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A VARIATIONAL FORMULATION OF TURBULENT COUETTE FLOW

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TROY DEAN REED

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A VARIATIONAL FORMULATION

OR TURBULENT COUETTE FLOW

APPROVED BY

Edward F. Blick

Raymond V. Kaser

Tom J. Bo...

John E. Francis

Jack C. Ha...

DISSERTATION COMMITTEE

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ABSTRACT

Couette flow was first examined experimentally in 1890 by Maurice Couette. Since that time, many theorists have studied the corresponding turbulent flow for the purpose of constructing a mathematical model which would predict the observed behavior. Although various semi-empirical theories were developed, the random, three-dimensional character of the fluid turbulence prevented the construction of a self-contained theory.

The author has obtained the first analytical solution for a turbulent, plane Couette flow via application of the recently developed general evolution criterion of Glansdorff and Prigogine. The resulting theory indicates that the flow obeys the well-known law of the wall and enables the calculation of the constants appearing in the law as a function of Reynolds number. The theoretical results also include an approximate solution for the distributions across the flow of the Reynolds stress and the rate of dissipation of turbulent kinetic energy by viscous effects. In addition, the relation between skin friction coefficient and Reynolds number is obtained.

ACKNOWLEDGEMENTS

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NOMENCLATURE

A_1, A_2	constants appearing in the series representation of (λ) , eq. (3.38)
B	constant appearing in the logarithmic law of the wall, eq. (3.33b)
C_f	skin friction coefficient,
C_1, C_2	constants appearing in the series representation of $(u^+)^2$, eq. (3.39)
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
e	internal energy per unit mass
H	enthalpy per unit mass
h	width of Couette flow
h^+	non-dimensional flow width, hu_τ/ν
J_i	generalized flux or rate of the ith transport process
K	Prandtl-von Karman constant appearing in the law of the wall
k	thermal conductivity
L_{ij}	phenomenological coefficients
ℓ	turbulent mixing length
N	unit normal vector
P	pressure
q	heat

$\dot{q}$	heat flux
S	total entropy of system
$\mathcal{S}$	surface area
S_v	entropy per unit volume
s	entropy per unit mass
T	temperature
t	time
U_j	components of mean velocity
U^+	non-dimensional mean velocity, U/u_τ
u_j	components of turbulent velocity fluctuations
u_τ	friction velocity, $(\tau_w/\rho)^{1/2}$
$\overline{(u')^2}$	mean-squared of the small-scale velocity fluctuations
$(u^+)^2$	non-dimensional velocity correlation, $\overline{(u')^2} / u_\tau^2$
V	volume of system
v	specific volume
V_j	components of total velocity
w	work
X_i	generalized force of the ith transport process
x_j	Cartesian coordinates
y	non-dimensional x_2 coordinate, $\frac{x_2 u_\tau}{\nu}$

Greek letters

α	an arbitrary constant
β	constant appearing in the law of the wall

ϵ	eddy viscosity
η	Lagrange multiplier
λ	non-dimensional dissipation scale,
λ_T	Taylor's dissipation scale
μ	molecular viscosity
ν	kinematic viscosity, μ/ρ
ξ	an arbitrary displacement
π_{ij}	components of fluid stress tensor, defined by $\tau_{ij} - p\delta_{ij}$
ρ	density
σ	rate of internal entropy production
τ_w	viscous shear stress at the boundary, $[dU/dx_2]\ x_2 = 0$
τ_{ij}	fluid shear stress tensor, defined as $\mu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)$
$\bar{\Phi}$	the local potential or generalized rate of entropy production
φ	entropy flow
$\overline{\chi}$	isothermal compressibility coefficient
ψ	Lagrangian associated with the local potential

subscripts

0	denotes stationary state values
1,2,3	denotes quantities measured in the x_1, x_2, x_3 directions, respectively
c	centerline value
m	maximum value
T	theoretical value

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INTRODUCTION

This work was initially undertaken in conjunction with an investigation of reduction of turbulent boundary layer skin friction via the use of flexible boundaries. Upon conduction of a literature search for relevant information, it was found that the firm of Bolt, Beranek and Newman, Inc., had conducted a similar investigation during the period 1962 - 1965 under contract to the Navy. Their work consisted of a limited experimental program and an analytical investigation of the interaction of both an externally excited surface and a passive, compliant surface with a turbulent shear flow.

In their final report, Jackson, et al [1965], they label the problem as being mathematically "intractable." Their approaches to the problem involved various applications of statistical methods to various forms of the Navier-Stokes equations. Apparently, a variational approach was not investigated, since prior to the publication of the work of Glansdorff and Prigogine [1964] no general variational procedure existed which could account for the non-linear convection terms in the Navier-Stokes equations.

Blick, et al [1969] have found experimentally that a very flexible boundary does indeed appear to reduce the skin friction produced by a turbulent boundary layer. First attempts to construct a mathematical model of this phenomenon met with only limited success, Blick [1969]. In subsequent efforts to simplify the problem to one which could be successfully modeled mathematically, it was decided to attempt to apply the variational procedure developed by Glansdorff and Prigogine to the simplest, realizable, turbulent shear flow, viz., Couette flow. Originally, it was planned that both rigid and flexible boundaries would be investigated. However, obtaining a solution for turbulent Couette flow with rigid boundaries proved to be a considerable problem and a study of the flexible case is relegated to future efforts. It is pertinent to note, however, that the method developed herein does appear to be general enough to account for the dissipative effects of a flexible boundary responding to the random, turbulent pressure fluctuations.

A VARIATIONAL FORMULATION OF TURBULENT COUETTE FLOW

CHAPTER I

DEFINITION AND DISCUSSION OF PROBLEM

Turbulent fluid motion is defined by Hinze [1959], p.2, as

> An irregular condition of flow in which the various quantities (such as velocity and pressure) show a random variation with time and space coordinates, so that statistically distinct average values can be discerned.

The vast majority of real flows are indeed turbulent and, furthermore,are usually associated with shear flows, i.e., flows which have a spatial gradient of the mean velocity normal to the streamlines. Turbulent flows are both difficult to understand physically and to model mathematically owing basically to:

1. the random, three-dimensional motion of a continuous medium (which negates application of the ideas of classical kinetic theory),
2. nonlinearity of the equations of motion which leads to the existence of more fluid motion properties than governing equations.

The fundamental problem is most aptly described by Nee and Kovasznay [1967].

> In turbulent flow, it is usually assumed that the fluid is regarded as a continuum while the instantaneous velocity components and the pressure obey the Navier-Stokes and continuity equations. Due to randomness of the turbulent motion, it is necessary to treat turbulence as a statistical phenomenon both in theory and in experiment, instead of considering the development of the instantaneous and local values of the fluid velocities and pressures. To state it precisely, we want to know only the statistical distribution of solutions when a statistical distribution of initial or boundary conditions are known or given. But, unfortunately, due to the nonlinearity of the Navier-Stokes equations the resulting governing equations for the statistical variables (such as the nth order correlations*, or spectrum functions*) are all indeterminate in the sense that they form a finite hierarchy of the equations where the number of the dependent variables grow more rapidly than the number of equations. The "closure" of such a hierarchy of equations by the use of some suitable hypothesis is the classical problem of the theory of turbulence.

The closure problem has led to speculation that the Navier-Stokes equations may constitute necessary but not sufficient conditions to describe a stationary field of turbulence, e.g. Stern [1968]. However, the present work demonstrates that the Navier-Stokes equations are at least sufficient to permit approximate analytical solutions for a stationary, turbulent flow.

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*For example, $\overline{u_1 u_2 p'}$ is a third order correlation and, in general, a spectrum function is the Fourier transform of the various correlations. As noted by Batchelor [1953],p.8, G. I. Taylor introduced the basic energy spectrum function in 1938 and is defined as the one-dimensional, Fourier transform of the correlation between two fluctuating velocities. The resulting function is an energy spectrum function in the sense that it describes the distribution of the components of kinetic energy over the constituent Fourier wave-numbers of the turbulence.

In order to illustrate the fundamentals of the problem, let us consider the conservation of momentum equation for flow of an incompressible fluid having constant viscosity and experiencing no external body forces which may be written as

$$\rho\left[\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial X_j}\right] = -\frac{\partial P}{\partial X_i} + \mu \frac{\partial^2 V_i}{\partial X_j \partial X_j} \tag{1.1}$$

For a turbulent flow one may introduce the so-called Reynolds decomposition of the velocity and pressure into a time averaged (or mean) component and a fluctuating component, i.e.,

$$\begin{aligned} V_i &= U_i + u_i \\ P &= \bar{P} + p' \end{aligned} \tag{1.2}$$

Substituting equations (1.2) into (1.1), we have

$$\rho\left[\frac{\partial (U_i+u_i)}{\partial t} + (U_j+u_j)\frac{\partial (U_i+u_i)}{\partial X_j}\right] = -\frac{\partial (\bar{P}+p')}{\partial X_j} + \mu \frac{\partial^2 (U_i+u_i)}{\partial X_j \partial X_j} \tag{1.3}$$

Reynolds simplified this equation by taking the time average and introducing the following rules for time averages of fluctuating quantities.

$$\begin{aligned} \overline{a} &= \overline{b} = 0 \\ \overline{A+B} &= \overline{A} + \overline{B} \\ \overline{AB} &= \overline{A}\,\overline{B} + \overline{ab} \\ \overline{\frac{\partial A}{\partial X}} &= \frac{\partial \overline{A}}{\partial X} \end{aligned} \tag{1.4}$$

Where a and b are fluctuating quantities, and $A(x,t) = \overline{A}(x) + a(x,t)$ and $B(x,t) = \overline{B}(x) + b(x,t)$. Upon applying these rules to the time average of equation (1.3), we obtain

$$\rho\left[\frac{\partial \overline{U}_i}{\partial t} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} + \overline{u_j \frac{\partial u_i}{\partial x_j}}\right] = -\frac{\partial \overline{P}}{\partial x_i} + \mu \frac{\partial^2 \overline{U}_i}{\partial x_j \partial x_j} \tag{1.5}$$

From conservation of mass, we have $\partial u_j / \partial x_j = 0$; therefore, equation (1.5) can be rearranged into the following form:

$$\rho\left[\frac{\partial \overline{U}_i}{\partial t} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j}\right] = -\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu \frac{\partial \overline{U}_i}{\partial x_j} - \rho \overline{u_i u_j}\right] \tag{1.6}$$

When the conservation equation for the mean properties of the turbulent flow is written in this form, it has the same appearance as the laminar flow equation except for an extra term added to the laminar stress term. The additional terms $-\rho \overline{u_i u_j}$ represent an effective shear stress produced by eddies which cause a macroscopic exchange of momentum between adjacent parts of the flow field having different mean velocities. These stresses are usually referred to as the Reynolds stresses.

1.1 Phenomenological Theories

If one attempts to solve equation (1.6) for the mean properties of the turbulent flow, it is found that the question of how to determine the Reynolds stresses prevents a straightforward solution. The object of the

so-called phenomenological theories of turbulent shear flow is to express the Reynolds stresses in terms of a function of the mean velocity profile in order to reduce the number of dependent variables and thereby allow a solution. These theories have introduced such pseudo-quantities as an eddy viscosity ϵ defined by

$$\epsilon \frac{\partial U_i}{\partial x_j} \equiv - \overline{u_i u_j} \qquad \text{Boussinesq, 1877, (1.7a)}$$

a momentum mixing length ℓ for two dimensional flows

$$-\rho \overline{u_1 u_2} \equiv \ell^2 \left| \frac{dU}{dx_2} \right| \frac{dU}{dx_2} \qquad \text{Prandtl, 1926, (1.7b)}$$

and

$$\ell \equiv \kappa \frac{dU}{dx_2} \Big/ \frac{d^2U}{dx_2^2} \qquad \text{Von Karman, 1930, (1.7c)}$$

and a vorticity transport theory due to G. I. Taylor which assumes the vorticity of each fluid "particle" is constant and results in

$$-\overline{u_1 u_2} = \tfrac{1}{2} \ell_v^2 \left| \frac{dU}{dx_2} \right| \frac{dU}{dx_2} \qquad \text{Taylor, 1932, (1.7d)}$$

A tabulation of these and other phenomenological theories can be found in Rotta [1962].

As pointed out by Nee and Kovasznay [1967], all the classical phenomenological theories are based on the idea of determining the Reynolds stress at a point in terms of the mean flow geometry at that point; whereas, the Navier-Stokes equations are of the elliptical type which means that the interior of a flow field is influenced by all the boundary conditions. Thus, $\overline{u_i u_j}$ is undoubtedly affected by the neighboring flow conditions. It is pertinent to note that the theory developed herein is based on an integral over the entire flow, and thus represents a significant departure from the phenomenological theories.

1.2 Couette Flow

Couette flow is the name given by fluid dynamicists to the fluid motion which occurs between two parallel surfaces moving relative to each other. The name honors M. Couette [1890] who first investigated the flow between two concentric cylinders. Couette's apparatus involved a fixed inner cylinder and a rotating outer shell. Later Taylor [1935] investigated the reversed case of a fixed outer cylinder and a rotating inner cylinder. Basic differences between the two types of flow have been found. For a discussion of this type of Couette flow, the reader may refer to Townsend [1956].

Owing to the attendant centrifugal forces, we expect rotational Couette flow to differ from the ideal

planar case of two infinite plane surfaces moving parallel to one another with some non-zero, relative velocity. For such a flow the momentum equation, eq. (1.6), reduces to a requirement that the total shear stress be a constant throughout the entire flow field. In the case of laminar flow, this means

$$d\tau/dx_2 = \text{constant} \qquad (1.8)$$

Of course, this is easily solved for the velocity distribution across the flow once the distance of separation and the relative velocity between the parallel boundaries are known, see Figure 1. However, if one wishes to consider turbulent Couette flow, he immediately encounters the classical problem of what to do with the Reynolds stress $-\rho\overline{u_i u_j}$. When the fluid properties (ρ, μ), the separation distance h and relative velocity U_m are given, and the Reynolds number is such that the flow is turbulent (Figure 2), there is no known way to directly calculate the turbulent shear stress for this very simple case! This is an amazing comment on the limitations of the present state of knowledge concerning mathematical modeling of turbulent shear flows.

