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A STUDY OF THE JOB STREAM THROUGH A CLINICAL LABORATORY

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CHAPTER I

INTRODUCTION

Overview of Clinical Laboratory Development

The modern clinical laboratory over the past ten years has witnessed an information explosion due to the technological breakthroughs in the development of new automated test equipment and a workload which has been doubling every five years for the past two decades.

In 1962, Peacock [28] reported a 10-25% yearly increase in most progressive laboratory test loads. He also reported that the technological solution of automatic instrumentation has provided the means of absorbing the rapid rate of growth over manual methods, but poses the related problem of handling large amounts of data generated by the automated test equipment.

Since that time, it has been well documented that the workload in the clinical laboratory has doubled approximately every five years during the past two decades. Kinney [22], Chairman of the 1966 Quail Roost Conference on Automation and Clinical Chemistry, identified the future trend in automation of clinical laboratories as that of incorporating automatic data processing and machine-readable data handling in all phases of the analytic process, from the physician's order, to the receipt of the patient's laboratory result.

Wattenburg [34] reports that the volume of clinical tests in the hospital laboratory is increasing at a rate of 20% yearly and while automated instruments have reduced the manual manipulation the technologist has to perform, he now spends an increased percentage of his time performing a clerical function of data handling.

This multifaceted set of events has a dramatic effect on the clinical laboratories. The once small, well-organized, efficient and properly staffed laboratory has now become a large, well-equipped laboratory, but inefficient in handling the surging volume of patient test data. Due to the heavy demand of the daily workload, most laboratories have purchased equipment based on their projected growth rate in number of tests per patient day, ignoring the concomitant increase for data collection, control and dissemination. The technological solution of automatic instrumentation poses the related problem of handling large amounts of data generated by automated instruments. The current dilemma of most modern laboratories is that they possess archaic systems to handle efficiently the increased pool of information as well as the rate in which information is derived from new clinical laboratory test equipment. Another dilemma is the lack of planning tools to aid the operation of the laboratory in optimizing the total equipment utilization in order to minimize job throughput time.

Since the work performed by the clinical laboratories provides valuable information toward confirmation of a physician's diagnosis, the time element in obtaining the desired information is a critical factor in the physician's ability to confirm the diagnosis and begin a therapeutic plan.

Even with highly sophisticated test equipment, Constandse [7] reports that 30-40% of the technologist's time is devoted to a clerical function of calculating test results, preparing worklists, transcribing of test results and generating patient reports. Paxton [27] confirms this percentage by stating that 3 to 3 1/2 hours per day, per technologist, is expended in working with pencil and paper in his hospital laboratory.

The logical questions which are facing clinical laboratory directors, pathologists and hospital administrators of today are:

1. Can an information handling system significantly reduce the clerical function of the laboratory technologist, thereby increasing his capacity to perform additional tests?
2. Can this system expedite accurate test information to attending physicians?

There exist many automated information systems in operation today in various hospitals and medical centers across the country, i.e., Johns Hopkins, University of Kentucky, University of Washington, University of Missouri, University of Wisconsin, Youngstown Hospital and others. A common criteria for justifying this type of system is as follows:

1. Laboratory demand for services will continue to rise 20% annually.
2. The pool of available trained technologists is not increasing.

3. Current Clerical demand of the technologist's time is about 30%.

4. Automated information handling techniques can provide significant reduction in the clerical function and increase the work capacity of the laboratory all within a constant labor force.

Ammer [1], et al, have investigated the productivity of clinical laboratories by relating skills and training of laboratory technologists to output. From a work sampling study, they were able to obtain basic data from which operational characteristics could be determined by the size of the laboratory. The results of this study indicated larger laboratories achieve a more efficient division of work, but that organizational structure and working environment have a strong influence on the laboratory's ability to produce. Another interesting result is that 80% of the laboratory technologist's time is spent on activities that cannot be directly related to any useful output.

Since 20% of their time is totally related to output, an increase in productivity can be realized by an increase in their capacity to process more work during their productive time without a degradation of accuracy or quality of work. We are interested in determining the measures of effectiveness by which job streams processed by a representative clinical laboratory model can be characterized.

Statement of the Problem

The scope of the problem is to develop a model representative of the functional elements whose behavior is characteristic of a large clinical laboratory. The model representing the large laboratory consists of both parallel and series service centers working in conjunction with each other to provide a prescribed sequence of services for each customer.

Classical queueing theory is used to develop an analytical formulation of the model and/or formulation of a simulation model. One of the problems is to measure the effect that automatic data acquisition will have on the job stream throughput. Another problem is to measure the operational characteristics of the various combinations of service channels found in clinical laboratories. Priorities of work are common constraints which can have an effect on the job stream throughput as well as overall operation. The behavior of the simulation models are observed after alteration of these constraints.

An attempt is made to derive the expected transient queueing time for the single server model with constant service time and relate this measure of effectiveness to the queueing time derived by methods of simulation. Another consideration is determining whether the behavior of a small segment of a complex system studied in isolation can contribute information about the total complex system.

Three dynamic simulation models representing the work flow through a clinical laboratory at different levels of complexity are also developed to identify those events or activities which may inhibit optimal throughput.

CHAPTER II

REVIEW OF PREVIOUS RESEARCH

The clinical laboratory can be envisioned as a network of independent service channels, each performing a separate function. Customers entering the system may require service from one or more of the service channels. Each service channel may have one or more service centers. Customers arriving at a service center constitute the input, and the amount of time spent in servicing the customer is a factor in determining the output rate from the service center. In most cases the output from a service center forms the input to future servers. In other cases the output forms the output of the total production process. In either case one is concerned with determining expressions for various measures of effectiveness related to both the input and output process.

The single server system is the smallest component of the total network whose behavior can be studied in isolation. A review of previous research reveals that measures of effectiveness for classical single server Markovian queueing models have been generally determined under conditions of steady state. Kendall's [20] notation is used when referencing Markovian type systems. Some of the measures which have been used to describe system behavior are expected arrival time, expected queueing time, expected queue length, expected service time, expected departure time, etc.

In determining the behavior of the queueing network under consideration the output process from the single server is an important characteristic to analyze, since the output of one server serves as the input to another.

Burke [4] and Reich [31] show that for an M/M/1 queueing system the output process is again a Poisson process with the same parameter as the input process. Finch [14] measures the output of the single server model in terms of departure intervals. He shows that, under the appropriate conditions, two successive departure intervals are independent, in the limit, and that the queue size left behind by a departing customer is independent, in the limit, of the length of the interval since the previous departure.

Disney [11] expands Burke's [4] and Finch's [14] work by showing that, under the appropriate conditions, the sequences of departure times are independent, identically distributed random variables, i.e., the process is a renewal process. He points out that in queueing networks, one cannot in general, treat the isolated server under the assumptions of classical queueing theory except perhaps in the case where all servers are Markovian and all waiting line capacities are infinite.

The behavioral problems of the M/M/1 system in series aim at providing some understanding in the study of a total complex queueing network. Jackson [18] considers a queueing system with K phases where the output from the (K-1st) phase forms the input to the Kth phase. He assumes a negative exponential service time distribution at each phase as well as a Poisson output and/or input at each phase. Using a differentialdifference technique, he develops closed form steady state expressions for the average number of customers waiting for service, the average number of customers being served, and the average total number of customers in the system. Steady state solutions are also determined for the case when the queue size is finite.

The expected waiting time in the queue is another measure of effectiveness which has been used in describing the behavior of series queueing systems. Reich [31] shows that if the first queue has an exponential service time and Poisson input and each following queue has an exponential service time, then subsequent queues have a waiting time distribution of the same type as that of the initial queue in equilibrium. Ghosal [15] obtains the same results when the service time for the first queue is a Gamma distribution and the second is a negative exponential distribution, i.e., if the service time in each of the first K-1 queues follows an exponential distribution, but the Kth queue is the same as would have been obtained if the service time in the first queue had been the same as that in the Kth queue. Sacks [33] shows the conditions for which the joint distribution of waiting times for customers waiting in two queues in series converges to a probability distribution.

Part of the queueing network that we are interested in consists of M/D/1 type systems in series. Prabhu [30] shows that the integral equation for the waiting time distribution, developed by Erlang [13] using a heuristic argument, can be transformed to a polynomial form. He also develops a closed form expression for the expected number of customers in the queue during the busy period and shows that if the traffic intensity is greater than one, there is a positive probability that the busy period will continue indefinitely, whereas if the traffic intensity is less than one it will terminate.

Our representative laboratory model consists of both series and parallel service centers which are inter-related in that a specific set of service centers are required to meet each customer demand. Similar systems which have been studied are multi-purpose production systems. In these systems, customers arrive from time to time, each with a routing, which is an ordered list of service channels and are provided a specific type of service. An arriving customer joins the queue of the first service channel on his routing, remains in the channel until his service is completed, then goes to the second channel on his routing and remains until service has been completed, so on, until all his specific routing assignments are completed, at which time he leaves the system. Jackson [17] considers job-shop like "networks of waiting lines", which are representative of multipurpose type production systems. He assumes that jobs may arrive from within the system or from outside the system and service center M is assigned to contain R_M servers of identical characteristics. Job arrivals from outside the system are assumed to be Poisson with mean λ_M. Service discipline is assumed to be first come, first serve. The service time at each service center is assumed to be negative exponential distribution with mean $1/\mu_M$. Upon completion of service at center M, the job is transferred instantaneously to another center K, where $K \neq M$ and with probability P_{MK}. A closed form expression is derived, under steady state conditions, for determining the probability that N_M jobs are at service center M. The results indicate that the centers behave independently since the probability of the total number in the system is a product of the individual probabilities. As shown also by Burke [4] and Reich [31], these results rest squarely on the assumption that if the arrival and service rates at each center are independent of arrivals and service rates at other centers, the system behaves as if they were independent elementary systems.

In the analysis of the various models reviewed, we clearly identify the importance of the basic queueing assumptions in developing closed form expressions for measures of effectiveness under steady state conditions; that is, the arrival process to the queue and the service process are each renewal processes and the arrival process and service process are independent of each other. Based on these assumptions, Disney [11] shows that for the M/G/1 queue with infinite capacity, the departure process is uncorrelated if and only if G = M. This lends some credence to his assertion that in queueing networks, one cannot view the isolated server as a classic queueing problem except in the case that all servers are of the M type and all waiting line capacities are infinite. This assertion was also confirmed by Jackson [19]. Disney [11] also explores the covariance structure of the M/G/1 queue with finite waiting capacity. He concludes that the results derived from the analysis of the departure process of these systems have implications with respect to the design and analysis of integrated systems. The implications he identified are as follows:

1. In queueing networks one cannot, in general, treat an isolated server under the assumptions of classic queueing theory. Except for the special case in which Burke's

theorem is valid, the departure process from a server is not a renewal process.

2. In general, the departure from a single server is a Markov Renewal Process.
3. In the case $N = \infty$ and $G = M$ one can use classical queueing theory to study the queueing behavior of a server whose arrivals are the output of the given server.
4. In systems in which storage is not possible, $N = 1$, the departure process is always a renewal process.
5. The $N \geq 1$ departure processes are not a renewal process, except possibly $N = 2$ and $G = D$; hence, for the finite capacity storage system, classic queueing results are not adequate.
6. For finite capacity service systems with negative exponentially distributed service time, the covariance for any pairs of inter-departure intervals is non-zero except for the special case $N = 2$; hence, the departure streams cannot be considered a collection of independent observations.

One of the characteristics of the arriving, servicing and departing of customers in a clinical laboratory is their dependence on time; therefore a time dependent analysis of the complex network of queues representative of a clinical laboratory would have more relative value in attempting to relate to the actual laboratory behavioral characteristics than would a steady state analysis.

Clarke [6], Bailey [2] and Saaty [32] consider transient behavior of the M/M/S type, whose solutions are derived by using the standard generating function technique. The work of Saaty [32] indicates that it is not feasible to obtain explicit transient solutions of even the simple M/M/S queueing model.

Neuts [26] and Prabhu [30] consider time dependent analysis for two queues in series for cases of finite and infinite waiting room. Neuts [26] shows that these systems can be studied in terms of an imbedded semi-Markov process. Equations for time dependent distributions are given. Neuts [25] discusses the transient and limiting behavior of a system of queues consisting of two service units in series. He studies this system in terms of a Markov Renewal Branching Process. His remarks on the computational procedures necessary for determining the queue length distribution and virtual waiting time distribution are worth restating. "The system of two units in series with a finite intermediate waiting room is one of the simplest queueing networks conceivable. Nevertheless, a complete discussion of the time-dependent behavior of such a system is extremely involved and few of the results obtained in our work can be called explicit. The theoretical results yielded properties of the limiting behavior of the system and one can, with a little extra effort, obtain moments of the asymtotic queue length distribution and the limiting virtual waiting time distribution. Asymptotic results of interest also follow from general limit theorems for functionals defined on Markov Renewal Processes. It is obvious that inversion of any of the complicated transforms, which we obtained for the transient and limiting distributions, is practically impossible."

Another technique used in the study of time dependent queueing systems is that of computer simulation. While the method of computer simulation cannot replace analytical formulation and solutions, it can provide characteristic information as to the behavior of a model for a particular set of variables when the results of the analytical solution becomes extremely complex and unwieldly. Our approach in analyzing the complex laboratory models considers a combination of analytical and simulation methods.

CHAPTER III

ANALYTICAL DEVELOPMENTS

In order to understand the complexity of mathematical solutions used in the analysis of a large scale clinical laboratory, it is necessary to develop some new results. It became clear; however, that even with the over-simplified assumptions of this discussion this approach is not practical except to check certain portions of the simulation. The assumptions of a Poisson distribution simplifies the computation, but in Chapter IV a discussion is given in which it is demonstrated that the actual case of input streams do not in general satisfy this condition.

Consider the M/D/1 system. This system has been studied extensively by Crommelin [9], Erlang [13], Khintchine [21] and Pollaczek [29]. It should be noted that in general, they consider only steady state cases. Since the behavior of the clinical laboratory is so highly time dependent, we are interested in obtaining transient results.

Assume that L customers are present in the queue at time t_o and that they are served on first come, first serve basis. We are interested in determining the expected transient queueing time in the system. Define

q_i = queueing time for the ith customer.

t_i = arrival time for the ith customer.

$T_i = t_{i+1}-t_i$ = inter-arrival time.

c = constant service time.

If L customers are in the system at $t_o = 0$, then

$t_1 = t_2 = t_3 \ldots t_L = 0$

$T_0 = T_1 = T_2 \ldots T_{L-1} = 0$

hence $T_L = t_{L+1}$

The queueing time for the ith customer is represented by two distinct states:

as

$$q_{i+1} = \begin{cases} t_i+q_i+c-t_{i+1} & \text{for } T_i<q_i+c \\ 0 & \text{for } T_i \geq q_i+c \end{cases} \qquad (1)$$

Let $\varepsilon_i = 1$, represent the state when $T_i<q_i+c$ for $i \geq L$

then

$q_{i+1} = q_i+c-T_i$

and $\varepsilon_i = 0$ be the state when $T_i \geq q_i+c$ for $i \geq L$

and define

$q_{i+1} = 0$

The total state of the system after L+n arrivals can be represented by a sequence of states

$\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n$

where ε_n is the state after the L+nth arrival.

