INTRODUCTION TO GREEN'S FUNCTIONS

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PREFACE

In 1828, George Green introduced a function which is in an integral solution of the potential equation. It is known as Green's function. Green's function is now associated with most boundary value problems. For example, the boundary value problem $u''(x) = f(x)$, $0 \leq x \leq 1$, with the boundary conditions $u(0) = u(1) = 0$ has the solution

$$u(x) = \int_0^1 G(x,t)f(t)dt.$$

The function $G(x,t)$ is called Green's function. This is the function which we want to determine.

The aim of this study is to present Green's function as a kernel of the integral for the solution of a boundary value problem in an ordinary differential equation (or a partial differential equation). Emphasis will be <u>on</u> <u>finding</u> Green's function. The purpose is to make this as easy and simple as possible by illustrating with examples. It is hoped that this study will be of interest to students of mathematics, physics, or engineering following a first course in both ordinary and partial differential equations.

This study has two parts. There is the ordinary differential equation part, which is in Chapter II, and the partial differential equation part, which is in Chapters III, IV and V.

Chapter I is concerned with background material used in this study, such as Green's second identity, Euler's theorem, and Leibnitz's rule.

It is hoped that the reader is already familiar with these things.

In Chapter II, Green's functions are introduced by means of boundary value problems associated with second-order ordinary differential equations. A number of examples serve to illustrate how Green's functions of the problems are found. A simple example is given first and more information is added in each succeeding example. In each example, properties of Green's function are given which lead to the properties of Green's function in more general cases. In particular, the case of unmixed boundary conditions, mixed boundary conditions, an initial condition, and boundedness as a boundary condition are considered. After giving the properties for a given case the problem is solved by using these properties. Then the other methods for finding Green's functions, namely, the use of a formula and the method of variation of parameters, are introduced. The example of unmixed boundary condition is used to show how to find Green's function by using the formula as well as the method of variation of parameters for mixed boundary conditions. After finding Green's function for the equations of the second order, an example of a third-order equation with unmixed conditions is given. This is then extended to the discussion of Green's function for an equation of order n. Most of the examples deal with homogeneous boundary conditions, since the problem of nonhomogeneous boundary conditions depend on it. Section 2.6 shows how to solve the problem with nonhomogeneous boundary conditions. In some special cases Green's functions cannot be found; then generalized Green's functions are necessary. Example of generalized Green's functions are discussed in Section 2.8. For the purpose of making things simple, the examples customarily deal with the interval [0,1]. For the case of the interval [a,b], it is shown that Green's function can be found in a similar way.

Chapters III through V are concerned with boundary value problems associated with second order partial differential equations. In particular, Laplace's equation, Helmholtz's equation, and the heat equation are considered. The Dirichlet boundary condition and the Neumann boundary condition are used in the examples. In the beginning of each chapter, the fundamental solution which is a part of Green's function is introduced. Then, the properties of Green's function are given. This leads to a method of finding Green's function, namely, the method of images. A number of examples show how to find Green's function by the method of images. The regions considered are those of a half-plane, a quarter-plane, an angular region with angle $\pi/3$, an angular region with angle π/k, a region between parallel lines, some intersections of parallel strips, a disk and a half-disk, some intersections between a disk and an angular region, a half-space, a sphere, and a hemisphere. Regions in one dimension, in particular, a line, a half-line, and a segment of a line are also considered for the heat equation. The method of images for each region is illustrated. Then, the method of images is extended to problems in n-space. The other method used in finding Green's function is the method of eigenfunctions, which is used for the problem of a region that is a half-disk, a quarter-disk, or a cube. The conformal mapping is used in two dimensions in finding Green's function for the Laplace's equation with a Dirichlet boundary condition. A number of examples illustrate this. In the case of Laplace's equation, the symmetry of Green's functions is shown.

By the time the reader completes Chapter V, it is hoped that he will be able to solve particular problems where Green's functions are

work out in detail.

The author has good reasons to be grateful to her adviser, Professor Eugene K. McLachlan, and she would like to express her sincere gratitude to him for his help, suggestions, and patience in making this study possible. The author also wishes to express her appreciation to the other members of her committee, Professors Marvin Keener, Dennis Bertholf, Douglas Aichele, and James Yelvington, for their understanding and suggestions. In addition, the author feels a debt of thanks to the Thai government which gave financial support for her education.

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CHAPTER I

INTRODUCTION

Green's function was first introduced by George Green in 1828. Green gave the solution V of the potential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

in a region $\mathcal{R}$ in xyz-space with boundary surface S as

$$4\pi V = -\iint_S V \frac{\partial U}{\partial n} \, d\sigma.$$

He introduced the function U with three properties:

(1). U must be 0 on the surface, S,

(2). U satisfies the potential equation in $\mathcal{R}$, and

(3). at a fixed undetermined point P in the interior, U becomes infinite as 1/r where r is the distance of any point from P.

The function U was later called Green's function by Riemann. Green's achievement was followed with additional work by Gauss in 1839 and Hilbert in 1904. Hilbert used Green's function to formulate the Sturm-Liouville boundary value problem. Thus, Green's function is associated with both ordinary and partial differential equations as will be shown in the later chapters.

1.1. Background

As a basis of understanding this study, it will be helpful to have handy for quick reference several things. These will be mentioned in brief detail with references to a larger treatment. The first two of these are used with ordinary differential equations.

1.1.1. Leibnitz's Rule

For any continuous function $f(x,t)$ whose derivative $f_x(x,t)$ is piecewise continuous, we have

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(x,t)\, dt = f(x, g_2(x))g_2'(x) - f(x,g_1(x))g_1'(x) + \int_{g_1(x)}^{g_2(x)} f_x(x,t)\, dt$$

where $g_1(x)$ and $g_2(x)$ are differentiable [16, p. 285].

1.1.2. Dirac Delta Function

The δ-function is zero for every value of x except the origin, where it is infinite in such a way that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Some properties of the δ-function that will be used are:

(1). $\displaystyle\int_{-\infty}^{\infty} \delta(x-x_0)\, f(x)\, dx = f(x_0)$

for every continuous function $f(x)$.

(2). $\delta(x) = \delta(-x)$.

$$(3).\quad \int_{-\infty}^{\infty} \delta^{(n)}(x)\ f(x)\ dx = (-1)^n \left.\frac{d^n f}{dx^n}\right|_{x=0}.$$

A good discussion of the δ-function may be found in [7, pp. 135-141].

For the later discussion of Green's functions for partial differential equations, the following background is used.

1.1.3. Euler's Theorem

If $f(y_1,y_2,\ldots,y_p)$ is a homogeneous function of the nth degree in the variables $y_1,y_2,\ldots,y_p$, that is,

$$f(\lambda y_1,\lambda y_2,\ldots,\lambda y_p) = \lambda^n f(y_1,y_2,\ldots,y_p)$$

and let $x_1 = y_1^{n_1}$, $x_2 = y_2^{n_2},\ldots,$ $x_p = y_p^{n_p}$

then

$$n_1 x_1 \frac{\partial f}{\partial x_1} + n_2 x_2 \frac{\partial f}{\partial x_2} + n_p x_p \frac{\partial f}{\partial x_p} = nf$$

[17, p. 10].

1.1.4. Kelvin's Inversion Theorem

If $\psi(\zeta,\eta,\xi)$ is a solution of

$$\Delta\psi = \frac{\partial^2\psi}{\partial\zeta^2} + \frac{\partial^2\psi}{\partial\eta^2} + \frac{\partial^2\psi}{\partial\xi^2} = 0$$

in a region $\mathcal{R}$, then $\frac{a}{r}\psi\left(\frac{a^2x}{r^2}, \frac{a^2y}{r^2}, \frac{a^2z}{r^2}\right)$ is a solution of

$$\Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0$$

in the region $\mathcal{R}'$, the region to which $\mathcal{R}$ is carried by the transformation

$$\zeta = \frac{a^2 x}{r^2},\ \eta = \frac{a^2 y}{r^2},\ \xi = \frac{a^2 z}{r^2}.$$

For the proof see [20, pp. 164-165].

1.1.5. Divergence Theorem

Let $\mathcal{R}$ be a region in xyz-space with boundary S, $\vec{F}$ be a continuous vector-valued function in $\bar{\mathcal{R}}$, the closure of $\mathcal{R}$ such that the first-order partial derivatives of the components are bounded and continuous on $\mathcal{R}$, then

$$\int_S (\vec{F}\cdot\vec{n})\ dS = \int_{\mathcal{R}} \vec{\nabla}\cdot\vec{F}\ dV \tag{1.1.1}$$

where $\vec{n}$ is the outward unit vector normal to S and $\vec{\nabla}\cdot\vec{F}$ is the divergence of $\vec{F}$. Precise details about the nature of $\mathcal{R}$ and its boundary are purposely omitted because of the difficulty in describing these in general. More information on the divergence theorem can be found in [16, pp. 329-335].

1.1.6. Green's Theorem

This is a special case of the divergence theorem in two dimensions. Let $\mathcal{R}$ be a bounded opened set in the plane whose boundary C is a simple closed curve. Let L and M be continuous on $\bar{\mathcal{R}}$ and let $\frac{\partial L}{\partial y}$, $\frac{\partial M}{\partial x}$ be bounded and continuous on $\mathcal{R}$, then

$$\int_C (\ L\ dx + M\ dy\ ) = \int_{\mathcal{R}} (\ \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\ )\ dv. \tag{1.1.2}$$

The proof of this theorem and more details about this theorem can be found in [16, Th. 6-5.1, p. 342].

1.1.7. Green's First Identity

In xyz-space suppose that the functions u and v are such that if $\vec{F}$ in the divergence theorem, (1.1.5), is $v\,\vec{\nabla}u$ then (1.1.1) holds. Then taking $\vec{F} = v\,\vec{\nabla}u$ in the divergence theorem gives

$$\int_{\mathcal{R}} v\,\Delta u\,dV + \int_{\mathcal{R}} (\vec{\nabla}u \cdot \vec{\nabla}v)\,dV = \int_S v\,\frac{\partial u}{\partial n}\,dS \qquad (1.1.3)$$

where $\vec{n}$ is the outward unit vector normal of S and Δu is the Laplace operator [12, p. 212]. This can be demonstrated as follows: Set

$$\vec{F} = v\,\vec{\nabla}u = v\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + v\,\frac{\partial u}{\partial z}$$

then the left-hand side of equation (1.1.1) becomes

$$\int_S \left(v\,\frac{\partial u}{\partial x}\cdot n_1 + v\frac{\partial u}{\partial y}\cdot n_2 + v\,\frac{\partial u}{\partial z}\cdot n_3\right)dS$$

where $\vec{n} = (n_1, n_2, n_3)$. Or, we can write $\int_S v\,\frac{\partial u}{\partial n}\,dS$, since $\vec{\nabla}u \cdot \vec{n} = \frac{\partial u}{\partial n}$, the normal derivative.

The right-hand side of equation (1.1.1) becomes

$$\int_{\mathcal{R}} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\cdot\left(v\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + v\,\frac{\partial u}{\partial z}\right)dV$$

or

$$\int_{\mathcal{R}} \left(v\,\frac{\partial^2 u}{\partial x^2} + v\,\frac{\partial^2 u}{\partial y^2} + v\,\frac{\partial^2 u}{\partial z^2}\right)dV + \int_{\mathcal{R}} \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}\right)dV$$

or finally

$$\int_{\mathcal{R}} v \, \Delta u \, dV + \int_{\mathcal{R}} \vec{\nabla} u \cdot \vec{\nabla} v \, dV.$$

Thus, we have (1.1.3).

1.1.8. Green's Second Identity

Suppose that both u and v are continuously differentiable in $\bar{\mathcal{R}}$ and have continuous partial derivatives of the second order in $\mathcal{R}$. Then

$$\int_{\mathcal{R}} (u \, \Delta v - v \, \Delta u) \, dV = \int_{S} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS \qquad (1.1.4)$$

where $\vec{n}$ is the outward unit vector normal to S and Δu is the Laplace operator [12, p. 215].

The derivation of equation (1.1.4) can be shown as follows: Applying Green's first identity, we have

$$\int_{\mathcal{R}} v \, \Delta u \, dV + \int_{\mathcal{R}} (\vec{\nabla} v \cdot \vec{\nabla} u) \, dV = \int_{S} v \frac{\partial u}{\partial n} \, dS \qquad (1.1.5)$$

Interchanging the roles of u and v we have

$$\int_{\mathcal{R}} u \, \Delta v \, dV + \int_{\mathcal{R}} (\vec{\nabla} u \cdot \vec{\nabla} v) \, dV = \int_{S} u \frac{\partial v}{\partial n} \, dS \qquad (1.1.6)$$

Subtracting (1.1.5) from (1.1.6) gives

$$\int_{\mathcal{R}} (u \, \Delta v - v \, \Delta u) \, dV = \int_{S} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

which is the same as (1.1.4).

If in Green's theorem we let $M = \frac{\partial u}{\partial x}$, $L = -\frac{\partial u}{\partial y}$, then we have

$$\int_{\mathcal{R}} \Delta u \, dV = \int_C - \frac{\partial u}{\partial y} \, dx + \frac{\partial u}{\partial x} \, dy$$

$$= \int_C \frac{\partial u}{\partial n} \, ds \qquad (1.1.7)$$

which is the special case of either Green's second identity or Green's first identity in two dimensions when v = 1.

1.2. Solutions of Laplace's Equation for a Compact Region

In Sections 3.1 and 3.7 the formulas for solutions to the Dirichlet's problem and Neumann's problem are given. In this section these formulas are derived in two dimensions. The following processes are also applicable to Helmholtz's equation and the heat equation.

Suppose P = (x,y) is a point in the interior of a compact region $\mathcal{R}$ with boundary C. The function u is assumed to satisfy $\Delta u = 0$ in the interior of $\mathcal{R}$ and to be such that it is continuous on $\mathcal{R}$. Its partial derivative of second orders are continuous within $\mathcal{R}$. Draw a circle Γ_ε of the center P and of radius ε small enough that the circle lies in the interior of $\mathcal{R}$ (cf. Figure 1.1). Let

$$v = \ln \frac{1}{|\vec{r}\,' - \vec{r}|}$$

where $\vec{r} = P = (x,y)$ and $\vec{r}\,' = (x',y')$. Taking partial derivatives, we have

$$\Delta v = \frac{\partial^2 v}{\partial x'^2} + \frac{\partial^2 v}{\partial y'^2} = 0.$$

Applying the Green's second identity to the region $\mathcal{R}_\varepsilon$ which consists $\mathcal{R}$ with the interior of Γ_ε removed, then

$$0 = \int_{\mathcal{R}_\varepsilon} (u\,\Delta v - v\,\Delta u)\,dv' = \left(\int_{\Gamma_\varepsilon} + \int_C\right)\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds' \qquad (1.2.1)$$

where Γ_ε and C are oriented in the positive sense (cf. Figure 1.1). As $\vec{r}\,'$ is on Γ_ε, $\frac{1}{|\vec{r}\,'-\vec{r}|} = \frac{1}{\varepsilon}$. Therefore,

$$\int_{\Gamma_\varepsilon} \ln\frac{1}{|\vec{r}\,'-\vec{r}|}\,\frac{\partial u}{\partial n}\,ds' = \ln\frac{1}{\varepsilon}\int_{\Gamma_\varepsilon}\frac{\partial u}{\partial n}\,ds',$$

and

$$\int_{\Gamma_\varepsilon} u\,\frac{\partial}{\partial n}\ln\frac{1}{\varepsilon}\,ds' = 2\pi\,u(\vec{r}\,''_\varepsilon)$$

where $\vec{r}\,''_\varepsilon$ is some point on Γ_ε. Hence

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} u\,\frac{\partial}{\partial n}\ln\frac{1}{\varepsilon}\,ds' = \lim_{\varepsilon\to 0} 2\pi\,u(\vec{r}\,''_\varepsilon) = 2\pi\,u(\vec{r}),$$

since u is continuous. Substituting into (1.2.1), then

$$u(\vec{r}) = \frac{1}{2\pi}\int_C\left(-u\,\frac{\partial}{\partial n}\ln\frac{1}{|\vec{r}\,'-\vec{r}|} + \ln\frac{1}{|\vec{r}\,'-\vec{r}|}\,\frac{\partial u}{\partial n}\right)ds'. \qquad (1.2.3)$$

This is called Green's third identity. Now apply Green's second identity to u and H(x,y;x',y') where H is such that

$$\Delta H = \frac{\partial^2 H}{\partial x'^2} + \frac{\partial^2 H}{\partial y'^2} = 0$$

in $\mathcal{R}$ and H has continuous second partial derivatives in $\mathcal{R}$. Therefore,

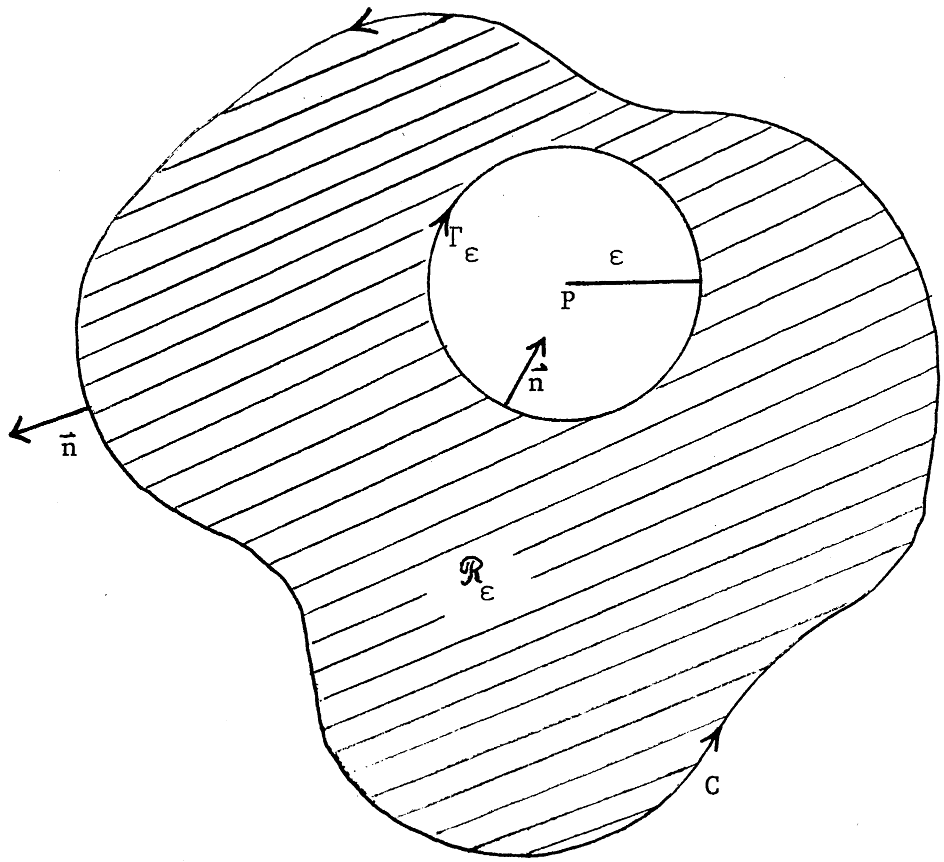


Figure 1.1. The Region $\mathcal{R}_\varepsilon$

$$\int_C \left(u \frac{\partial H}{\partial n} - H \frac{\partial u}{\partial n} \right) ds' = 0. \tag{1.2.4}$$

Multiply (1.2.4) by $1/2\pi$ and adding to (1.2.3), we have

$$u(\vec{r}) = \frac{1}{2\pi}\int_C \left[-u \frac{\partial}{\partial n}\left(H + \ln \frac{1}{|\vec{r}' - \vec{r}|} \right) + \left(H + \ln \frac{1}{|\vec{r}' - \vec{r}|} \right) \frac{\partial u}{\partial n} \right] ds'$$

$$= \frac{1}{2\pi}\int_C \left(-u \frac{\partial G}{\partial n} + G \frac{\partial u}{\partial n} \right) ds' \tag{1.2.5}$$

where

$$G(x,y;x',y') = H\ (x,y;x',y') + \ln \frac{1}{|\vec{r}' - \vec{r}|}\ . \tag{1.2.6}$$

If now $G = 0$ on C then G is called Green's function of the first kind. If $u(\vec{r}) = f(\vec{r})$ on C, then

$$u(\vec{r}) = -\frac{1}{2\pi}\int_C f(\vec{r}') \frac{\partial G}{\partial n}\, ds' \tag{1.2.7}$$

which is Poisson's formula. If instead $\frac{\partial G}{\partial n} = 0$ on C then G is called Green's function of the second kind. If in addition $\frac{\partial u}{\partial n} = f(\vec{r})$ on C, (1.2.5) becomes

$$u(\vec{r}) = \frac{1}{2\pi}\int_C G \cdot f(\vec{r}')\, ds', \tag{1.2.8}$$

which is also called Poisson's formula. Equation (1.2.7) and (1.2.8) give the solution to the Dirichlet's problem and the solution to Neumann's problem, respectively, for two dimensions.

In the case of three dimensions, the solution to the Dirichlet's problem and the solution to Neumann's problem are, respectively,

$$u(\vec{r}) = -\frac{1}{4\pi}\int_S f(\vec{r}')\frac{\partial G}{\partial n}\, dS' \tag{1.2.9}$$

and

$$u(\vec{r}) = \frac{1}{4\pi}\int_S G\cdot f(\vec{r}')\, dS' \tag{1.2.10}$$

where

$$G(\vec{r},\vec{r}') = H(\vec{r},\vec{r}') + \frac{1}{|\vec{r}'-\vec{r}|}\,, \tag{1.2.11}$$

$\vec{r} = (x,y,z)$, $\vec{r}' = (x',y',z')$ and H is such that $\Delta H = 0$ in $\mathcal{R}$. The derivation of (1.2.9) and (1.2.10) can be shown analogously to (1.2.7) and (1.2.8).

We have discussed the solution for a compact region in two and three dimensions. In the next section we will show how to get the solutions for the Dirichlet's problem and Neumann's problem in the exterior of a compact region.

1.3. Solutions for the Exterior of a Compact Region

Let u(x,y) be the solution of Laplace's equation whose partial derivatives of the second order are continuous outside a compact region $\mathcal{R}$. Assume that u is continuous on $\bar{\mathcal{R}}$. Suppose furthermore that for r sufficiently large there exists M such that $|u| < (M/r)$, $|u_x| < (M/r^2)$ and $|u_y| < (M/r^2)$. Let Γ_R be a circle containing $\mathcal{R}$, with $\vec{r}$ as center and the radius R (cf. Figure 1.2). Apply (1.2.3) to the compact region Γ_R shown in Figure 1.2. Then

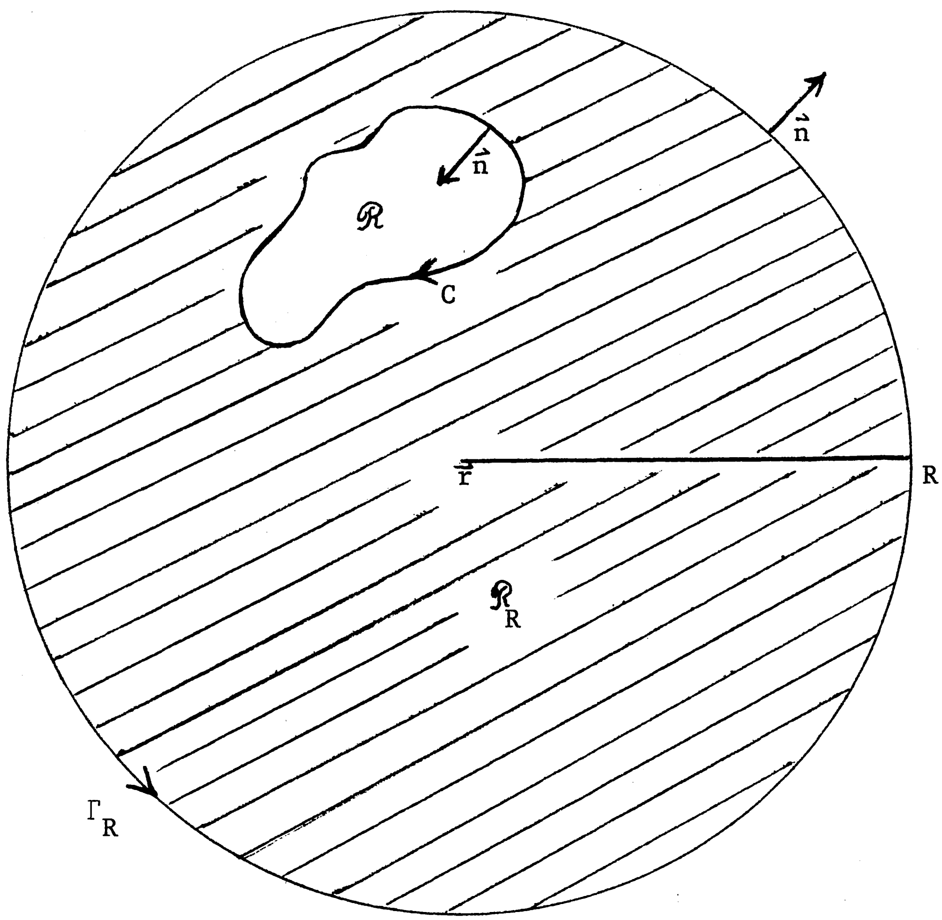


Figure 1.2. The circle Γ_R Containing the Compact Region $\mathscr{R}$

$$u(\vec{r}) = \frac{1}{2\pi}\left(\int_C + \int_R\right)\left(-u\frac{\partial}{\partial n}\ln\frac{1}{|\vec{r}\,' - \vec{r}\,|} + \ln\frac{1}{|\vec{r}\,' - \vec{r}|}\frac{\partial u}{\partial n}\right)ds' \tag{1.3.1}$$

where $\vec{n}$ is the unit outward normal of $\mathcal{R}_R$. Let $\vec{r}\,'$ be on Γ_R, therefore,

$$\int_{\Gamma_R}\left(-u\frac{\partial}{\partial n}\ln\frac{1}{|\vec{r}\,'-\vec{r}|} + \ln\frac{1}{|\vec{r}\,'-\vec{r}|}\frac{\partial u}{\partial n}\right)ds' =$$

$$\int_{\Gamma_R}\left(-u\frac{\partial}{\partial r'}\ln\frac{1}{r'} + \ln\frac{1}{r'}\frac{\partial u}{\partial n}\right)ds' \tag{1.3.2}$$

where $r' = |\vec{r}\,'-\vec{r}|$. Then using the mean-value theorem on the right-hand side of (1.3.2) gives

$$\int_{\Gamma_R}\left(-u\frac{\partial}{\partial n}\ln\frac{1}{|\vec{r}\,'-\vec{r}|} + \ln\frac{1}{|\vec{r}\,'-\vec{r}|}\frac{\partial u}{\partial n}\right)ds' =$$

$$u(\vec{r}\,'_1)\frac{1}{r'} + \ln\frac{1}{r'}\frac{\partial}{\partial n}u(\vec{r}\,'_2)\ 2\pi\ r' \tag{1.3.3}$$

where $\vec{r}\,'_1$ and $\vec{r}\,'_2$ are points on Γ_R. Giving the above conditions, then $|u(\vec{r}\,')| < \frac{M}{r'}$, and $|\frac{\partial}{\partial n}u(\vec{r}\,')| < 2M(\frac{1}{r'})^2$ as r' becomes large. Letting $r' \to \infty$, then (1.3.3) becomes

$$u(\vec{r}) = \frac{1}{2\pi}\int_C\left(-u(\vec{r}\,')\frac{\partial}{\partial n}\ln\frac{1}{|\vec{r}\,'-\vec{r}|} + \ln\frac{1}{|\vec{r}\,'-\vec{r}|}\frac{\partial u}{\partial n}\right)ds' \tag{1.3.4}$$

where $\vec{n}$ is the unit outward vector normal to C with respect to the exterior of $\mathcal{R}$. Equation (1.3.4) is the same as (1.2.3). Therefore, if we repeat the discussion after (1.2.3), we obtain (1.2.7) and (1.2.8) to be, respectively, the solutions of the Dirichlet's problem and Neumann's problem for the exterior problem.

In the case of three dimensions, (1.2.9) and (1.2.10) are, respectively, the solution of the Dirichlet's problem and Neumann's problem for the exterior problem if the conditions for r sufficient large, $|u| < (M/r)$, $|u_x| < (M/r^2)$, $|u_y| < (M/r^2)$ and $|u_z| < (M/r^2)$ are given.

We have discussed the solution of the Dirichlet's problem and Neumann's problem for a compact region and the exterior of a compact region. For the case of a region that is neither an interior nor an exterior problem, like an upper half-plane $\mathcal{R}$, we will find the solution analogously by drawing a circle with center $\vec{r}$ in $\mathcal{R}$ and the radius R such that the intersection of the circle and $\mathcal{R}$ is not an empty set. Let $\mathcal{R}_R$ be the intersection (cf. Figure 1.3). If $R \to \infty$ then $\mathcal{R}_R$ becomes $\mathcal{R}$. Apply (1.2.3) to $\mathcal{R}_R$. Then, let $R \to \infty$. Giving the same conditions as in the exterior problem in two dimensions, we will find that (1.2.7) and (1.2.8) are also, respectively, the solution of the Dirichlet's problem and Neumann's problem for a region that is neither an interior nor an exterior region in two dimensions.

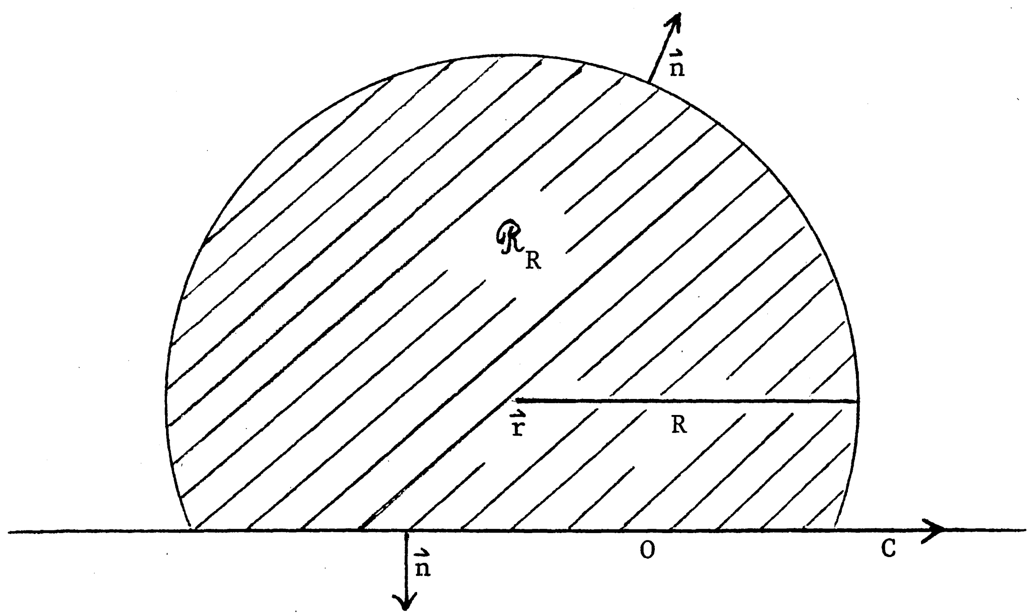


Figure 1.3. The Region $\mathcal{R}_R$ Being an Upper Half-Plane as $R \to \infty$

CHAPTER II

GREEN'S FUNCTIONS FOR ORDINARY DIFFERENTIAL EQUATIONS

2.1. Unmixed Boundary Conditions

The Green's function for a linear second order differential equation with boundary conditions is found directly in the following examples.

Example 2.1. Consider, first, the simple boundary value problem $u''(x) = f(x)$; $u(0) = u(1) = 0$, where f is the differentiable function in $[0,1]$.

The Green's function for this problem is $G(x,t)$, $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, defined by

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

such that G has the following properties:

(1). G is continuous on $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, in particular, it follows $G_2(t,t) - G_1(t,t) = 0$, and $(G_1)_{xx} = 0$ for $x < t$ and $(G_2)_{xx} = 0$ for $x > t$,

(2). G satisfies the boundary conditions, that is, $G_1(0,t) = G_2(1,t) = 0$,

(3). the partial derivative of G with respect to x has a jump discontinuity of magnitude 1 at $x = t$, that is, $\frac{\partial}{\partial x} G_2(t,t) - \frac{\partial}{\partial x} G_1(t,t) = 1$.

From (1), G_1 and G_2 are solutions of the differential equation $G_{xx} = 0$. Therefore, let

$$G(x,t) = \begin{cases} G_1(x,t) = a_1x + a_2, & x < t, \\ G_2(x,t) = b_1x + b_2, & x > t. \end{cases}$$

Properties (1) and (3) imply

$$(b_2-a_2) + (b_1-a_1)t = 0,$$

and

$$(b_1-a_1) = 1.$$

Therefore $b_2-a_2 = -t$. Property (2) implies $a_2 = 0$ and $b_1 + b_2 = 0$. Hence $b_2 = -t$ which implies $b_1 = t$. Then $a_1 = t - 1$. Substituting, we have

$$G(x,t) = \begin{cases} (t-1)x, & x < t, \\ (x-1)t, & x > t. \end{cases}$$

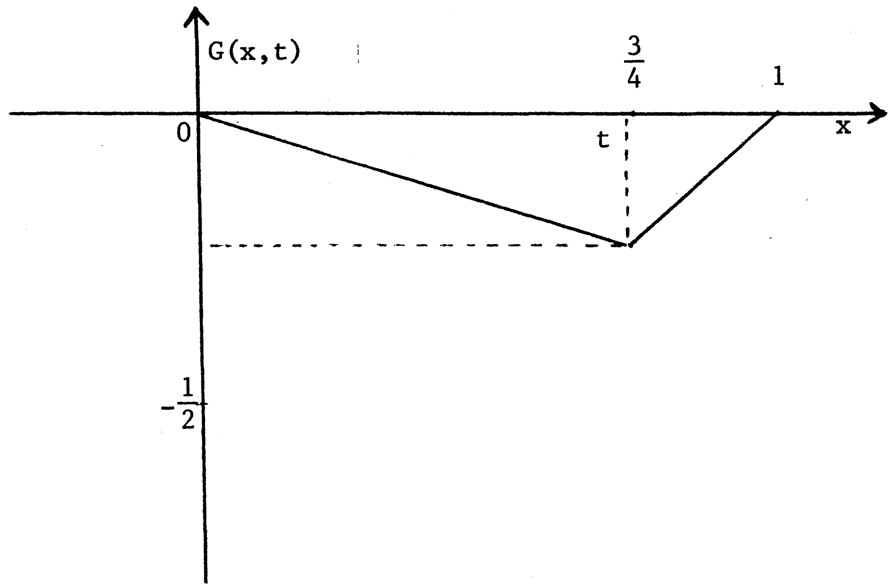
The graph of G is shown on Figure 2.1. The usefulness of the function G is that it provides a solution of the given differential equation with the other conditions. That is,

$$u(x) = \int_0^1 G(x,t)f(t)\,dt$$

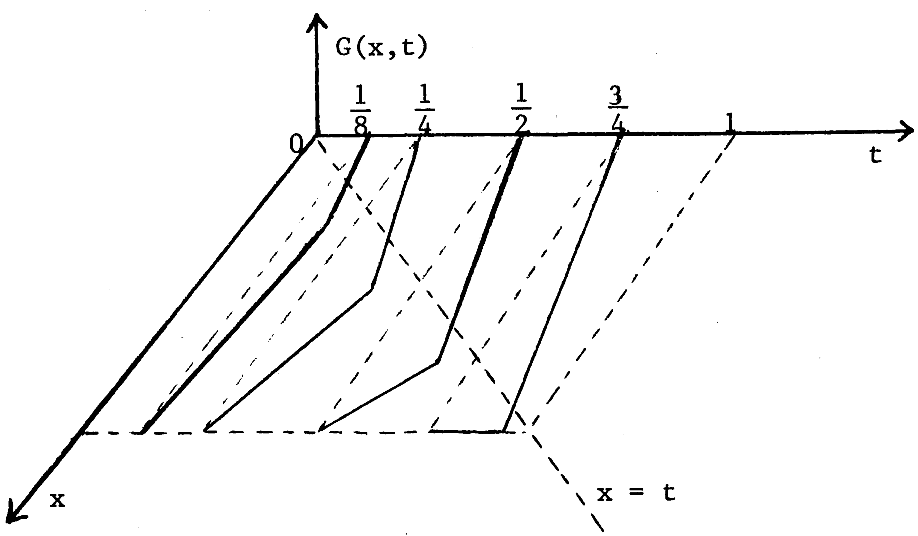
$$= \int_0^x (x-1)t\,f(t)\,dt + \int_x^1 (t-1)x\,f(t)\,dt$$

is such a solution.

We check that u is the solution as follows:



(a)



(b)

Figure 2.1. Green's Function for Example 2.1

$$u'(x) = \frac{d}{dx}\left[(x-1)\int_0^x t\ f(t)\ dt + x\int_x^1 (t-1)\ f(t)\ dt\right]$$

$$= (x-1)x\ f(x) + \int_0^x t\ f(t)\ dt - x(x-1)f(x) + \int_x^1 (t-1)\ f(t)\ dt$$

$$= \int_0^x t\ f(t)\ dt + \int_x^1 (t-1)\ f(t)\ dt$$

$$u''(x) = x\ f(x) - (x-1)\ f(x) = f(x).$$

Futhermore,

$$u(0) = 0$$

and

$$u(1) = (1-1)\int_0^1 t\ f(t)\ dt + 1\int_1^1 (1-t)\ f(t)\ dt = 0.$$

Thus u has the required properties. Now let us look at another example.

Example 2.2. Consider $u''(x) + 9u = f(x)$; $u(0) = u'(1) = 0$, where f is a differentiable function on $[0,1]$.

The Green's function $G(x,t)$, $0 \leq x \leq 1$, $0 \leq t \leq 1$, where

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

has the following properties:

(1). G is continuous on $0 \leq x \leq 1$, $0 \leq t \leq 1$. Thus, in particular, $G_2(t,t) - G_1(t,t) = 0$, and $(G_1)_{xx} + 9G_1 = 0$ for $x < t$ and

$(G_2)_{xx} + 9G_2 = 0$ for $x > t$,

(2). G satisfies the boundary conditions $G_1(0,t) = \frac{\partial}{\partial x} G_2(1,t) = 0$, and

(3). The partial derivative of G with respect to x has a jump of magnitude 1 at x = t, that is, $\frac{\partial}{\partial x} G_2(t,t) - \frac{\partial}{\partial x} G_1(t,t) = 1$.