The purpose of the work reported herein is to establish a self-contained, theoretical model for turbulent, plane Couette flow with smooth, rigid, and

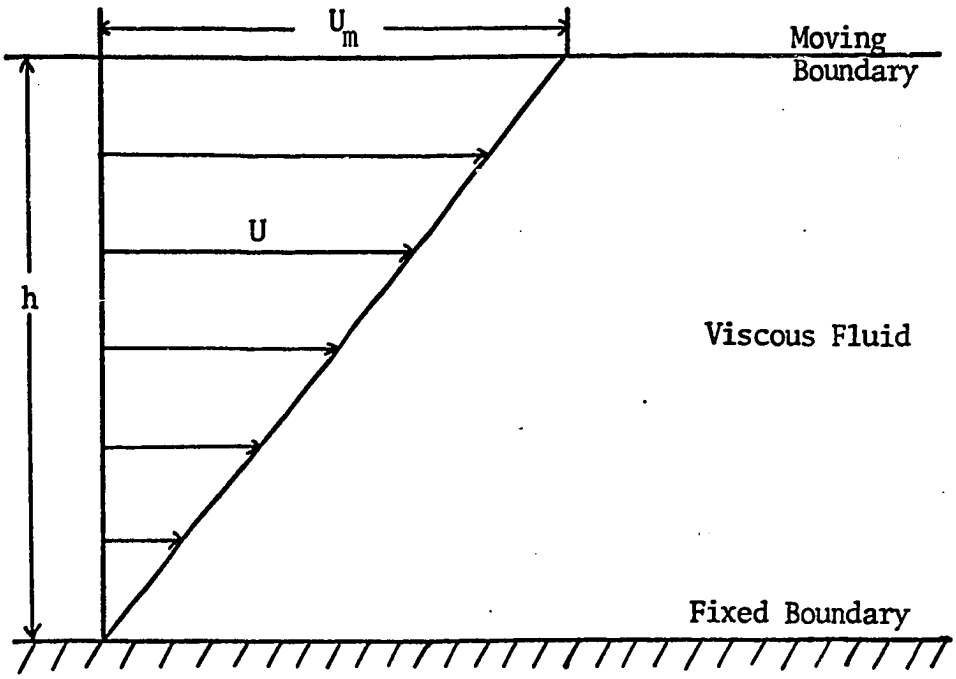

Fig. 1. LAMINAR, PLANE COUETTE FLOW

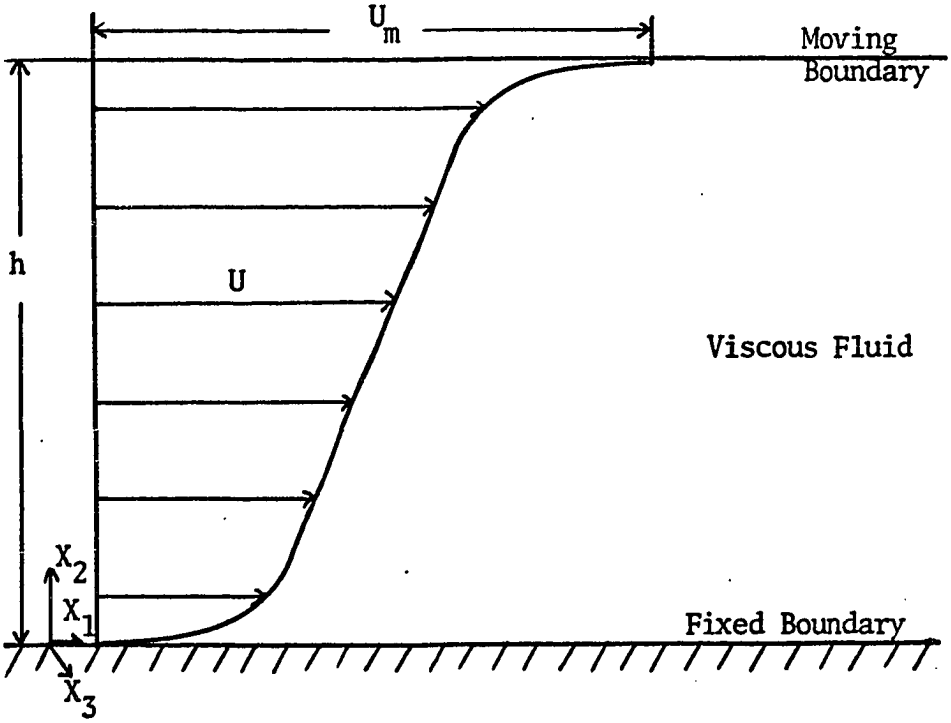

Fig. 2 TURBULENT, PLANE COUETTE FLOW

impermeable boundaries. The theory, which is developed, includes not only approximate, analytical solutions for the distributions of the mean velocity and Reynolds stress, but also the distribution of the rate of dissipation of turbulent kinetic energy.

CHAPTER II

THE GENERALIZED EVOLUTION CRITERION OF GLANSDORFF AND PRIGOGINE

A variational principle can be found for all linear differential equations which are self-adjoint or which can be put into such a form by multiplying by a "reducing factor," e.g. Hildebrand [1965], p. 216. The differential equation must be expressible in the self-adjoint form in order for it to have the same differential form as the Euler-Lagrange equation corresponding to the proposed stationary functional,* see Appendix I.

Glansdorff and Prigogine [1964] have developed a 'restricted' variational technique to handle nonlinear problems, which by definition are non-self-adjoint. In particular they purport to have developed a generalized evolution criterion for macroscopic physics based on the ideas of irreversible thermodynamics. The work reported herein is the first effort to apply their theory to a fully-turbulent shear flow, and it is found that the theory

*A functional is defined to be an integral whose integrand is a function of one or more functions of the independent variable.

does indeed appear to be applicable to turbulent shear flows provided the correct constraints can be found and applied.

However, before discussing the conceptually difficult theory of Glansdorff and Prigogine, it is necessary to set the scene with a brief discussion of variational theory and the principle of minimum rate of entropy production. These are discussed in section 2.1 and 2.2, respectively, and are followed by the development of the evolution criterion for a single-component fluid in section 2.3.

2.1 Variational Principles and Their Utility

The calculus of variations is that branch of mathematics which deals with the selection of an unknown function appearing in the integrand of an integral which will cause the value of the integral to be either a maximum or minimum. The modern theory began with Johann Bernoulli in 1696 who formulated and solved the famous brachistochrone (from the Greek meaning shortest time) problem*; and it was not until almost two hundred years later that Weierstrass established the sufficiency conditions for a relative (not absolute) minimum or maximum of a functional.

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*One is required to find the (frictionless) path between two points in a vertical plane along which a particle would move, under the action of gravity, in the shortest time.

> As the construction of the universe is the most perfect possible, being the handiwork of an all-wise Maker, nothing can be met with in the world in which some maximal or minimal property is not displayed. There is, consequently, no doubt but that all the effects of the world can be derived by the method of maxima and minima from their final causes as well as from their efficient ones. - L. Euler 1744, as quoted by Davis [1962].

This endorsement by the great Swiss mathematician of extremal principles for the mathematical modeling of nature has had far reaching and lasting effects in dynamics and quantum mechanics, e.g. Lanczos [1966], Goldstein [1959], and Yourgrau and Mandelstam [1960].

The success of such procedures, known as variational methods, in the mathematical analysis of many physical processes must be regarded as one of the most astonishing facts of science. The peculiar appeal of the idea of analyzing problems in terms of a minimal or maximal principle has persisted to the present, and indeed Einstein's Theory of General Relativity seems to endorse the application of variational theory to the description of nature* The immediate and most obvious objective of all mathematical modeling is to describe nature as closely as possible and obtain the best possible solution. In many cases variational methods seem to accomplish this objective most efficiently.

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*See Lanczos [1966] for an interesting historical discussion of variational methods and their relation to Relativity Theory.

In practice variational formulations of physical processes are quite useful, and one such formulation has enabled the present author to obtain the first successful, non-phenomenological solution for a turbulent shear flow. Before discussing the particular variational principle used herein, it seems appropriate to first define what one is and what are its possible advantages.

The following particularly clear and inclusive definition of a variational principle is given by Finlayson and Scriven [1967].

> A variational description of a physical system consists of a statement that the variation, or functional differential, of a specified functional is equal to some fixed value, which can be and customarily is chosen to be zero. The description is not complete without full specification of (1) the functions with respect to which the variation, or differential, is taken, and (2) any auxiliary conditions that must be satisfied as constraints when the variation is taken. The functional whose variation vanishes is said to be stationary relative to (1) those functions with respect to which the variation is taken, and (2) any constraints that are imposed. The stationary property of an integral functional implies by the calculus of variations one or more "Euler-Lagrange equations" and "natural boundary and initial conditions." If these match the equations of change, constitutive equations, boundary conditions, and so on, which describe the physical behavior of the system, then the variational formulation is indeed an alternative description, and is usually called a variational principle.

Many such variational principles have been labeled minimum (or maximum) principles, e.g. Hamilton's principle and the principle of least action, when in reality they only require some functional to be stationary which is only a necessary condition and <u>not</u> a sufficient condition for

the functional to be either a maximum or minimum. The question of a sufficient condition for a particular type of extremal is quite complex, e.g. Gelfand and Fomin [1963]. The search for sufficiency conditions extended from the middle 1700's to the late 1800's. This search uncovered three additional necessary conditions (the Euler-Lagrange being the first, see Appendix I) which must be considered in order to obtain a valid sufficiency proof, viz.,

1. Legendre's necessary condition, 1788
2. Jacobi's necessary condition, 1837
3. Weierstrass' necessary condition, 1879

Fortunately, in applications of variational principles the applied worker is seldom required to delve into the questions of sufficiency, primarily because the differential equations generated by the calculus of variations are consequences of the first variation. The comments of Dreyfus [1965] concerning sufficiency conditions and practical problems are relevant.

> For engineering purposes, necessary conditions are important than sufficient conditions. There are several reasons. Since the set of curves satisfying a valid sufficient condition may be vacuous, seeking a curve which satisfies a sufficient condition is akin to looking for a needle in a haystack which may not even contain a needle. This is a task not particularly appealing to a practical person with a pressing problem. Second, while necessary conditions are useful tests which can eliminate pretender curves, the violation of a sufficient condition by a curve proves little. Finally, various successive approximation algorithms that generate a sequence of curves converging through the space of nonminimizing curves toward a curve yielding a relative minimum can be formulated around necessary

conditions.

It should be noted here that all the necessary conditions concern relative, not absolute, extrema. Most sufficient conditions that exist are suitably strengthened combinations of the four fundamental necessary conditions and are of three types. Some guarantee weak relative minimization; others promise strong relative minimization; yet others, if satisfied, assure absolute minimization. These latter, unfortunately, are of a type that are rarely verifiable in practice. In dismissing conditions of the latter type as impractical, we relinquish all hope of isolating curves that yield anything provably better than relative minima. Such is the lamentable, but unavoidable, fate of practical applied variational theory.

Before concluding this discussion of variational principles, it is pertinent to note their advantages over analyses based just on differential equations. This is done with the following list of characteristics of variational principles which make them desirable for practical analyses.

Advantages of Variational Formulations:

1. Involve only physical quantities (e.g., kinetic energy, rate of entropy production, etc.) which can be defined without reference to a particular set of coordinates and are thus invariant with respect to the choice of coordinates for a system.

2. Can serve as the starting point for new mathematical formulations of physical processes.

3. Admit the possibility of obtaining approximate solutions via such methods as the Ritz Method, e.g., Schechter [1967], Chapter 3.

4. May suggest fruitful analogies and generalizations (as was the case in the present work).

5. A variational integral may represent a physical quantity of more use in a particular problem than the field given by the solution to the

Euler-Lagrange equation, and the variational integral is likely to approximate this integral more accurately than the solution.

6. If the principle is a minimum or maximum principle, the variational method provides upper or lower bounds on the variational integral.

7. If in addition a reciprocal variational principle (maximum or minimum) can be formulated, both upper and lower bounds can be found, and these may be most helpful in evaluating approximate solutions, see Schechter [1967], p. 93.

8. The direct method of the calculus of variations may yield proof of existence of solutions, a potential advantage when an exhaustive study of the mathematical aspects of a problem is needed.

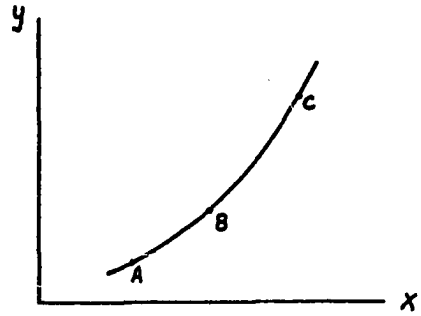
Thus, in view of these advantages it is not surprising that considerable effort has been expended in the search for variational principles.

2.2 The Principle of Minimum Entropy Production

In 1945 Prigogine introduced the theorem of minimum entropy production which is applicable to only linear problems in irreversible thermodynamics. Before discussing the assumptions of this theory, it is necessary to introduce some definitions and basic postulates of irreversible thermodynamics.

The International Dictionary of Applied Mathematics defines an irreversible process as the following:

> If a system undergoes a transformation ABC (see sketch), the change is said to be reversible if there exists a change CBA such that:
> (a) The variables characterizing the state of the system return through the same values, but in the reverse order;



(b) Exchanges of heat, matter and work with the surroundings are of the opposite sign and take place in the reverse order. Thus, for example, if in the trajectory ABC the system receives a quantity of heat Q, it must give up the same amount in the inverse trajectory CBA. All changes which do not satisfy these two conditions are irreversible.

All real, physical processes like heat transfer, diffusion, chemical reactions, etc. are irreversible. Such irreversible processes are characterized by the fact that they cannot be reversed without an expenditure of work by the surroundings greater than that done by the system.

In the same reference we find that the Onsager Reciprocal Relations are defined as relations which state that the matrix of phenomenological coefficients L_{kj} is symmetric provided a proper choice is made for the generalized rates of irreversible processes J_j and the corresponding generalized forces X_j, i.e.,

$$L_{kj} = L_{jk} \qquad (2.1)$$

The proper choice means that the rates J_j and forces X_j are such that the local entropy production per unit volume and unit time caused by internal irreversible processes is given by the product of these rates and forces, i.e.,

$$\sigma \equiv \frac{d_i S_v}{dt} = \sum_j J_j X_j \qquad (2.2)$$

$=0$ for reversible transport processes

>0 for irreversible " "

where S_v = entropy per unit volume

i denotes changes in entropy produced by changes inside the system, and the fluxes and forces are related by the so-called linear phenomenological laws which many transport processes appear to obey.

$$J_k = \sum_{j=1}^{r} L_{kj} X_j \qquad (2.3)$$

where r = number of processes involved. The coefficients L_{jj} (i.e. when K = j) may be proportional to thermal conductivity, electrical conductivity, the diffusion coefficient, etc.; whereas the coefficients L_{kj} (with $k \neq j$) describe the interference of the two irreversible processes k and j.

For example, Newton's 2nd Law, Fick's diffusion law, and Fourier's heat conduction law all have the structure of eq. (2.3) when k = j, e.g.

$$\dot{q}_x = -k \frac{\partial T}{\partial x} \sim \frac{\text{Heat}}{\text{Area-Time}} \tag{2.4}$$

and, as will be shown later, the corresponding volumetric rate of entropy production associated with this irreversible transport of internal energy caused by the temperature gradient is

$$\sigma = \dot{q}_x \frac{\partial}{\partial x}\left(\frac{1}{T}\right) \sim \frac{\text{E.U.}}{\text{Vol.-Time}} \tag{2.5}$$

where

$$\begin{aligned} X_j &= \partial T^{-1}/\partial x &&= \text{generalized force} \\ J_j &= \dot{q}_x &&= \text{generalized flux} \end{aligned} \tag{2.6}$$

Substituting eqs. (2.6) into eq. (2.3) with r = 1, we see that

$$\begin{aligned} \dot{q}_x &= L_{11} \frac{\partial T^{-1}}{\partial x} \\ &= -\frac{L_{11}}{T^2} \frac{\partial T}{\partial x} \end{aligned}$$

$$\therefore \quad L_{11} = k T^2 \quad \text{in accordance with Fourier's law}$$

When $k \neq j$, the Onsager reciprocity relations, eq. (2.1), state that when the flux J_k corresponding to the irreversible process k is influenced by the driving forces X_j of the other (r - 1) irreversible processes, then the individual fluxes J_j are also influenced by X_k through the same interaction coefficient L_{kj}. Coupling between various irreversible processes is the subject of various

specialities, e.g. thermo-elasticity (interaction of material displacements and temperature distribution), high temperature, real gasdynamics (interaction of temperature, velocity and chemical reactions), electro-mechanical phenomena (e.g. strain gauges), etc.

