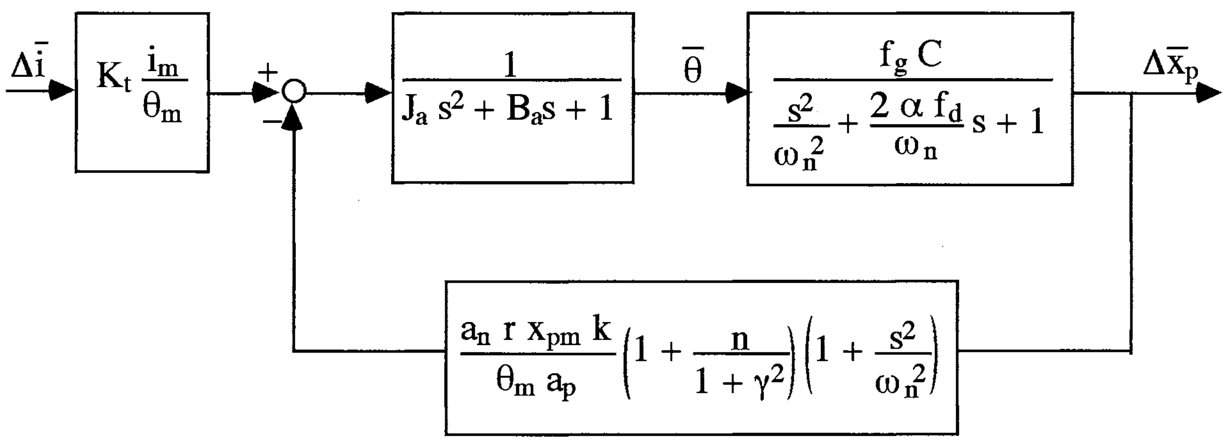
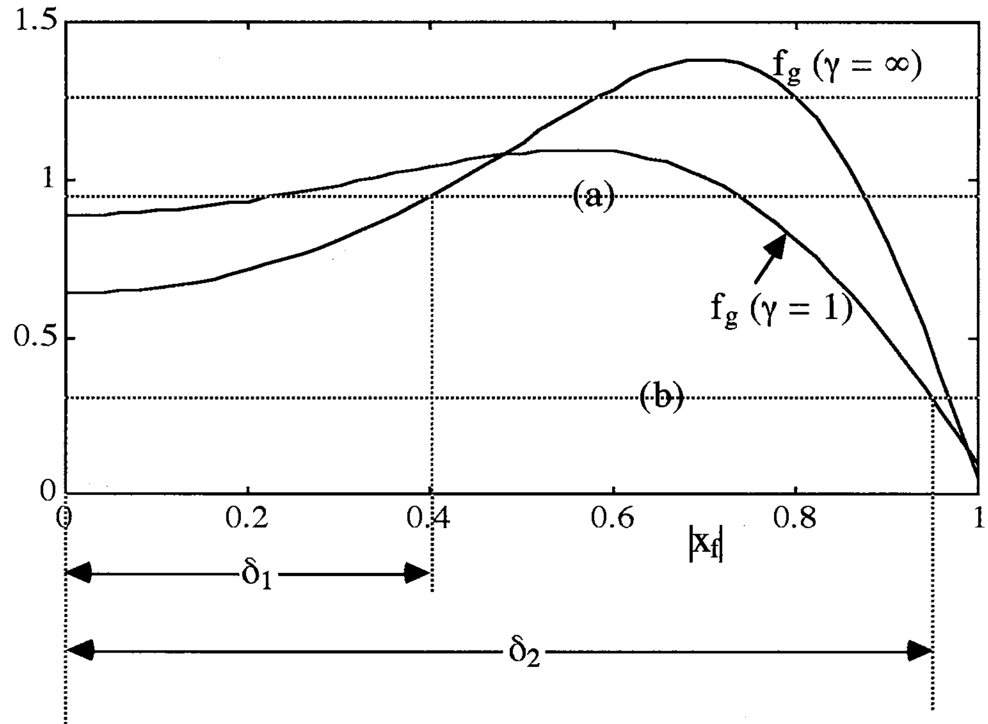


Figure D.12 Equivalent block diagram for the dynamics of a two-stage electrohydraulic servovalve



δ_1 : stable range for a servovalve without any damper

δ_2 : unstable range for a servovalve with drain orifice

(a) : when no damper is used. $\dfrac{B_a^2\, x_{fo}}{J_a\, P_s\, r^2\, a_n} = 0.961$

(b) : when variant drain orifice is used. $\dfrac{B_a^2\, x_{fo}}{J_a\, P_s\, r^2\, n\, a_n} = 0.32$

Figure D.13 Comparison of stable operation ranges of various servovalves

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