From (1), G_1 and G_2 are solutions of the differential equation $\frac{\partial^2}{\partial x^2} G + 9G = 0$. Hence

$$G(x,t) = \begin{cases} G_1(x,t) = a_1 \sin 3x + a_2 \cos 3x, & x < t, \\ G_2(x,t) = b_1 \sin 3x + b_2 \cos 3x, & x > t. \end{cases}$$

The coefficients will be determined by using the indicated properties. Properties (1) and (3) give

$$(b_1-a_1)\sin 3t + (b_2-a_2)\cos 3t = 0,$$

$$3(b_1-a_1)\cos 3t - 3(b_2-a_2)\sin 3t = 1.$$

Solving these two equations for b_1-a_1 and b_2-a_2 gives $b_1-a_1 = (\cos 3t)/3$ and $b_2-a_2 = -(\sin 3t)/3$. From (2), $G_1(0,t) = a_1 \sin 0 + a_2 \cos 0 = 0$ which implies that $a_2 = 0$. The condition $\frac{\partial}{\partial x} G_2(1,t) = 0$ gives $b_1 \cos 3 - b_2 \sin 3 = 0$. Hence, we have $a_2 = 0$, $b_2-a_2 = -(\sin 3t)/3$, $b_1 \cos 3 - b_2 \sin 3 = 0$ and $b_1-a_1 = (\cos 3t)/3$. solving these four equations gives $a_2 = 0$, $b_2 = -(\sin 3t)/3$, $b_1 = -\sin 3t(\sin 3)/3 \cos 3$ and $a_1 = -\cos(3t-3)/3 \cos 3$. Substituting we have

$$G(x,t) = \begin{cases} -\cos(3t-3)\sin 3x/3 \cos 3, & x < t, \\ -\cos(3x-3)\sin 3t/3 \cos 3, & x > t. \end{cases}$$

As in Example 2.1 the solution of the given problem is

$$u(x) = \int_0^1 G(x,t)\ f(t)\ dt =$$

$$-\int_0^x [\cos(3x-3)\sin 3t\ f(t)/3\cos 3]dt - \int_x^1 [\cos(3t-3)\sin 3x\ f(t)/3\cos 3]dt.$$

The graph of G is shown in Figure 2.2.

We shall now look at an example that is still more complicated.

Example 2.3. Consider $u''(x) - 6u' + 5u = f(x)$, or in self-adjoint form, $(e^{-6x}u')' + 5e^{-6x}u = f(x)e^{-6x}$, such that $u(0) + u'(0) = 0$, $2u(1) - u'(1) = 0$.

The Green's function $G(x,t)$, $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, is written

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

and has the following properties:

(1). G is continuous on $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, in particular $G_1(t,t) - G_2(t,t) = 0$ and G_1 and G_2 satisfy the homogeneous differential equation for $x < t$ and $x > t$, respectively,

(2). G satisfies the boundary conditions, that is $G_1(0,t) + \frac{\partial}{\partial x} G_1(0,t)=0$ and $2G_2(1,t) - \frac{\partial}{\partial x} G_2(1,t) = 0$,

(3). The x-derivative of G has a jump of magnitude e^{6t} at $x = t$, that is, $\frac{\partial}{\partial x} G_2(t,t) - \frac{\partial}{\partial x} G_1(t,t) = e^{6t}$.

From (1), G_1 and G_2 are solutions of the differential equation $\frac{\partial^2}{\partial x^2} G - 6\frac{\partial}{\partial x} G + 5G = 0$ and, therefore,

$$G(x,t) = \begin{cases} G_1(x,t) = a_1e^{5x} + a_2e^{x}, & x < t, \\ G_2(x,t) = b_1e^{5x} + b_2e^{x}, & x > t. \end{cases}$$

The coefficients are determined as follows: Properties (1) and (3) give

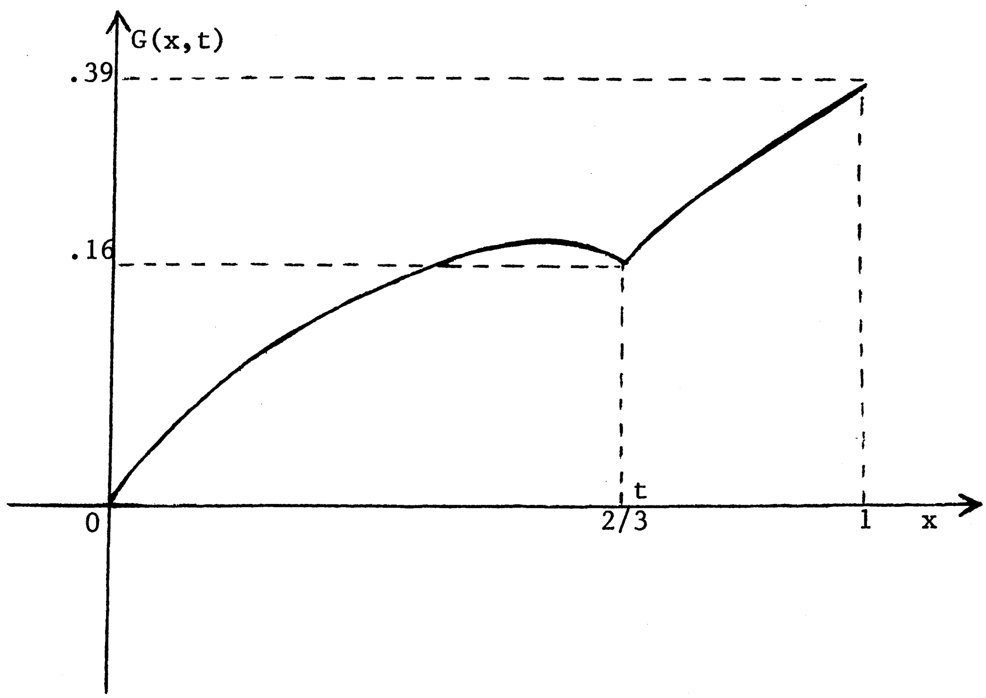


Figure 2.2. Green's Function for Example 2.2

$$(b_1-a_1)e^{5t} + (b_2-a_2)e^{t} = 0,$$

$$5(b_1-a_1)e^{5t} + (b_2-a_2)e^{t} = e^{6t}.$$

Solving these two equations for b_1-a_1 and b_2-a_2 we have $b_1-a_1 = e^{t}/4$ and $b_2-a_2 = -e^{5t}/4$. From (2), $G_1(0,t) + \frac{\partial}{\partial x} G_1(0,t) = 0$ implies $(a_1+a_2) + (5a_1 + a_2) = 0$, or $3a_1+a_2 = 0$. The condition $2G_2(1,t) - \frac{\partial}{\partial x} G_2(1,t) = 0$ implies $2(b_1e^5+b_2e) - (5b_1+b_2e) = 0$, or $-3b_1e^5 + b_2e = 0$. Solving $b_1-a_1 = e^{t}/4$, $b_2-a_2 = -e^{5t}/4$, $3a_1+a_2 = 0$ and $-3b_1e^5 + b_2e = 0$, we have

$$a_1 = -(e^{5t} + 3e^{t+4})/12(e^4 + 1),$$

$$a_2 = 3(e^{5t} + 3e^{t+4})/12(e^4 + 1),$$

$$b_1 = (3e^{t} - e^{5t})/12(e^4 + 1) \text{ and}$$

$$b_2 = 3e^4(3e^{t} - e^{5t})/12(e^4 + 1).$$

Substituting gives

$$G(x,t) = \begin{cases} (e^{5t} + 3e^{t+4})(3e^{x} - e^{5x})/12(e^4 + 1), & x < t, \\ (e^{5x} + 3e^{x+4})(3e^{t} - e^{5t})/12(e^4 + 1), & x > t. \end{cases}$$

Notice in each of these three examples $G(x,t) = G(t,x)$, that is, G is a symmetric function. Figure 2.3 shows the graph of G for $t = .75$.

<u>Summary</u>

The differential equations in Examples 2.1 through 2.3 can be written in the self-adjoint form

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x). \qquad (2.1.1)$$

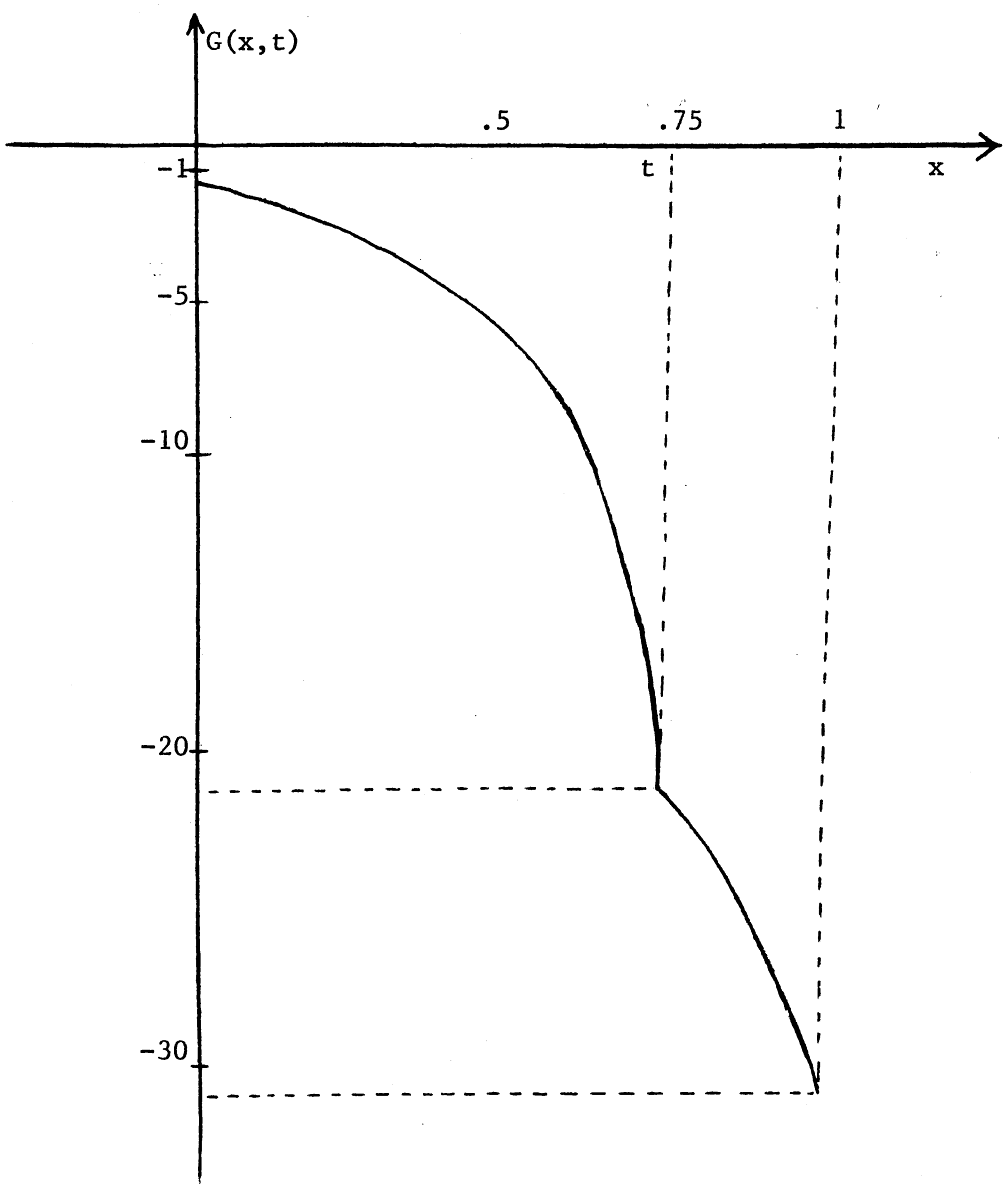


Figure 2.3. Green's Function for Example 2.3

In fact, any second degree differential equation

$$\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = g(x)$$

can be written in the form (2.1.1) by multiplying the equation with $e^{\int b(x)dx}$ and letting $p(x) = e^{\int b(x)dx}$, $q = cp$ and $f = gp$.

The boundary conditions in Example 2.1 through 2.3 are unmixed homogeneous conditions, that is, they are of the form

$$c_1u(0) + c_2u'(0) = 0,$$
$$d_1u(1) + d_2u'(1) = 0$$

where the coefficients are real numbers. The Green's functions

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

for Examples 2.1 through 2.3 have the following properties which are, in fact, properties of the Green's function for a second order equation of the form (2.1.1) with unmixed boundary conditions:

(1). G is continuous on $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$ and with respect to x. G satisfies the homogeneous differential equation for $0 \leqq x < t$ and for $t < x \leqq 1$,

(2). G satisfies the given boundary conditions, that is, $c_1G_1(0,t) + c_2 \frac{\partial}{\partial x} G_1(0,t) = 0$, $d_1G_2(1,t) + d_2 \frac{\partial}{\partial x} G_2(1,t) = 0$, and

(3). The partial of G with respect to x has a jump magnitude of $1/p(t)$ at $x = t$, that is,

$$\frac{\partial}{\partial x} G_2(t,t) - \frac{\partial}{\partial x} G_1(t,t) = 1/p(t).$$

The Green's function is found by the following steps:

1. Write the differential equation in self-adjoint equation, i.e.,

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x), \qquad (2.1.2)$$

2. Then

$$G(x,t) = \begin{cases} a_1u_1(x) + a_2u_2(x), & x < t, \\ b_1u_1(x) + b_2u_2(x), & x > t \end{cases}$$

where $u_1(x)$ and $u_2(x)$ are linearly independent solutions for (2.1.2).

3. Use (1) and (3), obtain

$$(b_1-a_1)u_1(t) + (b_2-a_2)u_2(t) = 0,$$

$$(b_1-a_1)u_1'(t) + (b_2-a_2)u_2'(t) = 1/p(t).$$

Then solve for b_1-a_1 and b_2-a_2.

4. Use (2) and the results of step 3 to solve for the a_1, a_2, b_1, and b_2

5. Substituting a_1, a_2, b_1 and b_2 into the equations of step 2 gives Green's function.

Formula for Finding Green's Function

Given the equation $(pu')' + qu = f(x)$ with boundary conditions

$$c_1u(0) + c_2u'(0) = 0,$$

$$d_1u(1) + d_2u'(1) = 0.$$

Let $w_1(x)$ be the solution of $(pw')' + qw = 0$, satisfying the unmixed boundary condition $c_1w(0) + c_2w'(0) = 0$ and let $w_2(x)$ be the solution of the same differential equation satisfying the unmixed condition $d_1w(1) + d_2w'(1) = 0$. Then Green's function is given by the following formula:

$$G(x,t) = \begin{cases} w_1(x)w_2(t)/J(w_2,w_1), & x < t, \\ w_2(x)w_1(t)/J(w_2,w_1), & x > t \end{cases}$$

where

$$J(w_2,w_1) = [w_2'(t)w_1(t) - w_1'(t)w_2(t)]p(t).$$

For example, the Green's function in Example 2.2 can be found by using the formula as follows:

Step 1. Find the solution of $w''(x) + 9w = 0$. The general solution is $c_1 \sin 3x + c_2 \cos 3x$.

Step 2. Find a solution $w_1(x)$ such that $w_1(0) = 0$. Take $w_1(x) = \sin 3x$ and therefore $w_1(0) = 0$.

Step 3. Find a solution $w_2(x)$ of the homogeneous equation such that $w_2'(1) = 0$. A general solution of the differential equation is $w_2(x) = \cos(3x+b)$. Therefore, $w_2'(x) = -3 \sin(3x+b)$. Since $w_2'(1) = -3 \sin(3+b) = 0$ we have $3 + b = 0$. Hence, $b = -3$, and thus $w_2(x) = \cos(3x-3)$.

Step 4. Find $J(w_2,w_1)$.

$$\begin{aligned} J(w_2,w_1) &= [w_2'(t)w_1(t) - w_1'(t)w_2(t)]p(t) \\ &= -3 \sin(3t-3)\sin 3t - 3 \cos 3t \cos(3t-3) \\ &= -3 \cos 3. \end{aligned}$$

Step 5. Use the formula. Then

$$G(x,t) = \begin{cases} -[\sin 3x \cos(3t-3)]/(3 \cos 3), & x < t, \\ -[\cos(3x-3)\sin 3t]/(3 \cos 3), & x > t \end{cases}$$

which is the same function as was found in Example 2.2.

From the Examples 2.1 through 2.3, notice that the differential equations can be written in the form $\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + qu = f(x)$ with unmixed homogeneous boundary conditions $c_1u(0) + c_2u'(0) = 0$ and $d_1u(1) + d_2u'(1) = 0$ where the coefficients are real constants. Substituting x for t and t for x gives $G(x,t) = G(t,x)$. That is, $G(x,t)$ is symmetric. For example, from Example 2.1

$$G(x,t) = \begin{cases} (1-t)x, & x < t, \\ (1-x)t, & x > t. \end{cases}$$

Then

$$G(t,x) = \begin{cases} (1-x)t, & t < x, \\ (1-t)x, & t > x. \end{cases}$$

After Green's function is known, the solution of the problem $\frac{d}{dx}\left(p(x)\,\frac{du}{dx}\right) + qu = f(x)$ with the same unmixed homogeneous boundary conditions is of the form

$$u(x) = \int_0^1 f(t)G(x,t)\,dt.$$

Recall that this was demonstrated in Example 2.1.

2.2. Mixed Boundary Conditions

In the first three examples each boundary condition involved only one of the endpoints. In this sense the boundary conditions are unmixed. This section will remove that requirement and allow the boundary conditions to involve a relationship between the two endpoints, that is, <u>mixed conditions</u>.

A second degree differential equation with mixed conditions can be solved as in the following examples. This first example illustrates <u>periodic</u> boundary conditions, that is, $u(0) = u(1)$, and $u'(0) = u'(1)$. The name of the conditions comes from the properties possessed by a differentiable function u of period one, that is, $u(x) = u(x+1)$.

<u>Example 2.4.</u> Consider $u'' + u = f(x)$; $u(0) = u(1)$, $u'(0) = u'(1)$.

The Green's function G where

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

for $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$ has the following properties:

(1). G_1 and G_2 are solutions with respect to x to the homogeneous differential equation for $0 \leqq x < t$ and $t < x \leqq 1$, respectively, and G is continuous on $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, that is,

$G_1(t,t) = G_2(t,t)$,

(2). G satisfies the boundary conditions

$G(1,t) - G(0,t) = 0$ and $G_x(0,t) - G_x(1,t) = 0$, and,

(3). G_x has a jump discontinuity at $x = t$ of magnitude 1.

The Green's function will be found by using these properties. Since G_1 and G_2 are solutions of $G_{xx} + G = 0$,

$$G(x,t) = \begin{cases} G_1(x,t) = a_1 \sin x + a_2 \cos x, & x < t, \\ G_2(x,t) = b_1 \sin x + b_2 \cos x, & x > t. \end{cases}$$

Using (1) and (3), we have

$$(b_1-a_1)\sin t + (b_2-a_2)\cos t = 0,$$

$$(b_1-a_1)\cos t - (b_2-a_2)\sin t = 1.$$

Solving these two equations gives,

$$b_1-a_1 = \cos t \qquad (2.2.1)$$

$$b_2-a_2 = -\sin t \qquad (2.2.2)$$

Using (2), $G_1(0,t) = G_2(1,t)$, we have

$$a_2 = b_1 \sin 1 + b_2 \cos 1 \qquad (2.2.3)$$

and $\frac{\partial}{\partial x} G_1(0,t) = \frac{\partial}{\partial x} G_2(1,t)$. We have

$$a_1 = b_1 \cos 1 - b_2 \sin 1. \qquad (2.2.4)$$

Solving equations (2.2.1) through (2.2.4) for the coefficients we have

$$a_1 = [\cos(1-t) - \cos t]/2(1-\cos 1),$$

$$a_2 = [\sin(1-t) + \sin t]/2(1-\cos 1),$$

$$b_1 = [-\cos(1+t) + \cos t]/2(1-\cos 1), \text{ and}$$

$$b_2 = [\sin(1+t) - \sin t]/2(1-\cos 1).$$

Substituting gives

$$G(x,t) = \begin{cases} [\sin(1-t+x) + \sin(t-x)]/2(1-\cos 1), & x < t, \\ [\sin(1+t-x) + \sin(x-t)]/2(1-\cos 1), & x > t. \end{cases}$$

Notice that G(x,t) is symmetric in x and t, that is, G(x,t) = G(t,x). This will always happen when the boundary conditions are periodic and when p(0) = p(1) where p comes from the self-adjoint form of the given differential equations.

The solution of the given problem is

$$u(x) = \int_0^1 f(t)G(x,t)\,dt$$

$$= \int_0^x f(t)[\sin(1+t-x) + \sin(x-t)]/2(1-\cos 1)\,dt +$$

$$\int_x^1 f(t)[\sin(1-t+x) + \sin(t-x)]/2(1-\cos 1)\,dt.$$

This can be demonstrated as being a solution as in Example 2.1. Notice that the boundary conditions for G are the same as for u. This is true since the boundary conditions are periodic and p(0) = p(1).

Determining the Boundary Conditions

In Example 2.4 the correct boundary conditions for G were given, but just how they arise from the given problem is not clear. For the moment let it just be said that we must make

$$p(x)\left[Gu' - u\frac{\partial}{\partial x}G\right]_{x=0}^{x=1} = 0. \qquad (2.2.5)$$

Later it will be shown why this is appropriate. In Example 2.4, the equation (2.2.5) implies that

$$G(1,t)u'(1) - G_x(1,t) - G(0,t)u'(0) + G_x(0,t)u(0) = 0. \quad (2.2.6)$$

Using the periodic boundary conditions, (2.2.6) can be written

$$[G(1,t) - G(0,t)]u'(1) + [G_x(0,t) - G_x(1,t)]u(1) = 0.$$

Thus the required condition will be met by taking

$$G(1,t) - G(0,t) = 0 \text{ and } G_x(0,t) - G_x(1,t) = 0, \text{ or}$$

$$G(1,t) = G(0,t) \text{ and } G_x(1,t) = G_x(0,t).$$

The next example shows for the first time that Green's function is not always symmetric. The boundary conditions of the given problem are periodic, but this time p is not such that $p(0) = p(1)$. This will make for some differences that should be noted.

<u>Example 2.5.</u> Consider $(x+1)^2u'' - 2(x+1)u' + 2u = f(x)$; $u(0) = u(1)$ and $u'(0) = u'(1)$.

The differential equation can be written in the self-adjoint form

$$[(x+1)^{-2}u']' + 2(x+1)^{-4}u = f(x)(x+1)^{-4}.$$

The condition (2.2.5) gives

$$(x+1)^{-2}\left(Gu' - u\frac{\partial}{\partial x}G\right)\Bigg|_{x=0}^{x=1} = 0$$

or

$$-G(0,t)u'(0) + G_x(0,t)u(0) + \frac{1}{4}G(1,t)u'(1) - \frac{1}{4}G_x(1,t)u(1) = 0.$$

Substituting from the given boundary conditions gives

$$\left[\frac{1}{4}G(1,t) - G(0,t)\right]u'(1) + \left[G_x(0,t) - \frac{1}{4}G_x(1,t)\right]u(1) = 0.$$

This requirement is met if the quantities in the brackets are zero.

Thus, take

$$4G(0,t) = G(1,t) \text{ and } 4G_x(0,t) = G_x(1,t).$$

Notice for the first time that these boundary conditions are different from those in the given problem. The Green's function

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

$0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$ has the following properties:

(1). $$(x+1)^2 \frac{\partial^2}{\partial x^2} G_1 - 2(x+1) \frac{\partial}{\partial x} G_1 + 2G_1 = 0 \text{ for } x < t,$$

and

$$(x+1)^2 \frac{\partial^2}{\partial x^2} G_2 - 2(x+1) \frac{\partial}{\partial x} G_2 + 2G_2 = 0 \text{ for } x > t,$$

and G is continuous on $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, that is, in particular, $G_2(t,t) = G_1(t,t)$,

(2). The boundary conditions for G are

$$4G(0,t) = G(1,t) \text{ and } 4G_x(0,t) = G_x(1,t).$$

(3). The derivative of G has the jump of magnitude $(t+1)^2$ at $x = t$, that is, $\frac{\partial}{\partial x} G_2(t,t) - \frac{\partial}{\partial x} G_1(t,t) = (t+1)^2$.

The Green's function will be found by using its properties. From (1), G_1 and G_2 are solutions of

$$(x+1)^2 \frac{\partial^2}{\partial x^2} G - 2(x+1) \frac{\partial}{\partial x} G + 2G = 0,$$

therefore,

$$G(x,t) = \begin{cases} a_1(x+1) + a_2(x+1)^2, & x < t, \\ b_1(x+1) + b_2(x+1)^2, & x > t. \end{cases}$$

Using (1) and (3), we have

$$(b_1-a_1)(t+1) + (b_2-a_2)(t+1)^2 = 0,$$

$$(b_1-a_1) + 2(b_2-a_2)(t+1) = (t+1)^2.$$

Solving these two equations for b_1-a_1 and b_2-a_2 gives $b_1-a_1 = -(t+1)^2$, $b_2-a_2 = t+1$. Then $4G(0,t) = G(1,t)$ and $4G_x(0,t) = G_x(1,t)$ give $4(a_1+a_2) = 2b_1 + 4b_2$ and $4(a_1+2a_2) = b_1 + 4b_2$. From the above equations $a_1 = (1+t)(1-t)$, $a_2 = \frac{1}{2}t(1+t)$, $b_1 = -2t(1+t)$ and $b_2 = \frac{1}{2}(1+t)(2+t)$. Substituting gives

$$G(x,t) = \begin{cases} (x+1)(1+t)(xt-t+2)/2, & x < t, \\ (x+1)(1+t)(xt+2x-3t+2)/2, & x > t. \end{cases}$$

Notice that $G(x,t)$ is not symmetric. This was due to the fact that $p(0) \neq p(1)$.

Again the solution of the given problem is

$$u(t) = \int_0^1 f(x)(x+1)^{-4}G(x,t)\,dx$$

$$= \int_0^t f(x)(x+1)^{-4}[(x+1)(1+t)(xt-t+2)/2]\,dx +$$

$$\int_t^1 f(x)(x+1)^{-4}[(x+1)(1+t)(xt+2x-3t+2)/2]\,dx.$$

Notice that in this case $u(x) = \int_0^1 f(t)(t+1)^{-4}G(x,t)\,dt$ is not the solution, since G is not symmetric, $G(x,t) \neq G(t,x)$. The fact that $u(t) = \int_0^t f(x)(x+1)^{-3}(1+t)(xt-t+2)/2\,dx + \int_t^1 f(x)(x+1)^{-3}(1+t)(xt+2x-3t+2)/2\,dx$ is a solution which can be shown directly as follows:

First, for the boundary conditions, we see that

$$u(0) = \int_0^1 f(x)(x+1)^{-3}(2x+2)/2 \, dx,$$

$$u(1) = \int_0^1 f(x)(x+1)^{-3}2(x+1)/2 \, dx.$$

Hence u(0) = u(1). Differentiating u(t) with respect to t and using Leibnitz's rule for differentiating integrals, we have

$$u'(t) = \int_0^t [f(x)(x+1)^{-3}(2xt+x-2t+1)/2] \, dx +$$

$$\int_t^1 f(x)(x+1)^{-3}(2xt+3x-6t-1)/2 \, dx.$$

Substituting t = 0 and t = 1, we have

$$u'(0) = u'(1) = \int_0^1 f(x)(x+1)^{-3}(3x-1)/2 \, dx.$$

Therefore, u(t) satisfies the boundary conditions. To check that u(t) satisfies the differential equation

$$(t+1)^2u'' - 2(t+1)u' + 2u = f(t),$$

we shall proceed as follows: Calculating u''(t) we have after using Leibnitz's rule

$$u''(t) = \int_0^t f(x)(x+1)^{-3}(x-1) \, dx + \int_t^1 f(x)(x+1)^{-3}(x-3) \, dx + f(t)(t+1)^{-2}.$$

Therefore,

$$(t+1)^2u'' = \int_0^t f(x)(x+1)^{-3}(t+1)^2(x+1)dx + \int_t^1 f(x)(x+1)^{-3}(x-3)(t+1)^2dx + f(t)$$

$$-2(t+1)u' = \int_0^t -f(x)(x+1)^{-3}(t+1)(2xt-2t+x+1)dx + \int_t^1 -f(x)(x+1)^{-3}(t+1)(2xt-6t+3x-1)dx$$

and

$$2u = \int_0^t f(x)(x+1)^{-3}(t+1)(2-t+xt)\ dx + \int_t^1 f(x)(x+1)^{-3}(t+1)(2+2x-3t+xt)\ dx.$$

Hence

$$(t+1)^2u'(t) - 2(t+1)u' + 2u = f(t).$$

2.3. The Method of Variation of Parameters

The Green's function for a second degree differential sequence with mixed boundary conditions can also be found by the method of variation of parameters. It will be illustrated by using Example 2.5,

$$(x+1)^2u'' - 2(x+1)u' + 2u = f(x),$$

$$u(0) = u(1) \text{ and } u'(0) = u'(1),$$

as follows:

First, we write the differential equation in self-adjoint form, that is,

$$[(x+1)^{-2}u']' + 2(x+1)^{-4}u = f(x)(x+1)^{-4}.$$

Recall that the boundary conditions on G were $4G(0,t) = G(1,t)$ and $4G_x(0,t) = G_x(1,t)$. For this method requires that Green's function satisfies

$$[(x+1)^{-2}G_x]_x + 2(x+1)^{-4}G = \delta(x-t),$$

where δ is the Dirac delta function. Then rewrite in the form

$$G_{xx} - 2(x+1)^{-1}G_x + 2(x+1)^{-2}G = (x+1)^2\delta(x-t),$$

thus making the coefficient of G_{xx} to be one. Let $r(x,t) = (x+1)^2(x-t)$. The Green's function will be found by writing

$$G(x,t) = \begin{cases} c_1u_1 + c_2u_2, & x < t, \\ c_1u_1 + c_2u_2 + G_p, & x > t, \end{cases}$$

where u_1 and u_2 are linearly independent solutions of

$$(x+1)^2u'' - 2(x+1)u' + 2u = 0 \text{ and } G_p = v_1u_1 + v_2u_2.$$

The functions v_1 and v_2 are variation parameters which are found by

$$v_1 = \int[-u_2r(x,t)/w]dx,$$

$$v_2 = \int[u_1r(x,t)/w]dx$$

where

$$w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}.$$

In this case, $u_1 = (x+1)$ and $u_2 = (x+1)^2$ which give $w = (x+1)^2$,

$$v_1 = \int -(x+1)^2\delta(x-t)dx = -(t+1)^2,$$

and

$$v_2 = \int(x+1)\delta(x-t)dx = t + 1.$$

Therefore,

$$G(x,t) = \begin{cases} c_1(x+1) + c_2(x+1)^2, & x < t, \\ c_1(x+1) + c_2(x+1)^2 - (t+1)^2(x+1) + (t+1)(x+1)^2, & x > t. \end{cases}$$

Using the boundary conditions for G, $4G(0,t) = G(1,t)$ and $4G_x(0,t) = G_x(1,t)$ give

$$4(c_1+c_2) = 2c_1 + 4c_2 + 2(t+1)(1-t)$$

and

$$4(c_1+2c_2) = c_1 + 4c_2 + (t+1)(3-t).$$

Solving these two equations for c_1 and c_2, we have

$$c_1 = (t+1)(1-t) \text{ and } c_2 = t(t+1)/2.$$

Substituting gives

$$G(x,t) = \begin{cases} (t+1)(x+1)(xt-t+2)/2, & x < t, \\ (t+1)(x+1)(xt+2x-3t+2)/2, & x > t, \end{cases}$$

which is the same as was found in Example 2.5. The method of variation of parameters also can be used for unmixed boundary conditions.

Summary

Green's function, $G(x,t)$ for a second degree equation

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + qu = f(x)$$

with mixed homogeneous boundary conditions is found by using the method of variation of parameters as in Section 2.3 or by using the direct method shown in Example 2.4. In these cases Green's function is not necessarily symmetric, i.e., $G(x,t) \neq G(t,x)$. After the Green's function is known, the solution of the given problem is

$$u(t) = \int_0^1 f(x)G(x,t)dx.$$

2.4. Green's Functions for Equations of Order n

So far the discussion has been related to ordinary differential equations of order two--a good starting point. However, for the sake of completeness higher order equations need to be considered as well. The following example illustrates the appropriate adjustments for the case of $n = 3$.

Example 2.6. Consider $u''' = f(x)$; $u(0) = u'(0) = u''(1) = 0$.

To find the Green's function, $G(x,t)$, $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$ we shall proceed as follows:

Integrating by parts we have

$$\int_0^1 Gu'''dx = \left(Gu' - G_x u' + G_{xx} u\right)\Big|_0^1 - \int_0^1 G_{xxx} u dx.$$

In order to get the solution of the given problem in the form

$$u(t) = \int_0^1 f(x)G(x,t)dx,$$

we have to put

$$\left(Gu'' - G_x u' + G_{xx} u\right)\Big|_0^1 = 0, \qquad (2,6,1)$$

$$-G_{xxx} = \delta(x-t),$$

and require that G, G_x are continuous on $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$. Using the boundary conditions, $u(0) = u'(0) = u''(1) = 0$ and (2.6.1) we get

$$-G(0,t)u''(0) - G_x(1,t)u'(1) + G_{xx}(1,t)u(1) = 0.$$

Hence take $G(0,t) = G_x(1,t) = G_{xx}(1,t) = 0$, and thus (2.6.1) is satisfied. Therefore, Green's function,

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

has the following properties:

(1). $\frac{\partial^3}{\partial x^3} G_1 = 0$ for $x < t$ and $\frac{\partial^3}{\partial x^3} G_2 = 0$ for $x > t$ and G, G_x are continuous at $x = t$, that is, in particular $G_2(t,t) - G_1(t,t) = 0$, $\frac{\partial}{\partial x} G_2(t,t) - \frac{\partial}{\partial x} G_1(t,t) = 0$,

(2). $G_1(0,t) = \frac{\partial}{\partial x} G_2(1,t) = \frac{\partial^2}{\partial x^2} G_2(1,t) = 0$,

(3). The second derivative of G with respect to x has a jump discontinuity of magnitude -1 at $x = t$, that is,

$$\frac{\partial^2}{\partial x^2} G_2(t,t) - \frac{\partial^2}{\partial x^2} G_1(t,t) = -1.$$

By (1)

$$G(x,t) = \begin{cases} a_1 + a_2 x + a_3 x^2, & x < t, \\ b_1 + b_2 x + b_3 x^2, & x > t. \end{cases}$$

Properties (1) and (3) give

$$(b_1-a_1) + (b_2-a_2)t + (b_3-a_3)t^2 = 0,$$
$$(b_2-a_2) + 2(b_3-a_3)t = 0,$$
$$2(b_3-a_3) = -1.$$

Solving these three equations for b_i-a_i, $i = 1,2,3$,

$$b_3-a_3 = -\frac{1}{2},\ b_2-a_2 = t \text{ and } b_1-a_1 = -t^2/2.$$

Using (2), $G_1(0,t) = 0$ implies $a_1 = 0$, and therefore $b_1 = -t^2/2$;

$\frac{\partial}{\partial x} G_2(1,t) = 0$ and $\frac{\partial^2}{\partial x^2} G_2(1,t) = 0$ imply, respectively, that $b_2+2b_3 = 0$ and $b_3 = 0$. Hence $b_2 = 0$. Then $a_2 = -t$ and $a_3 = \frac{1}{2}$. Therefore,

$$G(x,t) = \begin{cases} -tx + x^2/2, & x < t, \\ -t^2/2, & x > t. \end{cases}$$

Then the solution of the given problem becomes

$$\begin{aligned} u(t) &= \int_0^1 f(x)G(x,t)dx \\ &= \int_0^t f(x)G_1 dx + \int_t^1 f(x)G_2 dx \\ &= \int_0^t f(x)(-tx+x^2/2)dx + \int_t^1 -f(x)(t^2/2)dx. \end{aligned}$$

Generally speaking for a differential equation order n,

$$Lu = p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1}u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n u = f(x) \text{ in } [0,1]$$

with n boundary conditions, the Green's function, G(x,t) will be found as in Example 2.6. That is, first, we integrate

$$\int_0^1 GLudx = \int_0^1 G\left(p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1}u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n u \right) dx$$

by parts until we have the form

$$\int_0^1 GLu\, dx = p(u,G) \Big|_0^1 + \int_0^1 uL^*G\, dx$$

where L^*G is the adjoint differential operator on G with respect to x, in particular,

$$L^*G = (-1)^n \frac{d^n}{dx^n}(p_0G) + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(p_1G) + \dots - \frac{d}{dx}(p_{n-1}G) + p_nG$$

and

$$p(u,G) = u[p_{n-1}G - \frac{d}{dx}(p_{n-2}G) + \dots + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(p_0G)] +$$

$$u'[p_{n-2}G - \frac{d}{dx}(p_{n-3}G) + \dots + (-1)^{n-2}\frac{d^{n-2}}{dx^{n-2}}(p_0G)] +$$

$$\dots + u^{(n-1)}(p_0G).$$

The boundary conditions on G, B(G), are found by using $p(u,G)\Big|_0^1 = 0$ and the given boundary conditions on u.

The Green's function

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

has the following properties:

(1). $L^*G_1 = 0$ and $L^*G_2 = 0$, and $G, \frac{\partial}{\partial x}G, \dots, \frac{\partial^{n-2}}{\partial x^{n-2}}G$ are continuous on $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, in particular, the derivatives of G_1 and G_2 join up smoothly along $x = t$,

(2). G satisfies the boundary condition B(G),

(3). The jump discontinuity of $\frac{\partial^{n-1}}{\partial x^{n-1}}G$ at $x = t$ has magnitude $(-1)^n/p_0(t)$.

By using these properties of G we can find that for $x \in [0,1]$,

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

for an ordinary differential equation order n, Lu = f(x) with n homogeneous boundary conditions. Furthermore, one may prove that the solution to the problem is

$$u(t) = \int_0^t f(x)G_1(x,t)dx + \int_t^1 f(x)G_2(x,t)dx.$$

2.5. Initial Value Problems

Up until now the problems that have been considered are <u>two</u> point boundary value problems. That is, we have been seeking a solution to an ordinary linear differential equation on an interval [a,b], usually [0,1], such that the solution and its derivatives at the <u>two</u> endpoints satisfy certain linear homogeneous boundary conditions. In this section, a differential equation on the interval $[a,\infty)$ (usually a = 0) will be considered. At the point x = a, the solution and its derivatives take on certain given or specified values. These problems arise very naturally in physical circumstances where the independent variable is time. Thus, such a problem is called an <u>initial value problem</u>. Again a Green's function will be found and then it will be used to write a solution to the given problem.

<u>Example 2.7.</u> Consider u" - u = f(x); u(0) = u'(0) = 0.

The Green's function G(x,t), $0 \leq x < \infty$, $0 \leq t < \infty$ has the following properties:

(1). $G(x,t) = \begin{cases} 0, & 0 \leq x < t, \\ G_2(x,t), & x > t, \end{cases}$

where $\frac{\partial^2}{\partial x^2} G_2 - G_2 = 0$, G(x,t) is continuous on $0 \leq x < \infty$,

$0 \leq t < \infty$, meaning that $G_2(t,t) = 0$,

(2). $G(0,t) = G_x(0,t) = 0$, are the boundary conditions (automatically satisfied by (1)), and

(3). $\frac{\partial}{\partial x} G_2(t,t) = 1$, that is, the partial derivative of G with respect to x has a jump discontinuity of magnitude one at $x = t$.

Since G_2 is the solution of $\frac{\partial^2}{\partial x^2} G_2 - G_2 = 0$, $G_2(x,t) = c_1e^x - c_2e^{-x}$. Using (1) and (3),

$$G_2(t,t) = c_1e^t - c_2e^{-t} = 0,$$

and

$$\frac{\partial}{\partial x} G_2(t,t) = c_1e^t + c_2e^{-t} = 1.$$

Solving these two equations for c_1 and c_2, we have $c_1 = \frac{1}{2} e^{-t}$ and $c_2 = \frac{1}{2} e^t$. Substitution gives

$$G(x,t) = \begin{cases} 0, & 0 \leq x < t, \\ \frac{1}{2}(e^{x-t} - e^{t-x}), & x > t. \end{cases}$$

The solution to the given problem is

$$u(x) = \int_0^x \frac{1}{2}(e^{x-t} - e^{t-x})f(t)dt.$$

The following example with variable coefficients again illustrates this method.