There are four cases for which changes of states can be identified:

Case 1 represents k successive states each of which is $\varepsilon_i = 1$ for $i = 1, 2 \ldots k$.

Case 2 represents k successive states each of which is $\varepsilon_i = 0$ for $i = 1, 2 \ldots k$.

Case 3 represents k successive states each of which is $\varepsilon_i = 1$ for i = 1, 2 ... i and k+1 state of $\varepsilon_{k+1} = 0$.

Case 4 represents k successive states each of which is $\varepsilon_i = 0$ for i = 1, 2 ... k and k+1 state of $\varepsilon_{k+1} = 1$.

For Case 1 the total state of the system is represented as $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, ... $\varepsilon_k = 1$, which implies $T_i \leq q+c$ for i = 1, 2 ... k.

If the system contains L customers at t_o, then the queueing time for Lth customer is

$$q_L = (L-1)c. \tag{2}$$

Now for Case 1

$$q_{L+1} = (L-1)c+c-T_L \qquad \text{for} \quad T_L < q_L + c$$

$$= Lc-T_L \qquad T_L < Lc$$

$$q_{L+2} = Lc-T_L+c-T_{L+1} \qquad \text{for} \quad T_{L+1} < Lc-T_L+c$$

$$= (L+1)c-T_L+T_{L+1} \qquad T_{L+1} < (L+1)c-T_L$$

$$\vdots$$

$$q_{L+k} = (L+k-1)c-T_L-T_{L+1} \cdots T_{L+k-1} \qquad \text{for} \quad T_{L+k-1} < (L+k-1)c-T_L-T_{L+1} \cdots T_{L+k-2}. \tag{3}$$

The total queueing time is given by:

$$Q = \sum_{j=L}^{L+k} q_j \tag{4}$$

$$= \frac{(k+1)(2L+k-2)}{2}c-kT_L-(k-1)T_{L-1} \cdots -2T_{L+k-2}-T_{L+k-1}.$$

To find the probability for the total system to be in this sequence of states for Case 1 under the assumptions that the T's are independent and identically distributed with distribution function G(T) having density function g(T), it is convenient to make the following substitution.

$$\text{Let } T_L = cu_1 \qquad \text{for} \qquad 0 \le u_1 \le L \tag{5}$$

$$T_L + T_{L+1} = cu_2 \qquad \text{for} \qquad u_1 \le u_2 \le L+1$$

$$T_L + T_{L+1} \ldots T_{L+k-1} = cu_k \qquad \text{for} \qquad u_{k-1} < u_k < L+k-1.$$

conversely

$$T_L = cu_1$$

$$T_{L+1} = c(u_2 - u_1)$$

$$T_{L+2} = c(u_3 - u_2)$$

$$\vdots$$

$$T_{L+k-1} = c(u_k - u_{k-1}).$$

The probability of k successive states of type $\varepsilon_i = 1$ is given by

$$P(\varepsilon_1 = 1, \varepsilon_2 = 1, \ldots \varepsilon_k = 1) =$$

$$c^k \int_0^L \int_{u_1}^{L+1} \cdots \int_{u_{k-2}}^{L+k-2} \int_{u_{k-1}}^{L+k-1} g[c(u_k - u_{k-1})]\ g[c(u_{k-1} - u_{k-2})] \tag{6}$$

$$\ldots g[(cu_1)]du_k\, du_{k-1} \ldots du_1.$$

For the case of negative exponential distribution of inter-arrival time

$$g(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } 0 \leq x < \infty \\ 0 & \text{if } x < 0 \end{cases}$$

where λ is a positive constant.

Therefore,

$$g[c(u_k - u_{k-1})]g[c(u_{k-1} - u_{k-2})] \ldots g[cu_1] = \lambda\, e^{-\lambda c u_k}$$

then

$$P(\varepsilon_1 = 1,\ \varepsilon_2 = 1 \ldots \varepsilon_k = 1) = (\lambda c)^k \int_0^L \int_{u_1}^{L+1} \ldots \int_{u_{k-1}}^{L+k-1} e^{-cu_k} du_k\, du_{k-1} \ldots du_1 . \quad (7)$$

$$\text{Define } I_k(f) = \int_0^L \int_{u_1}^{L+1} \ldots \int_{u_{k-1}}^{L+k-1} f(u_k) du_k \ldots du_1 . \quad (8)$$

Let $f^{(-s)}$ be defined by

$$f^{(-s)}(x) = \int_{\delta}^{x} f^{(-s+1)}(t)dt$$ where δ is constant usually $\delta = 0, \infty, -\infty$

for $s>0$, and $f^{(0)} = f$

thus

$$\int_a^b f(t) = f^{(-1)}(b) - f^{(-1)}(a).$$

Let $f(u_k) = 1$ in (8) and integrating the Kth term of (7), we have

$$I_k(1) = \int_0^L \int_{u_1}^{L+1} \ldots \int_{u_{k-2}}^{L+k-2} [f^{(-1)}(L+k+1) - f^{(-1)}(u_{k-1})]du_{k-1} \ldots du_1 , \quad (9)$$

where 1 is used for the function defined by $1: x \to 1$ and X is defined by X; $x \to x$.

Hence, recursively

$$I_k(1) = (L+k-1)I_{k-1}(1) - I_{k-1}(x) \tag{10}$$

$$= (L+k-1)I_{k-1}(1) - \frac{(L+k-2)^2}{2} I_{k-2}(1) + I_{k-2}(\frac{x^2}{2})$$

$$= (L+k-1)I_{k-1}(1) - \frac{(L+k-2)}{2} I_{k-2}(1) + \frac{(L+k-3)^3}{3!} I_{k-3}(1)$$

$$\ldots (-1)^{k-2} \frac{(L+1)^{k-1}}{(k-1)!} I_1(1) + (-1)^{k-1} I_1 (\frac{x^{k-1}}{(k-1)!})$$

where

$$I_1 (\frac{x^{k-1}}{(k-1)!}) = \frac{L^k}{k!} .$$

Now let $c_k = I_k(1)$,

and then from (10)

$$c_k = (L+k-1)c_{k-1} - \frac{(L+k-2)}{2} c_{k-2} + \frac{(L+k-3)}{3!} \ldots (-1)^{k-1} \frac{L}{k!} c_0, \tag{11}$$

where $c_0 = 1$.

Hence from (11)

$$c_k = \sum_{s=1}^{k} \frac{(L+k-s)^s}{s!} (-1)^{s-1} c_{k-s} \tag{12}$$

It can be seen by a very simple induction argument that a solution to (12) when $c_0 = 1$ is unique. It will be shown that

$c_k = \frac{L(L+k)^{k-1}}{(k-1)!}$ is a solution and hence the solution. This solution was determined by observing the first few cases. It is more convenient to write (12) in the form

$$\sum_{s=0}^{k} \frac{(L+k-s)^s}{s!} (-1)^{s-1} c_{k-s} = 0. \tag{13}$$

Then if $c_{k-s} = \frac{L(L+k-s)^{s-k-1}}{(k-s)!}$ then

$$\sum_{s=0}^{k} \frac{(L+k-s)^s}{s!} (-1)^{s-1} c_{k-s} = \sum_{s=0}^{k} \frac{(L+k-s)^s}{s!} (-1)^{s-1} \frac{L(L+k-s)^{s-k-1}}{(k-s)!} \tag{14}$$

$$= \frac{L}{k!} \sum_{r=0} (-1)^{k-r-1} \binom{k}{r} (L+r)^{k-1} \quad \text{where } r = k-s.$$

In order to show that this sum is zero,

$$\text{let } F(x) = \sum_{r=0}^{k} (-1)^{k-r-1} \binom{k}{r} x^{L+r} \tag{15}$$

$$= (-1)^{k-1} x^L (1-x)^k.$$

Then since $(XD)^h X^t = t^h X^t$ where D is derivative operator and

$(XD)^h = \sum_{j=1}^{h} a_{hj} X^j D^j$ for some constants a_{hj}.

$$\text{It follows that } (XD)^{k-1} F(x) = \sum_{r=0}^{k} (-1)^{k-r-1} \binom{k}{r} (L+r)^{k-1} x^{L+r} \tag{16}$$

$$\text{and } (XD)^{k-1} F(x)\Big|_{x=1} = \sum_{r=0}^{k} (-1)^{k-r-1} \binom{k}{r} (L+r)^{k-1} \tag{17}$$

But $$(XD)^{k-1} F(x)\Big|_{x=1} = (-1)^{k-1} (XD)^{k-1} x^L (1-x)^k \Big|_{x=1} \qquad (18)$$

and all derivatives involved have time (1-x) to a positive power; hence, each term is zero at x = 1.

Therefore the prescribed values for c_{k-s} furnish the solution to (12) and $I_k(1) = c_k = \frac{L(L+k)^{k-1}}{k!}$.

For the general function, define

$$I_k(f) = \int_0^L \int_{u_1}^{L+k-1} \cdots \int_{u_{k-1}}^{L+k-1} f(u_k)du_k\, du_{k-1} \cdots du_1. \qquad (19)$$

Hence

$$I_k(f) = f^{(-1)}(L+k-1)\, I_{k-1}(1) - I_{k-1}(f^{(-1)})$$

$$\vdots$$

$$= f^{(-1)}(L+k-1) I_{k-1}(1) \ldots (-1)^{k-1} f^{(-k)}(L) I_0(1) + (-1)^k f^{(-k)}(0).$$

Since $I_k(1) = \frac{L(L+k)^{k-1}}{k!}$

then

$$I_k(f) = \sum_{s=1}^{k} (-1)^{s-1} \frac{L(L+k-s)^{k-s-1}}{(k-s)!} f^{(-s)}(L+k-s) + (-1)^k f^{(-k)}(0).$$

$$\text{Let } f(X) = \begin{cases} (\lambda c)^k e^{-\lambda cx} & \text{if } 0 \le x < \infty \\ 0 & \text{if } x < 0 \end{cases} \qquad (20)$$

where λc is a positive constant.

Then

$$f^{-s}(x) = (\lambda c)^{k-s}(-1)^{s}e^{-\lambda cx} \quad \text{and}$$

$$f^{-k}(0) = (-1)^{k}$$

using (20), (7) is seen to be

$$P(\varepsilon_1=1,\ \varepsilon_2=1\ \ldots\ \varepsilon_k=1) = 1 - \sum_{s=1}^{k} (\lambda c)^{k-s} \frac{L(L+k-s)^{k-s-1}}{(k-s)!} e^{-\lambda(L+k-s)c}. \qquad (21)$$

Let $j = k-s$

then

$$P(\varepsilon_1=1,\ \varepsilon_2=1\ \ldots\ \varepsilon_k=1) = 1 - e^{-\lambda cL} \sum_{j=0}^{k-1} \frac{L(L+j)}{j!} (\lambda ce^{-\lambda c})^{j}. \qquad (22)$$

The second case to be considered is where the system remains in state $\varepsilon_i = 0$ for k successive customers.

From (1)

$q_{i+1} = 0 \qquad$ for $T_i \geq q_i + c$ and $i \geq L$.

With L customers in the system, the queueing time is given as $q_L = (L-1)c$, and

$q_{L+1} = 0 \qquad$ for $T_L \geq (L-1)c+c$

$\qquad\qquad\qquad T_L \geq Lc$

$q_{L+2} = 0 \qquad$ for $T_{L+1} > c$

.

.

.

$q_{L+k} = \qquad$ for $T_{L+k-1} > c$.

Since $g(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } 0<x<\infty \\ 0 & \text{if } x<0 \end{cases}$

the distribution function is

$$G(x) = \begin{cases} 1-e^{-\lambda x} & \text{if } 0<x<\infty \\ 0 & \text{if } x<0 \end{cases}$$

The probability of $T_L \geq Lc$ or $q_{L+1} = 0$ is given as:

$$[1-G(Lc)]$$

and probability for the remaining k-1 terms being zero or

$T_{L+i} \geq c$ for $i = 2, 3 \ldots k-1$

is $[1-G(c)]^{k-1}$.

Hence the probability for k successive customers to be in $\varepsilon_i = 0$ state is given as

$$P(\varepsilon_1=0, \varepsilon_2=0 \ldots \varepsilon_k=0) = [1-G(Lc)][1-G(c)]^{k-1} \qquad (23)$$

The third case to be considered is a change of state after k customers in $\varepsilon_i = 1$ for $i = 1, 2, \ldots k$ to $\varepsilon_{k+1} = 0$ state.

The probability for the system to be in this sequence of states is given as

$$P(\varepsilon_1=0, \varepsilon_2=0 \ldots \varepsilon_k=0, \varepsilon_{k+1}=1) = [1-G(Lc)][1-G(c)]^{k-1} G(c). \qquad (24)$$

The remaining case to be considered is a change in state from

$\varepsilon_i =1$ for $i = 1, 2 \ldots k$ to $\varepsilon_{k+1} = 0$.

The queueing time for the k+1 customer is

$$q_{L+k+1} = 0 \qquad \text{for } T_{L+k} \geq q_{L+k}+c$$

$$T_{L+k} \geq (L+k)c - T_L \ldots T_{L+k-1}$$

Hence the probability is seen to be

$$P(\varepsilon_1=1,\ \varepsilon_2=1 \ldots \varepsilon_k=1,\ \varepsilon_{k+1} = 0) = \tag{25}$$

$$c^{k+1} \int_0^{L} \int_{u_1}^{L+1} \cdots \int_{u_{k-1}}^{L+k-1} \int_{L+k}^{\infty} g[c(u_{k+1} - u_k)]\ g(u_1) du_{k+1} \ldots du_1 .$$

Since $g(u) = \begin{cases} \lambda e^{-\lambda u} & 0 \leq u < \infty \\ 0 & u < 0 \end{cases}$

then $P(\varepsilon_1=1,\ \varepsilon_2=1 \ldots \varepsilon_k=1,\ \varepsilon_{k+1} = 0)$ (26)

$$= (\lambda c)^{k+1} \int_0^{L} \int_{u_1}^{L+1} \cdots \int_{u_{k-1}}^{L+k-1} \int_{L+k}^{\infty} e^{-\lambda c u_{k+1}} du_{k+1} \ldots du_1$$

$$= (\lambda c)^{k} e^{-\lambda c(L+k)} \int_0^{L} \int_{u_1}^{L+1} \cdots \int_{u_{k-1}}^{L+k-1} du_k \ldots du_1$$

and from (26) it follows that

$$P(\varepsilon_1=1,\ \varepsilon_2=1 \ldots \varepsilon_k=1,\ \varepsilon_{k+1} = 0) = (\lambda c)^{k} e^{-\lambda c(L+k)}\ I_k\ (1)$$

$$= (\lambda c)^{k} e^{-\lambda c(L+k)}\ \frac{L(L+k)}{k!} .$$

Thus probabilities are established for the fourth case.

