Name: David Logan Travis Date of Degree: August 5, 1961

Institution: Oklahoma State University Location: Stillwater, Okla.

Title of Study: PROBLEMS IN SOLUTION

Pages in Study: 39 Candidate for Degree of Master of Science

Major Field: Natural Science

Scope of Study: In practically all ages and times, man's ability to think and solve problems has been measured solely by the products of such thinking and solving. Many scholars have thought that it would be a good idea to set down the mental processes which are typically useful in solving problems of all kinds. This study is a brief survey and discussion of present and past attempts at systematic approaches to problem solving.

Findings and Conclusions: People who have studied the subject report with surprising unanimity that problem solving can be taught, and agree also on the general outlines of the procedure. Since the processes have been fairly well delineated and agreed upon, it would seem very valuble for a high school teacher to be familiar with some of the approaches. The teacher will not only improve his own abilities as a problem solver, but also will be able to assist his students to a greater understanding of the process by which a solution is reached. No claims are made for absoluteness; none were expected. Assistance only is the goal.

ADVISER'S APPROVAL James H. Zant

PROBLEMS IN SOLUTION

By

DAVID LOGAN TRAVIS

Bachelor of Science

New Mexico State University

University Park, New Mexico

1958

Submitted to the faculty of the Graduate School of
the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
MASTER OF SCIENCE
August, 1961

PROBLEMS IN SOLUTION

Report Approved:

Report Adviser

Dean of the Graduate School

PREFACE

In the summer of 1959 after one and one-half years of teaching experience, I applied for and received a General Electric Mathematics Fellowship to attend Stanford University. During the six weeks I was there I had the opportunity and pleasure of taking a seminar in problem solving under the direction of Professor G. Polya.

Professor Polya, though a man in his seventies, captivated the entire group with his enthusiasm and interest in everything connected with the teaching of mathematics. I suspect that all of us caught some of his enthusiasm; I know that I did, and I feel that my attitude toward teaching has been improved by that brief association.

I would like to acknowledge gratefully my debt to Professor G. Polya.

I would also like to acknowledge the valuble assistance of Miss Joanna Black of Eastern New Mexico University for proof-reading my manuscript; and Dr. James H. Zant, the Director of the NSF here at Oklahoma State University.

My gratitude also goes to the National Science Foundation whose financial assistance made the year of study culminating in this report possible.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Historically speaking, mathematics is somewhat singular among the sciences for the amount of important creative work which has been done by very young men. Many of them were in their twenties, and some were still in their teens. Possibly the most notable of these was Carl Friedrich Gauss, but many others could be noted. Most authorities agree that creative work in mathematics is dependent on "inspired" guesswork, intuition or inductive techniques which are generally frowned on by proponents of rigorous formal mathematics.

For a high school student to solve many of the problems of algebra and geometry with which he is presented, it is frequently more natural for him to use intuition or inductive reasoning than formal deduction. Many people feel that it is at least as important to develop the inventive facilities of young students as to teach them the basic algebra and geometry. As Professor Polya of Stanford puts it in the preface to his book, How to Solve It, A great discovery solves a great problem, but there is a grain of discovery in the solution of any problem.

In this study, the writer hopes to consider the teaching and learning of methods or approaches to solving problems. Most people in this field have suggested methods on the basis of their own experience and observation of other problem solvers. Brief results of two experimental studies in this area are included, but relatively little experimental work has been done. The reason for this lack is simple: very few people have any idea about how they solve a problem. Problem solving is a mental process, and until telepathy is discovered, there will be doubt about how it comes about.

A discussion of intuition and inductive reasoning is included, since a concensus of opinion is that these are two ways that past experience is linked to a present problem in order to reach a solution.

A history of problem solving, which is necessarily both brief and incomplete, is found in Chapter II, while modern approaches are found in ChapterV, which is, perhaps, the heart of the study.

CHAPTER II

HISTORICAL DEVELOPMENT

There are three "schools" of thought regarding the origin and nature of mathematics: Logisticism, Intuitionism, and Formalism. Formalism is probably best formulated in Principia Mathematica by Whitehead and Russell. Intuitionism is usually thought of as being founded by the Dutch mathematician, L. E. J. Brouwer and Formalism is probably best exemplified by the work of D. Hilbert. (1).

The three are not schools in the usual sense, but rather ways of thought classification; most mathematicians will partake of all three points of view.

The "kernel" of Intuitionism is the notion of mathematics as a construction of the "intuitively given" natural numbers. Intuitionism recognizes the ability of an individual to perform a series of mental acts consisting of a first act, then another and so on endlessly. In this way one attains fundamental series, the best known of which is the series of natural numbers. (1).

The high school student has no interest, function or business with the ideas of strict mathematical intuition, (and it is both strict and mathematical), but he knows the term from hearing about "woman's intuition", which is not too far away from Brouwer's idea. Many of the things which are "seen" as true, given in textual discussion as "obviously" or "clearly", etc., are actually examples of an individual's mental operations.

Intuitionism makes the point that mathematics is independent of language. For the communication of mathematics, the use of symbolic devices and ordinary language is necessary, but their only function is communication. (1).

Modern high school students have had experience in the type of mathematical construction implied by the Intuitionists, in their progressive development of the real and complex number systems. In this sense the idea of mathematical intuition has a real and valuble place in high school mathematics.

Intuitionist mathematics recognizes that mathematics has its origins in experience, but its modern abstract formulation is the product of the pure intellect, and has intuitive, not merely formal context, (1).

Without a doubt, all mathematical development has its psychological roots in more or less practical requirements. Once begun, however, it tends to forge ahead on its own, toward greater and greater abstraction. (2).

Greek mathematics had its roots in the Babylonian and Egyptian practical mathematics. The Greeks were the first to formulate it into a science. The deductive-postulational trend in mathematics originated at the time of Eudoxus and crystallized in Euclid's Elements. Eudoxus' theory of the geometrical continuum was an achievement paralleled only by the modern theory of irrational numbers more than two thousand years later. (2).