Example 2.8. Consider $(1+x)u'' + xu' - u = f(x)$; $u(0) = u'(0) = 0$.

The differential equation is first written in the form

$$u'' + x(1+x)^{-1}u' - (1+x)^{-1}u = (1+x)^{-1}f(x).$$

Green's function $G(x,t)$, $0 \leq x < \infty$, $0 \leq t < \infty$ has the following properties:

(1). $G(x,t) = \begin{cases} 0, & x < t, \\ G_2(x,t), & x > t, \end{cases}$

where

$$\frac{\partial^2}{\partial x^2} G_2 + x(1+x)^{-1} \frac{\partial}{\partial x} G_2 - (1+x)^{-1} G_2 = 0$$

and $G(x,t)$ is continuous on $0 \leqq x < \infty$, $0 \leqq t < \infty$, in particular, $G_2(t,t) = 0$,

(2). (The boundary conditions on G are automatically satisfied by the requirements of (1)),

(3). $\frac{\partial}{\partial x} G_2(t,t) = 1$.

Since by (1) the function $G_2(t,t)$ is a solution of the homogeneous differential equation, $G_2(x,t) = c_1 x + c_2 e^{-x}$. The continuity of G makes $c_1 t + c_2 e^{-t} = 0$ and (3) gives $c_1 - c_2 e^{-t} = 1$. Solving the last two equations for c_1 and c_2 we have $c_1 = (1+t)^{-1}$ and $c_2 = -te^{t}(1+t)^{-1}$. Substituting gives

$$G(x,t) = \begin{cases} 0, & x < t, \\ (1+t)^{-1}(x - te^{t-x}), & x > t. \end{cases}$$

The solution of the given problem is

$$u(x) = \int_0^x f(t)(1+t)^{-2}(x - te^{t-x})\,dt.$$

In Example 2.6 and 2.7 we have a second degree equation of the form

$$u'' + p(x)u' + q(x)u = r(x)$$

with initial conditions $u(0) = u'(0) = 0$. Then Green's function $G(x,t)$, $0 \leqq x < \infty$, $0 \leqq t < \infty$ has the following properties:

(1). $G(x,t) = \begin{cases} 0, & x < t, \\ G_2(x,t), & x > t, \end{cases}$

where

$$\frac{\partial^2}{\partial x^2} G_2 + p(x) \frac{\partial}{\partial x} G_2 + q(x) G_2 = 0 \text{ for } x > t$$

and $G(x,t)$ is continuous on $0 \leqq x < \infty$, $0 \leqq t < \infty$, in particular, $G_2(t,t) = 0$,

(2). (The boundary conditions for G are automatically satisfied by the requirements of (1)), and

(3). $\frac{\partial}{\partial x} G_2(t,t) = 1.$

The solution $u(x)$ of the differential equation

$$u'' + p(x)u' + q(x)u = r(x)$$

with the given initial conditions is

$$u(x) = \int_0^x r(t) G_2(x,t) dt.$$

If we consider the nth order differential equation

$$u^{(n)} + p_1 u^{(n-1)} + p_2 u^{(n-2)} + \ldots + p_n u = r(x)$$

with initial conditions

$$u(0) = u'(0) = u''(0) = u'''(0) = \ldots = u^{(n-1)}(0) = 0,$$

then Green's function, $G(x,t)$, $0 \leqq x < \infty$, $0 \leqq t < \infty$, has the following properties:

(1). $G(x,t) = \begin{cases} 0, & x < t, \\ G_2(x,t), & x > t, \end{cases}$

where $G_2(x,t)$ satisfies

$$\frac{\partial^n}{\partial x^n} G_2 + p_1 \frac{\partial^{n-1}}{\partial x^{n-1}} G_2 + \dots + p_n G_2 = 0$$

for $x > t$ and $G_2, \frac{\partial}{\partial x} G_2, \dots, \frac{\partial^{n-1}}{\partial x^{n-1}} G_2$ are continuous on $0 \leqq x < \infty$, $0 \leqq t < \infty$, which, in particular that

$$G_2(t,t) = \frac{\partial}{\partial x} G_2(t,t) = \dots = \frac{\partial^{n-2}}{\partial x^{n-2}} G_2(t,t) = 0,$$

(2). (The condition that $G(x,t) = 0$ for $x < t$, assures certain boundary conditions for G at $x = 0$.),

(3). $\frac{\partial^{n-1}}{\partial x^{n-1}} G_2(t,t) = 1$, that is, there is a jump discontinuity in $\frac{\partial^{n-1}}{\partial x^{n-1}} G(x,t)$ at $x = t$ of magnitude one.

The solution, $u(x)$, to the given nth order differential equation with the initial conditions is

$$u(x) = \int_0^x r(t) G_2(x,t)\,dt.$$

The following example illustrates this for $n = 4$.

Example 2.9. Consider $u^{(iv)} - u = r(x)$; $u(0) = u'(0) = u''(0) = u'''(0) = 0$.

The Green's function $G(x,t)$, $0 \leqq x < \infty$, $0 \leqq t < \infty$ can be obtained as follows: From (1)

$$G(x,t) = \begin{cases} 0, & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

where

$$\frac{\partial^{iv}}{\partial x^{iv}} G_2 - G_2 = 0, \text{ and}$$

$$G_2(t,t) = \frac{\partial}{\partial x} G_2(t,t) = \frac{\partial^2}{\partial x^2} G_2(t,t) = 0. \tag{2.5.1}$$

From (3)

$$\frac{\partial^3}{\partial x^3} G_2(t,t) = 1. \tag{2.5.2}$$

Since $G_2(x,t)$ is the solution of the differential equation then

$$G_2(x,t) = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x.$$

Then from (2.5.1) and (2.5.2) we have

$$c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t = 0$$

$$c_1 e^{t} - c_2 e^{-t} - c_3 \sin t + c_4 \cos t = 0$$

$$c_1 e^{t} + c_2 e^{-t} - c_3 \cos t - c_4 \sin t = 0$$

$$c_1 e^{t} - c_2 e^{-t} + c_3 \sin t - c_4 \cos t = 1.$$

Solving, we have $c_1 = -e^{-t}/4$, $c_2 = e^{t}/4$, $c_3 = -(\sin t)/2$ and $c_4 = (\cos t)/2$. Substituting gives

$$G(x,t) = \begin{cases} 0, & x < t, \\ [-e^{x-t} + e^{t-x} - 2\sin t \cos x + 2\cos t \sin x]/4, & x > t. \end{cases}$$

Then the solution, $u(x)$ to the problem is

$$u(x) = \int_0^x r(t)[-e^{x-t} + e^{t-x} - 2\sin t \cos x + 2\cos t \sin x]/4 \, dt.$$

2.6. Nonhomogeneous Conditions

A differential equation of order n with nonhomogeneous conditions can be solved by reducing it to the consideration of problem with homogeneous boundary values. For an example, consider $u''(x) = f(x)$ with nonhomogeneous conditions $u(0) = a$ and $u(1) = b$. If we let $u = u_1 + u_2$ where

$$u_1'' = f(x);\ u_1(0) = 0,\ u_1(1) = 0$$

and

$$u_2'' = 0;\ u_2(0) = a,\ u_2(1) = b.$$

It is easy to see that $u = u_1 + u_2$ satisfies

$$u'' = f(x);\ u(0) = a,\ u(1) = b.$$

Therefore, u is the solution of the problem.

In Example 2.1,

$$u_1 = \int_0^x (x-1)t\ f(t)dt + \int_x^1 (t-1)x\ f(t)dt.$$

The function $u_2(x)$ can be found directly as follows: The differential equation, $u_2'' = 0$ has the general solution $u_2 = c_1x + c_2$. The conditions $u_2(0) = a$ and $u_2(1) = b$ give $c_2 = a$ and $c_1 + c_2 = b$. Hence $c_2 = a$ and $c_1 = b - a$. Substituting, we have

$$u_2 = (b-a)x + a.$$

Therefore,

$$u(x) = \int_0^x (x-1)t\ f(t)dt + \int_x^1 (t-1)x\ f(t)dt + (b-a)x + a.$$

2.7. Boundedness as a Boundary Condition

One special type of nonhomogeneous condition is the condition that the solution u(x) of the differential equation is bounded as it approaches an endpoint. Green's function, in this case, is found directly as the following examples.

Example 2.10. Consider $xu'' + u' = f(x)$ where $u(x)$ is bounded as x approaches 0, and $u(1) = 0$.

The differential equation can be written in the form $(xu')' = f(x)$. Green's function for this problem, $G(x,t)$, $0 \leq x \leq 1$, $0 \leq t \leq 1$, is defined by

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

where G has the following properties:

(1). G is continuous on $0 \leq x \leq 1$, $0 \leq t \leq 1$, in particular, it follows $G_2(t,t) - G_1(t,t) = 0$, and $\left(x \frac{\partial}{\partial x} G_1\right)_x = 0$ for $x < t$, and $\left(x \frac{\partial}{\partial x} G_2\right)_x = 0$ for $x > t$,

(2). G satisfies the same boundary conditions as u, that is, $G_1(x,t)$ is bounded as x approaches 0 and $G_2(1,t) = 0$,

(3). The partial derivative of G with respect to x has a jump of magnitude t^{-1} at $x = t$, that is, $\frac{\partial}{\partial x} G_2(t,t) - \frac{\partial}{\partial x} G_1(t,t) = t^{-1}$.

From (1), G_1 and G_2 are the solution of $\left(x\frac{\partial}{\partial x} G\right)_x = 0$. Therefore,

$$G(x,t) = \begin{cases} G_1(x,t) = a_1 \ln x + a_2, & x < t, \\ G_2(x,t) = b_1 \ln x + b_2, & x > t. \end{cases}$$

Properties (1) and (3) give

$$(b_2-a_2) + (b_1-a_1)\ln t = 0,$$
$$(b_1-a_1)t^{-1} = t^{-1}.$$

From the last two equations, we have $b_1-a_1 = 1$, $b_2-a_2 = -\ln t$. By (2) $a_1 \ln x + a_2$ is bounded as $x \to 0$, hence $a_1 = 0$. Also (2) gives

$b_1 \ln 1 + b_2 = 0$ which implies $b_2 = 0$. Hence $a_1 = 0$, $b_2 = 0$ and $a_2 = \ln t$. Substituting gives

$$G(x,t) = \begin{cases} \ln t, & x < t, \\ \ln x, & x > t. \end{cases}$$

The solution $u(x)$ of the problem is

$$u(x) = \int_0^1 f(t)G(x,t)dt$$

$$= \int_0^x f(t)\ln x \, dt + \int_x^1 f(t)\ln t \, dt.$$

The next example has boundedness required at both endpoints.

Example 2.11. Consider $u'' - u = f(x)$ where $u(x)$ is bounded as $|x| \to \infty$.

The Green's function for this problem is $G(x,t)$, $-\infty < x < \infty$, $-\infty < t < \infty$ where

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

such that G has the following properties:

(1). G is continuous on $-\infty < x < \infty$, $-\infty < t < \infty$, in particular, at $x = t$, it follows that $G_1(t,t) = G_2(t,t)$. Also

$$\frac{\partial^2}{\partial x^2} G_1 - G_1 = 0$$

for $x < t$,

$$\frac{\partial^2}{\partial x^2} G_2 - G_2 = 0$$

for $x > t$,

(2). G satisfies the same boundary condtions as u, that is, G_1 is bounded as $x \to -\infty$, and G_2 is bounded as $x \to \infty$,

(3). The partial derivative of G with respect to x has a jump of magnitude 1 at x = t, that is,

$$\frac{\partial}{\partial x} G_2(t,t) - \frac{\partial}{\partial x} G_1(t,t) = 1.$$

From (1) G_1 and G_2 are solutions of

$$\frac{\partial^2}{\partial x^2} G - G = 0.$$

Hence

$$G(x,t) = \begin{cases} G_1(x,t) = a_1 e^{x} + a_2 e^{-x}, & x < t, \\ G_2(x,t) = b_1 e^{x} + b_2 e^{-x}, & x > t. \end{cases}$$

Properties (1) and (3) give

$$(b_1 - a_1)e^{t} + (b_2 - a_2)e^{-t} = 0,$$

$$(b_1 - a_1)e^{t} - (b_2 - a_2)e^{-t} = 1.$$

Solving the last two equations, we have $b_1 - a_1 = \frac{1}{2} e^{-t}$ and $b_2 - a_2 = -\frac{1}{2} e^{t}$. From (2) $a_1 e^{x} + a_2 e^{-x}$ is bounded as $x \to -\infty$ and $b_1 e^{x} + b_2 e^{-x}$ is bounded as $x \to \infty$ which imply, respectively, that $a_2 = 0$ and $b_1 = 0$. Since $b_1 - a_1 = \frac{1}{2} e^{-t}$ and $b_2 - a_2 = -\frac{1}{2} e^{t}$, it follows that $a_1 = -\frac{1}{2} e^{-t}$ and $b_2 = -\frac{1}{2} e^{t}$. Substituting gives

$$G(x,t) = \begin{cases} -\frac{1}{2} e^{x-t}, & x < t, \\ -\frac{1}{2} e^{t-x}, & x > t. \end{cases}$$

The solution of u(x) of the problem is

$$u(x) = \int_{-\infty}^{\infty} f(t)G(x,t)dt$$

$$= -\frac{1}{2}\int_{-\infty}^{x} f(t)e^{t-x}dt - \frac{1}{2}\int_{x}^{\infty} f(t)e^{x-t}dt.$$

2.8. Generalized Green's Functions

If there is a solution of second order homogeneous differential equation which satisfies both endpoint conditions, then a Green's function for the differential equation with these conditions does not exist. However, in order to have Poisson type solution, a generalized Green's function is defined, as illustrated by the following example.

Example 2.12. Consider $u'' + u = f(x)$; $u(0) = u(\pi) = 0$.

The general solution of $u'' + u = 0$ is

$$u(x) = c_1\sin x + c_2\cos x.$$

The conditions $u(0) = u(\pi) = 0$ imply only that $c_2 = 0$. Therefore, c_1 may be an arbitrary constant. Hence $u(x) = \sin x$ is a solution of $u'' + u = 0$; $u(0) = u(\pi) = 0$.

We shall now see that Green's function for the problem does not exist.

Define Green's function $G(x,t)$, $0 \leq x \leq \pi$, $0 \leq t \leq \pi$ as in Example 2.4 except that property 2 is

(2'). G satisfies the boundary conditions that u satisfies, that is,

$G_1(0,t) = 0$ and $G_2(\pi,t) = 0$.

Notice that the difference of this problem and Example 2.4 is that the right endpoint is π instead of one. Following Example 2.4, and by using (1) and (3), we have

$$G(x,t) = \begin{cases} a_1 \sin x + a_2 \cos x, & x < t, \\ b_1 \sin x + b_2 \cos x, & x > t, \end{cases}$$

and $(b_1-a_1)\cos t - (b_2-a_2)\sin t = 1$. Property (2') implies $a_2 = 0$ and $b_2 = 0$. Therefore, $b_2-a_2 = 0$ which implies $(b_1-a_1)\cos t = 1$, a contradiction since there is no constant b_1-a_1 that make $(b_1-a_1)\cos t = 1$ for all t. In such a case the generalized Green's function $H(x,t)$, $0 \leq x \leq \pi$, $0 \leq t \leq \pi$, is used. Define

$$H(x,t) = \begin{cases} H_1(x,t), & x < t, \\ H_2(x,t), & x > t, \end{cases}$$

such that H has the following properties:

(1). H is continuous on $0 \leq x \leq \pi$, $0 \leq t \leq \pi$. In particular, at $x = t$, we have $H_2(t,t) - H_1(t,t) = 0$. Also H_1 and H_2 satisfy the differential equation

$$\frac{\partial^2}{\partial x^2} H + H = C\, u(x)\, u(t)$$

where $u(x)$ is a solution of $u'' + u = 0$; $u(0) = u(\pi) = 0$, that is, in particular, $u(x) = \sin x$, C is a constant,

(2). H satisfies the same boundary conditions as u, that is, $H_1(0,t)=0$ and $H_2(\pi,t) = 0$,

(3). The partial derivative of H with respect to x has a jump of magnitude 1 at $x = t$, that is,

$$\frac{\partial}{\partial x} H_2(t,t) - \frac{\partial}{\partial x} H_1(t,t) = 1,$$

(4). H satisfies the condition

$$\int_0^{\pi} H(x,t)u(x)\ dx = 0,$$

that is,

$$\int_0^t H_1(x,t)u(x)dx + \int_t^{\pi} H_2(x,t)u(x)dx = 0.$$

One can prove that the solution of the problem is

$$u(x) = \int_0^{\pi} H(x,t)f(t)dt$$

$$= \int_0^x H_2(x,t)f(t)dt + \int_x^{\pi} H_1(x,t)f(t)dt.$$

The function H will be found directly as follows: From (1)

$$\frac{\partial^2}{\partial x^2} H + H = C \sin x \sin t.$$

The general solution H_g of the homogeneous equation is $H_g = a \sin x + b \cos x$. A particular solution $H_p = -(Cx/2)\cos x \sin t$ of

$$\frac{\partial^2}{\partial x^2} H + H = C \sin x \sin t$$

can be obtained by using the method of variations of parameters. Hence

$$H(x,t) = \begin{cases} a_1\sin x + a_2\cos x - (Cx/2)\cos x \sin t, & x < t, \\ b_1\sin x + b_2\cos x - (Cx/2)\cos x \sin t, & x > t. \end{cases}$$

From (1) and (3) we have

$$(b_1-a_1)\sin t + (b_2-a_2)\cos t = 0,$$

$$(b_1-a_1)\cos t - (b_2-a_2)\sin t = 1,$$

which imply that $b_1-a_1 = \cos t$ and $b_2-a_2 = -\sin t$. From (2), $H_1(0,t) = 0$

gives $a_2 = 0$ and $H(\pi,t) = 0$ gives $b_2 = (C\pi/2)\sin t$. Hence $-\sin t = (C\pi/2)\sin t$ which implies $C = -2/\pi$. At this point we have $a_2 = 0$, $b_2 = -\sin t$, $C = -2/\pi$ and $b_1 = a_1 + \cos t$. Substituting gives

$$H(x,t)=\begin{cases} \frac{x}{\pi}\cos x \sin t + a_1 \sin x, & x < t, \\ \left(\frac{x}{\pi} - 1\right)\cos x \sin t + (a_1+\cos t)\sin x, & x > t. \end{cases}$$

Using (4) we have

$$\int_0^t \left(\frac{x}{\pi}\cos x \sin t + a_1 \sin x\right)\sin x\, dx +$$

$$\int_t^\pi \left[\left(\frac{x}{\pi} - 1\right)\cos x \sin t + (a_1+\cos t)\sin x\right]\sin x\, dx = 0$$

which implies that

$$a_1 = (\sin t - 2\sin^3 t - 2\pi\cos t + 2t\cos t - \sin 2t \cos t)/2\pi.$$

Therefore,

$$H(x,t) = \begin{cases} [2x\cos x \sin t + \sin x \sin t \cos 2t + 2(t-\pi)\sin x \cos t - \\ \quad \sin x \sin 2t \cos t]/2\pi, \quad x < t \\ [2x \cos x \sin t + \sin x \sin t \cos 2t + 2t \sin x \cos t - \\ \quad \sin x \sin 2t \cos t - 2\pi \sin t \cos x]/2\pi, \quad x > t. \end{cases}$$

The solution of the problem is

$$u(x) = \int_0^\pi H(x,t)f(t)dt$$

$$= \frac{1}{2\pi}\int_0^\pi (2x\cos x \sin t + \sin x \sin t \cos 2t + 2t \sin x \cos t -$$

$$\sin x \sin 2t \cos t)f(t)dt - \sin x \int_x^\pi \cos t\, f(t)dt -$$

$$\cos x \int_0^x \sin t\ f(t)dt.$$

2.9. Interval [a,b]

Green's function for a differential equation of order n in the interval [a,b] with boundary conditions can be found by the same method as for [0,1]. The only difference is that the algebra is slightly more cumbersome.

Example 2.13. Consider $u'' + (b-a)^{-2}u = f(x)$; $u(a) = u(b) = 0$, where $a < b$.

Green's function $G(x,t)$, $a \leq x \leq b$, $a \leq t \leq b$ defined by

$$G(x,t) = \begin{cases} G_1(x,t), & x < t, \\ G_2(x,t), & x > t, \end{cases}$$

has the following properties:

(1). G is continuous on $a \leq x \leq b$, $a \leq t \leq b$. In particular, at $x = t$, it follows that $G_2(t,t) - G_1(t,t) = 0$. Also the functions G_1 and G_2 satisfy the equation

$$\frac{\partial^2}{\partial x^2} G + (b-a)^{-2}G = 0,$$

(2). $G_1(a,t) = 0$ and $G_2(b,t) = 0$,

(3). The jump of the derivative of G with respect to x has magnitude 1 at $x = t$, that is, $\frac{\partial}{\partial x} G_2(t,t) - \frac{\partial}{\partial x} G_1(t,t) = 1$.

From (1) G_1 and G_2 are solutions of $\frac{\partial^2}{\partial x^2} G + (b-a)^{-2}G = 0$, therefore

$$G(x,t) = \begin{cases} a_1\sin[(b-a)^{-1}x] + a_2\cos[(b-a)^{-1}x], & x < t, \\ b_1\sin[(b-a)^{-1}x] + b_2\cos[(b-a)^{-1}x], & x > t. \end{cases}$$

Properties (1) and (3) give

$$(b_2-a_2)\cos[t/(b-a)] + (b_1-a_1)\sin[t/(b-a)] = 0$$

$$-[(b_2-a_2)/(b-a)]\sin[t/(b-a)] + [(b_1-a_1)/(b-a)]\cos[t/(b-a)] = 1.$$

Solving these two equations for (b_1-a_1) and (b_2-a_2) gives

$$b_1 - a_1 = (b-a)\cos[t/(b-a)]$$

and

$$b_2 - a_2 = -(b-a)\sin[t/(b-a)].$$

Applying property (2) to $G(x,t)$, we have

$$a_1\sin[a/(b-a)] + a_2\cos[a/(b-a)] = 0$$

and

$$b_1\sin[b/(b-a)] + b_2\cos[b/(b-a)] = 0.$$

Solving the last four equations for the a and b coefficients, gives

$$a_1 = (b-a)\cos[a/(b-a)]\sin[(t-b)/(b-a)]/\sin 1,$$

$$a_2 = -(b-a)\sin[a/(b-a)]\sin[(t-b)/(b-a)]/\sin 1,$$

$$b_1 = (b-a)\cos[a/(b-a)]\sin[(t-b)/(b-a)] + \sin 1 \cos[t/(b-a)]/\sin 1,$$

$$b_2 = -(b-a)\sin[a/(b-a)]\sin[(t-b)/(b-a)] + \sin 1 \sin[t/(b-a)]/\sin 1.$$

Substituting gives

$$G(x,t) = \begin{cases} (b-a)\sin[(t-b)/(b-a)]\sin[(x-a)/(b-a)]/\sin 1, & x < t, \\ (b-a)\sin[(x-a)/(b-a)]\sin[(t-b)/(b-a)]/\sin 1 + \\ \quad (b-a)\sin[(x-t)/(b-a)], & x > t. \end{cases}$$

Warning: Green's function cannot be obtained by changing variables x,t to x', t' by the transformation $x' = (b-a)x + a$ and $t' = (b-a)t + a$ such that [0,1] maps onto [a,b]. This is illustrated as follows:

Let

$$x' = (b-a)x + a \text{ amd } t' = (b-a)t + a.$$

Then (0,0) maps onto (a,a) and (1,1) maps onto (b,b). Suppose we work Example 2.13 by changing variables. Then

$$\frac{du}{dx'} = \frac{du}{dx} \cdot \frac{dx}{dx'} = (b-a)^{-1} \frac{du}{dx}$$

$$\frac{d^2u}{dx'^2} = (b-a)^{-2} \frac{d^2u}{dx^2}.$$

Therefore, the differential equation of Example 2.13 becomes

$$u'' + u = (b-a)^2 f(x)$$

with boundary conditions $u(0) = u(1) = 0$. By the previous methods Green's function for [0,1] would be

$$G(x,t) = \begin{cases} \sin(t-1)\sin x/\sin 1, & x < t, \\ \sin(x-1)\sin t/\sin 1, & x > t. \end{cases}$$

Since $x = (x'-a)/(b-a)$ and $t = (t'-a)/(b-a)$ we would have by substituting

$$G(x,t) = \begin{cases} \sin[(t'-b)/(b-a)]\ \sin[(x'-a)/(b-a)]/\sin 1, & x' < t', \\ \sin[(x'-b)/(b-a)]\sin[(t'-a)/(b-a)]/\sin 1, & x' > t'. \end{cases}$$

Calculating $\frac{\partial}{\partial x} G(t'^{+},t') - \frac{\partial}{\partial x} G(t'^{-},t')$ gives, the jump of the derivative of $G(x',t')$ at $x' = t'$ to be $(b-a)^{-1}$ but, in fact the jump of the derivative of $G(x',t')$ at $x' = t'$ should be 1. Thus, Green's function cannot be obtained by simply changing the variables in the expected way. It is thus necessary to just solve for Green's function on the interval [a,b] in a way analogous to finding Green's function on [0,1].

CHAPTER III

GREEN'S FUNCTIONS FOR LAPLACE'S EQUATION

In a study of a variety of steady state problems, (oscillations, heat conduction, diffusion and others) one often arrives at equations of elliptic type. The most common equation of this type is Laplace's equation

$$\Delta u = 0.$$

3.1. Green's Function and the Dirichlet's Problem

Let $\mathcal{R}$ be a region in the n-space, n = 2,3,..., with boundary S for which Green's theorem is applicable for n = 2 and the divergence theorem is applicable for $n \geqq 3$. The solution $u(\vec{r})$ of the problem

$$\Delta u = 0 \text{ in } \mathcal{R} \tag{3.1.1}$$

with Dirichlet's boundary condition,

$$\begin{aligned} u(\vec{r}) &= g(\vec{r}) \text{ on C, } n = 2, \\ &= g(\vec{r}) \text{ on S, } n \geqq 3 \end{aligned} \tag{3.1.2}$$

is desired. This problem is called the Dirichlet's problem for the region $\mathcal{R}$. The solution of the Dirichlet's problem is given by the expression

$$u(\vec{r}) = -\frac{1}{2\pi}\int_C g(\vec{r}\,')\frac{\partial}{\partial n}G(\vec{r},\vec{r}\,')ds', \; n = 2$$

$$u(\vec{r}) = -\frac{1}{(n-2)\omega_n}\int_S g(\vec{r}')\frac{\partial}{\partial n}G(\vec{r},\vec{r}')dS', \quad n \geqq 3 \qquad (3.1.3)$$

where $\vec{n}$ is the outward-drawn normal to the boundary C or boundary curve S, and

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

$\Gamma(x)$ is the gamma function of x, $G(\vec{r},\vec{r}')$ is the Green's function which can be written as

$$G(\vec{r},\vec{r}') = W(\vec{r},\vec{r}') + w(\vec{r},\vec{r}') \qquad (3.1.4)$$

where W is a particular solution of $\Delta G = 0$ except when $\vec{r} = \vec{r}'$ which need not satisfy the required boundary condition, and w is a solution of the homogeneous equation $\Delta G = 0$, such that the combination W + w does satisfy those boundary conditions. The function W is called <u>a fundamental solution</u> or <u>a principal solution</u> or <u>an elementary solution</u>. It contains the basic singularity of the Green's function. Considering $\vec{r}$ in $\mathcal{R}$ as a fixed point and $\vec{r}'$ in $\overline{\mathcal{R}}$ as a variable point, the Green's function has the following properties:

(1). $\Delta G = 0$, $\vec{r}' \neq \vec{r}$,

(2). $G = 0$, $\vec{r}' \in C$ for $n = 2$ or $G = 0$, $\vec{r}' \in S$ for $n \geqq 3$,

(3). G assyptotically equals W as $\vec{r}' \to \vec{r}$, or $G \sim W$ as $\vec{r}' \to \vec{r}$.

3.2. Fundamental Solutions

Let us now determine the fundamental solutions for the Laplace's equations in n-dimensions, n = 2,3,..., so that, in constructing the Green's function, the function w is determined so as to satisfy the required boundary conditions.

3.2.1. Two Dimensions

Some particular solutions of Laplace's equation which are of great interest depend only on the one variable r, i.e., the distance from the origin.

From Laplace's equation in polar coordinates, we will see that if u(r) is the solution of the ordinary differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 0 \qquad (3.2.1)$$

then u(r) is a solution of

$$u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

Integrating (3.2.1), we find the general solution

$$u(r) = c_1 \ln r + c_2.$$

Choosing $c_1 = -1$ and $c_2 = 0$ we shall have

$$u(r) = \ln \frac{1}{r} \qquad (3.2.2)$$

The function $\ln \frac{1}{r}$ satisfies Laplace's equation everywhere except at the point r = 0 where it becomes infinite. The variable $r = \sqrt{x^2 + y^2}$ gives the distance between the point P(x,y) and the origin. If we let

$$R = \sqrt{(x'-x)^2 + (y'-y)^2}$$

be the distance between P(x,y) and Q(x',y'), then $\ln \frac{1}{R}$ also satisfies Laplace's equation everywhere except at the point R = 0, i.e. except at (x,y) = (x',y'). Let $\xi = x'-x$ and $\eta = y'-y$, then

$$W(x,y;x',y') = \ln \frac{1}{R}$$

satisfies $W_{\xi\xi} + W_{\eta\eta} = 0$. However, $W_{\xi\xi} + W_{\eta\eta} = W_{x'x'} + W_{y'y'}$, hence, $W = \ln \frac{1}{R}$ satisfies $\Delta W = 0$ except when $P = Q$. Thus $W = \ln \frac{1}{R}$ is a fundamental solution in the plane.

3.2.2. Three Dimensions and n-Dimensions

In three dimensional rectangular coordinates, Laplace's equation has the form

$$u_{xx} + u_{yy} + u_{zz} = 0$$

and in spherical coordinates it becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \phi^2} = 0.$$

The solution of Laplace's equation of the form $u = u(r)$ can be determined from

$$\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = 0.$$

Integrating this equation, we find that

$$u(r) = \frac{C_1}{r} + C_2$$

where C_1 and C_2 are arbitrary constants. Choosing $C_1 = 1$, $C_2 = 0$ we obtain

$$u(r) = \frac{1}{r}$$

which is called a <u>fundamental solution</u> of Laplace's equation in three dimensions. If we let

$$R = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$$

be distance between the point $P(x,y,z)$ and $Q(x',y',z')$, then following an argument analogous to that of two dimensions then

$$W(x,y,z;x',y',z') = \frac{1}{R}$$

is also a fundamental solution of Laplace's equation. The function $W = \frac{1}{R}$ satisfies Laplace's equation everywhere except when $R = 0$, i.e. when $(x,y,z) = (x',y',z')$ where it becomes infinite.

Similarly, in the case of Euclidean n-space E^n, $n = 4,5,\ldots$ the function

$$W = \frac{1}{R^{n-2}}$$

is the solution which satisfies Laplace's equation everywhere except $R = 0$. In this case, $R = |\vec{r}\,'-\vec{r}|$, is the distance from $P = \vec{r} = (x_1,\ldots,x_n)$ to $Q = \vec{r}\,' = (x_1',\ldots,x_n')$.

3.3. The Method of Images in the Plane

The following examples will show how Green's function for Dirichlet's problem in a region $\mathcal{R}$ is found by the method of images.

Example 3.1. Solve the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $x > 0$; $u = f(y)$ on $x = 0$ and $u \to 0$ as $x \to \infty$.

Notice that this region is unbounded. In this case a condition as $x \to \infty$ must be added, namely, $u \to 0$ as $x \to \infty$.

This is the Dirichlet's problem for the region $\mathcal{R}$ being the right half-plane. If P is the point (x,y), x > 0, then take P_1 to be (-x,y), the image point of P across the y-axis, and let Q be (x',y') (cf. Figure 3.1). Then

$$G(x,y;x',y') = \ln \frac{QP_1}{QP}$$

where QP_1 and QP is distance from Q to P_1 and Q to P, respectively. Hence

$$G(x,y;x',y') = \frac{1}{2} \ln\left[(x'+x)^2 + (y'-y)^2\right] + \ln \frac{1}{|\vec{r}'-\vec{r}|} .$$

The function G is Green's function for this problem. If we let

$$w(x,y;x',y') = \frac{1}{2} \ln\left[(x'+x)^2 + (y'-y)^2\right]$$

then its required properties can be shown as follows:

(1). $$\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w}{\partial y'^2} = 0$$

by directly calculating the partial derivatives.

(2). From Figure 3.1, if Q is on the boundary C, that is, x' = 0 then elementary geometry tells us that QP_1 = QP. Hence

$$G(x,y;0,y') = \ln \frac{QP}{QP} = \ln 1 = 0.$$

(3). $$\lim_{\vec{r}' \to \vec{r}} \left[G(x,y;x',y') - \ln \frac{1}{|\vec{r}'-\vec{r}|}\right] = \lim_{\vec{r}' \to \vec{r}} \ln |\vec{r}'-\vec{r}_1| = \ln |\vec{r}'-\vec{r}|$$

where $\vec{r}_1$ = (-x,y).

The solution of this problem by using (3.1.3) is

$$u(x,y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} f(y') \frac{\partial G}{\partial n} dy'$$

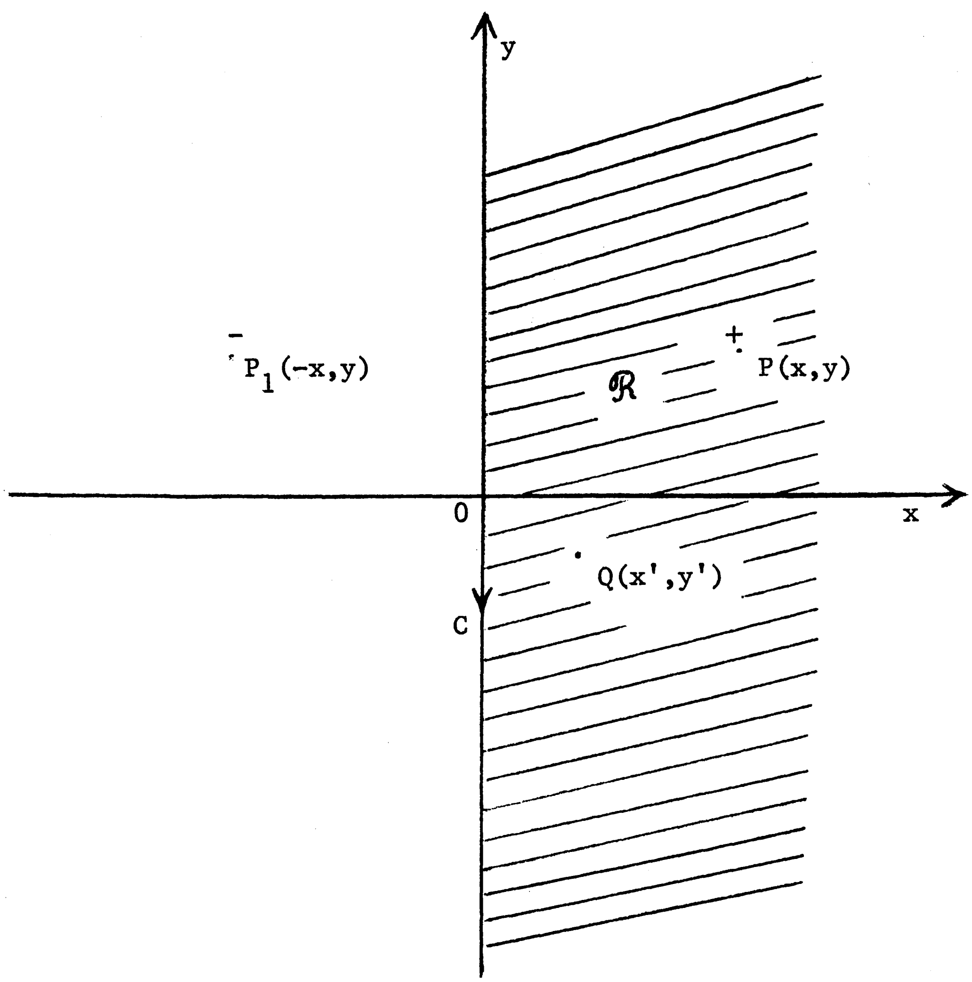


Figure 3.1. The Image of a Point in a Half-Plane

where

$$\frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial x'}\right)_{x'=0} = -\frac{2x}{x^2 + (y'-y)^2} .$$

Substituting, gives

$$u(x,y) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{f(y')}{x^2 + (y'-y)^2}\, dy' .$$

Notice that $u \to 0$ as $x \to \infty$ as required.

We will consider the region which gives images of P across the x-axis and the y-axis in the following example.

Example 3.2. Consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $x > 0$, $y > 0$; $u = f(y)$ on $x = 0$, $u = g(x)$ on $y = 0$.