The expected value of the total queueing time for the first case can be shown as follows

from (4) and using transformations in (5)

$$\sum_{j=L}^{L+k} q_j = (L-1)c+Lc-cu_1 + (L+1)c-cu_2 + \ldots (L+k+1)c-cu_k$$

$$= c[\frac{(k+1)(2L+k-2)}{2} - (u_1 + u_2 + \ldots u_k)]$$

Using the probability of k successive states of type $\varepsilon_i = 1$ from (7) we have

$$E(Q) = c^{k+1}\lambda^k \int_0^L \cdots \int_{k-1}^{L+k-1} e^{-\lambda c u_k}\left[\frac{(k+1)(2L+k-2)}{2} - (u_1 + u_2 \cdots u_k)\right]$$

$$du_k \cdots du_1. \tag{27}$$

Let $A = \frac{(k+1)(2L+k-2)}{2}$ and

simplifying (27) and using results obtained in (22), we have

$$E(Q) = c^{k+1}\lambda^k A[1-e^{-\lambda cL} \sum_{j=0}^{k-1} \frac{L(L+j)^{j-1}}{j} (\lambda c e^{-\lambda c})^j] -$$

$$c^{k+1}\lambda^k \int_0^L \cdots \int_{u_{k-1}}^{k+k-1} e^{-\lambda c u_k}(u_1+u_2+ \cdots u_k)\, du_k \cdots du_1. \tag{28}$$

From (28) the integral may be expressed as follows:

$$\sum_{h=1}^{k} (\lambda c)^k \int_0^L \cdots \int_{u_{k-1}}^{L+k-1} u_h\, e^{-\lambda c u_k}\, du_k\, du_{k-1} \cdots du_1 \tag{29}$$

For $\delta = 0$ it is well known that

$$f^{(-s)}(x) = \int_0^x \frac{(x-u)^{s-1}}{(s-1)!} f(u)du$$

when

$$f(x) = \begin{cases} (\lambda c)^k x e^{-\lambda cx} & \text{for } 0<x<\infty \\ 0 & x<0 \end{cases}$$

$$f^{(-s)}(x) = (\lambda c)^k \int_0^x \frac{(x-u)^{s-1}}{(s-1)!} u e^{-\lambda cu} du.$$

From the commutativity of the convolution product

$$f^{(-s)}(x) = (\lambda c)^k \int_0^x \frac{u^{s-1}}{(s-1)!} (x-u)\ e^{-\lambda c(x-u)} du \tag{30}$$

then

$$f^{(-s)}(x) = \frac{(\lambda c)^k}{(s-1)!} e^{-\lambda cx}[x\int_0^x u^{s-1}\ e^{\lambda u}\ du - \int_0^x u^s e^{\lambda cu}\ du].$$

To integrate the functions within the brackets of (30)

$$\text{Let } \phi_k(x) = \int_0^x u^k\ e^{au}\ du \tag{31}$$

and integrate by parts, to obtain

$$\int_0^x u^k\ e^{au}\ du = \int_0^x u^k\ \frac{de^{au}}{a}$$

$$= \frac{x^k e^{ax}}{a} - \frac{k}{a} \int_0^x u^{k-1} e^{au}\ du.$$

Then

$$\phi_k(x) = \frac{x^k e^{ax}}{a} - \frac{k}{a} \phi_{k-1}(x) \tag{32}$$

$$= \frac{x^k e^{ax}}{a} - \frac{kx^{k-1} e^{ax}}{a^2} + \frac{k(k-1)x^{k-2} e^{-ax}}{a^3} \cdots \frac{(-1)^{k-1} k(k-1)\cdots 2}{a^{k-1}} \phi_1(x).$$

For k = 1

$$\phi_1(x) = \frac{xe^{ax}}{a} - \frac{1}{a} \int_0^x e^{au} du$$

$$= \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2} + \frac{1}{a^2}$$

and define $(k)_r = k(k)_{r-1}$ where $(k)_0 = 1$

and from (32)

$$\phi_k(x) = \sum_{s=0}^{k} (-1)^{-s} \frac{(k)_s X^{k-s}}{a^{s+1}} e^{ax} + (-1)^{k+1} \frac{k!}{a^{k+1}} . \tag{33}$$

Let s = r and k = s-1

then from (30)

$$f^{(-s)}(x) = \frac{(\lambda c)^k e^{-\lambda cx}}{(s-1)!} \left[\sum_{r=0}^{s} (-1)^r \frac{[(s-1)_r - (s)_r]}{(\lambda c)^{r+1}} x^{s-r} e^{\lambda cx} + \frac{(-1)^s (s-1)!}{(\lambda c)^s} - \frac{(1)^{s+1} s!}{(\lambda c)^{s+1}} \right] . \tag{34}$$

Substituting (34) into (20) and let x = L+k-s

then I_k (f) can now be written as:

$$I_k(f) = \sum_{s=1}^{k} (-1)^{s-1} \frac{L(L+k-s)^{k-s-1}}{(k-s)!} (-1)^s [(\lambda c)^{k-s} (L+k-s) e^{-\lambda c(L+k-s)} \tag{35}$$

$$+ s(\lambda c)^{k-s-1} e^{-\lambda c(L+k-s)}] + \sum_{s=1}^{k} \sum_{r=0}^{s} (-1)^{s-1} \frac{L(L+k-s)^{k-s-1}}{(k-s)!} [$$

$$\frac{(\lambda c)^{k-r-1}}{(s-1)!} (-1)^r [(s-1)_r - (s)_r] (L-k-s)^{s-r}].$$

To simplify (35)

let $\frac{(s-1)_r - (s)_r}{(s-1)!} = \frac{-(s-1)_{r-1} r}{(s-1)_{r-1} (s-r)!}$

and $s - r = t$

hence the double sum of (35) can be written as

$$\sum_{s=1}^{k} L\left[\frac{(L+k-s)\lambda c}{(k-s)!}\right]^{k-s-1} \sum_{t=0}^{s} (-1)^{t} \frac{\lambda c(L+k-s)^{t}}{t!}.$$

Therefore

$$I_k(f) = \sum_{s=1}^{k} (-1)^{s-1} L(L+k-s)^{k-s-1} (-1)^{s} [(\lambda c)^{k-s} (L+k-s) e^{-\lambda c(L+k-s)} \tag{36}$$

$$+ s(\lambda c)^{k-s-1} e^{-\lambda c(L+k-s)}] + \sum_{s=1}^{k} \frac{L[L+k-s)\ c]}{(k-1)!}^{k-s-1} \sum_{t=0}^{} (-1)^{t} \frac{\lambda c(L+k-s)^{t}}{t!}.$$

The remaining terms of the multiple integral in (28) can now be integrated as follows

(29) can be represented as follows:

$$(\lambda c)^{k} \int_{0}^{L} \int_{u_{h-1}}^{L+h-1} u_h \left[\int_{u_h}^{L+h} u_{h+1} \cdots \int_{u_{k-1}}^{L+k-1} e^{-\lambda c u_k} du \ldots du_{h+1} \right] du_h \ldots du_1. \tag{37}$$

Now let

$$J_{h+1}^{k}(f) = \int_{u_h}^{L+h} \int_{u_{h+1}}^{L+h+1} \cdots \int_{u_{k-1}}^{L+k-1} f(u_k)\, du_k \ldots du_{h+1} \tag{38}$$

$$= \int_{u_h}^{L+h} \cdots \int_{u_{k-2}}^{L+k-2} [f^{(-1)}(L+k-1) - f^{(-1)}(u_{k-1})]\, du_{k-1} \ldots du_{h+1}$$

$$= f^{(-1)}(L+k-1)\ J_{h+1}^{k-1}(1) - J_{h+1}^{k-1}(f^{(-1)})$$

$$J_{h+1}^{k}(f) = \sum_{s=1}^{k-h} (-1)^{s-1} f^{(-s)}(L+k-1)\; J_{h+1}^{k-s}(1) + (-1)^{k-h} f^{-(k-h)}(u_h),$$

where $J_{h+1}^{h}(1)$ is defined to be one

and

$$J_{h+1}^{h+1}(f^{-(k-h-1)}) = \int_{u_h}^{L+h} f^{-(k-h-1)}(u_{h+1})du_{h+1} = f^{-(k-h)}(L+h) - f^{-(k-h)}(u_h).$$

when $f = 1$

$$f^{(-s)}(L+k-1) = \frac{(L+k-s)^s}{s!}.$$

Hence

$$J_{h+1}^{k}(1) = \sum_{s=1}^{k-h} (-1)^{s-1} \frac{(L+k-s)^s}{s!}\; J_{h+1}^{k-s}(1) + (-1)^{k-h} \frac{u_h^{k-h}}{(k-h)!}. \tag{39}$$

In simplifying and combining terms in (39),

let

$d = k-h$

and

$r = k-h-s = d-s$

and

$X_r = J_{h+1}^{h+r}(1)$

and

$u = u_h$

then

$$\sum_{r=0}^{d} (-1)^{d-r} \frac{(L+h+r)^{d-r}}{(d-r)!} X_r = (-1)^d \frac{u^d}{d!}. \tag{40}$$

Now, let L+h = a, and

multiply the left side of (40) by $t^{d-r}\ t^{r}$ and right side by t^{d} and summing from r = 0 to ∞ yields

$$\sum_{r=0}^{\infty} e^{-(a+r)t}\ t^{r}\ X_r = e^{-ut}, \tag{41}$$

and rearranging terms gives

$$\sum_{r=0}^{\infty} (t^{r} e^{-rt})\ X_r\ (u) = e^{(a-u)t}.$$

From Leibnitz's formula,

$$D_t^{k}\ t^{r}\ e^{-rt} = \sum_{j=0}^{k} \binom{k}{j} (-1)^{j} r^{j} (r)_{k-j}\ t^{r-k+j}\ e^{-rt} \tag{42}$$

and therefore at t = 0

$$D_t^{k}\ t^{r}\ e^{-rt} = \begin{cases} 0 & k<r \\ \binom{k}{k-r} (-1)^{k-r}\ r^{k-r}\ r! & k\geq r\ . \end{cases}$$

If both sides of (41) are differentiated k times with respect to t and the results evaluated at t = 0 then

$$D_t^{k}\ t^{r} = \sum_{r=0}^{k} \binom{k}{k-r} (-1)^{(k-r)}\ r^{k-r}\ r!\ X_r = (a-u)^{k}. \tag{43}$$

The sum for k>0 can run from 1 to k.

Now, define

$$a_{kr} = \binom{k}{r} r^{k} \qquad \text{for } k,r\geq 1,$$

and

$$Y_r = (-1)^{r}\ \frac{r!\ X_r}{r^{r}}$$

For the matrix A whose elements are a_{kr} let b_{rj} be the element of the inverse A^{-1}.

Then

$$\sum_{j\le k\le r} b_{jk}\, a_{kr} = \delta_{jr} \tag{44}$$

where

$$\delta_{jr} = \begin{cases} 0 \text{ if } r \ne j \\ 1 \text{ if } r = j \end{cases}$$

From (43)

$$Y_j = \sum_{k=1}^{j} b_{jk}\, (u-a)^k .$$

Since A is lower triangular with non-zero diagonal terms, the inverse exist and is unique.

Now assume $b_{rj} = \binom{r-1}{j-1} (-1)^{r-j} r^{-j}$ (45)

then

$$\sum_{r=1}^{k} a_{kr} b_{rj} = \sum_{r=1}^{k} \binom{k}{r}\binom{r-1}{j-1} r^k (-1)^{r-j} r^{-j} . \tag{46}$$

The expression is zero if r<k. If k = j the only non-zero term occurs when

r = k = j, and the expression $\binom{k}{k}\binom{k-1}{k-1} k^k (-1)^{k-k} k^{-k} = 1.$

If k>j then

$$\binom{k}{r}\binom{r-1}{j-1} = \left[\frac{k!}{r!(k-r)!}\right]\left[\frac{(r-1)!}{(j-1)!(r-j)!}\right]\left[\frac{(k-j)!j}{(k-j)!j}\right] = \frac{j}{r}\binom{k}{j}\binom{k-j}{k-r}$$

and the sum is

$$j\binom{k}{j} \sum_{j\le r\le k} \binom{k-j}{k-r} r^{k-j-1} (-1)^{r-j} .$$

and setting r-j = i. Then as in the process for solving (22)

$$j\binom{k}{j}\sum_{i=0}^{k-j}\binom{k-j}{i}(j+1)^{k-j-1}(-1)^{i} =$$

$$j\binom{k}{j}(xD)^{k-j-1}\sum_{i=0}^{k-j}\binom{k-j}{i}(-1)^{i}x^{j+i} \quad \text{for } x = 1$$

which reduces to

$$j\binom{k}{j}(xD)^{k-j-1}x^{j}(1-x)^{k-j} \quad \text{for } x = 1.$$

As before with $k-j>0$ every term involves a positive power and thus the expression vanishes at $x = 1$. Therefore $\sum a_{kr}b_{rj} = \delta_{kj}$ and B is the inverse of A.

From the definition of u_r, X_r and (45) thus

$$J_{h+1}^{h+r}(1) = X_r = \frac{r}{r!}^{r}(-1)^{r}Y_r = \frac{1}{r!}\sum_{j=1}^{r}\binom{r-1}{j-1}r^{r-j}(a-u)^{j}. \qquad (47)$$

Since

$$f(X) = \begin{cases} e^{-\lambda cx} & 0 \le x < \infty \\ 0 & x < 0 \end{cases}$$

and

$$f^{-s}(x) = (-1)^{s}(\lambda c)^{-s}e^{-\lambda cx}$$

and let $x = L+k-1$

then from (45), (38) reduces to

$$J_{h+1}^{k}(f) = \sum_{s=1}^{k-h}(-1)^{s-1}\frac{(\lambda c)^{k-s}e^{-\lambda c(L+k-1)}}{(k-h-s)!}\Bigg[\qquad (48)$$

$$\sum_{j=1}^{r} \binom{k-h-s-1}{j-1} r^{k-h-s-j} [(L+h-u_h)^j] + (-1)^{k-h} (\lambda c)^h e^{-\lambda c u_h} \Big].$$

The formulations for determining the probability of the T's when the system is in each of the four cases are given by equations (22), (23), (24) and (26).

