AN INTRODUCTION TO THE THEORY OF PARTITIONS

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CHAPTER I

INTRODUCTION AND STATEMENT OF THE PROBLEM

Introduction

The natural numbers or positive rational integers,

$$1, 2, 3, \ldots, n, \ldots$$

play an important role in mathematics. All the real and complex numbers can be derived from them. The set of natural numbers is the domain of the arithmetic functions of number theory. The natural numbers even play a role in the complex function e^z which can be seen in the relation,

$$e^{2\pi i z} = e^{2\pi i(z \pm n)},$$

which holds for any natural number n and all complex values z. This study is not to find applications of the natural numbers in mathematics, however, but to apply mathematics in the study of the natural numbers.

The theory of additive arithmetic involves expressing an arbitrary natural number n in the form

$$n = a_1 + a_2 + \ldots + a_s$$

where $a_i \in A$ $(i = 1, 2, \ldots, s)$. The set A might be; the set of natural numbers, the set of prime numbers, the set of even natural numbers, the set of squares, etc.. If A is the set of natural numbers then the study of these representations is referred to as the theory of partitions. If A is the set of natural numbers; no restrictions are placed on s, repetitions are allowed, and order is irrelevant, then this is the study of unrestricted partitions. The number of such representations of n is denoted by p(n).

For example if $n = 5$, the possible representations are,

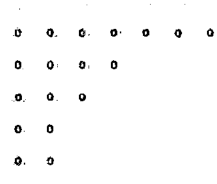
$$\begin{aligned} 5 &= 5 \\ &= 4 + 1 \\ &= 3 + 2 \\ &= 3 + 1 + 1 \\ &= 2 + 2 + 1 \\ &= 2 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1 \end{aligned}$$

and consequently $p(5) = 7$.

The study of partitions is basically done by combinatorial methods or analytic methods. The first method is aided by graphs introduced by Drs. Ferrers and Sylvester. The second method, which will be used in this paper, is aided by the generating function introduced by Euler.

Graphs

Partitions can also be represented graphically. Since, in unrestricted partitions order is irrelevant, it is convenient to arrange the summands a_i such that $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_s$. Then a geometric representation of $n = a_1 + a_2 + \ldots + a_s$ is the array of points with a_1 points in the first row, a_2 points in the second row, and on down to the last row. For example



represents the partition 19 = 7 + 4 + 3 + 2 + 1. The graph could also represent the partition 19 = 6 + 5 + 3 + 2 + 1 + 1 + 1 if it was read by columns instead of rows. Partitions related in this manner are said to be conjugate.

Another graphical representation of 19 is the following:

```
o o o o o o
o o o o o
o o o
o o
o o
o
```

If the graph is read by rows it represents the partition 19 = 6 + 5 + 3 + 2 + 2 + 1. The same representation is obtained if the graph is read by columns. Representations of this type are considered self-conjugate.

Many of the theorems in the theory of partitions can be proved by combinatorial methods which use the idea of 1-1 correspondence. This method is greatly enhanced by the graphic representation. Since this method is not used in this paper it will be illustrated by the elementary Theorem 1.1.

Consider the partitions of n in which the summands are $\leq m$. The number of all such partitions will be denoted by $p(n;m)$. Let the number of partitions of n with no more than m summands be denoted by $p^m(n)$.

<u>Theorem 1.1.</u> $$p^m(n) = p(n;m).$$

Proof: Represent the partitions of n graphically. If the graphs are read by rows, then $p(n;m)$ is the number of such graphs with maximum number of columns m. On the other hand, if the graphs are read by columns, then the number of graphs with a maximum of m columns will be $p^m(n)$. But, since each graph can be read by rows or by columns, there is a 1-1 correspondence between the graphs representing $p(n;m)$ and the graphs representing $p^m(n)$ and the theorem is proved.

Generating Functions

Generating functions are extremely useful in the study of the theory of partitions. It will be shown in Chapter IV that

$$P(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1} = \sum_{n=0}^{\infty} p(n)\, x^n.$$

The function $P(x)$ is said to be the generating function for $p(n)$, the number of unrestricted partitions of n. Since

$$\prod_{n=1}^{\infty} (1 - x^n)^{-1} = (1 - x)^{-1}(1 - x^2)^{-1}(1 - x^3)^{-1} \cdots$$
$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \cdots$$

the coefficients generate values of $p(n)$. That is, the coefficient of x^n is equal to $p(n)$. Evidently $p(n)$ is 1, 2, 3, 5, 7, 11 and 15 for $n = 1, 2, 3, 4, 5, 6$ and 7 respectively. It will also be convenient to define $p(0) = 1$, the coefficient of x^0.

The size of $p(n)$ increases very fast as n increases. For example it is known that $p(300) = 9,253,082,936,723,602$ and that

$$p(600) = 4580\ 04788\ 00814\ 43085\ 53622.$$

The values of $p(n)$, for all $n \leq 600$, can be found in Gupta's tables [21] and [22]. A few isolated values of $p(n)$ for $n > 600$ have been computed by D. H. Lehmer.

Until recent years $p(n)$ was not known for very large n. With the aid of generating functions, asymptotic formulae can be developed. Generating functions are also useful for developing arithmetic properties for $p(n)$ which are still few in number.

Purpose and Significance of Study

The purpose of this paper is to give a brief introduction and historical development of the theory of partitions. A second purpose is to illustrate the various branches of mathematics used in attempting to solve the problems in the theory of partitions. Thirdly, it will be the purpose of this paper to give a brief report, in an expository manner, on the recent work involving the asymptotic and arithmetic properties of $p(n)$.

It is hoped that this dissertation may be of use to college students, to gain insight into some procedures and techniques of research in mathematics, doing independent study or work in a seminar course. The author has attempted to show how one generalizes concepts and ideas in mathematics. The importance of trying to simplify known proofs is also significant and the author feels he has particularly accomplished this in the proof of Theorem 5.7.

CHAPTER II

CONCEPTS FROM NUMBER THEORY AND ANALYSIS

This chapter is only for the purpose of making the paper more self-contained. It includes the topics from number theory and analysis which are needed in the following chapters. The definitions and theorems (stated here without proof) can be found in most elementary books on the subject and in particular Apostol [1], Grosswald [23] and Pennisi [39].

If one has a good background in elementary number theory and analysis, including series and complex variables, it is possible to skip this chapter. One may refer back to it as the need arises.

The material on elementary number theory should be sufficient for Chapters III, IV and V, however, Chapter VI requires the more advanced analysis.

Topics from Elementary Number Theory

Definition 2.1. For any integer $m \neq 0$, a is congruent to b modulo m if and only if m divides a − b. We write $a \equiv b \pmod{m}$. If a − b is not divisible by m, we say that a is not congruent to b modulo m, with the notation $a \not\equiv b \pmod{m}$.

For example, $22 \equiv 1 \pmod{7}$ since $7 \mid (22-1)$. Note that the remainder of 1 is obtained when 22 is divided by 7. This observation leads to an equivalent definition of congruence in terms of remainders (or residues). The set $A = \{0, 1, 2, 3, 4, 5, 6\}$, for example, will be considered the least residue system modulo 7 since $n \equiv a \pmod 7$ for some a in A.

Definition 2.2. If $a = nq + r$ with $0 \leq r < m$, then r is called the least residue of a modulo m.

Definition 2.3. The set of integers 0, 1, 2, ..., m-1 is called the least residue system modulo m. Any set of m integers, no two of which are congruent modulo m, is called a complete residue system modulo m.

Theorem 2.4. For every integer m, the congruence modulo m is an equivalence relation.

Because of Theorem 2.4, a residue system is sometimes called a residue class.

Theorem 2.5. The following statements hold where all congruences without indication of a modulus are mod m:

(1) $a \equiv b$ implies $ca \equiv cb$;

(2) $a \equiv b$, $c \equiv d$ implies $a + c \equiv b + d$;

(3) $a \equiv b$, $c \equiv d$ implies $ac \equiv bd$;

(4) $a \equiv b$ implies $a^n \equiv b^n$; and

(5) $a \equiv b \pmod{mn}$ implies $a \equiv b \pmod m$.

Theorem 2.6. If p(x) is a polynomial with integer coefficients and $a \equiv b \pmod m$, then $p(a) \equiv p(b) \pmod m$.

Definition 2.7. $\sum_{n=0}^{\infty} p(n)\, x^n \equiv \sum_{n=0}^{\infty} q(n)\, x^n \pmod m$ if and only if $p(n) \equiv q(n) \pmod m$ for each n.

Definition 2.8. If $(a,b) = 1$, then a and b are said to be relatively prime or coprime where (a,b) represents the greatest common divisor of a and b.

Theorem 2.9. $ca \equiv cb \pmod{m}$ implies $a \equiv b \pmod{m/(m,c)}$.

Theorem 2.10. If $(m_1,m_2) = 1$, $a \equiv b \pmod{m_1}$, and $a \equiv b \pmod{m_2}$, then $a \equiv b \pmod{m_1 m_2}$.

Theorem 2.11. If R is a prime, then $(1 - x)^R \equiv 1 - x^R \pmod{R}$.

Theorem 2.12. If R is a prime, then $(1 - x)^{-R} \equiv (1 - x^R)^{-1} \pmod{R}$.

Definition 2.13. The number of positive integers r, not exceeding m and coprime with m is denoted by $\phi(m)$, called Euler ϕ function, i.e.

$$\phi(m) = \sum_{\substack{0<r\leq m \\ (m,r)=1}} 1$$

Definition 2.14. Any set of $\phi(m)$ integers which are coprime with m and which are mutually incongruent (no two are congruent), modulo m, is called a reduced residue system modulo m.

Theorem 2.15. (The Euler-Fermat Theorem). If $(a,m) = 1$, then

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

Definition 2.16. Let $(r,m) = 1$; then r is said to be a quadratic residue modulo m, if there exists some integer x, such that $x^2 \equiv r \pmod{m}$. We call n a quadratic non-residue modulo m, if the congruence $x^2 \equiv n \pmod{m}$ has no solutions.

Definition 2.17. For prime $p \nmid a$ we define the Legendre symbol $(\frac{a}{p})$ as follows: $(\frac{a}{p}) = 1$ if a is a quadratic residue and $(\frac{a}{p}) = -1$ if a is a quadratic non-residue modulo p.

Definition 2.18. The greatest integer function, in symbols $[x]$, is defined as the largest rational integer not to exceed x.

Definition 2.19. The sum of the k-th powers of the divisors of the integer n is denoted by $\sigma_k(n)$. In particular, $\sigma_0(n)$ (also denoted $\tau(n)$) is the number of divisors of n and $\sigma_1(n)$ (also denoted $\sigma(n)$) is the sum of the divisors of n. In symbols, $\sigma_k(n) = \sum_{d|n} d^k$.

Theorem 2.20. $$\sum_{d|n} \frac{n}{d} = \sum_{d|n} d = \sigma(n).$$

Some notation, which is perhaps restricted almost exclusively to analytic number theory, is given in Definitions 2.21, 2.22 and 2.23.

Definition 2.21. If f and g are two functions (real or complex) defined in a neighborhood of c (finite or infinite), we say f is asymptotic to g, written $f \sim g$, if $\lim_{x \to c} f/g = 1$.

Definition 2.22. If f and g are two functions (real or complex) defined in a neighborhood of c (finite or infinite), we say f is "big O of g", written $f = O(g)$, if there exists a constant $K > 0$ and a neighborhood N(c) such that $|f(x)| \leq K\,|g(x)|$ for all x in N(c).

Definition 2.23. If f and g are two functions (real or complex) defined in a neighborhood of c with $g(x) > 0$, then we say f is "little o of g", written $f = o(g)$, if $\lim_{x \to c} f/g = 0$.

Equivalent definitions for 2.21, 2.22 and 2.23 could be given for sequences.

Topics From Advanced Calculus and Complex Variables

Theorem 2.24. If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both absolutely convergent with sums A and B respectively, then

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k b_{n-k} \right)$$

in the sense that the series on the right is also absolutely convergent and has sum AB.

Definition 2.25. (Double series) Let f be a double sequence. The double series s defined by the equation

$$s(p,q) = \sum_{m=1}^{p} \sum_{n=1}^{q} f(m,n)$$

is called a double series. The double series is said to converge to the sum a if $\lim_{p,q\to\infty} s(p,q) = a$. The convergent double series is denoted by

$$\sum_{n,m=1}^{\infty} f(m,n).$$

Theorem 2.26. Let $\sum_{n,m=1}^{\infty} f(m,n)$ be an absolutely convergent double series, then

(a) $\sum_{n=1}^{\infty} f(m,n)$ and $\sum_{m=1}^{\infty} f(m,n)$ both converge absolutely and

(b) $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n).$

Definition 2.27. The infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ is said to converge, if $a_n \neq -1$ at least for $n \geq n_0$ and if $\lim_{k\to\infty} \prod_{n=n_0}^{k} (1 + a_n)$ exists and is different from zero.

Theorem 2.28. The infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ with $a_n \neq -1$ for all n is absolutely convergent if and only if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Definition 2.29. (Gamma function) $\Gamma(x) = \int_0^{\infty} e^{-\beta} \beta^{x-1} \, d\beta, \; x > 0.$

Theorem 2.30. (Riemann zeta function) $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, where $s = u + iv$, converges for $u > 1$. In particular $\zeta(2) = \pi^2/6$.

Theorem 2.31. (a Tauberian theorem) If $a_n \geq 0$ and $\sum_{n=0}^{\infty} a_n z^n \sim \frac{\alpha}{1-z}$ (as $z \to 1^-$) then $\sum_{k \leq u} a_k \sim \alpha u$.

Theorem 2.32. (Abel's theorem) If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ $(-1 < x < 1)$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n$. In particular

$$\lim_{x \to 1} \sum_{m=0}^{\infty} \frac{x^m}{m^2} = \zeta(2) = \pi^2/6.$$

Definition 2.33. Let f be a complex-valued function defined on an open set S and let z_o be any fixed point in S. Then f is said to have a derivative $f'(z_o)$ at z_o if the limit

$$\lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_o} = f'(z_o)$$

exists.

Functions which possess a continuous derivative at each point of an open set are called analytic functions. It has been shown that the existence of f' on S automatically implies continuity of f' on S.

Definition 2.34. A complex-valued function f is said to be analytic (or an analytic function) on an open set S if it has a derivative at every point in S. The function f is said to be analytic at a point z_o if there exists a neighborhood $N(z_o)$ on which f is analytic.

Definition 2.35. A contour C represented by $z = f(t)$ on the interval I: $\alpha \leq t \leq \beta$, is said to be closed if $f(\alpha) = f(\beta)$. A contour C is said to be simple if for any two points $t_1 \neq t_2$ in I we have $f(t_1) \neq f(t_2)$, except possible when $t_1 = \alpha$ and $t_2 = \beta$, that is the contour doesn't cross itself. A contour which is both simple and closed is called a simple closed contour.

Theorem 2.36. (Cauchy's Integral Formula) Let f(z) be analytic within and on a simple closed contour C. If z_o is any point interior to C, then

$$f(z_o) = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{z - z_o},$$

where the integral along C is taken in the positive (counterclockwise) direction.

Theorem 2.37. (Laurent's Series) Let S be the region bounded by the concentric circles C_1 and C_2 with center at z_o and radii r_1 and r_2 respectively, $r_1 < r_2$. Let f(z) be analytic within S and on C_1 and C_2. Then at each point z in the interior of S, f(z) can be represented by a convergent series of positive and negative powers of $(z-z_o)$,

(2.1) $$f(z) = \sum_{n=0}^{\infty} a_n(z-z_o)^n + \sum_{n=1}^{\infty} b_n(z-z_o)^{-n},$$

where

(2.2) $$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)\, dz}{(z-z_o)^{n+1}}, \quad n = 0,1,2,\ldots$$

(2.3) $$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)\, dz}{(z-z_o)^{-n+1}}, \quad n = 1,2,3,\ldots$$

and the integral along C_1 and C_2 being in the positive direction.

Definition 2.38. A point z_o is called a singular point or a singularity of the function f(z), if f is not analytic at z_o, but every neighborhood of z_o contains at least one point at which f is analytic. The series in negative powers of $(z-z_o)$ in (2.1) is called the principal part of f at the isolated singular point z_o. If an infinite number of the b_n do not vanish in (2.3), then z_o is said to be an essential singular point.

Definition 2.39. If z_o is an isolated singular point of $f(z)$, then

$$b_1 = \frac{1}{2\pi i} \int_{C_1} f(z)\, dz,$$

the coefficient of (2.3) for $n = 1$, is called the residue of f at $z = z_o$, and will be denoted by Res $[f(z), z_o]$.

Theorem 2.40. (Cauchy's residue theorem) Let C be a simple closed contour, and let f be analytic on C and in the interior of C except at a finite number of singular points $z_1, z_2, \ldots, z_k$ contained in the interior of C. Then

$$\int_C f(z)\, dz = 2\pi i \sum_{n=1}^{k} \text{Res}\,[f(z), z_n],$$

where the integral along C is taken in the positive direction.

CHAPTER III

HISTORICAL DEVELOPMENT

Early Beginnings

The study of the natural numbers is but a small part of the field called the theory of numbers or number theory. Even this study can be divided into two major divisions, multiplicative number theory and additive number theory.

Multiplicative number theory, which deals with questions of factorization, divisibility, prime number, and so on, goes back more than 2000 years to Euclid. Additive number theory, on the other hand, began less than 250 years ago with Leonard Euler (1707-1783, Swiss). In his famous treatise, Introductio in Analysin Infinitorum (1748), Euler devotes the sixteenth chapter, "De partitione numerorum," to problems of additive number theory. A "partition" is, after Euler a decomposition of a natural number into summands (parts) which are natural numbers.

According to Dickson [14], G. W. Leibniz (1646-1716, German) asked Bernoulli (1654-1705, Swiss) if he had investigated the number of ways a given number can be separated into two, three, or many parts, and remarked that the problem seemed difficult but important. Leibniz saw the relation between the number of ways a given integer could be expressed as a sum of smaller integers, as 3, 2 + 1, 1 + 1 + 1, and the number of symmetric functions of a given degree as $\Sigma\, a^3$, $\Sigma\, a^2b$, $\Sigma\, abc$.

The first real contributions to the theory of partitions, however, were made by Euler. The great algorist that he was, Euler developed many formulae by the device of comparing coefficients in two or more expressions of a given function by different algorithms. By the use of symmetric functions and their relationships, he was able to find the number of ways n is a sum of a given number of distinct parts. He also noted in 1748 that p(n), the number of unrestricted partitions of n, is the coefficient of x^n in the expansion of $\prod_{k=1}^{\infty} (1 - x^k)^{-1}$ into power series in x, hence this is called a generating (or enumerating) function of p(n). More will be said about the generating function after discussing the theory of formal power series.