The bilinear form for the local entropy production, eq. (2.2), arises via the use of Gibbs thermostatic* equation.

$$Tds = de + Pdv \tag{2.7}$$

where s = entropy per unit mass

e = internal energy per unit mass

v = specific volume = $1/\rho$

This equation is obtained from the first and second law of thermodynamics and is subject to the following assumptions:

1. The only mode of work exchange between the system and its surroundings are of the PdV type. Furthermore, the volume changes associated with this work must be performed "slowly," so that at any instant P has a definite, unique value (e.g., during a rapid expansion of a gas, it becomes non-uniform so that there is no unique pressure). In addition, there must be no frictional work which would require additional pressure to accomplish a given volume change, Pippard [1961], p. 21.

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*According to Tribus [1961], thermostatics is concerned with equilibrium processes which do not depend on time as an explicit variable and, basically, tell us in what direction a process will go, but not how rapidly. Whereas, thermodynamics deals with the rates at which non-equilibrium processes tend to equilibrium. The reader is referred to Truesdell [1965] for an interesting discussion of these two subjects.

2. Gibbs' equation is based on the assumption of equilibrium states, i.e., changes in a thermodynamic system occur slowly enough so that the system passes from one state to another without appreciable deviation from a state of local equilibrium, which means that in the case of a single-component system its state at any instant is completely specified by two state variables such as temperature and pressure.

Since Gibbs' equation is applied to irreversible processes it is of interest to note the comments of Denbigh [1961], p. 44.

> As regards the application of the equation to irreversible paths the following may be said. The derivation was based on a reversible change, since it is only for such a change that we can, in general, write $dq = TdS$. The resultant equation gives the change, de, in the internal energy of the system, which is at a particular temperature T and a particular pressure P, in terms of the corresponding changes of entropy and volume. All of these quantities are functions of state. Moreover, provided that there are no irreversible changes of composition*, the choice of any two of the variables will determine the state of the system and therefore will determine the values of the other three variables. Thus, if we consider a change between a defined initial state and a defined final state, the integral of the equation must be valid even if the path is not a reversible one (but excluding irreversible changes of composition). Thus, as we go from an initial state (P_A, T_A) to a final state (P_B, T_B), the changes of e, V and S all have definite values, depending only on these states, and it is of no interest how this change takes place.
>
> Despite what has just been said, the terms TdS and PdV can be identified as heat absorbed and work done, only in the case of a reversible path. Thus we can write
>
> $$dq - dw = de = TdS - PdV,$$

---

*Such as would be caused either by diffusion or chemical reaction taking place within the body; in such cases the concentrations of the various species must also be specified in order to determine the local equilibrium state.

but whenever the process is irreversible dq is less than TdS and dw is less than PdV. (It is to be emphasized that P in the above equation refers to the pressure of the system itself, and therefore, in an irreversible expansion, it is larger than P_{ext}, the external pressure against which the work is done.)

The relation between dq and TdS referred to by Denbigh follows from the second law, $dS \geq dq/T$, which clearly states (since $dS = S_B - S_A$ is determined only by the initial and final equilibrium states A and B) that the heat intake of a system during an irreversible path is less than along a reversible one, or in the case where heat is released by the system (negative q), it is greater. The consequence of this difference in heat transfer is that the two types of paths do not give rise to the same changes of state in the surroundings.

With this introduction to Gibbs' equation and some of the definitions used in irreversible thermodynamics, we are ready to briefly discuss the principle of minimum entropy production. In this principle Gibbs' equation is assumed to apply equally well to fluid flow problems and is used in such problems with the differentials of eq. (2.7) replaced by the material derivatives of fluid mechanics*, i.e.,

$$T\frac{DS}{Dt} = \frac{De}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt} \qquad (2.8)$$

---

* $\frac{D}{Dt} = \frac{\partial}{\partial t} + V_j\frac{\partial}{\partial X_j}$ = local rate of change + changes caused by convection

By substituting the conservation equation for mass and energy into eq. (2.8), one can obtain a very general expression for the local, volumetric rate of entropy production. If we use the following forms of the continuity and energy equations for a single-component fluid experiencing no external forces:

$$\frac{D\rho}{Dt} = -\rho \frac{\partial V_i}{\partial X_i} \tag{2.9}$$

$$\rho \frac{De}{Dt} = \pi_{ij} \frac{\partial V_i}{\partial X_j} - \frac{\partial \dot{q}_i}{\partial X_i} \tag{2.10}$$

where π_{ij} = components of fluid stress tensor = $\tau_{ij} - P\delta_{ij}$

and substitute into the appropriate terms of eq. (2.8), we obtain an expression for the local time rate of increase of entropy per unit volume.

$$\rho \frac{Ds}{Dt} = \frac{1}{T}\left\{ \tau_{ij} \frac{\partial V_i}{\partial X_j} - \frac{\partial \dot{q}_i}{\partial X_i} \right\}$$

$$= -\frac{\partial (\dot{q}_i/T)}{\partial X_i} + \dot{q}_i \frac{\partial T^{-1}}{\partial X_i} + \frac{\tau_{ij}}{T} \frac{\partial V_i}{\partial X_j}$$

$$= -\operatorname{div} \varphi + \sigma \tag{2.10}$$

where $\operatorname{div}\varphi$ = the net flow of entropy per unit volume and time out of a fixed differential volume element of the fluid.

σ = the entropy production per unit volume and time.

$\dot{q}_i \frac{\partial T^{-1}}{\partial X_i}$ = entropy produced by the transport of energy in a finite temperature gradient.

$\frac{\tau_{ij}}{T} \frac{\partial V_i}{\partial X_j}$ = entropy produced by viscous dissipation of fluid motion into heat.

The principle of minimum entropy production requires that the volume integral of σ over the region of interest be stationary, i.e.

$$\delta \int_V \sigma \, dV = 0 \tag{2.11}$$

The interesting comments of Schechter [1967], p. 144, concerning this principle bear repeating here.

> Upon some reflection, it would seem intuitively correct that a principle of minimum entropy production should be valid. We have seen that the volumetric rate of entropy production is a measure of the irreversibilities associated with a process, and we know that in the absence of external intrusions a system will approach an equilibrium state, that is, a state in which the entropy productions vanishes. In the presence of certain imposed constraints it appears reasonable to assume that the system will come as closely as possible to equilibrium while satisfying the imposed constraints. Thus, we feel intuitively that the system in the stationary state* will generate as little entropy as possible by approaching the equilibrium state as closely as possible.

In actuality, very few physical systems obey this requirement. One can show this by comparing the Euler-Lagrange equations associated with eq. (2.11) with the appropriate balance equations which are known to correctly describe a system. If these two sets of equations are not identical, then the principle cannot lead to a correct description of a system's behavior (recall the definition of a valid variational principle given in section 2.1).

As an elementary example, consider the case of

---

*A time-independent state.

steady-state heat conduction in a body with a prescribed, time-independent, nonuniform temperature distribution. For such a case

$$\sigma = \dot{q}_i \frac{\partial T^{-1}}{\partial X_i} = J_i X_i \tag{2.12}$$

and the corresponding linear phenomenological law is

$$J_i = \dot{q}_i = L \frac{\partial T^{-1}}{\partial X_i} = -\frac{L}{T^2} \frac{\partial T}{\partial X_i} \tag{2.13}$$

where $L = k T^2$

Therefore, the total rate of entropy production is

$$\int_V \sigma(X_i, T, T') \, dV = \int_V L \frac{\partial}{\partial X_i}\left(\frac{1}{T}\right) \frac{\partial}{\partial X_i}\left(\frac{1}{T}\right) dV \tag{2.14}$$

where V is the volume of the system. Now if this integral is to be an extremal, we must require that its first variation with respect to the function $T(X_i)$ be zero, i.e.,

$$0 = \frac{\partial \sigma}{\partial T} - \frac{\partial}{\partial X_i}\left(\frac{\partial \sigma}{\partial T'}\right) = (T')^2 \frac{\partial L}{\partial T} + \frac{2}{T^2} \frac{\partial \dot{q}_i}{\partial X_i} \tag{2.15}$$

which is the Euler-Lagrange equation. The known, correct balance equation for this problem is

$$0 = \frac{\partial \dot{q}_i}{\partial X_i} \tag{2.16}$$

Thus, if eq. (2.15) and (2.16) are to be identical, we must require

$$0 = \frac{\partial L}{\partial T} \tag{2.17}$$

or kT^2= constant which can only be approximately true for nearly isothermal systems. Eq. (2.17) means that the phenomenological coefficients of a system must be constant in order for the principle of minimum rate of entropy production to be applicable.

Via similar considerations of more general problems it can be shown, e.g. Schechter [1967], p. 144-148, that the principle is a valid description if and only if the system meets the following requirements.

1. Linear phenomenological laws,

$$J_i = \sum_j L_{ij} X_j \qquad (2.18)$$

2. The Onsager reciprocity relations are valid,

$$L_{ij} = L_{ji} \qquad (2.19)$$

3. The phenomenological coefficients are constants,

$$L_{ij} = \text{constants} \qquad (2.20)$$

4. Nonlinear convection terms are negligible, i.e., no mechanical convection is considered, but purely dissipative processes of thermal, mechanical, or chemical origin are.

5. Boundary conditions and external forces are time independent.

For those linear systems which satisfy the above five conditions, the stationary state, i.e., the time-independent state, of the system is a state of minimum entropy production.

Since most physical systems do not satisfy these

severe restrictions, Glansdorff and Prigogine undertook the project of developing a more general principle which would be valid for nonlinear and time dependent processes.

2.3 The Local Potential: A Generalization of Thermodynamic Entropy Production

The fundamental conservation of mass, momentum and energy equations provide the starting point for mathematical analysis of flow problems. For specific problems, one often can simplify these partial differential equations somewhat, but usually the forms of the equations which are of interest remain coupled and nonlinear. In general the integration of such a set of equations is quite difficult, if not impossible. Typical contemporary approaches to such problems involve setting up a large number of finite-difference equations to approximate the differential equations and using a digital computer to solve the resulting horrendous set of algebraic equations. Not only is this an expensive route to follow, but it is plagued with many problems associated with minimizing errors and obtaining satisfactory solutions.

An alternate and less expensive approach to the problem of solving the fluid conservation equations is offered by the general evolution criterion proposed by Glandorff and Prigogine (1964). It will be shown that the resulting "restricted" variational principle has the complete nonlinear conservation equations as its Euler-Lagrange

equations; although no new physical information is introduced, the problem is thus made amenable to various approximate techniques such as the Ritz method and the method of partial integration, for examples see Schechter [1967], Chapter 3.

The thermodynamic system considered by Glansdorff and Prigogine is unrestricted, except that there exists at every point at each instant of time a state of local thermodynamic equilibrium as required by the Gibbs equation. However, the system as a whole need not be in equilibrium either thermally, mechanically or chemically; in other words, microscopic equilibrium is required, but macroscopic nonequilibrium is admissible and indeed is the source of entropy production. This assumption permits system evolution to be described in terms of macroscopic thermodynamics and fluid mechanics without any explicit introduction of molecular concepts, Prigogine [1965].

According to Prigogine [1965], the development of a general evolution criterion began with the observation that something still remains valid in the principle of minimum entropy production even when the various restrictions on that principle are not met. Namely, it was observed that when one splits the local volumetric rate of entropy production into two parts, viz.,

$$d\sigma = \sum X_i \, dJ_i + \sum J_i \, dX_i \qquad (2.21)$$

the last term is always negative for time-independent boundary conditions, irrespective of the phenomenological relations between J_i and X_i, (which may be nonlinear), i.e.,

$$\Sigma\, J_i\, dX_i \leq 0 \tag{2.22a}$$

$$\therefore \quad \int_V \left\{ \Sigma\, J_i\, dX_i \right\} dV \leq 0 \tag{2.22b}$$

For a proof of eq. (2.22) in a simple case, consider the heat conduction problem of an isothermal solid body which at time t = 0 suddenly has time-independent, but nonuniform, temperatures imposed at the boundaries. The appropriate form of the energy equation is

$$\rho \frac{\partial e}{\partial t} = -\frac{\partial q_i}{\partial x_i} \tag{2.23}$$

If we now multiply this equation by $\partial\,(1/T)/\partial t$ and use the product rule, the results are

$$\Psi = \rho \frac{\partial e}{\partial t} \frac{\partial}{\partial t}\left(\frac{1}{T}\right) = -\frac{\partial}{\partial x_i}\left[q_i \frac{\partial}{\partial t}\left(\frac{1}{T}\right)\right] + q_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial t}\left(\frac{1}{T}\right) \tag{2.24}$$

Now, since the internal energy is dependent only on the temperature, we have

$$de = C_v dT \tag{2.25}$$

where C_v = the specific heat at constant volume. Substituting eq. (2.25) into the left hand side of eq. (2.24), we see that it must be negative since $C_v > 0$ is one of the

requirements for thermodynamic stability*, i.e.,

$$-\rho \frac{\partial e}{\partial t} \frac{1}{T^2} \frac{\partial T}{\partial t} = -\frac{\rho C_v}{T^2} \left(\frac{\partial T}{\partial t}\right) \leq 0 \qquad (2.26)$$

Thus, it follows from eq. (2.24) that

$$-\frac{\partial}{\partial x_i}\left[q_i \frac{\partial}{\partial t}\left(\frac{1}{T}\right)\right] + q_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial t}\left(\frac{1}{T}\right) \leq 0 \qquad (2.27)$$

If we now integrate ψ over the volume of the system and recall that we have specified time-independent boundary conditions, i.e., $\delta T = 0$ at the boundaries, we obtain after applying the divergence theorem

$$\int_V \left[q_i \frac{\partial}{\partial t} \frac{\partial}{\partial x_i}\left(\frac{1}{T}\right)\right] dV \leq 0 \qquad (2.28)$$

which is a special case of eq. (2.22) and thus completes our proof. Eq. (2.28) is a measure of the deviation from the stationary state, i.e.,it will decay to zero as the system approaches the equilibrium state of a constant temperature distribution $T(X_i)$.

---

*$C_v > 0$ simply requires that the addition of heat to a closed, stable system must increase its temperature. The condition that the entropy be a maximum is the classical requirement for stability of an equilibrium state. For discussions of thermodynamic stability one may refer to Callen [1960], Chap. 8 and Kirkwood and Oppenheim [1961], Chap. 6.