Let $P(T_{ck})$ be the probability of the T's in the state defined by ck where c = 1,2,3,4 defines the four cases and k the state after n arrivals. Since we assume the T's are independent and identically distributed and $T_L = 0$, then the probability for the system to be in any sequence of k states can be determined from the product of the probabilities representing the four types of cases. For example, to determine the probability for the T's for which the system is in the following sequence of states

$\varepsilon_1=0,\ \varepsilon_2=0,\ \varepsilon_3=0,\ \varepsilon_4=1,\ \varepsilon_5=1,\ \varepsilon_6=0,\ \varepsilon_7=1,\ \varepsilon_8=0,\ \varepsilon_9=0,\ \varepsilon_{10}=1,$

we examine the sequence beginning at the kth state. The partitioning of the sequence of states into the four types of cases is shown as follows:

1. The states $\varepsilon_9\ \varepsilon_{10}$ are equal to case 3 with k = 1.
2. The states $\varepsilon_7\ \varepsilon_8$ are equal to case 4 with k = 1.
3. The states $\varepsilon_4\ \varepsilon_5\ \varepsilon_6$ are equal to case 4 with k = 2.
4. The states $\varepsilon_1\ \varepsilon_2\ \varepsilon_3$ are equal to case 2 with k = 3.

The probability of the T's for which the system is in this sequence is given by

$P(T_{10}) = P(T_{23})\ P(T_{42})\ P(T_{41})\ P(T_{31})$ which can be determined from (49)

equations (23) and (26).

Let Q_r represent the total queueing time for a sequence of k states and let Q_{ck} represent the total queueing time under the conditions represented by ck.

The conditional expected value of the queueing time if the system is in the sequence of states expressed by case 1 is derived from equations (28), (36) and (46) and is given by

$$E(Q/\varepsilon_1=1,\ \varepsilon_2=1\ \dots\ \varepsilon_{k=1}) = \tag{50}$$

$$- C^{k+1}\lambda^k \frac{(k+1)(2L+k-2)}{2}\left[1 - e^{-\lambda cL}\sum_{j=0}^{k-1} L\frac{(L+j)^{j-1}}{j}(\lambda ce^{-\lambda c})^j\right]$$

$$- C\left[\sum_{s=1}^{k}(-1)^{s-1}L(L+k-s)^{k-s-1}(-1)^s[(\lambda c)^{k-s}(L+k-s)e^{-\lambda c(L+k-s)}\right.$$

$$\left. + S(\lambda c)^{k-s-1}e^{-\lambda c(L+k-s)}] + \sum_{s=1}^{k} L\frac{[(L+k-s)c]^{k-s-1}}{(k-1)!}\sum_{t=0}^{s}(-1)\frac{\lambda c(L+k-s)^t}{t!}\right]$$

$$- \sum_{h=1}^{k-1} I_h\left[U_h\sum_{s=1}^{k-h}(-1)^{s-1}\frac{(\lambda c)^{k-s}e^{-\lambda c(L+k-1)}}{(k-h-s)!}\sum_{j=1}^{r}\binom{k-h-s-1}{j-1}r^{k-h-s-1}\right.$$

$$\left.(L+h-u_n)^j + (-1)^{k-h}(\lambda c)^h e^{-\lambda cu_h}\right] \Bigg/ 1 - e^{-\lambda cL}\sum_{j=0}^{k-1}\frac{L(L+j)}{j!}(\lambda ce^{-\lambda c})^j$$

To simplify the numerator in (50)

let $J^k_{h+1}(f) = m_{kh} + (-1)^{k-h}(\lambda c)^h e^{-\lambda cu_h}$

represent equation (50) with m_{kh} denoting the constant terms.

From (11)

$I_k(1) = (L+k-1)I_{k-1}(1) - I_{k-1}(x)$

and let $x = u_h$ and $I_k(x) = \frac{L(L+k)^{k-1}}{k!}$

hence

$$I_h(u_h) = \frac{(L+h)L(L+h)^{h-1}}{h!} - \frac{L(L+h+1)^h}{(h+1)!}$$

$$= \frac{L}{(h+1)!}\,[(L+h)^L(h+1) - (L+h+1)^h] \tag{51}$$

The $I_h(x)$ term in (46) can be represented as

$$I_h\,(m_{kh}u_h + (-1)^{k-h}(\lambda c)^h u_h e^{-\lambda c u_h}). \tag{52}$$

(51) and (52) are used to simplify (50) as follows

$$E(Q/\varepsilon_1=1,\varepsilon_2=1 \ldots \varepsilon_k=1) = C^{k+1}\lambda^k\,(k+1)\frac{(2L+k-2)}{2}\,[1 - e^{-\lambda cL} \tag{53}$$

$$\sum_{j=0}^{k-1}\frac{L(L+j)^{j-1}}{j}(\lambda ce^{-\lambda c})^j] - \sum_{h=1}^{k-1}\left[\sum_{s=1}^{k-h}(-1)^{s-1}\frac{(\lambda c)^{k-s}e^{-\lambda c(L+k-1)}}{(k-h-s)!}\right.$$

$$\left.\sum_{j=1}^{r}\binom{k-h-s-1}{j-1} r^{k-h-s-j}[L+h-u_h]^j\frac{L}{(L+1)!}\left((L+h)(h+1) - (L+h+1)^h\right)\right]$$

$$-\sum_{h=1}^{k}(-1)^{k-h}\sum_{s=1}^{h}(-1)^{s-1}L(L+h-s)^{h-s-1}(-1)^s[(\lambda c)^{h-s}(L+h-s)e^{-\lambda c(L+h-s)}$$

$$+S(\lambda c)^{h-s-1}e^{-\lambda c(L+h-s)}] + \sum_{s=1}^{h}\frac{L(L+h-s)c^{h-s-1}}{(h-1)!}\sum_{t=0}^{s}(-1)^t\frac{\lambda c(L+k-s)^t}{t!} \Bigg/$$

$$1 - e^{-\lambda cL}\sum_{j=0}^{k-1}\frac{L(L+j)}{j!}(\lambda ce^{-\lambda c})^j$$

The expected queueing time for case 2 is given

$$E(Q/\varepsilon_1=0,\varepsilon_2=0 \ldots \varepsilon_{k=0}) = \frac{Q[1-G(Lc)][1 - G(c)]^k}{[1-G(Lc)][1 - G(c)]^k}$$

$$= Q$$

$$= (L-1)c$$

The expected queueing time for case 3

$$E(Q/\varepsilon_1=0 \ldots \varepsilon_k=0,\varepsilon_{k+1}=1) = \int_{Lc}^{\infty}\int_{c}^{\infty}\cdots\int_{c}^{\infty}\int_{0}^{c} (Lc-T_{L+k}) \tag{54}$$

$$g(T_{L+k})g(T_{L+k-1}) \ldots g(T_1)\, dT_{L+k} \ldots dT_1 \Big/ P(\varepsilon_1=0 \ldots \varepsilon_k=0,\varepsilon_{k+1}=1)$$

$$= Lc - \frac{\int_0^c ug(u)du}{\int_0^c g(u)du}$$

If $g(x) = \lambda c e^{-\lambda cx}$

then $G(x) = 1 - e^{-\lambda cx}$

hence

$$\int_0^c udG(u) = cG(c) - \int_0^c G(u)du$$

$$= c(1-e^{-\lambda c^2}) - \int_0^c (1 - e^{-\lambda cx})dx$$

$$= \frac{1}{\lambda c} - \frac{1+\lambda c}{\lambda c} e^{-\lambda c^2}$$

therefore

$$E(Q/\varepsilon_1=0 \ldots \varepsilon_k=0,\varepsilon_{k+1}=1) = Lc - \frac{1}{\lambda c} + \frac{1+\lambda c^2}{\lambda c} e^{-\lambda c^2} \tag{55}$$

For case four

$$E(Q/\varepsilon_1=1 \ldots \varepsilon_k=1,\varepsilon_{k+1}=0) = c(\lambda c)^k \int_0^L \cdots \int_{u_{k-1}}^{L+k-1} \int_{L+k}^{\infty} (A - \sum_{L=1}^{k} u_i) e^{-\lambda c u_{k+1}}$$

$$du_{k+1}\, du_k \ldots du_1 \Big/ P(\varepsilon_1=1,\varepsilon_2=1 \ldots \varepsilon_{k+1}=0) \tag{56}$$

$$E(Q/\varepsilon_1=1 \ldots \varepsilon_k=1,\varepsilon_{k+1}=0) = c\left(1 - e^{-\lambda c(L+k)}\right)(\lambda c)^{k-1}\left[AI_k(1) - \sum_{h+1}^{k} I_h(u_h) J_{h+1}^{k}(1)\right] \Big/ (\lambda c)^k e^{-\lambda c}\frac{(L+k)}{k!}^{k-1} .$$

From the example Q_{10} represents the total queueing time given this sequence of states, hence

$$E(Q_{10}/\varepsilon_1=0\ \varepsilon_2=0\ \varepsilon_3=0\ \varepsilon_4=1\ \varepsilon_5=1\ \varepsilon_6=0\ \varepsilon_7=1\ \varepsilon_8=0\ \varepsilon_9=0\ \varepsilon_{10}=1) = \frac{\int Q_{10}f(T)}{\int f(T)dT}$$

$$= \frac{\int Q_{23}f(T_{23}) + \int Q_{42}f(T_{42}) + \int Q_{41}f(T_{41}) + \int Q_{32}f(T_{32})}{P(T_{23})\ P(T_{42})\ P(T_{41})\ P(T_{32})}$$

$$= \frac{E(Q_{23}) + E(Q_{42}) + E(Q_{41}) + E(Q_{32})}{P(T_{23})\ P(T_{42})\ P(T_{41})\ P(T_{32})} .$$

All the formulas necessary to calculate $E(Q_k/\varepsilon_{11}\ \varepsilon_{12} \ldots \varepsilon_{ij})$ have been previously derived; hence, the total queueing time can be obtained as follows:

$$E(Q_k) = \sum_{\bar{\varepsilon}} E(Q_k/\varepsilon)$$

where $\bar{\varepsilon} = \left[\varepsilon/\varepsilon \text{ is a possible state of the system}\right]$.

CHAPTER IV

SIMULATION METHODS AND TECHNIQUES

The concept of simulation involves the construction and study of a model of an operating system without any direct action on the system. Simulation attempts to describe properties or behavior of a system without being its exact analog. The first step in defining the behavior of any system is to isolate the system's elements and formulate the logical rules governing their interaction. These rules may be a set of mathematical expressions for which the solution determines the behavior of the system. These types of models must be capable of being described in mathematical terms and the equation developed capable of solution.

There is, however, a large class of systems which cannot be modeled by using physical and mathematical techniques without making simplifying assumptions to reduce the complexity of formulation into more manageable terms. Some large complex systems such as complex manufacturing systems, transportation networks, large computer systems and management information systems are some examples. Digital simulation is a powerful tool for the evaluation of such systems. In using the digital computer simulation, the investigator can describe the various components and rules governing the interaction of these components to a computer by means of a computer program. Special simulation languages such as GPSS

(General Purpose System Simulator), Simscript or Simula, GASP (Generalized Activity Simulation Program), and many others have been developed to aid in the translation of problem definition into computer executable form.

GPSS is a simulation language which has a rigidly defined data structure in terms of special purpose blocks which are used to construct flow charts describing the system structure and decision rules. The language utilizes queueing theory to measure the various attributes of the system modeled.

The mechanics of the flow of work through a clinical laboratory are not unlike those of a multi-purpose production system composed of special-purpose service centers. These systems have been extensively studied by Jackson [16], Saaty [32], Elmaghraby [12] and others.

The major difference between Jackson's [16] jobshop-like queueing network and the actual behavorial characteristics of the laboratory lies in the assumptions that:

1. The service time is distributed as Poisson.
2. The output from each service facility form the input to the next service facility in the routing and is distributed as Poisson.
3. There exist a probability associated with choosing the next service facility.
4. Only one routing is required to determine the total customer service required.
5. The system is in steady state.

The input into the clinical laboratory is a function of the time the physician sees his patient. In a teaching institution such as the University of Oklahoma Hospital, physician contact with the patient is usually during the morning and the evening hours. Table 1 reflects the distribution of the request for service by type of clinical test at one hour intervals. The tests are divided into two groups, those whose processing requires automated techniques and those requiring manual methods. Since requests for several tests may be incorporated in one request for service form, a breakout of the types of request for services and the priority of the request is shown under the request for service section. During the hours 0800 to 1700, the actual distribution of the total number of tests was tested against a Poisson distribution to determine the Goodness of Fit. The probabilities for a Poisson distribution are given by:

$$P(x=k) = \frac{e^{-\mu k}}{k!} \quad \text{where } k = 0, 1, 2 \ldots .$$

Let $\overline{x} = \frac{\Sigma fx}{\Sigma f}$ be the estimate of the mean μ and variance σ^2, where f is the observed frequency and x the observed value.

To test the Goodness of Fit, χ^2 is an appropriate test criterion where $\chi^2_{N-1} = \sum \frac{(\text{Observed - Expected})^2}{\text{Expected}}$ for N-1 degrees of freedom. From Table 2, the degrees of freedom are 9, and the estimate of the mean $\overline{x} = 1.95$. Since the χ^2 value for the first term is so large, the computations were terminated. The critical value for the 5% error level is $\chi^2_9 = 15.5$; hence, we reject the hypothesis that the unknown distribution is equal to a Poisson.