If P is the point (x,y), $x > 0$, $y > 0$, then take P_1 to be $(-x,y)$, P_2 to be $(-x,-y)$, P_3 to be $(x,-y)$ and Q to be (x',y') where $(x',y') \in \mathcal{R}$ (cf. Figure 3.2). Notice that P_1 and P_2 are images of P and P_3, respectively, across the y-axis. Let P have a plus sign and P_1 have a minus sign. Signs for P, P_1, P_2 and P_3 are then +, −, + and −, respectively. Note that pairs of image points across each axis have opposite signs. Then

$$G(x,y;x',y') = \ln \frac{QP_1 \cdot QP_3}{QP \cdot QP_2}$$

$$= \frac{1}{2} \ln \frac{\left[(x'-x)^2+(y'-y)^2\right]\left[(x'-x)^2+(y'+y)^2\right]}{(x'+x)^2+(y'-y)^2} + \ln \frac{1}{|\vec{r}\,'-\vec{r}|} .$$

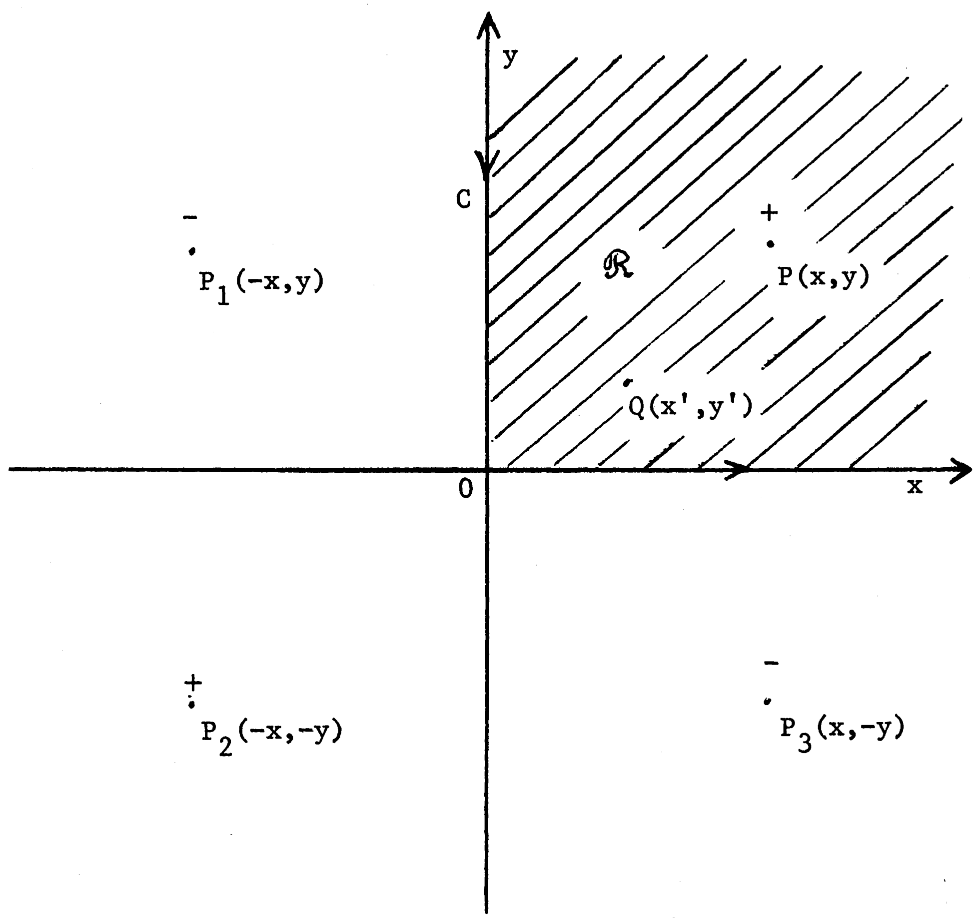


Figure 3.2. The Image of a Point in Quarter-Plane

If we let

$$w(x,y;x',y') = \frac{1}{2} \ln \frac{[(x'+x)^2+(y'-y)^2][(x'-x)^2+(y'+y)^2]}{(x'+x)^2+(y'-y)^2}$$

then again G has properties (1), (2) and (3). On the boundary C, we can find $\frac{\partial G}{\partial n}$ as follows:

On $x' = 0$,

$$\frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial x'}\right)_{x'=0} = -\frac{2x}{x^2+(y'-y)^2} + \frac{2x}{x^2+(y'+y)^2}.$$

On $y' = 0$,

$$\frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial y'}\right)_{y'=0} = -\frac{2y}{(x'+x)^2+y^2} + \frac{2y}{(x'-x)^2+y^2}.$$

Therefore, by using (3.1.3), the solution for the problem is

$$u(x,y) = -\frac{1}{2\pi}\int_0^\infty f(y')\frac{\partial G}{\partial n}\,dy' - \frac{1}{2\pi}\int_0^\infty g(x')\frac{\partial G}{\partial n}\,dx'$$

$$= -\frac{x}{\pi}\int_0^\infty f(y')\left[-\frac{1}{x^2+(y'-y)^2} + \frac{1}{x^2+(y'+y)^2}\right]dy' -$$

$$\frac{y}{\pi}\int_0^\infty g(x')\left[-\frac{1}{(x'+x)^2+y^2} + \frac{1}{(x'-x)^2+y^2}\right]dx'.$$

Now notice how Green's function was obtained in these two examples. In Example 3.1, the point with the - sign is P_1 and

$$G = \ln \frac{QP_1}{QP},$$

and in Example 3.2, points with minus signs are P_1 and P_3 and

$$G = \ln \frac{QP_1 \cdot QP_3}{QP \cdot QP_2}.$$

The following example is of the region which gives more images of P and across the x-axis and the line through the origin with inclination $\frac{\pi}{3}$.

Example 3.3. Find u such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for a region $\mathfrak{R} = \left\{ re^{i\theta} : 0 \leq \theta \leq \pi/3,\ r > 0 \right\}$ and $u = f(x,y)$ on the boundary C.

Let P be the point (r,θ), $0 < r < \infty$, $0 < \theta < \pi/3$, let $P_1, P_2, \ldots,$ P_5 be the points $(r, 2\pi/3 - \theta)$, $(r, 2\pi/3 + \theta)$, $(r, 4\pi/3 - \theta)$, $(r, 4\pi/3 + \theta)$ and $(r, 6\pi/3 - \theta)$, respectively, and Q be the point $(x',y') \in \mathfrak{R}$ (cf. Figure 3.3).

The points P_i, $i = 1,2,3,4,5$ are obtained by beginning with P and successively generating all possible points by reflections across either the x-axis or the line through the origin with inclination $\pi/3$. Label P with a plus sign, label with a minus sign the two images of P and successively alternate signs for each new image point. Thus P, P_2 and P_4 are labeled plus and the others are labeled minus. Then

$$G(x,y;x',y') = \ln \frac{QP_1 \cdot QP_3 \cdot QP_5}{QP \cdot QP_2 \cdot QP_4}$$

$$= \frac{1}{2} \sum_{n=1}^{3} \ln \frac{[x'-r\cos(2n\pi/3-\theta)]^2 + [y'-r\sin(2n\pi/3-\theta)]^2}{[x'-r\cos(2n\pi/3+\theta)]^2 + [y'-r\sin(2n\pi/3+\theta)]^2}.$$

As before let

$$w(x,y;x',y') = \frac{1}{2} \sum_{n=1}^{3} \ln[(x'-r\cos(2n\pi/3-\theta))^2 + (y'-r\sin(2n\pi/3-\theta))^2] -$$

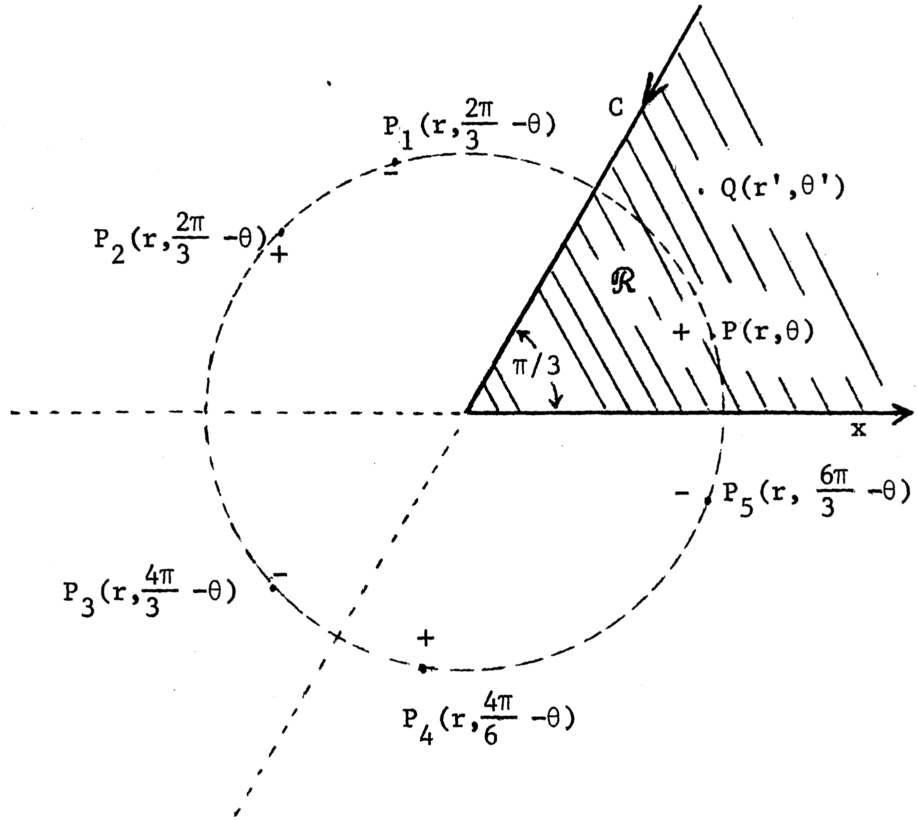


Figure 3.3. Images of a Point in a Region with Angle $\pi/3$

$$\frac{1}{2}\sum_{n=1}^{2} \ln[(x'-r\cos(2n\pi/3+\theta))^2 + (y'-r\sin(2n\pi/3+\theta))^2],$$

then

$$G = w + \ln\frac{1}{|\vec{r}\,'-\vec{r}|}.$$

One can show that G has properties (1), (2), and (3).

We have used the method of images in finding Green's functions in Examples 3.1-3.3. In general, Green's function for Dirichlet's problem for a region $\mathcal{R}$ with angle $\psi = \pi/k$, k = 1,2,... can be found by using the method of images. The next example illustrates this.

Example 3.4. Consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for a region

$$\mathcal{R} = \left\{ r^{i\theta}: 0 < \theta < \pi/k,\ r > 0,\ k = 1,2,\ldots ,\right\}$$

u = f(x,y) on the boundary C.

Let P be the point (r,θ), $0 < r < \infty$, $0 < \theta < \pi/k$, P_i be a point (r,θ_i), i = 1,2,...,2k-1 where

$$\theta_i = \begin{cases} (i+1)\psi - \theta, & i = 1,3,5,\ldots,2k-1 \\ i\psi + \theta, & i = 2,4,6,\ldots,2k-2 \end{cases}$$

and Q be $(x',y') \in \mathcal{R}$ (cf. Figure 3.4).

Following Example 3.3 Green's function is given by

$$G(x,y;x',y') = \ln\frac{QP_1\cdot QP_3\cdot QP_5\cdot\ldots\cdot QP_{2k-1}}{QP\cdot QP_2\cdot QP_4\cdot\ldots\cdot QP_{2k-2}}$$

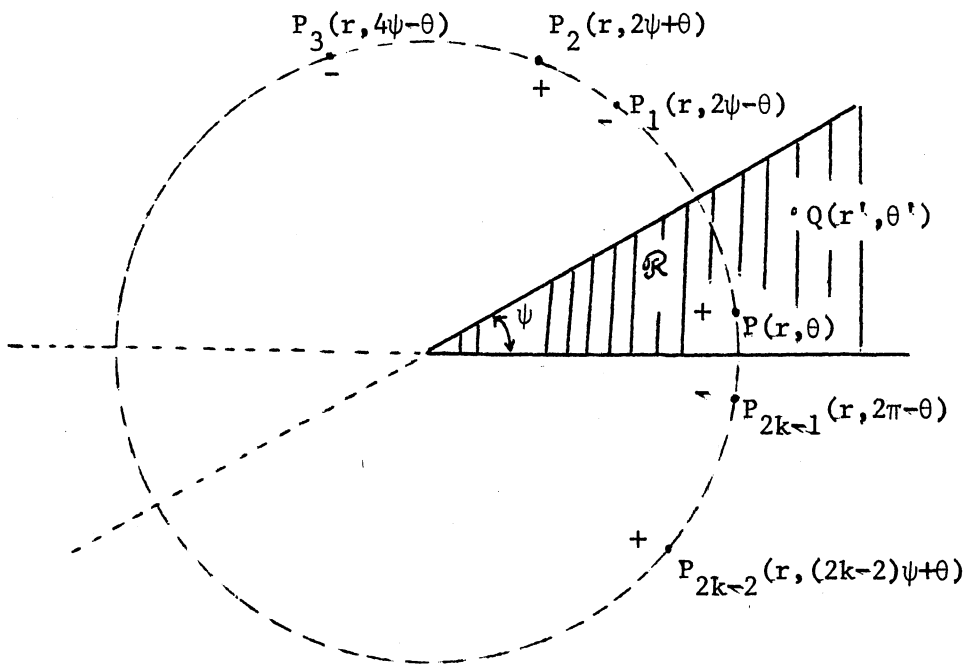


Figure 3.4. Images of a Point in a Region with Angle $\psi = \pi/k$

$$= \frac{1}{2}\sum_{i=1}^{k} \ln \frac{[x'-r\cos(2i\psi-\theta)]^2+[y'-r\sin(2i\psi-\theta)]^2}{[x'-r\cos(2i\psi+\theta)]^2+[y'-r\sin(2i\psi+\theta)]^2}$$

and

$$w(x,y;x',y') = \frac{1}{2}\sum_{i=1}^{k} \ln\left\{[x'-r\cos(2i\psi-\theta)]^2+[y'-r\sin(2i\psi-\theta)]^2\right\} -$$

$$\frac{1}{2}\sum_{i=1}^{k-1} \ln\left\{[x'-r\cos(2i\psi+\theta)]^2+[y'-r\sin(2i\psi+\theta)]^2\right\}.$$

Again G has properties (1), (2) and (3).

Example 3.4 gives us the form of Green's function for the region between the x-axis and a line through the origin with inclination ψ, $\psi = \pi/k$, $k = 1,2,\ldots.$ The number of points P_i are $2k-1$. Generally, for the case of $\psi = r\pi$, where r is a rational number, the number of points p_i is finite. It is not neccessary to be unique for a given region. For example, if $\psi = 7\pi/18$ and $\theta = \pi/12$ there are 35 points P_i but if $\theta = \pi/6$ then there are 17 points P_i. In these cases the method of images also can be used. In the other case when $\psi = q\pi$, q is an irrational number, the number of points P_i will be infinite. For this type of region, we have a lot of difficulties in using the method of images. However, the method of images can be used for the special type of region which gives an infinite number of images. The following example is of a region bounded by two parallel lines. In this case there will be two infinite sequences of images.

Example 3.5. Find u such that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for the partial strips $\mathfrak{R} = \left\{(x,y);\ 0 < x < a\right\}$ such that $u = f(x,y)$ on the boundary C.

Let P be $(x,0)$, $0 < x < a$, and let Q be a point $(x',y') \in \mathfrak{R}$

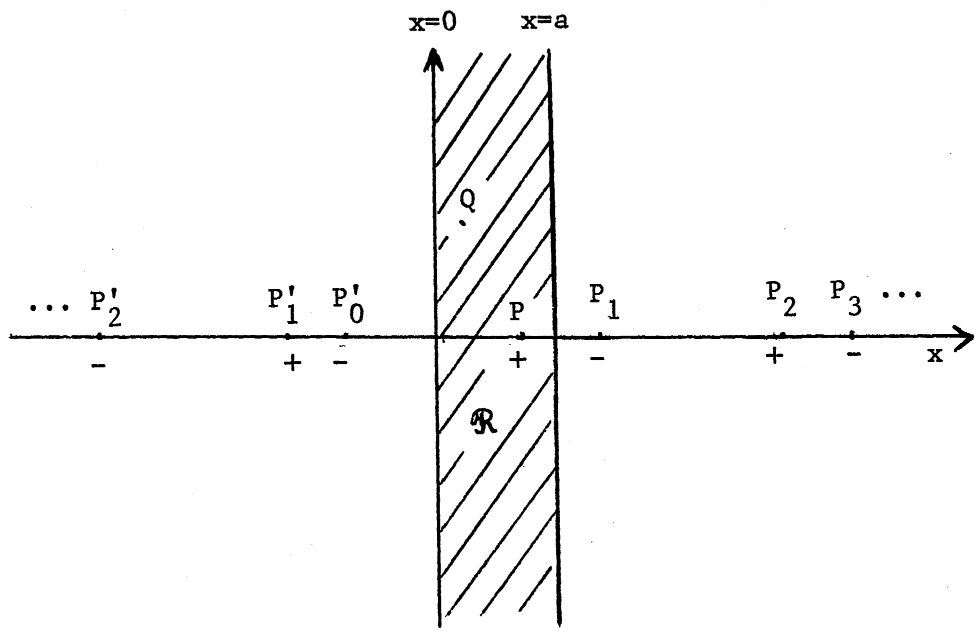


Figure 3.5. Images of a Point between Parallel Lines

(cf. Figure 3.5). Signs for the points P_i are shown in Figure 3.5. Therefore

$$G(x,y;x',y') = \ln \frac{QP_1 \cdot QP_0' \cdot QP_3 \cdot QP_2' \cdot QP_5 \cdot QP_4' \cdots}{QP \cdot QP_1' \cdot QP_2 \cdot QP_3' \cdot QP_4 \cdot QP_5' \cdots}$$

$$= \frac{1}{2} \ln \frac{(x'+x)^2+(y'-y)^2}{(x'-x)^2+(y'-y)^2} + \frac{1}{2} \sum_{n=1}^{\infty} \ln \frac{(2na-x'-x)^2+(y'-y)^2}{(2na+x'-x)^2+(y'-y)^2} +$$

$$\frac{1}{2} \sum_{n=1}^{\infty} \ln \frac{(2na+x'+x)^2+(y'-y)^2}{(2na-x'+x)^2+(y'-y)^2}$$

which we can shown to have properties (1), (2) and (3).

Following Example 3.5, we can solve Dirichlet's problem in a region that is an intersections of parallel strips, e.g. rectangles, parallelograms, hexagons, etc. (cf. Figure 3.6). In such cases the number of infinite sequences is twice the number of parallel strips involved.

The method of images can also be applied to a region that is a disk. The following example will show this.

Example 3.6. Find u such that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

$0 \leqq r < a$, $0 \leqq \theta \leqq 2\pi$ such that $u = f(\theta)$ for $r = a$.

Let P be the point (r,θ), Q be (r',θ'), such that P and Q belong to $\mathcal{R} = \left\{(r,\theta)\colon\ 0 \leqq r < a\right\}$ and let P_1 be $(a^2/r,\theta)$, the inverse point (or image point) to P with respect to C, the boundary of $\mathcal{R}$ (cf. Figure 3.7). If we let

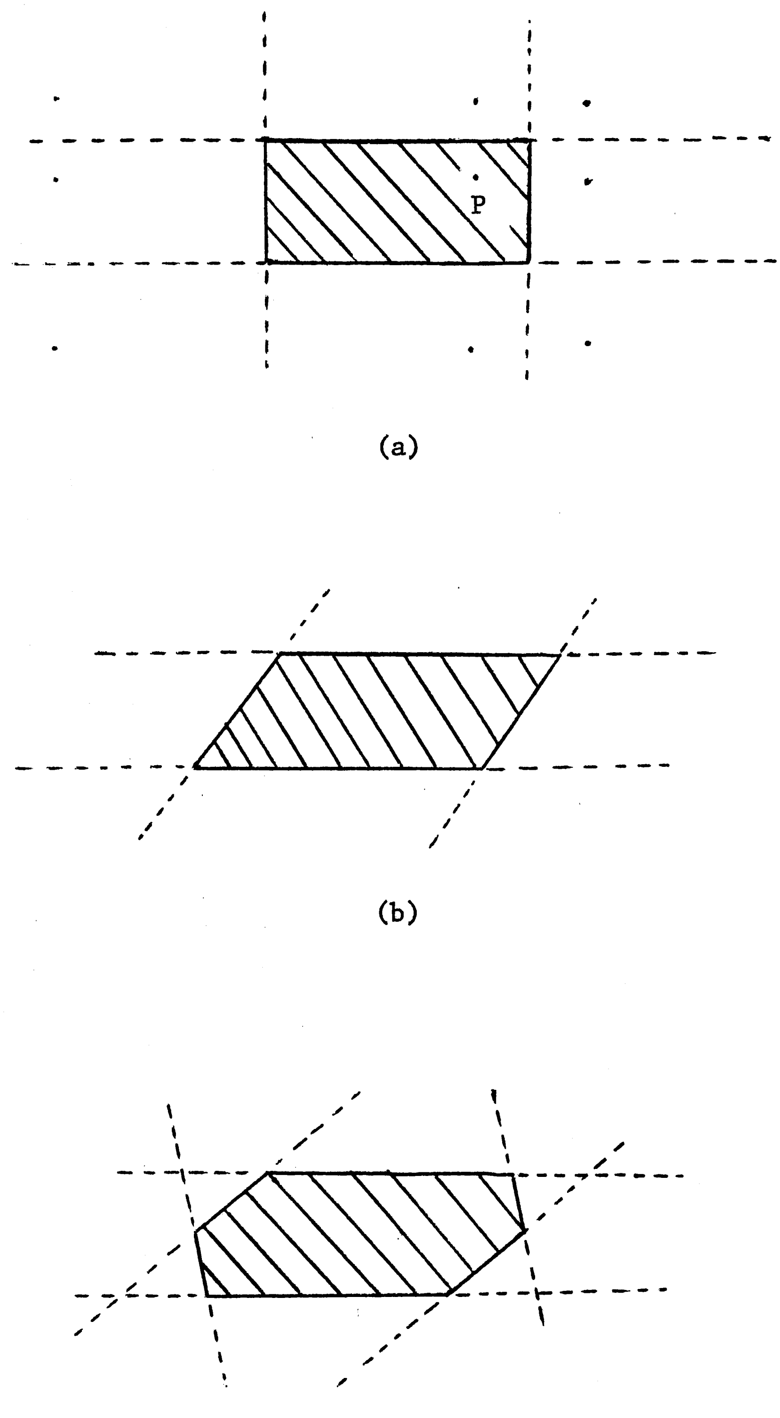


Figure 3.6. Some Intersections of Parallel Strips

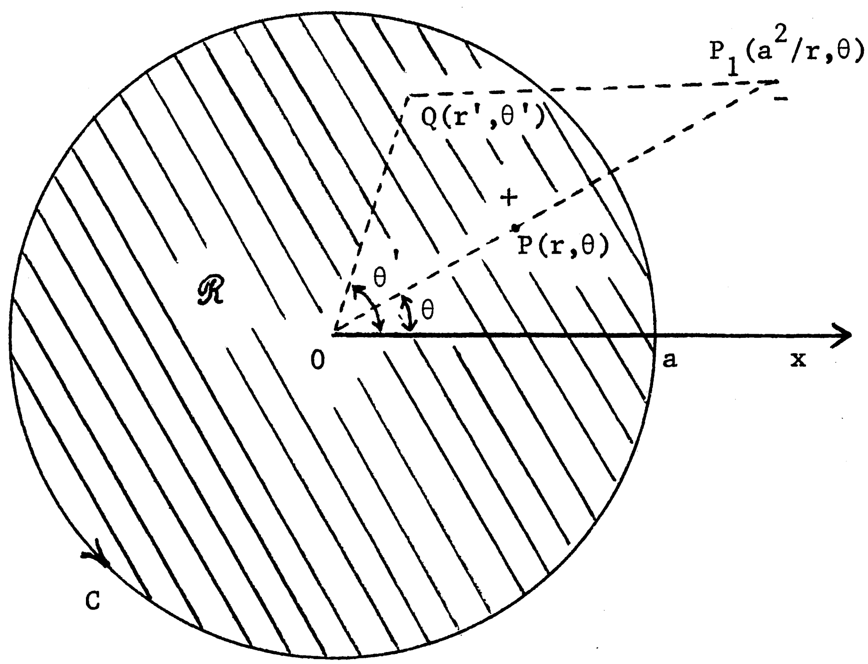


Figure 3.7. Image of a Point in a Disk

$$G(r,\theta;r',\theta') = \ln \frac{1}{QP} - \ln \frac{a/r}{QP_1}$$

$$= \ln \frac{r \cdot QP_1}{a \cdot QP},$$

and take

$$w(r,\theta;r',\theta') = \ln \frac{r \cdot QP_1}{a}$$

$$= \ln \frac{r}{a} + \frac{1}{2} \ln\left(r'^2 + \frac{a^4}{r^2} - \frac{2r'a^2}{r} \cos(\theta'-\theta)\right)$$

$$= \frac{1}{2} \ln\left(\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos(\theta'-\theta)\right),$$

then

$$G(x,y;x',y') = w + \ln \frac{1}{|\vec{r}\,'-\vec{r}|} .$$

We can show that G has the following properties:

(1). $\dfrac{\partial^2 w}{\partial r'^2} + \dfrac{1}{r'} \dfrac{\partial w}{\partial r'} + \dfrac{1}{r'^2} \dfrac{\partial^2 w}{\partial \theta'^2} = 0$ within the circle except at the point P,

(2). $G = 0$ on the circle $r' = a$,

(3). $G \sim \ln \dfrac{1}{|\vec{r}\,'-\vec{r}|}$ as $\vec{r}\,' \to \vec{r}$.

Therefore, using (3.1.3) the solution to the problem is

$$u(r,\theta) = -\frac{1}{2\pi} \int_0^{2\pi} f(\theta') \frac{\partial G}{\partial n} a \, d\theta'$$

where

$$\frac{\partial G}{\partial n} = \left(\frac{\partial G}{\partial r'}\right)_{r'=a}$$

$$= \frac{-(a^2 - r^2)}{a[a^2 - 2ar \cos(\theta'-\theta) + r^2]} .$$

Hence

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\theta')d\theta'}{a^2 - 2ar\cos(\theta'-\theta) + r^2}. \tag{3.3.2}$$

This is the well-known Poisson integral for a disk.

We have seen how to use the method of images for a half-plane in Example 3.1 and for a disk in Example 3.6. The following example of a semidisk will combine the techniques of the other two examples.

Example 3.7. Find u such that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

$0 \leqq r < a$, $0 \leqq \theta \leqq \pi$; and $u = f(\theta)$ for $r = a$.

Let P be the point (r,θ), Q be (r', θ'), such that P and Q belong to $\mathcal{R} = \left\{(r,\theta)\colon 0 \leqq r < a,\ 0 \leqq \theta \leqq \pi\right\}$ and let P_1 be $(a^2/r,\theta)$, P_2 be $(a^2/r,-\theta)$, and P_3 be $(r,-\theta)$. Notice that P_2, P_3 are images of P_1 and P, respectively, across the x-axis. The points P_1, P_2 are images of P, P_3, respectively, with respect to the disk (cf. Figure 3.8). Signs for P, P_1, P_2 and P_3 are shown in Figure 3.8. If we let

$$G(r,\theta;r',\theta') = \ln\frac{1}{QP} - \ln\frac{a/r}{QP_1} - \ln\frac{1}{QP_3} + \ln\frac{a/r}{QP_2}$$

$$= \ln\frac{QP_1\cdot QP_3}{QP\cdot QP_2} \tag{3.3.3}$$

$$= \frac{1}{2}\ln\frac{[r^2r'^2+a^4-2rr'a^2\cos(\theta'-\theta)][r'^2+r^2-2rr'\cos(\theta'+\theta)]}{[r^2r'^2+a^4-2rr'a^2\cos(\theta'+\theta)][r'^2+r^2-2rr'\cos(\theta'-\theta)]}$$

then G is the Green's function for this problem.

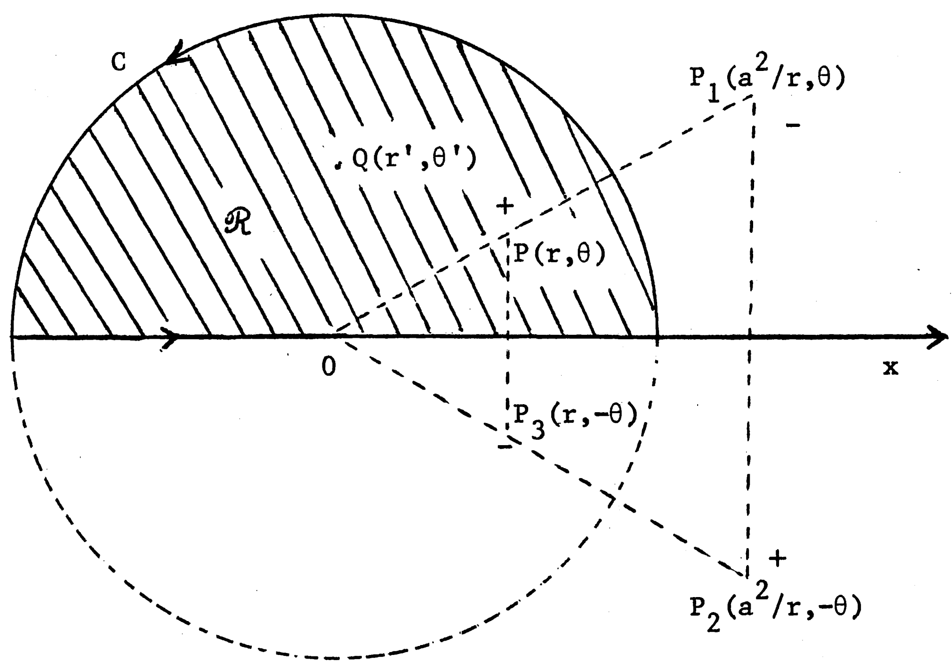


Figure 3.8. Images of a Point in Half-Disk

Notice that equation (3.3.3) is obtained by considering the images across the x-axis and the image in the disk. Note also that P_1 and P_3 have minus signs. For showing G = 0 on the circular part of the boundary, we should use the first form of (3.3.3).

Now if we wish to find Green's functions for a region that is the intersection of a disk and an angular region, e.g., a quarter-disk, one-sixth disk, etc. (cf. Figure 3.9). The following steps will be used:

Step 1. Find all possible images across the boundary of the disk and lines forming the straight portions of the boundary. Then assign either a minus or a plus sign such that each pair of image points has opposite signs. The point P is always assigned the plus sign.

Step 2. Construct Green's function by comparing with (3.3.3). Thus, those P_i with a minus sign produce a factor QP_i in the numerator and those P_i with a plus sign produce a factor QP_i in the denominator. Thus, Green's function

$$G(r,\theta:r',\theta') = \ln \frac{QP_1 \cdot QP_3 \cdot QP_5 \cdot QP_7}{QP \cdot QP_2 \cdot QP_4 \cdot QP_6} .$$

Notice that the method of images has been applied only to regions of a plane that is an intersection of a half-space and a disk. It is generally not applicable to a general plane region. To be a problem where an infinite series is avoided, the number of generated image points must be finite.

3.4. Conformal Mapping and Green's Function

The relationship between Green's function for a simply connected domain and the conformal mapping of that domain onto the unit circle can be used in solving the two-dimensional Dirichlet's problem. Some examples in Section 3.3 will be used for the purpose of finding Green's

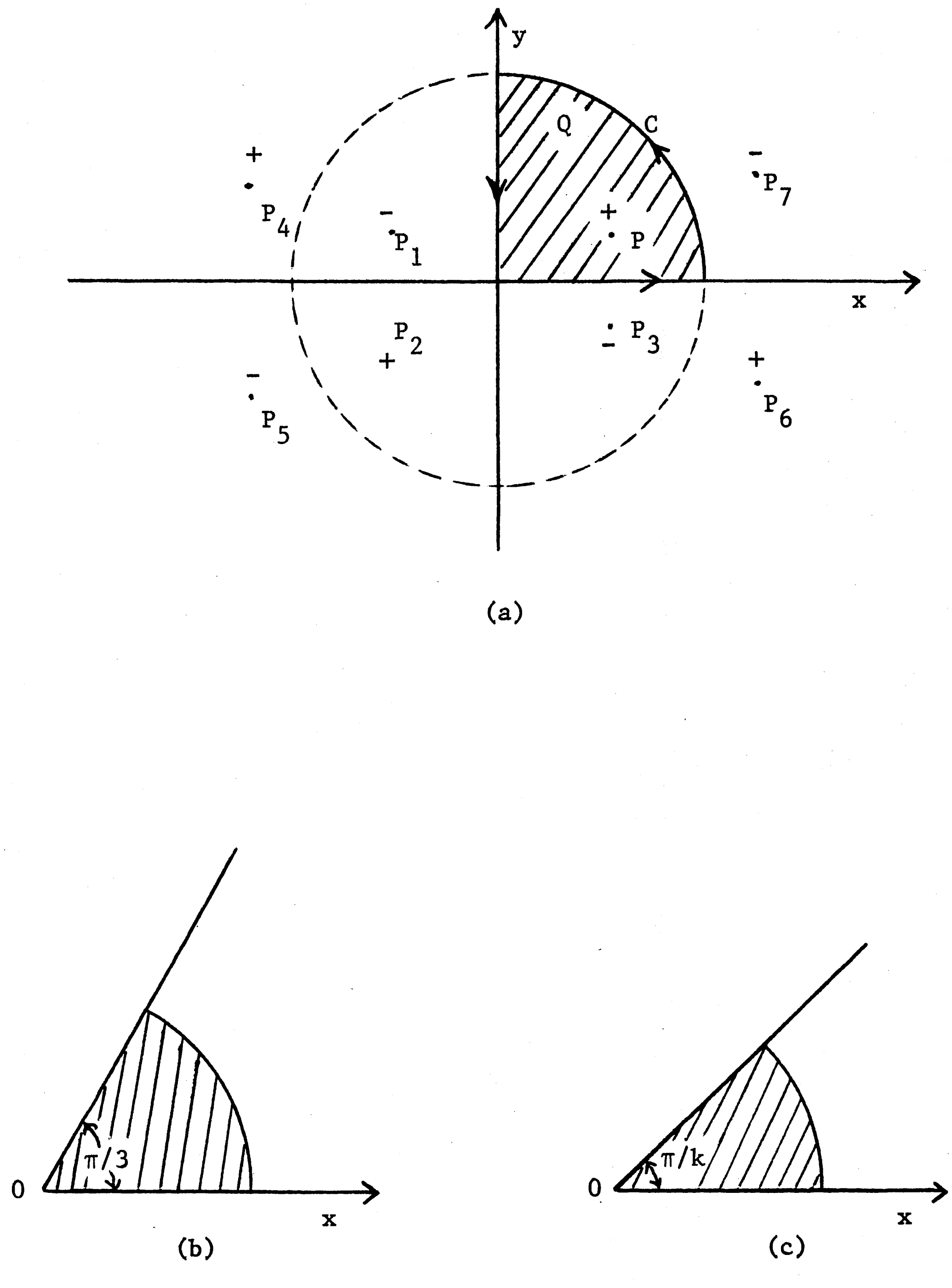


Figure 3.9. Some Intersections between a Disk and an Angular Region

function by the method of conformal mapping. Then we can compare answers with the answers by the method of images.

Example 3.8. The same as Example 3.1, that is, find u such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $x > 0$; $u = f(y)$ on $x = 0$ and $u \to 0$ as $x \to \infty$.

Let P be $z = x + iy$, $x > 0$, Q be $z' = x' + iy'$ and P_1 be $z_1 = -x + iy$, then

$$G(z;z') = -\ln |f(z,z')| \tag{3.4.1}$$

where $f(z,z')$ maps the right half-plane, $x \geqq 0$ in the z-plane onto the interior of the unit circle in w-plane in a one-to-one conformal manner such that $f(z,z) = 0$ and $|f(z,z')| = 1$ for z' on the boundary, $x' = 0$, (cf. Figure 3.10). Such a function is

$$f(z.z') = \frac{z' - z}{z' - z_1}.$$

Then

$$|f(z.z')| = \frac{|z' - z|}{|z' - z_1|} = \sqrt{\frac{(x'-x)^2+(y'-y)^2}{(x+x')^2+(y'-y)^2}}$$

and hence by using (3.4.1)

$$G(x,y;x',y') = \frac{1}{2} \ln \frac{(x'+x)^2+(y'-y)^2}{(x'-x)^2+(y'-y)^2}.$$

This is the same Green's function as found in Example 3.1.

To get the conformal mapping of the region $\mathcal{R}$ onto the interior of the unit circle we may, first, find the comformal mapping of the

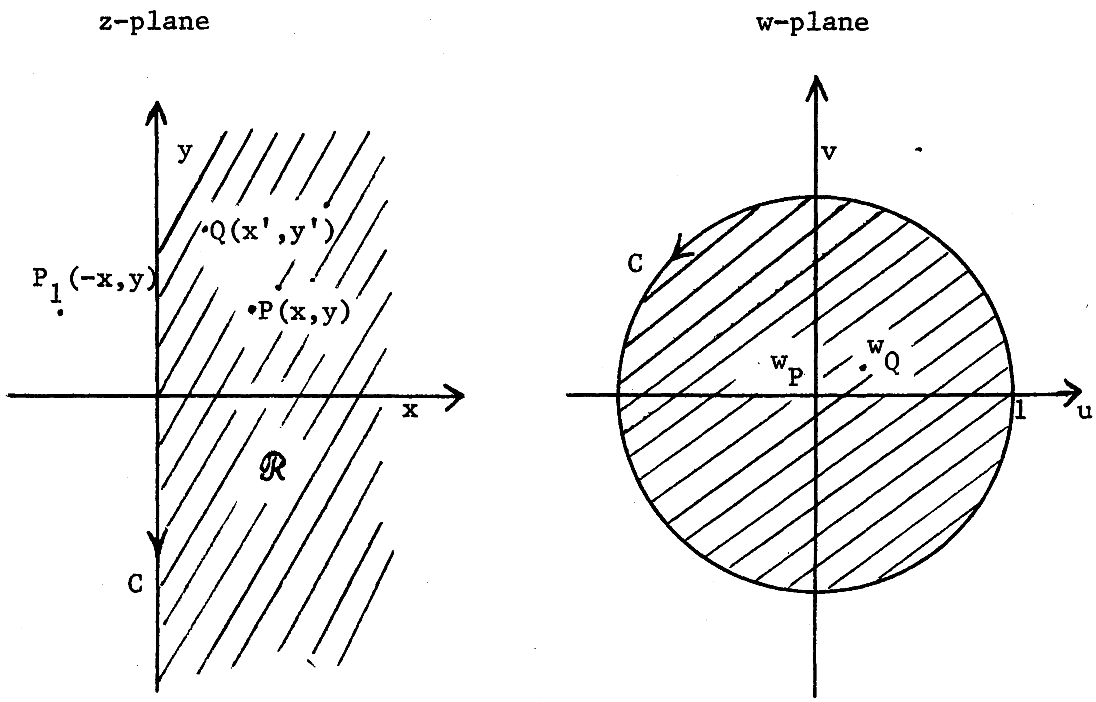


Figure 3.10. Conformal Mapping of a Half-Plane to a Unit Circle

region $\mathcal{R}$ onto an upper half-plane by using a table of conformal transformations of regions, for examples [3, pp. 284-291], and the mapping, f(z.z') which maps the upper half-plane onto the interior of the unit circle such that $f(z,z) = 0$ and $|f(z,z')| = 1$ as z' is on $y = 0$. The following examples illustrate the mapping process.

Example 3.9. Find $u(r,\theta)$ such that $\Delta u = 0$ in $\mathcal{R}$ where $\mathcal{R} = \left\{(r,\theta): 0 < r,\ 0 < \theta < \pi/n\right\}$, $n = 1,2,\ldots$ and $u(\vec{r}) = f(\vec{r})$ where $\vec{r}$ belong to C, the boundary of $\mathcal{R}$.

Let P be the point $z = re^{i\theta}$ and Q be the point $z' = r'e^{i\theta'}$ as shown in Figure 3.11(a).