Now, introduce the linear phenomenological law into eq. (2.28) to obtain

$$\int_V \left[L \frac{\partial}{\partial X_i}\left(\frac{1}{T}\right)\right]\left[\frac{\partial}{\partial t}\frac{\partial}{\partial X_i}\left(\frac{1}{T}\right)\right] dV \leq 0 \qquad (2.29)$$

Since L (= kT^2) is a function of temperature, assume that $[L\,\partial T^{-1}/\partial X_i] = L(T_o)\,\partial T_o^{-1}/\partial X_i$ where $T_o(X_i)$ is the time-independent temperature distribution at the stationary state. Using this assumption, one can write eq. (2.29) in the form

$$\frac{\partial \Phi(T,T)}{\partial t} = \frac{\partial}{\partial t}\int_V L(T_o)\frac{\partial T_o^{-1}}{\partial X_i}\frac{\partial T^{-1}}{\partial X_i} dV \leq 0 \qquad (2.30)$$

This is the defining equation for the <u>local potential</u> for steady state heat conduction. The main properties of the local potential are:

1. $\Phi(T,T_o)$ decreases in time until it reaches its minimum value $\Phi(T_o,T_o)$.
2. $\Phi(T_o,T_o) = 1/2(d_iS/dt)$ = one half the total rate of entropy production at the stationary state $T_o(X_i)$, since the phenomenological coefficients are time-independent in the stationary state.

Since the local potential is a generalization of the classical thermodynamic entropy production, it is also called the <u>generalized entropy production</u>.

Before demonstrating that it is legitimate, and indeed necessary, to replace the generalized flux with its stationary state values and thus is not subject

to variation, we must keep in mind the definition of a valid variational principle, viz., it must be equivalent to the conservation equations. Thus, if we wish to build a variational principle around the local potential, we must require that the first variation of $\Phi(T,T_o)$ with respect to variations in T be zero, and further, it must reproduce the correct form of the energy equation.

$$0 = (\delta\Phi)_{T_o} = -\int_V L(T_o)\frac{\partial T_o^{-1}}{\partial X_i}\frac{\partial}{\partial X_i}\left(\frac{\delta T}{T^2}\right) dV$$

$$= \int_V \frac{\delta T}{T^2}\frac{\partial}{\partial X_i}\left[L(T_o)\frac{\partial T_o^{-1}}{\partial X_i}\right] dV \qquad (2.31)$$

Eq. (2.31) follows from the divergence theorem and the fact that $\delta T = 0$ on the boundaries. If eq. (2.31) is to hold for arbitrary δT, it follows that the rest of the integrand must be zero.

$$\therefore \frac{\partial}{\partial X_i}\left[L(T_o)\frac{\partial T_o^{-1}}{\partial X_i}\right] = 0 \qquad (2.32)$$

At this point, i.e., after we have taken the variation, we assume that the temperature T_o which satisfies eq. (2.32) is the desired stationary temperature; thus

$$\frac{\partial}{\partial X_i}\left[L(T_o)\frac{\partial T_o^{-1}}{\partial X_i}\right] = \frac{\partial q_i}{\partial X_i} = 0 \qquad (2.33)$$

which is indeed the appropriate form of the energy equation for steady-state heat conduction.

The "name of the game" in defining (T,T_o) is to construct a functional which has:

1. Euler-Lagrange equations and boundary conditions identical to the appropriate form of the conservation equations.

2. A physical meaning within the framework of irreversible thermodynamics in hopes of having a unique relation between the variational principle and the conservation equations, i.e., we do not wish to have a functional which might have a relative extremal for some thermodynamic state other than the one which satisfies the conservation equations.

However, the local potential cannot be a true extremum since it is not actually stationary as defined in the calculus of variations theory. By this we mean that the defining functional

$$\Phi(T, T_0) = \int_V L(T_0) \frac{\partial T_0^{-1}}{\partial X_i} \frac{\partial T^{-1}}{\partial X_i} \, dV \tag{2.34}$$

is not stationary with respect to variations in T because we have set $L\,\partial T^{-1}/\partial x_i = L(T_0)\,\partial T_0^{-1}/\partial x_i$ and have treated it as a constant in the variation process. This procedure, i.e., defining certain parts of an integrand to be constant functions during a variation and afterwards requiring the corresponding varied and unvaried functions to be identical, is known as a "restricted" variational principle and was first introduced by Rosen [1953] in his studies of irreversible processes.

In order to clarify the distinction between a true stationary functional and the requirement that the first variation of the local potential be zero, it is helpful to consider an example given by Schechter [1967], p. 152.

Suppose we have a large sheet of material of unit thickness with time-independent face temperatures of 1 and 2, and let X be the position coordinate perpendicular to the faces. The entropy production for this problem is, see eq. (2.10),

$$\frac{d_i S}{dt} = \int_0^1 L \left(\frac{\partial T^{-1}}{\partial X_i}\right)^2 dV \tag{2.35}$$

If we arbitrarily let $L = kT^2 = \alpha/T$, we can calculate the temperature distribution which makes the entropy production be stationary. Making this substitution for L, the Euler-Lagrange equation is

$$\alpha \left(\frac{dT^{-1}}{dX}\right)^2 + \frac{d}{dX}\left(\frac{2\alpha}{T^3}\frac{dT}{dX}\right) = 0 \tag{2.36}$$

The solution of eq. (2.36), which satisfies the boundary conditions T = 1 at x = 0 and T = 2 at x = 1, is

$$T^{-3/2} = -\left(1 - \frac{1}{2^{3/2}}\right)X + 1 \tag{2.37}$$

Substituting this solution into eq. (2.35), we find the stationary value of the entropy production, viz.,

$$\frac{d_i S}{dt} = \int_0^1 \frac{\alpha}{T}\left[\frac{dT^{-1}}{dX}\right]^2 dX = \alpha\left(1 - 2^{3/2}\right)^2 \frac{4}{9} = 0.1857\,\alpha \tag{2.38}$$

Now we wish to compare this temperature distribution with the solution to the steady-state heat conduction equation.

$$0 = \frac{\partial q_i}{\partial x_i} = -\frac{d}{dx}\left[\frac{\alpha}{T^3}\frac{dT}{dx}\right] \tag{2.39}$$

the solution of which is

$$T = \left(1 - \frac{3x}{4}\right)^{-1/2} = T_0 \tag{2.40}$$

If we substitute this solution into eq. (2.35), we find that the entropy production is

$$\frac{d_i S}{dt} = \frac{3\alpha}{1.6} = 1.875\,\alpha \tag{2.41}$$

which is an order of magnitude larger than the stationary value, eq. (2.38). Thus, we have shown that the state of minimum entropy production does <u>not</u> correspond to the observable, stationary state of this nonlinear transport process.

However, the local potential for this problem

$$\Phi(T, T_0) = \int_0^1 \frac{\alpha}{T_0}\frac{\partial T_0^{-1}}{\partial x}\frac{\partial T^{-1}}{\partial x}\,dx$$

$$= \int_0^1 -\frac{3}{8}\,\alpha\,\frac{\partial T^{-1}}{\partial x}\,dx \tag{2.42}$$

is a minimum when $dT^{-1}/dx = dT_0^{-1}/dx$ is substituted from eq. (2.40) into this equation, since one can show that all other functions T(x) produce larger values of Φ. Thus, the temperature distribution which causes the local potential, with $J = J(T_0)$, to be an extremal is the observable,

steady-state temperature which satisfies the energy equation. If the solution T_o is not known a priori, as it was here, one must exercise caution in using the local potential, since calculations based on it being an extremal are only valid when the unvaried, stationary state functions (e.g., T_o) are handled properly. Since a practical application of the local potential, in which the stationary state is the unknown which we wish to solve for, is presented in Chapter III, we shall relegate further discussion of applications to that section.

Since the present work is concerned with only a single-component fluid, we shall derive a restricted form of the local potential which is applicable to such a system. For the general case of a system consisting of a mixture of various species, which are interacting via diffusion and chemical reactions, the reader is referred to the original work of Glansdorff and Prigogine [1964], and for discussions of time-dependent systems one may refer to Glansdorff [1965] and Hays [1965].

The derivation begins with the conservation equations which may be written in the following form for a single-component fluid.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i}(\rho v_i) \tag{2.43a}$$

$$\rho \frac{\partial v_i}{\partial t} = -\rho v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial \pi_{ij}}{\partial x_j} \tag{2.43b}$$

$$\frac{\partial(\rho e)}{\partial t} = -\frac{\partial}{\partial x_i}(\rho v_i e) - \frac{\partial q_i}{\partial x_i} + \pi_{ij}\frac{\partial v_i}{\partial x_j}$$

(2.43c)

Now multiply the continuity equation by $-\left[H\frac{\partial T^{-1}}{\partial t} + \frac{1}{\rho T}\frac{\partial \rho}{\partial t}\right]$, the momentum equation by $-\frac{1}{T}\frac{\partial v_i}{\partial t}$ and the energy equation by $\partial T^{-1}/\partial t$, and define Ψ as the sum of the left hand sides of the resultant equations*.

$$\Psi \equiv -\left[H\frac{\partial T^{-1}}{\partial t} + \frac{1}{\rho T}\frac{\partial \rho}{\partial t}\right]\frac{\partial \rho}{\partial t} - \frac{\rho}{T}\left(\frac{\partial v_i}{\partial t}\right)^2 + \frac{\partial T^{-1}}{\partial t}\frac{\partial(\rho e)}{\partial t}$$

(2.44)

We wish to prove that Ψ is always nonpositive. In order to accomplish this task, it is necessary to use some of the results of classical thermostatics which are strictly applicable only to systems in local equilibrium. The comments of Schechter [1967], p. 156, concerning the justification for this procedure is pertinent.

> Thus this represents a limitation on the validity of the work to follow. However, it is worth repeating that a variational formulation can be shown to be correct by insuring that the Euler-Lagrange equations are identical (including boundary conditions) with the appropriate form of the balance equations. If this latter condition is satisfied, one is not required to justify in a rigorous fashion the derivation of the evolution criterion.

---

*The justification for these multipliers are that they work, i.e., they lead to a local potential whose Euler-Lagrange equations are the conservation equations.

Therefore, with this thought in mind, we shall follow the procedure developed by Glansdorff and Prigogine to derive a specialized form of their general local potential.

To begin, we expand the last term of eq. (2.44) in terms of the pressure and enthalpy,

$$\frac{\partial(\rho e)}{\partial t} = \frac{\partial}{\partial t}\left(-P+\rho H\right) = -\frac{\partial P}{\partial t} + \rho\frac{\partial H}{\partial t} + H\frac{\partial \rho}{\partial t} \tag{2.45}$$

which follows from the definition $H \equiv e + P/\rho$. Since the enthalpy for a single-component fluid in equilibrium is a function only of the temperature and pressure, we may write

$$\frac{\partial H}{\partial t} = \left(\frac{\partial H}{\partial T}\right)_P \frac{\partial T}{\partial t} + \left(\frac{\partial H}{\partial P}\right)_T \frac{\partial P}{\partial t} \tag{2.46}$$

If we multiply by ρ and substitute the relations

$$\left(\frac{\partial H}{\partial T}\right)_P = C_P \tag{2.47a}$$

$$\left(\frac{\partial H}{\partial P}\right)_T = \frac{1}{\rho} + \frac{T}{\rho^2}\frac{\partial \rho}{\partial T} \tag{2.47b}$$

into eq. (2.46), we obtain

$$\rho\frac{\partial H}{\partial t} = \rho C_P \frac{\partial T}{\partial t} + \frac{\partial P}{\partial t} + \frac{T}{\rho}\frac{\partial \rho}{\partial T}\frac{\partial P}{\partial t} \tag{2.48}$$

Now inserting eq. (2.48) into (2.45), we have

$$\frac{\partial(\rho e)}{\partial t} = \rho C_P \frac{\partial T}{\partial t} + \frac{T}{\rho}\frac{\partial \rho}{\partial T}\frac{\partial P}{\partial t} + H\frac{\partial \rho}{\partial t} \tag{2.49}$$

and putting this into eq. (2.44) leads to

$$\Psi = \left[\frac{H}{T^2}\frac{\partial T}{\partial t} - \frac{1}{\rho T}\frac{\partial P}{\partial t}\right]\frac{\partial \rho}{\partial t} - \frac{\rho}{T}\left(\frac{\partial V_i}{\partial t}\right)^2$$

$$-\frac{1}{T^2}\left\{\rho C_p\left(\frac{\partial T}{\partial t}\right)^2 + \frac{T}{\rho}\frac{\partial \rho}{\partial T}\frac{\partial P}{\partial t}\frac{\partial T}{\partial t} + H\frac{\partial \rho}{\partial t}\frac{\partial T}{\partial t}\right\} \tag{2.50}$$

Next, observe that the first and last terms cancel, and the density ρ is a state function.

$$\therefore \frac{\partial \rho}{\partial t} = -\frac{1}{v^2}\frac{\partial v}{\partial t} = -\rho^2\left[\left(\frac{\partial v}{\partial T}\right)_P\frac{\partial T}{\partial t} + \left(\frac{\partial v}{\partial P}\right)_T\frac{\partial P}{\partial t}\right] \tag{2.51}$$

where $v \equiv 1/\rho$ = specific volume.

Thus, eq. (2.50) can be written as

$$\Psi = \frac{\rho}{T}\left[\frac{\partial v}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial v}{\partial P}\frac{\partial P}{\partial t}\right]\frac{\partial P}{\partial t} - \frac{\rho}{T}\left(\frac{\partial V_i}{\partial t}\right)^2$$

$$-\frac{\rho C_p}{T^2}\left(\frac{\partial T}{\partial t}\right)^2 - \frac{1}{\rho T}\frac{\partial \rho}{\partial T}\frac{\partial P}{\partial t}\frac{\partial T}{\partial t} \tag{2.52}$$

If we now use the relation

$$C_v - C_p = T\left(\frac{\partial v}{\partial T}\right)_P^2 \Big/ \left(\frac{\partial v}{\partial P}\right)_T \tag{2.53}$$

and the definition of the isothermal compressibility coefficient

$$\overline{\chi} = -\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T \tag{2.54}$$

eq. (2.52) can be written in the form

$$\Psi = -\frac{1}{T}\left\{\rho\left(\frac{\partial V_i}{\partial t}\right)^2 + \frac{\rho C_p}{T}\left(\frac{\partial T}{\partial t}\right)^2 + \frac{\rho^2}{\bar{\chi}}\left(\frac{\partial \nu}{\partial t}\right)^2\right\} \tag{2.55}$$

Now since thermodynamic stability about a state of local equilibrium requires $C_V > 0$ and $\bar{\chi} > 0$*, it follows that

$$\Psi \leq 0 \tag{2.56}$$

which is the desired results.