The Greeks were aware of the existence of "incommensurable quantities, and it may be that the difficulties connected with these quantities deterred the Greeks from developing the art of numerical reckoning achieved previously in the Orient. Instead, they forced their way through the thicket of pure axiomatic geometry.

Since the Greek mathematical thinkers were to play such an important role in the development of mathematics, we can say that this was the beginning of a two-thousand year "detour" in the history of science.

The weight of Greek geometrical tradition retarded the inevitable evolution of the number concept and of algebraic manipulation, which may be said to form the basis of modern science. (2).

After a long period of preparation, the revolution in mathematical thought began its vigorous phase in the seventeenth and eighteenth centuries. Greek geometry retained an important place, but the Greek ideal of axiomatic crystallization and systematic deduction seemed to disappear in a veritable orgy of intuitive guesswork and cogent reasoning, interwoven with nonsensical mysticism. (2).

In the ninteenth century, a need was felt for consolidation and greater security in the extensions of higher learning. This naturally led to a revision of the foundations of the new mathematics, particularly of the integral and differential calculus and the underlying concept of limit. The ninteenth century was thus characterized by a successful return to the classical ideal of precision and rigorous proof. Once more, the pendulum swung toward the side of logical purity and abstraction. (2).

At present we seem to be still in this period, though there are signs of disatisfaction among some mathematicians with the resulting unfortunate separation between pure mathematics and the vital applications.

The writer does not have the background, ability or inclination for a detailed philosophical or psychological analysis of mathematics; in any case, this is not the place for such a discussion. A number of people who have been studying mathematical creativity see a great danger in the prevailing overemphasis on the deductive-postulational approach to mathematics, feeling that the current trend will tend to stifle creative work.

Solving problems has, of course, fascinated many men in the history of mathematics and science. A few such men whose ideas seem appropriate to this study will now be discussed briefly, (in chronological order).

PAPPUS, (circa A. D. 300)

In the seventh book of his Collectiones, Pappus reports on a branch of study which he calls Analyomenos, which can be translated as "Treasury of Analysis" or "Art of Solving Problems". (3). Pappus is not the author of this study, which he ascribes to Euclid, Apollonius of Perga, and Aristaeus the elder. The techniques described are those of Synthesis and Analysis.

An excellent English translation of Pappus' text is available (4), but for our purposes, a paraphrase and condensation is perhaps more desirable.

Briefly, the two techniques (Analysis and Syntheses) are described as being useful for those who have studied the ordinary Euclid's elements and are desirous of acquiring the ability to solve mathematical problems.

In Analysis, we begin with what is required, take it for granted and draw consequences from it, and consequences from the consequences, until we reach a consequence which is already known or admittedly true. This is called Analysis or "solution backward" or "regressive reasoning". (4).

To put it more concretely, suppose we have a theorem A, to prove or disprove. We derive from A another theorem B. From B, another, C, and so on, until we come to a last theorem L, about which we have definite knowledge. If L is true, A is true. If L is false, A is false, provided all our derivations are reversible.

The same procedure can be applied to a "problem to find". We are asked for an unknown x satisfying a clearly stated condition. Assuming such an x exists, we derive a y, and so on, until we come to a last unknown z, which we can find, or which clearly does not exist. If z exists, we can retrace our steps and find x; if z does not exist, x does not exist. Again, all derivations must be reversible.

This point on the reversibility of the derivations is carefully pointed out as a consequence of Aristotelian logic that correct conclusions can come from false hypotheses. (4).

In Synthesis, we start from the point which we reached last of all in the Analysis, the thing already known or admittedly true. We derive from it what preceeded it in the analysis, and go on making derivations until we arrive at what is required. This procedure is called Synthesis, or "constructive solution" or "progressive reasoning".

For a more complete discussion of Pappus' manuscript, with examples, the reader should see reference (3).

KEPLER, (1571 - 1630)

Kepler does not fit the usual picture of a scientist who makes world-famous discoveries. He was a slow thinking man, and an indifferent calculator; his successes came about through a number of facilities, chief of which was unending patience. The thinking processes in problem solving according to Kepler are:

1. Bold guessing as the basis for fertile suggestions.
2. Erroneous guessing, for "all who discover truths must have reasoned upon many errors to discover each truth."
3. Skill in devising means of testing the truth of guesses.
4. Willingness to abandon an erroneous guess or hypothesis. (5).

DESCARTES, (1596 - 1650)

René Descartes was a philosopher, as well as a mathematician. It was his intention, it is thought, to give a universal method for solving problems. This plan never came to fruition, but parts of it are evident in several of his manuscripts and personal writings. Best known of his writings in this area is probably Discours de la Méthode, but ideas more directly applicable to problem solving may be found in Regulae ad Directionem Ingenii, or Rules for the Direction of the Mind, a somewhat earlier manuscript. (4).

The following lines of Descartes seem to describe the origin of the "rules":

"As a young man, when I heard about ingenious inventions, I tried to invent them by myself, even without reading the author. In so doing, I perceived, by degrees, that I was making use of certain rules." (6).

There were originally to be thirty-six rules, but only twenty-three exist, and some lack explanatory espositions. The work is probably unfinished because of its defects such as rambling, repetition, and inconsistency, but it was written when Descartes was thirty-two years old, and he was still feeling his way. (6). Of the twenty-three existing rules, only I, II, III, IV, and VI actually need concern us here. The others are somewhat abstract philosophy.

Rule I: The ultimate aim of study should be to guide the mind so that it can pass solid and true judgements on all that comes before it. (6).

Rule II: We ought to study exclusively subjects which our mind seems competent to know with a certainty beyond all doubt. (6).

Rule II makes it clear that Descartes is interested only in intellectual activity, or science in the abstract. A far different view is ascribed to Professor Einstein:

"Insofar as the propositions of mathematics refer to reality, they are uncertain; and insofar as they are certain, they do not refer to reality." (7).

Rule III: In whatever subject we thus propose, we must enquire not what others believe, but only what can be clearly perceived or with certainty inferred, these being the only ways in which genuine knowledge can be acquired. (6).

The essence of Descartes method is: to admit no step which is not self-evident, and in moving from step to step, to follow the inevitable logical order. (6).