After the morning rounds by the physician, the laboratory receives requests for services based on patient demand which can be assumed to be random and independent. The Goodness of Fit Test was

TABLE 1

DISTRIBUTION OF TESTS ORDERED BY TIME OF DAY AT THE
UNIVERSITY OF OKLAHOMA CLINICAL LABORATORIES
FOR THE MONTH OF NOVEMBER 1969

	Automated Tests										
Time	Elect.	Bun	Glu.	CA	Phos.	Phos.	SGOT	CPK	LDH	Bili.	Uric Acid
7-8a.m.	27	18	6	1	1	5	9	3	2	9	1
8-9a.m.	462	445	250	73	58	144	166	23	41	215	33
9-10a.m.	61	53	62	15	12	9	13	1	3	21	12
10-11a.m.	73	70	43	20	11	14	17	1	3	25	11
11-12 Noon	60	47	38	10	8	18	18	3	6	30	8
12-1p.m.	29	20	15	7	5	6	9	0	0	12	5
1-2p.m.	53	66	64	17	14	21	22	2	7	23	19
2-3p.m.	62	62	44	13	6	22	23	5	5	35	10
3-4p.m.	39	51	51	19	17	12	15	0	1	20	13
4-5p.m.	46	44	51	11	12	15	12	0	0	13	9
5-6p.m.	25	26	15	6	6	4	1	0	1	3	1
6-7p.m.	22	15	15	4	3	7	7	0	0	15	1
7-8p.m.	16	9	19	1	0	3	2	0	0	7	0
8-9p.m.	12	7	9	0	0	0	0	0	0	5	0
9-10p.m.	14	10	7	1	1	0	1	1	1	6	0
10-11p.m.	16	13	13	1	1	2	2	0	0	5	0
11-12 Mid.	6	3	5	0	0	0	0	0	0	1	0
12-1a.m.	8	6	5	0	0	0	0	0	0	4	0
1-2a.m.	8	3	4	1	0	0	0	0	0	2	0
2-3a.m.	0	0	1	0	0	0	0	0	0	2	0
3-4a.m.	2	2	2	1	0	1	0	0	0	2	0
4-5a.m.	2	0	0	0	0	0	0	0	1	2	0
5-6a.m.	2	3	2	0	0	0	0	0	0	1	0
6-7a.m.	1	1	1	0	0	0	0	0	0	1	0
Total	1,046	974	722	201	155	283	317	39	71	459	123

TABLE 1--Continued

Manual Tests					Request for Service					
BSP	Chol.	Acid Phos.	Amy.	Total Tests	Auto	Manual	Both	Stat	Today	Total
0	1	0	2	85	38	5	8	7	1	51
0	37	4	40	1,991	1,025	75	70	67	2	1,170
0	7	0	15	284	140	34	18	42	1	192
2	10	0	10	310	155	32	24	50	1	211
3	5	0	5	259	117	26	21	43	0	164
0	2	0	2	112	51	11	24	31	1	86
1	11	3	4	327	150	24	35	28	2	209
3	10	1	6	307	134	35	31	32	5	200
2	8	1	6	255	115	23	23	24	2	161
0	7	3	6	229	121	12	21	32	1	154
0	0	1	2	91	51	4	2	15	3	57
0	2	0	3	94	39	8	9	24	2	56
0	0	0	1	58	35	3	5	22	0	43
0	0	0	0	33	20	6	1	20	2	27
0	0	0	0	42	25	6	0	18	0	31
0	0	0	1	54	29	4	3	26	0	36
0	1	0	3	19	10	4	1	8	0	15
0	0	0	0	23	16	1	0	15	0	17
0	0	0	0	18	10	1	1	10	0	12
0	0	0	0	3	1	2	0	3	0	3
1	0	0	0	11	3	1	2	4	0	6
0	0	0	0	5	2	2	0	4	0	4
0	0	0	1	9	5	2	0	2	0	7
0	0	0	0	4	2	1	0	3	1	3
12	101	13	107	4,623	2,294	322	299	530	23	2,915

TABLE 2

GOODNESS OF FIT TEST
CASE 1

Interval	Coded Interval	Observed Frequency	Probability	Expected Frequency	$\frac{(0-E)^2}{E}$
8-9	0	1170	$P_0 = e^{-\mu} = .1422$	362.2	1800.0
9-10	1	192	$P_1 = \frac{\mu}{1}P_0 = .2772$	706.0	374.2
10-11	2	211	$P_2 = \frac{\mu}{2}P_1 = .2702$	688.2	330.9
11-12	3	164	$P_3 = \frac{\mu}{3}P_2 = .1756$	447.2	179.3
12-13	4	86	$P_4 = \frac{\mu}{4}P_3 = .0856$	218.0	79.9
13-14	5	209	$P_5 = \frac{\mu}{5}P_4 = .0333$	84.8	181.9
14-15	6	200	$P_6 = \frac{\mu}{6}P_5 = .0108$	27.5	
15-16	7	161	$P_7 = \frac{\mu}{7}P_6 = .0030$	7.64	
16-17	8	154	$P_8 = \frac{\mu}{8}P_7 = .0003$	.764	
17	9	0	$1 - \sum_{i=0}^{9} P$ .0000	0	

applied to the distribution of input for 1000 to 1700 Hrs. The critical $\chi^2 = 14.1$ for 6 degrees of freedom at the 5% error level. From Table 3, we again reject the null hypothesis that the unknown distribution is equal to a Poisson distribution.

A second difference in basic assumption is that the service time for most service centers in the clinical laboratory is essentially constant for those fully automated centers and more likely normally distributed for those centers requiring human intervention.

The third difference is that customers entering into the laboratory have a prescribed route through the laboratory and usually require several service centers to meet the total service demands. In most modern laboratories the expense of multiple service centers performing the same function is prohibitive; hence, either the customer remains queued up to that service center or his only alternative is a manual method which has entirely different service characteristics. While it is desirable to develop mathematical models whose characteristics describe the behavior of the desired system and whose mathematical formulation renders analytical solutions, simulation does provide an alternate means of determining characteristic behavior for a particular system for which some restrictive assumptions can be relaxed.

The results of simulation give a direct qualitative impression of what the behavior of the system should look like under the conditions postulated.

The simulation model must take into consideration the varieties of conditions present in the system under study. The various conditions in the clinical laboratory can best be defined by following a customer

TABLE 3

GOODNESS OF FIT TEST
CASE 2

Interval	Coded Interval	Observed Frequency	Probability	Expected Frequency	$\frac{(O-E)^2}{E}$
10-11	0	211	$P_0 = e^{-\mu} = .0528$	62.56	353.29
11-12	1	164	$P_1 = \frac{\mu}{1} P_0 = .1552$	183.91	2.15
12-13	2	86	$P_2 \quad \frac{\mu}{2} P_1 = .2281$	270.29	125.65
13-14	3	209	$P_3 = \frac{\mu}{3} P_2 = .2235$	264.84	11.75
14-15	4	200	$P_4 = \frac{\mu}{4} P_3 = .1642$	194.57	1.51
15-16	5	161	$P_5 = \frac{\mu}{5} P_4 = .0965$	114.35	19.0
16-17	6	154	$P_6 = \frac{\mu}{6} P_5 = .0472$	55.93	32.9
17	7	0	$1 - \sum_{i=0}^{9} P = .0350$	41.46	41.46
					587.71

through the various stages within the laboratory operation.

Customer arrival patterns can best be described by referring to Table 1. After the customer arrives at the laboratory, his total service demands are determined and a route through the required service channels is established. Each service channel is composed of one or more service centers in series. Since the service channels perform independent functions, the customer is partitioned into segments required by the particular service rendered at the various service channels. A customer number is assigned to correlate all the partitions as they proceed through each service center within the service channel. Customers are then dispatched to their respective service channels. There are basically two types of service channels, which possess high speed test equipment capable of processing large numbers of customers in a relatively short period of time and those which provide service predominantly by manual means. Due to the nature of some clinical laboratory tests, technology has not yet developed to the point of being economically feasible to automate these procedures.

At the service channel human intervention can be defined by three distinct activities.

1. Identify and rank the customers into some prescribed sequence or priority.
2. Monitor the test instrument during operation.
3. Record and convert raw test determinations into final determination results.

The time required to perform these activities is a function of human variability and can be attributed to difference in training, skill and motivation. A human factors study would have to be performed within each laboratory in order to determine valid estimates of these variables. However, a constant time per activity per customer is considered a reasonable estimation in the development of our simulation model.

After the customer leaves his respective service channel, he enters a data collection service channel in which all the various services rendered to any one customer are recorded. After his total service has been completed and recorded, he exits the system.

The GPSS simulation language is designed for systems whose operation can be described as special purpose blocks. These blocks can function as decision rules or activities which relate to the desired system structure. A sequence of these blocks can be developed to describe the many service centers of various types and the conditions by which customers utilize these service centers within the operation of a clinical laboratory.

The attributes which are measured during the operation of a simulation are defined as follows:

1. Block Counts
   - A. Current - Number of transactions in each block which the system is currently processing at the time of print-out.
   - B. Total - Total number of transactions which have been processed by the block.
2. Facility Attributes
   - A. Average Utilization - $\dfrac{\text{Cumulative time integral}}{\text{Relative clock time since last reset.}}$

B. Number Entries - Number of transactions processed by facility.

C. Average Time/Transaction - $\frac{\text{Cumulative time integral}}{\text{Number of entries.}}$

D. Cumulative Time Integral - The sum of time the facility is in use.

3. Queue Attributes

A. Maximum Contents - Maximum number of transactions in the queue during simulation.

B. Average Contents - $\frac{\text{Cumulative time integral}}{\text{Relative clock time since reset.}}$

C. Average Time/Transaction - $\frac{\text{Cumulative time integral}}{\text{Total entries}}$

D. $ Average Time/Transaction - $\frac{\text{Cumulative time integral}}{\text{Number of non-zero delay entries}}$

E. Total Entries - Total entries into the queue.

F. Zero Entries - Number of entries which had zero delay time in the queues.

4. Mean Waiting Time - Defines the mean of waiting time distribution of entries in the queue.

From this point on in the discussion of the simulation language, transactions will be synonymous with customers and/or job streams. The transactions are first generated at the Generate block and are moved from block to block in a similar manner in which the units of traffic that they represent progress in the real laboratory system.

Each such movement is an event that is due to occur at some point in time. The program maintains a record of the times at which these events are due to occur, then proceeds by executing the events in their correct sequence. An invariant sequential clock is maintained in order that the total simulation time can be determined and related to the time of the real system. Customer arrival time and service time can be determined by use of a random number generator and user defined distributions which are randomly sampled.

CHAPTER V

SIMULATION MODELS

In this chapter, we formulate three different models representative of a clinical laboratory which vary in complexity, the series queue model, the simple model and the complex model. The series queue model is an over-simplification of a total laboratory system but could realistically represent one type of service function. The purposes of this model are to illustrate the methodology used in constructing a simulation model and to demonstrate the type of parameters used in determining the behavioral characteristics of the model. The model is run in various modes to determine the effect of starting the system empty or starting the system with customers already present in the system.

The simple model addresses its formulation to the basic functions found in clinical laboratories. This model demonstrates the power of simulation over analytical techniques when the analytical formulation is beyond those characteristics of the classical single server models with Poisson input and negative exponential service time. These analytical limitations have been clearly demonstrated by Disney [11], Cox [8] and Neuts [25]. The simple model requires that each customer has a predetermined route through the system. This model allows customers to be partitioned into sub-customers and independently routed through the required service centers while maintaining queue facility statistics on the sub-customers as well as on regrouped sub-customers. A priority system is introduced into the model to determine the difference between the behavior of models with priority and those without priority.

The complex model represents an extension of the simple model by increasing the magnitude of the various functions as well as routes available to any customer in the system. The data collection function at the various service centers is altered to reflect a high speed data acquisition function relative to a computer. The effect these changes have on the overall behavior of the model is evaluated.

Notations and terminology used in this chapter:

Terminology

Transaction or Customer - a specimen for which service is rendered in the laboratory.

Queue - a line of customers or transactions waiting for service.

Service Center - function which performs or renders some type of service on one customer at a time.

Service Channel - a type of service rendered. It may have one or more service centers.

Automated Service Channel - a special class of services rendered especially by those instruments in the laboratory which are capable of processing several customers per hour.

Manual Service Channel - a special class of services rendered by techniques requiring predominantly human intervention in the service rendered.

Data Collection - function which represents the manipulation of, or acquisition of data.

Processing - function which is necessary to determine and prepare customers for cycling through the laboratory system in relation to their service requirements.

Notations

Q1 - Queue - number or type.

Q1A1 - Queue - service channel - type of service - queue number.

SC1 - Service center - number or type.

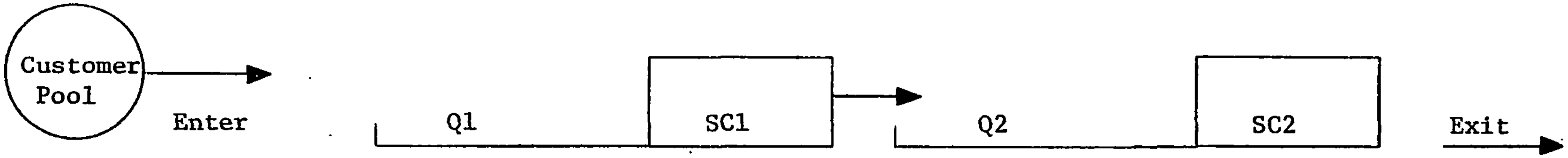
SC1M1 - Service center - service channel - type of service - service center number.

T - denotes the number of customers or transactions.

Series Queues

The construction of the model for two service centers in series represent one of the simplest configurations in the complex model and is presented here to illustrate the simulation methodology and techniques used. Figure 1 represents a series queue system with T customers arriving at service center SC1 to be immediately serviced or waiting in Q1 to be serviced. Upon completion of service at SC1, the customer enters the second queue to be immediately serviced or must wait in Q2 to be serviced. After completion of SC2, he exits the system.

The construction of the model for the two service centers in series is used to illustrate the power and flexibility of the GPSS language in simulation as well as the behavior of a common combination of service centers found in the clinical laboratories. The formulation



Series Queues

Figure 1

of the model is shown in Figure 2.

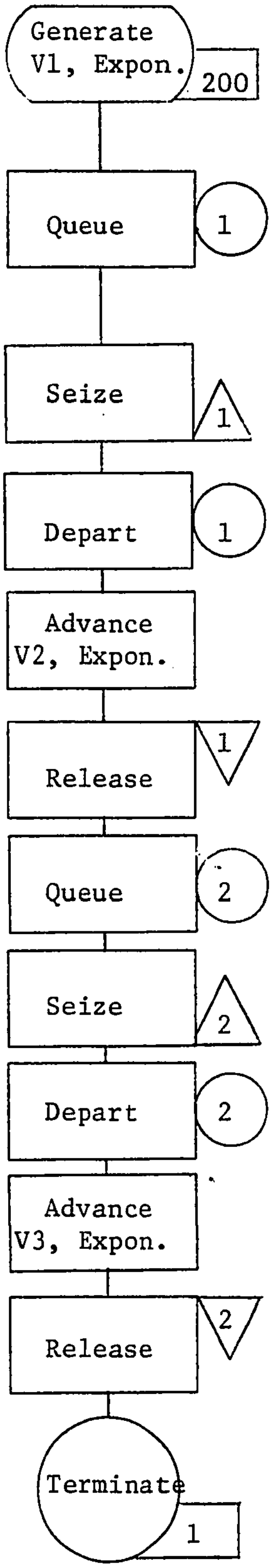
Assume 200 customers represent a normal day of operation for the modeled laboratory. The inter-arrival time between customers in exponentially distributed with parameter V_1, and the service time is also exponentially distributed with parameters V_2 and V_3. Let the parameters assume the following values:

V_1 = 144 seconds which represents an arrival rate of 25 customers/hour.

V_2 = 60 seconds which represents mean service time for SC1.

V_3 = 90 seconds which represents mean service time for SC2.

The model was run in three modes. Mode 1 partitioned the input customers into groups. The first group of 25 customers is run through the model, and then a group of 25 customers is added to the original group, and the model is rerun. The customers completing the service are counted as they exit the system. Each successive group acquired an additional 25 customers until the group size reaches 200 customers. Table 4 reflects the queue and facility attributes resulting from each group run. Mode 2 partitions the customers into larger groups of 100, 150 and 200 customers, and the model is reinitialized after each group is run. Reinitialization has an effect on the average time per transaction in that the cumulative time integral is set to zero, but the number in the queue at the time of reset is maintained; therefore, the average time per transaction in the queue will be less than or equal to the mean waiting time which reflects the mean of the actual waiting time distribution which is not effected by the reinitialization condition. Table 5 indicates the run statistics for Mode 2.