Euler made the simple remark that, since we have $x^m x^n = x^{m+n}$, exponents of powers can easily be combined in an additive manner, and therefore products of power series can be used as "generating function."

Formal Power Series

The method of formal power series was introduced by Euler around the middle of the eighteenth century and is one of the basic tools in additive number theory.

A formal power series is an expression

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots$$

where the symbol x is an indeterminate symbol, i.e. it is never assigned a numerical value. Consequently, all questions of convergence are irrelevant.

Consider the power series $A = \sum_{n=0}^{\infty} a_n x^n$ and $B = \sum_{n=0}^{\infty} b_n x^n$, where a_n and b_n are elements of some algebraic system with the definitions:

$A = B$ if and only if $a_n = b_n$ for all n.

$A + B = \sum_{n=0}^{\infty}(a_n + b_n)x^n$ (A formal power series.)

$AB = \sum_{n=0}^{\infty}(a_0b_n + a_1b_{n-1} + \ldots + a_{n-1}b_1 + a_nb_0)x^n$
(A formal power series.)

$A = 0$ if and only if $a_n = 0$ for all n.

Many properties can be verified and all the following fifteen have been verified by Rademacher [41]. If the coefficients are elements of a ring, then the formal power series form a commutative ring. That is, the identity relation (=) and the two binary relations of addition and multiplication which have been defined satisfy:

(1) $A + B = B + A$ (2) $AB = BA$

(3) $(A + B) + C = A + (B + C)$ (4) $(AB)C = A(BC)$

(5) $A(B + C) = AB + AC$

(6) $A + 0 = 0 + A = A$, where 0 is the zero power series,

(7) $A + (-A) = 0$, where $-A = \sum_{n=0}^{\infty}(-a_n)x^n$.

If the coefficients have multiplicative inverses, then the additional properties follow:

(8) $AB = 0$ implies $A = 0$ or $B = 0$

(9) $A \neq 0$ and $AB = AC$ implies $B = C$

(10) $a_o \neq 0$ implies there exists a B such that $B = 1/A$ is a power series.

Consider $A' = \sum_{n=0}^{\infty}na_nx^{n-1}$ to be the derivative of A. The following properties follow:

(11) $(A + B)' = A' + B'$ (12) $(AB)' = AB' + A'B$

Both of these can be extended by induction to a finite number of sums or products.

If AB has a reciprocal (i.e. $a_ob_o \neq 0$), $(AB)'/AB$ is called the logarithmic derivative of AB. It follows that:

(13) $\frac{(AB)'}{AB} = \frac{A'B}{AB} + \frac{AB'}{AB} = \frac{A'}{A} + \frac{B'}{B}.$

By induction it follows that:

(14) $\frac{(A_1A_2 \ldots A_n)'}{A_1A_2 \ldots A_n} = \frac{A_1'}{A_1} + \ldots + \frac{A_n'}{A_n}$.

A formal power series will result from the product of a finite number of power series, $\prod_{i=1}^{n} A_i$, since the coefficient of the kth power of x can be found by adding only a finite number of terms. If we consider products such that only a finite number of terms are needed to compute the term x^n, then it also follows that:

(15) $\frac{(\prod_{n=1}^{\infty} A_n)'}{\prod_{n=1}^{\infty} A_n} = \sum_{n=1}^{\infty} \frac{A_n'}{A_n}$.

Formal power series will frequently be expressed as products. It will thus be necessary to work with products, particularly of the type $\prod_{k=1}^{\infty}(a + bx^k)$. Only products in which there exists a finite number of terms involving x^n will be considered.

Identities of Euler and Jacobi

Euler used formal power series to develop the generating function of p(n). A given partition of n,

(3.1) $$n = n_1 + n_2 + n_3 + \ldots + n_s ,$$

can be written in the more systematic form

(3.2) $$n = n_1 + 2n_2 + 3n_3 + \ldots \ (n_i \geq 0) ,$$

where n_1 is the number of 1's, n_2 is the number of 2's, n_3 is the number of 3's, etc. Thus p(n) is the number of solutions of (3.2).

On the other hand,

$$\prod_{k=1}^{\infty}(1 - x^k)^{-1} = \frac{1}{1 - x} \cdot \frac{1}{1 - x^2} \cdot \frac{1}{1 - x^3} \cdots$$

$$= \sum_{n_1=0}^{\infty} x^{n_1} \cdot \sum_{n_2=0}^{\infty} x^{2n_2} \cdot \sum_{n_3=0}^{\infty} x^{3n_3} \cdots$$

$$= \sum_{n=0}^{\infty} C(n)\, x^n ,$$

where $n = n_1 + 2n_2 + 3n_3 + \ldots$ and $C(n) = \Sigma\, 1$, where the sum is over all n_i such that $n = n_1 + 2n_2 + 3n_3 + \ldots$. Thus $C(n)$ is the number of solutions of (3.2) and one formally concludes:

(3.3) $$P(x) = \prod_{k=1}^{\infty}(1 - x^k)^{-1} = \sum_{n=0}^{\infty} p(n)\, x^n.$$

Relation (3.3) will be stated and proved in Theorem 4.5 for all real x such that $|x| < 1$.

In order to gain knowledge of $p(n)$, Euler studied the reciprocal of the generating function, i.e. $\Phi(x) = \frac{1}{P(x)} = \prod_{k=1}^{\infty}(1 - x^k)$.

After expanding the product of a number of factors, he obtained

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - - + + \ldots$$

He wrote out the powers of x, odd terms to the right and even terms to the left as follows:

$$\ldots 26,\ 15,\ 7,\ 2,\ 0,\ 1,\ 5,\ 12,\ 22,\ \ldots$$

Euler then took the difference of each pair of consecutive numbers in the sequence to obtain the new sequence,

$$\ldots -11,\ -8,\ -5,\ -2,\ 1,\ 4,\ 7,\ 10,\ \ldots$$

He again took the difference of each pair of consecutive numbers to get the constant 3 in each case. It was now evident that the original sequence must be of the form $an^2 + bn + c$. He was led to conjecture:

(3.4) $$\Phi(x) = \prod_{k=1}^{\infty}(1 - x^k) = \sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n-1)/2} ,$$

and he was unable to prove this result until 9 years later in 1750. The relation (3.4) is known as Euler's identity and will be stated and proved in Theorem 4.7. Euler's identity was also proved by Carl G. J. Jacobi (1804–1851, German) in his "Fundamenta Nova" of 1829, where he made important applications of elliptic functions to the theory of partitions.

Jacobi's well-known theta formula:

$$(3.5) \qquad \prod_{n=1}^{\infty}(1 - x^{2n})(1 + x^{2n-1}z)(1 + x^{2n-1}z^{-1}) = \sum_{n=-\infty}^{\infty} x^{n^2} z^n$$

was proved by Jacobi in 1829. James J. Sylvester (1814–1897, English) [47] gave a direct proof by elementary means and E. Maitland Wright (1906–, Britian) [50] gave an enumerative or combinatorial proof of Jacobi's theta formula in 1963 from which not only Euler's identity but other results in partition theory can be deduced. F. Franklin gave a proof of Euler's formula (3.4) with combinatorial arguments. A more recent algebraic proof was given by Daniel Shanks [45], which is the method of proof given in Chapter IV.

With unsurpassed manipulative skill, Euler derived numerous identities from his relations (3.3) and (3.4). Among the many results of Euler's were the elegant formulae whereby $p(n)$ could be calculated recursively.

Recursion Formulae

The result of multiplying the relations (3.3) and (3.4) together yields

$$1 = \sum_{n=0}^{\infty} p(n)\, x^n \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2}.$$

Upon equating coefficients, Euler obtained the recursion formula

$$(3.6) \qquad p(n) = \sum_{0<w_k\leq n} (-1)^{k-1} p(n - w_k),$$

where $$w_k = k(3k - 1)/2, \quad (k = 0, \pm 1, \pm 2, \ldots).$$

The recursion formula (3.6) makes possible the computation of $p(n)$ by inductive steps. Of course the number of terms in (3.6) increases as n increases. The number of terms needed to find $p(n)$ may be determined approximately as follows. For $k > 0$,

$$w_k = \frac{3k^2}{2} - \frac{k}{2} < n,$$

and for large k, the term $k/2$ is small compared to $(3k^2)/2$ and hence may be neglected. Thus $(3k^2)/2 < n$ and hence $k < \sqrt{2n/3}$. Similarly, for $k < 0$; or approximately $2\sqrt{2n/3}$ terms altogether.

Another of Euler's recursion formulae was

$$(3.7) \qquad p_m(n) = p_{m-1}(n) + p_m(n-m),$$

where $p_m(n)$ is the number of partitions of n into summands not greater than m. By the use of this recursion formula, Euler computed a table of values of $p_m(n)$ for $n \leq 69$, $m \leq 11$, [14]. A proof of the recursion formula (3.7) is given by Grosswald [23]. However, this recursion formula is useless as an aid to practical computation for any but inconsiderably small numbers [7].

One peculiarity of Euler's formal analysis (The interested reader can refer to Rademacher's lecture [41] Chapter I, Formal Power Series, to see a more complete development of formal analysis.) is that it can lead to absurdities if not used properly. He recognized that if an infinite series is not convergent it is unsafe to use unless the variable is used as an indeterminate. For example, by long division one obtains

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k.$$

It is known that this series converges for $|x| < 1$, but for $x = 2$ the absured result of $-1 = 1 + 2 + 4 + 8 + \ldots$ is obtained.

P. A. MacMahon (1854–1929, Britian), used Euler's formula (3.6) to compute a table of values of p(n) for values of n up to 200. This table was published at the end of a paper by Hardy and Ramanujan [25].

Hansraj Gupta [21] added to the list another recursion formula,

$$(n,m) = (n-m,m) + (n+1,\ m+1),$$

where (n,m) denotes the number of those partitions of n in which the smallest element that occurs is m. This helped him to construct tables [21] of p(n) for values of n up to 300 and later for values of n up to 600 [22].

Asymptotic Results

The generating function and the formulae of Euler and Jacobi were used to develop many interesting recursion formulae. No general independent representation of p(n) was known until 1937 when Hans Rademacher (1892–, German) [42] developed a convergent series representation. According to Rademacher the order of magnitude of p(n) for large n was never examined until 1917, when G. H. Hardy (1877–, English) and S. Ramanujan (1887–1920, Indian) [25] applied their new analytic methods and derived an asymptotic formula for p(n).

The value p(200) = 397 29990 29388, computed from the recursion formula by MacMahon, can be computed with only six terms of the asymptotic formula with an error of .004.

The asymptotic formula obtained by Hardy and Ramanujan is

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3}), \tag{3.8}$$

or more precisely,

$$(3.9)\qquad p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k \leq \alpha\sqrt{n}} A_k(n)\sqrt{k}\,\frac{d}{dn}\left(\frac{\exp(\pi/k\sqrt{2/3}\,\lambda_n)}{\lambda_n}\right) + O(n^{-1/4}),$$

with α an arbitrary constant, $\lambda_n = \sqrt{n - 1/24}$ and

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k) = 1}} W_{h,k} \exp(-2\pi i h n/k).$$

The symbol $W_{h,k}$ means a 24k-th root of unity given for odd h by

$W_{h,k} = (-k/h)\exp[-\{(2 - hk - h)/4 + (k - 1/k)(2h - h' + h^2h')/12\}\pi i]$,

and for odd k by

$$W_{h,k} = (-h/k)\exp[-\{(k - 1)/4 + (k - 1/k)(2h - h' + h^2h')\}\pi i],$$

where (a/b) denotes the Legendre-Jacobi symbol and h' is any solution of the congruence

$$hh' \equiv -1 \pmod{k}.$$

Rademacher has also shown that $W_{h,k}$ has the following representation

$$W_{h,k} = \exp \pi i \sum_{x=1}^{k-1}\left(\frac{x}{k} - \frac{1}{2}\right)\left(\frac{hx}{k} - \left[\frac{hx}{k}\right] - \frac{1}{2}\right),$$

where [x] means the greatest integer function.

D. H. Lehmer (1905-, American) [30] used the series (3.9) to develop

p(599) = 4353 50207 84301 70000,

p(721) = 16 10617 55750 17947 34762,

p(2052), and the 127 digit number for p(14031), the largest known [40]. Lehmer [29] later proved that the series (3.9) is not convergent but divergent. His results for p(599) and p(721) are easily justified, however, by convergent series due to Hans Rademacher (1892-, German) [42].

Rademacher simplified and perfected the original analysis of Hardy and Ramanujan to obtain the convergent series

$$(3.10)\qquad p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\;\frac{d}{dn}\,\frac{\sinh C\lambda_n/k}{\lambda_n}.$$

He also showed [42] that the series (3.9) can be derived as a corollary from his series (3.10).

Congruence Properties

Ramanujan [24], upon observation of MacMahon's table of p(n) for n up to 200, discovered and proved the congruence properties

$$\text{(3.11)} \quad \begin{aligned} &\text{(a)} \quad p(5m + 4) \equiv 0 \pmod{5} \\ &\text{(b)} \quad p(7m + 5) \equiv 0 \pmod{7} \\ &\text{(c)} \quad p(11m + 6) \equiv 0 \pmod{11}. \end{aligned}$$

He made the general conjecture that if

$$\delta = 5^a\, 7^b\, 11^c$$

and $24n \equiv 1 \pmod{\delta}$,

then $p(n) \equiv 0 \pmod{\delta}$.

The proof would only need to involve the cases $\delta = 5^a$, 7^b, and 11^c since all others would follow as corollaries. Ramanujan proved the result for $\delta = 5, 7, 11, 5^2$, and 7^2. It was later observed by S. Chowla [13] from Gupta's table, with values of n up to 300, that there was a contradiction to the conjecture. A contradiction occurs with $n = 243$ since $24(243) \equiv 1 \pmod{7^3}$ but 7^3 does not divide $p(243) = 13397\ 82593\ 44888$.

Watson [48] found and proved the appropriate modification of Ramanujan's conjecture for $\delta = 7^b$, vis. that $p(n) \equiv 0 \pmod{7^b}$ if $b > 1$ and $24n \equiv 1 \pmod{7^{2b-2}}$. He also proved the congruence for the case 5^a.

Lehmer [28] used the celebrated Ramanujan series to compute p(599) and p(721) to check the conjecture for $\delta = 5^4$ and 11^3 respectively. However, he could not be sure of these values and this led Gupta [22] to extend his tables for p(n) up to n = 600. This confirmed his value for p(599) and also for p(721) with respect to the modulus 247.

However, after Rademacher's development of his convergent series for p(n), Lehmer [28] established the following facts:

$$p(1224) \equiv 0 \pmod{5^4}$$
$$p(2052) \equiv 0 \pmod{11^3}$$
$$p(2474) \equiv 0 \pmod{5^3}$$
$$p(14031) \equiv 0 \pmod{11^4},$$

which are all in accord with Ramanujan's conjecture.

In 1943, Joseph Lehner (1912-, American) [31] proved the Ramanujan Conjecture for 11 and 11^2 but said nothing is known concerning the conjecture for higher powers of 11 except for the affirmative test of Lehmer's above. Seven years later Lehner [32] also proved the conjecture for 11^3.

Atkin and Swinnerton-Dyer proved some results for which the congruences (3.11) are immediate corollaries. More general theorems were also proved by J. M. Gandhi which will be given in Chapter IV.

In spite of the simplicity of the definition of p(n), very few arithmetic properties are known. There is no simple criterion to determine whether p(n) is even or odd for example. Ramanujan had inquired of MacMahon if he had a simple way of determining the parity of p(n). Since Ramnujan was successful in discovering some congruences from a table of values for p(n) for n up to 300, MacMahon [34] obtained the 470 values of p(n) which are even for $n \leq 1000$. More on congruences will be given in Chapter V.

Generalizations

The usual thing in mathematics, if possible, is to generalize what you already have. This is exactly what happened in the theory of partitions and continues to happen. For example a partition of 10 can be written with the summands in non-ascending order of magnitude along a line as follows: 5 3 2. This is regarded as a one-dimensional or line partition.

A. Plane Partitions and higher dimensions

MacMahon [33] was the first to take a line partition of an integer n and arrange the summands in rows and columns with non-ascending order of magnitude in each row from left to right, and in each column from top to bottom. Such an obvious generalization is called a two-dimensional or plane partition.

Thus from the line partition 5 3 2 of the number 10, we would have the plane partitions 532, 53, 52, 5.

532,	53,	52,	5.
	2	3	3
			2

MacMahon was now able to place restrictions on (i) the size of each summand, (ii) the number of rows, and (iii) the number of columns. If the number of rows is k, ($k \leq n$), then the representation is referred to as a k-rowed or k-line partition. He was able to obtain generating functions for the plane partitions with and without restrictions. However, his developments were neither intuitive nor easy. He used the intricate and beautiful analysis based on his theory of Lattice Functions.

Later T. W. Chaundy [10] devised an algebraic technique which gives the result more rapidly. Other proofs were given for $k = 2$ by Forsyth [15] and the first one based on the purely combinatorial method of one-to-one correspondence by Cheema and Gordon [12].

More recently, Gordon and Houten [19] have given a much more simple proof of the generating function for the plane partition function.

They also developed a generating function for the k-rowed partition function with restrictions. However, the proofs still fall short of the intuitive development which was first the goal of MacMahon and which still remains perhaps an impossible challenge.

Wright [54] uses the notation p(n), q(n), and r(n) for the number of linear, plane and solid partitions of n respectively. He claims it has long been conjectured that the generating function for r(n) is

$\prod_{k=1}^{\infty}(1 - x^k)^{-k(k+1)/2} = \sum_{n=0}^{\infty} r(n)\, x^n$, but it is now known to be false.

An asymptotic result for the plane partition function was first obtained in 1931 by E. M. Wright [51]. He applies Cauchy's integral theorem to the generating function and applies the method of 'steepest descent' in evaluating the integral. He mentions in this paper the improbability of an asymptotic result with as small an order for the error as in the case of p(n).

In 1964, published in 1966 [52], Wright used the reciprocal of the generating function for plane partitions, i.e.,

$$\prod_{k=1}^{\infty}(1 - x^k)^k = \sum_{n=0}^{\infty} c(n)\, x^n, \ (c(0) = 1)$$

and developed asymptotic results for c(n). Before this was published, Haskell [27] also developed the asymptotic results which includes the famous Hardy-Ramanujan results for p(n) as a special case. Haskell's paper also has a crude asymptotic formula for the k-line partition function.