Rule IV: We have next to consider why a method is necessary for investigating the truth of things, what it can hope to do, and on what rules it should proceed.

In the exposition for this rule, Descartes warns that the most intricate problems should not be attacked until the elementary difficulties have been resolved.

Rule VI: If we are to distinguish the most simple things from the most complex and to advance in the right order, we must proceed as follows: taking any sequence of truthsdeduced one from another, we must observe which is the simples, and how the rest are related to it - whether more, or less, or equally removed from it. (6).

The exposition for this rule brings us back to the original form of Descartes' method. It remains as always, the resolution of a complex into absolute simples, simply related. (6).

LEIBNITZ, (1646 - 1716)

Leibnitz planned to write an "Art of Invention", but never carried through his plan. Numbers of fragments from his works show that he had interesting ideas about the subject; the importance he attached to this field can be seen in this quotation:

"Nothing is more important than to see the sources of invention, which are, in my opinion, more interesting than the inventions themselves." (3).

BOLZANO, (1781 - 1848)

Bolzano devoted an extensive part of his comprehensive presentation of logic, Wissenshaftslehre, to the subject of heuristic (vol. 3, pp. 293-575). He writes of this section:

"I do not think at all that I am able to present here any procedure of investigation that was not perceived long ago by all men of tallent.... I shall take pains to state in clear words the rules and ways of investigation which are followed by all able men, who in most cases are not even conscious of following them." (3).

In other words, a person who has not discovered these universal procedures may be helped by consciously learning them; perhaps as a series of questions to be asked of oneself or the student, as suggested by Professor Polya (3).

CHAPTER III

INDUCTIVE AND INTUITIVE REASONING

It is important to separate the two terms inductive reasoning, and mathematical induction. The similarity in names in somewhat regrettable, because there is little logical connection between the two.

Induction is the process of discovering general laws by the observation and combination of particular instances. It is upon induction or inductive reasoning, that almost all actual life problems are solved. Mathematical induction is a method of proving certain mathematical theorems. When correctly applied it gives as hard and fast a demonstration as any method of formal proof. Another name for this procedure is "proof from n to n+1" or "passage to the next integer." (3).

The ideas of inductive reasoning can, perhaps, best be explained by a few examples. First, let us be clear; inductive reasoning in no way proves anything. All such reasoning can do absolutely is disprove something. Disproving something by induction is known as finding a counter-example. As far as positive proofs are concerned, induction can present evidence of greater or lesser weight in favor of a particular conclusion. The point at which the evidence becomes strong enough to convince will vary greatly. A strict, formal mathematician will never be convinced. The ordinary person will probably be convinced rather quickly, and rather too quickly many times. A balance between the extremes is necessary.

Perhaps one reason why formal mathematicians will not recognize inductive evidence is summed up by the following:

> The element of constructive invention, of directing and motivating intuition is apt to evade a simple philosophical formulation; but it remains the core of any mathematical achievement, even in the most abstract fields. If the crystallized deductive form is the goal, intuition and construction are at least the driving forces. (2).

Over and over again in historical and modern writings the theme is repeated:

> Mathematical truth is <u>discovered</u> inductively, requiring imagination and insight. This truth is then <u>established</u> deductively, requiring rigor and care. (8).

One must be open-minded about possibilities, and tough-minded about proof.

An example of inductive reasoning might be found in a person who picks up a magnet for the first time. Assuming that he is in a place where a number of ferrous metal objects are fixed in place, he would soon discover that the magnet would cling to each of them. Since the objects are all fixed, he might inductively conclude that metals attract magnets. He would, of course, be incorrect. <u>He is reasoning from too few and too special cases</u>. When he finds a piece of small, unattached ferrous metal, he will discover the error, and will be forced to modify his conclusion to magnets attract metals. He would again be incorrect, for when he came upon a piece of metal containing only, say, copper and tin, no attraction would take place. Again, a modification must take place in his conclusion. If he has sufficient knolwedge of metallurgy, he may soon reach a conclusion which is generally correct.

This example, simple and simplified as it is, brings out a number of important features about induction. First: the conclusion based on a small number of cases is weak. Yet, mere numbers of cases are not the answer, because his conclusions based on trying ten-thousand steel objects would still be wrong. Second: the more different the instances examined, the stronger the conclusion, and conversely. Third: the baseball advice to "stay loose" is valuble, because the next case may necessitate a change in the theory.

In general, the conclusion, once arrived at from a given number of cases, must be tested against new instances. The more different, new instances for which the conclusion holds, the more evidence for the conclusion.

EULER

As mentioned before, there was in the seventeenth and eighteenth centuries, among mathematicians "a veritable orgy" of intuitive guesswork. It is the contention of Professor Polya that guesswork, If backed up by a good weight of plausible reasoning deserves very serious attention from creative mathematicians.

As a positive example of this type of research, we might look at some of the work of Leonhard Euler (1707 - 1783). A master of inductive research,

> He made important discoveries in theory of numbers, infinite series, and other branches of mathematics by induction, that is, by observation, daring guess, and shrewd verification. (3).

This does not, of course make his work unique, but he is almost unique in that he takes pains to present the relevant inductive evidence carefully, in detail and in good order. He was satisfied as to the truth of a proposition by a weight of inductive evidence and said so, while still trying for a demonstrative proof. In one exposition of such a discovery, for example, he notes:

> "We have thus discovered that these two infinite expressions are equal even though it has not been possible to demonstrate their equality. All conclusions which may be deduced from it will be the same, that is, true but not demonstrated." (3).

AN "ALGEBRA" OF PLAUSIBLE REASONING

Some of the objections of rigorous, demonstrative mathematicians to non-demonstrative methods of discovery and proof may be overcome by the development of an "algebra" of plausible reasoning, or Heuristic. (A discussion of the meaning of the word heuristic will be given in Chapter V. Here it is simply a convenient title.) A development of this in any detail is both impossible and out of place in a discussion of this sort; however, a brief look seems indicated. The interested reader will find a thorough presentation in reference (9), chapter XIII.