From Figure 3.11 the mappings are $w = z^n$ and

$$\zeta = \frac{w_Q - w_P}{w_Q - \overline{w}_P} = \frac{z'^n - z^n}{z'^n - \overline{z}^n},$$

where w_Q and w_P are the images of P and Q, respectively, in the w-plane. The conformal map $\zeta = f(z,z')$ is such that $f(z,z) = 0$ and $|f(z,z')| = 1$ for z' on the line $y = 0$. The mapping from z-plane to ζ-plane is

$$f(z,z') = \frac{(r'e^{i\theta'})^n - (re^{i\theta})^n}{(r'e^{i\theta'})^n - (re^{-i\theta})^n}$$

$$= \frac{r'^n e^{in\theta'} - r^n e^{in\theta}}{r'^n e^{in\theta'} - r^n e^{-in\theta}}.$$

Then

$$|f(z,z')| = \sqrt{\frac{r'^{2n} + r^{2n} - 2r'^n r^n \cos(n\theta' - n\theta)}{r'^{2n} + r^{2n} - 2r'^n r^n \cos(n\theta' + n\theta)}}.$$

Therefore, using equation (3.4.1) we have

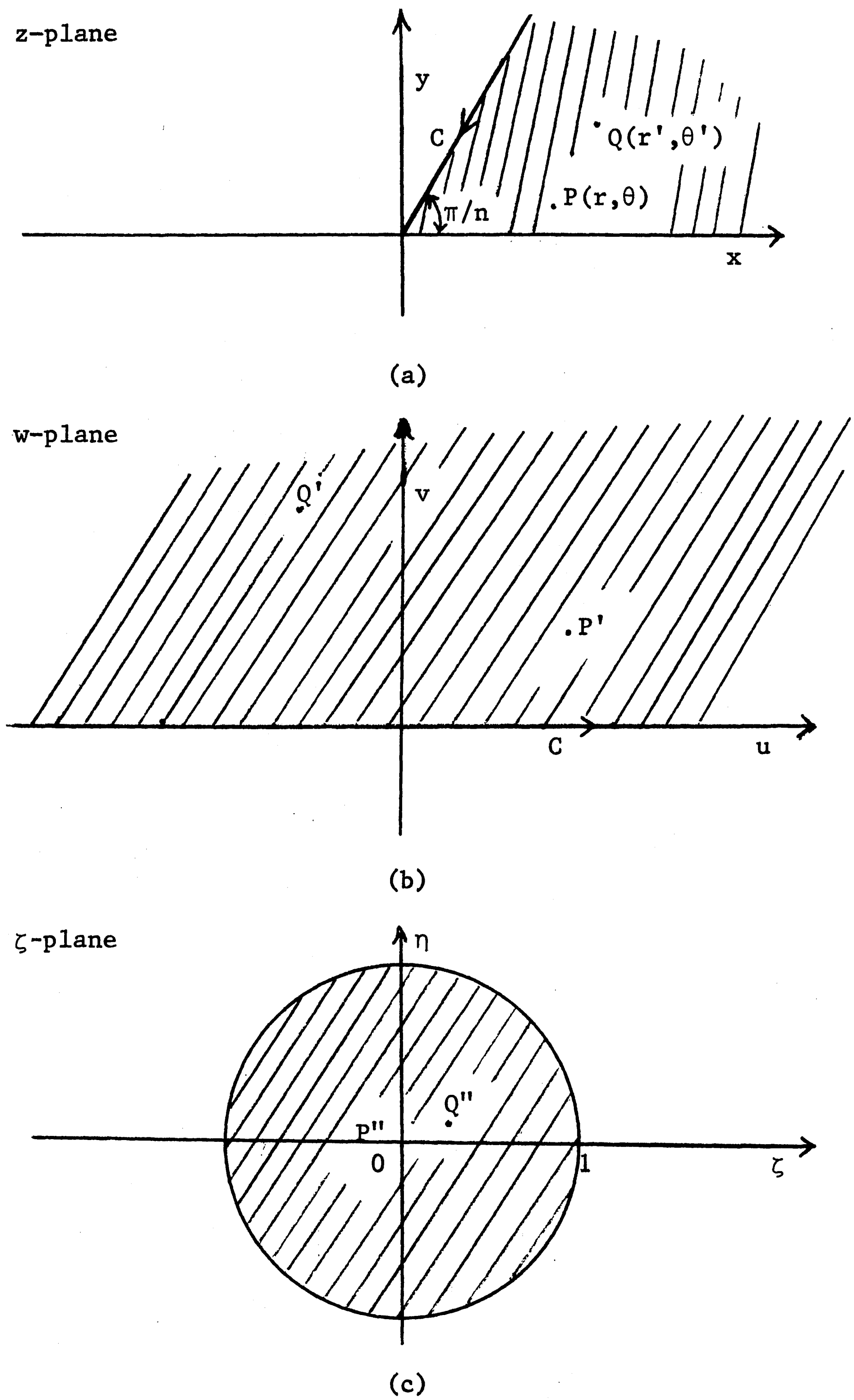


Figure 3.11. Conformal Mapping of Angular Region of Angle π/n onto a Unit Circle

$$G(r,\theta;r',\theta') = \frac{1}{2}\ln\frac{r'^{2n} + r^{2n} - 2r'^n r^n\cos(n\theta'+n\theta)}{r'^{2n} + r^{2n} - 2r'^n r^n\cos(n\theta'-n\theta)}.$$

Example 3.10. Find u such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in $\mathfrak{R}$; $u = f_1(y)$ on $x = 0$ and $u = f_2(y)$ on $x = a$. The region $\mathcal{R}$ is the region between the lines $x = 0$ and $x = a$.

Let P be $z = x + iy$ and Q be $z' = x' + iy'$ in $\mathcal{R}$. The desired map will be obtained by the conformal maps (cf. Figure 3.12),

$$w = iz,\ w' = \frac{\pi}{a}w,\ w'' = e^{w'},\ \zeta = \frac{w''_Q - w''_P}{w''_Q - w''_1}$$

where w''_P, w''_Q are the images of P, Q in the w"-plane, and w''_1 is the image of w''_P with respect to the u"-axis. From the composite function we obtain the mapping of the points P and Q in w"-plane to be

$$e^{\frac{\pi iz}{a}} \text{ and } e^{\frac{\pi iz'}{a}},$$

respectively. Now

$$w_P = e^{\frac{\pi}{a}(-y+ix)} \text{ and } w''_1 = e^{\frac{\pi}{a}(-y-ix)}.$$

Hence

$$\zeta = \frac{e^{\frac{\pi}{a}(-y'+ix')} - e^{\frac{\pi}{a}(-y+ix)}}{e^{\frac{\pi}{a}(-y'-ix')} - e^{\frac{\pi}{a}(-y-ix)}}.$$

Therefore

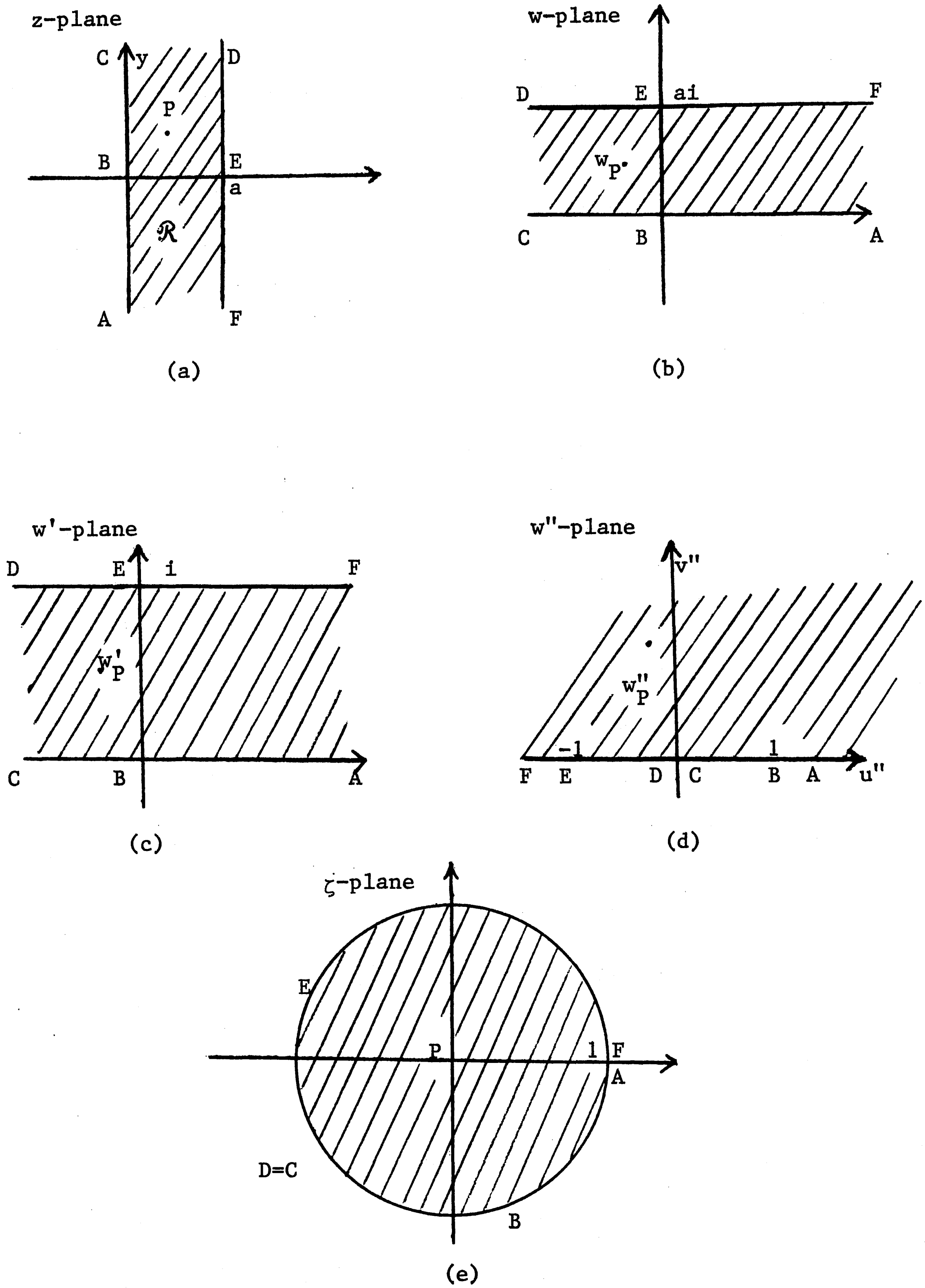


Figure 3.12. Conformal Mapping for Example 3.10

$$G(x,y;x',y') = -\ln|\zeta|$$

$$= \frac{1}{2}\ln \frac{e^{\frac{-2\pi y'}{a}} + e^{\frac{-2\pi y}{a}} - 2e^{\frac{-\pi(y'+y)}{a}}\cos\frac{\pi}{a}(x'+x)}{e^{\frac{-2\pi y'}{a}} + e^{\frac{-2\pi y}{a}} - 2e^{\frac{-\pi(y'+y)}{a}}\cos\frac{\pi}{a}(x'-x)} .$$

Example 3.11. Find u such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in $\mathcal{R}$ and $u = f(\theta)$ on C where $\mathcal{R} = \left\{re^{i\theta};\ 0 < r < a,\ 0 < \theta < \pi\right\}$ and C is the boundary of the region $\mathcal{R}$.

Let P be $z = e^{i\theta}$ and Q be $z' = e^{i\theta'}$ in $\mathcal{R}$. The desired map will be obtained by the conformal maps (cf. Figure 3.13)

$$w = \frac{z}{a},\ w' = -w - \frac{1}{w},\ \text{and}\ \zeta = \frac{w'_Q - w'_P}{w'_Q - w'_1}$$

where w'_Q, w'_P are the images of Q and P in the w'-plane and w'_1 is the image of w'_P with respect to the real axis in the w'-plane. From the composite function, we obtain the points w'_P and w'_Q in the w'-plane to be

$$-\left[\left(\frac{r}{a} + \frac{a}{r}\right)\cos\theta + i\left(\frac{r}{a} - \frac{a}{r}\right)\sin\theta\right]$$

and

$$-\left[\left(\frac{r'}{a} + \frac{a}{r'}\right)\cos\theta' + i\left(\frac{r'}{a} - \frac{a}{r'}\right)\sin\theta'\right].$$

Therefore

$$w'_1 = -\left[\left(\frac{r}{a} + \frac{a}{r}\right)\cos\theta - i\left(\frac{r}{a} - \frac{a}{r}\right)\sin\theta\right].$$

Hence

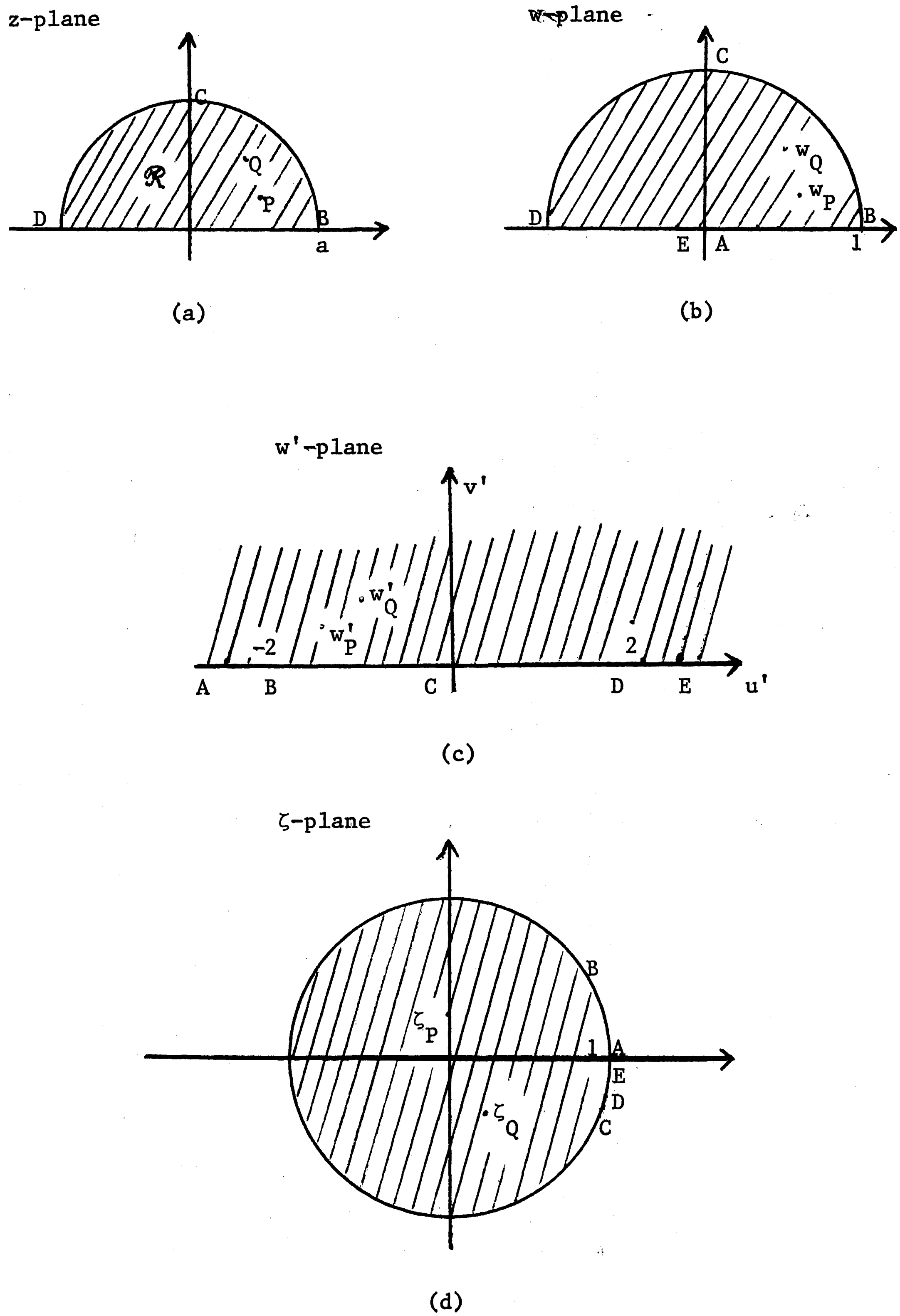


Figure 3.13. Conformal Mapping for Example 3.11

$$\zeta = \frac{\alpha + \beta i}{\alpha + \delta i}$$

where

$$\alpha = r'(r^2+a^2)\cos\theta - r(r'^2+a^2)\cos\theta',$$

$$\beta = r'(r^2-a^2)\sin\theta - r(r'^2-a^2)\sin\theta',$$

$$\delta = -r'(r^2-a^2)\sin\theta - r(r'^2-a^2)\sin\theta'.$$

Since $G(r,\theta;r',\theta') = -\ln|\zeta|$, therefore

$$G = \frac{1}{2}\ln\frac{\alpha^2+\delta^2}{\alpha^2+\beta^2}$$

$$= \frac{1}{2}\ln\Big[\Big[r'^2(r^4+a^4) + r^2(r'^4+a^4) + 4a^2r'^2r^2\cos(\theta'+\theta)\cos(\theta'-\theta) - 2rr'\Big\{(r^2r'^2+a^4)\cos(\theta'+\theta) + (a^2r^2+a^2r'^2\cos(\theta'-\theta)\Big\}\Big]/$$

$$\Big[r'^2(r^4+a^4) + r^2(r'^4+a^4) + 4a^2r'^2r^2\cos(\theta'+\theta)\cos(\theta'-\theta) - 2rr'\Big\{(r^2r'^2+a^4)\cos(\theta'-\theta) + (a^2r^2+a^2r'^2)\cos(\theta'+\theta)\Big\}\Big]\Big]$$

$$= \frac{1}{2}\ln\frac{[r^2r'^2+a^4-2rr'a^2\cos(\theta'-\theta)][r'^2+r^2-2rr'\cos(\theta'+\theta)]}{[r^2r'^2+a^4-2rr'a^2\cos(\theta'+\theta)][r'^2+r^2-2rr'\cos(\theta'-\theta)]}.$$

3.5. The Method of Images in n-Space, $n \geq 3$

Suppose we wish to find the solution of Laplace's equation of 3-space, that is,

$$\frac{\partial^2 u}{\partial x^2}u(x,y,z) + \frac{\partial^2}{\partial y^2}u(x,y,z) + \frac{\partial^2}{\partial z^2}u(x,y,z) = 0$$

in $\mathcal{R}$ with the Dirichlet boundary condition

$$u(x,y,z) = f(x,y,z)$$

on S the boundary of $\mathcal{R}$. The solution, u(x,y,z) to this problem is

Poisson's formula, namely,

$$u(x,y,z) = -\frac{1}{4\pi}\int_S f(x',y',z')\frac{\partial}{\partial n}G(x,y,z;x',y',z')dS' \quad (3.5.1)$$

where $\vec{n}$ is the outward-drawn unit normal to the boundary surface S. The function G is the Green's function for $\mathcal{R}$ which can be written as

$$G(\vec{r},\vec{r}') = w(\vec{r},\vec{r}') + \frac{1}{|\vec{r}'-\vec{r}|}$$

where $\vec{r} = (x,y,z)$ and $\vec{r}' = (x',y',z')$ both of which belong to $\mathcal{R}$. The function G has the following properties:

(1). $\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w}{\partial y'^2} + \frac{\partial^2 w}{\partial z'^2} = 0$ for $(x',y',z') \varepsilon \quad , (x',y',z') \neq (x,y,z)$,

(2). $G = 0$ for $(x',y',z') \varepsilon S$,

(3). $G \sim \frac{1}{|\vec{r}'-\vec{r}|}$ as $\vec{r}' \to \vec{r}$.

The following examples will show how to find Green's function for Dirichlet's problem on a half-space and a sphere.

Example 3.12. Find u(x,y,z) such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

for $x \geq 0$, $u = f(y,z)$ on $x = 0$ and $u \to 0$ as $x \to \infty$.

Let P be the point (x,y,z), Q be the point (x',y',z'), and P_1 be the image of P across the yz-plane (cf. Figure 3.14). If Q is on the plane $x = 0$, then $QP = QP_1$. Therefore, let

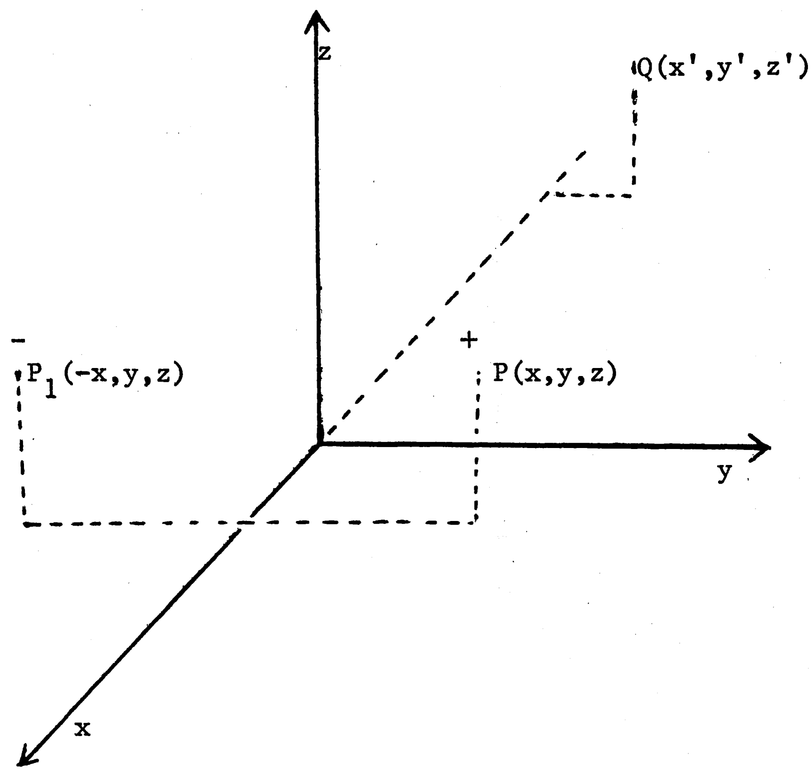


Figure 3.14. Image of a Point in a Half-Space

$$G = -\frac{1}{QP_1} + \frac{1}{QP}$$

$$= -\frac{1}{\sqrt{(x'+x)^2+(y'-y)^2+(z'-z)^2}} + \frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}},$$

we can show that G has the properties (1), (2) and (3). We have

$$\frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial x'}\right)_{x'=0} = \frac{2x}{\left[x^2+(y'-y)^2+(z'-z)^2\right]^{3/2}},$$

hence, using the equation (3.5.1), the solution to the given problem is

$$u(x,y,z) = -\frac{x}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(y',z')}{\left[x^2+(y'-y)^2+(z'-z)^2\right]^{3/2}}\,dy'dz'.$$

<u>Example 3.13.</u> (Dirichlet's problem for a sphere) Find $u(r,\theta,\phi)$ such that $\Delta u = 0$ in $\mathcal{R}$, $u = f(\theta,\phi)$ on S, where $\mathcal{R}$ is a sphere with center at the origin and its radius is a.

Let P, Q be the points (r,θ,ϕ), (r',θ',ϕ'), respectively, in $\mathcal{R}$, and $P_1(a^2/r,\theta,\phi)$ be the image of P with respect to S (cf. Figure 3.15). From a proposition of Euclidean geometry, if Q lies on S then

$$\frac{QP}{QP_1} = \frac{r}{a}.$$

Therefore, let

$$G = -\frac{a}{r}\cdot\frac{1}{QP_1} + \frac{1}{QP},$$

$$= -\frac{a}{r}\frac{1}{\sqrt{\frac{a^4}{r^2} + r'^2 - 2\frac{a^2}{r}r'\cos\Theta}} + \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\Theta}},$$

where $\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$, then we can show that

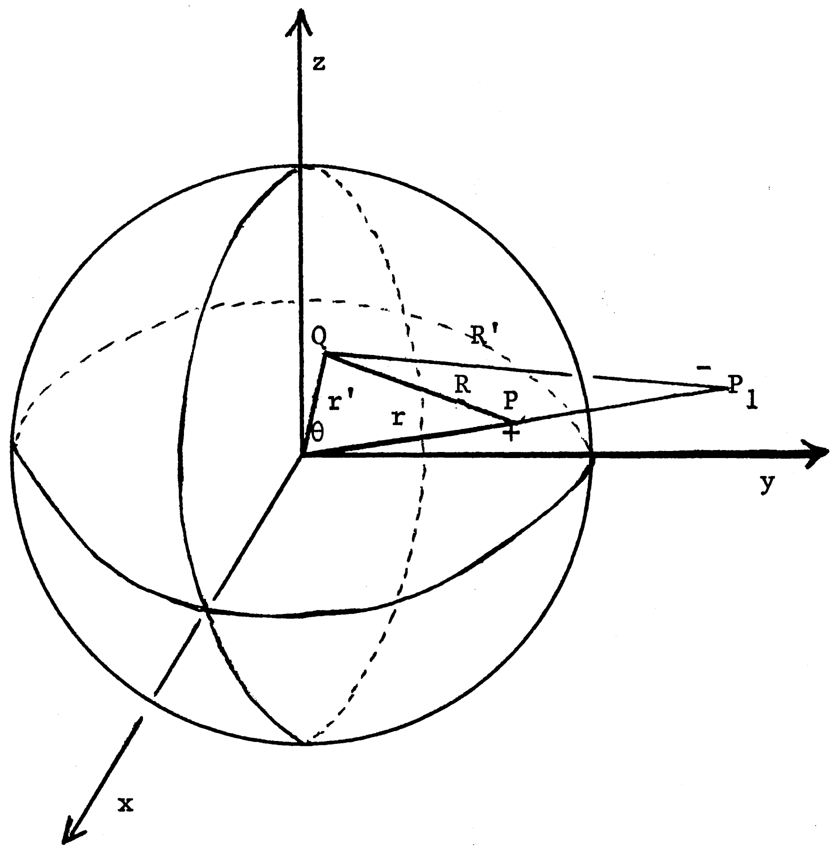


Figure 3.15. Image of a Point in a Sphere

G has properties (1), (2) and (3). Property (1) can be shown by Kelvin's inversion theorem [20, pp. 164-165]. We have

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r'}\Big|_{r'=a} = \frac{r^2-a^2}{aR^3}$$

where $R^2 = r^2+a^2-2ar\cos\Theta$. Hence, using the equation (3.5.1), the solution to the given problem is

$$u(r,\theta,\phi) = \frac{-1}{4\pi}\int_S f(\theta',\phi')\,\frac{\partial G}{\partial r'}\,dS'$$

$$= \frac{a(a^2-r^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{f(\theta',\phi')\sin\theta'}{R^3}\,d\theta'\,d\phi'.$$

If we consider a region which is the intersection of a sphere and a finite number of half-spaces the Green's function can be found by using the method of images and by forming Green's functions in a manner similar to that of Examples 3.12 and 3.13. For example, let the region be a hemisphere.

Example 3.14. Find $u(r,\theta,\phi)$ such that $\Delta u = 0$ in $\mathcal{R}$ and $u = f(r,\theta,\phi)$ on S, where $\mathcal{R}$ is a hemisphere with center at origin and of radius a.

Let P, Q, be points (r,θ,ϕ), (r',θ',ϕ') in $\mathcal{R}$, respectively, P_1, P_2 are images of P, P_3, respectively, across the surface of the sphere and P_2, P_3 are images of P_1 and P, respectively, across the plane of the base of the sphere (cf. Figure 3.16). Assign plus and minus signs such that P is a plus and image point pairs have opposite signs. Form the function G as

$$G(\vec{r},\vec{r}\,') = \frac{1}{QP} - \frac{a/r}{QP_1} + \frac{a/r}{QP_2} - \frac{1}{QP_3}, \qquad (3.5.2)$$

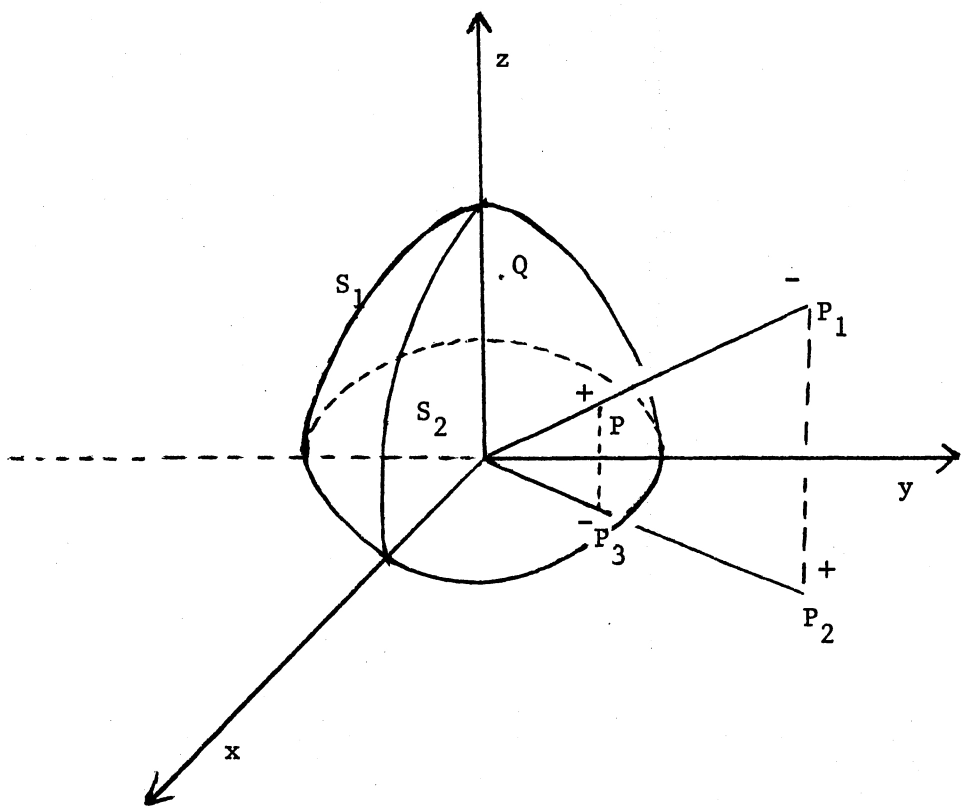


Figure 3.16. Images of a Point in a Hemisphere

then G has properties (1), (2) and (3). Note that the signs in the expression (3.5.2) are the same as the signs of P, P_1, P_2 and P_3.

The fundamental solution, the idea of images points and an analogy to Kelvin's inversion theorem will be used in guessing Green's function in n-space. For example, in 4-dimensions, a fundamental solution for Dirichlet's problem for a region $\mathcal{R}$ is

$$W(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}\,'-\vec{r}|^2}.$$

Let P, P_1, P_2, P_3 and Q be the points shown in Figure 3.2. Analogous to Kelvin's inversion theorem we can show that if $\Psi(\xi_1,\xi_2,\xi_3,\xi_4)$ is a harmonic function with respect ot the variables ξ_i in a domain $\mathcal{R}$, then the

$$\frac{a^2}{r^2}\Psi\left(\frac{a^2x_1}{r^2}, \frac{a^2x_2}{r^2}, \frac{a^2x_3}{r^2}, \frac{a^2x_4}{r^2}\right)$$

is a harmonic function with respect to the variables x_i in the domain $\mathcal{R}'$, where $\mathcal{R}$ and $\mathcal{R}'$ are related by the transformation

$$\xi_1 = \frac{a^2x_1}{r^2},\quad \xi_2 = \frac{a^2x_2}{r^2},\quad \xi_3 = \frac{a^2x_3}{r^2},\quad \xi_4 = \frac{a^2x_4}{r^2}$$

and

$$r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Green's function would be

$$G(\vec{r},\vec{r}\,') = \frac{1}{|QP|^2} - \frac{1}{|QP_1|^2},$$

if $\mathcal{R}$ is a half-4-space,

$$G(\vec{r},\vec{r}\,') = \frac{1}{|QP|^2} - \frac{a^2}{r^2}\frac{1}{|QP_1|^2},$$

if $\mathcal{R}$ is a 4-sphere, or

$$G(\vec{r},\vec{r}\,') = \frac{1}{|QP|^2} - \frac{a^2}{r^2}\frac{1}{|QP_1|^2} + \frac{a^2}{r^2}\frac{1}{|QP_2|^2} - \frac{1}{|QP_3|^2},$$

if $\mathcal{R}$ is a semi-4-sphere.

The solution, $u(\vec{r})$, to the Dirichlet's problem of a region of $\mathcal{R}$ in 4-space is

$$u(\vec{r}) = -\frac{1}{2\omega_4}\int_S f(\vec{r}\,')\frac{\partial G}{\partial n}\,dS'$$

$$= -\frac{1}{4\pi^2}\int_S f(\vec{r}\,')\frac{\partial G}{\partial n}\,dS', \qquad (3.5.3)$$

where S is the boundary of $\mathcal{R}$, $\vec{n}$ is the unit outward normal and ω_4 is the area at a unit sphere in 4-space.

Similarly, the solution $u(\vec{r})$ to the Dirichlet's problem for a region $\mathcal{R}$ in n-dimensions is

$$u(\vec{r}) = -\frac{1}{2\pi}\int_C f(\vec{r}\,')\frac{\partial G}{\partial n}\,ds', \quad n = 2$$

$$= -\frac{1}{(n-2)\omega_n}\int_S f(\vec{r}\,')\frac{\partial G}{\partial n}\,dS', \quad n \geq 3 \qquad (3.5.4)$$

where $\omega_n = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$, $\Gamma(x)$ is the gamma function of x.

3.6. Symmetry of Green's Function

Green's function $G(\vec{r},\vec{r}\,')$ for Dirichlet's problem is symmetric, that is, $G(\vec{r},\vec{r}\,') = G(\vec{r}\,',\vec{r})$. We can show the symmetry directly, as for an example, Green's function from Example 3.6 is

$$G(\vec{r},\vec{r}') = \ln \frac{1}{QP} - \ln \frac{a/r}{QP_1}$$

or

$$G(\vec{r},\vec{r}') = \ln \frac{1}{|\vec{r}'-\vec{r}|} - \ln \frac{a}{r} \frac{1}{\left|\frac{a^2}{r^2}\vec{r}-\vec{r}'\right|}$$

(cf. Figure 3.17). Substituting $\vec{r}$ for $\vec{r}'$ and $\vec{r}'$ for $\vec{r}$, we have

$$G(\vec{r}',\vec{r}) = \ln \frac{1}{|\vec{r}-\vec{r}'|} - \ln \frac{a}{r'} \frac{1}{\left|\frac{a^2}{r^2}\vec{r}'-\vec{r}\right|}$$

$$= \ln \frac{1}{QP} - \ln \frac{a}{r'} \cdot \frac{1}{QP_1} .$$

Since ΔOPQ_1 and ΔOP_1Q are similar, we have

$$\frac{QP_1}{Q_1P} = \frac{r'}{r}, \text{ or } \frac{1}{Q_1P} = \frac{r'}{r} QP_1 .$$

By substituting

$$G(\vec{r}',\vec{r}) = \ln \frac{1}{QP} - \ln \frac{a}{r} \frac{1}{QP_1} = G(\vec{r},\vec{r}').$$

In general, we can prove that Green's function for Dirichlet's problem in n-space is symmetric by using Green's second identity, namely,

$$\int_{\mathcal{R}} [u\Delta v - v\Delta u]dV = \int_S [u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}]dS,$$

and let $u = G(\vec{r}',\vec{r}_1)$ and $v = G(\vec{r}',\vec{r}_2)$ where $\vec{r}_1$ and $\vec{r}_2$ are the singular points [6, p. 158].

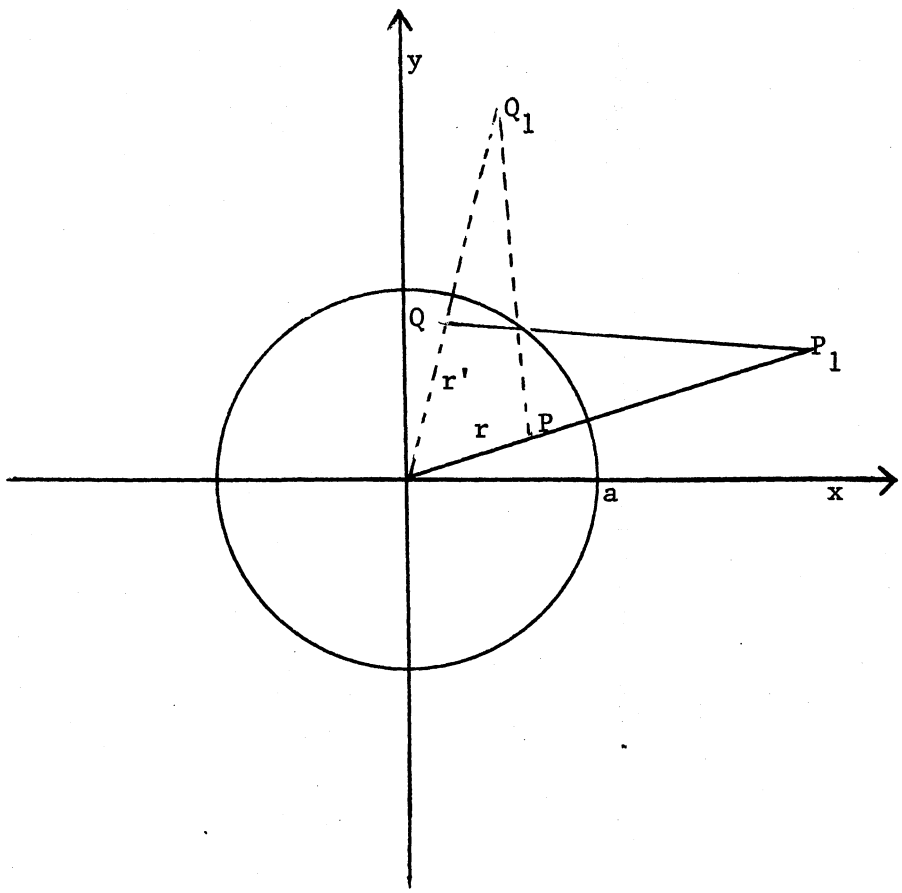


Figure 3.17. Images of P and Q in a Circle

3.7. Neumann's Problem

Suppose we wish to find the solution $u(x_1, x_2, \ldots, x_n)$ of Laplace's equation in an n-dimensional region $\mathcal{R}$, that is, find u such that

$$u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

in $\mathcal{R}$ such that u satisfies the Neumann's boundary condition, namely,

$$\frac{\partial u}{\partial n} = f(\vec{r}) \text{ on } C, \; n = 2,$$

$$= f(\vec{r}) \text{ on } S, \; n \geqq 3,$$

where C or S is the boundary of $\mathcal{R}$ for $n = 2$ or $n \geqq 3$. This problem is called Neumann's problem. The solution to this problem is given by

$$u(\vec{r}) = \frac{1}{2\pi} \int_C G(\vec{r}, \vec{r}\,') f(\vec{r}\,') ds', \; n = 2,$$

$$= \frac{1}{(n-2)\omega_n} \int_S G(\vec{r}, \vec{r}\,') f(\vec{r}\,') dS', \; n \geqq 3, \qquad (3.7.1)$$

where $\omega_n = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$.

The function $G(\vec{r}, \vec{r}\,')$ is called Green's function of the second kind which is expressed as

$$G(\vec{r}, \vec{r}\,') = w(\vec{r}, \vec{r}\,') + \ln \frac{1}{|\vec{r}\,' - \vec{r}|}, \; n = 2,$$

$$= w(\vec{r}, \vec{r}\,') + \frac{1}{|\vec{r}\,' - \vec{r}|^{n-2}}, \; n \geqq 3, \qquad (3.7.2)$$

with the following properties:

(1). $\Delta w(\vec{r},\vec{r}') = 0$, $\vec{r}' \in \mathcal{R}$ except at $\vec{r}' = \vec{r}$,

(2). $\frac{\partial G}{\partial n}$ is a constant when $\vec{r}'$ is on the boundary C or S, $\vec{n}$ is the unit outward-drawn normal to the boundary C or S,

(3). $G \sim \ln \dfrac{1}{|\vec{r}'-\vec{r}|}$, if $n = 2$,

$G \sim \dfrac{1}{|\vec{r}'-\vec{r}|^{n-2}}$, if $n \geqq 3$.

By using Neumann's condition and the divergence theorem, we have

$$\int_S f(\vec{r}')dS' = \int_S \frac{\partial u}{\partial n}\, dS' = \int_S \vec{\nabla}u\cdot\vec{n}\; dS' = \int_{\mathcal{R}} \vec{\nabla}\cdot\vec{\nabla}u dV' = \int_{\mathcal{R}} \Delta u dV' = 0.$$

Therefore, for Neumann's problem to have a solution, it is necessary that the function $f(\vec{r})$ satisfies the condition

$$\int_S f(\vec{r}')dS' = 0,$$

[4, p. 100].

By the method of images, we shall obtain Green's function for $\mathcal{R}$ where $\mathcal{R}$ is a half-space and for an n-sphere in E^n, $n = 2,3,\ldots$.

3.7.1. Half-Space in E^n

Suppose we wish to find a solution of Neumann's problem in the right half-plane, that is, we want to find a solution u(x,y) such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $x > 0$; $\frac{\partial u}{\partial n} = f(y)$ on $x = 0$.