There are very few results concerning congruence properties of k-line and plane partition functions. Cheema and Gordon [12] gave some for $t_2(n)$ and $t_3(n)$, the two-line and three-line partitions. In 1967, Gandhi [18] extended this to $t_k(n)$ for k = 4 and 5 and claims to have some for k = 6, 7, 8 and 9 which have not appeared in print at this time.

The partitions of n have also been extended in a logical way to n-dimensional or multi-dimensional. MacMahon refers to the number of 3-dimensional partitions of n as the number of 'solid graphs' of n nodes.

The generating functions, some asymptotic and arithmetic results will be given in a later chapter.

B. Vector Partitions

Another extension of the theory of partitions is to consider the 'multipartite' numbers. A multipartite number of order s is a s dimensional vector, the components of which are non-negative rational integers. A vector partition of $(n_1, n_2, \ldots n_s)$ is a solution of the vector equation

$$\sum_k (n_{1k}, n_{2k}, \ldots n_{sk}) = (n_1, n_2, \ldots n_s)$$

in multipartite numbers other than (0, 0, ... 0). The number of such partitions without restrictions (order of vectors not significant) is generally denoted by $p(n_1, n_2, \ldots n_2)$. The generating function is

$$\prod_{k_i \geq 0} (1 - x_1^{k_1} x_1^{k_1} x_2^{k_2} \ldots x_s^{k_s})^{-1} = \sum_{n_i=0}^{\infty} p(n_1, n_2, \ldots n_s)\, x_1^{n_1} x_2^{n_2} \ldots x_s^{n_s},$$

where $k_1 + k_2 + \ldots + k_s > 0$.

The generating function for the partitions of bipartites (m,n) without restrictions was already known by MacMahon [33]. However, L. Carlitz [8] obtained explicit generating functions for the bipartite with restrictions. Let u(m,n) denote the number of partitions of (m,n) into parts (m_i,n_i) such that min $(m_i,n_i) \geq$ max (m_i,n_i) for (i=1,2,3,...). Then he obtains,

$$\prod_{n=1}^{\infty} (1 - x^n y^{n-1})^{-1} (1 - x^{n-1} y^n)^{-1} (1 - x^{2n} y^{2n})^{-1} = \sum_{\substack{m,n=0 \\ (m+n>0)}}^{\infty} u(m,n) x^m y^n.$$

He noticed the close similarity with the Jacobi theta functions and was able to prove

$$\prod_{n=1}^{\infty}(1 - x^n y^{n-1})(1 - x^{n-1}y^n)(1 - x^{2n}y^{2n})$$
$$= \prod_{n=1}^{\infty}(1 + x^n y^n)\sum_{r=-\infty}^{\infty}(-1)^r x^{r(r+1)/2} y^{r(r-1)/2}.$$

L. Carlitz [9] was then able to develop the following two relations.

$$v(m,n) = p[m - (m - n)(m - n + 1)/2],$$

where $p(n)$ is the number of unrestricted partitions of n with $p(-m) = 0$ for $m > 0$ and $v(m,n)$ is the number of partitions of (m,n) into distinct parts $(a,a - 1)$, $(b - 1,b)$ $(a,b = 1, 2, 3 \ldots)$.

$$w(m,n) = \sum_{r=0}^{\min(m,n)} p(r)y(m - r,n - r),$$

where $w(m,n)$ denotes the number of partitions of (m,n) into (not necessarily distinct) parts (a,a), $(b,b - 1)$, $(c - 2,c)$ $(a,b,c = 1, 2, 3 \ldots)$ and $y(m,n)$ is the number of partitions of (m,n) into (not necessarily distinct) parts $(a,a - 1)$, $(b - 1,b)$ $(a,b = 1, 2, 3 \ldots)$.

In 1953 Auluck [5] obtained an asymptotic formula for $p(n_1, n_2)$ when n_2 is fixed and n_1 large. Nanda [35] shows this same result applies for fixed n_2 of order $n_1^{1/4}$ and large n_1. Robertson [44] extended Nanda's method to asymptotic formula for the number of unrestricted partitions of an s-dimensional vector. Cheema [11] obtains some asymptotic results of $p(n_1, \ldots n_s)$ with restrictions and establishes a relation between vector partitions and the multi-dimensional partitions.

The problem of congruences, i.e. to show $p(n_1, \ldots n_s) \equiv a \pmod m$ has an infinite number of solutions for all m, a, and fixed s, is an open problem even in the case $s = 1$.

The problem of finding certain generating functions is still open. Cheema [11] states the conjecture that the generating function for bipartite plane partitions (with non-ascending order of magnitude in both directions) is

$$\prod_{\substack{k_1 k_2 \geq 0 \\ k_1+k_2>0}} (1 - x_1^{k_1} x_2^{k_2})^{-(k_1+k_2)}.$$

Wright [47] refers to a conjecture for the generating function of d-dimensional partitions of n, namely

$$R_d(x) = \prod_{k=1}^{\infty} (1 - x^k)^{-\binom{d+k-2}{k-1}}$$

where $\binom{d+k-1}{k-1}$ takes the value 1 for k=1 and otherwise denotes the usual binomial coefficient. The proofs for d=1 and 2 have already been alluded to, but it has been disproved for d=3 by Atkin et al. [3]. E. M. Wright [53] makes the following comment, "It is interesting to learn that $R_3(x)$ is not the generating function of $q(3,\infty;n)$ and it would be of some interest to have a more plausible conjecture as to what is the correct generating function." He denotes the number of unrestricted 3-dimensional partitions of n by $q(3,\infty;n)$.

C. The partition function $p_r(n)$

Another generalization has been to study the function $p_r(n)$ which is defined by the relation

$$P^r(x) = \prod_{k=1}^{\infty} (1 - x^k)^r = \sum_{n=0}^{\infty} p_r(n)\, x^n.$$

Thus $p_{-1}(n) = p(n)$ is just the unrestricted partitions of n.

K. G. Ramanathan, in 1950, proved congruence properties modulo powers of 5, 7 and 11 for $p_r(n)$ similar to what had already been done for p(n). However, Atkin [2] in 1966 found an error in his lemma 4 on which his main theorem depends and unfortunately his results are incorrect. Atkin does prove a theorem for the congruence properties modulo 5 and 7 for $p_{-k}(n)$, $(1 < k < 8)$.

The function $p_r(n)$ is very useful in obtaining elementary proofs for the congruences of $p(n)$. In 1964 Gandhi [17] used it to obtain very simple proofs for $p(5m + 4) \equiv 0 \pmod 5$ and $p(7m + 5) \equiv 0 \pmod 7$. The author has used it in Chapter V of this paper to develop a simple proof for $p(11m + 6) \equiv 0 \pmod{11}$.

The only applications of the theory of partitions which will be indicated in this paper will now be given as a quote from V. S. Nanda with reference given by Wright [52].

> The close similarity between the basic problems in statistical thermodynamics and the partition theory of numbers is now well recognized. In either case one is concerned with partitioning a large integer, under certain restrictions, which in effect means that the 'Zustandsumme' of a thermodynamic assembly is identical with the generating function of partitions appropriate to that assembly. ... Asymptotic expressions are deduced which constitute a generalization of the Hardy-Ramanujan formula for $p(n)$ which corresponds to an assembly of linear oscillators. ... It is remarkable that the Zustandsumme of an assembly of a variable number of two-dimensional oscillators is identical with the generating function of plane partitions. ... Further, it is noticed that a study of two-dimensional oscillator assembly is connected with the partitions of bi-partite numbers. ...

CHAPTER IV

GENERAL DEFINITIONS AND THEOREMS OF PARTITIONS

Partition Functions

A function which is defined on the natural numbers or positive rational integers is said to be an arithmetic function. Many arithmetic functions exist in number theory and the partition function to be discussed in this chapter is such a function.

Definition 4.1. A partition of the positive integer n is a representation of n as a sum of positive integers referred to as the summands or parts. Sums which differ only in the order of summands are considered the same partition.

Definition 4.2. The number of partitions of the positive integer n is the partition function $p(n)$.

Other partition functions of n have been defined by placing various restrictions on the summands and/or the number of summands. Some of these will be introduced and illustrated here. There are different symbols in the literature for certain functions (as you have perhaps noticed in the previous chapter), but to avoid duplication of symbols from this point on the following definition will be adhered to.

Definition 4.3. The functions $p(n;m)$, $p_A(n)$, $q(n)$, $q^e(n)$ and $q^o(n)$ represent the number of partitions of n with summands not exceeding m, summands from the set A, mutually distinct summands, an even number of mutually distinct summands and an odd number of mutually distinct summands, respectively.

To illustrate the partition functions, consider the following partitions of the number 5.

(1) 5

(2) 4 + 1

(3) 3 + 2

(4) 3 + 1 + 1

(5) 2 + 2 + 1

(6) 2 + 1 + 1 + 1

(7) 1 + 1 + 1 + 1 + 1

There are a total of seven partitions and hence $p(5) = 7$. The three partitions in (5) - (7) are the only ones with summands not exceeding 2 and consequently $p(5;2) = 3$. The partitions in (1), (4) and (7) have odd summands, with the set of odd integers represented by 0, we have $p_0(5) = 3$. Since no partition has only even (E) summands, $p_E(5) = 0$. The partitions with mutually distinct summands are (1), (2) and (3) so $q(5) = 3$. There are two partitions with an even number of mutually distinct summands and there is only one with an odd number of mutually distinct summands, thus $q^e(5) = 2$ and $q^o(5) = 1$.

An almost evident result relating the partition functions is given by the following theorems.

<u>Theorem 4.4.</u> (a) $p(n;m) = p(n)$ if $n \leq m$,

(b) $p(n;m) \leq p(n)$ for all $m \geq 1$,

(c) $q(n) = q^e(n) + q^o(n)$,

(d) $p(n) \geq p_E(n) + p_0(n)$ and

(e) $p_E(2n+1) = 0$.

Proof: (a) follows since each summand of n must be less than or equal to n. If it happens that m < n, then it is evident from the definition that the inequality will hold in (b), otherwise the equality will hold. The result in (c) follows since every partition with distinct summands must have either an even number or an odd number of summands. A partition of n must be one of the three types: summands all even, all odd, or some even and some odd, thus (d) follows. Since no odd integer can be expressed as the sum of even positive integers the result (e) follows.

Generating Functions

Power series are extremely useful in additive number theory because of the additive property in the laws of exponents. Much of the work in the theory of partitions has been done with the aid of power series. Certain functions represented by power series can be used to define the partition functions.

If a real valued function F can be expressed in a power series

$$F(x) = \sum_{n=0}^{\infty} f(n)\, x^n = f(0) + f(1)\, x + f(2)x^2 + \ldots + f(n)x^n + \ldots,$$

then the coefficient of x^n is the function value of f at n. Thus it is possible to use series to define an arithmetic function and it will be convenient to extend the domain of the function to include zero. Since the generating function (defined in Definition 4.6) for the partition functions will have constant terms equal 1, it is convenient to make the following definition.

Definition 4.5. $p(0) = p(0;m) = p_A(0) = q(0) = q^e(0) = q^0(0) = 1$.
That is, all partition functions are defined to be 1 for $n = 0$.

Definition 4.6. Any function $F(x) = \sum_{n=0}^{\infty} f(n)x^n$ is called a generating function of $f(n)$.

The convergence of the series is not really significant in terms of defining the arithmetic function $f(n)$ since it is the coefficients which give the function values. However, in most of the work convergence is needed and the series representing the partition functions converge for all real x numerically less than 1.

First a formal development of the generating functions will be given without regard to the question of convergence. The generating functions will be represented by the same letter but it will be capital. The proof will require Lemma 4.7.

Lemma 4.7. $p_A(n)$ is the number of distinct solutions of the diophantine equation

$$k_1a_1 + k_2a_2 + \ldots + k_ia_i + \ldots = n \tag{4.1}$$

in positive integers k_i and distinct elements a_i of A.

Proof: Any solution of (4.1) is by definition a partition of n, since the value of k_i indicates the number of times a_i occurs as a summand. Hence, the number of distinct solutions corresponds to the number of partitions of n with summands that are elements of A, i.e. $p_A(n)$.

Theorem 4.8. $p_A(x) = \prod_{a \in A} (1 - x^a)^{-1}$ and $Q(x) = \prod_{k=1}^{\infty} (1 + x^k)$ are generating functions for $p_A(n)$ and $q(n)$ respectively.

Proof: $(1 - x^a)^{-1} = \sum_{k=0}^{\infty} (x^a)^k$ (The interval of convergence will be considered in Theorem 4.11). From this it follows that

$$\begin{aligned} p_A(x) &= \prod_{a\in A} (1 - x^a)^{-1} \\ &= \prod_{a\in A} \sum_{k=0}^{\infty} x^{ak} \\ &= \prod_{a\in A} (1 + x^a + x^{2a} + \ldots + x^{ka} + \ldots) \\ &= \sum_{n=0}^{\infty} c_n x^n, \end{aligned}$$

where c_n is the number of distinct positive integral solutions of the diophantine equation (4.1). By Lemma 4.7, $c_n = p_A(n)$ and thus $P_A(x)$ is the generating function for $p_A(n)$. Similarly,

$$\begin{aligned} \prod_{k=1}^{\infty} (1 + x^k) &= (1 + x)(1 + x^2)(1 + x^3)\ldots \\ &= \sum_{n=0}^{\infty} d_n x^n, \end{aligned}$$

where d_n is the number of distinct solutions of the diophantine equation

$1k_1 + 2k_2 + 3k_3 + \ldots = n$, where k_i $(i = 1, 2, 3, \ldots)$ is 0 or 1.

Hence, d_n is the number of times that n can be expressed as the sum of distinct positive integers, i.e. $q(n)$. Thus, $Q(x)$ is the generating function for $q(n)$ and the theorem is proved.

Other generating functions follow as a consequence of Theorem 4.8.

Corollary 4.9.

(a) $P(x) = \prod_{k=1}^{\infty} (1 - x^k)^{-1} = \sum_{n=0}^{\infty} p(n)\, x^n,$

(b) $P_E(x) = \prod_{k=1}^{\infty} (1 - x^{2k})^{-1} = \sum_{n=0}^{\infty} p_E(n)\, x^n,$

(c) $P_0(x) = \prod_{k=1}^{\infty} (1 - x^{2k-1})^{-1} = \sum_{n=0}^{\infty} p_0(n)\, x^n,$ and

(d) $P(x;m) = \prod_{k=1}^{m} (1 - x^k)^{-1} = \sum_{n=0}^{\infty} p(n;m)\, x^n.$

That is, $P(x)$, $P_E(x)$, $P_0(x)$, and $P(x;m)$ are generating functions for $p(n)$, $p_E(n)$, and $p(n;m)$ respectively.

Proof: This follows immediately from Theorem 4.8 where A is the set of positive integers, even positive integers, odd positive integers, and positive integers not exceeding m.

The next theorem shows that the generating function P(x) actually converges for all x such that $|x| < 1$. All the other generating functions converge for $|x| < 1$ with similar proofs which will not be included here. This implies that they represent analytic functions at least inside the unit circle.

A convenient notation which is used through-out this paper will be stated in the following definition.

Definition 4.10. For all x (particularly x = 0),

$$\sum_{n=0}^{\infty} p(n)\, x^n = 1 + \sum_{n=1}^{\infty} p(n)\, x^n.$$

This definition applies for other partition functions as well.

Theorem 4.11. [23] For $|x| < 1$, $P(x) = \prod_{k=1}^{\infty} (1 - x^k)^{-1} = \sum_{n=0}^{\infty} p(n)\, x^n$.

Proof: Let z be a complex number such that $|z| = r < 1$. Then

$$\begin{aligned} |P(z)| &\leq P(r) \\ &= \prod_{n=1}^{\infty} (1 - r^n)^{-1} \\ &= \prod_{n=1}^{\infty} (1 + a_n), \end{aligned}$$

where $a_n = \sum_{k=1}^{\infty} (r^{nk})$, since $(1 - r^n)^{-1} = \sum_{k=0}^{\infty} (r^n)^k$. But with $0 < r < 1$, the geometric series a_n converges absolutely and has sum $\frac{r^n}{1 - r^n}$. The product of absolutely convergent series converges by Theorem 2.24 and thus,

$$|P(z)| \leq \prod_{n=1}^{\infty} \left(1 + \frac{r^n}{1 - r^n}\right),\ 0 < r < 1,$$

which converges by Theorem 2.28 since $\sum_{n=1}^{\infty} \frac{r^n}{1 - r^n}$ converges absolutely for $0 < r < 1$. In view of Corollary 4.9(a) this completes the proof. A longer proof with less advanced analysis is given by Niven and Zuckerman [38].

It should be observed here that the convergence of the generating function is uniform for $|z| \leq 1 - \epsilon$ (for arbitrarily small $\epsilon > 0$). It can also be observed from the product representation of the generating function that it fails to converge on or outside the unit circle.

A simple application of the generating functions will be illustrated in the proof of Theorem 4.12.

Theorem 4.12. The number of partitions of n whose summands are odd integers is equal to the number of partitions of n with distinct summands. That is, $p_0(n) = q(n)$.

Proof: It will only be necessary to show that they have the same generating functions since a given partition function is represented (or enumerated) by a unique series.

From Theorem 4.8,

$$\begin{aligned} Q(x) &= \prod_{k=1}^{\infty} (1 + x^k) \\ &= (1 + x)(1 + x^2)(1 + x^3) \ldots (1 + x^k) \ldots \\ &= \frac{1 - x^2}{1 - x} \; \frac{1 - x^4}{1 - x^2} \; \frac{1 - x^6}{1 - x^3} \ldots \frac{1 - x^{2k}}{1 - x^k} \ldots . \end{aligned}$$

Since the factors in the numerators, $(1 - x^{2k})$ for all $k \in N$ (natural numbers), also occur in the denominator leaving only the factors $(1 - x^{2k-1})$ for $k \in N$ in the denominator, it follows that

$$Q(x) = \prod_{k=1}^{\infty} (1 - x^{2k-1})^{-1}$$

which is the generating function for $P_0(x)$ given in Corollary 4.9.

An interesting alternative proof of Theorem 4.12 which uses the base two representation of the natural numbers is given by Hardy and Wright [26].

The next theorem will be given not only to illustrate the use of generating functions in obtaining a relation between two partition functions, but also to illustrate a technique used by Euler of introducing a second variable. This theorem involves $p(n;r;m)$, the number of partitions of n into r summands with each summand greater than or equal to m, and $p(n;r)$ given in Definition 4.3. This proof, with slightly different notation, is given by Haskell [27].