Returning to eqs. (2.43), multiplying by the appropriate multipliers used to obtain eq. (2.44), and summing the right-hand sides, we must have

$$\Psi = \left[H\frac{\partial T^{-1}}{\partial t} + \frac{1}{\rho T}\frac{\partial P}{\partial t}\right]\frac{\partial(\rho V_i)}{\partial X_i} + \frac{\rho V_j}{T}\frac{\partial V_i}{\partial t}\frac{\partial V_i}{\partial X_j}$$

$$-\frac{1}{T}\frac{\partial V_i}{\partial t}\frac{\partial \pi_{ij}}{\partial X_j} - \frac{\partial T^{-1}}{\partial t}\left[\frac{\partial(\rho V_i e)}{\partial X_i} + \frac{\partial q_i}{\partial X_i}\right]$$

$$+ \frac{\partial T^{-1}}{\partial t}\pi_{ij}\frac{\partial V_i}{\partial X_j} \leq 0 \tag{2.57}$$

Upon using the product rule for differentiation and rearranging, eq. (2.57) can be written in the form

---

*$\bar{\chi} > 0$ simply requires that an isothermal expansion of a stable system must cause a decrease in its pressure. The stability criteria constitute Le Chatelier's Principle which states that spontaneous processes induced by a deviation from equilibrium be in a direction to restore the system to equilibrium.

$$\Psi = \frac{\partial}{\partial x_i}\left[\rho v_i\left(H\frac{\partial T^{-1}}{\partial t} + \frac{1}{\rho T}\frac{\partial P}{\partial t}\right) - \frac{\pi_{ij}}{T}\frac{\partial v_j}{\partial t} - \frac{\partial T^{-1}}{\partial t}\rho v_i e - q_i\frac{\partial T^{-1}}{\partial t}\right] - \rho v_i \frac{\partial}{\partial x_i}\left(H\frac{\partial T^{-1}}{\partial t} + \frac{1}{\rho T}\frac{\partial P}{\partial t}\right)$$

$$+ \left[q_i + \rho v_i\left(e + \frac{v^2}{2}\right) - \pi_{ij} v_j\right]\frac{\partial}{\partial t}\frac{\partial T^{-1}}{\partial x_i}$$

$$+ \left[\pi_{ij} - \rho v_i v_j\right]\frac{\partial}{\partial t}\frac{\partial}{\partial x_j}\left(\frac{v_i}{T}\right) + \rho v_j \frac{\partial}{\partial t}\frac{\partial}{\partial x_j}\left(\frac{v^2}{2T}\right) \leq 0 \qquad (2.58)$$

Eq. (2.58) now has the same form as eq. (2.10), i.e., a flow term and a source term related to the internal evolution of the system. It is also important to note that the source has the form $\sum J_i' \frac{\partial X_i'}{\partial t}$ where the forces and fluxes for the generalized entropy production are:

X_i'	J_i'
$\frac{\partial}{\partial x_i}\left(H\frac{\partial T^{-1}}{\partial t} + \frac{1}{\rho T}\frac{\partial P}{\partial t}\right)$	$-\rho v_i$
$\partial T^{-1}/\partial x_i$	$q_i + \rho v_i (e + v^2/2) - \pi_{ij} v_j$
$\partial(v_i T^{-1})/\partial x_j$	$\pi_{ij} - \rho v_i v_j$
$\partial(\frac{v^2}{2} T^{-1})/\partial x_j$	ρv_j

It is of primary importance to notice that these fluxes and forces admit contributions from <u>both</u> thermodynamic and mechanical convection and reduce to those which appear in the principle of minimum entropy production when mechanical equilibrium prevails, i.e., when all macroscopic motions relative to the boundaries cease, and the various other

restrictions on the principle apply.

Integrating Ψ over the volume of the system, as was done in arriving at eq. (2.30), we obtain the time derivative of the local potential and a generalized evolution criterion.

$$\frac{\partial \Phi}{\partial t} = \int_V \Psi \, dV = \int dV \sum J_i' \frac{\partial X_i'}{\partial t} \leq 0 \tag{2.59}$$

This derivative is negative during the evolution of an arbitrary macroscopic system* and goes to zero when the system reaches a steady state consistent with the boundary conditions. This general criterion provides an important new tool for analyzing nonlinear transport phenomena.

We have seen in the special case of steady, nonlinear heat conduction that if requiring the local potential to be stationary is to restore the appropriate balance equation, it is necessary to write Φ in terms of a functional which has the fluxes treated as fixed functions corresponding to the steady state. Thus, in the general case, it follows from eq. (2.59) that we should define the local potential as

$$\Phi = \int_V \left[\sum (J_i')_0 X_i' \right] dV \tag{2.60}$$

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*i.e., eq. (2.59) is independent of any kinetic laws such as non-newtonian fluid laws, nonlinear dependency of thermal conductivity on temperature, etc.

The $(J_i')_o$ denote the values of the fluxes at the stationary thermodynamic state and are NOT subject to variations. When one requires this functional to be stationary (with $(J_i')_o$ = fixed functions) with respect to variations in $\left[H \frac{\partial T^{-1}}{\partial t} + \frac{1}{\rho T} \frac{\partial \rho}{\partial t}\right]$, V_i, and T^{-1}, the time-independent conservation equations of mass, momentum and energy are restored, respectively.

Equation (2.60) leads to the double appearance in the local potential of such macroscopic variables as temperature, pressure, velocity, etc., i.e. as zero subscripted and non-subscripted variables. Originally the local potential was "introduced in a heuristic way as a tool for calculations." However, Prigogine and Glansdorff have found a posteriori that the local potential is related to macroscopic fluctuation theory and in fact expresses the stability of an arbitrary macroscopic state with respect to small fluctuations. In actuality, the local potentials might better be called: "fluctuation potentials which determine the probability of deviations of thermodynamic variables from their average value at the steady state. This is the physical reason why each variable appears twice in the local potential, once as a fluctuating quantity and once as an average value.", Prigogine and Glansdorff [1965].

For example, $T = T_o + \delta T$ is the fluctuating temperature and T_o is the average temperature at the steady state.

Requiring $(\delta\bar{\Phi}/\delta T)_{T_o} = 0$ determines the most probable distribution of the fluctuating quantity T for a given average distribution T_o which in turn is required by the classical fluctuation theory of Einstein to be identical with the average T_o, Prigogine [1965]. It was noted by Glansdorff and Prigogine that this basic procedure is exactly what the theory of the local potential requires in order to restore the correct form of the conservation equations, i.e.,

1. first require $\left(\frac{\delta\bar{\Phi}}{\delta T}\right)_{T_o} = 0 \Longrightarrow$ most probable T with respect to T_o

2. after which we set

$$T = T_o$$

which is called the self-consistency condition in the local potential theory and is equivalent to requiring the most probable distribution to equal the average distribution.

Thus, the variational technique is applied to the fluctuations about the local equilibrium state and the admissible functions which the fluctuations can have consist of those distributions which differ from the average distribution by the effect of molecular fluctuations.

Therefore, in attempting to apply the general evolution criterion to turbulent flow, it is relevant to note that the time scale of even the smallest turbulent fluctuations are considerably larger than those associated with

molecular fluctuations, Hinze [1959], p. 8. Thus, it is expected that the variations used in the evolution criterion and resultant conclusions apply equally well to the description of turbulent flows.

CHAPTER III

APPLICATION OF THE LOCAL POTENTIAL TO TURBULENT, PLANE COUETTE FLOW

Apparently, up to the time of this work, the local potential had only been applied to laminar flows and investigations of the critical Reynolds number at which such flows become unstable and transition to turbulent flow begins. One may refer to the book by Schechter [1967] for a variety of examples of such applications. The purpose of the present work is to investigate the application of the local potential to the simplest type of realizable turbulent shear flow. In particular, we shall consider the case of fully-turbulent, plane Couette flow of a single-component fluid which is:

1. steady
2. incompressible
3. isothermal

3.1 Reduction of the Local Potential

We begin by using the isothermal assumption (T = const.) to reduce eq. (2.57) to

$$\psi = \frac{1}{T}\frac{\partial}{\partial x_i}\left\{v_i\frac{\partial P}{\partial t} - \pi_{ij}\frac{\partial v_j}{\partial t}\right\} - \frac{\rho v_i}{T}\frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial P}{\partial t}\right) +$$

$$\frac{\pi_{ij}}{T}\frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial t} + \frac{\rho v_j}{T}\frac{\partial v_i}{\partial t}\frac{\partial v_i}{\partial x_j} \qquad (3.1)$$

If we substitute $\pi_{ij} = \tau_{ij} - p\,\delta_{ij}$, this becomes

$$T\psi = \frac{\partial}{\partial x_i}\left\{\frac{\partial}{\partial t}(v_i P)\right\} - \frac{\partial \tau_{ij}}{\partial x_i}\frac{\partial v_j}{\partial t} - \rho v_i \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial P}{\partial t}\right)$$

$$- P\delta_{ij}\frac{\partial}{\partial t}\frac{\partial v_j}{\partial x_i} + \rho v_j \frac{\partial v_i}{\partial t}\frac{\partial v_i}{\partial x_j} \qquad (3.2)$$

where $T\psi$ can be interpreted as the volumetric rate of dissipation which is more convenient to work with when analyzing isothermal problems. Now introducing the assumption of an incompressible flow (ρ = const.), eq. (3.2) reduces to

$$T\psi = \frac{\partial P}{\partial x_i}\frac{\partial v_i}{\partial t} - \frac{\partial \tau_{ij}}{\partial x_j}\frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial t} \qquad (3.3)$$

If this is substituted into the defining equation for the local potential, eq. (2.59), we obtain

$$\frac{\partial(T\Phi)}{\partial t} = \int_V (T\psi)\,dV = \int_V \left\{\frac{\partial P}{\partial x_i}\frac{\partial v_i}{\partial t} - \frac{\partial \tau_{ij}}{\partial x_j}\frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial t}\right\} \qquad (3.4)$$

This leads to a functional for Φ corresponding to eq. (2.60), i.e.,

$$T\Phi = \int_V \left\{\left[\frac{\partial P}{\partial x_i}\right]_o - \left[\frac{\partial \tau_{ij}}{\partial x_j}\right]_o + \left[\rho v_j \frac{\partial v_i}{\partial x_j}\right]_o\right\} v_i\,dV \qquad (3.5)$$

where the zero subscript denotes:

1. functions which are treated as constants in all variational processes,
2. the appropriate values of the functions corresponding to the stationary thermodynamic state.

Requiring Φ to be stationary with respect to V_i yields

$$\rho V_j \frac{\partial V_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 V_i}{\partial x_j^2} \qquad (3.6)$$

where we have dropped the subscript <u>after</u> the variation is taken, since this is a self-consistent requirement or corollary of the theory of the local potential. Equation (3.6) is indeed the Navier-Stokes equations for steady, incompressible flow.

In the case of turbulent flow, the corresponding form of eq. (3.6) is

$$\rho U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial U_i}{\partial x_j} - \rho \overline{u_i u_j}\right) \qquad (3.7)$$

as shown in Hinze [1959], p. 19. The overbars denote time averaged properties, and

$$V_i = U_i + u_i$$
$$P = \overline{P} + P' \qquad \text{Reynolds decomposition}$$

U_i = components of time averaged velocity

u_i = components of turbulent velocity fluctuations

This leads one to ask: What will be obtained from the

integral of eq. (3.5) if one introduces a Reynolds decomposition and assumes the zero subscripts can be interpreted as denoting time averages? Doing this and using the rules for time averages of fluctuating quantities*, one obtains

$$T\bar{\Phi} = \int_V \left\{ \left[\frac{\partial \bar{P}}{\partial x_i} + \rho V_j \frac{\partial V_i}{\partial x_j} + \rho \overline{u_j \frac{\partial u_i}{\partial x_j}} - \frac{\partial \overline{\tau_{ij}}}{\partial x_j} \right]_0 (V_i + u_i) dV \right.$$ (3.8)†

If the Euler-Lagrange equation for this functional is to be identical with eq. (3.7), we see that we must take the variation with respect to the <u>total</u> instantaneous velocity $V_i = U_i + u_i$, in which case

$$0 = \frac{\partial \bar{P}}{\partial x_i} + \rho V_j \frac{\partial V_i}{\partial x_j} + \rho \overline{u_j \frac{\partial u_i}{\partial x_j}} - \frac{\partial \overline{\tau_{ij}}}{\partial x_j} \qquad (3.9)$$

Since the time average viscous shear stress $\overline{\tau_{ij}}$ has the same form as for laminar flow, one can easily note the identification of eq. (3.9) with (3.7) by using the continuity relation $\partial u_j / \partial x_j = 0$.

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* $\overline{u_i} = \overline{P'} = 0$

$$\overline{(V_i + u_i)(V_j + u_j)} = V_i V_j + \overline{u_i} V_j + V_i \overline{u_j} + \overline{u_i u_j} = V_i V_j + \overline{u_i u_j}$$

†It is of interest to note that in the case of products for turbulent flow the zero superscript clearly denotes the stationary state of the product and <u>not</u> the product of the individual terms evaluated at this reference state.

With this verification of the capability of the local potential to restore the appropriate form of the momentum eq. for turbulent flow, let us return to eq. (3.3) and rearrange it to obtain

$$T\Psi = \frac{\partial}{\partial t}\frac{\partial}{\partial x_i}\left(V_i P - \tau_{ij} V_j\right) - V_i \frac{\partial}{\partial t}\left(\frac{\partial P}{\partial x_i}\right) + V_i \frac{\partial}{\partial t}\frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial}{\partial t}\left(\tau_{ij}\frac{\partial V_i}{\partial x_j}\right) + \rho V_j \frac{\partial V_i}{\partial x_j}\frac{\partial V_i}{\partial t} \qquad (3.10)$$

Now using the divergence theorem and the definition of the local potential

$$T\Phi = \int_S \left(V_i P - \tau_{ij} V_j\right) N_i dS + \int_V \left\{-\left[V_i\right]_o \frac{\partial P}{\partial x_i} + \left[V_i\right]_o \mu \frac{\partial^2 V_i}{\partial x_j^2} + \mu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)\frac{\partial V_i}{\partial x_j} + \left[\rho V_j \frac{\partial V_i}{\partial x_j}\right]_o V_i\right\} dV \qquad (3.11)$$

We are now ready to introduce the Reynolds decomposition and specialize this functional to describe turbulent, plane Couette flow. However, before proceeding, it is appropriate at this point to introduce the concept of ergodic turbulence.

3.2 Ergodic Turbulence

Statistical mechanics attempts to describe the time averaged or equilibrium properties of isolated systems which have so many degrees of freedom that a complete description of all the internal motions of the system is impossible. According to classical mechanics one may picture

the motion of a system with N degrees of freedom as an orbit in a 2N-dimensional space known as phase space whose coordinates consists of N generalized coordinates and N generalized momenta. At any instant of time the instantaneous state of the system is completely specified by the position of a representative point in this space. Since the total energy of an isolated system is constant, we can imagine a multi-dimensional surface in phase space which the system would be constrained to move on. An ergodic* system is defined to be one whose representative point in phase space passes through every point of the constant-energy surface, corresponding to the energy of the system, before returning to its original position.

Using the ergodic hypothesis, Boltzmann was able to demonstrate the equivalence of time and representative ensemble averages. A representative ensemble being defined as a large number of similar systems whose representative points in phase space lie on a constant-energy surface and are suitably distributed so that every state accessible to the actual system (as it moves along the constant-energy surface) is represented by at least one system of the ensemble which has an identical state. However, one can construct an argument which appears to exclude real, physical systems from being ergodic, e.g., ter Haar [1954], p. 357.

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*From the Greek meaning "energy path."