Traditional algebraic logic (demonstrative logic) has a pattern which is:

$$\begin{array}{c} \text{A implies B} \\ \underline{\text{B false}} \\ \text{A false} \end{array}$$

Now suppose we have A implies B, and B is true. It is certainly logical to say that A, while not necessarily either true or false, is more credible. Let us put that as the conclusion: A more credible.

Other comparisons between demonstrative and heuristic patterns, (demonstration on the left, heuristic on the right), are:

$$\begin{array}{c} \text{A implied by B} \\ \underline{\text{B true}} \\ \text{A true} \end{array} \qquad\qquad \begin{array}{c} \text{A implied by B} \\ \underline{\text{B false}} \\ \text{A less credible} \end{array}$$

Our confidence in a conjecture can only diminish when a possible ground for the conjecture is exploded. (9).

$$\begin{array}{c} \text{A incompatible with B} \\ \underline{\text{B true}} \\ \text{A false} \end{array} \qquad\qquad \begin{array}{c} \text{A incompatible with B} \\ \underline{\text{B false}} \\ \text{A more credible} \end{array}$$

Our confidence in a conjecture can only increase when an incompatible rival conjecture is exploded. (9).

Important to the development of classical logic are the equivalence statements: A false equivalent non-A true; A implies B equivalent non-B implies non-A and others. (9). The pattern excludes the two heuristic conclusions, more credible and less credible. However, by widening the domain of formal logic to include the equivalence "non-A more credible equivalent A less credible", a pattern of shaded inference can be developed which is weaker than the demonstrative, but still useful. (9).

CHAPTER IV

EXPERIMENTAL STUDIES

In general, problem solving is judged on the basis of the products, which are visible, tangible, and easily checked.

> To study productive thinking where it is most conspicuous in great achievements is certainly a great temptation. On the other hand, lightning is the most striking example of electrical discharge, but in the laboratory the laws behind such a display are better investigated as sparks. (7).

In a study of problem solving, we are not interested in the solution; we are interested in the mental processes which lead to the solution, and comparing these successful processes to unsuccessful ones. Professor Polya's list of questions, which will be discussed in detail later, are designed to be indicative of certain mental processes which will frequently lead to correct results.

Many times these mental processes have been put forth on the basis of experience and observation. There have also been a few experimental studies designed to "get at" these processes.

One of these studies was made by Burack (10), who studied nine methods of attacking problems in reasoning. (A problem non requiring reasoning is not a problem, and can be handled by a machine.) The nine are:

1. Clear formulation of the problem.
2. Preliminary survey of all aspects of the presented material.
3. Analysis into major variables.
4. Locating a crucial aspect of the problem.
5. Application of past experience.
6. Varied trials.
7. Control - holding one or more variables constant.
8. Elimination of sources of error.
9. Visualization.

Three problems were used. One, and"induction" problem, involving selecting the groups of five letters which are not like the others. Two, a deduction problem, involving a detective story, and three, a geometric figure puzzle problem. The idea was to discover which methods were used the most on these problems, and how effective they were.

Three of the methods were used on the induction problem. From three to eight were used on the deduction problem and from two to seven on the geometric figure problem. (10).

This experiment was carried out on an undergraduate class in advanced psychology, and it was thus expected and found that the percentages and numbers of correct solutions were high. All of the methods studied were used at least once with some success. The least successful students were those who used only a few of the methods of attack.

One very possible cause for this is that repeatedly starting a problem in a way which will either not help or actually hinder solution will quickly lead to boredom, giving up and wild guessing.

Another revealing experimental study of problem solving was carried out by Bloom and Broder (11).

This study, which is perhaps a little closer to home for high-school students because the subjects were freshmen and sophomores in college, used the well-known and widely used technique of having the students verbalize their mental processes as they attack a problem. The method is not, of course, highly satisfactory because of the inability to make such verbalizations rapidly and in some cases at all.

Nevertheless, some interesting results comparing the successful with the non-successful problem solvers are brought out. These comparisons are made in four categories:

1. Understanding the nature of the problem.
2. Understanding of the ideas contained in the problem.
3. General approach to the solution of the problem.
4. Attitude toward the solution of the problem.

In category one, differences were found in two major points: in their ability to start a problem and in their ability to solve a problem in its own terms. A non-successful problem-solver would be unable to pinpoint a key word or phrase to start with. In addition, the non-successful solver would frequently present an acceptable or correct solution to the problem he attacked which was not the given problem. In general these two difficulties arose because of inability or unwillingness to read the directions with care and understanding.

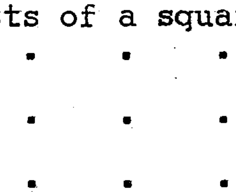
In category two, it was found that the non-successful problem-solvers frequently had all the background and information necessary to the solution, but seemed unable to relate the information to the problem, especially if the problem material was in a form different from the form they had encountered previously. This ability to put relevant knowledge to use is apparently related to the individual's self-confidence.

In category three, three basic differences were found: extent of thought about the problem, care and system in thought about the problem, and the ability to follow through on a process of reasoning. The non-successful group tended to be completely passive in their thinking about a problem and to select an answer because none of the others appeared attractive. Their attack tended to be random, and with no basic plan. In addition, they would frequently start correctly and, for no apparent reason, suddenly stop and give up.

In category four, several differences came up. The non-successful problem-solvers were inclined to believe that a person knows an answer at once or not at all. They had little or no confidence in their ability, refusing to attack problems which appeared complex or abstract. Further, they were unable to maintain an objective attitude on certain problems because of personal opinions. (11).

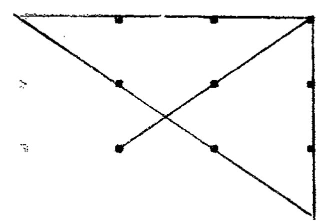
As an example of a problem which illustrates a difficulty for many people, let us look at the following.

The problem consists of a square array of nine points.



The instructions say to start at one of the points and connect all nine of them without raising the pencil and using only four straight line segments.

Here is the correct solution.



The average person will assume that the lines must be contained within the area of the square. There is no basis for such an assumption! Self-imposed restrictions are often the most weakening and frustrating of all.