Theorem 4.13. $$p(n;r;m) = p(n-rm;r).$$

Proof: From Corollary 4.9(d) the generating function of $p(n;r)$ is

(1) $$\prod_{k=1}^{r} (1 - x^k)^{-1} = \sum_{n=0}^{\infty} p(n;r)\, x^n.$$

The function $p(n;r;m)$, already known by MacMahon [32], is the coefficient of $a^r x^n$ in the expansion of

(2) $$F(a;x;m) = \prod_{j=0}^{\infty} (1 - ax^{m+j})^{-1}$$

$$= \sum_{r=0}^{\infty} g_r(x;m)\, a^r, \text{ where}$$

(3) $$g_0(x;m) = 1 \text{ and } g_r(x;m) = \sum_{n=0}^{\infty} p(n;r;m)\, x^n.$$

Replace a by ax in (2) to obtain

$$F(ax;x;m) = \frac{1}{(1-ax^{m+1})(1-ax^{m+2})(1-ax^{m+3})\cdots}$$

$$= \frac{(1-ax^m)}{(1-ax^m)(1-ax^{m+1})(1-ax^{m+2})\cdots}$$

$$= (1-ax^m)\, F(a;x;m).$$

Thus,

$$F(ax;x;m) = (1-ax^m)\, F(a;x;m),$$

or from (2),

$$\sum_{r=0}^{\infty} g_r(x;m)\, a^r x^r = \sum_{r=0}^{\infty} g_r(x;m)\, a^r - ax^m \sum_{r=0}^{\infty} g_r(x;m)\, a^r.$$

Equate coefficients of a^r to obtain,

$$x^r\, g_r(x;m) = g_r(x;m) - x^m\, g_{r-1}(x;m).$$

Hence,

$$g_r(x;m) = \frac{x^m}{1 - x^r}\, g_{r-1}(x;m).$$

Thus, with $g_0(x;m) = 1$, it follows that,

$$g_1(x;m) = \frac{x^m}{1 - x},$$

$$g_2(x;m) = \frac{x^m}{1 - x^2}\ \frac{x^m}{1 - x},$$ and by induction

(4)
$$g_r(x;m) = \frac{x^m}{1 - x}\ \frac{x^m}{1 - x^2} \cdots \frac{x^m}{1 - x^r}$$

$$= \frac{x^{rm}}{(1 - x)(1 - x^2) \cdots (1 - x^r)}.$$

Therefore, from (1) and (3), (4) becomes

$$\sum_{n=0}^{\infty} p(n;r;m)\, x^n = x^{rm} \sum_{n=0}^{\infty} p(n;r)\, x^n.$$

Equate the coefficients to obtain the results of the theorem.

Theorems of Euler and Jacobi

Perhaps the most useful theorems in the theory of partitions are Theorems 4.15 and 4.16, due to Euler and Jacobi, respectively. Euler considered the function defined by the reciprocal of the generating function $P(x)$ which will be stated for reference purposes in the following definition.

Definition 4.14. $[P(x)]^{-1} = \prod_{k=1}^{\infty} (1 - x^k)$ and denoted by $\Phi(x)$.

Euler was able to express this function in a series represented in equation (4.2). This will be proved by the algebraic method of Shanks [45].

Theorem 4.15. (Euler's identity)

(4.2) $$\Phi(x) = \prod_{k=1}^{\infty} (1 - x^k) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n+1)/2}.$$

Proof: First notice that the identity may be written in the form

(4.3) $$\prod_{k=1}^{\infty} (1 - x^k) = 1 + \sum_{n=1}^{\infty} (-1)^n [x^{n(3n-1)/2} + x^{n(3n+1)/2}].$$

Let the partial products and partial sums of (4.3) be

$$P_0 = 1, \; P_n = \prod_{s=1}^{n} (1 - x^s)$$

and

$$S_n = 1 + \sum_{s=1}^{n} (-1)^s [x^{s(3s-1)/2} + x^{s(3s+1)/2}].$$

It is important to observe that,

(4.4) $$\frac{P_n}{P_{n-1}} = 1 - x^n$$

and

(4.5) $$S_n - S_{n-1} = (-1)^n [x^{n(3n-1)/2} + x^{n(3n+1)/2}].$$

It will first be shown by mathematical induction that

(4.6) $$S_n = F_n, \text{ where } F_n = \sum_{s=0}^{n} (-1)^s \frac{P_n}{P_s} x^{sn+s(s+1)/2}.$$

i) If $n = 1$, then

$$S_1 = 1 - (x + x^2)$$
$$= (1 - x) + x^2 = F_1.$$

Thus, (4.6) is true for $n = 1$.

ii) Now it will be shown that $S_{k-1} = F_{k-1}$ implies $S_k = F_k$. From the definition of F_n in (4.6), and detaching the last term of the summation,

$$F_k = \sum_{s=0}^{k-1} (-1)^s \frac{P_k}{P_s} x^{sk+s(s+1)/2} + (-1)^k x^{k(3k+1)/2}.$$

Use (4.4) in the form $P_k = P_{k-1}(1 - x^k)$ to split the summation into two parts. This gives,

$$F_k = \sum_{s=0}^{k-1} (-1)^s \frac{P_{k-1}}{P_s} x^{sk+s(s+1)/2} + \sum_{r=0}^{k-1} (-1)^{r+1} \frac{P_{k-1}}{P_r} x^{k+rk+r(r+1)/2} + (-1)^k x^{k(3k+1)/2}.$$

Now detach the first term, $s = 0$, in the first summation and detach the last term, $r = k-1$, in the second summation to obtain,

$$F_k = \frac{P_k}{P_0} + \sum_{s=1}^{k-1} (-1)^s \frac{P_{k-1}}{P_s} x^{sk+s(s+1)/2} + \sum_{r=0}^{k-2} (-1)^{r+1} \frac{P_{k-1}}{P_r} x^{k(r+1)+r(r+1)/2} + (-1)^k \left[x^{k(3k-1)/2} + x^{k(3k+1)/2}\right].$$

Let $r = s-1$ in the second summation in order to re-combine summations to give,

$$F_k = \frac{P_k}{P_0} + \sum_{s=1}^{k-1} (-1)^s \frac{P_{k-1}}{P_s} \left[x^{sk+s(s+1)/2} + \frac{P_s}{P_{s-1}} x^{ks+s(s-1)/2}\right] + (-1)^k \left[x^{k(3k-1)/2} + x^{k(3k+1)/2}\right].$$

From (4.4), it follows that $\frac{P_s}{P_{s-1}} = 1 - x^s$. This substitution in the brackets of the summation and including the first term in the summation for $s = 0$ gives,

$$F_k = \sum_{s=0}^{k-1} (-1)^s \frac{P_{k-1}}{P_s} x^{s(k-1)+s(s+1)/2} + (-1)^k \left[x^{k(3k-1)/2} + x^{k(3k+1)/2}\right].$$

$$= F_{k-1} + (S_k - S_{k-1}),$$

by the use of (4.5) with $n = k$. This can be written in the form,

$$S_{k-1} - F_{k-1} = S_k - F_k.$$

Thus, if $S_{k-1} = F_{k-1}$, then $S_k = F_k$ and equation (4.6) follows by induction. Now to compare F_n and P_n, we observe that the first term in definition (4.6) of F_n gives P_n. That is,

$$F_n = P_n - \frac{P_n}{P_1} x^{n+1} + \dots + (-1)^s \frac{P_n}{P_s} x^{sn+s(s+1)/2} + \dots + (-1)^{n(3n+1)/2}.$$

Since the degree of P_s is $s(s+1)/2$, it follows that all the terms of F_n after the first are of degree $n+1$ or greater. This implies that all the terms of degree less than $n+1$ agree for both F_n and P_n. Let $n \to \infty$ and the series representation for both numbers of (4.3) are the same and the theorem is proved.

Hardy and Wright [26] have proved several theorems which use Euler's device of the introduction of a second parameter which Euler used in proving his identity (4.2). However, Hardy and Wright prove the identity (4.2) as a special case of Jacobi's theta formula (3.5). Since this formula belongs properly to the theory of elliptic functions, it will not be proven here. It can also be used to prove [26], as a special case, the very useful identity of Jacobi's.

Theorem 4.16. (Jacobi's identity)

$$(4.7) \qquad \Phi^3(x) = \prod_{k=1}^{\infty} (1 - x^k)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1)\, x^{n(n+1)/2}.$$

Proof: Can be found on page 285 of Hardy and Wright [26].

Some Generalizations and Relations

Euler was the first to consider the reciprocal of the generating function for $p(n)$, namely $\Phi(x)$. Many have since considered the more general definition.

Definition 4.17. Define $p_r(n)$, for all integral r and positive integral n, by the equation,

$$\Phi^r(x) = \prod_{k=1}^{\infty} (1 - x^k)^r = \sum_{n=0}^{\infty} p_r(n)\, x^n.$$

For convenience define $p_r(0) = 1$ and $p_r(-n) = 0$.

It is important for the reader to note that $p_{-1}(n)$ is identical to $p(n)$, the ordinary partition function of n, i.e. the number of unrestricted linear partitions of n. Another function closely related to $p_r(n)$ is Ramanujan's τ function [16] which is of no particular interest here.

It is interesting to note that the only power series representation of $\Phi^r(x)$ are those for r = 1 and 3 given by Euler's identity (4.2) and Jacobi's identity (4.7) respectively. The function $p_r(n)$, for different values of r, is related to the k-line partition function. The real significance of this function will be evident in Chapter V.

Definition 4.18. A k-line partition of the positive integer n is a representation of n in the form,

$$n = \sum_{i=1}^{k} \sum_{j=1}^{\infty} a_{i,j} \tag{4.8}$$

where the $a_{i,j}$ (summands) are non-negative integers which satisfy the conditions $a_{i,j} \geq a_{i,j+1}$ and $a_{i,j} \geq a_{i+1,j}$.

The representation of n in (4.8) is considered an unrestricted k-line partition. The reason for the conditions on $a_{i,j}$ is simply for the convenience of arranging the summands in a decreasing order.

Definition 4.19. The number of k-line partitions of the positive integer n is the k-line partition function $t_k(n)$. For convenience we define $t_k(0) = 1$ and $t_k(-n) = 0$.

A k-line partition of n may be conveniently written down by arranging the summands in k rows (lines) with $a_{i,j}$ as the j-th member of i-th row. It will be convenient to omit zero summands and the plus signs. Thus, the three-line partitions of 4 are:

4, 31, 22, 211, 1111,

3	2	21	111	11
1,	2,	1 ,	1 ,	11,

2	11
1	1
1,	1 ,

and evidently $t_3(4) = 12$. It can also be observed that $p(4) = t_1(4) = 5$, $t_2(4) = 10$, and since there would only be one four-line partition of 4, $t_4(4) = 13$.

Again the reader can notice that a one-line (linear) partition is identical to the ordinary partition and we have $p(n) = t_1(n) = p_{-1}(n)$.

An interesting result which shows the relation between the number of two-line partitions of n and the ordinary unrestricted partitions of n is given by Theorem 4.20.

Theorem 4.20. The number of solutions of (4.8) with $k = 2$ and with the additional restriction that $a_{i,j} > a_{i,j+1}$ (i.e. strictly decreasing along rows) is $p(n)$.

Proof: A direct proof by combinatorial methods has been given by Sudler [46]. His references list a proof by Gordon who used the generating function.

It was pointed out in Chapter III that there are several different proofs of the generating function for $t_k(n)$.

Theroem 4.21. The generating function for $t_k(n)$ is given by

$$\prod_{m=1}^{\infty} (1 - x^m)^{-\min(m,k)} = \sum_{n=0}^{\infty} t_k(n)\, x^n.$$

Proof: A proof is given by Chaundy [10].

The plane partition function, $t(k)$ (sometimes denoted by $q(n)$), is the k-line partition function where $k \to \infty$. The generating function for plane partitions is given by Theorem 4.22.

Theorem 4.22. $$\prod_{m=1}^{\infty} (1 - x^m)^{-m} = \sum_{n=0}^{\infty} t(n)\, x^n.$$

Proof: A proof is given by Chaundy [10].

The remaining theorems in this chapter, except for Theorems 4.34 and 4.35, will be given not only to see the relation between $t_k(n)$ and $p_k(n)$, but also because they are needed in Chapter V.

Theorem 4.23. [12] $t_2(n) = p_{-2}(n) - p_{-2}(n-1)$ for integral n.

Proof: The result is immediate for non-positive integers n by Definitions 4.17 and 4.19. For positive n, we have, first by the use of Theorem 4.21 and later the use of Definition 4.17:

$$\begin{aligned}\sum_{n=0}^{\infty} t_2(n)\, x^n &= (1 - x)^{-1} \prod_{m=2}^{\infty} (1 - x^m)^{-2} \\ &= (1 - x)(1 - x)^{-2} \prod_{m=2}^{\infty} (1 - x^m)^{-2} \\ &= (1 - x) \prod_{m=1}^{\infty} (1 - x^m)^{-2} \\ &= (1 - x) \sum_{n=0}^{\infty} p_{-2}(n)\, x^n \\ &= \sum_{n=0}^{\infty} p_{-2}(n)\, x^n - \sum_{n=0}^{\infty} p_{-2}(n)\, x^{n+1}.\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} t_2(n)\, x^n = \sum_{n=0}^{\infty} [p_{-2}(n) - p_{-2}(n-1)]\, x^n.$$

Now to equate the coefficients of x^n we obtain,

$t_2(n) = p_{-2}(n) - p_{-2}(n-1)$ and the theorem is proved.

Theorem 4.24. $t_3(n) = p_{-3}(n) - 2p_{-3}(n-1) + 2p_{-3}(n-3) - p_{-3}(n-4)$.

Proof: From Theorem 4.21 and later Definition 4.17,

$$\sum_{n=0}^{\infty} t_3(n)\, x^n = \prod_{m=1}^{\infty} (1 - x^m)^{-\min(m,3)}$$

$$= \frac{1}{(1-x)} \frac{1}{(1-x^2)^2} \prod_{m=3}^{\infty} (1 - x^m)^{-3}$$

$$= \frac{(1-x)^2}{(1-x)^3} \frac{(1-x^2)}{(1-x^2)^3} \prod_{m=3}^{\infty} (1 - x^m)^{-3}$$

$$= (1-x)^2(1-x^2) \prod_{m=1}^{\infty} (1 - x^m)^{-3}$$

$$= (1-2x+2x^3-x^4) \sum_{n=0}^{\infty} p_3(n)\, x^n.$$

Hence,

$$\sum_{n=0}^{\infty} t_3(n)\, x^n = \sum_{n=0}^{\infty} [p_{-3}(n) - 2p_{-3}(n-1) + 2p_{-3}(n-3) - p_{-3}(n-4)]x^n.$$

Equate the coefficients of x^n to obtain the results of the theorem.

Actually, as indicated by Cheema and Gordon [12], the technique used in Theorems 4.22 and 4.23 can be extended arbitrarily to give $p_{-k}(n)$ in terms of $t_k(n)$ and conversely. For example,

$$\sum_{n=0}^{\infty} t_k(n)\, x^n = \prod_{m=1}^{\infty} (1 - x^m)^{-\min(m,k)}$$

$$= \prod_{m=1}^{k-1} (1 - x^m)^{k-m} \prod_{m=1}^{\infty} (1 - x^m)^{-k}$$

$$= \sum_{n=0}^{v} a_k(n)\, x^n \sum_{n=0}^{\infty} p_{-k}(n)\, x^n,$$

where $v = v(k)$ will be determined by k and $a_k(n)$ is determined by the finite product in equation (4.9).

(4.9) $$\sum_{n=0}^{v} a_k(n)\, x^n = \prod_{m=1}^{k-1} (1 - x^m)^{k-m}.$$

Therefore we have,

$$\sum_{n=0}^{\infty} t_k(n)\, x^n = \sum_{n=0}^{\infty} [\sum_{s=0}^{v} a_k(s)\, p_{-k}(n-s)]\, x^n.$$

Equate the coefficients of both members to obtain Theorem 4.25.

<u>Theorem 4.25</u>. $t_k(n) = \sum_{s=0}^{v} a_k(s)\, p_{-k}(n-s),$

where v and $a_k(s)$ are given by equation (4.9).

Theorem 4.26. [13] $p_{-2}(n) = \sum_{k=0}^{n} t_2(k)$.

Proof: From Theorems 4.21 and Definition 4.17, with k = 2, we have,

$$\sum_{n=0}^{\infty} t_2(n)\, x^n = (1-x) \prod_{m=1}^{\infty} (1 - x^m)^{-2}$$

$$= (1-x) \sum_{n=0}^{\infty} p_{-2}(n)\, x^n.$$

Thus,

$$\sum_{n=0}^{\infty} p_{-2}(n)\, x^n = \frac{1}{1-x} \sum_{n=0}^{\infty} t_2(n)\, x^n$$

$$= \sum_{k=0}^{\infty} x^k \sum_{n=0}^{\infty} t_2(n)\, x^n$$

$$= \sum_{n=0}^{\infty} \Big[\sum_{k=0}^{n} t_2(n-k)\Big]\, x^n.$$

Equate the coefficients of x^n to obtain the results.

The proof of the next theorem can be simplified by use of the following lemma.

Lemma 4.27. $\log \Phi(x) = -\sum_{n=1}^{\infty} \sigma(n)\, x^n/n.$

Proof: From elementary analysis,

$$\log (1-y) = - y - y^2/2 - y^3/3 - \dots, \quad -1 \leq y < 1,$$

and hence,

$$\log (1-x^k) = - x^k - x^{2k}/2 - x^{3k}/3 - \dots = -\sum_{m=1}^{\infty} x^{mk}/m.$$

Now from the definition of $\Phi(x)$, Definition 4.14 and some straightforward manipulations we obtain,

$$\log \Phi(x) = \log \prod_{k=1}^{\infty} (1-x^k)$$

$$= \sum_{k=1}^{\infty} \log (1-x^k)$$

$$= \sum_{k=1}^{\infty} \Big[-\sum_{m=1}^{\infty} x^{mk}/m\Big]$$

$$= -\sum_{k=1}^{\infty} [x^k + x^{2k}/2 + x^{3k}/3 + \ldots + x^{mk}/m + \ldots]$$

$$= -\sum_{n=1}^{\infty} [\sum_{rk=n} 1/r]\ x^n$$

$$= -\sum_{n=1}^{\infty} [\sum_{d|n} 1/d]\ x^n$$

$$= -\sum_{n=1}^{\infty} [1/n \sum_{d|n} n/d]\ x^n$$

$$= -\sum_{n=1}^{\infty} [1/n \sum_{d|n} d]\ x^n$$

$$= -\sum_{n=1}^{\infty} \sigma(n)\ /n\ x^n$$

and consequently,

$$\log \Phi(x) = -\sum_{n=1}^{\infty} \sigma(n)/n\ x^n.$$

Theorem 4.28. $p_r(n) = -\ r/n \sum_{j=1}^{n} \sigma(j)\ p_r(n-j).$

Proof: Take the logarithm of each member of,

$$\Phi^r(x) = \sum_{n=0}^{\infty} p_r(n)\ x^n$$

to obtain,

$$r \log \Phi(x) = \log \sum_{n=0}^{\infty} p_r(n)\ x^n.$$

Substitution for $\log \Phi(x)$ from Lemma 4.27 yields,

$$-\ r \sum_{n=1}^{\infty} \sigma(n)\ x^n/n = \log \sum_{n=0}^{\infty} p_r(n)\ x^n.$$

Differentiate each side with respect to x and then multiply both members by $\sum_{n=0}^{\infty} p_r(n)\ x^n$ to obtain,

$$-\ r \sum_{n=1}^{\infty} \sigma(n)\ x^{n-1} \sum_{n=0}^{\infty} p_r(n)\ x^n = \sum_{n=0}^{\infty} n\ p_r(n)\ x^{n-1}$$

or

$$-\ r \sum_{n=1}^{\infty} \sum_{j=1}^{n} \sigma(j)\ p_r(n-j)\ x^{n-1} = \sum_{n=0}^{\infty} n\ p_r(n)\ x^{n-1}.$$

Equate the coefficients of x^{n-1} to obtain,

$$-\ r \sum_{j=1}^{n} \sigma(j)\ p_r(n-j) = n\ p_r(n)$$

or

$$p_r(n) = - r/n \sum_{j=1}^{n} \sigma(j)\, p_r(n-j).$$

The more common relation is obtained by letting $r = -1$ and recalling $p_{-1}(n) = p(n)$ to yield Corollary 4.29.