If P is the point (x,y), $x > 0$, then take P_1 to be $(-x,y)$, the image point of P across the y-axis, and let Q be $(x',y') \in \mathcal{R}$ (cf. Figure 3.1). From (3.7.2)

$$G(x,y;x',y') = w(x,y;x',y') + \ln \frac{1}{QP}.$$

If we let

$$G(x,y;x',y') = \ln \frac{1}{QP_1} + \ln \frac{1}{QP}$$

$$= -\frac{1}{2} \ln \left[\left[(x'-x)^2+(y'-y)^2\right] \left[(x'+x)^2+(y'-y)^2\right]\right], \quad (3.7.3)$$

then

$$w(x,y;x',y') = -\frac{1}{2} \ln\left[(x'+x)^2+(y'-y)^2\right].$$

We can show that G has properties (1), (2), and (3). They follow from earlier demonstrations. To show property (2), that is, $\frac{\partial G}{\partial n}$ = constant on the boundary, differentiate directly. Then we have

$$\frac{\partial G}{\partial n} = -\left.\frac{\partial G}{\partial x'}\right|_{x'=0} = 0.$$

The solution to the problem is obtained from (3.7.1)

$$u(x,y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} f(y') \ln\left[x^2+(y'-y)^2\right] dy'.$$

Generally, suppose we consider Neumann's problem in a right half-space in E^n, $n \geq 3$. Let

$$G(\vec{r},\vec{r}\,') = \frac{1}{QP^{n-2}} + \frac{1}{QP_1^{\,n-2}}$$

where $P = \vec{r} = (x_1,x_2,\ldots,x_n)$, $Q = \vec{r}\,' = (x_1',x_2',\ldots,x_n')$, and $P_1 = (-x_1,x_2,\ldots,x_n)$, the image of P across the hyperplane $x_1 = 0$ (cf. Figure 3.14). Figure 3.14 shows P, Q and P_1 in E^3. We can show that G has properties (1), (2), and (3). The solution to the problem by using (3.7.1) is

$$u(\vec{r}) = \frac{2}{(n-2)\omega_n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{f(x_2',\ldots,x_n')}{[x_1^2+(x_2'-x_2)^2+\ldots+(x_n'-x_n)^2]^{n-2}}\, dx_2'\, dx_3' \ldots dx_n'.$$

3.7.2. n-Sphere in E^n

First, we will consider Neumann's problem in a disk, that is, we want to find a solution $u(r,\theta)$ such that $\Delta u = 0$ in $\mathcal{R}$ and $\frac{\partial u}{\partial n} = f(\theta)$ on C, where $\mathcal{R} = \left\{ re^{i\theta};\ 0 < r < a,\ 0 < \theta < 2\pi \right\}$ and C is the boundary of $\mathcal{R}$.

If P is the point $(r,\theta) \in \mathcal{R}$, then take P_1 to be $(a^2/r,\theta)$, the image point of P with respect to C, and let Q be $(r',\theta') \in \mathcal{R}$ (cf. Figure 3.7). From (3.7.2), if we let

$$G(r,\theta;r',\theta') = \ln \frac{1}{QP} + \ln \frac{a^3}{r'QP_1}$$

$$= \ln \frac{a^3 r}{r'\sqrt{[r'^2+r^2-2rr'\cos(\theta'-\theta)]\cdot[r'^2r^2+a^4-2a^2r'r\ \cos(\theta'-\theta)]}}$$

then

$$w(r,\theta;r',\theta') = \ln \frac{a^3}{r'QP_1}$$

$$= \ln \frac{a^3 r}{r'\sqrt{r'^2r^2+a^4-2a^2rr'\cos(\theta'-\theta)}}$$

We can show that G has properties (1), (2) and (3). The normal derivative, $\frac{\partial G}{\partial n} = \left.\frac{\partial G}{\partial r'}\right|_{r'=a} = -\frac{2}{a}$. By using (3.7.1), the solution to the problem is

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^a f(\theta')\left(\ln \frac{ar}{a^2+r^2-2ar\cos(\theta'-\theta)}\right) a\, dr'd\theta'.$$

Generally, Green's function for the n-sphere in E^n, $n \geqq 3$, is

$$G(\vec{r},\vec{r}') = \frac{1}{QP^{n-2}} + \left(\frac{a}{r\cdot QP_1}\right)^{n-2} + \frac{2(n-2)}{a^{n-2}}\int_a^{\infty} t^{n-3} - \left[\frac{1}{t^{n-2}}\left(\frac{t}{rR}\right)^{n-2}\right]dt$$

where $P = \vec{r}$, $Q = \vec{r}'$, $P_1 = a^2\vec{r}/r^2$, the image of P across the boundary of the n-sphere, $r = |\vec{r}|$ and $R = |\vec{r}'-t^2\vec{r}/r^2|$ (cf. Figure 3.15). Figure 3.15 shows P, Q, P_1 in E^3. It can be shown that G satisfies Laplace's equation by using Euler's theorem and it can be shown that

$$\frac{\partial G}{\partial n} = \left.\frac{\partial G}{\partial r'}\right|_{r'=a} = -\frac{n-2}{a^{n-2}}.$$

3.8. Green's Function in Terms of Eigenfunctions

We will consider the following problem in n-dimensions, n = 2,3,...

$$\Delta u(\vec{r}) = f(\vec{r}),\ \vec{r} \text{ in } \mathcal{R},$$

$$B(u) = g(\vec{r}),\ \vec{r} \text{ on S or C if } n = 2,$$

where B(u) is Dirichlet or Neumann boundary condition and $\mathcal{R}$ be a bounded region with boundary S or C if n = 2. Green's function for this problem is

$$G(\vec{r},\vec{r}') = -\sum_{n=1}^{\infty} \frac{\psi_n(\vec{r}')\psi_n(\vec{r})}{\lambda_n|\psi_n|^2} \qquad (3.8.1)$$

where ψ_n and λ_n are eigenfunction and corresponding eigenvalue, respectively, of the following eigenvalue problem:

$$\Delta\psi + \lambda\psi = 0 \text{ in } \mathcal{R},$$
$$B(\psi) = 0 \text{ on } S, \text{ or } C \text{ if } n = 2,$$

[4, pp. 132-133].

The solution is

$$u(\vec{r}) = \int_{\mathcal{R}} G(\vec{r},\vec{r}\,')f(\vec{r}\,')dV' + \int_S g(\vec{r}\,') \frac{\partial G}{\partial n} dS', \qquad (3.8.2)$$

if B(u) is Dirichlet boundary condition,

$$u(\vec{r}) = \int_{\mathcal{R}} G(\vec{r},\vec{r}\,')f(\vec{r}\,')dV' - \int_S g(\vec{r}\,')G(\vec{r},\vec{r}\,')dS', \qquad (3.8.3)$$

if B(u) is Neumann boundary condition, [4, pp. 140-141]. For $n = 2$, S is replaced by C, the boundary of $\mathcal{R}$.

Example 3.15. Find u such that $\Delta u = f(r,\theta)$ in $\mathcal{R}$ and $u = g(r,\theta)$ on C where $\mathcal{R} = \left\{ re^{i\theta};\ 0 < r < a,\ 0 < \theta < \pi \right\}$, and C is the boundary of $\mathcal{R}$.

The associated eigenvalue problem is

$$\Delta\psi + \lambda\psi = 0 \text{ in } \mathcal{R} \text{ and } \psi = 0 \text{ on } C.$$

By the method of separation of variables, we have

$$\psi(r,\theta) = R(r)\Theta(\theta),$$

hence

$$R(r) = AJ_\nu(\alpha r) + BY_\nu(\alpha r),$$
$$\Theta(\theta) = E\cos\nu\theta + D\sin\nu\theta,$$

A, B, E, and D are arbitrary constants, $\nu = \sqrt{\mu}$, μ is the separation constant, $\alpha = \sqrt{\lambda}$, and J_ν, Y_ν are, respectively, the Bessel functions of the first and second kind, [4, pp. 110-112]. Since $\psi(r,\theta) = 0$ on C, therefore $\psi(a,\theta) = \psi(r,0) = \psi(r,\pi) = 0$ for $0 < \theta < \pi$, $0 < r < a$, which imply $R(a) = 0$, $\Theta(0) = \Theta(\pi) = 0$, (cf. Figure 3.18). Hence $E = 0, \nu = n$, $n = 1,2,3,\ldots$. Since the Bessel function of the second kind, Y_ν is unbounded near $r = 0$, it is necessary to choose $B = 0$. The condition $R(a) = 0$ implies $J_n(\alpha_n a) = 0$ or $\alpha_n a$ are the positive zeroes of J_n, that is $J_n(\xi_{mn}) = 0$ for $m = 1,2,\ldots$. Therefore, we have

$$\psi_{mn}(r,\theta) = \sin n\theta \; J_n\left(\frac{\xi_{mn} r}{a}\right)$$

and

$$\lambda_{mn} = \left(\frac{\xi_{mn}}{a}\right)^2, \quad m = 1,2,\ldots; \; n = 1,2,\ldots.$$

The magnitude of

$$\|\psi_{mn}\|^2 = \int_0^\pi \int_0^a |\psi_{mn}|^2 \, r \, dr \, d\theta$$

$$= \int_0^a J_n^2\left(\frac{\xi_{mn} r}{a}\right) r \, dr \int_0^\pi \sin^2 n\theta \, d\theta$$

$$= \frac{\pi a^2 J_{n+1}^2(\xi_{mn})}{4}$$

[4, pp. 345-346]. Using (3.8.1), we obtain Green's function

$$G(r,\theta;r'\theta') = -4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin n\theta \; J_n(\xi_{mn} r/a) \sin n\theta' \; J_n(\xi_{mn} r'/a)}{\pi \xi_{mn}^2 \; J_{n+1}^2(\xi_{mn})}.$$

The following example illustrate the adjustments that would be made if the region were a quarter-disk instead of a semi-disk.

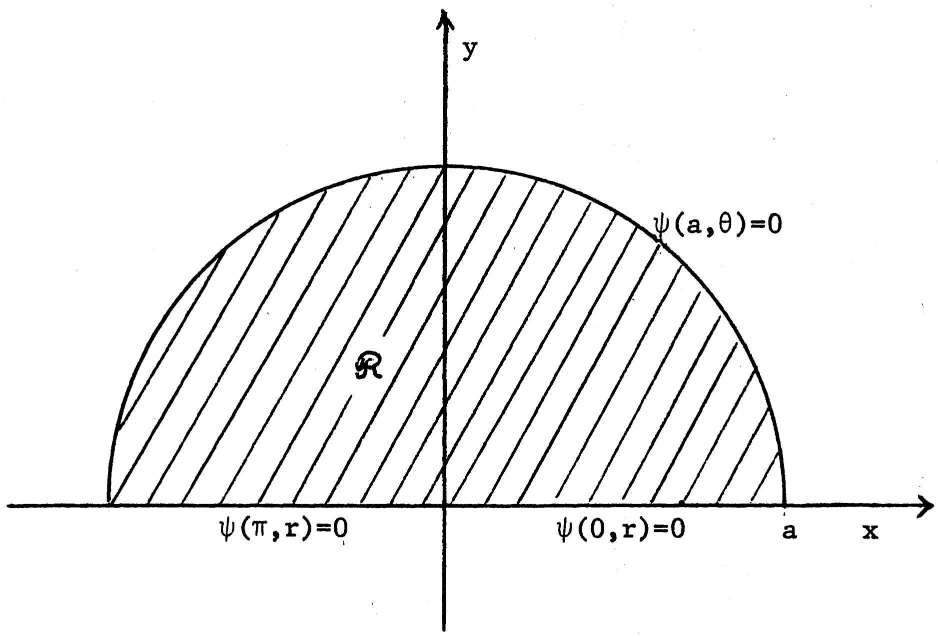


Figure 3.18. Boundary Conditions for Associated Eigenvalue Problem in Example 3.15

<u>Example 3.16.</u> Find u such that $\Delta u = f(r,\theta)$ in $\mathcal{R}$ and $u = g(r,\theta)$ on C where $\mathcal{R} = \left\{ re^{i\theta};\ 0 < r < a,\ 0 < \theta < \pi/2 \right\}$, and C is the boundary of $\mathcal{R}$.

The associated eigenvalue problem is again

$$\Delta\psi + \lambda\psi = 0 \text{ in } \mathcal{R} \text{ and } \psi = 0 \text{ on C.}$$

By the method of separation of variables, we have

$$\psi(r,\theta) = R(r)\Theta(\theta)$$

and

$$R(r) = AJ_\nu(\alpha r),$$

$$\Theta(\theta) = E \cos \nu\theta + D \sin \nu\theta,$$

(cf. Example 3.16). Since $\psi(r,\theta) = 0$ on C, therefore $\psi(a,\theta) = \psi(r,0) = \psi(r,\pi/2) = 0$, for $0 < \theta < \pi/4$, $0 < r < a$, (cf. Figure 3.19). Hence $R(a) = \Theta(0) = \Theta(\pi/2) = 0$. This gives $E = 0$, $\nu = 2n$, $n = 1,2,\ldots$, and $J_{2n}(\alpha_{2n}a) = 0$ or $\alpha_{2n}a = \xi_{m(2n)}$ are the positive zeroes of J_n, for $m = 1,2,\ldots$. Therefore, we have

$$\psi_{m(2n)}(r,\theta) = \sin 2n\theta\ J_{2n}(\xi_{m(2n)}r/a)$$

and $\lambda_{m(2n)} = \alpha^2_{m(2n)} = (\xi_{m(2n)}/a)^2$, $m = 1,2,\ldots;\ n = 1,2,\ldots$.

Then

$$\|\psi_{m(2n)}\|^2 = \int_0^{\frac{\pi}{2}} \int_0^a |\psi_{m(2n)}|^2 r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 2n\theta\, d\theta \int_0^a J^2_{2n}\left(\xi_{m(2n)}r/a\right) r\, dr$$

$$= \frac{\pi a^2}{8} J^2_{2n+1}\left(\xi_{m(2n)}\right).$$

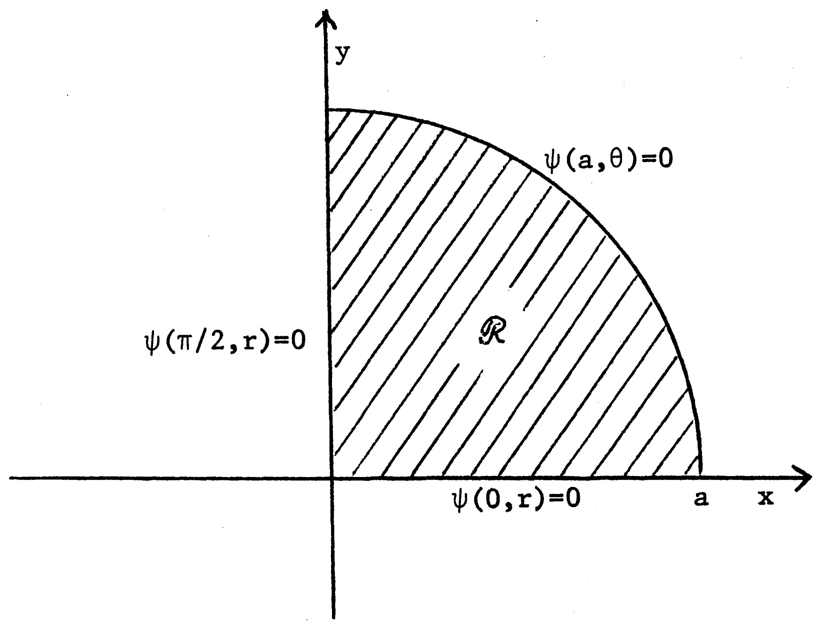


Figure 3.19. Boundary Conditions for Associated Eigenvalue Problem in Example 3.16

Using (3.8.1), Green's function is

$$G(r,\theta;r',\theta') = -8 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin 2n\theta \; J_{2n}(\xi_{m(2n)} r/a) \sin 2n\theta' \; J_{2n}(\xi_{m(2n)} r'/a)}{\pi \xi_{m(2n)}^2 \; J_{2n+1}^2(\xi_{m(2n)})}.$$

Lastly, the method of finding Green's function in terms of eigenfunctions will be applied to a closed parallelepiped in xyz-space.

Example 3.17. Find u such that $\Delta u = f(x,y,z)$ in $\mathcal{R}$ and $u = g(x,y,z)$ on S where $\mathcal{R} = \{(x,y,z);\ 0 < x < a,\ 0 < y < b,\ 0 < z < c\}$ and S is the boundary of $\mathcal{R}$.

The associated eigenvalue problem is

$$\Delta\psi + \lambda\psi = 0 \text{ in } \mathcal{R} \text{ and } \psi = 0 \text{ on S.}$$

By separation of variables, let

$$\psi(x,y,z) = X(x)Y(y)Z(z).$$

Substituting gives

$$YZX'' + XZY'' + XYZ'' + \lambda XYZ = 0,$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + \lambda = 0,$$

or

$$\frac{X''}{X} = -\left(\frac{Y''}{Y} + \frac{Z''}{Z} + \lambda\right) = -\nu,$$

ν is a separation constant. Since $\psi = 0$ on S, we have $\psi(a,y,z) = \psi(0,y,z) = \psi(x,b,z) = \psi(x,0,z) = \psi(x,y,0) = \psi(x,y,c) = 0$ (cf. Figure 3.20). These conditions imply that $X(0) = X(a) = Y(0) = Y(b) = Z(0) = Z(c) = 0$. Hence, the function X satisfies $X'' + \nu X = 0$ and $X(0) = X(a) = 0$.

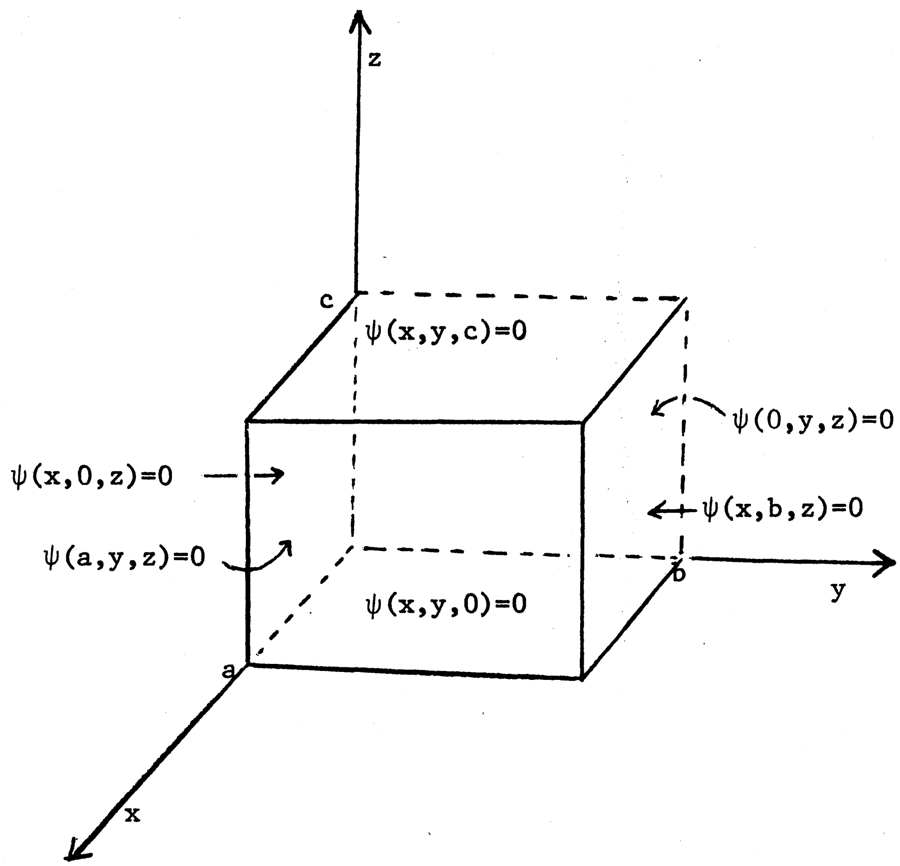


Figure 3.20. Boundary Conditions for Associated Eigenvalue Problem in Example 3.17

Therefore, $X = \sin \sqrt{\nu_n} x$ and $\nu_n = (n\pi/a)^2$, $n = 1,2,\ldots.$ Since

$$\frac{Y''}{Y} + \frac{Z''}{Z} + \lambda = \nu,$$

therefore,

$$\frac{Y''}{Y} = -\mu, \text{ and } \frac{Z''}{Z} = -\omega$$

where μ and ω are positive constants. The function Y satisfies

$$\frac{Y''}{Y} = -\mu \text{ and } Y(0) = Y(b) = 0.$$

Therefore $Y = \sin \sqrt{\mu_n} y$ and $\mu_n = (m\pi/b)^2$, $m = 1,2,\ldots.$ The function Z satisfies

$$\frac{Z''}{Z} = -\omega \text{ and } Z(0) = Z(c) = 0.$$

Hence $Z = \sin \sqrt{\omega_n} z$ and $\omega_n = (k\pi/c)^2$, $k = 1,2,\ldots.$ Therefore,

$$\psi_{n,m,k} = \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{k\pi z}{c}$$

and

$$\lambda_{n,m,k} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{k^2}{c^2} \right).$$

Hence

$$\|\psi_{n,m,k}\|^2 = \int_0^c \int_0^b \int_0^a \sin^2 \frac{n\pi x}{a} \sin^2 \frac{m\pi y}{b} \sin^2 \frac{k\pi z}{c} \, dx \, dy \, dz = \frac{abc}{8}.$$

Using (3.8.1), Green's function, $G(x,y,z;x',y',z')$, is

$$G = -\frac{8abc}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{\sin \frac{n\pi x'}{a} \sin \frac{m\pi y'}{b} \sin \frac{k\pi z'}{c} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{k\pi z}{c}}{n^2 b^2 c^2 + m^2 a^2 c^2 + k^2 a^2 b^2}.$$

It should be obvious that we cannot find Green's function by the method of eigenfunctions if there are no eigenfunctions for the associated eigenvalue problem. Such a region is, for example, $\mathcal{R} = \left\{ (x,y);\ 0 < x < a,\ 0 < y < \infty \right\}$. The associated eigenvalue problem has only the solution $\psi = 0$.

CHAPTER IV

GREEN'S FUNCTIONS FOR HELMHOLTZ'S EQUATION

From the homogeneous wave equation

$$\Delta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

we assume the solution is in the form

$$u(\vec{r},t) = \psi(\vec{r})T(t).$$

If the partial derivatives are calculated and substitution is made into the wave equation, the relation

$$\psi T'' = c^2 T\Delta\psi$$

is obtained. Therefore,

$$\frac{\Delta\psi}{\psi} = \frac{T''}{c^2 T} = -k^2,$$

where k^2 is a separation constant. Thus the functions ψ and T must, respectively, satisfy

$$\Delta\psi + k^2\psi = 0 \text{ and } T'' + c^2k^2T = 0.$$

The equation $\Delta\psi + k^2\psi = 0$ is called the space form of the wave equation or Helmholtz's equation.

4.1. Fundamental Solutions

4.1.1. Two Dimensions

Consider the homogeneous Helmoltz's equation in polar coordinates

$$\Delta u + k^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + k^2 u = 0. \tag{4.1.1}$$

Similarly as was done in Section 3.2, we want to find a solution of (4.1.1) that depends only on r, that is, find u(r) of

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} + k^2 u = 0. \tag{4.1.2}$$

Let $\eta = kr$ that $u(r) = u(\eta/k) = v(\eta)$. Then (4.1.2) is transformed into

$$\frac{d^2 v}{d\eta^2} + \frac{1}{\eta}\frac{dv}{d\eta} + v = 0$$

which is Bessel's equation of zero order. Therefore, two linearly independent solutions of the equation are $J_0(kr)$ and $N_0(kr)$ where J_0 and N_0 are Bessel's and Neumann's function, respectively. If we consider the Hankel function of the first kind, namely,

$$H_0^{(1)}(kr) = J_0(kr) + iN_0(kr),$$

then $H_0^{(1)}(kr)$ is a solution of (4.1.2), hence it is also a solution of (4.1.1). It is unbounded near zero since N_0 is unbounded near zero. The function $H_0^{(1)}(kr)$ is called the fundamental solution of Helmholtz's equation in two dimensions.

Consider the function $H_0^{(1)}(k|\vec{r}'-\vec{r}|)$. Since $H_0^{(1)}(kr)$ is a solution of (4.1.1) except at r = 0, we can show that $H_0^{(1)}(k|\vec{r}'-\vec{r}|)$, which is a translation of the origin to $\vec{r}'$ of the original function $H_0^{(1)}$, is also a solution of (4.1.1) except at $\vec{r}'$.

Weber's Theorem. We are now going to find the solution of Helmholtz's equation with boundary conditions. Weber's theorem and Green's second identity are used. Weber's theorem, [20, p. 241], is that, if $u(\vec{r})$ is a solution of the two dimensional Helmholtz's equation $\Delta u + k^2 u = 0$ whose partial derivatives of the first and second orders are continuous within a region $\mathcal{R}$ and on the closed curve C, the boundary of $\mathcal{R}$, then

$$\frac{1}{4i}\int_C\left(u(\vec{r}')\frac{\partial}{\partial n} H_0^{(1)}(k|\vec{r}'-\vec{r}|) - H_0^{(1)}(k|\vec{r}'-\vec{r}|)\frac{\partial}{\partial n}u(\vec{r}')\right)ds' = \begin{cases} u(\vec{r}), & \vec{r} \in \mathring{\mathcal{R}}, \\ 0, & \vec{r} \notin \bar{\mathcal{R}}, \end{cases} \tag{4.1.3}$$

where $\mathring{\mathcal{R}}$ is the interior of $\mathcal{R}$ and $\vec{n}$ is the outward normal to C. Now, define

$$G(\vec{r},\vec{r}') = H_0^{(1)}(k|\vec{r}'-\vec{r}|) + G_1(\vec{r},\vec{r}') \tag{4.1.4}$$

where

$$\Delta G_1 + k^2 G_1 = 0$$

for $\vec{r}' \in \mathring{\mathcal{R}}$. By using Weber's theorem and Green's second identinty, we can find the solution of the Helmholtz's equation in the integral form with a part of the kernel as Green's function as follows:

From Green's second identity, we have

$$\int_{\mathcal{R}}\left(u(\vec{r}')\Delta G_1 - G_1\Delta u(\vec{r}')\right)dv' = \int_C\left(u(\vec{r}')\frac{\partial G_1}{\partial n} - G_1\frac{\partial}{\partial n}u(\vec{r}')\right)ds'$$

[12, p. 215]. Substituting into the left-hand side gives

$$0 = \int_{\mathcal{R}}\left(-u(\vec{r}')k^2G_1 + G_1k^2u(\vec{r}')\right)dv' = \int_C\left(u(\vec{r}')\frac{\partial G_1}{\partial n} - G_1\frac{\partial}{\partial n}u(\vec{r}')\right)ds'.$$

Then

$$\frac{1}{4i}\int_C\left(u(\vec{r}\,')\,\frac{\partial G_1}{\partial n} - G_1\,\frac{\partial}{\partial n}\,u(\vec{r}\,')\right)ds' = 0. \tag{4.1.5}$$

Adding (4.1.5) and (4.1.3) and using (4.1.4) gives

$$\frac{1}{4i}\int_C\left(u(\vec{r}\,')\,\frac{\partial G}{\partial n} - G\,\frac{\partial}{\partial n}\,u(\vec{r}\,')\right)ds' = u(\vec{r}),\ \text{if } \vec{r} \in \mathring{\mathcal{R}}. \tag{4.1.6}$$

If $G(\vec{r},\vec{r}\,') = 0$ and $u(\vec{r}\,') = f(\vec{r}\,')$ for $\vec{r}\,'$ on C then (4.1.6) becomes

$$u(\vec{r}) = \frac{1}{4i}\int_C f(\vec{r}\,')\,\frac{\partial G}{\partial n}\,ds',\ \text{if } \vec{r} \in \mathring{\mathcal{R}}. \tag{4.1.7}$$

One can show that (4.1.7) is the solution of the Helmholtz's equation in two dimensions with the Dirichlet boundary condition, that is, (4.1.7) is the solution of

$$\Delta u(x,y) + k^2 u(x,y) = 0,\ (x,y) \in \mathcal{R},$$

$$u(x,y) = f(x,y),\ (x,y) \in C.$$

The function G in (4.1.7) is called the Green's function of the first kind. It has the form (4.1.4) and it has the following properties:

(1). $\Delta G_1 + k^2 G_1 = 0$,

(2). G satisfies the homogeneous boundary condition, $G(x,y;x',y') = 0$, if (x',y') is on C,

(3). $G \sim H_0^{(1)}(k|\vec{r}\,'-\vec{r}|)$, if $\vec{r}\,' \to \vec{r}$.

If $\frac{\partial G}{\partial n} = 0$ and $\frac{\partial}{\partial n}\,u(\vec{r}\,') = f(\vec{r}\,')$ for $\vec{r}\,'$ on C then (4.1.6) becomes

$$u(\vec{r}) = -\frac{1}{4i}\int_C G\,f(\vec{r}\,')\,ds',\ \text{if } \vec{r} \in \mathring{\mathcal{R}} \tag{4.1.8}$$

which is the solution of the Helmholtz's equation in two dimensions with the Neumann boundary condition, that is, (4.1.8) is the solution of

$$\Delta u(x,y) + k^2 u(x,y) = 0, \; (x,y) \in \mathcal{R}$$
$$\frac{\partial}{\partial n} u(x,y) = f(x,y), \; (x,y) \in C.$$

The function G in (4.1.8) is called the Green's function of the second kind. It has the form (4.1.4) and the same properties as the Green's function of the first kind except property (2) is changed to

(2'). G satisfies the homogeneous boundary condition,

$\frac{\partial G}{\partial n} = 0$ if (x',y') is on C.

Weber's theorem applies to a compact region $\mathcal{R}$. If the region is the complement of the region $\mathcal{R}$, then Weber's theorem can be used by applying to the circle containing $\mathcal{R}$ and of center the origin with a very large radius r. Letting $r \to \infty$ and adding the conditions $\sqrt{r}\, u(r)$ be bounded and $\sqrt{r}\left(\frac{\partial u}{\partial n} - iku\right) \to 0$ uniformly, we have

$$\frac{1}{4i} \int_C \left[H_0^{(1)}(k|\vec{r}\,'-\vec{r}|) \frac{\partial}{\partial n} u(\vec{r}\,') - u(\vec{r}\,') \frac{\partial}{\partial n} H_0^{(1)}(k|\vec{r}\,'-\vec{r}|) \right] ds' = \begin{cases} u(\vec{r}), & \vec{r} \notin \overline{\mathcal{R}}, \\ 0, & \vec{r} \in \mathcal{R}^{\circ}, \end{cases}$$

where $\vec{n}$ is the outward normal to C with respect to the exterior region of $\mathcal{R}$.

The results of Weber's theorem and the application to the exterior of a compact region can be extend to include the regions which are neither the interior or the exterior of the compact region when the conditions at infinity are maintained.

4.1.2. Three Dimensions

As in two dimensions, we consider the homogeneous Helmholtz's equation in spherical coordinates, namely,

$$\Delta u + k^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + k^2 u = 0. \tag{4.1.9}$$

We want to find a solution of (4.1.9) that depends only on r. That is, find u(r) a solution of

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) + k^2 u = 0$$

or of

$$\frac{d^2}{dr^2}(ru) + k^2(ru) = 0. \tag{4.1.10}$$

A solution of (4.1.10) is

$$u_1 = \frac{e^{ikr}}{r},$$

hence u_1 is also a solution of (4.1.9). Consider the function

$$u_2 = \frac{e^{ik|\vec{r}\,'-\vec{r}|}}{|\vec{r}\,'-\vec{r}|}.$$

Since u_1 is a non-zero solution of (4.1.9) except at $r = 0$, we can show that u_2 which is a translation of the origin to $\vec{r}\,'$ of the original function, is also a non-zero solution of (4.1.9) except at $\vec{r} = \vec{r}\,'$. The function u_2 is called the fundamental solution of Helmholtz's equation in three dimensions.

<u>Helmholtz's First Theorem.</u> Helmholtz's first theorem [20, pp. 239-240] and Green's second identity are used in writing the solution of

Helmholtz's equation in the integral form. Such a solution can be used as a solution of boundary value problem. Helmholtz's first theorem is that, if $u(\vec{r})$ is a solution of the space form of the wave equation $\Delta u + k^2 u = 0$ whose partial derivatives of the first and second orders are continuous within $\bar{\mathcal{R}}$ and on the closed surface S bounding $\mathcal{R}$, then

$$\frac{1}{4\pi}\int_S \left[\frac{e^{ik|\vec{r}'-\vec{r}|}}{|\vec{r}'-\vec{r}|}\frac{\partial}{\partial n}u(\vec{r}') - u(\vec{r}')\frac{\partial}{\partial n}\frac{e^{ik|\vec{r}'-\vec{r}|}}{|\vec{r}'-\vec{r}|}\right] dS' = \begin{cases} u(\vec{r}), & \vec{r} \in \mathring{\mathcal{R}}, \\ 0, & \vec{r} \notin \bar{\mathcal{R}}, \end{cases}$$

where $\vec{n}$ is the outward normal to S.

Define

$$G(\vec{r},\vec{r}') = \frac{e^{ik|\vec{r}'-\vec{r}|}}{|\vec{r}'-\vec{r}|} + G_1(\vec{r},\vec{r}') \qquad (4.1.11)$$

where

$$\Delta G_1 + k^2 G_1 = 0$$

for $\vec{r}' \in \mathring{\mathcal{R}}$. By using Helmholtz's first theorem and Green's second identity on G_1, we obtain

$$u(\vec{r}) = \frac{1}{4\pi}\int_S \left[G\frac{\partial}{\partial n}u(\vec{r}') - u(\vec{r}')\frac{\partial G}{\partial n}\right] dS', \qquad (4.1.12)$$

if $\vec{r} \in \mathring{\mathcal{R}}$.

If $G = 0$ and $u(\vec{r}') = f(\vec{r}')$, $\vec{r}'$ on S, then (4.1.12) becomes

$$u(\vec{r}) = -\frac{1}{4\pi}\int_S f(\vec{r}')\frac{\partial G}{\partial n}\, dS', \text{ if } \vec{r} \in \mathring{\mathcal{R}}. \qquad (4.1.13)$$

One can show that (4.1.13) is the solution of the Helmholtz's equation in three dimensions with the Dirichlet boundary condition, that is, (4.1.13) is the solution of

$$\Delta u + k^2 u = 0, \ \vec{r} \in \mathcal{R},$$

$$u(\vec{r}) = f(\vec{r}), \ \vec{r} \text{ on } S.$$

The function G in (4.1.13) has the form (4.1.11) and the following properties:

(1). $\Delta G_1 + k^2 G_1 = 0$,

(2). G satisfies the homogeneous boundary condition, $G(\vec{r},\vec{r}\,') = 0$, $\vec{r}\,'$ on S,

(3). $G \sim \dfrac{e^{ik|\vec{r}\,'-\vec{r}|}}{|\vec{r}\,'-\vec{r}|}$, if $\vec{r}\,' \to \vec{r}$.

The function G is called Green's function of the first kind.

If $\dfrac{\partial G}{\partial n} = 0$ and $\dfrac{\partial}{\partial n} u(\vec{r}\,') = f(\vec{r}\,')$, $\vec{r}\,'$ on S, then (4.1.12) becomes

$$u(\vec{r}) = \frac{1}{4\pi} \int_S G \, f(\vec{r}\,') \, dS', \text{ if } \vec{r} \in \mathring{\mathcal{R}}. \tag{4.1.14}$$

which is the solution of the Helmholtz's equation in three dimensions with the Neumann boundary condition, that is, (4.1.14) is the solution of

$$\Delta u + k^2 u = 0, \ \vec{r} \in \mathcal{R},$$

$$\frac{\partial}{\partial n} u(\vec{r}) = f(\vec{r}), \ \vec{r} \text{ on } S.$$

The function G in (4.1.14) is Green's function of the second kind which has the form (4.1.11) and the same properties as G in (4.1.13) except property (2) is changed to

(2'). G satisfies the homogeneous boundary condition, $\dfrac{\partial}{\partial n} G(\vec{r},\vec{r}\,') = 0$, $\vec{r}\,'$ on S.

Helmholtz's first theorem is for a compact region. If the region is the exterior of a compact region Helmholtz's second theorem

[20, p. 240] is used in the similar way. The theorem is for the complement of a compact region $\mathcal{R}$ with surface S bounding $\mathcal{R}$ and the condition $ru(\vec{r})$ is bounded and $r(\frac{\partial u}{\partial r} - iku) \to 0$ uniformly with respect to the angle variables as $r \to \infty$ is added. The result of these two theorems can be extended to include regions that are neither the interior or the exterior of a compact region when the conditions at infinity are maintained.

4.2. The Method of Images

Finding Green's function by the method of images for Helmholtz's equation is similar to the method of finding Green's function by the method of images for Laplace's equation (cf. Sections 3.3 and 3.5). The following example will show how to find Green's function for a half-space. Compare this example with Example 3.11.

Example 4.1. Solve the problem

$$\Delta u + k^2 u = 0 \text{ in } \mathcal{R}, \; u = f(y,z) \text{ on } S,$$

The region $\mathcal{R} = \left\{ (x,y,z);\ 0 < x < \infty,\ -\infty < y,z < \infty \right\}$, and S is the boundary of $\mathcal{R}$.

Green's function must be of the form:

$$G(\vec{r},\vec{r}\,') = \frac{e^{ik|\vec{r}\,'-\vec{r}|}}{|\vec{r}\,'-\vec{r}|} + G_1(\vec{r},\vec{r}\,'),$$

where $\vec{r} = (x,y,z)$ and $\vec{r}\,' = (x',y',z')$. Consider

$$G(\vec{r},\vec{r}\,') = \frac{e^{ik|\vec{r}\,'-\vec{r}|}}{|\vec{r}\,'-\vec{r}|} - \frac{e^{ik|\vec{r}\,'-\vec{r}_0|}}{|\vec{r}\,'-\vec{r}_0|}$$

where $\vec{r}_0 = (-x,y,z)$, the image of $\vec{r}$ across the yz-plane. Then

$$G(\vec{r},\vec{r}\,') = \frac{e^{ik\cdot QP}}{QP} - \frac{e^{ik\cdot QP_1}}{QP_1}$$

(cf. Figure 3.14). It is easy to see that

$$G(\vec{r},\vec{r}\,')\Big|_{x'=0} = 0.$$

Property (2) can be shown by recalling that $v(r) = e^{ikr}/r$ is the solution of $\Delta v + k^2 v = 0$. Hence, $e^{ik|\vec{r}-\vec{r}_0|}/|\vec{r}-\vec{r}_0|$, the translation of the origin to $\vec{r}_0$ satisfies Helmholtz's equation. The normal derivative of G gives

$$\frac{\partial G}{\partial n} = - \frac{\partial G}{\partial x'}\Big|_{x'=0} = 2\,\frac{\partial}{\partial x'}\left(\frac{e^{ikR}}{R}\right)$$

where $R^2 = x^2 + (y'-y)^2 + (z'-z)^2$.

Using (4.1.13), the solution to the problem is

$$u(x,y,z) = - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y',z')\,\frac{\partial}{\partial x'}\,\frac{e^{ikR}}{R}\,dy'dz'.$$

Table I provides Green's function of Helmholtz's equation with Dirichlet boundary condition for some regions in two and three dimensions. Green's functions in this table can be found by using the method of images (cf. Sections 3.3 and 3.5).