CHAPTER V

CAN PROBLEM SOLVING BE TAUGHT?

People who have worked in this field agree that problem solving can be taught - subject to a number of limitations. There is no way to teach a person with an IQ of 75 to solve problems in integral calculus. There are, however, techniques by which the problem solving ability of any person not already working at full capacity can be improved; in some cases quite markedly. As far as the researcher could discover, there is unanimous agreement up to this point.

When we attempt to consider just how problem solving ability can be increased, there is still agreement, but in a more general way. The agreement is that problem solving abilities are increased by consciously learning and applying the methods which successful problem-solvers have found themselv using through the years; the amount of improvement will be roughly equivalent to the extent the subject was not already using these principles.

What are these principles? Well, as Hamlet said, that is the question. We will consider later two systems outlining such processes, which use two somewhat different viewpoints. Just not, however, let us look at three general approaches found in (12), (5) and (3).

Correct and skillful problem solving can be brought about through four mental attitudes. No one knows an absolute process for bringing about these attitudes, but recognizing them should be of assistance. The four: incentive, self-confidence, interest, and common sense. (12).

These four points are certainly logical. Unless a person wants to solve a problem and has an interest in the solution, he is unlikely to be successful. It is also fairly certain that a person who doesn't think he can solve a problem, can't, because he won't start; the self-confidence must be tempered by critical judgement, naturally.

Parker (5), gives the following suggestions for a teacher conducting problem solving lessons.

1. Aid the pupils to define the problem clearly.
2. Keep the problem clearly in mind.
3. Stimulate suggestions:
   a) Analyze the problem into parts or elements.
   b) Recall previously known similar cases.
   c) Formulate definite hypotheses or tentative plans from vague guesses.

Providing that pupils are seriously concerned with their problems, courageous guessing or leaps into mental darkness are to be encouraged. When a number of such guesses are accumulated, they can be called multiple hypotheses. These should then be investigated with an eye to verification or disproof. A good idea is to use "Occam's Razor"*, that is to prefer the simpler hypothesis until additional evidence one way or the other arises. (5).

4. Evaluate suggestions:
   a) Be open-minded, maintain the state of doubt.
   b) Criticize all suggestions.
   c) Verify or discard solutions by reference to facts as revealed by miniature experiments or in authoritative sources of information.

The teacher must be a model of impartiality in taking and evaluating suggestions.

One of the most valuble results of problem solving lessons

---

* William of Occam, (1280 - 1349). English Schoolman, known as <u>Venerablis Inceptor</u>.

can be the teaching of the student to criticize and anticipate mentally the consequences of a suggestion or scheme. (5).

5. Keep the discussion organized.

The teacher must have a broad, flexible grasp of the problem example being considered and keep suggestions from going too far afield. (5).

Polya (3) gives the most complete analysis of the role of a teacher trying to teach problem solving. The main points are:

First, and most important, give the student a desire to solve the problem.
Second, the teacher should help, but not obtrusively, and not too much; the student should have a fair share of the work.
Third, the help is best given by asking questions which could have occured to the student alone.
Fourth, problem solving is a practical skill, such as swimming. Any practical skill is learned by imitation and practice. (3).

Dadourian (12) also compares problem solving to swimming, commenting on the analogy between fear of water before learning to swim, and fear of problems, before learning to solve them.

Below are two quotes from (3), which pretty well sup up Professor Polya's ideas about teaching problem solving.

The purposes of such teaching are two-fold: first, to help the student solve the problem at hand, and second, to develop the student's ability so that he may solve future problems by himself. The questions and suggestions given have two characteristics in common - common sense and generality.... If a student succeeds in solving the problem at hand, he adds a little to his ability to solve problems.

If the teacher wishes to develop in his students the mental operations which correspond to the questions and suggestions on our list, he puts these questions and suggestions to the students as often as he can do so naturally. Moreover, when the teacher solves a problem before the class he should dramatize his ideas a little and put to himself the same questions and suggestions he uses with the students. Thanks to such guidance, the student will eventually discover the right use of these questions and suggestions, and doing so he will acquire something that is more important than the knowledge of any particular mathematical fact. (3).

The writer would like to recommend highly the book by Professor Polya, How to Solve It, (3), for all teachers. He gives a number of examples of dialog between teacher and students going over a problem. The idea is not new, but these dialogs are unique in the sense that Professor Polya recognizes that a teacher will quite often get no answer to his most provocative and well thought out question but a blank stare.

The teacher trying to lead a class to a solution through questions must be prepared to meet the disconcerting silence of the students. To devise a successful plan takes former knowledge, good mental habits, concentration, and - good luck. (:

SYSTEM NO. 1. H. M. DADOURIAN (5)

General directions for solving problems.

1. Don't be afraid of the problem. (Which is excellent advice, but perhaps a bit hard to follow.)
2. Read the problem carefully and determine what are given and what are required.
3. Restate the problem in its bare outlines, leaving out incidental details.
4. Formulate a plan of action, a strategy.
5. Using an appropriate system of notation, assign a symbol to each of the given and required magnitudes.
6. If the problem admits of a figure, draw a suitable one and label each of the given and required magnitudes with its symbol.

7. Make a table of the given and required magnitudes.

8. Write all of the principal equations necessary for the solution of the problem before manipulating any of them.

9. Solve the equations simultaneously for the required magnitudes, and obtain an expression for each in terms of the given, and only the given, magnitudes.

10. In case there is only one, or a principal required magnitude, start with its mathematical definition or expression when ever convenient.

11. Write successive expressions of a required magnitude in the "columnar" form. Follow the straight-line method; avoid the zig-zag method.

12. Before taking a new step in the analysis, put the last expression in as simple for as possible.

13. Discuss the final literal equation which gives a required magnitude, in terms of the symbols of the given magnitudes.

14. Solve numerical problems as litteral problems first, and introduce the numerical data after the final expressions of the required magnitudes are obtained in terms of the symbols of the given magnitudes.

15. Use your common sense at every step of the analysis. At each step ask yourself if the step you have taken is sensible and then, at the end, whether the final result is reasonable.