Corollary 4.29. $$p(n) = 1/n \sum_{j=1}^{n} \sigma(j)\, p(n-j).$$

This result is interesting in that it relates the arithmetic function, $\sigma(n)$, of multiplicative number theory and the partition function, $p(n)$, of additive number theory.

It will be more convenient to use the recursion formula, of Corollary 4.29, in Theorem 6.3, if it is put in a slightly different form.

Theorem 4.30. $$n\, p(n) = \sum_{m=1}^{n} \sum_{k=1}^{n} m\, p(n-km).$$

Proof: From Corollary 4.29, after multiplying both members by n and then applying the definition for $\sigma(j)$, we obtain,

$$n\, p(n) = \sum_{j=1}^{n} \sigma(j)\, p(n-j)$$

$$= \sum_{j=1}^{n} p(n-j) \sum_{m \mid j} m.$$

Now replace j by km so that,

$$\sum_{j=1}^{n} p(n-j) \sum_{m \mid j} m = \sum_{m=1}^{n} \sum_{k=1}^{n/m} p(n-km)m$$

$$= \sum_{m=1}^{n} \sum_{k=1}^{n} m\, p(n-km)$$

and the theorem is proved.

A similar recursion formula for the plane partition function, $t(n)$, can also be given.

Theorem 4.31. [27] $$n\, t(n) = \sum_{j=0}^{n-1} t(j)\, \sigma_2(n-j).$$

Proof: Let the $\tau(n)$ divisors of n, refer to Definition 2.19, be represented as follows: $n = d_1^n > d_2^n > \dots > d_{\tau(n)-1}^n > d_{\tau(n)}^n = 1$. Take the natural logarithm of both members of Theorem 4.21, the generating function of t(n), and carry out the following manipulations:

$$\sum_{n=0}^{\infty} t(n)\, x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-n}$$

$$\ln\left(\sum_{n=0}^{\infty} t(n)\, x^n\right) = -\sum_{n=1}^{\infty} n \ln (1 - x^n)$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{n\, x^{nj}}{j}$$

$$= (x + x^2/2 + x^3/3 + x^4/4 + \dots + x^j/j + \dots) + 2(x^2 + x^4/2 + \dots + x^{2j}/j + \dots) + 3(x^3 + x^6/2 + \dots + x^{3j}/j + \dots) + \dots + n(x^n + x^{2n}/2 + \dots + x^{nj}/j + \dots) + \dots$$

$$= x + (1/2 + 2\cdot 1)x^2 + (1/3 + 3\cdot 1)x^3 + (1/4 + 2\cdot 1/2 + 4\cdot 1)x^4 + \dots + \left(1/n + d_{\tau(n)-1}^n \cdot 1/d_2^n + d_{\tau(n)-2}^n \cdot 1/d_3^n + \dots + n\cdot 1\right)x^n + \dots$$

$$= x + \frac{(1+2^2)}{2} x^2 + \frac{(1+3^2)}{3} x^3 + \frac{(1+2^2+4^2)}{4} x^4 + \dots + \frac{\left[1 + \left(d_{\tau(n)-1}^n\right)^2 + \left(d_{\tau(n)-2}^n\right)^2 + \dots + n^2\right]}{n} x^n + \dots$$

$$= \sum_{n=1}^{\infty} \frac{\sigma_2(n)}{n} x^n.$$

Now take the derivative with respect to x of both members, then multiply by x and $\sum_{n=0}^{\infty} t(n)\, x^n$ to obtain,

$$\frac{\sum_{n=1}^{\infty} n\, t(n)\, x^n}{\sum_{n=0}^{\infty} t(n)\, x^n} = \sum_{n=1}^{\infty} \sigma_2(n)\, x^n,$$

and hence,

$$\sum_{n=1}^{\infty} n\, t(n)\, x^n = \sum_{n=0}^{\infty} t(n)\, x^n \sum_{n=1}^{\infty} \sigma_2(n)\, x^n$$

$$= \sum_{n=1}^{\infty} \left(\sum_{j=0}^{n-1} t(j)\, \sigma_2(n-j)\right) x^n.$$

Equate the coefficients of x^n which completes the proof.

Let $(n_1, n_2, \dots, n_s)$ be a non-negative s-vector, i.e. an ordered s-tuple of non-negative integers.

Definition 4.32. A vector partition of a vector will mean a representation as a sum of non-zero vectors called parts or summands. Sums which differ only in the order of parts are regarded as the same partition.

Definition 4.33. The number of vector partitions of the vector $(n_1, n_2, \ldots, n_s)$ is the vector partition function $p(n_1, n_2, \ldots, n_s)$. We again define $p(0, 0, \ldots, 0) = 1$.

To illustrate, consider the following partitions of (2,1):

$$\begin{aligned}(2,1) &= (2,1)\\ &= (2,0) + (0,1)\\ &= (1,1) + (1,0)\\ &= (1,0) + (1,0) + (0,1),\end{aligned}$$

and consequently $p(2,1) = 4$.

The generating function for the unrestricted vector partition function was first known by MacMahon [33].

Theorem 4.34. The generating function for $p(n_1, n_2, \ldots, n_s)$ is given by

$$(4.9)\quad \prod_{k_i \geq 0} (1 - x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s})^{-1} = \sum_{n_i = 0}^{\infty} p(n_1, n_2, \ldots, n_s)\, x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s},$$

where $k_1 + k_2 + \cdots + k_s > 0$ and $n_i = n_1, n_2, \ldots, n_s$.

A recursion formula for the vector partition, similar to Theorem 4.30 for ordinary partitions and Theorem 4.31 for plane partitions, was obtained by Cheema [11] and is given in the following theorem.

Theorem 4.35. $n_1 p(n_1, n_2, \ldots, n_s)$

$$= \sum_{k_i = 0}^{\infty} \left(\sum_{t \mid (k_1, \ldots, k_s)} k_1/t \right) p(n_1 - k_1, \ldots, n_s - k_s).$$

Proof: Take the logarithm of each member of (4.9) and simplify to obtain,

$$\log \sum_{n_i=0}^{\infty} p(n_1,\ldots,n_s)\, x_1^{n_1} \ldots x_s^{n_s} = \log \prod_{k_i\geq 0} (1 - x_1^{k_1} \ldots x_s^{k_s})^{-1}$$

$$= \sum_{k_i\geq 0} \log (1 - x_1^{k_1} \ldots x_s^{k_s})^{-1}$$

$$= -\sum_{k_i\geq 0} \log (1 - x_1^{k_1} \ldots x_s^{k_s})$$

$$= \sum_{n,k_i\geq 0} \frac{(x_1^{k_1} \ldots x_s^{k_s})^n}{n},$$

where $(k_1,\ldots,k_s) \neq (0,\ldots,0)$. Hence,

$$\log \sum_{n_i=0}^{\infty} p(n_1,\ldots,n_s)\, x_1^{n_1} x_2^{n_2} \ldots x_s^{n_s} = \sum_{n,k_i\geq 0} \frac{x_1^{nk_1} \ldots x_s^{nk_s}}{n}.$$

Now take the partial derivative with respect to x_1 of each member to obtain,

$$\frac{\sum_{n_i\geq 0} n_1 p(n_1,\ldots,n_s)\, x_1^{n_1-1} x_2^{n_2} \ldots x_s^{n_s}}{\sum_{n_i\geq 0} p(n_1,\ldots,n_s)\, x_1^{n_1} x_2^{n_2} \ldots x_s^{n_s}} = \sum_{n,k_i\geq 0} k_1\, x_1^{nk_1-1} x_2^{nk_2} \ldots x_s^{nk_s}$$

which yields, after multiplying by x_1 and the denominator,

$$\sum_{n_i\geq 0} n_1 p(n_1,\ldots,n_s)\, x_1^{n_1} x_2^{n_2} \ldots x_s^{n_s}$$

$$= \left(\sum_{n_i\geq 0} p(n_1,\ldots,n_s)\, x_1^{n_1} \ldots x_s^{n_s}\right)\left(\sum_{n,k_i\geq 0} k_1\, x_1^{nk_1} \ldots x_s^{nk_s}\right)$$

$$= \sum_{n_i\geq 0} \left[\left(\sum_{k_i\geq 0} p(n_1-k_1,\, n_2-k_2,\, \ldots,\, n_s-k_s)\right)\left(\sum_{t \| (k_1,\ldots,k_s)} k_1/t\right)\right] x_1^{n_1} \ldots x_s^{n_s}$$

Equate the coefficients and the theorem is proved.

One may perhaps most easily use the recursive formulae given in Theorems 4.25, 4.28 and 4.31 to compute $t_r(n)$, $p_r(n)$ and $t(n)$ respectively. Values for $t_1(n) = p_{-1}(n) = p(n)$, $t_2(n)$, $t_3(n)$, $t_5(n)$, $t(n)$, and $p_{-2}(n)$ are given for $n \leq 34$ in appendix A. Haskell [27], to my knowledge, has the only printed table for $p_{-2}(n)$, $t_5(n)$ and $t(n)$ for all values of $n \leq 299$. His values for $t_k(n)$ (k=2,3,5 and 25) and $t(n)$ were computed by Lippman on the IBM 7072 computer at the University of Arizona in 1963. Values for $p_{-2}(n)$ were computed on the IBM 1620 at California State Polytechnic College, San Luis Obsispo by Messrs. Kay and Arndt.

CHAPTER V

CONGRUENCE PROPERTIES

Congruences of $p(n)$ and $p_r(n)$

The purpose of this chapter is to investigate arithmetic properties of the partition functions. For example, it would be good to know if n can be determined for which $p(n)$ is even, odd, a multiple of 5, a multiple of 7, etc.

The first congruences to be considered are the well-known Ramanujan's congruences:

(5.1) $$p(5m + 4) \equiv 0 \pmod 5,$$

(5.2) $$p(7m + 5) \equiv 0 \pmod 7 \text{ and}$$

(5.3) $$p(11m + 6) \equiv 0 \pmod{11}.$$

The proofs of (5.1) and (5.2) have been much easier than the proof of (5.3). The proof of (5.3) has required much more advanced analysis requiring complex integration and a modular transformation which of course will require the convergence of the series used. Gandhi [17] has given a remarkably simple proof of (5.1) and (5.2) which is a special case of a congruence for $p_r(n)$. The author will give this proof for (5.1) and then apply this method, thus giving a simple proof, to (5.3).

Before proving the congruences it will be necessary to prove some preliminary theorems which have been developed by Gandhi [16].

<u>Theorem 5.1</u>. If $r/n = m/t$, $(m,t) = 1$, then $p_r(n) \equiv 0 \pmod m$, for

integral values of r.

Proof: Substitute m/t in place of r/n in Theorem 4.28 to obtain,

$$p_r(n) = - m/t \sum_{j=1}^{\infty} \sigma(j)\, p(n-j).$$

Since $m/t \equiv 0 \pmod{m}$, the theorem follows.

It will be convenient to consider $r = \pm R$ (R a prime) in Theorem 5.1. Also, if n and R are coprime, we have the following corollary.

Corollary 5.2. If R is a prime and coprime with n, then

$$p_{\pm R}(n) \equiv 0 \pmod{R}.$$

Theorem 5.3. If R is a prime, then $p_{kR}(mR) \equiv p_k(m) \pmod{R}$, for integral values of k.

Proof: From Theorems 2.11 and 2.12, it follows, for all integral k and prime R,

$$[(1 - x)^R]^k \equiv [(1 - x^R)]^k \pmod{R}.$$

Now replacing x by x^m for any positive integer m,

$$(1 - x^m)^{Rk} \equiv (1 - x^{mR})^k \pmod{R}.$$

Hence,

$$\prod_{m=1}^{\infty} (1 - x^m)^{Rk} \equiv \prod_{m=1}^{\infty} (1 - x^{mR})^k \pmod{R}.$$

But, by Definition 4.17, this is equivalent to;

$$\sum_{n=0}^{\infty} p_{kR}(n)\, x^n \equiv \sum_{n=0}^{\infty} p_k(n)\, x^{Rn} \pmod{R}.$$

Now compare the coefficients of x^{Rm} and use Definition 2.7 to obtain,

$$p_{kR}(mR) \equiv p_k(m) \pmod{R},$$

and the theorem is proved.

Theorem 5.4. [17] If R is a prime, then

$$p_{-(R-4)}(mR + t) \equiv 0 \pmod{R},$$

where $t \not\equiv i(i+1)/2 + j(3j-1)/2 \pmod R$ for any i and j except
$i \equiv (R-1)/2 \pmod R$.

Proof: From Definition 4.17 of $p_r(n)$, Euler's identity (4.2) and Jacobi's identity (4.7),

$$\sum_{n=0}^{\infty} p_{-(R-4)}(n)\, x^n = \prod_{k=1}^{\infty} (1 - x^k)^{-(R-4)}$$

$$= \prod_{k=1}^{\infty} (1 - x^k)^{-R} \prod_{k=1}^{\infty} (1 - x^k)^3 \prod_{k=1}^{\infty} (1 - x^k)$$

$$= \sum_{n=0}^{\infty} p_{-R}(n)\, x^n \sum_{i=0}^{\infty} (-1)^i (2i+1)\, x^{i(i+1)/2} \sum_{j=-\infty}^{\infty} (-1)^j\, x^{j(3j-1)/2}.$$

But, from Corollary 5.2 and Theorem 5.3 with $k = -1$, respectively,

$$\sum_{n=0}^{\infty} p_{-R}(n)\, x^n \equiv \sum_{n=0}^{\infty} p_{-R}(Rn)\, x^{Rn} \pmod R$$

$$\equiv \sum_{n=0}^{\infty} p_{-1}(n)\, x^{Rn} \pmod R.$$

Therefore it follows (note that $p_{-1}(n) = p(n)$) that,

$$\sum_{n=0}^{\infty} p_{-(R-4)}(n)\, x^n \equiv \sum_{n=0}^{\infty} p(n)\, x^{Rn} \sum_{i=0}^{\infty} (-1)^i (2i+1)\, x^{i(i+1)/2} \sum_{j=-\infty}^{\infty} (-1)^j\, x^{j(3j-1)/2} \pmod R,$$

or

$$(1) \quad \sum_{n=0}^{\infty} p_{-(R-4)}(n)\, x^n \equiv \sum_{\substack{n,i=0 \\ j=-\infty}}^{\infty} (-1)^{i+j}\, p(n)\, (2i+1)\, x^u \pmod R,$$

where

$$u = Rn + i(i+1)/2 + j(3j-1)/2.$$

Now $p_{-(R-4)}(n)$ is divisible by R for all values of n for which x^n does not occur in the right member of (1); or, if it does occur but its coefficient is divisible by R. The coefficient of x^u in the right member of (1) will be divisible by R if $2i+1$ is divisible by R or equivalently,

$$(2) \qquad i \equiv (R-1)/2 \pmod R.$$

Thus, one can conclude $p_{-(R-4)}(n)$ will be divisible by R for those values of $n = mR + t \neq Rn + i(i+1)/2 + j(3j-1)/2$ for all i and j which do not satisfy (2). This is to say $p_{-(R-4)}(mR + t) \equiv 0 \pmod{R}$ for all t such that $t \not\equiv i(i+1)/2 + j(3j-1)/2$ (mod R for all i and j with the restriction $i \not\equiv (R-1)/2 \pmod{R}$. This completes the proof.

Corollary 5.5. $\quad p(5m + 4) \equiv 0 \pmod{5}. \qquad (5.1)$

Proof: Let R = 5 in Theorem 5.4. Since $(5-1)/2 = 2$, consider all $i \not\equiv 2 \pmod{5}$. The only possible least residues of $i(i+1)/2 \pmod{5}$ are 0 and 1. The least residues for $j(3j-1)/2 \pmod{5}$ are 0, 1 and 2. Therefore, the only possible least residues for $i(i+1)/2 + j(3j-1)/2$ are 0, 1, 2 and 3, and hence $t = 4 \not\equiv i(i+1)/2 + j(3j-1)/2 \pmod{5}$ and the congruence (5.1) follows.

The proof of congruence (5.2) is similar and is also given by Gandhi [17]. The author would like to use the same method to prove the congruence (5.3). To shorten the proof, a lemma will be stated without proof which is given by Winquist [49].