4.3. Green's Function in Terms of Eigenfunctions

We have used the method of images to find Green's function for the homogeneous Helmholtz's equation with nonhomogeneous boundary condition in the preceeding section. We are now consider the different problem, the nonhomogeneous Helmholtz's equation with the homogeneous boundary condition, that is,

TABLE I

GREEN'S FUNCTION FOR HELMHOLTZ'S EQUATION WITH THE DIRICHLET BOUNDARY CONDITION

Region	Green's Function	Reference
$\left\{(x,y) : 0 < x < \infty, -\infty < y < \infty\right\}$	$H_0^{(1)}(k\cdot QP) - H_0^{(1)}(k\cdot QP_1)$	Figure 3.1, Example 3.1
$\left\{(x,y) ; 0 < x < \infty, 0 < y < \infty\right\}$	$H_0^{(1)}(k\cdot QP) - H_0^{(1)}(k\cdot QP_1) + H_0^{(1)}(k\cdot QP_2) - H_0^{(1)}(k\cdot QP_3)$	Figure 3.2, Example 3.2
$\left\{re^{i\theta} : 0 < \theta < \frac{\pi}{3}, r > 0\right\}$	$H_0^{(1)}(k\cdot QP) - H_0^{(1)}(k\cdot QP_1) + H_0^{(1)}(k\cdot QP_2) - H_0^{(1)}(k\cdot QP_3) + H_0^{(1)}(k\cdot QP_4) - H_0^{(1)}(k\cdot QP_5)$	Figure 3.3, Example 3.3
$\left\{(x,y) : 0 < x < a, -\infty < y < \infty\right\}$	$H_0^{(1)}(k\cdot QP) - H_0^{(1)}(k\cdot QP_1) + H_0^{(1)}(k\cdot QP_1') - H_0^{(1)}(k\cdot QP_0') + H_0^{(1)}(k\cdot QP_2) - H_0^{(1)}(k\cdot QP_3) + \ldots$	Figure 3.5, Example 3.5
$\left\{re^{i\theta} : 0 < \theta < 2\pi, 0 < r < a\right\}$	$H_0^{(1)}(k\cdot QP) - H_0^{(1)}\left(k\,\frac{r\cdot QP_1}{a}\right)$	Figure 3.7, Example 3.6
$\left\{re^{i\theta} : 0 < \theta < \pi, 0 < r < a\right\}$	$H_0^{(1)}(k\cdot QP) - H_0^{(1)}\left(k\,\frac{r\cdot QP_1}{a}\right) + H_0^{(1)}\left(k\,\frac{r\cdot QP_2}{a}\right) - H_0^{(1)}(k\cdot QP_3)$	Figure 3.8, Example 3.7

TABLE I (Continued)

Region	Green's Function	Reference
A sphere with center at (0,0,0) and radius a	$\frac{e^{ik\cdot QP}}{QP} - \frac{e^{ik(r\cdot QP_1/a)}}{r\cdot QP_1/a}$	Figure 3.15, Example 3.13
A hemisphere with center at (0,0,0) and radius a and z>0	$\frac{e^{ik\cdot QP}}{QP} - \frac{e^{ik(r\cdot QP_1/a)}}{r\cdot QP_1/a} + \frac{e^{ik(r\cdot QP_2/a)}}{r\cdot QP_2/a} - \frac{e^{ik(r\cdot QP_3/a)}}{r\cdot QP_3/a}$	Figure 3.16, Example 3.14

$$\Delta u + k^2 u = F, \ \vec{r} \ \varepsilon \ \mathcal{R},$$

$$B(u) = 0, \ \vec{r} \text{ on } S \text{ or on } C \text{ if } n = 2,$$

where B(u) is Dirichlet or Neumann boundary condition, that is,

$$u = 0 \text{ or } \frac{\partial u}{\partial n} = 0, \text{ respectively, on } S \text{ or } C \text{ if } n = 2.$$

In this case Green's function is found by using the associated eigenvalue problem.

Let ψ_n and λ_n be eigenfunctions and the corresponding eigenvalues of the eigenvalue problem

$$\Delta\psi + \lambda\psi = 0, \ \vec{r} \text{ in } \mathcal{R},$$

$$B(\psi) = 0, \ \vec{r} \text{ on } S.$$

Write

$$u(\vec{r}) = \sum_{n=1}^{\infty} C_n \psi_n(\vec{r})$$

where C_n is a constant to be determined. Let m be fixed, m = 1,2,..., then we have

$$\int_{\mathcal{R}} u(\vec{r}\,')\psi_m(\vec{r}\,') \ dV' = \int_{\mathcal{R}} \sum_{n=1}^{\infty} C_n \psi_n(\vec{r}\,')\psi_m(\vec{r}\,')dV'$$

By the orthogonal property of eigenfunctions, we obtain

$$C_n = \frac{1}{\|\psi_n\|^2} \int_{\mathcal{R}} u(\vec{r}\,')\psi_n(\vec{r}\,')dV'$$

where

$$\|\psi_n\|^2 = \int_{\mathcal{R}} \psi_n^2 \ dV'.$$

Now,

$$\int_{\mathcal{R}} u\,\psi_m\,dV' = \frac{1}{\lambda_m}\int_{\mathcal{R}} u\lambda_m\psi_m\,dV'$$

$$= -\frac{1}{\lambda_m}\int_{\mathcal{R}} u\Delta\psi_m\,dV'.$$

Using Green's second identity, the right-hand side equals

$$-\frac{1}{\lambda_m}\int_{\mathcal{R}}\psi_m\Delta u\,dV' - \frac{1}{\lambda_m}\int_S \left(u\,\frac{\partial\psi_m}{\partial n} - \psi_m\,\frac{\partial u}{\partial n}\right) dS'$$

Since $B(u) = 0$ and $B(\psi_m) = 0$ on S, then

$$\int_{\mathcal{R}} u\psi_m\,dV' = -\frac{1}{\lambda_m}\int_{\mathcal{R}}\psi_m(F - k^2u)\,dV'.$$

Therefore,

$$\int_{\mathcal{R}} u\psi_m\,dV' = -\frac{\int_{\mathcal{R}}\psi_m F\,dV'}{\lambda_m - k^2},$$

and thus

$$C_m = \frac{-1}{\|\psi_m\|^2(\lambda_m - k^2)}\int_{\mathcal{R}}\psi_m F\,dV'.$$

Hence,

$$u(\vec{r}) = \int_{\mathcal{R}} F(\vec{r}\,')\left(-\sum_{m=1}^{\infty}\frac{\psi_m(\vec{r}\,')\psi_m(\vec{r})}{|\psi_m|^2(\lambda_m - k^2)}\right) dV'$$

or,

$$u(\vec{r}) = \int_{\mathcal{R}} F(\vec{r}\,')G(\vec{r},\vec{r}\,')\,dV'. \qquad (4.3.1)$$

The function,

$$G(\vec{r},\vec{r}\,') = \sum_{m=1}^{\infty} \frac{\psi_m(\vec{r}\,')\psi_m(\vec{r})}{|\psi_m|^2(k^2-\lambda_m)} \tag{4.3.2}$$

is called Green's function.

Table II gives Green's functions in terms of eigenfunctions for a half-circle, a quarter-circle and

$$\left\{(x,y,z):\ 0 < x < a,\ 0 < y < b,\ 0 < z < c\right\}$$

for the Dirichlet boundary condition. They are obtained by using the solutions of the associated eigenvalue problems of Examples 3.15-3.17 and using (4.3.2).

TABLE II

GREEN'S FUNCTION IN TERMS OF EIGENFUNCTIONS

Region	Green's Function	References
$\left\{re^{i\theta}: 0 < \theta < \pi,\ 0 < r < a\right\}$	$4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin n\theta\, J_n(\xi_{mn}r/a)\sin n\theta'\, J_n(\xi_{mn}r'/a)}{\pi J_{n+1}^2(\xi_{mn})\left(a^2k^2 - \xi_{mn}^2\right)}$	Figure 3.18, Example 3.15
$\left\{re^{i\theta}: 0 < \theta < \frac{\pi}{2},\ 0 < r < a\right\}$	$8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin 2n\theta\, J_{2n}(\xi_{m(2n)}r/a)\sin 2n\theta' J_{2n}(\xi_{m(2n)}r'/a)}{\pi J_{2n+1}^2\left(\xi_{m(2n)}\right)\left(a^2k^2 - \xi_{m(2n)}^2\right)}$	Figure 3.19, Example 3.16
$\left\{(x,y,z): 0 < x < a,\ 0 < y < b,\ 0 < z < c\right\}$	$8abc\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sin\frac{n\pi x'}{a}\sin\frac{m\pi y'}{b}\sin\frac{k\pi z'}{c}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{k\pi z}{c}}{a^2b^2c^2k^2 - \pi^2(b^2c^2n^2+a^2c^2m^2+a^2b^2k^2)}$	Figure 3.20, Example 3.17

CHAPTER V

GREEN'S FUNCTIONS FOR THE HEAT EQUATION

5.1. Fundamental Solutions

Consider the homogeneous heat equation with an initial condition and zero boundary conditions,

$$u_t = \kappa u_{xx},\ 0 < x < L,\ t > 0,\ \kappa \text{ is a positive constant},$$

$$u(x,0) = \Psi(x),\ 0 < x < L,$$

$$u(0,t) = 0,\ u(L,t) = 0.$$

Using the method of the separation of variables, the solution of the problem is

$$u(x,t) = \int_0^L \left[\frac{2}{L}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{L}\right)^2 \kappa t} \sin\frac{n\pi x'}{L}\sin\frac{n\pi x}{L}\right]\Psi(x')\,dx'.$$

Define

$$H(x,x',t) = \frac{2}{L}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{L}\right)^2 \kappa t} \sin\frac{n\pi x'}{L}\sin\frac{n\pi x}{L}.$$

Let $v = x - \frac{L}{2}$ and $v' = x' - \frac{L}{2}$. Therefore,

$$H(v,v',t) = \frac{2}{L}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{L}\right)^2 \kappa t} \sin\frac{n\pi}{L}\left(v+\frac{L}{2}\right)\sin\frac{n\pi}{L}\left(v'+\frac{L}{2}\right)$$

$$= \frac{2}{L}\sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t} \sin\left(\frac{n\pi v}{L}+\frac{n\pi}{2}\right)\sin\left(\frac{n\pi v'}{L}+\frac{n\pi}{2}\right)$$

$$= \frac{2}{L}\sum_{n=0}^{\infty}{}'' e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t} \sin\frac{n\pi v}{L}\sin\frac{n\pi v'}{L} + \frac{2}{L}\sum_{n=1}^{\infty}{}' e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t}\cos\frac{n\pi v}{L}\cos\frac{n\pi v'}{L}$$

where $\sum_{n=0}^{\infty}{}'' \left(\sum_{n=1}^{\infty}{}'\right)$ indicates that the summation is carried out over even (odd) values of n. Let

$$\lambda_n = \frac{n\pi}{L},\ \Delta\lambda = \frac{2\pi}{L},\ f_1(\lambda_n) = e^{-\lambda_n^2 \kappa t}\cos\lambda_n v\cos\lambda_n v'$$

and

$$f_2(\lambda_n) = e^{-\lambda_n^2 \kappa t}\sin\lambda_n v\sin\lambda_n v'.$$

Therefore,

$$H(v,v',t) = \frac{1}{\pi}\sum_{n=1}^{\infty}{}' f_1(\lambda_n)\Delta\lambda + \frac{1}{\pi}\sum_{n=0}^{\infty}{}'' f_2(\lambda_n)\Delta\lambda.$$

As $L \to \infty$, $\Delta\lambda \to 0$. It can be shown that

$$\lim_{\Delta\lambda \to 0}\sum_{n=0}^{\infty}{}'' f_2(\lambda_n)\Delta\lambda = \int_0^{\infty} f_2(\lambda)d\lambda$$

as follows:

We have

$$|f_2(\lambda)| = |e^{-\lambda^2\kappa t}\sin\lambda v'\sin\lambda v| \leqq e^{-\lambda^2\kappa t}.$$

Let $g(\lambda) = e^{-\lambda^2\kappa t}$. Applying the limit test to $g(\lambda)$, it follows that

$$\int_0^{\infty} g(\lambda)d\lambda < \infty.$$

Using the comparison test, then

$$\int_0^\infty |f_2(\lambda)|d\lambda < \infty$$

which implies that $\int_0^\infty f_2(\lambda)d\lambda$ is convergent. Therefore, given $\varepsilon > 0$, there exists a $\lambda^{(1)} > 0$ such that for $\lambda \geq \lambda^{(1)}$ then

$$\left|\int_\lambda^\infty f_2(\lambda)d\lambda\right| < \frac{\varepsilon}{12}.$$

Now $\sum_{n=0}^{\infty}{}'' e^{-\lambda_n^2 \kappa t} \Delta\lambda$ is convergent since by the ratio test

$$\frac{e^{-\lambda_{2i+2}^2 \kappa t} \Delta\lambda}{e^{-\lambda_{2i}^2 \kappa t} \Delta\lambda} = e^{-\left(\lambda_{2i+2}^2 - \lambda_{2i}^2\right)\kappa t} < 1.$$

Therefore, choose $\Delta\lambda = \Delta'\lambda$, there exists an even integer k_1 such that

$$\sum_{n=k}^{\infty}{}'' e^{-\lambda_n^2 \kappa t} \Delta'\lambda < \frac{\varepsilon}{12}$$

for an even integer $k \geq k_1$. Let $\left\{\lambda_0', \lambda_2', \lambda_4', \ldots\right\}$ be such that $\lambda_0' = 0$, $\lambda_{m+2}' - \lambda_m' = \Delta'\lambda$, $m = 0,2,4,\ldots$. Let k be an even integer that $\lambda_k' \geq \lambda^{(1)}$ and $k \geq k_1$. Since $f_2(\lambda)$ is a continuous function on $[0,\infty)$, therefore $f_2(\lambda)$ is Riemann integrable over $[0,\lambda_k']$. Therefore, there exists a $\delta_1 > 0$ for which

$$\left|\int_0^{\lambda_k'} f_2(\lambda)d\lambda - \text{Riemann sum of } f_2 \text{ over } [0,\lambda_k']\right| < \frac{\varepsilon}{12}$$

for all partitions such that $\Delta\lambda > \delta_1$.

Choose $\Delta^{*}\lambda = 2^{-i}\Delta^{'}\lambda$ such that $\Delta^{*}\lambda \leqq \frac{\varepsilon}{12}$ and $\Delta^{*}\lambda \leqq \delta_1$. Let $\left\{\lambda_0^*,\lambda_2^*,\lambda_4^*,\ldots\right\}$ be such that, $\lambda_0^* = 0$, $\lambda_{m+2}^* - \lambda_m^* = \Delta^*\lambda$, $m = 0,2,4,\ldots.$ Let $M = 2^i k$ therefore $\lambda_M^* = \lambda_k^{'}$. We have

$$\sum_{n=M}^{\infty}{}'' e^{-\lambda_n^2 \kappa t}\, \Delta^*\lambda < \frac{\varepsilon}{12}$$

since

$$\sum_{n=2k}^{\infty}{}'' e^{-\lambda_n^{(1)2}\kappa t}\, \frac{\Delta^{'}\lambda}{2} \leqq \sum_{n=k}^{\infty}{}'' e^{-\lambda_n^2 \kappa t}\, \Delta^{'}\lambda \tag{5.1.1}$$

(cf. Figure 5.1). Similarly

$$\sum_{n=M}^{\infty}{}'' e^{-\lambda_n^* \kappa t}\, \Delta^*\lambda = \sum_{n=2^i k}^{\infty}{}'' e^{-\left(\lambda_n^{(i)}\right)^2 \kappa t}\, \frac{\Delta^{'}\lambda}{2^i} \leqq \cdots \leqq$$

$$\sum_{n=2^2 k}^{\infty}{}'' e^{-\left(\lambda_n^{(2)}\right)^2 \kappa t}\frac{\Delta\lambda}{2^2} \leqq \sum_{n=2k}^{\infty}{}'' e^{-\left(\lambda^{(1)}\right)^2 \kappa t}\, \frac{\Delta^{'}\lambda}{2} \leqq \sum_{n=k}^{\infty}{}'' e^{-\lambda_n^2 \kappa t}\, \Delta'\lambda$$

where $\left\{\lambda_0^{(i)},\lambda_2^{(i)},\ldots,\lambda_{2n}^{(i)},\ i = 1,2,3,\ldots\right\}$ is such that $\lambda_0^{(i)} = 0$ and $\lambda_{m+2}^{(i)} = \lambda_m^{(i)} = 2^{-i}\Delta\lambda^{(1)}$, $m = 0,2,4,\ldots.$

Let $\delta = \Delta^*\lambda$. Consider $\Delta\lambda < \delta$, let $\left\{\lambda_0,\lambda_2,\lambda_4,\ldots\right\}$ be the partition over $[0,\infty)$ such that $\lambda_{m+2} - \lambda_m = \Delta\lambda$, $m = 0,2,4,\ldots.$ Let K be the least even integer such that $\lambda_K \geqq \lambda_M^*$. We have

$$e^{-\lambda_n^2 \kappa t} < e^{-\left(\lambda_{n+2} - \Delta^*\lambda\right)^2 \kappa t},\ n = 0,2,4,\ldots$$

since $\lambda_{n+2} = \lambda_n + \Delta\lambda$ and $\Delta\lambda < \Delta^*\lambda$. It follows that

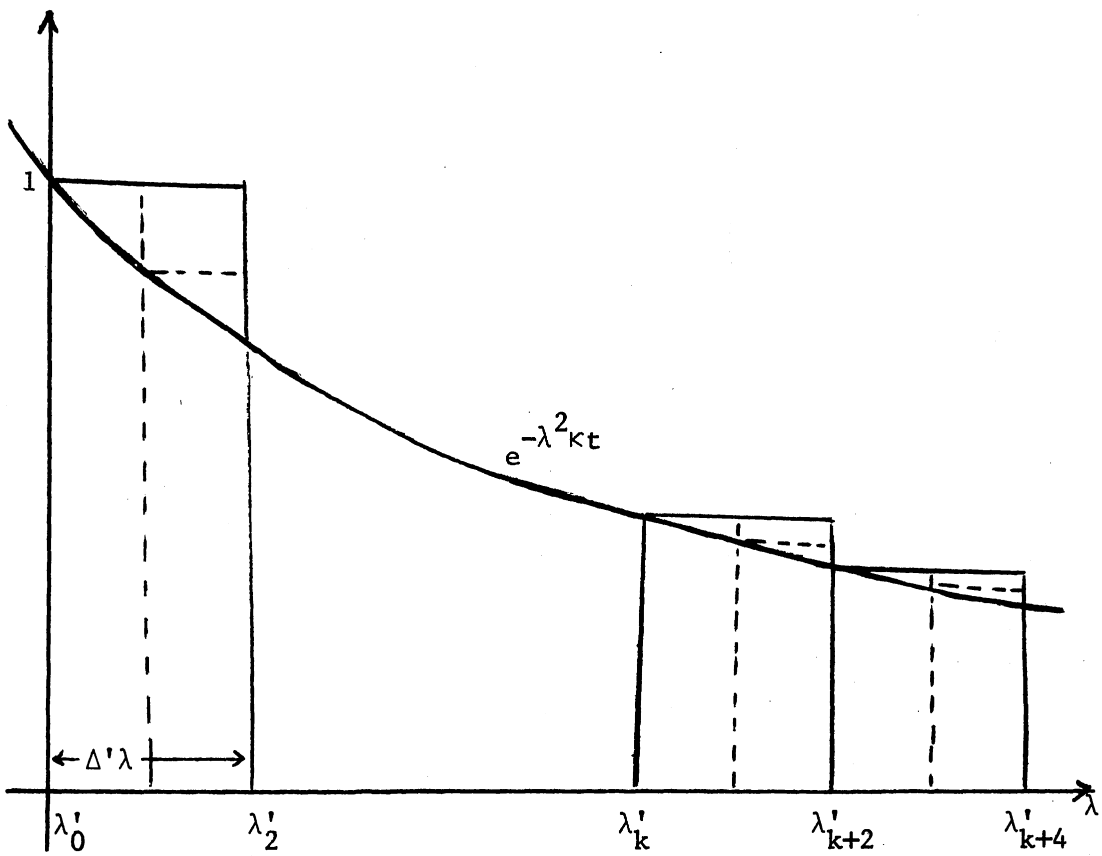


Figure 5.1. Illustration for Inequality (5.1.1)

$$\sum_{n=K+2}^{\infty}{}'' e^{-\lambda_n^2 \kappa t} \Delta\lambda < \int_{\lambda_{K+2}}^{\infty} e^{-(\lambda_n - \Delta^*\lambda)^2 \kappa t} d\lambda \tag{5.1.2}$$

(cf. Figure 5.2). The right-hand side of (5.1.2) equals

$$\int_{\lambda_K}^{\infty} e^{-\lambda^2 \kappa t}\, d\lambda \leqq \sum_{n=M}^{\infty}{}'' e^{-\lambda_n^2 \kappa t} \Delta^*\lambda < \frac{\varepsilon}{12}.$$

Then

$$\left|\sum_{n=K+2}^{\infty}{}'' f_2(\lambda_n)\Delta\lambda\right| \leqq \sum_{n=K+2}^{\infty}{}'' | f_2(\lambda_n)|\Delta\lambda \leqq \sum_{n=K+2}^{\infty}{}'' e^{-\lambda_n^2 \kappa t} \Delta\lambda < \frac{\varepsilon}{12}.$$

Consider

$$\left|\int_{\lambda_{K+2}}^{\infty} f_2(\lambda)d\lambda\right| = \left|\int_{\lambda_K}^{\infty} f_2(\lambda)d\lambda - \int_{\lambda_K}^{\lambda_{K+2}} f_2(\lambda)d\lambda\right|$$

$$\leqq \left|\int_{\lambda_K}^{\infty} f_2(\lambda)d\lambda\right| + \left|\int_{\lambda_K}^{\lambda_{K+2}} f_2(\lambda)d\lambda\right|$$

$$< \frac{\varepsilon}{12} + \max |f_2(\lambda)| \ (\lambda_{K+2} - \lambda_K)$$

$$\leqq \frac{\varepsilon}{12} + \Delta\lambda \leqq \frac{\varepsilon}{12} + \frac{\varepsilon}{12} = \frac{\varepsilon}{6}$$

since $\lambda_K \geqq \lambda_M^* > \lambda^{(1)}$, $\max|f_2(\lambda)| \leqq 1$ and $\Delta\lambda \leqq \Delta^*\lambda \leqq \frac{\varepsilon}{12}$.

On the interval $[0,\lambda_{K+2}]$,

$$\left|\int_0^{\lambda_{K+2}} f_2(\lambda)d\lambda - \sum_{n=0}^{K}{}'' f_2(\lambda_n)\Delta\lambda\right| =$$

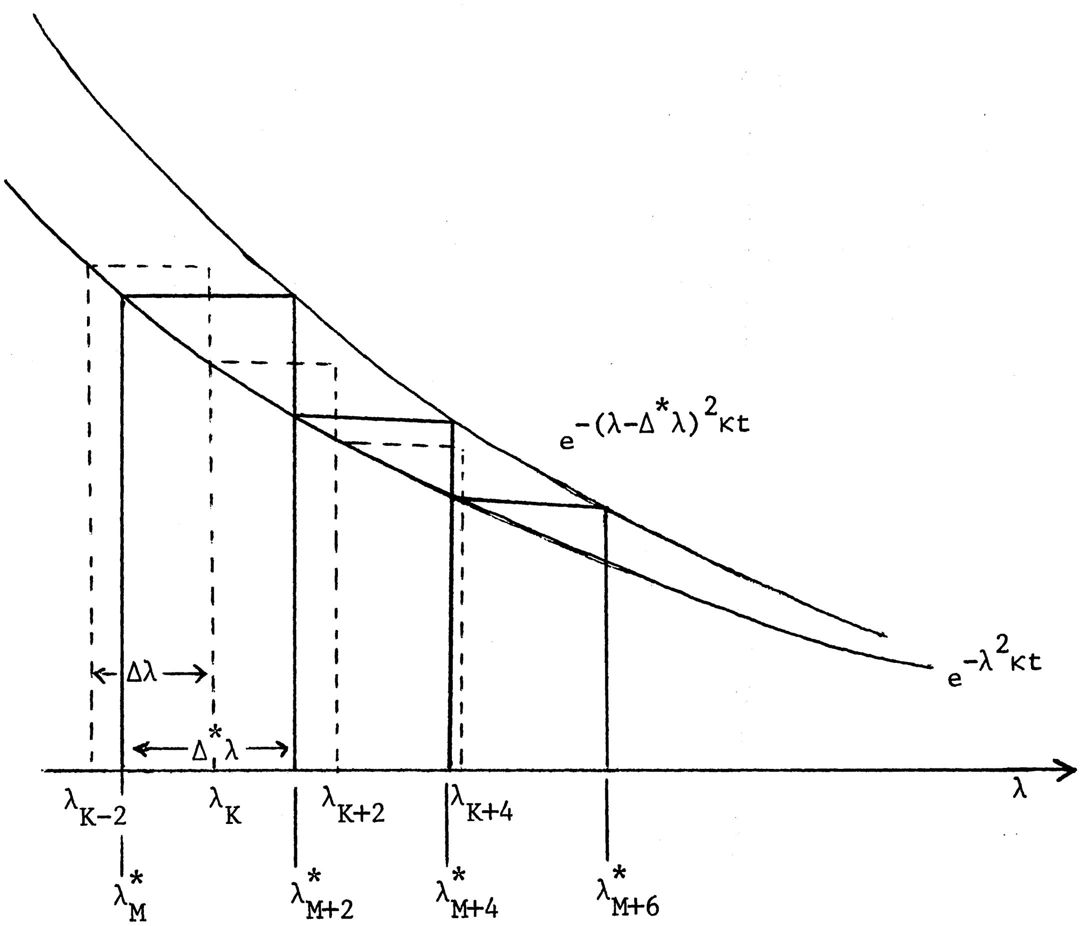


Figure 5.2. Illustration for Inequality (5.1.2)

$$\left| \int_0^{\lambda_M^*} f_2(\lambda)d\lambda + \int_{\lambda_M^*}^{\lambda_{K+2}} f_2(\lambda)d\lambda - \sum_{n=0}^{K-4}{}'' f_2(\lambda_n)\Delta\lambda - f_2(\lambda_{K-2})(\lambda_M^*-\lambda_{K-2}) - \right.$$

$$\left. f_2(\lambda_{K-2})(\lambda_K-\lambda_M^*) - f_2(\lambda_K)\Delta\lambda \right| \leqq$$

$$\left| \int_0^{\lambda_M^*} f_2(\lambda)d\lambda - \sum_{n=0}^{K-4}{}'' f_2(\lambda_n)\Delta\lambda - f_2(\lambda_{K-2})(\lambda_M^*-\lambda_{K-2}) \right| + \left| \int_{\lambda_M^*}^{\lambda_{K+2}} f_2(\lambda)d\lambda \right| +$$

$$\left| f_2(\lambda_K)\Delta\lambda \right| \leqq$$

$$\frac{\varepsilon}{12} + \max |f_2(\lambda)| (\lambda_{K+2}-\lambda_M^*) + |f_2(\lambda_K)|\Delta\lambda \leqq$$

$$\frac{\varepsilon}{12} + (\lambda_{K+2}-\lambda_M^*) + \Delta\lambda \leqq \frac{\varepsilon}{12} + 3\,\Delta\lambda \leqq \frac{\varepsilon}{3}.$$

We have

$$\left| \int_0^{\infty} f_2(\lambda)d\lambda - \sum_{n=0}^{\infty}{}'' f_2(\lambda_n)\Delta\lambda \right| =$$

$$\left| \int_0^{\lambda_{K+2}} f_2(\lambda)d\lambda + \int_{\lambda_{K+2}}^{\infty} f_2(\lambda)d\lambda - \sum_{n=0}^{K}{}'' f_2(\lambda_n)\Delta\lambda - \sum_{n=K+2}^{\infty}{}'' f_2(\lambda_n)\Delta\lambda \right| \leqq$$

$$\left| \int_0^{\lambda_{K+2}} f_2(\lambda)d\lambda - \sum_{n=0}^{K}{}'' f_2(\lambda_n)\Delta\lambda \right| + \left| \int_{\lambda_{K+2}}^{\infty} f_2(\lambda)d\lambda \right| + \left| \sum_{n=K+2}^{\infty}{}'' f_2(\lambda_n)\Delta\lambda \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{12} <$$

ε.

Therefore,

$$\lim_{\Delta\lambda\to 0} \sum_{n=0}^{\infty}{}'' f_2(\lambda_n)\Delta\lambda = \int_0^{\infty} f_2(\lambda)d\lambda.$$

Similarly, we can show that

$$\lim_{\Delta\lambda\to 0} \sum_{n=1}^{\infty}{}' f_1(\lambda_n)\Delta\lambda = \int_0^{\infty} f_1(\lambda)d\lambda.$$

Substituting, we have

$$\lim_{L \to \infty} H(v,v',t) = \frac{1}{\pi}\int_0^{\infty} f_1(\lambda)d\lambda + \frac{1}{\pi}\int_0^{\infty} f_2(\lambda)d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty} e^{-\lambda^2\kappa t} \cos\lambda v \cos\lambda v' d\lambda +$$

$$\frac{1}{\pi}\int_0^{\infty} e^{-\lambda^2\kappa t} \sin\lambda v \sin\lambda v' \, d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty} e^{-\lambda^2\kappa t} \cos\lambda(v'-v)d\lambda. \qquad (5.1.3)$$

Consider $\int_0^{\infty} e^{-\lambda^2\kappa t} \cos\lambda(v'-v)d\lambda$ as the function on the variable $\beta = v'-v$. Let

$$I(\beta) = \int_0^{\infty} e^{-\lambda^2\alpha}\cos\lambda\beta \, d\lambda, \qquad (5.1.4)$$

where $\alpha = \kappa t$. Using Leibnitz's rule,

$$\frac{dI}{d\beta} = \int_0^{\infty} -\lambda e^{-\lambda^2\alpha} \sin\lambda\beta \, d\lambda.$$

Integrate the right-hand side by parts, we have

$$\frac{dI}{d\beta} = \frac{1}{2\alpha} \sin \lambda\beta e^{-\lambda^2\alpha}\Big|_0^{\infty} - \int_0^{\infty} \beta e^{-\lambda^2\alpha}\cos\lambda\beta \, d\lambda$$

$$= -\frac{\beta}{2\alpha} I(\beta) \qquad (5.1.5)$$

The general solution of (5.1.5) is

$$I = Ce^{-\beta^2/4\alpha}. \tag{5.1.6}$$

From (5.1.4),

$$I(0) = \int_0^\infty e^{-\lambda^2\alpha}d\lambda = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\ .$$

Substituting into (5.1.6), we have

$$C = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} = \frac{\sqrt{\pi}}{2\sqrt{\kappa t}}.$$

Therefore,

$$I(\beta) = \frac{\sqrt{\pi}}{2\sqrt{\kappa t}}\ e^{-(v'-v)^2/4\kappa t}. \tag{5.1.7}$$

Define

$$G(v,v',t) = \lim_{L\to\infty} H(v,v',t).$$

Substituting (5.1.7) into (5.1.3), we have

$$G(v,v',t) = \frac{1}{\sqrt{4\pi\kappa t}}\ e^{-(v'-v)^2/4\kappa t}. \tag{5.1.8}$$

This function is called the fundamental solution of the heat equation in one dimension, that is, of

$$u_t = \kappa u_{v'v'},\ -\infty < v < \infty,\ t > 0.$$

The function $G(x,x',t)$ of (5.1.8) has the following properties:

(1). G satisfies the heat equation, that is,

$$G_t = \kappa G_{x'x'},\ -\infty < x < \infty,\ -\infty < x' < \infty,\ t > 0.$$

This property can be shown directly by taking partial derivative.

(2). G satisfies the initial condition, that is,

$$G(x,x',0) = 0 \text{ for } x' \neq x.$$

This property can be shown as follows:

Let $\theta = 1/t$ and $c = (x'-x)^2/4\kappa$, then

$$\lim_{t \to 0} G(x,x',t) = \lim_{\theta\to\infty} \frac{\sqrt{\theta}}{\sqrt{4\pi\kappa}e^{c\theta}}$$

$$= \lim_{\theta\to\infty} \frac{\frac{1}{2}\cdot\frac{1}{\theta}}{\sqrt{4\pi\kappa}c\theta e^{c\theta}}$$

$$= \lim_{\theta\to\infty} \frac{1}{2c\sqrt{4\pi\kappa\theta}e^{c\theta}}$$

$$= 0.$$

Therefore,

$$G(x,x',0) = 0,\ x' \neq x.$$

(3). G has discontinuity at $t = 0$, and $x' = x$, that is,

$$G(x,x,0) = \infty.$$

This property can be shown directly as follows:

Since $x' = x$, then

$$G(x,x,t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-(x-x)^2/4\kappa t}$$

$$= \frac{1}{\sqrt{4\pi\kappa t}}.$$

Therefore,

$$G(x,x,0) = \infty.$$

The expressions for the fundamental solutions of the heat equations in two dimensions and three dimensions are obtained in similar way as the fundamental solution of the heat equation in one dimension. The fundamental solutions of the heat equations in two, three and n-dimensions, respectively, are

$$G(x,y,x',y',t) = \frac{1}{4\pi\kappa t}\, e^{-\frac{(x'-x)^2+(y'-y)^2}{4\kappa t}},$$

$$G(x,y,z,x',y',z',t) = \frac{1}{(4\pi\kappa t)^{3/2}}\, e^{-\frac{(x'-x)^2+(y'-y)^2+(z'-z)^2}{4\kappa t}},$$

and

$$G(\vec{r},\vec{r}',t) = \frac{1}{(4\pi\kappa t)^{n/2}}\, e^{-|\vec{r}'-\vec{r}|^2/4\kappa t}$$

for $n = 1,2,\ldots.$

The fundamental solution, $G(\vec{r},\vec{r}',t)$, of the heat equation in n-dimensions, $n = 2,3,\ldots$ has properties as in one dimension, that is,

(1). G satisfies the heat equation, that is,

$$G_t = \kappa\Delta G,\ -\infty < x_1',\ x_2',\ \ldots,\ x_n' < \infty,\ t > 0$$

where Δ is the Laplace operator with respect to $\vec{r}'$,

(2). G satisfies the initial condition, that is,

$$G(\vec{r},\vec{r}',0) = 0,\ \vec{r}' \neq \vec{r},$$

(3). G has discontinuity at $t = 0$, and $\vec{r}' = \vec{r}$, that is,

$$G(\vec{r},\vec{r},0) = \infty.$$

Let $t' > 0$ and consider

$$G(\vec{r},\vec{r}',t-t') = \frac{1}{[4\pi\kappa(t-t')]^{n/2}}\, e^{-|\vec{r}'-\vec{r}|^2/4\kappa(t-t')},\quad t-t' > 0. \tag{5.1.9}$$

Let $T = t-t'$. Since $G(\vec{r},\vec{r}',T)$ satisfies $G_T = \kappa\Delta G$

and

$$G_T = \frac{\partial G}{\partial t}\cdot\frac{\partial t}{\partial T},\ \text{so}\ G_T = \frac{\partial G}{\partial t}$$

and

$$G_T = \frac{\partial G}{\partial t'} \frac{\partial t'}{\partial T}, \text{ so } G_T = -\frac{\partial G}{\partial t'}.$$

Hence

$$\frac{\partial}{\partial t} G(\vec{r},\vec{r}',t-t') = \kappa\Delta G(\vec{r},\vec{r}',t-t') \tag{5.1.10}$$

and

$$-\frac{\partial}{\partial t'} G(\vec{r},\vec{r}',t-t') = \kappa\Delta G(\vec{r},\vec{r}',t-t'). \tag{5.1.11}$$

5.2. Green's Function for the Whole Space

Suppose we want to find the solution of the heat equation with initial value condition given in the whole space for the case of one dimension. That is, consider

$$\begin{aligned} u_t - \kappa u_{xx} &= F(x,t),\ -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x),\ -\infty < x < \infty. \end{aligned} \tag{5.2.1}$$

From (5.1.7) and (5.1.9), the fundamental solution

$$G(x,x',t-t') = \frac{1}{[4\pi\kappa(t-t')]^{1/2}} e^{-(x'-x)^2/4\kappa(t-t')}$$

satisfies

$$\frac{\partial G}{\partial t'} + \kappa G_{x'x'} = 0,\ t > t'. \tag{5.2.2}$$

Using (5.2.1) with variables x',t' instead of x,t and (5.2.2), we have after multiplying the equations by G and u, respectively, and adding

$$u\frac{\partial G}{\partial t'} + G\frac{\partial u}{\partial t'} = GF + \kappa G u_{x'x'} - u\kappa G_{x'x'}$$

or

$$\frac{\partial}{\partial t'}(uG) = GF + \kappa(Gu_{x'x'} - uG_{x'x'}).$$

Let ε be an arbitrary small positive constant. Then

$$\int_0^{t-\varepsilon}\int_{-\infty}^{\infty}\frac{\partial}{\partial t'}(uG)\,dx'dt' = \int_0^{t-\varepsilon}\int_{-\infty}^{\infty}\left[GF + \kappa(Gu_{x'x'}-uG_{x'x'})\right]dx'dt' \tag{5.2.3}$$

On the left-hand side, we interchange the order of integrations and integrate, that is,

$$\int_{-\infty}^{\infty}\int_0^{t-\varepsilon}\frac{\partial}{\partial t'}(uG)\,dt'dx' = \int_{-\infty}^{\infty}(uG)\Big|_{t'=0}^{t'=t-\varepsilon}dx'$$

$$= \int_{-\infty}^{\infty}G(x,x',\varepsilon)u(x',t-\varepsilon)\,dx' - \int_{-\infty}^{\infty}G(x,x',t)u(x',0)\,dx'$$

$$= \int_{-\infty}^{\infty}G(x,x',\varepsilon)u(x',t-\varepsilon)\,dx' - \int_{-\infty}^{\infty}G(x,x',t)f(x')\,dx',$$

where in the last step the initial condition of (5.2.1) is used. Consider

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}G(x,x',\varepsilon)u(x',t-\varepsilon)\,dx'$$

where

$$G(x,x',\varepsilon) = \frac{1}{(4\pi\kappa\varepsilon)^{1/2}}\,e^{-(x'-x)^2/4\kappa\varepsilon}.$$

Let $\eta^2 = (x'-x)^2/4\kappa\varepsilon$. Therefore, $x' = x + 2\eta\sqrt{\kappa\varepsilon}$. Changing variable of integration from x' to η, we have

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}G(x,x',\varepsilon)u(x',t-\varepsilon)\,dx' = \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\eta^2}u(x+2\eta\sqrt{\kappa\varepsilon},t-\varepsilon)\,d\eta =$$

$$u(x,t)\,\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\eta^2}d\eta = u(x,t). \tag{5.2.4}$$

Integrating

$$\int_{-\infty}^{\infty} (Gu_{x'x'} - uG_{x'x'})\,dx'$$

by parts, gives

$$(Gu_{x'} - uG_{x'})\Big|_{x' = -\infty}^{x' = \infty}$$

Assume u and $u_{x'}$ are bounded functions. Since

$$\lim_{x' \to -\infty} G(x,x',t-t') = 0, \quad \lim_{x' \to \infty} G(x,x',t-t') = 0$$

and

$$\lim_{x' \to -\infty} G_{x'}(x,x',t-t') = 0, \quad \lim_{x' \to \infty} G_{x'}(x,x',t-t') = 0.$$

Therefore,

$$\int_{-\infty}^{\infty} (Gu_{x'x'} - uG_{x'x'})\,dx' = 0 \tag{5.2.5}$$

Substituting (5.2.4), (5.2.5) into (5.2.3), we have

$$u(x,t) = \int_{-\infty}^{\infty} f(x')G(x,x',t)\,dx' + \int_{0}^{t}\int_{-\infty}^{\infty} G(x,x',t-t')F(x',t')\,dx'dt' \tag{5.2.6}$$

Similarly, the solution of the heat equation with initial condition for the whole space in two dimensions is

$$u(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')G(x,y,x',y',t)\,dx'dy' + \int_{0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,x',y',t-t')F(x',y',t')\,dx'dy'dt' \tag{5.2.7}$$

where

$$G(x,y,x',y',t-t') = \frac{1}{4\pi\kappa(t-t')} e^{-[(x'-x)^2+(y'-y)^2]/4\kappa(t-t')}.$$

Also, in n-dimensions

$$u(\vec{r},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\vec{r}')G(\vec{r},\vec{r}',t)\,dx_1'dx_2'\ldots dx_n' +$$

$$\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} G(\vec{r},\vec{r}',t-t')F(\vec{r}',t')\,dx_1'dx_2'\ldots dx_n'dt' \tag{5.2.8}$$

where

$$G(\vec{r},\vec{r}',t) = \frac{1}{(4\pi\kappa t)^{n/2}} e^{-|\vec{r}'-\vec{r}|^2/4\kappa t},$$

$$\vec{r} = (x_1,x_2,\ldots,x_n) \text{ and } \vec{r}' = (x_1',x_2',\ldots,x_n')$$

is the solution of

$$\frac{\partial u}{\partial t} - \kappa\Delta u = F(\vec{r},t),\ -\infty < x_1,x_2,\ldots,x_n < \infty,\ t > 0,$$

$$u(\vec{r},0) = f(\vec{r}),\ -\infty < x_1,x_2,\ldots,x_n < \infty.$$

Here Δ is the Laplace operator with respect to the variables $x_1,x_2,\ldots,x_n$. The function $G(\vec{r},\vec{r}',t)$ is the Green's function of the given problem.