200 transactions will be generated with the inter-departure time being exponentially distributed with mean V_1.

Transaction enters queue block and queueing time is recorded if queue contents is not zero. If zero, the transaction proceeds to next sequential block immediately.

Seizes the first facility so that no other transaction can enter until processing is complete.

Calculates time the transaction spent in first queue.

Assigns positive time delay to each transaction entering the block based on exponential service time of mean V_2.

Releases the advance block in order that other transactions may seize the facility.

Same as above except defines second queue.

Same as above except for second service center.

Same as above except for second queue.

Second service center.

Same as above except for second center.

Terminates simulation when the number of transactions specified by generated block has entered.

Figure 2

TABLE 4

SERIES QUEUE
MODE 1

	Number of Transactions Completed							
	25	50	75	100	125	150	175	200
Clock Time	4146	5529	11983	15379	16834	20872	23704	27388
Facility Attributes								
Avg. Utilization								
SC1	.476	.458	.390	.366	.407	.480	.495	.412
SC2	.580	.869	.534	.592	.764	.652	.666	.575
Avg. Time/Trans.								
SC1	68.172	52.833	61.500	56.349	54.007	67.716	66.401	55.886
SC2	96.199	96.099	85.333	91.099	102.959	90.839	90.331	78.789
Queue Attributes								
Avg. Cont.								
Q1	.477	.259	.408	.220	.165	.618	.363	.409
Q2	.584	3.578	.389	1.265	1.384	1.183	1.239	.825
Zero Entries								
Q1	13	26	41	61	74	78	92	113
Q2	12	2	29	39	33	53	55	86
Avg. Time/Trans.								
Q1	61.843	30.574	64.355	34.252	22.000	87.216	48.909	55.731
Q2	86.535	388.000	61.447	192.693	182.046	164.659	166.948	111.866
$ Avg. Time/Trans.								
Q1	104.157	68.428	139.742	89.236	52.716	184.399	102.476	127.295
Q2	151.431	439.733	99.361	313.903	245.284	254.628	242.834	194.801
Mean Waiting Time Q1	63.896	33.255	64.355	34.252	22.000	87.216	48.909	55.731
Mean Waiting Time Q2	86.359	399.299	63.306	192.569	181.543	168.813	167.417	110.244

TABLE 5

SERIES QUEUE
MODE 2

	Number of Transactions		
	100	150	200
Clock Time	13813	23224	26377
Facility Attributes			
Average Utilization			
SC1	.433	.380	.461
SC2	.679	.574	.755
Average Time/Transaction			
SC1	58.72	58.87	60.85
SC2	93.83	88.90	99.649
Queue Attributes			
Maximum Contents			
Q1	4	4	11
Q2	8	11	11
Average Contents			
Q1	.286	.333	.497
Q2	1.678	.997	1.643
Zero Entries			
Q1	56	85	109
Q2	30	56	52
Average Time/Transaction			
Q1	38.44	51.93	65.32
Q2	229.53	153.37	216.73
$Average Time/Transaction			
Q1	84.25	120.90	142.72
Q2	326.52	243.78	292.88
Mean Waiting Time			
Q1	38.19	52.36	65.59
Q2	231.48	154.53	216.80

Mode 3 changes the service time distribution from negative exponential to a constant service time. Again input customers of group sizes 100, 150 and 200 are run against the model and Table 6 reflects the statistical attributes for this mode of operation.

Discussion of Series Queues Simulation Model

Review of Table 4 indicates no significant change in the results which would identify a change from the transient state to the steady state. By the time the model has processed 100 transactions, we assume that steady state has been reached. As seen from the Table, the queue and facility attributes for the second service facility in the case of 50 transactions is significantly greater than any of the other cases. This large difference could be attributed to random number generation selecting clustered samples from the extreme ends of the exponential distribution. Above average service time in SC2 would cause the output from SC1 to find SC2 busy more of the time. This can be seen from the fact that only 4% of the customers leaving SC1 found SC2 idle while 52% of the customers entering the system found SC1 idle.

Comparing the results for simulation Modes 1 and 2 from Tables 4 and 5 at the levels of 100, 150 and 200 transactions indicates the difference can be attributed to sampling variation, the number of transactions remaining in the system at the time of termination, and the effects of reinitialization.

Comparing Mode 3 with Mode 2, one can see the effect that constant service time has on the queue attributes. More customers arriving at the first and second queues find the queues empty; hence, the

TABLE 6

SERIES QUEUE
MODE 3

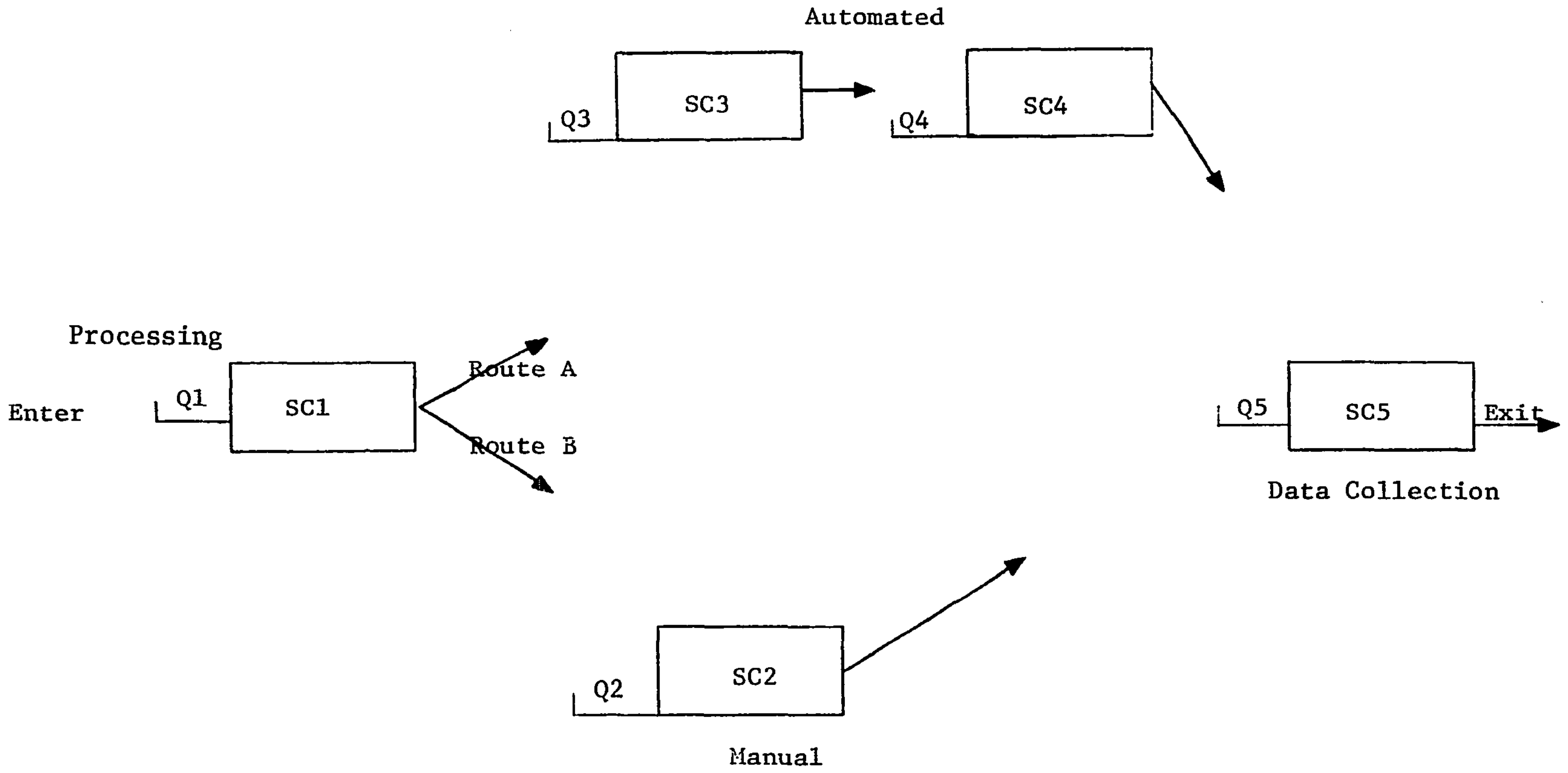
	Number of Transactions		
	100	150	200
Clock Time	14406	22847	26698
Facility Attributes			
Average Utilization			
SC1	.416	.396	.449
SC2	.624	.590	.674
Average Time/Transaction			
SC1	60.	60.	60.
SC2	90.	90.	90.
Queue Attributes			
Maximum Contents			
Q1	4	3	4
Q2	3	2	5
Average Contents			
Q1	.161	.106	.245
Q2	.272	.184	.708
Zero Entries			
Q1	60	103	98
Q2	39	72	55
Average Time/Transaction			
Q1	23.20	16.09	32.76
Q2	39.20	27.90	94.04
$Average Time/Transaction			
Q1	68.04	50.64	64.25
Q2	64.36	53.33	129.47
Mean Waiting Time			
Q1	23.21	16.09	32.76
Q2	39.25	27.88	94.63

number of zero entries is increased under Mode 3 type operation. A significant decrease in average time per transaction and mean waiting time can also be attributed to an improved processing time due to the constant service rate.

Simple Model

The purpose of the simple model shown in Figure 3 is to increase the complexity of the series queue model and approach the basic characteristics of the clinical laboratory. Customers arrive at the first test channel, SC1, which represents a processing station. If the service channel is busy, they wait in Q1 until it is free. In SC1, they are assigned routes A, B, or A and B through the remaining service channels which are required to meet their total service demands. There are two service channels, one represents the automated and the second represents the manual test stations. Customers may require service at any one or both channels. The manual channel, SC2, consists of one service center which represents test setup and the instrument operation function. The automated channel consists of two service centers, SC3 and SC4, the first of which represents the instrument operation function and the second, the data collection function. Customers completing service at SC3 must enter SC4. After the customer completes either or both of these service channels, he enters the final service channel, SC5, which represents the data collection function for his total service. After completion of this channel, he exits the system. The service time of all service centers is assumed to be constant.

Customers are generated based on a random sampling from a negative exponential density function, with a mean inter-arrival time of $1/\lambda$



Simple Model

Figure 3

or an arrival rate of λ customers/hour. The model was run under the set of conditions shown in Table 7.

One condition which is inherent in every clinical laboratory is that of servicing priority customers. A priority customer is defined to be that customer whose requirement for service is ranked above another customer's. Three priority levels were established in this model to represent:

1. Immediate service required - highest priority.
2. Service required today - 2nd highest priority.
3. Routine service required - lowest priority.

The number of transactions assigned to each priority level is as follows:

10% - highest priority.

5% - 2nd highest priority.

85% - lowest priority.

Customers arriving at SC1 were assigned Routes A, B, or A and B based on distribution estimates derived from Table 1, which reflects the number of customers requiring automated service only (Route A), manual service only (Route B) and both automated and manual service Routes A and B. The allocation of routes is accomplished by randomly selecting a route from a table of routes which weights the routes accordingly:

37.5% of total routes are for Route A.

37.5% of total routes are for Route A and B.

26% of total routes are for Route B.

The simulation was run in normal mode (no priorities) and priority mode. Both simulation runs use a 100 customer pool to warm up the model before processing the desired 200 customers. The results of the attributes

TABLE 7

SIMPLE MODEL RUN CONDITIONS

Run Number	Arrival Rate/Hr.	Service Time Parameters in Seconds				
		SC1	SC2	SC3	SC4	SC5
1	25	90	58	58	43	237
2	25	90	58	58	43	144
3	25	90	58	58	43	90
4	25	90	58	58	43	60
5	40	90	58	58	43	144

measured on each run in normal mode are shown in Tables 8 and 9. Tables 10 and 11 reflect the priority mode results.

Discussion of Simple Model Simulation Results

The results of this model clearly indicate that the input and output service centers apply the only constraints on the transaction throughput. Transactions receive immediate service from the service channels and delays only occur at the processing and data collection service centers. The utilization of the service centers remains approximately constant except when the arrival rate is increased by 60%; then the utilization increases for each service center by 30-40%. Figure 4 approximates a negative exponential reduction in queueing time as the service rate is increased. By decreasing the service time at SC5 by 65%, a 98.7% decrease in waiting time can be acquired. This basically is a balance of the input which is 25 transactions per hour; therefore, the mixed service rates of the other service centers have a degrading effect on the arrival rate to the last service center even though transactions are not required to wait for service. When the input rate and the service rate are increased for SC5, in run #5, the service center is always busy, whereas for the same service rate and reduced input rate as in run #2, the service center has available idle time.

The output from the model run in priority mode was not different from the normal mode except in run #5 where the arrival rate is increased. The priorities appear to have the effect of prolonging the queueing time.