Lemma 5.6.
$$\prod_{k=1}^{\infty} (1 - x^k)^{10} = \sum_{\substack{i=0 \\ j=-\infty}}^{\infty} (-1)^{i+j}(2i+1)(6j+1) \left[(3i+1)(3i+2)/2 - 3j(3j+1)/2\right] x^{3i(i+1)/2 + j(3j+1)/2}.$$

Theorem 5.7. If R is a prime, then
$$p_{-(R-10)}(mR + t) \equiv 0 \pmod{R},$$
where $t \not\equiv 3i(i+1)/2 + j(3j+1)/2 \pmod{R}$, for all i and j except
$$i \not\equiv (R-1)/2 \pmod{R} \text{ and } j \not\equiv (5R-1)/6 \pmod{R}.$$

Proof: From Definition 4.17 and Lemma 5.6,

$$\sum_{n=0}^{\infty} p_{-(R-10)}(n)\, x^n = \prod_{k=1}^{\infty} (1 - x^k)^{-(R-10)}$$

$$= \prod_{k=1}^{\infty} (1 - x^k)^{-R} \prod_{k=1}^{\infty} (1 - x^k)^{10}$$

$$= \sum_{n=0}^{\infty} p_{-R}(n)\, x^n \sum_{\substack{i=0\\ j=-\infty}}^{\infty} (-1)^{i+j}(2i+1)(6j+1)$$

$$[(3i+1)/2 - 3j(3j+1)/2]\, x^{3i(i+1)/2 + j(3j+1)/2}.$$

But,

$$\sum_{n=0}^{\infty} p_{-R}(n)\, x^n \equiv \sum_{s=0}^{\infty} p(s)\, x^{sR} \pmod{R},$$

by Theorem 5.3 and Corollary 5.2. Therefore,

(1) $$\sum_{n=0}^{\infty} p_{-(R-10)}(n)\, x^n = \sum_{\substack{s,i=0\\ j=-\infty}}^{\infty} (-1)^{i+j}\, p(s)\, (2i+1)(6j+1)$$

$$[(3i+1)/2 - 3j+1)/2]\, x^u,$$

where $u = sR + 3i(i+1)/2 + j(3j+1)/2$. Now $p_{-(R-10)}(n)$ is divisible by R for all values of n for which x^n does not occur in the right member of (1); or, if it does occur but its coefficient is divisible by R. The coefficient of x^u in the right member of (1) will be divisible by R, for example, if $2i+1 \equiv 0 \pmod{R}$ or if $6j+1 \equiv 0 \pmod{R}$ or equivalently;

(2) $$i \equiv (R-1)/2 \pmod{R} \text{ or } j \equiv (5R-1)/6 \pmod{R}.$$

Thus, $p_{-(R-10)}(n)$ will be divisible by R for those values of $n = mR + t \not\equiv sR + 3i(i+1)/2 + j(3j+1)/2$ for all i and j which do not satisfy (2). Therefore, $p_{-(R-10)}(mR + t) \equiv 0 \pmod{R}$ for all t such that, $t \not\equiv 3i(i+1)/2 + j(3j+1)/2 \pmod{R}$ for all i and j such that $i \not\equiv (R-1)/2 \pmod{R}$ and $j \not\equiv (5R-1)/6 \pmod{R}$ and the theorem is proved.

<u>Corollary 5.8.</u> $p(11m + 6) \equiv 0 \pmod{11}$. (5.3)

Proof: Let R = 11 in Theorem 5.7. Since $(11-1)/2 = 5$ and $(55-1)/6 = 9$, consider all i and j such that $i \not\equiv 5 \pmod{11}$ and $j \not\equiv 9 \pmod{11}$. For these values of i and j, the least residues of $3i(i+1)/2 + j(3j+1)/2$

(mod 11) are 0, 1, 2, 3, 4, 5, 7, 8, 9 and 10. Hence, for $t = 6$ and $R = 11$, the result (5.3) follows.

Other congruences for $p_r(n)$, of no real significance for this paper, are given by Gandhi [16].

Congruences of $t(n)$ and $t_k(n)$

Several congruences will now be developed for the k-line partition function, $t_k(n)$, for $k = 2, 3, 4, 5, 6, 7$ and 8. These results are basically due to Cheema and Gordon [12] and Gandhi [18].

Theorem 5.9. $t_2(5n+3) \equiv t_2(5n+4) \equiv 0 \pmod 5$.

Proof: From Definition 4.17 of $p_3(n)$ and Jacobi's identity (4.7),

(1) $$\sum_{n=0}^{\infty} p_3(n)\, x^n = \prod_{m=1}^{\infty} (1 - x^m)^3$$

$$= \sum_{j=0}^{\infty} (-1)^j (2j+1)\, x^{j(j+1)/2}.$$

But, $2j+1 \equiv 0 \pmod 5$ for $j \equiv 2 \pmod 5$. Hence, if $j \not\equiv 2 \pmod 5$, then the only residues for $j(j+1)/2 \pmod 5$ are 0 and 1. Therefore, by comparing coefficients in (1),

(2) $$p_3(n) \equiv 0 \pmod 5 \text{ for } n \equiv 2, 3, \text{ or } 4 \pmod 5.$$

Now, since $\prod_{m=1}^{\infty} (1 - x^m)^{-2} = \prod_{m=1}^{\infty} (1 - x^m)^{-5} \prod_{m=1}^{\infty} (1 - x^m)^3$,

$$\sum_{n=0}^{\infty} p_{-2}(n)\, x^n = \sum_{n=0}^{\infty} p_{-5}(n)\, x^n \sum_{n=0}^{\infty} p_3(n)\, x^n$$

$$= \sum_{n=0}^{\infty} \left[\sum_{j=0}^{n} p_{-5}(j)\, p_3(n-j) \right] x^n.$$

Equate the coefficients to obtain,

(3) $$p_{-2}(n) = \sum_{j=0}^{n} p_{-5}(j)\, p_3(n-j).$$

From Corollary 5.2, $p_{-5}(j) \equiv 0 \pmod 5$ for all $j \not\equiv 0 \pmod 5$.

Thus, for $n \equiv 2, 3$, or $4 \pmod 5$, it follows that each term in the right member of (3) is divisible by 5 and hence,

(5.4) $$p_{-2}(n) \equiv 0 \pmod 5 \text{ for } n \equiv 2, 3 \text{ or } 4 \pmod 5.$$

Hence, if $n \equiv 3$ or $4 \pmod 5$, then $p_{-2}(n) \equiv p_{-2}(n-1) \equiv 0 \pmod 5$.

From the results of Theorem 4.23, $t_2(n) = p_{-2}(n) - p_{-2}(n-1)$, it follows that $t_2(n) \equiv 0 \pmod 5$ for $n \equiv 3$ or $4 \pmod 5$ and the theorem is proved.

Theorem 5.10. If $n \equiv 2$, 3 or $4 \pmod 5$, then $\sum_{k=0}^{n} t_2(k) \equiv 0 \pmod 5$.

Proof: From Theorem 4.26,

$$p_{-2}(n) = \sum_{k=0}^{n} t_2(k).$$

But, $p_{-2}(n) \equiv 0 \pmod 5$ for $n \equiv 2$, 3 or $4 \pmod 5$ from equation (5.4) and the theorem follows.

The next theorem states that $p(n)$, $t_2(2n)$ and $t_2(2n+1)$ have the same parity. That is, for all values of n such that $p(n)$ is even, it follows that $t_2(2n)$ and $t_2(2n+1)$ are also even and the values of n for which $p(n)$ is odd are the values for which $t_2(2n)$ and $t_2(2n+1)$ are odd.

Theorem 5.11. $$p(n) \equiv t_2(2n) \equiv t_2(2n+1) \pmod 2.$$

Proof: Replace n by $2n$ and $2n+1$ respectively in the results of Theorem 4.23 to obtain:

(1) $$t_2(2n) = p_{-2}(2n) - p_{-2}(2n-1) \text{ and}$$

(2) $$t_2(2n+1) = p_{-2}(2n+1) - p_{-2}(2n).$$

But, from Theorem 5.3 with $R = 2$ and $k = -1$,

$$p_{-2}(2n) \equiv p_{-1}(n) \pmod 2.$$

Also, from Corollary 5.2 with $k = 2$, since $2n+1$ and $2n-1$ are odd,

$$p_{-2}(2n-1) \equiv p_{-2}(2n+1) \equiv 0 \pmod 2.$$

Therefore, it follows from (1) and (2) that:

$$t_2(2n) \equiv p_{-1}(n) \pmod 2 \text{ and}$$

$$t_2(2n+1) \equiv -p_{-2}(2n) \equiv p_{-1}(n) \pmod 2.$$

But, since $p_{-1}(n) = p(n)$, $t_2(2n) \equiv p(n) \equiv t_2(2n+1) \pmod 2$ which proves the theorem.

There is no convenient way of determining which values of n will make p(n) even or odd. However, with the restriction that summands of the partitions be unique, the problem becomes easy.

Theorem 5.12. If n is a pentagonal number, i.e. of the form $j(3j\pm1)/2$, then q(n) is odd. For n not of this type, q(n) is even.

Proof: From Definition 4.3, Theorem 4.8 and Euler's identity (4.2),

$$\sum_{n=0}^{\infty} q(n)\, x^n = \prod_{k=1}^{\infty} (1 - x^k)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n\, x^{n(3n+1)/2}.$$

Another form of Euler's series is $\sum_{j=0}^{\infty} (-1)^j\, x^{j(3j\pm1)/2}$ which gives,

$$\sum_{n=0}^{\infty} q(n)\, x^n = \sum_{j=0}^{\infty} (-1)^j\, x^{j(3j\pm1)/2}$$

$$\equiv \sum_{j=0}^{\infty} x^{j(3j\pm1)/2} \pmod 2$$

and the theorem follows.

Theorem 5.13. $t_3(3n+2) \equiv 0 \pmod 3$.

Proof: Replace n by 3n+2 in the results of Theorem 4.24 to obtain:

(1) $t_3(3n+2) = p_{-3}(3n+2) - 2p_{-3}(3n+1) + 2p_{-3}(3n-1) - p_{-3}(3n-2)$.

But, with R = 3 in Corollary 5.2,

(2) $p_{-3}(3n+2) \equiv p_{-3}(3n+1) \equiv p_{-3}(3n-1) \equiv p_{-3}(3n-2) \equiv 0 \pmod 3$.

The results of (2) used in (1) gives the result of the theorem.

Theorem 5.14. $t_3(3n+3) \equiv t_3(3n+4) \pmod 3$.

Proof: Replace n by 3n+1 and 3n+3 respectively in the results of Theorem 4.24 to obtain:

(1) $t_3(3n+1) = p_{-3}(3n+1) - 2p_{-3}(3n) + 2p_{-3}(3n-2) - p_{-3}(3n-3)$ and

(2) $t_3(3n+3) = p_{-3}(3n+3) - 2p_{-3}(3n+2) + 2p_{-3}(3n) - p_{-3}(3n-1)$.

But, from Corollary 5.2 with R = 3,

(3) $p_{-3}(3n+1) \equiv p_{-3}(3n-2) \equiv p_{-3}(3n+2) \equiv p_{-3}(3n-1) \equiv 0 \pmod 3$.

Also, from Theorem 5.3,

(4) $p_{-3}(3n) \equiv p_{-1}(n)$, $p_{-3}(3n+3) \equiv p_{-1}(n+1)$ and $p_{-3}(3n-3) \equiv p_{-1}(n-1)$

with each congruence (mod 3). Therefore, from (1) and (2), with (3) and (4):

(5) $t_3(3n+1) \equiv -2p_{-1}(n) - p_{-1}(n-1) \equiv p_{-1}(n) - p_{-1}(n-1) \pmod 3$ and

(6) $t_3(3n+3) \equiv p_{-1}(n+1) + 2p_{-1}(n) \equiv p_{-1}(n+1) - p_{-1}(n) \pmod 3$.

Replace n by n+1 in (5) which gives,

(7) $t_3(3n+4) \equiv p_{-1}(n+1) - p_{-1}(n) \pmod 3$.

Since the right member of (6) and (7) are the same, the theorem follows.

Gandhi [18] has stated the following congruences concerning t_k:

$$t_4(4n) \equiv t_4(4n+1) \equiv t_4(4n+2) \pmod 2,$$

$$t_4(4n+3) \equiv 0 \pmod 2,$$

$$t_5(5n+1) \equiv t_5(5n+3) \pmod 5 \text{ and}$$

$$t_5(5n+2) \equiv t_5(5n+4) \pmod 5.$$

Gandhi claims he has congruences for t_k (k = 6, 7, 8 and 9) which will soon be published. The author hasn't seen any in print and includes, in the next three theorems, some congruences for t_6, t_7 and t_8.

<u>Theorem 5.15</u>. $t_6(6n+1) + t_6(6n+3) \equiv t_6(6n+2) + t_6(6n+4) \pmod 3$.

Proof: From Theorem 4.25 with $k = 6$,

(1) $$t_6(n) = \sum_{s=0}^{v} a_6(s)\, p_{-6}(n-s),$$

where

(2) $$\sum_{s=0}^{v} a_6(s)\, x^s = \prod_{m=1}^{5} (1 - x^m)^{6-m}.$$

After computing $a_6(s)$ $(s = 1, 2, \ldots, 35)$ (mod 3) in equation (2) and substituting into (1), it follows that:

(3) $$\begin{aligned} t_6(n) \equiv\ & p_{-6}(n) + p_{-6}(n-1) + p_{-6}(n-3) - p_{-6}(n-4) - p_{-6}(n-6) \\ & + p_{-6}(n-7) + p_{-6}(n-8) - p_{-6}(n-9) - p_{-6}(n-10) + p_{-6}(n-11) \\ & + p_{-6}(n-13) - p_{-6}(n-14) - p_{-6}(n-16) - p_{-6}(n-17) - p_{-6}(n-18) \\ & - p_{-6}(n-21) + p_{-6}(n-22) + p_{-6}(n-24) - p_{-6}(n-25) - p_{-6}(n-26) \\ & + p_{-6}(n-27) + p_{-6}(n-28) - p_{-6}(n-29) - p_{-6}(n-31) + p_{-6}(n-32) \\ & + p_{-6}(n-34) + p_{-6}(n-35) \pmod 3. \end{aligned}$$

Now replace n by $6n+1$, $6n+2$, $6n+3$ and $6n+4$ respectively in (3) to obtain, after applying Theorems 5.1 and 5.3,

(4) $$\begin{aligned} t_6(6n+1) \equiv\ & p_{-2}(2n) - p_{-2}(2n-1) + p_{-2}(2n-2) - p_{-2}(2n-3) + p_{-2}(2n-4) \\ & - p_{-2}(2n-6) + p_{-2}(2n-7) - p_{-2}(2n-8) + p_{-2}(2n-9) \\ & - p_{-2}(2n-10) + p_{-2}(2n-11) \pmod 3, \end{aligned}$$

(5) $$\begin{aligned} t_6(6n+2) \equiv\ & p_{-2}(2n-2) + p_{-2}(2n-3) - p_{-2}(2n-4) - p_{-2}(2n-8) \\ & - p_{-2}(2n-9) + p_{-2}(2n-10) + p_{-2}(2n-11) \pmod 3, \end{aligned}$$

(6) $$\begin{aligned} t_6(6n+3 \equiv\ & p_{-2}(2n+1) + p_{-2}(2n) - p_{-2}(2n-1) - p_{-2}(2n-2) - p_{-2}(2n-6) \\ & + p_{-2}(2n-7) + p_{-2}(2n-8) \pmod 3 \text{ and} \end{aligned}$$

(7) $$\begin{aligned} t_6(6n+4) \equiv\ & p_{-2}(2n+1) - p_{-2}(2n) + p_{-2}(2n-1) - p_{-2}(2n-2) + p_{-2}(2n-3) \\ & - p_{-2}(2n-4) + p_{-2}(2n-6) - p_{-2}(2n-7) + p_{-2}(2n-8) \end{aligned}$$

$$- p_{-2}(2n-9) + p_{-2}(2n-10) \pmod 3.$$

The theorem follows by comparing the sum of equations (4) and (6) with the sum of equations (5) and (7).

<u>Theorem 5.16</u>. $t_7(7n+2) + t_7(7n+3) \equiv t_7(7n+4) + t_7(7n+5) \pmod 7$.

Proof: The use of Theorem 4.24 with $k = 7$ gives 57 terms for $t_7(n)$. After replacing n by 7n+2, 7n+3, 7n+4 and 7n+5 respectively and then using Corollary 5.2 and Theorem 5.3 gives:

(1) $t_7(7n+2) \equiv 3p(n) + 3p(n-1) + p(n-3) + 2p(n-4) - p(n-5) + 2p(n-6) - 3p(n-7) \pmod 7$,

(2) $t_7(7n+3) \equiv - p(n) + 3p(n-1) - 3p(n-2) - p(n-3) - 2p(n-4) - 3p(n-5) - p(n-6) + p(n-7) \pmod 7$,

(3) $t_7(7n+4) \equiv - p(n) + p(n-1) + 3p(n-2) + 2p(n-3) + p(n-4) + 3p(n-5) - 3p(n-6) + p(n-7) \pmod 7$ and

(4) $t_7(7n+5) \equiv 3p(n - 2p(n-1) + p(n-2) - 2p(n-3) - p(n-4) - 3p(n-6) - 3p(n-7) \pmod 7$.

Since the sum of equations (1) and (2) is the same as the sum of equations (3) and (4), the theorem follows.

<u>Theorem 5.17</u>. (a) $t_8(8n) \equiv t_8(8n+4) \pmod 2$ and

(b) $t_8(8n+5) \equiv t_8(8n+6) \equiv t_8(8n+7) \equiv 0 \pmod 2$.

Proof: From Theorem 4.25 with $k = 8$,

(1) $$t_8(n) = \sum_{s=0}^{v} a_8(s)\, p_{-8}(n-s),$$

where

(2) $$\sum_{s=0}^{v} a_8(s)\, x^s = \prod_{m=1}^{7} (1 - x^m)^{8-m}$$
$$\equiv 1 + x + x^2 + x^4 + x^8 + x^{11} + x^{12} + x^{16} + x^{17} + x^{20} + x^{25} + x^{27} + x^{33} + x^{34} + x^{35} + x^{41} + x^{43}$$

$$+ x^{49} + x^{50} + x^{51} + x^{57} + x^{59} + x^{64} + x^{67}$$

$$+ x^{68} + x^{72} + x^{73} + x^{76} + x^{80} + x^{82} + x^{83}$$

$$+ x^{84} \pmod{2}.$$

After substituting the values of $a_8(s)$ $(s = 0,1,\ldots,84)$ into equation (1), replacing n by 8n, 8n+4, 8n+5, 8n+6 and 8n+7, respectively, and observing from Theorem 5.1 that $p_{-8}(n) \not\equiv 0 \pmod{2}$ only if 8 divides n, it follows that:

(3) $$t_8(8n) \equiv p_{-8}(8n) + p_{-8}(8n-8) + p_{-8}(8n-16) + p_{-8}(8n-64) + p_{-8}(8n-72) + p_{-8}(8n-80) \pmod{2},$$

(4) $$t_8(8n+4) \equiv p_{-8}(8n) + p_{-8}(8n-8) + p_{-8}(8n-16) + p_{-8}(8n-64) + p_{-8}(8n-72) + p_{-8}(8n-80) \pmod{2},$$

(5) $$t_8(8n+5) \equiv 0 \pmod{2},$$

(6) $$t_8(8n+6) \equiv 0 \pmod{2} \text{ and}$$

(7) $$t_8(8n+7) \equiv 0 \pmod{2}.$$

Part (a) follows from (3) and (4), part (b) follows from (5), (6) and (7) and the theorem is proved.