<u>Futher comments on Solutions</u>

In the case where $F(\vec{r},t) = 0$, we will have the homogeneous heat equation with the given initial condition, that is,

$$u_t - \kappa\Delta u = 0,\ -\infty < x_1,x_2,\ldots,x_n < \infty,\ t > 0,$$

$$u(\vec{r},0) = f(\vec{r}),\ -\infty < x_1,x_2,\ldots,x_n < \infty.$$

Green's function for the problem is

$$G(\vec{r},\vec{r}',t) = \frac{1}{(4\pi\kappa t)^{n/2}} e^{-|\vec{r}'-\vec{r}|^2/4\kappa t},$$

and the solution from (5.2.8) is

$$u(\vec{r},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\vec{r}')G(\vec{r},\vec{r}',t)\,dx_1'dx_2'\ldots dx_n \tag{5.2.9}$$

For example, in one dimension, by using (5.2.9)

$$u(x,t) = \int_{-\infty}^{\infty} f(x')G(x,x',t)\,dx'$$

where

$$G(x,x',t) = \frac{1}{(4\pi\kappa t)^{1/2}} e^{-(x'-x)^2/4\kappa t}$$

is the solution of

$$u_t - \kappa u_{xx} = 0, \; -\infty < x < \infty, \; t > 0,$$

$$u(x,0) = f(x), \; -\infty < x < \infty.$$

In the other case, when $f(\vec{r}) = 0$, we will have the nonhomogeneous heat equation with the zero initial condition. Therefore, by using (5.2.8)

$$u(\vec{r},t) = \int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} G(\vec{r},\vec{r}',t-t')F(\vec{r}',t')\,dx_1'dx_2'\ldots dx_n'dt' \tag{5.2.10}$$

is the solution of

$$u_t - \kappa\Delta u = F(\vec{r},t), \; -\infty < x_1,x_2,\ldots,x_n < \infty, \; t > 0,$$

$$u(\vec{r},0) = 0, \; -\infty < x_1,x_2,\ldots,x_n < \infty.$$

For example, in one dimension, by using (5.2.10)

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,x',t-t')F(x',t')\,dx'dt'$$

where

$$G(x,x',t) = \frac{1}{(4\pi\kappa t)^{1/2}} e^{-(x'-x)^2/4\kappa t}$$

is the solution of

$$u_t - \kappa u_{xx} = F(x,t),\ -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = 0,\ -\infty < x < \infty.$$

5.3. Green's Function for a Half-Space

Suppose we want to find the solution of the heat equation with a Dirichlet boundary condition and an initial condition on a semi-infinite straight line. That is, we want to solve the problem

$$u_t - \kappa u_{xx} = F(x,t),\ 0 < x < \infty,\ t > 0, \tag{5.3.1}$$
$$u(0,t) = h(t),\ t > 0,$$
$$u(x,0) = f(x),\ 0 < x < \infty.$$

Green's function $w(x,x',t-t')$ for the problem has the following properties:

(1). w satisfies the equation

$$w_{t'} + \kappa w_{x'x'} = 0,\ 0 < x,x' < \infty,\ t,t' > 0,$$

(2). w satisfies the zero boundary condition and the zero initial condition, that is,

$$w(x,0,t-t') = 0,$$

and

$$w(x,x',0) = 0$$

when $x' \neq x$,

(3). w is continuous everywhere, $x \geqq 0$, $t \geqq t'$, at $t' = t$ and $x' = x$ where w has an infinite discontinuity. In particular,

$$w(x,x,0) = \infty.$$

Define

$$w(x,x',t-t') = G(x,x',t-t') - G(-x,x',t-t') \tag{5.3.2}$$

where G is the function (5.1.6). Notice that we used -x, the image of x across the boundary x = 0, in defining the function w. The function w has property (1) since G(x,x',t-t') and G(-x,x',t-t') have property (1). To show w has property (2), we have

$$w(x,x',t-t') = \frac{1}{\sqrt{4\pi\kappa(t-t')}}\left[e^{-(x'-x)^2/4\kappa(t-t')} - e^{-(x'+x)^2/4\kappa(t-t')}\right]. \tag{5.3.3}$$

Substituting x' = 0, then w(x,0,t-t') = 0. We have G(x,x',0) = 0, when $x' \neq x$ and G(-x,x',0) = 0 when $x' \neq -x$. It follows that w(x,x',0) = 0 when $x' \neq x$. Therefore, w satisfies (2).

Since G(x,x',t-t') is continuous everywhere except when x' = x, t = t' where $G \to \infty$ it follows that w has property (3).

The solution to the problem is

$$u(x,t) = \int_0^\infty f(x')w(x,x',t)\,dx' + \kappa\int_0^t h(t')w_{x'}(x,0,t-t')\,dt' + \int_0^t\int_0^\infty F(x',t')w(x,x',t-t')\,dx'dt'. \tag{5.3.4}$$

The solution can be derived as follows: Using (5.3.1) with variables x',t' instead of x,t, respectively, and (5.3.2), we have

$$\frac{\partial}{\partial t'}(uw) = uw_{t'} + wu_{t'} = wF + \kappa(wu_{x'x'} - uw_{x'x'}).$$

Let ε be an arbitrary small positive number. Then

$$\int_0^{t-\varepsilon}\int_0^\infty \frac{\partial}{\partial t'}(uw)\,dx'dt' = \int_0^{t-\varepsilon}\int_0^\infty\left[wF + \kappa(wu_{x'x'}-uw_{x'x'})\right]dx'dt'$$

Interchanging the order of integrations, the left-hand side is

$$\int_0^\infty uw\Big|_{t'=0}^{t'=t-\varepsilon} dx' = \int_0^\infty u(x',t-\varepsilon)w(x,x',\varepsilon)\,dx' - \int_0^\infty f(x')w(x,x',t)\,dx' =$$

$$\int_0^\infty u(x',t-\varepsilon)\,G(x,x',\varepsilon)\,dx' - \int_0^\infty u(x',t-\varepsilon)\,G(-x,x',\varepsilon)\,dx' -$$

$$\int_0^\infty f(x')w(x,x',t)\,dx'.$$

Let $\eta = (x'-x)/\sqrt{4\kappa t}$, we have

$$\lim_{\varepsilon\to 0}\int_0^\infty u(x',t-\varepsilon)\,G(x,x',\varepsilon)\,dx' = \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_{\frac{-x}{2\sqrt{\kappa\varepsilon}}}^{\infty} e^{-\eta^2}\,u(x+2\eta\sqrt{\kappa\varepsilon},t-\varepsilon)\,d\eta$$

$$= u(x,t)$$

(cf. Section 5.2). Let $\beta = (x+x')/\sqrt{4\kappa t}$, we have

$$\lim_{\varepsilon\to 0}\int_0^\infty u(x',t-\varepsilon)\,G(-x,x',\varepsilon)\,dx' = \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\kappa\varepsilon}}}^{\infty} e^{-\beta^2}\,u(-x+2\beta\sqrt{\kappa\varepsilon},t-\varepsilon)\,d\beta$$

$$= 0.$$

Therefore, as $\varepsilon \to 0$ the left-hand side becomes

$$u(x,t) - \int_0^\infty f(x')w(x,x',t)\,dx'$$

Integrating

$$\int_0^\infty (wu_{x'x'} - uw_{x'x'})\,dx' = (wu_{x'} - uw_{x'})\Big|_0^\infty$$

$$= -w(x,0,t-t')u_{x'}(0,t') + u(0,t')w_{x'}(x,0,t-t') +$$

$$w(x,\infty,t-t')u_{x'}(\infty,t') - u(\infty,t')w_{x'}(x,\infty,t-t')$$

$$= u(0,t')w_{x'}(x,0,t-t')$$

As $x' \to \infty$, $w(x,\infty,t-t') = 0$, and $w_{x'}(x,\infty,t-t') = 0$. Assume u and u_x be bounded at $x = \infty$. Hence, as $\varepsilon \to 0$, the right-hand side is

$$\kappa\int_0^t h(t')w_{x'}(x,0,t-t')dt' + \int_0^t \int_0^\infty wF\, dx'dt'.$$

Hence,

$$u(x,t) = \int_0^\infty f(x')w(x,x',t)dx' + \kappa\int_0^t h(t')w_{x'}(x,0,t-t')dt' + \int_0^t \int_0^\infty w(x,x',t-t')F(x',t')dx'dt'.$$

which is (5.3.4).

In the case of Neumann condition, $\frac{\partial}{\partial n} u(0,t) = h(t)$, then Green's function, $w(x,x',t-t')$ has property (1), (3) and instead of property (2) (2'). The function w satisfies the boundary condition, that is,

$$\frac{\partial}{\partial n} w(x,0,t-t') = w_{x'}(x,0,t-t') = 0.$$

In this case define

$$w(x,x',t-t') = G(x,x',t-t') + G(-x,x',t-t').$$

Then w has property (1), (2') and (3).

$$u(x,t) = \int_0^\infty f(x')w(x,x',t)dx' - \kappa\int_0^t h(t')w(x,0,t-t')dt' + \int_0^t \int_0^\infty w(x,x',t-t')F(x',t')dx'dt'. \qquad (5.3.5)$$

The solution can be derived similarly to (5.3.4).

We will see that the solution (5.3.4) and (5.3.5) can be applied to the problem which has $F(x,t) = 0$ or $h(t) = 0$ or $f(x) = 0$ but not all of them are zero as is shown in the following examples:

Example 5.1. Assume that a homogeneous conductor occupies the semi-infinite rod $x \geq 0$. The temperature at $x = 0$ is zero at time $t > 0$. The initial temperature in the rod is u_0 (u_0 a positive constant). Assume there are no heat sources within the rod. Determine the temperature.

In this case $F(x,t) = 0$, $h(t) = 0$ and $f(x) = u_0$. Applying (5.3.4), we have

$$u(x,t) = u_0 \int_0^\infty w(x,x',t)\,dx'$$

$$= \frac{u_0}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{4\kappa t}} \left[e^{-(x'-x)^2/4\kappa(t-t')} - e^{-(x'+x)^2/4\kappa(t-t')} \right].$$

Let $\eta = (x'-x)/\sqrt{4\kappa(t-t')}$ and $\beta = (x'+x)/\sqrt{4\kappa(t-t')}$. Then

$$u(x,t) = \frac{u_0}{\sqrt{\pi}} \left[\int_{\frac{-x}{\sqrt{4\kappa t}}}^\infty e^{-\eta^2} d\eta - \int_{\frac{x}{\sqrt{4\kappa t}}}^\infty e^{-\beta^2} d\beta \right]$$

$$= \frac{u_0}{2} \left[\operatorname{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right) - \operatorname{erf}\left(\frac{-x}{\sqrt{4\kappa t}}\right) \right]$$

$$= u_0 \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right)$$

where the error function is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2} d\eta, \quad 0 \leq x < \infty.$$

We shall now apply (5.3.4) to the homogeneous heat equation with a nonzero Dirichlet boundary condition and a zero initial condition.

Example 5.2. Given the same conditions as in Example 5.1 except that the temperature at $x = 0$ is u_1 (u_1 a nonzero constant) and the initial temperature is zero.

In this case $h(t) = u_1$ and $f(x) = 0$. Applying (5.3.4), we have

$$u(x,t) = \kappa u_1 \int_0^t w_{x'}(x,0,t-t')\,dt'$$

$$= \kappa u_1 \int_0^t \frac{x}{\kappa(t-t')\sqrt{4\pi\kappa(t-t')}}\, e^{-x^2/4\kappa(t-t')}\,dt'.$$

Let $\eta = x/\sqrt{4\kappa(t-t')}$. Then

$$u(x,t) = \frac{2u_1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} e^{-\eta^2}\,d\eta$$

$$= u_1\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right)\right].$$

The next example is again dealing with the homogeneous heat equation with a nonzero Dirichlet boundary condition but the initial condition is also not zero.

Example 5.3. Given the same conditions as in Example 5.1 but with the temperature at $x = 0$ is u_1 for $t > 0$ and the initial temperature is u_2 (u_1, u_2 are nonzero constants).

This is the combination problem of Example 5.1 and 5.2. We have $F(x,t) = 0$, $h(t) = u_1$ and $f(x) = u_2$. Applying (5.3.4), then

$$u(x,t) = u_2 \int_0^{\infty} w(x,x',t)\,dx' + \kappa u_1 \int_0^{t} w_{x'}(x,0,t-t')\,dt'$$

$$= u_0 \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right) + u_1\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right)\right]$$

$$= (u_2 - u_1) \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right) + u_1$$

(cf. Examples 5.1 and 5.2).

The following example will show the application of (5.3.5) to the homogeneous heat equation with a nonzero Neumann boundary condition and a zero initial condition.

Example 5.4. Find the solution of the equation

$$u_t - \kappa\Delta u = 0,\ 0 < x < \infty,\ t > 0$$

satisfying the initial condition

$$u(x,0) = 0$$

and the Neumann boundary condition

$$\frac{\partial}{\partial n} u(x,t)\Big|_{x=0} = u_0 \quad (u_0 \text{ a nonzero constant}).$$

We have $F(x,t) = 0$, $f(x) = 0$ and $h(t) = u_0$. Applying (5.3.5), then

$$u(x,t) = -\kappa \int_0^{t} u_0 w(x,0,t-t')\,dt'$$

where

$$w(x,x',t-t') = \frac{1}{\sqrt{4\pi\kappa(t-t')}}\left[e^{-(x'-x)^2/4\kappa(t-t')} + e^{-(x'+x)^2/4\kappa(t-t')}\right].$$

Therefore

$$u(x,t) = -\frac{2\kappa u_0}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{4\kappa(t-t')}} e^{-x^2/4\kappa(t-t')} dt'.$$

Let $\eta = x/\sqrt{4\kappa(t-t')}$. Then

$$u(x,t) = -\frac{u_0 x}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} \frac{1}{\eta^2} e^{-\eta^2} d\eta$$

$$= \frac{u_0 x}{\sqrt{\pi}} \left[\frac{e^{-\eta^2}}{\eta} \Bigg|_{\eta = \frac{x}{\sqrt{4\kappa t}}}^{\eta = \infty} + 2 \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} e^{-\eta^2} d\eta \right]$$

$$= \frac{u_0 x}{\sqrt{\pi}} \left[-\frac{\sqrt{4\kappa t}}{x} e^{-x^2/4\kappa t} + \sqrt{\pi} - \sqrt{\pi}\, \mathrm{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right) \right].$$

Now, suppose we want to find the solution of the heat equation with Dirichlet boundary condition and an initial condition for a half space in n-dimensions, $n = 2,3,\ldots$. That is, we want to solve the problem,

$$u_t - \kappa\Delta u = F(\vec{r},t),\ \vec{r} \in \mathcal{R},\ t > 0,$$
$$u(\vec{r},t) = h(\vec{r},t),\ \vec{r} \in S,\ t > 0,$$
$$u(\vec{r},0) = f(\vec{r}),\ \vec{r} \in \mathcal{R}$$

where $\vec{r} = (x_1,x_2,\ldots,x_n)$,

$$\mathcal{R} = \{(x_1,x_2,\ldots,x_n);\ 0 < x_1 < \infty,\ -\infty < x_2,x_3,\ldots,x_n < \infty\},$$

and S is the boundary of $\mathcal{R}$.

Green's function $w(\vec{r},\vec{r}',t-t')$ for the problem has the following properties:

(1). w satisfies the equation $w_{t'} + \kappa\Delta w = 0$ where Δ is an operator in the prime variables,

(2). w satisfies the zero Dirichlet boundary condition and zero initial condition, that is,

$$w(\vec{r},\vec{r}',t-t') = 0,\ \vec{r}' \in S,\ \vec{r} \in \mathcal{R},\ t-t' > 0,$$

$$w(\vec{r},\vec{r}',0) = 0,\ \vec{r},\ \vec{r}' \in \mathcal{R},$$

(3). w is continuous everywhere in $\mathcal{R}$ and $t \geq t'$ except at $t' = t$, $\vec{r}' = \vec{r}$ where w has an infinite discontinuity. In particular,

$$w(\vec{r},\vec{r},0) = \infty.$$

Define

$$w(\vec{r},\vec{r}',t-t') = G(\vec{r},\vec{r}',t-t') - G_1(\vec{r}_1,\vec{r}',t-t')$$

where $\vec{r}_1$ is the image of $\vec{r}$ across the boundary $x_1 = 0$, and $G(\vec{r},\vec{r}',t-t')$ is the function in equation (5.1.9). One can show that w has properties (1), (2) and (3). To find the solution $u(\vec{r},t)$, we have

$$\int_0^{t-\varepsilon}\int_{\mathcal{R}} (wu_{t'} + uw_{t'})dV'dt' = \int_0^{t-\varepsilon}\int_{\mathcal{R}} \left[wF + \kappa(w\Delta u - u\Delta w)\right]dV'dt\ .$$

Applying Green's second identity and the method used in one dimension, obtain

$$u(\vec{r},t) = \int_{\mathcal{R}} f(\vec{r}')w(\vec{r},\vec{r}',t)dV' - \kappa\int_0^t\int_S h(\vec{r}',t')\ \frac{\partial}{\partial n}\ w(\vec{r},\vec{r}',t-t')\Big|_{\vec{r}' \in S} dS'dt' +$$

$$\int_0^t\int_{\mathcal{R}} w(\vec{r},\vec{r}',t-t')F(\vec{r}',t')dV'dt'. \qquad (5.3.6)$$

Similarly, in the case of the Neumann boundary condition,

$$\frac{\partial}{\partial n}\ u(\vec{r},t) = h(\vec{r},t),\ \vec{r} \in S,\ t > 0.$$

Green's function is

$$w(\vec{r},\vec{r}',t-t') = G(\vec{r},\vec{r}',t-t') + G(\vec{r}_1,\vec{r}',t-t')$$

which has properties (1), (3) and

(2'). $\frac{\partial}{\partial n} w(\vec{r},\vec{r}',t-t')\Big|_{\vec{r}'\varepsilon S} = 0.$

The solution to the problem is

$$u(\vec{r},t) = f(\vec{r}')w(\vec{r},\vec{r}',t)dV' + \kappa\int_0^t \int_S h(\vec{r}',t')w(\vec{r},\vec{r}',t-t')\Big|_{\vec{r}'\varepsilon S} dS'dt' +$$

$$\int_0^t \int_{\mathcal{R}} F(\vec{r}',t')w(\vec{r},\vec{r}',t-t')dV'dt'. \qquad (5.3.7)$$

Particularly in two dimensions, suppose we want to solve the problem in a half-plane,

$$u_t(x,y,t) - \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0,\ 0 < x < \infty,\ -\infty < y < \infty,\ t > 0,$$

$$u(0,y,t) = h(y,t),\ -\infty < y < \infty,\ t > 0,$$

$$u(x,y,0) = f(x,y),\ 0 < x < \infty,\ -\infty < y < \infty.$$

Green's function for the problem is

$$w(x,y,x',y',t-t') = G(x,y,x',y',t-t') - G(-x,y,x',y',t-t')$$

$$= \frac{1}{4\pi\kappa(t-t')} e^{-[(x'-x)^2+(y'-y)^2]/4\kappa(t-t')} -$$

$$\frac{1}{4\pi\kappa(t-t')} e^{-[(x'+x)^2+(y'-y)^2]/4\kappa(t-t')}$$

which has the following properties:

(1). $w_{t'} + \kappa\left(\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w}{\partial y'^2}\right) = 0,$

(2). $w(x,y,0,y',t-t') = 0$, and $w(x,y,x',y',0) = 0$,

(3). w is continuous everywhere in the half-plane except at $t' = t$ and $(x',y') = (x,y)$ where w has an infinite discontinuity. That is, $w(x,y,x,y,0) = \infty$.

Now,

$$\frac{\partial w}{\partial n} = -\left.\frac{\partial w}{\partial x'}\right|_{x' = 0}$$

$$= -\frac{e^{-[x^2+(y'-y)^2]/4\kappa(t-t')}}{4\pi\kappa^2(t-t')^2}.$$

Applying (5.3.6), the solution is

$$u(x,y,t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} f(x',y')w(x,y,x',y',t)dx'dy' +$$

$$\kappa\int_{0}^{t}\frac{1}{4\pi\kappa^2(t-t')^2}\int_{-\infty}^{\infty} h(y',t')\, e^{-[x^2+(y'-y)^2]/4\kappa(t-t')}dy'dt'.$$

5.4. The Method of Images

Finding Green's function for the heat equation by the method of images is similar to the method of finding Green's function by the method of images for Laplace's equation and Helmholtz's equation (cf. Sections 3.3, 3.5, and 4.2). The next example will show how to find Green's function by the method of images for the heat equation in a finite interval.

<u>Example 5.5.</u> Find Green's function for the heat equation

$$u_t - \kappa u_{xx} = F(x,t),\ 0 < x < a,\ t > 0$$

satisfying the initial condition $u(x,0) = f(x)$, and the Dirichlet boundary condition $u(0,t) = h_1(t)$ and $u(a,t) = h_2(t)$, $t > 0$.

Green's function, $w(x,x',t-t')$ of the problem has the following properties:

(1). $w_{t'} + \kappa w_{x'x'} = 0$, $0 < x, x' < a$, $t > t'$,

(2). w satisfies the zero initial condition and the zero boundary condition, that is, $w(x,x',0) = 0$ and $w(x,0,t-t') = w(x,a,t-t')=0$,

(3). w is continuous everywhere, $0 < x,x' < a$, $t > t'$ except at $t' = t$ and $x' = x$ where x has an infinite discontinuity. In particular, $w(x,x,0) = \infty$.

The images of x across x = 0 and x = a are shown in Figure 5.3.

Define

$$w(x,x',t-t') = \sum_{n=-\infty}^{\infty} \left[G(2na+x,x',t-t') - G(2na-x,x',t-t')\right]$$

where G is the fundamental solution,

$$G(x,x',t-t') = \frac{1}{\sqrt{4\pi\kappa(t-t')}}\, e^{-(x'-x)^2/4\kappa(t-t')} .$$

It is not difficult to show that w has properties (1), (2) and (3). The solution to the problem is

$$u(x,t) = \int_0^t \int_0^a F(x',t')w(x,x',t-t')dx'dt' - \kappa\int_0^t \left[h_1(t') \left.\frac{\partial w}{\partial n}\right|_{x'=0} + h_2(t') \left.\frac{\partial w}{\partial n}\right|_{x'=a}\right]dt' + \int_0^a f(x')w(x,x',t)dx' \qquad (5.4.1)$$

The derivation of the solution can be done similarly to the derivation of the solution for a semi-infinite straight line (cf. Section 5.3).

The other form of Green's function for this problem is

$$w(x,x',t-t') = \frac{2}{a}\sum_{n=1}^{\infty} e^{-(\pi n/a)^2\kappa(t-t')} \sin\frac{n\pi x'}{a} \sin\frac{n\pi x}{a}$$

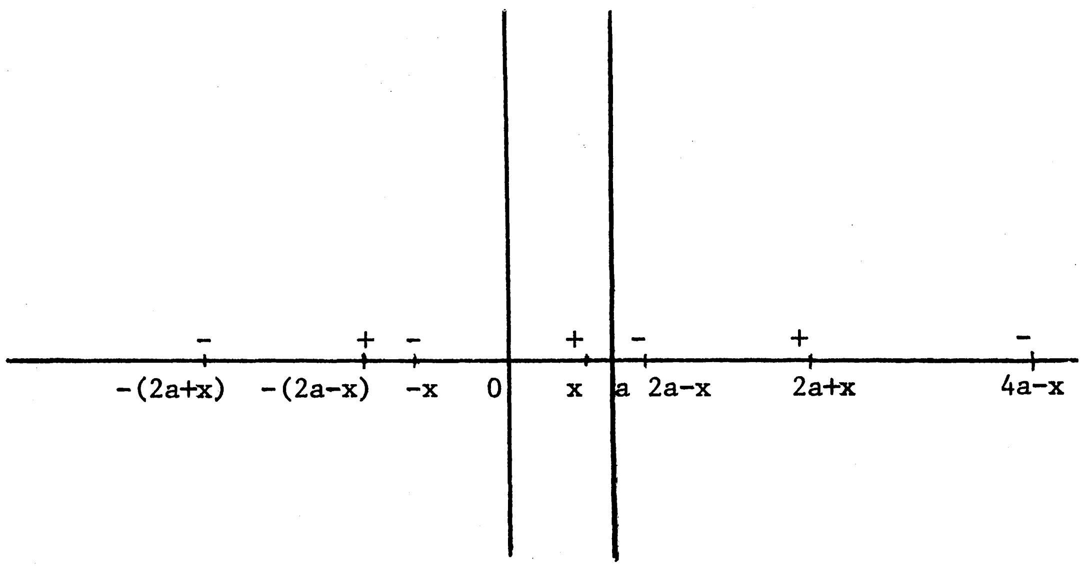


Figure 5.3. Images of x Across x = 0 and x = a

which is obtained by using the method of eigenfunctions. This method will be discussed in the next section.

In the case of the Neumann boundary condition,

$$\left.\frac{\partial u}{\partial n}\right|_{x=0} = h_1(t) \text{ and } \left.\frac{\partial u}{\partial n}\right|_{x=a} = h_2(t),$$

Green's function has properties (1), (3) and (2') w satisfies the zero boundary condition, that is, $\left.\frac{\partial w}{\partial n}\right|_{x'=0} = \left.\frac{\partial w}{\partial n}\right|_{x'=a} = 0$ and the zero initial condition, $w(x,x',0) = 0$. If we define

$$w(x,x',t-t') = \sum_{n=-\infty}^{\infty}\left[G(2na+x,x',t-t') + G(2na-x,x',t-t')\right]$$

then w has properties (1), (2') and (3). The solution is

$$u(x,t) = \int_0^t \int_0^a F(x',t')w(x,x',t-t')dx'dt' + \kappa\int_0^t \Big[h_1(t')w(x,0,t-t') +$$

$$h_2(t')w(x,a,t-t')\Big]dt' + \int_0^a f(x')w(x,x',t)dx'. \quad (5.4.2)$$

Following Example 5.5, we can find Green's function for a region which is an intersection of parallel strips (cf. Figure 3.6). The method of images can be used for the region whose boundary is composed of pairs of parallel lines as it has been used in the preceding examples. Again, the next example shows how to find Green's function for the heat equation in a quarter-plane.

Example 5.6. Suppose we want to solve the problem of the heat equation

$$u_t - \kappa\Delta u = F(x,y,t),\ 0 < x < \infty,\ 0 < y < \infty,\ t > 0$$

satisfying the initial condition $u(x,y,0) = f(x,y)$ and the Dirichlet boundary condition $u(0,y,t) = h_1(y,t)$ and $u(x,0,t) = h_2(x,t)$.

Green's function $w(x,y,x',y',t-t')$ has the following properties:

(1). $w_{t'} + \kappa\Delta w = 0$ where Δ is an operator with respect to the prime variables,

(2). w satisfies the zero initial condition and the zero boundary condition, that is, $w(x,y,x',y',0) = 0$ and $w(x,y,0,y',t-t') = w(x,y,x',0,t-t') = 0$,

(3). w is continuous everywhere, $0 < x,x' < \infty$, $0 < y,y' < \infty$, $t \geq t'$ except at $t' = t$ and $(x',y') = (x,y)$ where w has an infinite discontinuity.

The images of $P(x,y)$ across $x = 0$ and $y = 0$ are shown in Figure 3.2. Define

$$w(x,y,x',y',t-t') = G(P,Q,t-t')-G(P_1,Q,t-t')+G(P_2,Q,t-t')-G(P_3,Q,t-t')$$

where G is the fundamental solution

$$G(P,Q,t-t') = \frac{1}{4\pi\kappa(t-t')} e^{-QP^2/4\kappa(t-t')} .$$

One can show that w has properties (1), (2) and (3). Applying (5.3.6), the solution to the problem is

$$u(x,y,x',y',t) = \int_0^\infty \int_0^\infty f(x',y')w(x,y,x',y',t)dx'dy' -$$

$$\kappa\int_0^t \int_0^\infty h_1(y',t') \frac{\partial}{\partial n} w(x,y,x',y',t-t')\Big|_{x'=0} dy'dt' -$$

$$\kappa\int_0^t \int_0^\infty h_2(x',t')\frac{\partial}{\partial n} w(x,y,x',y',t-t')\Big|_{y'=0} dx'dt' +$$

$$\int_0^t \int_0^\infty \int_0^\infty w(x,y,x',y',t-t')F(x',y',t')dx'dy'dt'.$$

The next example will show how to determine Green's function for a radially dependent problem.

<u>Example 5.7.</u> Let (r,θ,ϕ) be sperical coordinates. A temperature distribution $u(r,t)$ which is purely radially dependent satisfies

$$u_t - \frac{\kappa(r^2 u_r)_r}{r^2} = F(r,t), \quad r > 0, \quad t > 0$$

and the initial condition $u(r,0) = f(r)$ for $r > 0$.

Let $v(r,t) = ru(r,t)$, then

$$v_t - \kappa v_{rr} = rF(r,t), \quad r > 0, \quad t > 0,$$

$$v(0,t) = 0, \quad t > 0,$$

$$v(r,0) = rf(r), \quad r > 0.$$

This is a kind of problem like (5.3.1). Therefore, applying (5.3.4) and using $v = ru$, the solution is

$$u(r,t) = \frac{1}{r}\int_0^\infty r'f(r')w(r,r',t)dr' + \frac{1}{r}\int_0^t \int_0^\infty r'F(r',t')w(r,r',t-t')dr'dt' \tag{5.4.3}$$

where $w(r,r',t) = G(r,r',t) - G(-r,r',t)$ and $G(r,r',t)$ is the fundamental solution for one dimension. Using (5.3.6) for the region $\mathcal{R} = \left\{(r,\theta,\phi);\ 0 < r < \infty,\ 0 < \theta < \pi,\ 0 < \phi < 2\pi\right\}$, the solution is

$$u(r,t) = \int_{\mathcal{R}} f(r')H(r,r',t)dV' + \int_0^t \int_{\mathcal{R}} F(r',t')H(r,r',t-t')dV'dt'$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} f(r')H(r,r',t)r'^2 \sin\theta' d\phi' d\theta' dr' +$$

$$\int_0^t \int_0^\infty \int_0^\pi \int_0^{2\pi} F(r',t')H(r,r',t-t')r'^2 \sin\theta' d\phi' d\theta' dr' dt'$$

$$= \int_0^\infty 4\pi r'^2 f(r')H(r,r',t)dr' + \int_0^t \int_0^\infty 4\pi r'^2 F(r',t')H(r,r',t-t')dr'dt' \qquad (5.4.4)$$

where $H(r,r',t-t')$ is the Green's function for the problem. Comparing (5.4.4) to (5.4.3), we find that

$$H(r,r',t-t') = \frac{w(r,r',t-t')}{4\pi r r'}.$$

In the case of a disk or a sector of a disk, Green's function cannot be determined by using the method of images. The circumstance is different from that of Laplace's equation and Helmholtz's equation since Kelvin's theorem is not applicable. Green's function can be found by using Laplace transforms [20, p. 297] or by using the method of eigenfunctions which is mentioned in the following section.

5.5. Green's Function in Terms of Eigenfunctions

We have used the method of images to find Green's functions in the preceding sections. We are now obtaining Green's functions in terms of eigenfunctions for the heat equation of the function $u(\vec{r},t)$ in the region $\mathcal{R}$ of the boundary S, with the boundary condition $B(u) = h(\vec{r},t)$, the Dirichlet boundary condition or the Neumann boundary condition. Green's function is

$$w(\vec{r},\vec{r}',t-t') = \sum_{n=1}^{\infty} \frac{\psi_n(\vec{r})\psi_n(\vec{r}')}{|\psi_n|^2} e^{-\kappa\lambda_n(t-t')}, \quad t-t' > 0 \qquad (5.5.1)$$

where λ_n and ψ_n are, respectively, eigenvalues and corresponding eigenfunctions of the associated eigenvalue problem,

$$\Delta\psi + \lambda\psi = 0 \text{ in } \mathcal{R},$$

$$B[\psi(\vec{r})] = 0 \text{ on } S$$

[4, pp. 297-298].

Table III provides Green's functions in terms of eigenfunctions for the heat equation with the Dirichlet boundary condition in a semi-circle, a quarter-circle, and $\left\{(x,y,z)\colon 0 < x < a,\ 0 < y < b,\ 0 < z < c\right\}$. Green's functions are obtained by finding eigenfunctions and eigenvalues of the associated eigenvalue problems then using (5.5.1) to write out Green's functions.

TABLE III

GREEN'S FUNCTION FOR THE HEAT EQUATION WITH THE DIRICHLET BOUNDARY CONDITION

Region	Green's Function	References
$\left\{re^{i\theta}: 0 < r < a,\ 0 < \theta < \pi\right\}$	$\frac{4}{\pi a^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin n\theta\, J_n(\xi_{mn}r/a)\sin n\theta'\, J_n(\xi_{mn}r'/a)}{J_{n+1}^2(\xi_{mn})}\, e^{-\kappa(\xi_{mn}/a)^2(t-t')}$	Figure 3.18, Example 3.15
$\left\{re^{i\theta}: 0 < r < a,\ 0 < \theta < \pi/2\right\}$	$\frac{8}{\pi a^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin 2n\theta\, J_{2n}(\xi_{m(2n)}r/a)\sin 2n\theta'\, J_{2n}(\xi_{m(2n)}r'/a)}{J_{2n+1}^2(\xi_{m(2n)})}\cdot e^{-\kappa(\xi_{m(2n)}/a)^2(t-t')}$	Figure 3.19, Example 3.16
$\left\{(x,y,z): 0 < x < a,\ 0 < y < b,\ 0 < z < c\right\}$	$\frac{8}{abc}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{k\pi z}{c}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y'}{b}\sin\frac{k\pi z'}{c}\cdot e^{-\kappa\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{k^2}{c^2}\right)(t-t')}$	Figure 3.20, Example 3.17

CHAPTER VI

SUMMARY

In this thesis a number of boundary value problems have been solved in an integral form which has a function as a kernel of the integral. This function is called Green's function. If the Green's function of the problem is known then the solution for the problem is found. Thus, we are interested in finding Green's function of the problem. It all started in 1828 when George Green first found Green's function for the solution of a potential equation. In order to find Green's function, a definition of the function is needed. From this study, knowing its properties is very helpful in finding the function. This thesis shows that there seems to be three properties in common for each Green's function. The following general definition of Green's function points out these three properties.

<u>Definition</u>

For a given differential equation with boundary conditions on region $\mathfrak{R}$ in Euclidean n-space, a function $G(\vec{r},\vec{r}')$ (or $G(\vec{r},\vec{r}',t-t')$) where $\vec{r}$ and $\vec{r}'$ belong to $\mathfrak{R}$ with $\vec{r}$ fixed but $\vec{r}'$ arbitrary (and $t-t' > 0$) is a Green's function for the differential equation with respect to the region $\mathfrak{R}$ if

(1). G satisfies certain continuity conditions on the region and G with respect to $\vec{r}'$ satisfies a certain form of the given differential equation,

(2). G satisfies certain boundary (and/or initial) conditions on $\mathcal{R}$, and

(3). G satisfies certain discontinuity conditions at $\vec{r}$ in the $\vec{r}'$ variables.

The many Green's functions discussed througout the thesis satisfy this definition. For example, notice that Green's function, G(x,t), for a second order differential equation $(p(x)u')' + qu = f(x)$, $a \leqq x \leqq b$ with unmixed homogeneous conditions has properties

(1). G is continuous in [a,b], and G satisfies the homogeneous form of the given equation,

(2). G satisfies the given boundary conditions, and

(3). The partial derivative of Green's function with respect to x has a jump discontinuity at x = t of the magnitude 1/p(t).

Furthermore, note that for the case of mixed conditions, Green's function has properties (1), (3) and

(2'). G satisfies the conditions which are obtained from the given conditions and

$$p(Gu' - uG_x)\Big|_{x=a}^{x=b} = 0.$$

The pattern continues for Green's functions for partial differential equations. In particular, Green's function, $G(\vec{r},\vec{r}')$ for Laplace's equation (or Helmholtz's equation) with a boundary condition on a region $\mathcal{R}$ has the following properties:

(1). G is continuous in $\overline{\mathcal{R}}$, $\vec{r}' \neq \vec{r}$, and G satisfies Laplace's equation (or Helmholtz's equation), (2). G satisfies the homogeneous boundary condition, and

(3). G has infinite discontinuity as $\vec{r}' \to \vec{r}$.

Finally, note that Green's function $G(\vec{r},\vec{r}',t-t')$ for the heat equation $u_t - \kappa\Delta u = 0$ with an initial condition and a boundary condition in a region $\mathcal{R}$ has the following properties:

(1). G is continuous in $\overline{\mathcal{R}}$, except at $t-t' = 0$ and $\vec{r}' = \vec{r}$ and G satisfies $G_{t'} + \kappa\Delta G = 0$,

(2). G satisfies the zero initial condition and the homogeneous boundary condition, and

(3). G has an infinite discontinuity at $t-t' = 0$ and $\vec{r}' = \vec{r}$.

Green's function for an ordinary differential equation with mixed or unmixed conditions has been found by using several methods: direct use of its properties, a formula, the method of variation of parameters or in some cases, generalized Green's function. These methods have been illustrated by examples. This study has applied the method of variation of parameters in finding Green's function for second order ordinary differential equations with either mixed or unmixed conditions. Warning and notices about shortcoming of the method are made througout the study. In the partial differential equation part of the thesis, Green's function has been found by the method of images, conformal mapping, or the method of eigenfunctions. Many regions in one, two, and three dimensions are considered. In addition, the form of Green's function for Dirichlet's problem in an angular region of angle π/k, $k = 1,2,3,\ldots$ and the form of Green's function for Neumann's problem in n-sphere were given. Tables of Green's functions for Helmholtz's equation and the heat equation with Dirichlet boundary condition in some regions are constructed.

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