Complex Model

In the complex model, we represent the clinical laboratory as a

TABLE 8

SIMPLE MODEL FACILITY ATTRIBUTES

Facility	Service Time	Average Utilization	Number Entries	Average Time/Trans.	Run No.
SC1	90	.621	329	89.537	1
SC1	90	.637	212	89.311	2
SC1	90	.641	203	89.467	3
SC1	90	.641	202	89.905	4
SC1	90	.981	315	89.704	5
SC2	58	.212	175	57.67	1
SC2	58	.216	111	58.00	2
SC2	58	.221	108	58.00	3
SC2	58	.221	108	58.00	4
SC2	58	.341	170	57.85	5
SC3	58	.367	301	57.81	1
SC3	58	.378	194	58.00	2
SC3	58	.379	186	57.747	3
SC3	58	.378	185	58.00	4
SC3	58	.578	288	57.87	5
SC4	43	.272	301	42.95	1
SC4	43	.280	194	43.00	2
SC4	43	.280	185	43.00	3
SC4	43	.279	185	42.83	4
SC4	43	.429	288	43.00	5
SC5	237	1.0	200	237.00	1
SC5	144	.969	200	144.00	2
SC5	90	.635	200	90.00	3
SC5	60	.423	200	60.000	4
SC5	144	1.000	200	144.00	5

TABLE 9

SIMPLE MODEL QUEUE ATTRIBUTES

Queue	Max. Contents	Avg. Contents	Total Entries	Zero Entries	Average Time/Tran.	$ Average Time/Tran.	Mean Waiting Time	Run No.
Q1	6	.638	328	118	92.20	144.01	92.33	1
Q1	6	.552	211	81	77.76	126.22	77.76	2
Q1	6	.552	205	79	76.32	124.17	75.53	3
Q1	6	.548	206	80	75.42	123.30	73.70	4
Q1	22	11.07	331	6	963.42	981.21	982.42	5
Q2	1	0	175	175	0	0	0	1
Q2	1	0	111	111	0	0	0	2
Q2	1	0	108	108	0	0	0	3
Q2	1	0	108	108	0	0	0	4
Q2	1	0	170	170	0	0	0	5
Q3	1	0	301	301	0	0	0	1
Q3	1	0	194	194	0	0	0	2
Q3	1	0	186	186	0	0	0	3
Q3	1	0	185	185	0	0	0	4
Q3	1	0	287	287	0	0	0	5
Q4	1	0	300	300	0	0	0	1
Q4	1	0	194	194	0	0	0	2
Q4	1	0	185	185	0	0	0	3
Q4	1	0	185	185	0	0	0	4
Q4	1	0	288	288	0	0	0	5
Q5	197	130.58	397	0	15590.73	15590.73	18551.35	1
Q5	17	6.55	215	6	905.54	931.54	894.614	2
Q5	1	.013	200	191	1.93	43.00	1.93	3
Q5	1	.003	200	192	.519	13.00	.519	4
Q5	171	111.65	371	0	8667.8	8667.8	10241.81	5

TABLE 10

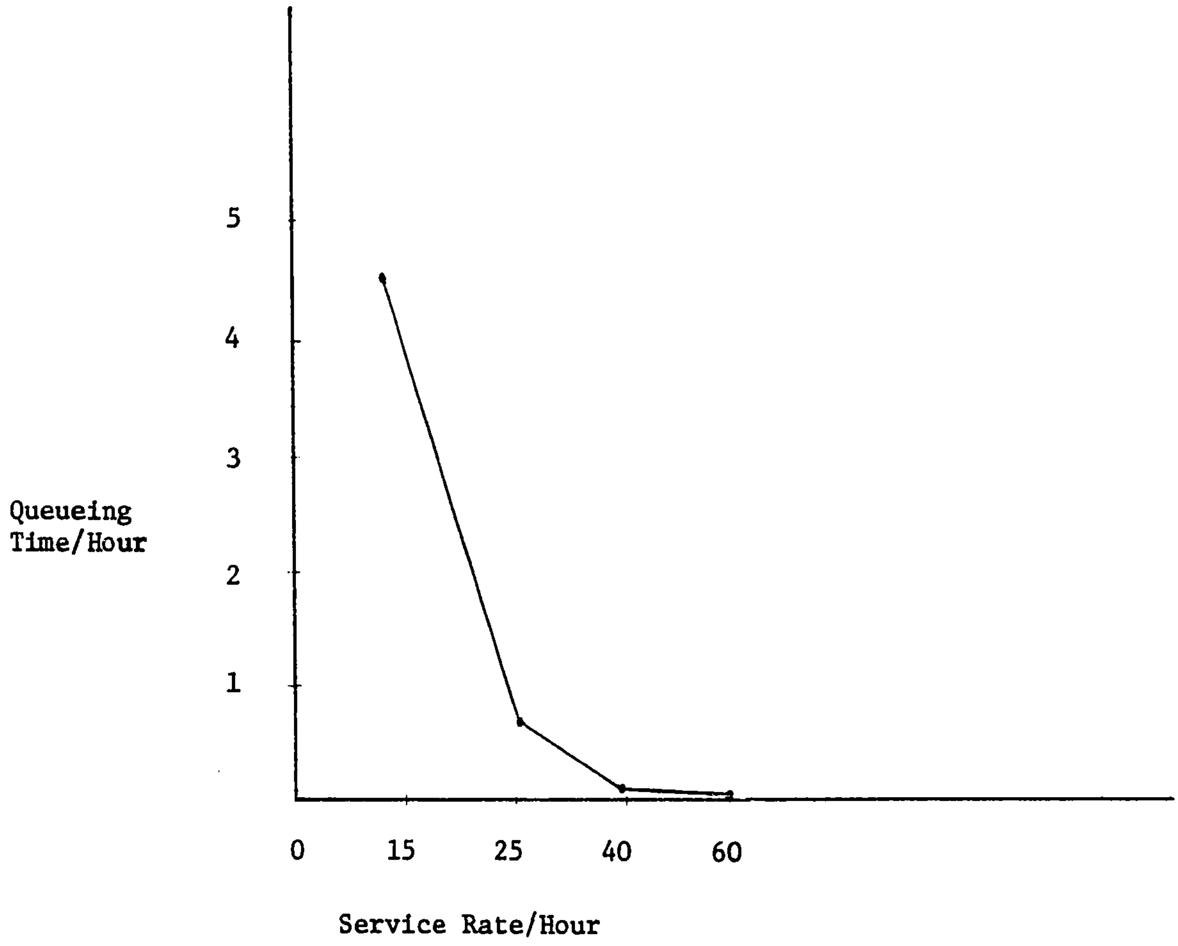
SIMPLE MODEL FACILITY ATTRIBUTES WITH PRIORITY

Facility	Service Time	Average Utilization	Number Entries	Average Time/Trans.	Run No.
SC1	90	.672	355	89.78	1
SC1	90	.681	219	89.65	2
SC1	90	.663	203	89.61	3
SC1	90	.662	202	89.90	4
SC1	90	1.000	321	89.71	5
SC2	58	.233	191	58.00	1
SC2	58	.233	116	58.00	2
SC2	58	.228	108	58.00	3
SC2	58	.228	108	58.00	4
SC2	58	.350	174	58.00	5
SC3	58	.399	327	57.94	1
SC3	58	.403	201	57.75	2
SC3	58	.391	186	57.74	3
SC3	58	.391	185	58.00	4
SC3	58	.592	295	57.80	5
SC4	43	.296	327	42.90	1
SC4	43	.300	201	43.00	2
SC4	43	.290	185	43.00	3
SC4	43	.289	185	42.83	4
SC4	43	.438	294	43.00	5
SC5	237	1.000	200	237.00	1
SC5	144	1.000	200	144.00	2
SC5	90	.656	200	90.00	3
SC5	60	.437	200	60.00	4
SC5	144	1.000	200	144.00	5

TABLE 11

SIMPLE MODEL QUEUE ATTRIBUTES WITH PRIORITY

Queue	Max. Contents	Avg. Contents	Total Entries	Zero Entries	Average Time/Tran.	$ Average Time/Tran.	Mean Waiting Time	Run No.
Q1	5	.558	354	128	74.73	117.05	74.73	1
Q1	4	.585	219	78	77.05	119.68	77.22	2
Q1	4	.523	205	77	70.07	112.22	69.88	3
Q1	4	.521	204	77	70.09	112.59	68.82	4
Q1	49	30.48	356	0	2466.26	2466.26	2524.54	5
Q2	1	0	191	191	0	0	0	1
Q2	1	0	116	116	0	0	0	2
Q2	1	0	108	108	0	0	0	3
Q2	1	0	108	108	0	0	0	4
Q2	1	0	174	174	0	0	0	5
Q3	1	0	327	327	0	0	0	1
Q3	1	0	200	200	0	0	0	2
Q3	1	0	186	186	0	0	0	3
Q3	1	0	185	185	0	0	0	4
Q3	1	0	294	294	0	0	0	5
Q4	1	0	326	326	0	0	0	1
Q4	1	0	201	201	0	0	0	2
Q4	1	0	185	185	0	0	0	3
Q4	1	0	185	185	0	0	0	4
Q4	1	0	294	294	0	0	0	5
Q5	222	148.94	422	0	16730.05	16730.05	16722.30	1
Q5	31	17.80	231	0	2219.98	2219.98	2300.00	2
Q5	1	.019	201	187	2.67	38.42	2.68	3
Q5	1	.005	200	189	.71	13.00	.71	4
Q5	171	110.32	371	0	8564.51	8564.51	8655.05	5



SC5 Service Rates Related to Queueing Time

Figure 4

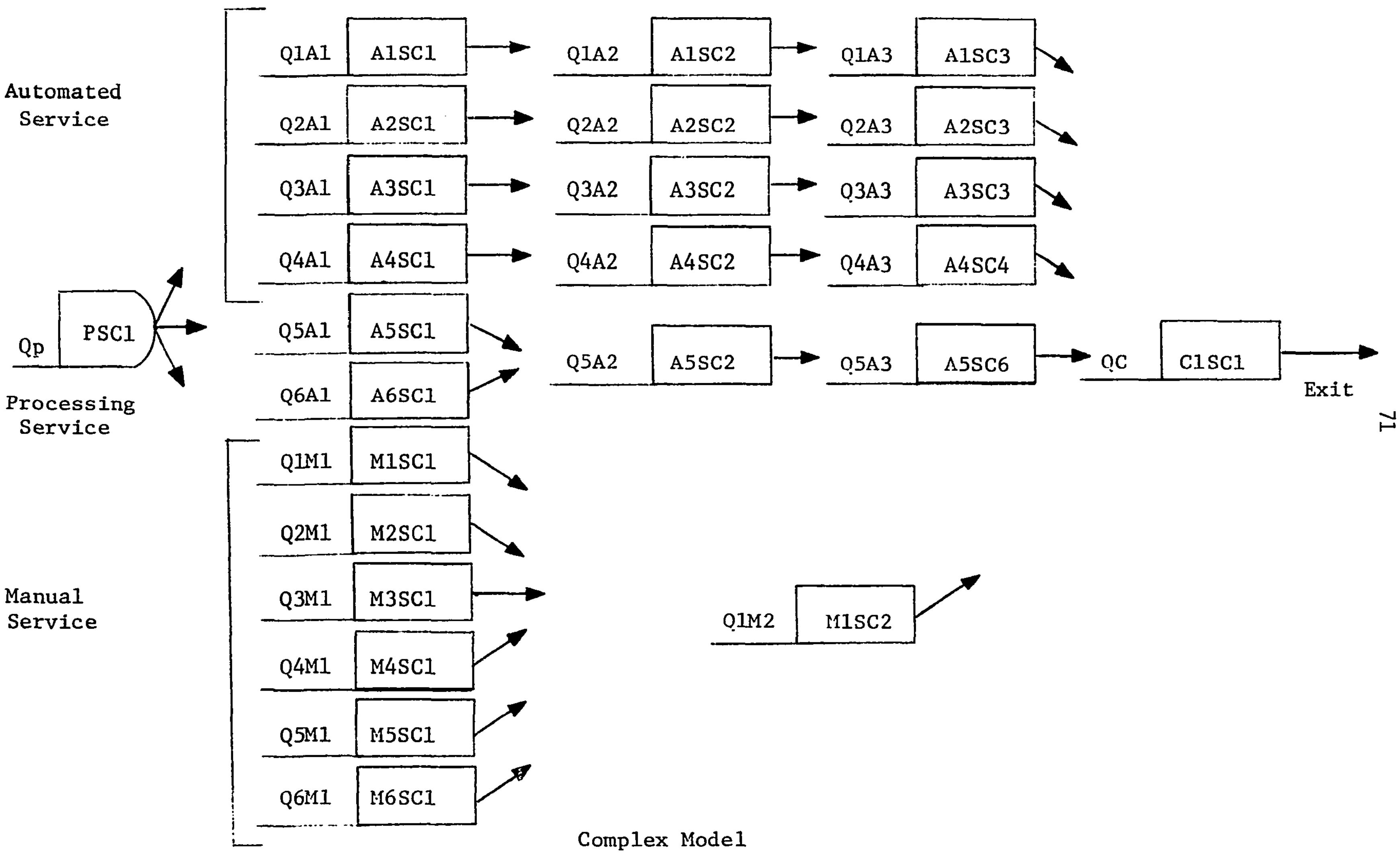
network of both series and parallel service centers. Customers arrive at the first service center for preprocessing and to obtain their service route through the system. There are five service channels in parallel, each composed of three service centers. These service channels represent the functions incorporated in the simulation of the automated instrumentation found in the clinical laboratory such as continuous flow analyzers and spectrophotometers and other laboratory instrumentation. The three service centers in each channel represent the functions, test setup, processing and data collection.

In parallel to the automated service channels there exist six service centers which represent the different manual test equipment such as colormeters, flame photometers, pH meters, etc. The test setup function is different for each type of test, but the processing and data collection function is common to all manual tests.

Customers completing either manual, automated or both service channels must enter the final data collection service center for recording of total service demands. Figure 5 illustrates the total complex model indicating the customer flow through the system as well as the various queues a customer may encounter within the system.

Determination of Parameters Used in the Model

The arrival pattern of customers to the laboratory was based on Table 1, which reflects a relatively constant arrival during the first 1 1/2 to 2 hours and an undetermined arrival pattern during the remainder of the day. The arrival rate was set at 60 second intervals up to the first simulated 1 1/2 hours or a maximum of 70 customers during the time interval. After this time, a Poisson arrival rate of approximately 12



Complex Model

Figure 5

customers per hour was assumed.

Prior to customers arriving at the first service centers, customers are assigned one of three levels of priority - 0, 1, and 2 with 2 being highest priority. These priority levels represent the normal range of priorities found in most laboratories, i.e., routine, emergency, and as soon as possible. The percentage of priorities assigned at each level is based on the actual percentages reflected in Table 1. This is as follows:

18% were assigned priority 2

5% were assigned priority 1

77% were assigned priority 0

At the first service center, the customer's route through the system is established. Customers may be routed through any one service channel or a combination of channels. Customers entering a service channel must be processed by all service centers within the channel before entering another channel. Customers whose service requires multi-channel use are partitioned into the desired number and sent simultaneously to each required service channel. Partitioned customers are re-grouped at the data collection service center before service is determined complete. Customers may require service from automated channels, manual channels or both channels. The percentage of types of channels required for the total customer input was based on data from Table 1, and is as follows:

11% assigned to manual services only.

79% assigned to automated services only.