With the realization around the turn of the century that real systems could not be strictly ergodic, much effort was directed toward proving the existence of quasi-ergodic systems for which the representative points in phase space would come arbitrarily close to all points of the constant-energy surface if sufficient time were allowed. Although no completely satisfactory proof that real systems obey such a hypothesis has been obtained, the entire theory of classical statistical mechanics is based on the assumption of the interchangeability of ensemble and time averages.

Beginning with a paper by G. D. Birkhoff [1931], ergodic theory has developed into a complex mathematical theory and has been the subject of much recent work, e.g., Arnold and Avez [1968] and Billingsley [1965]. It is relevant to the present work to note that recently, Birkhoff and Kampé de Fériet [1962] have applied the ergodic theory in their analysis of homogeneous turbulence, i.e., all the properties of the turbulence are independent of absolute spatial position in the turbulent field.

Concerning application of the hypothesis to fluid turbulence, Batchelor [1953] was apparently first to suggest that statistically homogeneous turbulence would obey the ergodic hypothesis:

> The property of spatial homogeneity says, in effect, that all regions of space are similar so far as the statistical properties of u are concerned, and this

> suggests that the result of averaging over a large number of realizations* or trials could be obtained equally well by averaging over a large region of space for <u>one</u>+ realization.

Again, this has never been rigorously proven, Beran [1968], p. 41.

As Khinchin [1949], p. 53, suggests, in the absence of a formal proof it is much easier to simply apply such a hypothesis and judge the theory constructed on it by its practical success or failure. This is the approach adopted here. In particular, in order to proceed with the evolution criterion and avoid random functions of time, it is necessary to postulate the existence of plane, ergodic turbulence which we shall define as follows.

$$\overline{fg} \equiv \left.\frac{\int_0^\infty fg\,dt}{\int_0^\infty dt}\right]_{x_i = \text{const.}} = \left.\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f g\,dX_1\,dX_3}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dX_1\,dX_3}\right]_{\substack{X_2 = \text{const.} \\ t = \text{const.}}} \qquad (3.12)$$

where $f = f(X_i, t)$ and $g = g(X_i, t)$ are fluctuating quantities associated with the turbulence. The product $\overline{fg}$ is defined experimentally either at one or two points, e.g., $f = (X_i, t)$ and $g(X_i + \xi i, t)$, in the flow field. Experiments show that it is a function only of the coordinate along which the mean velocity varies -- herein defined as X_2, the coordinate normal to the boundaries of the Couette

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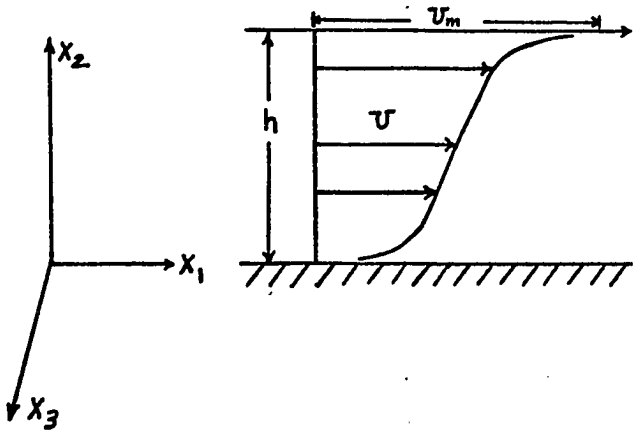
*time average

+instantaneous

flow. The basic assumption is that the turbulence is statistically homogeneous in planes parallel to the boundaries, and furthermore, the time average of a turbulent quantity at any point on such a plane is equal to the corresponding spatial average over the (infinite) plane at any instant of time.

3.3 Couette Flow

The nomenclature and boundary conditions used herein to describe plane Couette flow are as follows.



define

$$U = U_1(X_2)$$
$$U_2 = U_3 = 0$$
$$\bar{P} = \text{const.}$$
$$\tau = \text{const. [from eq. (3.7)]} \qquad (3.13a)$$

B.C.:

$$U = u_i = 0 \quad @ \; X_2 = 0$$
$$U = U_m = \text{velocity of moving boundary} \Big\} @ X_2 = h$$
$$u_i = 0 \qquad (3.13b)$$

We begin the process of specializing the local potential to describe this problem by introducing the Reynolds decompositions into the surface integral of eq. (3.11).

$$\int_S \left\{ (V_i + u_i)(\bar{P} + p') - (V_j + u_j)\mu\left[\frac{\partial(V_i + u_i)}{\partial X_j} + \frac{\partial(V_j + u_j)}{\partial X_i}\right]\right\} N_i \, dS \tag{3.14}$$

If we invoke the ergodic hypothesis and note that $\overline{V_j u_j} = \overline{V_j p'} = 0$, then eq. (3.14) reduces to

$$\int_S \left\{ V_i \bar{P} + u_i p' - \mu V_j\left(\frac{\partial V_i}{\partial X_j} + \frac{\partial V_j}{\partial X_i}\right) - \mu u_j\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)\right\} N_i \, dS \tag{3.15}$$

(see footnote)

Now applying the boundary conditions, eq. (3.13b), we obtain

$$\int_{X_2=h} \left\{-\mu V_m \frac{dV}{dX_2}\right\} N_2 \, dS = -\int_{-\infty}^{+\infty}\int \mu V_m \frac{dV}{dX_2}\Big]_{X_2=h} dX_1 \, dX_3 \tag{3.16}$$

It is of interest to note that eq. (3.15) appears to be the appropriate integral to account for a flexible boundary. An extension of this work to include the dissipative effects of a flexible boundary, i.e., $\int_S (u_i p') N_i dS$ would be most interesting and would provide additional information concerning the possibilities of obtaining drag reductions via such boundaries as has been found experimentally by Blick, et. al. [1969].

This integral represents the rate of work which the moving boundary is exerting on the fluid via the action of viscosity and as such must have a negative sign since dissipation terms are positive in the definition of the local potential.

Now directing our attention to the volume integral of eq. (3.11), we again introduce a Reynolds decomposition to obtain

$$\int_V \Big\{ -\Big[\overline{U_i + u_i}\Big]_o \frac{\partial(\bar{P} + P')}{\partial X_i} + \Big[\overline{U_i + u_i}\Big]_o \mu \frac{\partial^2 (U_i + u_i)}{\partial X_j^2} + \mu\Big(\frac{\partial(U_i + u_i)}{\partial X_j} + \frac{\partial(U_j + u_j)}{\partial X_i}\Big) \frac{\partial(U_i + u_i)}{\partial X_j} + \rho_o \Big[(U_j + u_j)\frac{\partial(U_i + u_i)}{\partial X_j}\Big]_o (U_i + u_i)\Big\} dV \tag{3.17}$$

Now using the definition of the time average and the ergodic hypothesis to eliminate terms linear in a fluctuating quantity, eq. (3.17) becomes

$$\int_V \Big\{ -\Big[U_i\Big]_o \frac{\partial \bar{P}}{\partial X_i} + \mu\Big[U_i\Big]_o \frac{\partial^2 U_i}{\partial X_j^2} + \mu\Big(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\Big)\frac{\partial U_i}{\partial X_j} + \mu\Big(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\Big)\frac{\partial u_i}{\partial X_j} + \rho_o\Big[U_j \frac{\partial U_i}{\partial X_j} + u_j \frac{\partial u_i}{\partial X_j}\Big]_o U_i\Big\} dV \tag{3.18}$$

We can now particularize this equation to the case of Couette flow by applying eqs. (3.13) which produces

$$\int_V \Big\{ \mu U_o \frac{d^2 U}{dX_2^2} + \mu\Big(\overline{\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}}\Big)\frac{\partial u_i}{\partial X_j} + \mu\Big(\frac{dU}{dX_2}\Big)^2 + \rho_o\Big[u_j \frac{\partial u_i}{\partial X_j}\Big]_o U\Big\} dV \tag{3.19}$$

The last term of eq. (3.19) may be simplified by using the following argument.

$$\overline{u_j \frac{\partial u_1}{\partial x_j}} = \frac{\partial (\overline{u_j u_1})}{\partial x_j} \tag{3.20}$$

which follows from continuity. Furthermore, the conservation of momentum equation, eq. (3.7), for Couette is simply

$$\tau = \mu \frac{dU}{dX_2} - \rho \overline{u_1 u_2} = \text{const.} \tag{3.21}$$

Thus, when dU/dX_2 is constant across a region of the flow, it follows that the time average properties of the turbulence are also constant in the same region which is consistent with our assumption of statistical homogeneity in planes X_2 = constant. Therefore, since the averaged turbulence properties can only vary in the X_2 direction, it follows that eq. (3.20) must reduce to

$$\overline{u_j \frac{\partial u_1}{\partial x_j}} = \frac{\partial (\overline{u_2 u_1})}{\partial X_2} \tag{3.22}$$

In addition, from eq. (3.21), we have

$$\rho \frac{d(u_1 u_2)}{dX_2} = \mu \frac{d^2 U}{dX_2^2} \tag{3.23}$$

$$\therefore \left[\rho u_j \frac{\partial u_1}{\partial x_j}\right]_o = \left[\mu \frac{d^2 U}{dX_2^2}\right]_o \tag{3.24}$$

3.4 Isotropic Dissipation of Turbulent Kinetic Energy

The second term of eq. (3.19) represents the dissipation of turbulent kinetic energy into heat via the action of viscosity. In the general case the dissipation consists of twelve different terms, and it has not yet been completely measured experimentally (Hinze [1959], p. 496) nor successfully predicted by theoretical methods. Thus to bypass this difficulty, Taylor [1935] introduced the assumption of isotropy for the small scale eddies responsible for viscous dissipation and a dissipation or microscale scale λ_T as a measure of the average dimension of eddies which are being dissipated.

Isotropy requires that all relations between turbulent quantities be invariant with respect to rotation of the coordinate system. One of the consequences of this definition is that there can be no average turbulent shear stress since it requires

$$\overline{u_i u_j} = -\overline{u_i u_j} = 0 \qquad \text{(isotropic)} \qquad (3.25)$$

which obviously does not apply to a turbulent shear flow. However, theoretical consideration (Hinze [1959], p. 183) and experimental evidence (e.g. Laufer [1951]) show that the small-scale structure of most actual nonisotropic flows is nearly isotropic. Thus, it is important to note that the assumption of isotropy of the motions associated with

dissipation indicates only that the small eddies are isotropic and does not restrict the larger scale eddies which are the primary cause of the turbulent shear stresses, i.e., $\overline{u_i u_j} \neq 0$, and are by definition nonisotropic.

If we expand the dissipation term of eq. (3.19) and use the isotropic relations:

$$\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} = \overline{\left(\frac{\partial u_2}{\partial x_2}\right)^2} = \overline{\left(\frac{\partial u_3}{\partial x_3}\right)^2} \tag{3.26a}$$

$$\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2} = \overline{\left(\frac{\partial u_1}{\partial x_3}\right)^2} = \overline{\left(\frac{\partial u_2}{\partial x_1}\right)^2} = \overline{\left(\frac{\partial u_2}{\partial x_3}\right)^2} = \overline{\left(\frac{\partial u_3}{\partial x_1}\right)^2} = \overline{\left(\frac{\partial u_3}{\partial x_2}\right)^2} \tag{3.26b}$$

$$\overline{\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}} = \overline{\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}} = \overline{\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}} \tag{3.26c}$$

the dissipation may be expressed as

$$\mu\overline{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}} = 6\mu\left[\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} + \overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2} + \overline{\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}}\right] \tag{3.27}$$

To further reduce this expression Taylor introduced the dissipation scale which is related to the above derivatives by

$$\frac{\overline{(u')^2}}{\lambda_T^2} = \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} \tag{3.28a}$$

$$2\frac{\overline{(u')^2}}{\lambda_T^2} = \overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2} \tag{3.28b}$$

$$-\frac{1}{2}\frac{\overline{(u')^2}}{\lambda_T^2} = \overline{\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}} \tag{3.28c}$$

where $\overline{(u')^2} \equiv \overline{(u_1)^2} = \overline{(u_2)^2} = \overline{(u_3)^2}$ for isotropy. The reader may refer either to the original work of Taylor [1935] or Hinze [1959], pp. 143-154, for a derivation of these relations. Substituting eqs. (3.28) into (3.27), the isotropic dissipation of turbulent kinetic energy into heat becomes simply:

$$\text{isotropic dissipation} = 15\mu\frac{\overline{(u')^2}}{\lambda_T^2} \tag{3.29}$$

The amount by which the actual dissipation for a particular flow varies from the isotropic value of 15 is a measure of anisotropy of the small-scale motion.

3.5 Nondimensionalization of Functional

Inserting eqs. (3.29) and (3.24) into (3.19) leads to

$$\int_V \left\{\mu U_0 \frac{d^2U}{dX_2^2} + 15\mu\frac{(u')^2}{\lambda_T^2} + \mu\left(\frac{dv}{dX_2}\right)^2 + \left[\mu\frac{d^2v}{dX_2^2}\right]_0 v\right\} dV \tag{3.30}$$

Now adding eqs. (3.30) and (3.16), we obtain the appropriate form of the local potential, eq. (3.11), for turbulent Couette flow.

$$\tau\Phi = -\int_{-\infty}^{+\infty}\int\int \mu U_m \frac{dU}{dX_2}\Big]_{X_2=h} dX_1 dX_3 + \int_{-\infty}^{+\infty}\int\int \int_0^h \Big\{\mu U_0 \frac{d^2U}{dX_2^2}$$

$$+\mu\left(\frac{dU}{dX_2}\right)^2 + \Big[\mu \frac{d^2U}{dX_2^2}\Big]_0 U + 15\mu\frac{\overline{(u')^2}}{\lambda_T^2}\Big\} dX_1 dX_2 dX_3$$

(3.31)

In order to generalize the results and reduce the number of independent parameters required to describe the problem, it is necessary to nondimensionalize this equation before attempting a solution.