CHAPTER VI

ASYMPTOTIC PROPERTIES

Introduction

Chapter V dealt with the arithmetic properties of p(n) and other more general functions. This chapter is concerned with the behavior of the partition function for large values of n. It is obvious that p(n) increases rapidly as n increases, but just how fast. Since the recursion formulae will be of no use for large n, the problem is how to determine p(n) for large n.

It is apparently beyond the present resources of mathematics to give a simple expression for p(n), hence all one can hope for is a simple function f which approximates p(n) for large n. That is, it would be desirable to find a function f such that,

$$p(n) = f(n) + r(n)$$

where f would be as simple a function as possible and r would approach zero as n gets large. This is to say, with reference to Definitions 2.21 and 2.23 respectively, that,

$$p(n) \sim f(n) \text{ or } p(n) = f(n)[1 + o(1)].$$

It might first be of interest to get some idea of the relative size of p(n) and the function values of some well-known functions for large n. This has been done by Ayoub [6] with the functions n^{c_1} and $\exp(c_2 n)$ with results given in Theorem 6.2. To facilitate the proof, however, one must first look at the generating function of p(n), (6.1) $$P(x) = \prod_{k=1}^{\infty} (1 - x^k)^{-1} = \sum_{n=0}^{\infty} p(n)\, x^n, \text{ for } |x| < 1.$$

According to the Tauberian Theorem 2.31 and the equation (6.1), since $p(n) \geq 0$ and the series in (6.1) converges for $|x| < 1$, it follows that the order of magnitude of $p(n)$ will determine the behavior of $P(x)$ in the neighborhood of $x = 1$ and conversely, the behavior of $P(x)$ as $x \to 1$ yields information concerning $p(n)$. A result concerning $P(x)$ which will be needed for Theorem 6.2 will now be given.

<u>Lemma 6.1</u>. $$\log P(x) \sim \frac{\pi^2}{6(1-x)}, \text{ as } x \to 1.$$

Proof: Take the logarithm of $P(x)$ in equation (6.1) to obtain,

$$\begin{aligned} \log P(x) &= \log \prod_{k=1}^{\infty} (1 - x^k)^{-1} \\ &= \sum_{k=1}^{\infty} \log (1 - x^k)^{-1} \\ &= \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} (x^k)^m/m \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} (x^m)^k/m. \end{aligned}$$

But with $|x| < 1$, hence $|x^m| < 1$, the geometric series $(x^m)^k$ has a sum which gives,

(1) $$\log P(x) = \sum_{m=1}^{\infty} \frac{x^m}{m(1-x^m)}.$$

Also, with $0 < x < 1$, hence $0 < x^k < 1$ for $k = 1, 2, \ldots, m-1$ and $x^{m-1} < x^{m-2} < \ldots < x < 1$, it follows that,

$$mx^{m-1} < 1 + x + \ldots + x^{m-1} < m.$$

Multiply each member by $(1-x)$ to obtain,

$$mx^{m-1}(1-x) < 1-x^m < m(1-x)$$

or

$$\frac{1}{m(1-x)} < \frac{1}{1-x^m} < \frac{1}{mx^{m-1}(1-x)}.$$

Now from equation (1),

$$(2) \qquad \frac{1}{1-x} \sum_{m=1}^{\infty} x^m/m^2 < \log P(x) < \frac{1}{1-x} \sum_{m=1}^{\infty} x/m^2.$$

But, from Theorem 2.32,

$$\lim_{x \to 1} \sum_{m=1}^{\infty} x^m/m^2 = \pi^2/6.$$

Therefore, after dividing all members of (2) by the left member and letting $x \to 1$, it follows that,

$$\log P(x) \sim \frac{\pi^2}{6(1-x)}$$

and the lemma is proved.

Theorem 6.2. For n sufficiently large,

(1) $\quad n^{c_1} < p(n) < \exp(c_2 n)$, ($c_1$, $c_2 \geq 1$ are arbitrary constants).

Proof: Let (1) be replaced by the two equivalent inequalities:

$$(2) \qquad n^{c_1} < p(n) \text{ and}$$

$$(3) \qquad p(n) < \exp(c_2 n).$$

Assume $p(n) = n^{c_1}$. Then for $x = e^{-y}$,

$$P(x) = \sum_{n=0}^{\infty} n^{c_1} x^n = \sum_{n=1}^{\infty} n^{c_1} e^{-ny}.$$

But,

$$\sum_{n=1}^{\infty} n^{c_1} e^{-ny} \sim \int_0^{\infty} t^{c_1} e^{-ty} \, dt = \frac{\Gamma(c_1+1)}{y^{c_1+1}}.$$

Since $y = -\log x \sim (1-x)$ as $x \to 1$, it follows that,

$$P(x) \sim \frac{\Gamma(c_1+1)}{(1-x)^{c_1+1}}$$

which implies that,

$$\log P(x) = \log \Gamma(c_1+1) - (c_1+1)\log(1-x) + o(1).$$

This contradicts Lemma 6.1, since,

$$\lim_{x \to 1} \frac{\log (c_1+1) - (c_1+1)\log(1-x) + o(1)}{\pi^2/[6(1-x)]} = 0 \neq 1.$$

Therefore, inequality (2) follows.

To prove inequality (3), assume $p(n) = \exp(c_2 n)$. Then there exists a constant d such that $|d| < 1$, namely $d = \exp((1-c_2)n)$, for which $\sum_{n=0}^{\infty} \exp(c_2 n)(d)^n$ diverges. This contradicts equation (6.1) since that series converges for all $|x| < 1$. This proves inequality (3) and hence the theorem.

In the light of Theorem 6.2, one might conjecture that $p(n) = \exp(an^b)$, for the appropriate constants a and b, yet to be determined. In order to estimate a and b; let $p(n) = \exp(an^b)$ and $x = e^{-y}$. Then the series,

$$\sum_{n=0}^{\infty} p(n)\, x^n = \sum_{n=0}^{\infty} \exp(an^b - ny)$$

has its largest term when the derivative is zero, i.e. $abn^{b-1} - y = 0$. This gives $n = [y/(ab)]^{1/(b-1)}$ and for this n, the term which will determine the order of magnitude of $P(x)$, is

$$\exp\left(a[y/(ab)]^{b/(b-1)} - y[y/(ab)]^{1/(b-1)}\right)$$

$$= \exp\left(a^{1/(1-b)} b^{b/(1-b)} y^{b/(1-b)} - a^{1/(1-b)} b^{1/(1-b)} y^{b/(b-1)}\right).$$

Now, as $x \to 1$, $y = -\log x \sim 1 - x$ and therefore,

$$P(x) \sim \exp\left(a^{1/(1-b)} b^{b/(1-b)} (1-x)^{b/(1-b)} - a^{1/(1-b)} b^{1/(1-b)} (1-x)^{b/(10b)}\right).$$

To choose a and b so this agrees with the result (6.2), one may select $b = 1/2$ and $a = \pi\sqrt{2/3}$. Hence, assume $p(n)$, for large n, is approximately $\exp(\pi\sqrt{2n/3})$. Theorem 6.3 is a very similar result.

Asymptotic Formula for log p(n)

<u>Theorem 6.3.</u> [6] (Hardy and Ramanujan's asymptotic formula for $\log p(n)$).

$$\log p(n) \sim \pi\sqrt{2n/3}. \tag{6.3}$$

Proof: The theorem follows from the two inequalities:

(1) $$p(n) < \exp(\pi\sqrt{2n/3})$$

and for every $\epsilon > 0$, there exists a constant A_ϵ such that,

(2) $$p(n) > (1/A_\epsilon)\ \exp[(a-\epsilon)\ \sqrt{n}\ ].$$

This is seen by combining the inequalities (1) and (2), dividing by $\pi\sqrt{2n/3}$ and then letting $n \to \infty$ to obtain,

$$1 - \epsilon/(\pi\sqrt{2/3}) \leq \lim_{n\to\infty} \frac{\log p(n)}{\pi\sqrt{2n/3}} \leq 1.$$

By Definition 2.21 of asymptotic, the result follows.

Now to prove inequality (1) by induction on n:

(i) $p(1) = 1 < \exp(\pi\sqrt{2/3})$.

(ii) Suppose that for all $m < n$, $p(m) < \exp(\pi\sqrt{2m/3})$. Now from Theorem 4.28,

$$n\ p(n) = \sum_{m=1}^{n} \sum_{k=1}^{n} m\ p(n-km)$$

$$< \sum_{m=1}^{n} \sum_{k=1}^{n} m\ \exp(\pi\sqrt{2(n-km)/3})$$

$$= \sum_{m=1}^{n} \sum_{k=1}^{n} m\ \exp(\pi\sqrt{2n/3}\ \sqrt{1-km/n}).$$

Since $(1-x)^a \leq 1 - ax$, it follows that,

$$n\ p(n) < \sum_{m=1}^{n} \sum_{k=1}^{n} m\ \exp[\pi\sqrt{2n/3}\ (1-\frac{km}{2n})]$$

$$< \exp(\pi\sqrt{2n/3}) \sum_{k=1}^{n} \sum_{m=1}^{\infty} m\ \exp\left(\frac{-km\pi}{\sqrt{6n}}\right).$$

From the fact that,

$$\sum_{m=1}^{\infty} m\ e^{-mx} = \frac{1}{4\ \sinh^2 x/2} < 1/x^2, \text{ for } x < 0,$$

which can easily be verified, it follows with $x = k\pi/\sqrt{6n}$ that,

$$n\ p(n) < \exp(\pi\sqrt{2}n/3) \sum_{k=1}^{n} 6n/(k\pi)^2$$

$$< \exp(\pi\sqrt{2n/3})\ 6n/\pi^2 \sum_{k=1}^{\infty} 1/k^2$$

$$= \exp(\pi\sqrt{2n/3})\ (6n/\pi^2)(\pi^2/6)$$

$$= n \exp(\pi\sqrt{2n/3}).$$

Division by n gives,

$$p(n) < \exp(\pi\sqrt{2n/3})$$

and by induction, inequality (1) holds for all n. The proof of inequality (2) is similar and will be omitted.

The result of the theorem is not as significant as one might hope since an asymptotic property of log p(n) does not give very accurate results concerning p(n). For example with the results of Theorem 6.3, one could just as well have:

$$p(n) \sim n^{1000} \exp(\pi\sqrt{2n/3})$$

$$p(n) \sim n^{-1000} \exp(\pi\sqrt{2n/3}).$$

Asymptotic Formula for p(n)

The next step in the development of trying to evaluate p(n), or at least approximate it, is to prove the asymptotic formula (6.4). Only a sketch of the proof will be given. For a complete proof, the reader can refer to Hardy and Ramanujan [25].

<u>Theorem 6.4</u>. (Hardy and Ramanujan's asymptotic formula for p(n)).

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3}). \tag{6.4}$$

The proofs given for Theorem 6.4 requires much more advanced mathematics than for any of the previous results. In order to prove this theorem, at least to our present knowledge, it will require the theory of analytic functions, in particular Cauchy's integral (6.5), and the theory of modular functions, in particular (6.6).

Cauchy's integral Theorem 2.40 can be applied to the generating function (6.1) in a natural way. Clearly P(x) is analytic inside the unit circle with center at the origin. Thus, since

$$\frac{P(x)}{x^{n+1}} = \sum_{k=0}^{\infty} p(k)\, x^{k-n-1}$$

$$= p(0)\, x^{-n-1} + p(1)\, x^{-n} + \dots + p(n)\, x^{-1} + p(n+1) + p(n+2)\, x + \dots,$$

it follows that p(n) is the residue at x = 0 of the function $\frac{P(x)}{x^{n+1}}$.

Now, from Cauchy's Theorem 2.40,

$$p(n) = \frac{1}{2\pi i} \int_C \frac{P(x)}{x^{n+1}}\, dx, \tag{6.5}$$

where C is a simple closed contour enclosing the origin and lying entirely inside the unit circle. The difficulty now comes, of course, in evaluating the integral.

The reason for the difficulty is that,

$$P(x) = \frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^3} \cdots \frac{1}{1-x^k} \cdots$$

has zero denominators in every factor when x = 1, every second when x = -1, every third when $x = \exp(2\pi i/3)$, and in general every k-th when $x = \exp(2\pi ih/k)$, (h,k) = 1. Hence, $x = \exp(2\pi ih/k)$ for every rational number is an essential signularity of P(x). However, since the rational points are dense, any irrational point will have singularities arbitrarily close to it and therefore will itself be a singularity. Therefore, all points on the circle are singular points and there is no possibility of integrating across the singularities.

The difficulty seemed unsurmountable until G. H. Hardy recognized that, quote, "P(x) belongs to a class of functions called elliptic modular functions whose properties have been intensively studied and whose behavior is well-known." Riemann, Dedekind and others have studied the important modular function,

$$h(t) = \exp(\pi it/12) \prod_{n=1}^{\infty} (1-e^{2\pi int}), \tag{6.6}$$

which is very nearly the reciprocal of P(x). To be exact, if one lets $x = e^{2\pi it}$, Im (t) > 0 to give $|x| < 1$, then from (6.1) and (6.6),

$$h(t) = \frac{\exp(\pi it/12)}{P(e^{2\pi int})}.$$

Now, the 'heaviest' singularity of P(x) occurs at x = 1. Hence, suppose that the greatest contribution to the integral of (6.5) would come at x = 1. In fact, if f(n) is the contribution to p(n) from the point x = 1, one might expect (as it turns out) that the contribution from x = 1 is $O(\sqrt{f(n)})$ and so on. Therefore, one would like to find a function g(x) which is analytic at all points of the unit circle except x = 1, and has there a singularity of a type as near as possible to that of the singularity of P(x). Cauchy's theorem would then be applied to P-g instead of P.

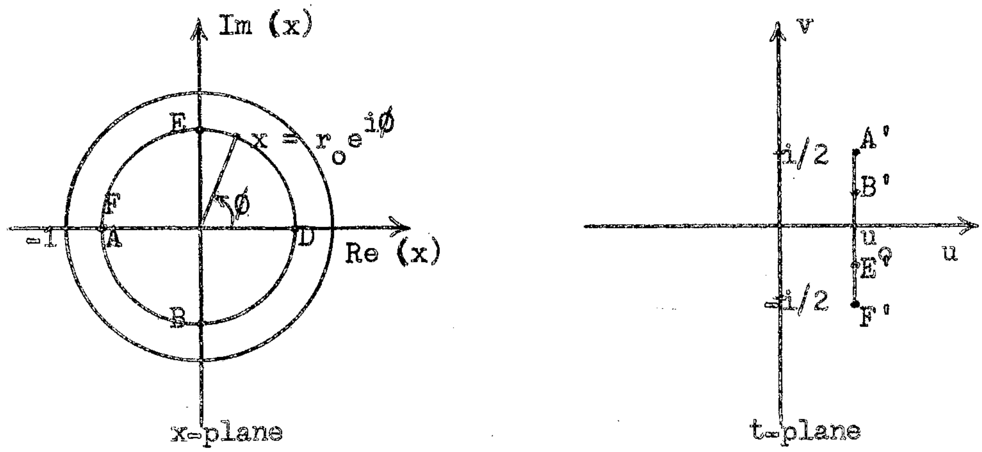
A function g can be found from the properties of the modular function h(t). The substitution $x = e^{-2\pi t}$, Re(t) > 0 to give $|x| < 1$, into (6.6) gives,

$$P(e^{-2\pi t}) = \sqrt{t}\,\exp[\pi(1/t - t)/12]\; P[\exp(-2\pi/t)]$$

$$= g(t)\; P[\exp(-2\pi/t), \tag{6.7}$$

where $g(t) = \sqrt{t}\, \exp[\pi(1/t - t)/12]$ is the desired function. Since $P[\exp(-2\pi/t)] = \sum_{n=0}^{\infty} p(n) \exp(-2\pi n/t)$ approaches 1 very rapidly as $t \to 0$, i.e. $x \to 1$, it follows that $P(e^{-2\pi t}) \to g(t)$ as $t \to 0$.

The first approximation will now be found by choosing the contour C of (6.5) to be the circle C_o: $x = re^{i\phi}$ $(-\pi \leq \phi \leq \pi)$, in the complex x-plane with a radius of $|x| = r_o < 1$ but extremely close to 1. Now make the transformation, $x = e^{-2\pi t}$, where $t = u+iv$, $u > 0$. Then $t = -\frac{1}{2\pi}(\ln r + i\phi)$, so that C_o is mapped onto the line L, in the t-plane: $u_o = -\frac{1}{2\pi} \ln r_o$ with endpoints $t_o = u_o + i/2$ and $t_1 = u_o - i/2$ as sketched below:



Substitution of $x = e^{-2\pi t}$ into (6.5) with $C = C_o$ gives,

$$p(n) = \frac{1}{2\pi i} \int_L P(e^{-2\pi t})\, e^{2\pi t(n+1)}\, e^{-2\pi t}\, (-2\pi)\, dt$$

$$= -\frac{1}{i} \int_L P(e^{-2\pi t})\, e^{2\pi n t}\, dt.$$

Now the integral is divided into two parts by the identity,

$$P(e^{-2\pi t}) = g(t) + P(e^{-2\pi t}) - g(t),$$

so that,

$$(6.8) \qquad p(n) = -\frac{1}{i} \int_L g(t)\, e^{2\pi n t} dt - \frac{1}{i} \int_L [P(e^{-2\pi t}) - g(t)]\, dt,$$

where the first integral will contribute the principal part and the second integral will be investigated only to determine its magnitude.

The evaluation of (6.8), still very difficult and long, is given by Ayoub [6] to be,

(6.9) $$p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} e^{a\lambda_n}/\lambda_n + O[\exp(5a\lambda_n/8)],$$

where $\lambda_n = \sqrt{n - 1/24}$, $a = \pi\sqrt{2/3}$ and O refers to $n\to\infty$. The asymptotic result (6.4) can now be obtained if we carry out the differentiation to obtain:

$$p(n) = \frac{1}{2\pi\sqrt{2}} \frac{e^{a\lambda_n}(a/2) - e^{a\lambda_n}[1/(2\lambda_n)]}{\lambda_n^2} + O[\exp(5a\lambda_n/8)]$$

$$= \frac{a}{4\pi\sqrt{2}} \frac{e^{a\lambda_n}}{n-1/24} (1 - \frac{1}{a\lambda_n}) + O[\exp(5a\lambda_n/8)]$$

$$= \frac{1}{4\sqrt{3}} \frac{e^{a\lambda_n}}{n-1/24} (1 - \frac{1}{a\lambda_n}) + O[\exp(5a\lambda_n/8)].$$

Now, as $n\to\infty$, $a\lambda_n\to\infty$, $n-1/24\to n$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3}).$$

The next step is to consider the singular point at $x = -1$ and to subtract from $P(x)$ a second auxiliary function related to this point as $g(x)$ is to $x = 1$. Similar developments by Hardy and Ramanujan gave,

$$p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} e^{a\lambda_n}/\lambda_n + \frac{(-1)^n}{2\pi} \frac{d}{dn} \frac{e^{a\lambda_n/2}}{\lambda_n} + O(e^{D\sqrt{n}}),$$

where $D > \frac{1}{3} a$. One might also notice that in equation (6.9); $5a\lambda_n/8 > \frac{1}{2} a\sqrt{n}$. Therefore, since the accuracy is improving, Hardy and Ramanujan [25] continued the process with the rational singularities and were able to obtain the more precise results given in (3.9) so that p(n) is, for sufficiently large values of n, the integer nearest the value of the first term in the right member of (3.9).