SYSTEM NO. 2. G. POLYA

Professor Polya's book, How to Solve It (3) is the result of his attempt to revive heuristic "in a modern and modest form".

Heuristic or heuretic, or "ars inveniendi" was the name of a certain branch of study, not very clearly circum scribed, belonging to logic, or to philosophy, often outlined, seldom presented in detail, and as good as forgotten today. [Some important past studies were reviewed in chapter II._7 The aim of heuristic is to study the methods and rules of discovery and invention. (3).

Heuristic, as an adjective, means "serving to discover."

Heuristic reasoning is reasoning not regarded as final and strict, but as provisional and plausible only, whose purpose is to discover the solution of the present problem. (3).

Heuristic reasoning is good; everyone uses it to a greater or lesser extent. What is bad is an attempt to substitute heuristic reasoning for rigorous proof. Very few mathematicians present or past, recognize the existence of heuristic arguments; Euler (see chapter III) was a notable exception.

The study of heuristic should lead to a better understanding of the mental processes typically useful in solving problems. It should, therefore, "exert some good influence on teaching, especially on the teaching of mathematics." (3). It should be noted that the generality of the subject applies to problems of any sort, up to and including crossword puzzles. If interested, the reader can find examples in (3).

Professor Polya (3) divides the work in solving a problem into four parts; understanding the problem; devising a plan; carrying out the plan; and looking back at the solution. (3).

Every teacher has had experience with students who are able to give an answer to a problem seemingly without intervening steps. Such "bright ideas" are, of course, extremely desirable as long as our interest is in the solution. However, so often a classroom situation teaching mathematics is somewhat artificial in the sense that we may be interested not in the answer, but the process by which we arrive at the answer. Therefore it would seem wise to bypass the "bright ideas" (without discouraging them, of course) and follow all the steps.

> The worst may happen if the student embarks upon computations or constructions without having understood the problem. It is generally useless to carry out details without having seem the main connection, or having made a sort of plan. (3).

The most difficult part of the solution is found in devising a plan. In assisting, the teacher must be careful not to simply give a plan to the student. Unless he has had at least a part in devising the plan himself, he is likely to forget it, especially in a long and involved problem. In the list of suggested questions which follows, the idea of varying the problem, or changing the approach is very important.

> 'Insistent' analyses of the situation, especially the endeavor to vary appropriate elements meaningfully sub-specie of the goal, must belong to the essential nature of a solution through thinking. We may call such relatively general procedures Heuristic methods of thinking. (7).

We again find agreement on looking back at the solution. Professor Polya is more insistent on this point than the others. It is valuble to consolidate our knowledge by trying to solve it another way, to check each step, to find uses and applications for the problem in new problems.

> Good problems and mushrooms of certain kinds have something in common; they grow in clusters. Having found one, you should look around; there is a good chance that there are some more near. (3).

HOW TO SOLVE IT LIST (3).

Understanding the problem

First: You have to understand the problem.

What is the unknown? What are the data? What is the condition?

Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce suitable notation.

Separate the various parts of the condition. Can you write them down?

Devising a plan

Second: Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should eventually obtain a plan of the solution.

Have you seen it before? Or have you seen the problem in a slightly different form?

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem? Could you restate it still differently? Go back to definitions.

If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined? How can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are

are nearer to each other?

Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

Carrying out the plan

Third: Carry out your plan.

Carrying out your plan of the solution, check each step. Can you see clearly that the step is correct? Can you prove that it is correct?

Looking Back

Fourth: Examine the solution obtained.

Can you check the result? Can you check the argument?

Can you derive the result differently? Can you see it at a glance?

Can you use the result, or the method for some other problem?

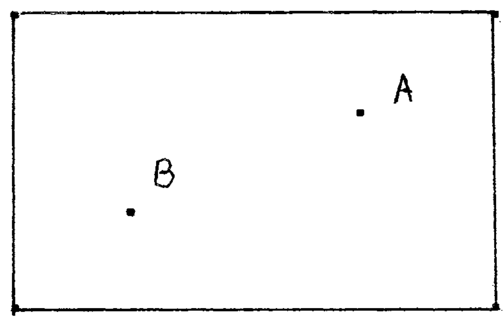
EXAMPLE

It is the purpose here to show the solution of a problem by conscious application of the techniques so graphically described in How to Solve It (3).

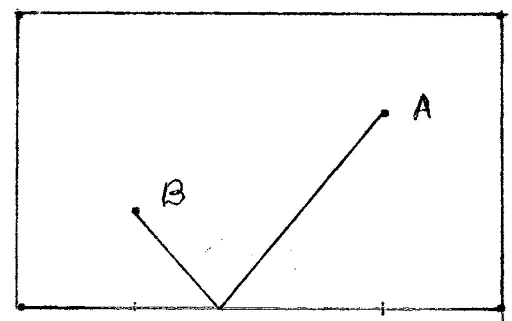
The problem is one taken from Plane Geometry by Welchons and krickenberger, fourth edition. The writer was once asked about the problem by a student in a class of plane geometry, but was unable to dwell on it at that time. Thus, the solution was unknown at the beginning of this study.

The problem is: there are two balls on a billiard table. Find the point on one bank that one ball must hit in order to rebound and hit the other. (Figure 1)

For our purposes the ball can be represented by a point, and the bank by a straight line. The solution will then be a point on that straight line. It is necessary to assume the table perfectly level, and that no "english" is to be imparted to the ball.