Since it is known from experiments that the constant total shear stress portions of turbulent shear flows with zero pressure gradient obey the well-known "law of the wall," eq. (3.31) is nondimensionalized with respect to quantities which make the mean velocity profile $U(X_2)$ amenable to such a description. For this purpose we define the following quantities.

$$y \equiv \frac{X_2 u_\tau \rho}{\mu} \quad , \quad U^+ = \frac{U}{u_\tau}$$

$$h^+ \equiv \frac{h u_\tau \rho}{\mu} \quad , \quad (u^+)^2 \equiv \frac{\overline{(u')^2}}{u_\tau^2}$$

$$\lambda \equiv \frac{\lambda_T u_\tau \rho}{\mu}$$

(3.32)

where $u_\tau \equiv [\tau_w/\rho]^{1/2}$ is the so-called friction velocity and $\tau_w \equiv \mu\,(dU/dX_2)_{X_2=0}$ is the viscous shear stress at the boundary. Experiments with turbulent shear flows with zero pressure gradient, e.g. Kline, et. al. [1967], indicate that the mean velocity can be represented by

$$U^+ = y \quad \text{for} \quad 0 \le y \le 5 \tag{3.33a}$$

and

$$U^+ = \frac{1}{K} \ln y + B \quad \text{for} \quad 30 < y < 400 \tag{3.33b}$$

where the constants K (known as the Prandtl-von Karman constant) and B are empirically determined. For boundary layer type flows $K \approx 0.4$ and $B \approx 5$. In between the two regions where eqs (3.33) correlate the data, there is a smooth transition between the two relations. Various authors, e.g., Mellor and Gibson [1966], Kleinstein [1966], and Rasmussen and Karamcheti [1965], have proposed equations for the law of the wall which provide a smooth mean velocity profile, i.e., a continuous dU/dX_2. The form obtained by Rasmussen and Karamcheti was chosen for the present work. Their equation expresses the distance from the boundary X_2 in terms of a function of U, viz.,

$$y \equiv \frac{X_2 u_\tau \rho}{\mu} = U^+ + \frac{2}{\beta}\left[\cosh(KU^+) - \frac{(KU^+)^2}{2} - 1.0\right] \tag{3.34}$$

Using this equation we can evaluate the derivatives appearing in eq. (3.31) and make a change of variable so as to convert the integral over X_2 of eq. (3.31) into an integral over U, since

$$dy = dU^+ + \frac{2K}{\beta}\left[\sinh(KU^+) - KU^+\right] \tag{3.35}$$

Introducing the nondimensional variables into eq. (3.31), we obtain

$$T\Phi = -\rho u_\tau^3 \int_{-\infty}^{+\infty}\int U_m^+ \left.\frac{dU^+}{dy}\right|_{y=h^+} dX_1 dX_3 \quad +$$

$$\rho u_\tau^3 \int_{-\infty}^{+\infty}\int \int_0^{h^+} \left\{ \left[U^+\right]_0 \frac{d^2U^+}{dy^2} + \left[\frac{d^2U^+}{dy^2}\right]_0 U^+ + \right.$$

$$\left. \left(\frac{dU^+}{dy}\right)^2 + 15\frac{(u^+)^2}{\lambda^2} \right\} dy\, dX_1 dX_3 \tag{3.36}$$

We are still faced with the problem of finding a relation between the dissipation and the mean velocity profile in order to evaluate the integral over y. The only alternative is to assume a relation which appears to be compatible with experimental data, eq. (3.21), and the appropriate boundary conditions.

3.6 Assumed Solution for Dissipation

Since the momentum equation, eq. (3.21), requires

$$-\rho\overline{u_1 u_2} + \mu\frac{dU}{dX_2} = \text{constant},$$

it appears to be reasonable to assume that when dU/dX_2 is constant that the turbulence is also constant; in particular, we shall assume when dU/dX_2 = constant that:

$\frac{\rho}{2}\overline{(u')^2}$ = turbulent kinetic energy/mass of the isotropic, small-scale motion = const.

λ_T = const.

dissipation = const. (3.37)

We shall use the experimental results of Robertson [1959] to guide the selection of a specific functional relation between these quantities and the derivative of the mean velocity.

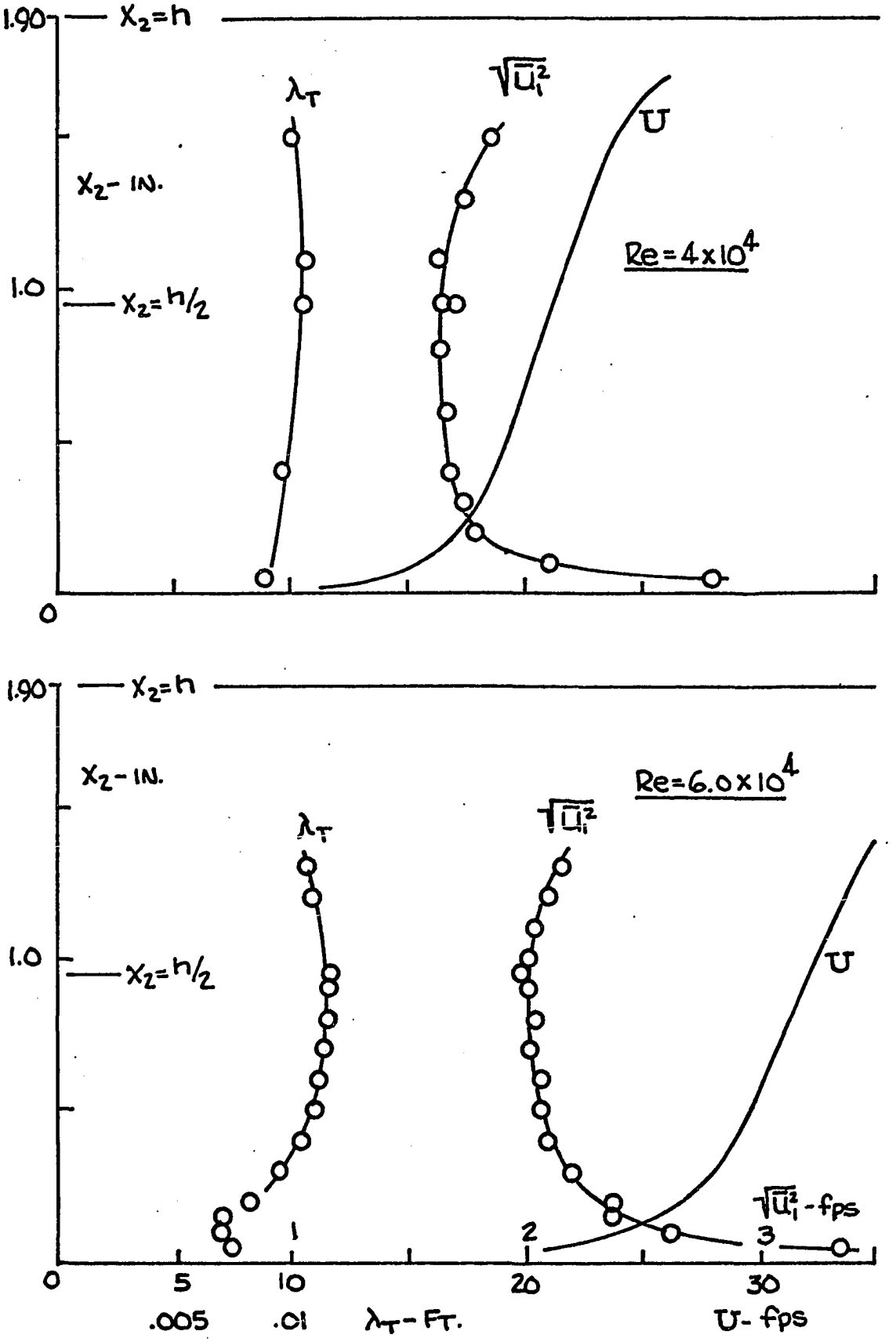
The measurements of Robertson, see Figure 3, do indeed indicate a nearly constant dissipation scale in the core of the flow where dU/dX_2 = constant. As a rigid boundary is approached, the dissipation scale decreases to a non-zero value (see Laufer [1951]) while dU/dX_2 increases. Thus the following functional form is assumed.

$$\lambda^2 = \left[A_1 + A_2 \frac{dU^+}{dy} \right]^{-1} \tag{3.38}$$

where the constants A_1 and A_2 are to be determined by requiring the local potential to be stationary.

In addition, the measurements of Robertson indicate $\overline{(u_1)^2}$ is nearly constant in the core and increases as one approaches a boundary. However, since a rigid boundary

Fig. 3. TURBULENCE INTENSITY DATA AND CALCULATED MICROSCALE, ROBERTSON [1959]



requires $u_1 = u_2 = u_3 = 0$ @ $x_2 = 0$ we expect $\overline{(u_1)^2}$ and the turbulent kinetic energy to reach a maximum and then fall to zero at $x_2 = 0$. The measurements of Klebanoff in a flat-plate, turbulent boundary layer clearly show such a behavior, Hinze [1959], p. 489. Thus, as a first approximation, it is assumed that

$$(u^+)^2 = C_1\left(1 - \frac{dU^+}{dy}\right) + C_2\left(1 - \frac{dU^+}{dy}\right)^2 \tag{3.39}$$

which satisfies the boundary conditions for rigid boundaries, i.e.

$$(u^+)^2 = 0 \quad @ \quad \begin{cases} U = 0 \\ U = U_m \end{cases} \tag{3.40}$$

Upon substituting eqs. (3.38 and (3.39) into eq. (3.36), we obtain the following approximation for the local potential.

$$T\Phi = -\rho u_\tau^3 \int_{-\infty}^{+\infty}\int U_m^+ \left.\frac{dU^+}{dy}\right|_{y=h^+} dx_1 dx_3 \; +$$

$$\rho u_\tau^3 \int_{-\infty}^{+\infty}\int \int_0^{h^+} \left\{ U_o^+ \frac{d^2U^+}{dy^2} + \left[\frac{d^2U^+}{dy^2}\right]_0 U^+ + \left(\frac{dU^+}{dy}\right)^2 + \right.$$

$$\left. 15\left(A_1 + A_2\frac{dU^+}{dy}\right)\cdot\left[C_1\left(1 - \frac{dU^+}{dy}\right) + C_2\left(1 - \frac{dU^+}{dy}\right)^2\right]\right\} dy\, dx_1 dx_3 \tag{3.41}$$

Using eq. (3.35) to evaluate the derivatives, the integral over X_2 or $\mathcal{U}$ can be numerically integrated once values of β and K are known.

3.7 The Ritz Method

The Ritz method is basically a procedure for approximating functions which are required to make some functional stationary. For example, suppose we are given some functional I(U) which is required to be stationary with respect to U(y). The Ritz method consists of representing U(y) in terms of a series which:

1. satisfies the boundary conditions imposed on U(y) and
2. forms a complete set, i.e., as the number of terms go to infinity the series can represent an arbitrary function. This is expressed mathematically for an arbitrary function f(y) as

$$\int [f(y) - f_n(y)]^2 dy \to 0 \quad \text{as} \quad n \to \infty$$

where $f_n(y) = \sum_{i=1}^{n} A_i \varphi_i(y)$ and usually the $\varphi_i(y)$ are required to be continuous and differentiable functions, e.g.,

$$f_n(y) = \sum_{i=1}^{n} A_i y^i \quad = \text{a power series}$$

which can be shown to have the above property. The Fourier series is another example of a complete set.

In practice the function U(y) is represented by the truncated series and the constants are determined by the set of equations resulting from requiring I(U) to be stationary with respect to these constants, i.e.,

$$I(U) \approx I\left(\sum_{i=1}^{n} A_i \varphi_i\right)$$

$$\frac{\partial I}{\partial A_i} = 0$$

(3.42)

The convergence of this procedure is usually rapid and often requires only a few terms to obtain a satisfactory approximation. However, the admonitions of Schechter [1967], p. 93 should be recognized.

> . . . convergence of the Ritz method is assured under certain conditions; however, the quality of a given approximation depends largely on the skill of the computer in selecting "good" functional forms. In most cases one can assess the goodness of it only by introducing a larger and larger number of free parameters until the answer no longer varies significantly. Although this test is often reliable, it is by no means rigorous. Indeed we can construct certain examples for which this test would fail. Consider the case in which the exact solution includes every fourth member of an orthogonal set of functions. Suppose further that we first use a single member of the orthogonal set as an initial approximation. On adding a second member of the orthogonal set to the approximating function, we would find that this member contributes nothing to the first set. Can we then conclude that convergence is complete? Obviously not. Thus there are certain dangers associated with the process of testing convergency by introducing a larger number of free parameters, and the reader should be cautious. However, the test is most often reliable and certainly is the best that can be done in many cases.

Referring back to eq. (3.41), it may be seen that we have constructed a functional which can be solved via the Ritz procedure. In particular, we can determine the unknown constants A_1, A_2, C_1, C_2, β and K by attempting to solve

the set of equations associated with the conditions

$$\frac{\partial \bar{\Phi}}{\partial \alpha_i} = 0 \tag{3.43}$$

where α_i = the desired constants.

If we blindly apply eq. (3.43) to eq. (3.41) and recall that the zero subscripts denote quantities which do not vary, we see that the first three terms within the volume integral form an exact differential; for example, taking the derivative with respect to β

$$\frac{\partial}{\partial \beta}\left\{ U_0^+ \frac{d^2U^+}{dy^2} + \left[\frac{d^2U^+}{dy^2}\right]_0 U^+ + \left(\frac{dU^+}{dy}\right)^2 \right\} =$$

$$U^+ \frac{\partial}{\partial \beta}\frac{d^2U^+}{dy^2} + \frac{d^2U^+}{dy^2}\frac{\partial U^+}{\partial \beta} + \frac{\partial}{\partial \beta}\left(\frac{dU^+}{dy}\right)^2 \tag{3.44}$$

where we have dropped the subscript <u>after</u> the derivative is applied in accord with the <u>self-consistency condition</u>. Eq. (3.44) can be further reduced and integrated over y to obtain

$$\int_0^{h^+} \frac{\partial}{\partial \beta}\frac{d}{dy}\left(U^+ \frac{dU^+}{dy}\right) dy = \frac{\partial}{\partial \beta} U_m^+ \left.\frac{dU^+}{dy}\right]_{y=h^+} = 0 \tag{3.45}$$

Thus, when we apply eq. (3.43) to the local

potential given by eq. (3.41), we are in effect requiring only that the integral of the volumetric rate of dissipation be stationary. This leads to two difficulties:

1. The set of equations resulting from requiring $\bar{\Phi}$ to be stationary with respect to the constants leads to a set of 4 homogeneous, linear equations in A_1, A_2, C_1, C_2 which has the trival solution $A_1 = A_2 = C_1 = C_2 = 0$.

2. Requiring the dissipation to be an extremum leads to the laminar flow solution, for which the dissipation of turbulent kinetic energy is zero, since none would exist!

The simplest remedy for the first of these problems is to make the local potential quadratic in A_1 or A_2 and C_1 or C_2. We wish to do this without restricting the admissible functions allowed by our assumed solutions for $(\mu^+)^2$ and λ^2, since it follows that we will have a greater probability of obtaining an optimum solution from a larger range of possible solutions. This can be accomplished a number of ways, but the author chose to make the following substitutions.

$$A_2 \longrightarrow A_2C_1$$
$$C_2 \longrightarrow A_1C_2$$

Upon again applying eq. (3.43), we obtain a solvable set of nonlinear algebraic equations.

The second problem can only be dealt with by introducing the appropriate constraints to find the desired relative extremum corresponding to a stationary field of turbulence.

3.8 Additional Constraints Necessary To Describe Turbulent Flows

The inability of the unconstrained local potential to describe a stationary field of turbulence seems to be traceable to the isothermal assumption, eq. (3.1). For laminar, incompressible flows this means the energy equation may be ignored and contributes nothing to the mathematical modeling of the flow. However, in the case of a turbulent flow the mechanical energy equation, obtained by multiplying the momentum equation by the total velocity, can be divided into a mean-energy equation and a turbulence-energy equation (e.g. Hinze [1959], p. 64). The turbulence-energy equation for a steady, incompressible flow is

$$U_j \frac{\partial \overline{q^2/2}}{\partial x_j} = -\frac{\partial}{\partial x_i}\overline{u_i\left(\frac{p'}{\rho}+\frac{q^2}{2}\right)} - \overline{u_i u_j}\frac{\partial U_j}{\partial x_i} + \nu\frac{\partial}{\partial x_i}\overline{u_j\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)} - \nu\overline{\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_j}{\partial x_i}} \tag{3.46}$$

Where $q^2 = \sum_{i=1}^{3} u_i u_i$ = 2 (turbulent kinetic energy/mass)

There terms have the following physical significance.