It was later shown, however, that the series in (3.9) was divergent. Rademacher simplified and perfected the original analysis of Hardy and Ramanujan to obtain a convergent series representation of p(n).

Convergent Series Representation of p(n)

Rademacher [42] proved the convergent series representation of p(n) given in the next theorem.

Theorem 6.5. $$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \frac{d}{dn} \frac{\sinh \pi/k \sqrt{2/3}\lambda_n}{\lambda_n},$$

where,

(6.10) $$\lambda_n = \sqrt{n - 1/24}, \quad A_k(n) = \sum_{\substack{0 \le h < k \\ (h,k)=1}} W_{h,k} \exp(-2\pi i hn/k),$$

$$W_{h,k} = \exp[\pi i S(h,k)] \text{ and,}$$

$$S(h,k) = \sum_{u=1}^{k-1} \left(\frac{u}{k} - \frac{1}{2}\right)\left(\frac{hu}{k} - \left[\frac{hu}{k}\right] - \frac{1}{2}\right),$$ (Brackets refer to the bracket function).

In 1938, L. R. Ford discovered some geometric properties of the Farey sequences to be defined in Definition 6.1 below. Ford represented each Farey fraction by a circle, now called 'Ford circles', given in Definition 6.3. This permitted Rademacher to replace his previous path of integration, done by dissecting the circle into 'Farey arcs', by the new path along the Ford circles. It is this method which we shall briefly outline. The reader may furnish the missing details and proofs or find them in Rademacher's lecture notes [41]. Ayoub [6] also gives a proof, due to J. V. Uspensky, which differs very little from Rademacher's proof.

First the substitution $x = e^{2\pi i t}$, $\text{Im}(t) > 0$, is made in the integral (6.5) to obtain,

$$(6.11) \qquad p(n) = \int_i^{i+1} P(e^{2\pi it})\, e^{-2\pi int} dt,$$

where the path of integration can be any curve in the upper half-plane connecting the points i and i+1. Now, to determine a convenient path, it will be necessary to discuss the Farey sequences and Ford circles.

Definition 6.1. By a Farey sequence F_n of order n, is meant the set of all fractions h/k with $0 \leq h \leq k$, $(h,k) = 1$, $1 \leq k \leq n$ and arranged in ascending order of magnitude. For example, the Farey sequences F_n, n = 1, 2, 3 and 4 are:

0/1, 1/1

0/1, 1/2, 1/1

0/1, 1/3, 1/2, 2/3, 1/1

0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1

Farey sequences have many interesting properties. Only those properties needed to prove the necessary results of the Ford circles will now be stated with proofs found in [41].

Properties:

(1.a) If h/k and h'/k' are two successive fractions in F_n, then $k + k' > n$.

(2.a) If h/k and h'/k' are two successive fractions in F_n, then $k \neq k'$.

(3.a) If h/k and h'/k' are two successive fractions in F_n, then $h'k - hk' = 1$.

Definition 6.2. The mediant u of h/k and h'/k', two successive fractions in F_n, is $u = (h + h')/(k + k')$. Observe that $h/k < u < h'/k'$.

Definition 6.3. Let h/k be a fraction in F_n. Consider the complex τ-plane and suppose that there is a circle $C_{h,k}$ with center at the point $\sigma_{h,k} = h/k + i/(2k^2)$ and radius $1/(2k^2)$. Such a circle is called a Ford circle.

The Ford circles corresponding to the Farey sequence F_n have the following properties which can be found in [41].

Properties:

(1.b) All the circles are tangent to the real axis.

(2.b) No two circles intersect.

(3.b) Two circles $C_{h,k}$ and $C_{h',k'}$ are tangent if and only if h/k and h'/k' are neighbors in some Farey sequence of the same order.

The Ford circles corresponding to F_4 are illustrated in Fig. 1.

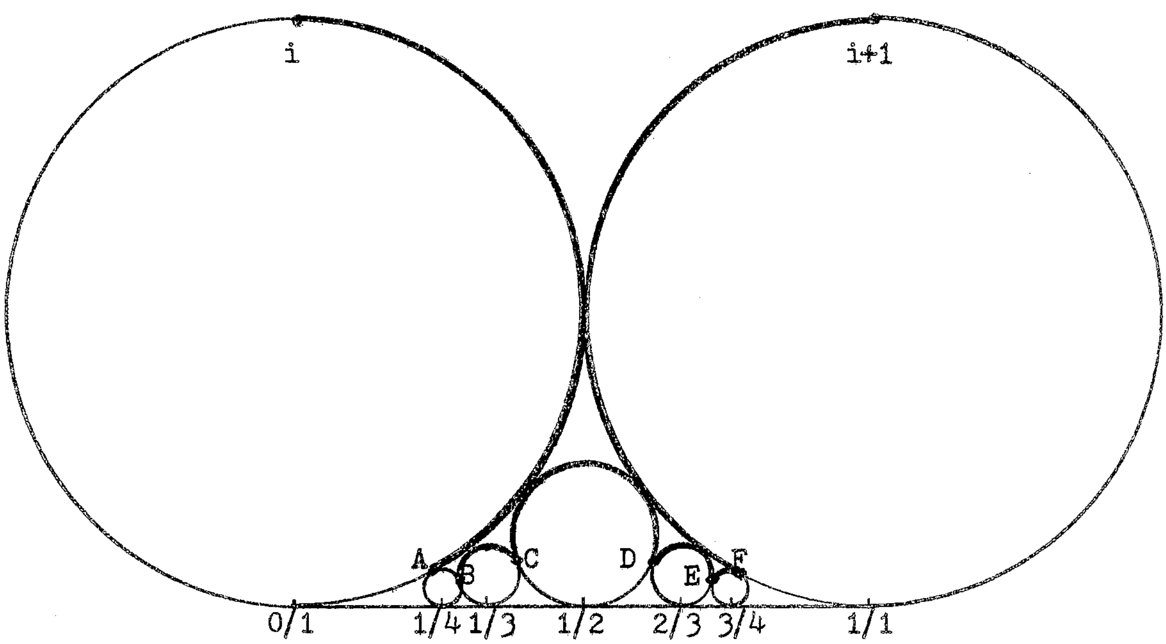


Fig. 1.

Now consider any fixed positive integer N (later we let $N \to \infty$) and the Ford circles corresponding to F_N. By property (3.b), consecutive circles are tangent. The points of tangency divide each circle into upper and lower arcs. For the circle $C_{h,k}$, denote the points of tangency by $h/k + S'$ and $h/k + S''$ and the upper arc by $g_{h,k}$. For $N = 4$, the points A and B in Fig. 1 illustrate $1/4 + S'$ and $1/4 + S''$ with the upper arc of AB corresponding to $g_{1,4}$.

Let the path of integration, P_N, for the integral in (6.11) be the row of arcs $g_{h,k}$ starting at the point i and ending at the point i+1. Because of the periodicity of the integrand, that part of the arc of $g_{0,1}$ to the left of the imaginary axis is replaced by the part of $g_{1,1}$ to the left of the line Re(t) = 1. The heavy line in Fig. 1 illustrates the path P_4. The integral (6.11) now becomes,

$$(6.12) \qquad p(n) = \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} \int_{g_{h,k}} P(e^{2\pi it})\, e^{-2\pi int}\, dt,$$

where $g_{h,k}$ runs clockwise from $h/k + S'_{h,k}$ to $h/k + S''_{h,k}$.

Now the change of variable $t = h/k + S$ in each integral gives,

$$(6.13) \qquad p(n) = \sum \int_{S'_{h,k}}^{S''_{h,k}} P(e^{2\pi i(h/k + S)})\, e^{-2\pi in(h/k + S)}\, dS,$$

where the summation here and henceforth will be over all h and k such that: $0 \leq h < k \leq N$ and $(h,k) = 1$, except where otherwise indicated. This transformation had the effect of transforming each of the circles so that the new centers are at the points $i/(2k^2)$. Now, to transform the circles to the circle K with radius 1/2 and center at the point 1/2, let $S = iz/k^2$. The integral (6.13) now becomes.

$$(6.14) \quad p(n) = \sum i/k^2\, e^{-2\pi inh/k} \int_{z'_{h,k}}^{z''_{h,k}} P(e^{2\pi i(h/k + iz/k^2)})\, e^{2\pi nz/k^2}\, dz,$$

where $z'_{h,k}$ and $z''_{h,k}$ are the images of $S'_{h,k}$ and $S''_{h,k}$ respectively. The path of integration is illustrated in Fig. 2.

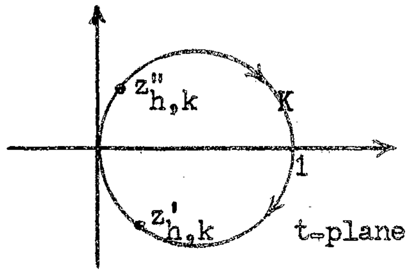


Fig. 2.

Now apply to P the functional equation derived from the theory of the elliptic modular functions by Hardy and Ramanujan [25]:

(6.14) $$P(x) = W_{h,k}\, G_k(z)\, P(x'),$$

where $x = \exp[2\pi i(h/k + iz/k^2)]$, $x' = \exp[2\pi i(h'/k + i/z)]$, $hh' \equiv -1 \pmod k$, $W_{h,k}$ is given in (6.10) and $G_k(z) = \sqrt{z}\,\exp(\frac{\pi}{12z} - \frac{\pi z}{12k^2})$.

The integral in (6.13) now becomes,

(6.15) $$p(n) = \Sigma\, ik^{-5/2} A_k(n) \int_{z'_{h,k}}^{z''_{h,k}} G_k(z)\, P(e^{2\pi ih'/k - 2\pi/z}) e^{2\pi nz/k^2} dz,$$

where $A_k(n)$ is defined in (6.10).

Divide each integral into two integrals by the identity,

$$G_k(z)\, P(x') = G_k(z)[P(x') - 1] + G_k(z),$$

to obtain,

(6.16) $$p(n) = \Sigma\, ik^{-5/2} A_k(n)\, I'_{h,k} + \Sigma\, ik^{-5/2} A_k(n)\, I''_{h,k},$$

where

$$I'_{h,k} = \int_{z'_{h,k}}^{z''_{h,k}} G_k(z)\, e^{2\pi nz/k^2} dz \text{ and}$$

$$I''_{h,k} = \int_{z'_{h,k}}^{z''_{h,k}} G_k(z)[P(e^{2\pi ih'/k - 2\pi/z}) - 1]\, e^{2\pi nz/k^2} dz,$$

where the path of integration is an arc of the circle K in the clockwise (negative) direction.

Estimation of $I''_{h,k}$, found by replacing the arc z' to z" by the chord (integrand is analytic away from the origin), and using the properties of Ford circles gives,

$$\left|I''_{h,k}\right| \le C(k/N)^{3/2},$$

where C is a constant which depends on the fixed integer n.

Now the second sum in (6.16) can be estimated for size to show that it is less than or equal to $C/N^{1/2}$ which gives,

$$p(n) = \Sigma\, ik^{-5/2}\, A_k(n)\, I'_{h,k} + O(N^{-1/2}). \tag{6.17}$$

The evaluation of $I'_{h,k}$ is carried out by using the entire circle K as follows:

$$I'_{h,k} = \int_K G_k(z)\, e^{2\pi nz/k^2} dz - \int_0^{z'_{h,k}} G_k(z)\, e^{2n\pi z/k^2} dz - \int_{z''_{h,k}}^{0} G_k(z)\, e^{2n\pi z/k^2} dz, \tag{6.18}$$

where K is traversed in a clockwise direction. The last two integrals in (6.18) can be written as,

$$\int_{z'_{h,k}}^{z''_{h,k}} G_k(z)\, e^{2n\pi z/k^2} dz. \tag{6.19}$$

As before, it can be shown that the integral in (6.19) is of order $O(k/N)^{3/2}$ and also $O(\Sigma\, k^{-5/2}(k/N)^{3/2}) = O(N^{-1/2})$. Substitution of this result into (6.17) gives,

$$p(n) = i\, \Sigma\, A_k(n)\, k^{-5/2} \int_K G_k(z)\, e^{2\pi nz/k^2} dz + O(N^{-1/2}). \tag{6.20}$$

Notice that the summands are independent of N. Let $N \to \infty$ in (6.20) to obtain the infinite series:

$$p(n) = i \sum_{k=1}^{\infty} A_k(n)\, k^{-5/2} \int_K G_k(z)\, e^{2\pi nz/k^2} dz. \tag{6.21}$$

It is now possible to evaluate the integral in (6.21) by the use of the substitution $w = 1/z$ and application of Bessel functions to yield the result of the theorem.

The techniques of Rademacher have been used to obtain series representations of other partition functions. In particular, Haskell [27], and independently Wright, have obtained series representations for $p_{-k}(n)$ which includes Rademacher's for $p(n)$ as a special case.

CHAPTER VII

SUMMARY

The purpose of this thesis is given in Chapter I. Briefly, it is to introduce to the advanced undergraduate student the rather unfamiliar but important theory of partitions of a number which involves the number-theoretic partition function, $p(n)$. After giving the necessary background in Chapter II, the origin and a brief historical development is given in Chapter III. The introduction, including basic definitions and theorems of partition theory, is given in Chapter IV.

One of the major problems, to determine the congruence properties of the partition functions, is considered in Chapter V. Several congruences are obtained. A simple proof of the congruence $p(11m + 6) \equiv 0 \pmod{11}$ is given. Except for a proof in 1969 by Winquist [49], this is the only proof to the author's knowledge which has not required the more advanced analysis, including modular functions, which is needed and used in Chapter VI.

Congruences are also obtained for the k-line partition function, $t_k(n)$, for k = 1, 2, 3, 4, 5, 6, 7 and 8. The congruences for $t_6(n)$, $t_7(n)$ and $t_8(n)$ are found by the method of Cheema [12] which, to the author's knowledge, have never appeared in print.

Chapter VI, requiring more advanced analysis and mathematical maturity, gives a brief development of the asymptotic and series representation of $p(n)$ which were found by Hardy and Ramanujan [25] in 1918 and Rademacher [42] in 1937, respectively.

An interesting problem for further study would be an attempt to solve the conjecture posed by Morris Newman [36] in 1960.

Conjecture. $p(n)$ fills all residue classes modulo m infinitely often, that is, that if r is any integer such that $0 \leq r \leq m-1$, then the congruence $p(n) \equiv r \pmod{m}$ has infinitely many solutions in non-negative integers n.

The known congruences of Ramanujan's conjecture already give a partial answer for all powers of 5 and 7, for 11, 11^2 and 11^3. Newman [36], in an attempt to prove the conjecture, has shown the conjecture to be true for $m = 5$ and 13.

Another interesting problem is to determine the (asymptotic) density, if it exists, of the integers n such that m divides $p(n)$ for a given m. For example, the congruence $p(5m + 4) \equiv 0 \pmod{m}$ implies that 5 divides $p(n)$ for infinitely many n. Newman [37] showed that,

$$\liminf_{x \to \infty} \frac{s(x)}{x} > \frac{1}{5} + \frac{2}{5.19^4},$$

where,

$$s(x) = \sum_{n \leq x;\ p(n) \equiv 0 \pmod{5}} 1.$$

Whether the limit exists, hence the density, still seems a difficult question.

The need for an explicit generating function for the d-dimendional partition function, more fully explained in Chapter III, is certainly a challenge to anyone capable of even coming up with a worthy conjecture.

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APPENDIX

A Table of Partitions

The following table gives the values of $p(n)$, $t_2(n)$, $t_3(n)$, $t_5(n)$, $t(n)$ and $p_{-2}(n)$ for all $n \leq 34$. These values can easily be computed from the recursion formulae of Chapter IV.

A more complete table for $p(n)$, values for $n \leq 600$, is given by Gupta [21,22]. The values of $t_k(n)$ for $k = 2, 3, 5, 25$, $t(n)$ and $p_{-2}(n)$ for all $n \leq 299$ are given by Haskell [27].

A TABLE OF PARTITIONS

n	p(n)	$t_2(n)$	$t_3(n)$	$t_5(n)$	t(n)	$p_{-2}(n)$
1	1	1	1	1	1	2
2	2	3	3	3	3	5
3	3	5	6	6	6	10
4	5	10	12	13	13	20
5	7	16	21	24	24	36
6	11	29	40	47	48	65
7	15	45	67	83	86	110
8	22	75	117	152	160	185
9	30	115	193	263	282	300
10	42	181	319	457	500	481
11	56	271	510	768	859	752
12	77	413	818	1292	1479	1165
13	101	605	1274	2118	2485	1770
14	135	895	1983	3462	4167	2665
15	176	1291	3032	5564	6879	3956
16	231	1866	4610	8888	11297	5822
17	297	2648	6915	14016	18334	8470
18	385	3760	10324	21937	29601	12230
19	490	5260	15235	34081	47330	17490
20	627	7352	22371	52552	75278	24842
21	792	10160	32554	80331	1 18794	35002
22	1002	14008	47119	1 22078	1 86475	49010
23	1255	19140	67689	1 84161	2 90783	68150
24	1575	26085	96763	2 76303	4 51194	94235
25	1958	35277	1 37404	4 11870	6 96033	1 29512
26	2436	47575	1 94211	6 10818	10 68745	1 77087
27	3010	63753	2 72939	9 00721	16 32658	2 40840
28	3718	85175	3 81872	13 21848	24 83234	3 26015
29	4565	1 13175	5 31576	19 29981	37 59612	4 39190
30	5604	1 49938	7 36923	28 05338	56 68963	5 89128
31	6842	1 97686	10 16904	40 58812	85 12309	7 86814
32	8349	2 59891	13 97853	58 47966	127 33429	10 46705
33	10143	3 40225	19 13561	83 90097	189 74973	13 86930
34	12310	4 44135	26 10023	119 90531	281 75955	18 31065

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