The problem

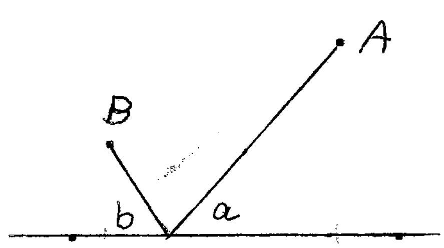


The unknown

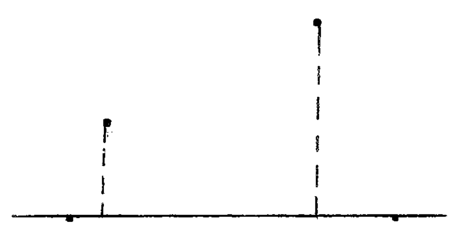
Figure 1

Do you know a theorem that can be useful/ Not exactly a theorem, but we do know that a ball will rebound from a bank in an angle equal to the angle at which it hits the bank. (Figure 2)

Can we introduce some auxiliary elements to help in the solution? The point lies on a line segment formed by perpendiculars to the bank from each ball. (figure 2)



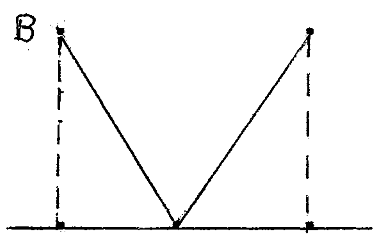
Angle a = angle b



Auxiliary elements

Figure 2

Consider the extreme cases. These are two; one with the balls on a line parallel to the bank and another with one ball on the bank. (Figure 3)



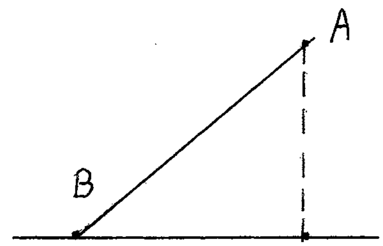


Figure 3

It is obvious from these two cases that the point is midway between the feet of the perpendiculars in case I, and "moves" toward the foot of the perpendicular from the ball which is closer to the bank.

We can find the mid-point or divide the segment up into any given ratio, but the variation here is continous and there is no apparent way to find the correct ratio for a given position of B. Is there another way to find the midpoint? Yes, we can drop a perpendicular from the intersection of the diagonals of the rectangle ABCD. (Figure 4)

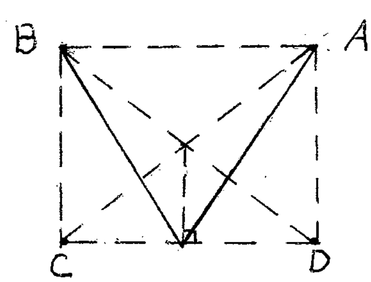
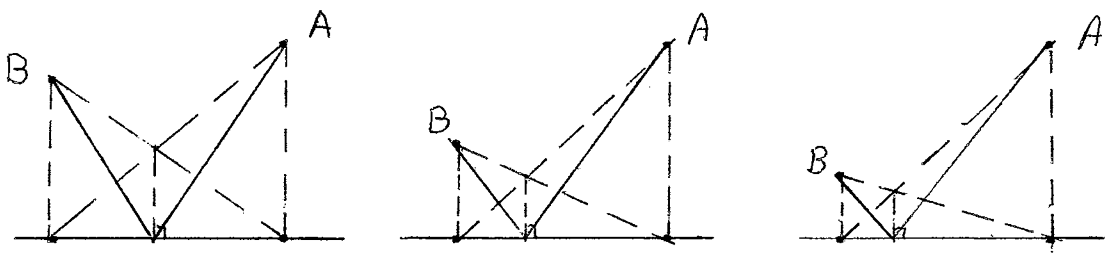


Figure 4

Can we make a conjecture applying to the general case? Well, the lines corresponding to the diagonals can be drawn, and their point of intersection used to find a point on the bank. (Figure 5)



Three trials of our conjecture

Figure 5

Is this the solution? Well, it seems logical. It solves the extreme cases, and the point moves in the right direction as we move away from the extreme case.

Is it proved? No. Trial and a scale drawing will give additional evidence, but they are not proof, of course.

However, a geometric proof can be obtained. Figure 6 gives the picture. It is possible to prove that triangles ADS and BCS are similar and therefore angle ASD is equal to angle BSC. The algebraic steps are not pertinent to our discussion are are omitted.

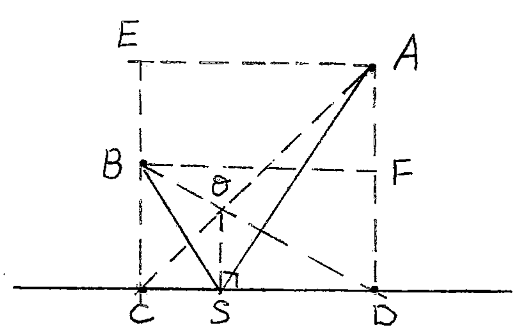


Figure 6

Is there another way of finding the solution? Yes, there is. It can be found by trying to imagine a problem which is similar to ours, and with a similar solution. The rebound of the ball and the equality of the angles strongly suggest the angles of incidence and reflection of a ray of light from a reflecting surface. (Figure 7)

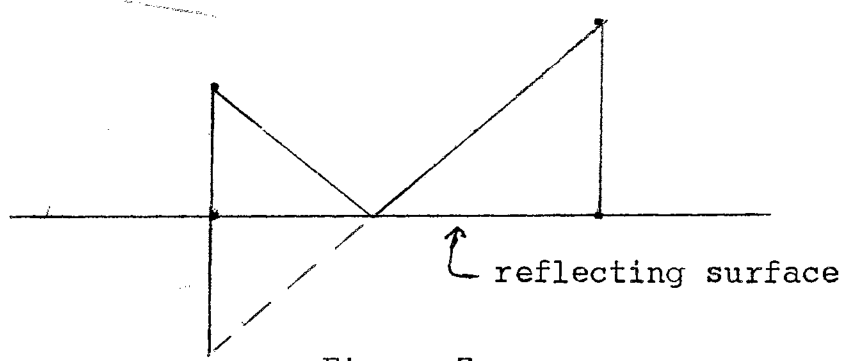


Figure 7

This situation applys perfectly to our billiard table problem, and is actually an easier method.