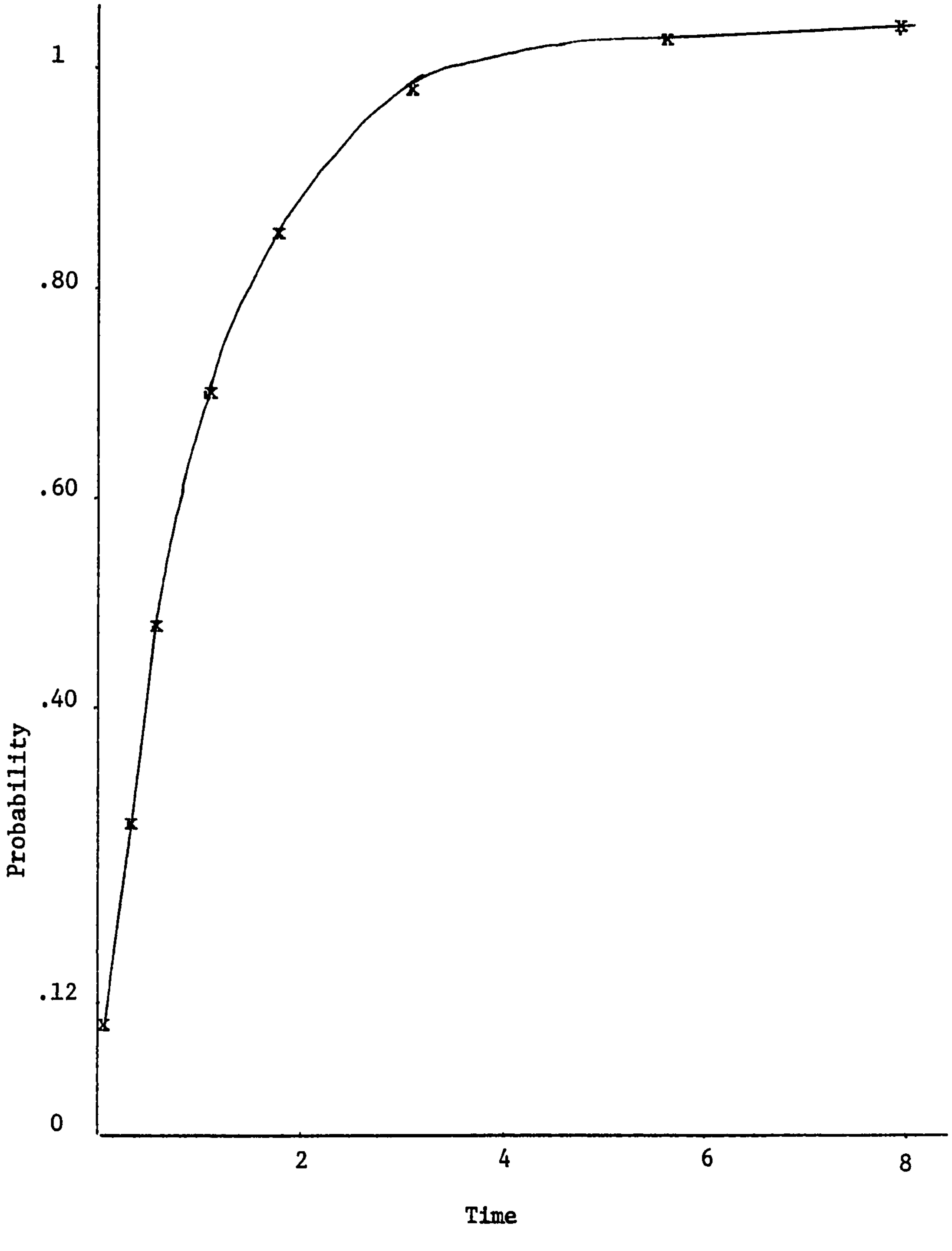
10% required both types of service.

The customer directed to either or both types of services can use one or a combination of service channels within the specific service. The number of channels used by each customer is based on a weighted distribution of the tests reflected in each type of service indicated in Table 1. A function was developed based on these weights which randomly assigns the number and type of service channels required to meet its specific service demand. The customer's total service demand is the service demand rendered by both the automated and manual service sections. A negative exponential distribution function shown in Figure 6 was randomly sampled to determine the inter-arrival time of the next customer after the generation of the first 70 customers. The variables used to estimate service time are shown in Table 12.

Customers are generated from an infinite source, but when 150 customers have completed service, the running is terminated and the facility and queue attributes are calculated. One hundred and fifty customers were used to estimate the number of customers processed during the operation of the real laboratory. These attributes are shown in Tables 13 and 14.

In order to determine the effect of an automated data acquisition system on system throughput, the data collection variables were changed as follows:

Service Center	Normal Service Time	Data Acquisition Service Time
A1SC3	90, 10	10, 0
A2SC3	90, 10	10, 0
A3SC3	90, 10	10, 0
A4SC3	90, 10	10, 0
A5SC3	120, 30	10, 0
C1SC1	120, 60	30, 0



Negative Exponential Distribution

Figure 6

TABLE 12

SERVICE CENTER VARIABLES

Type Service	Service Center	Service Time in Seconds Mean	Service Time in Seconds S.D.	Tests/ Hour	Function	Test Simulated
Processing	PSC1	180	60	20	--	All
Automated	A1SC1	120	30	30	Setup	Electrolytes
	A1SC2	90	0	40	Processing	Electrolytes
	A1SC3	90	10	40	Data Collection	Electrolytes
	A2SC1	120	30	30	Setup	Bun/Glucose
	A2SC2	60	0	60	Processing	Bun/Glucose
	A2SC3	90	10	40	Data Collection	Bun/Glucose
	A3SC1	120	30	30	Setup	Ca/Phos.
	A3SC2	60	0	40	Processing	Ca/Phos.
	A3SC3	90	10	40	Data Collection	Ca/Phos.
	A4SC1	120	30	30	Setup	Alk/Phos/SGOT
	A4SC2	60	0	60	Processing	Alk/Phos/SGOT
	A4SC3	90	10	40	Data Collection	Alk/Phos/SGOT
	A5SC1	120	30	30	Setup	CPK
	A5SC2	120	30	30	Processing	CPK, LDH
	A5SC3	120	30	30	Data Collection	CPK, LDH
	A6SC1	120	30	30	Setup	LDH
Manual	M1SC1	150	20	25	Setup	Bilirubin
	M2SC1	240	30	15	Setup	Uric Acid
	M3SC1	180	30	20	Setup	BSP
	M4SC1	120	20	30	Setup	Cholesterol
	M5SC1	90	10	40	Setup	Acid Phos.
	M6SC1	120	30	30	Setup	Amylase
	M1SC2	60	10	60	Process/Data Collection	(All Manual Tests)
Data Collection	C1SC1	120	60	30	Total Data Collection	All

TABLE 13

COMPLEX MODEL FACILITY ATTRIBUTES

Facility	Average Utilization	Number Entries	Average Time/Tran.
PSC1	.977	150	182.253
M1SC1	.095	18	148.222
M2SC1	.149	17	246.294
M3SC1	.006	1	177.000
M4SC1	.033	8	117.375
M5SC1	.029	9	90.555
M6SC1	.034	8	121.375
M1SC2	.070	33	59.363
A1SC1	.588	136	121.117
A1SC2	.437	136	90.000
A1SC3	.441	136	90.720
A2SC1	.398	92	121.293
A2SC2	.197	92	60.000
A2SC3	.295	92	89.967
A3SC1	.408	95	120.178
A3SC2	.203	95	60.000
A3SC3	.306	95	90.157
A4SC1	.249	58	120.310
A4SC2	.124	58	60.000
A4SC3	.187	58	90.517
A5SC1	.112	27	117.037
A6SC1	.116	27	120.555
A1SC2	.117	27	121.703
C1SC1	.670	150	125.086

TABLE 14

COMPLEX MODEL QUEUE ATTRIBUTES

Queue	Max. Cont.	Avg. Cont.	Total Entries	Zero Entries	Average Time/Tran.	$ Average Time/Tran.	Mean Queueing Time
QP	50	25.776	150	1	4807.058	4839.320	4807.058
Q1M1	1	.011	18	13	17.111	61.599	17.111
Q2M1	1	.025	17	14	42.411	240.333	42.111
Q3M1	1	.000	1	1	.000	.000	.000
Q4M1	1	.000	8	8	.000	.000	.000
Q5M1	1	.003	9	8	10.777	97.000	10.777
Q6M1	1	.004	8	7	16.750	134.000	16.750
Q1M2	1	.002	33	30	2.090	23.000	2.090
Q1A1	1	.003	136	128	.742	12.625	.742
Q1A2	1	.000	136	136	.000	.000	.000
Q1A3	1	.000	136	136	.000	.000	.000
Q2A1	3	.122	92	63	37.097	117.689	37.097
Q2A2	1	.000	92	92	.000	.000	.000
Q2A3	1	.000	92	91	.032	3.000	.032
Q3A1	4	.216	95	53	63.884	144.500	63.884
Q3A2	1	.000	95	95	.000	.000	.000
Q3A3	1	.000	95	94	.010	1.000	.010
Q4A1	2	.037	58	49	18.051	116.333	18.051
Q4A2	1	.000	58	58	.000	.000	.000
Q4A3	1	.000	58	58	.000	.000	.000
Q5A1	1	.009	27	25	10.000	135.000	10.000
Q6A1	1	.001	27	26	1.629	44.000	1.629
Q5A2	1	.000	27	26	.555	15.000	.555
QC	2	.166	150	90	30.986	77.466	30.986

The model terminated after 150 customers had been serviced. Tables 15 and 16 reflect the system's attributes with these variables changed.

Discussion of Results

The data reflected in Table 13 indicates that the processing and total data collection service centers are busy more than 50% of the time which is representative of a normal laboratory operation. Even though M1SC2 received input from six different service centers, the idle time remains greater than 93%. The automated service channels are busy less than 60% of the time. From observing the overall utilization of the complex model, one can infer that input for the automated service center could be increased by 60% before saturation and the manual service center could support a twenty-fold increase. The only major constraints are the first and last service centers whose service time could be reduced by increasing the number of servers performing this service.

Table 14 indicates that most transactions are not held up waiting in the queue except for OP and OC. Some transactions are held up for more than one hour in QP. This is due to the high influx of customers during the first two hours of operation and the low service time in relation to input.

The effect of imposing conditions representative of data acquisition at the various data collection service points is reflected in Tables 15 and 16. The utilization of these service centers was drastically reduced which was to be expected. This reduction provides equal increase in the idle time to those functions requiring manual intervention or increasing the workload at the service center to compensate for the increased idle time. The two service centers preceding the data

TABLE 15

COMPLEX MODEL FACILITY ATTRIBUTES
WITH DATA ACQUISITION

Facility	Average Utilization	Number Entries	Average Time/Trans.
PSC1	.997	152	179.144
M1SC1	.116	21	151.904
M2SC1	.168	19	242.000
M3SC1	.006	1	183.000
M4SC1	.049	11	122.909
M5SC1	.036	11	90.636
M6SC1	.037	9	112.222
M1SC2	.075	35	58.571
A1SC1	.596	133	122.368
A1SC2	.435	132	90.000
A1SC3	.048	132	10.000
A2SC1	.390	89	119.842
A2SC2	.193	88	60.000
A2SC3	.032	88	10.000
A3SC1	.414	93	121.688
A3SC2	.204	93	60.000
A3SC3	.034	93	10.000
A4SC1	.251	58	118.327
A4SC2	.125	57	60.000
A4SC3	.020	57	10.000
A5SC1	.118	26	124.730
A6SC1	.118	26	124.346
A5SC2	.009	26	10.000
C1SC1	.164	150	30.000

TABLE 16

COMPLEX MODEL QUEUE ATTRIBUTES
WITH DATA ACQUISITION

Queue	Max. Cont.	Avg. Cont.	Total Ent.	Zero Ent.	Average Time/Trans.	$ Average Time/Trans.	Mean Queueing Time
QP	46	28.971	164	1	4821.003	4850.582	5115.957
Q1M1	1	.005	21	19	6.571	69.000	6.571
Q2M1	2	.043	19	13	61.947	196.166	61.947
Q3M1	1	.000	1	1	.000	.000	0.000
Q4M1	1	.004	11	9	10.454	57.500	10.454
Q5M1	1	.002	11	10	7.272	80.000	7.272
Q6M1	1	.003	9	8	10.333	93.000	10.333
Q1M2	1	.002	35	32	1.571	18.333	1.571
Q1A1	1	.004	133	124	.849	12.555	.849
Q1A2	1	.000	132	132	.000	.000	.000
Q1A3	1	.000	132	132	.000	.000	.000
Q2A1	3	.125	89	66	38.337	148.347	38.337
Q2A2	1	.000	88	88	.000	.000	.000
Q2A3	1	.000	88	88	.000	.000	.000
Q3A1	4	.291	93	55	85.473	209.184	85.473
Q3A2	1	.000	93	93	.000	.000	.000
Q3A3	1	.000	93	93	.000	.000	.000
Q4A1	2	.038	58	49	18.000	116.000	18.000
Q4A2	1	.000	57	57	.000	.000	.000
Q4A3	1	.000	57	57	.000	.000	.000
Q5A1	2	.009	26	23	9.769	84.666	9.769
Q6A1	1	.002	26	22	2.730	17.750	2.730
Q5A2	1	.000	26	26	.000	.000	.000
AC	1	.003	150	144	.639	16.000	.639

collection service center on the automated service channel can be totally saturated and the data collection function will still have idle time. This is due to the fact that a 50% utilization in the first two service centers only requires a 5% utilization in the data collection service center.

The data acquisition conditions had a 30 to 1 reduction in the waiting time a Qc, hence 96% of the customers entering C1SC1 did not have to wait for service.

We can conclude from the results shown in Tables 15 and 16 that the average throughput time is not significantly improved, but the average idle time per service center is increased.

CHAPTER VI

SUMMARY

Our first approach was to develop an analytical formulation which could provide some measure of effectiveness for a complex queueing model. In reducing the complex system to the classical case of the single server system with Poisson Input but constant service time in lieu of negative exponential distribution of service time, we were able to develop a complex expression for the expected total queueing time. Neuts [25], Cox [8], Disney [11], and others have previously discussed the difficulty in obtaining closed form expressions for measures of effectiveness when one deviates from the classical Poisson Input and service time conditions or when transient solutions are desired. Since the analytical approach proved to be unmanageable in the formulation as a means of formulating the rules governing the behavior of the system, the GPSS simulation language was used to translate the problem definition into computer executable form by which measured service center and queue attributes could be quantized.

Three simulation models were developed. The first model represents the simplest queueing subsystem, i.e., two service centers in series, within the total complex network. Customer input into the system was augmented by a group of 25 and the model run for each size group up to 200 customers. There appeared no significant change in the measured attributes which would identify a change from transient state to steady state. Customers were then grouped in sizes of 100, 150 and 200 and compared with the first run. The differences indicated can be attributed to sampling variation and the effects of reinitialization. The service time was changed from value sampled from a negative exponential distribution to a constant service time. The measured attributes indicated a significant decrease in mean waiting time as well as queueing time per transaction.

The second model represents the basic subsystem in combination of series and parallel configurations which increases the complexity of the first model. The results of this model indicate that the input and output service centers provide the only significant constraints on the customer throughput. A 65% increase in output service center service rate decreased the queueing time per transaction by 98%. The model was also run in priority mode and the effect of priorities appears to be related to an increase in queueing time.

The third model is the complex model which represents the operational characteristics of a clinical laboratory. The model consists of multiple service channels whose service centers are in both series and parallel. Each service channel can be related to a particular service function found in the clinical laboratory. The parameters used in the model were derived from samples taken from the operation of the University of Oklahoma Clinical Laboratory. The results of running the model clearly indicated that certain key points in the job processing cycle require customers to nearly always wait before receiving service and other points are primarily idle. A bottle neck appears at the input center which can be related to the distribution of the input job stream which is constant for the first two hours and then approximates a Poisson arrival rate for the remaining period.

The effect of imposing conditions representative of automatic data acquisition at various data collection points in the processing cycle, reduced service center utilization but did not significantly improve overall throughput time. This indicates that the laboratory could improve its capacity to handle more jobs without sacrificing job throughput time.

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