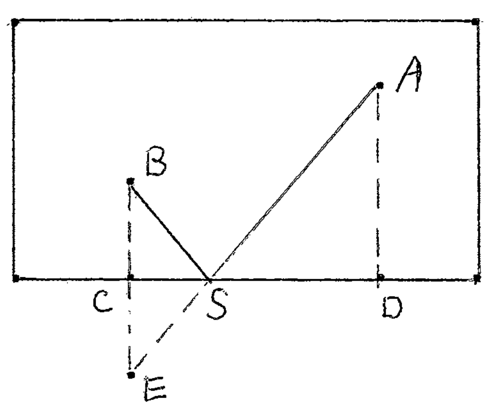


Figure 8

The proof this time is comparatively simple, and again omitted.

CHAPTER VI

SUMMARY AND CONCLUSION

Any animal with any brain or facsimile therof has experience in and a history of solving problems. Existence itself is a problem; survival is a problem.

> Life is, of course, among other things, a sum total of solution-processes which refer to immediate problems, great and small. (7).

Museums are full of fossil remains of creatures that do not today exist because they did not yesterday solve the problem of survival.

The most successful solutions to the problem of survival have been turned in by certain members of the so-called "lower orders". The turtle, the shark, and insects of various kinds are examples. These successful problem solvers lack, however, that quality which is commonly called intelligence; a term which will not be defined herein. This would seem to indicate that a "final" solution to the problem of survival, leads to a static existence, and either halts or greatly retards evolution.

The animal that has evolved most rapidly is man himself. He has never become a victim of the final solution to the problem of survival, but has evolved, (or obtained, or been given), an instrument especially for the purpose of providing solutions to any problems that come up. Presumably this instrument, (called the mind), can provide long-range solutions and short term solutions. As we know, it can provide both good and bad solutions. The keyword in describing this mind's success as a problem-solver is Flexibility.

It may be constructive to consider the procedure by which an ant attempts to solve a problem. The ant is an extremely well-adjusted animal, and is occasionally pointed out in imaginative fiction as man's successor.

Be that as it may, let us suppose that an ant is carrying some morsel of food back to the nest when he comes to a barrier over which he cannot climb without dropping the food. The persistent insect will try again and again, but each try will be a repeat of the last. It may occur to him to try to go around the barrier; even so, he will often go back and again try to climb. Eventually, the ant will be able to solve most such problems, but the key word in his procedure is patience. If a man were willing to expend the same amount of patience on a comparable problem he would be as successful. However, man's strong point is not patience. On this we can agree, without necessarily agreeing about what his strong points are.

It is true that man tends to shy away from intellectual difficulties more than physical. There is a logical, though probably unsound, biological reason advanced for this. The brain is the last of the organs to evolve; therefore, man is not as efficient in the use of his brain as his hands, because he has not had as much practice with the former. (12).

Be that as it may, a man faced with a difficult problem, one that is not readily solved, with frequently give up, unless there are some definite signs of progress. We have looked briefly at some systems designed to give a problem-solver tools with which to attack a problem. With a wide repertoire of such devices, there is better chance for success because there will usually be some new way of starting. The lists of questions and rules contained herein were designed from observation, personal experience and occasionally from experimental data. Their aim is to lead the user to the correct mental attitudes that have been found successful in solving problems. No claims are made for certainty; the concensus of opinion is that none is possible.

Certain great men of science and/or mathematics who recognized the importance and the possibilities inherent in a systematic method for solving problems have been briefly reviewed. Several modern authors have been discussed. The unusual point about these attempts at systematizing problem solving is how similar they all are: no major disagreements anywhere, and complete agreement in all basic ideas.

The individual system which seems to the writer to be the best is that outlined by Professor Polya. This opinion which is purely subjective of course, seems substantiated by the depth of thought, the aptness of application and the universality of the methods.

That is, however, one of the basic tenets: the solution of most problems can be approached through many paths; though there is a longest and shortest, there is no right way or wrong way.

BIBLIOGRAPHY

(1) Wilder, Raymond L., Introduction to the Foundations of Mathematics. New York: John Wiley and Sons, Inc., 1952.

(2) Courant, R. and Robbins, H., What is Mathematics? New York: The Oxford University Press, 1941.

(3) Polya, G., How to Solve It. Garden City, New York: Doubleday and Co., Inc., 1957.

(4) Heath, T. L., The Thirteen Books of Euclid's Elements. New York: Dover Publications, Inc., 1956.

(5) Parker, S. C., "Problem Solving or Practice in Thinking", The Elementary School Journal, Vol. XXI, 1921.

(6) Joachim, H. H., Descartes Rules for the Direction of the Mind. London: George Allen and Unwin Ltd., 1957.

(7) Duncker, K., "On Problem Solving", Psychological Monographs, No. 270, 1945.

(8) McCreery, Louis R., "Let Us Teach Our Students About Mathematics", Mathematics Magazine, Vol. XXXIV, (Jan. - Feb., 1961)

(9) Polya, G., Mathematics and Plausible Reasoning (two vols.) Princeton, New Jersey: Princeton University Press, 1954.

(10) Burack, Benjamin., "The Nature and Efficacy of Methods of Attack on Reasoning Problems", Psychological Monographs No. 313, 1950

(11) Bloom, B. S. and Broder, L. J., Problem Solving Processes of College Students. Chicago: The University of Chicago Press, 1950

(12) Dadourian, H. M., How to Study: How to Solve. Cambridge: Addison-Wesley Press, Inc., 1949.

VITA

David Logan Travis

Candidate for the Degree of

Master of Science

Report: Problems in Solution

Major Field: Natural Science

Biographical:

Personal Data: Born in Clovis, New Mexico, December 30, 1935, the son of R. L. and Katie Pearl Travis.

Education: Attended grade school in Clovin, New Mexico; graduated from Clovis High School in 1953; received Bachelor of Science degree from New Mexico State University, University Park, New Mexico, with a major in secondary education, in January, 1958; attended Eastern New Mexico University, and STanford University; completed requirements for the Master of Science degree at Oklahoma State University in August, 1961.

Professional experience: Mathematics instructor in Grants High School, Grants, New Mexico, Spring, 1958; Mathematics instructor in Tombstone High School, Tombstone, Arizona, 1958 - 1960; Mathematics instructor for the Department of the Army, Fort Huachuca, Arizona, (evenings) January, 1959 to June, 1960.