1. rate of change of turbulent kinetic energy per unit mass caused by variations from point to point of the flow field $U_j(x_j)$
2. convective diffusion by turbulence of the total turbulent energy

3. production of turbulent energy caused by the extraction of energy from the mean motion via the action of the turbulent shear stresses.

4. rate of work per unit mass done by the viscous shear stresses associated with the turbulent motion.

5. the rate of viscous dissipation of turbulent kinetic energy per unit mass.

If we integrate eq. (3.46) over the volume of plane Couette flow and use the divergence theorem and the boundary conditions appropriate for rigid, impermeable boundaries, we obtain

$$0 = \int_V \left\{ -\rho \overline{u_i u_j} \frac{\partial U_j}{\partial x_i} - \mu \overline{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_j}{\partial x_i}} \right\} dV \tag{3.47}$$

This equation simply requires that the stationary field of turbulent flow must obey the basic energy balance, viz., the net production of turbulence must equal the net dissipation.

Upon specializing eq. (3.47) to the case presently being considered, we have an integral constraint which restricts the variations of the local potential.

$$0 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{h^+} \left\{ \left(1 - \frac{dU^+}{dy}\right)\frac{dU^+}{dy} - 15\frac{(u^+)^2}{\lambda^2} \right\} dy\, dx_1\, dx_3 \tag{3.48}$$

One can apply this constraint to eq. (3.41) by

using the method of Lagrange multipliers, e.g. Schechter [1967], p. 16. This basically consists of introducing an undetermined multiplier η which is to be determined by the requirements

$$\delta \Phi_c \equiv \delta \Phi + \eta \, \delta J = 0 \tag{3.49}$$

and $J = \text{const.}$,

where J is defined to be the constraint, eq. (3.48). Thus, in place of eq. (3.43), we now have eq. (3.48) and the following set of nonlinear algebraic equations to solve:

$$\frac{\partial \Phi_c}{\partial \alpha_i} = 0 \tag{3.50}$$

where $\alpha_i = A_1, A_2, C_1, C_2, \beta, K$

Eqs. (3.48) and (3.50) constitute a set of seven equations for seven unknowns.

Upon solving for the unknown constants, we will obtain an approximate solution for:

1. the mean velocity profile, eq. (3.34)
2. the distribution across the flow of the: dissipation scale λ, eq. (3.38), turbulent kinetic energy per unit mass, eq. (3.39), the dissipation, eq. (3.29).

With the exception of the theory of Malkus [1956], which has been disproven by Reynolds and Tiedermann [1967], this represents the first entirely analytical solution for a turbulent shear flow.

CHAPTER IV

THEORETICAL RESULTS AND COMPARISONS WITH EXPERIMENTAL DATA

Before discussing the numerical results, it is of interest to note that a priori one does not know that eq. (3.34) will lead to $U = U_m/2$ at $X_2 = h/2$ as is found experimentally. Thus, it appears that formally the Ritz method requires that β and K be constrained so as to satisfy this boundary condition. However, it has been found that numerically this has negligible effect; i.e., less than 3%, on the resulting mean velocity profiles, which lends additional credulence to the assumption that the "law of the wall," eq. (3.33), is a valid representation for the mean velocity distribution in a turbulent, plane Couette flow.

4.1 Experiments

Although plane Couette flow is conceptually simple, it is extremely difficult to realize experimentally, and apparently there have been only two attempts to obtain such a flow in the laboratory, viz., Reichardt [1956] and

Robertson [1959]. Both experimental set-ups involved a continuous, moving belt. The degree to which flows so-obtained approximate the ideal, two dimensional case is a function of three basic variables:

1. ratio of the belt width to the distance h between the two primary flow boundaries, which may be defined as the flow aspect ratio,
2. constraints imposed at the sides of the flow,
3. length of the flow.

Reichardt studied the motions of oil and water inside a continuous belt loop; his basic experimental set-up is shown in Figure 4. The belt was 80 cm. wide and the spacing between the two runs of the belt was 16 cm. which provided as aspect ratio of five,* and the distance between the mounting pulleys was two meters.

On the upper side of the belt the liquid (water or oil) formed a free surface with the ambient air; however, from Reichardt's [1956] paper it is not clear exactly what type of boundary existed on the bottom side.

Some of the mean velocity profiles obtained by Reichardt in his apparatus are presented in Figure 4 along with corresponding numerical results from the subject theory. It may be seen that the theory indicates smaller values of dU/dX_2 in the middle of the flow and larger values near the boundaries. This is indicative of a basic

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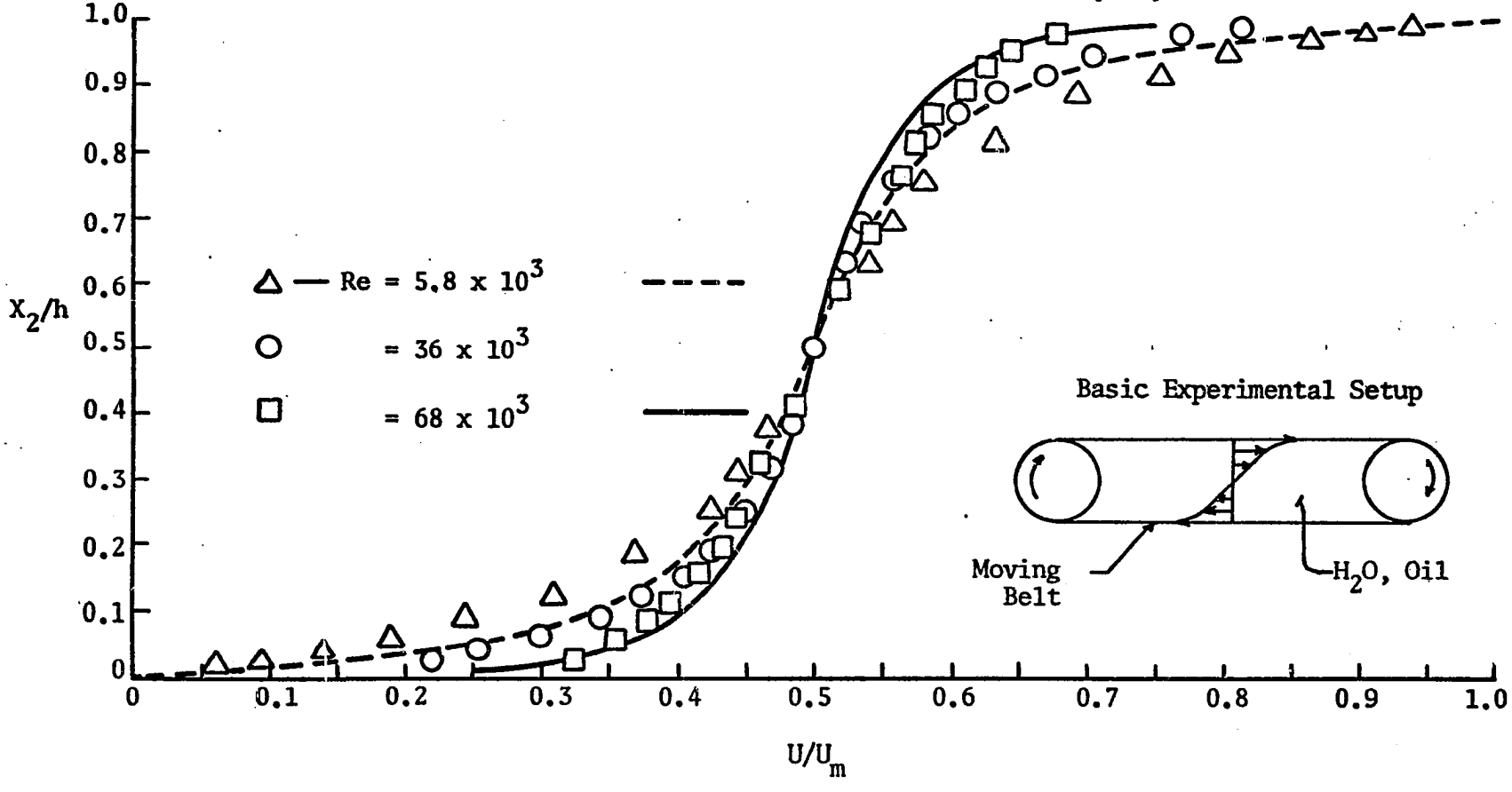
*One expects the larger the aspect ratio the more nearly the flow will approximate the two-dimensional case.

Fig. 4 COMPARISON OF THEORY WITH EXPERIMENTAL VELOCITY PROFILES OBTAINED BY REICHARDT [1956]



difference in shear stress level with the theory predicting larger values.

These data of Reichardt are not unquestionable since as suggested by Robertson [1959], p. 3, end effects were possible. In addition, these data were obtained with an unconventional arrangement whereby a cylindrical stick, suspended by a steel wire from a movable "wagon," was lowered into the flow, and the flow velocity was obtained by varying the speed of the wagon until the wire was perpendicular to the fluid-air interface during a traversal along the flow. Neither the details of how this was done nor an estimate of possible errors was discussed.

Robertson [1959] conducted his experiments with air so that some of the turbulence properties could be measured with a hot-wire anemometer. The basic experimental apparatus consisted of a 20-inch wide belt mounted on pulleys above a fixed, metal surface through which various measurement probes could be inserted into the flow. The distance h between the belt and fixed surface was variable, but most of the data were taken at a spacing $h \approx 2$ inches which provided a flow aspect ratio of ten.

The sidewalls were fixed and were found to be a source of trouble owing to the additional drag imposed on the flow. This additional drag extracted momentum from the fluid motion caused by the moving belt and thus prevented the midstream velocity from attaining $U_m/2$.

The solution employed was to simply set-up a fan to force air down the tunnel at a uniform speed equal to one half the belt speed so as to compensate for sidewall resistance.

Twelve-inch diameter belt pulleys were located ninety inches apart; this spacing was based on boundary layer calculations which indicated the boundary layers on the moving belt and fixed surface would join well ahead of the test section located twelve inches from the rear pulley.

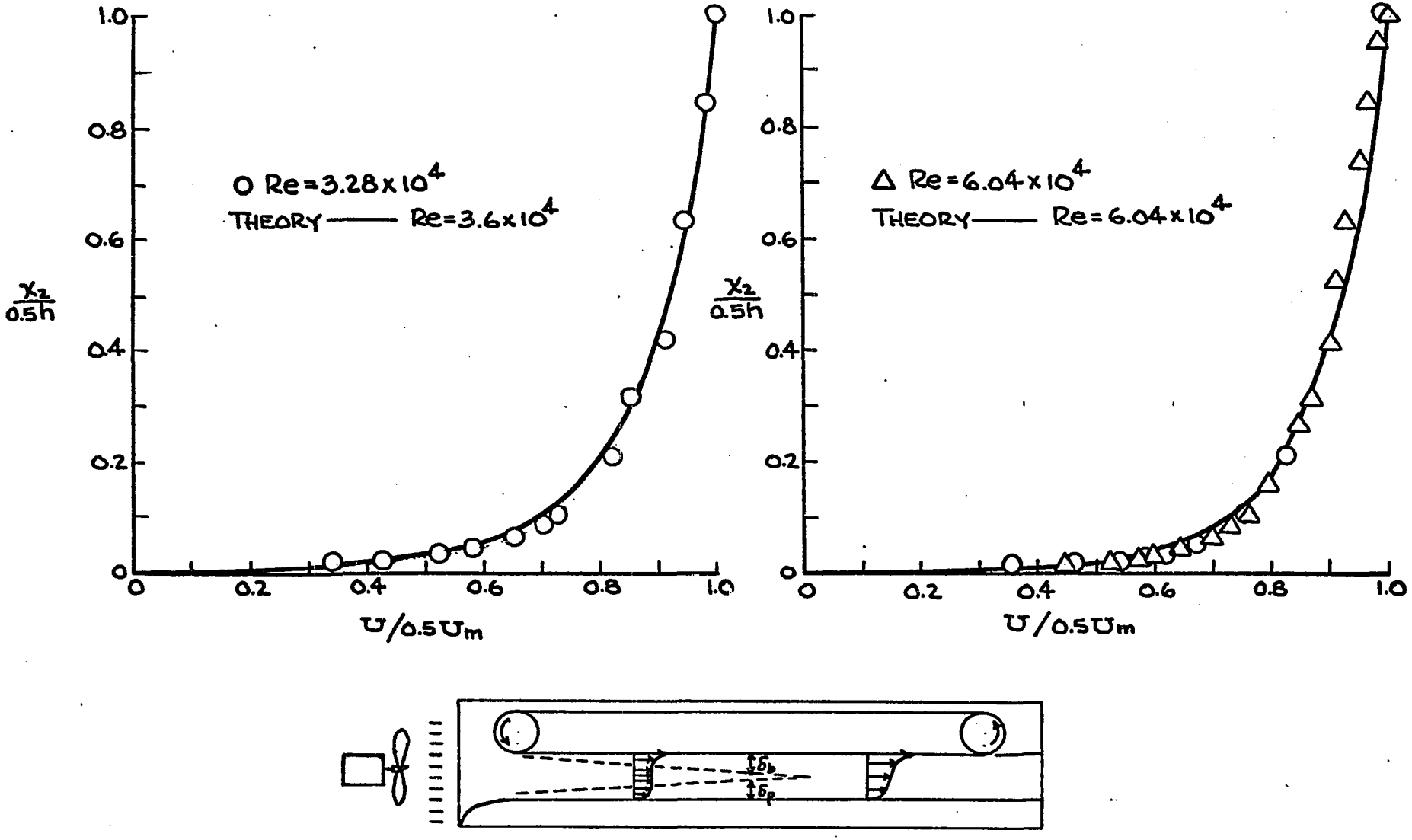
In order to measure the two-dimensionality of the flow, Robertson made longitudinal and transverse pressure and velocity measurements in the midstream about the test station and found the variations to be less then ten per cent. In addition, the midpoint velocity was found to be within one per cent of $U_m/2$.

Some typical data obtained by Robertson are shown in Figure 5 together with relevant theoretical predictions. It is readily seen that an excellent correlation of the data is obtained.

It is unfortunate that neither Reichardt nor Robertson attempted to measure the shear stress directly, since a graph of the mean velocity profile in terms of the law of the wall is quite sensitive to the value of skin friction used (since $u_\tau = \sqrt{\tau_w/\rho}$).

Reichardt obtained a value of u_τ by arbitrarily requiring a particular form of the law of the wall to produce the correct centerline velocity, i.e.

Fig. 5. COMPARISON OF THEORY WITH EXPERIMENTAL VELOCITY PROFILES OBTAINED BY ROBERTSON [1959]



$$\frac{U_c}{u_\tau} = 2.5 \ln \frac{u_\tau h/2}{\nu} + 5.5 \qquad (4.1)$$

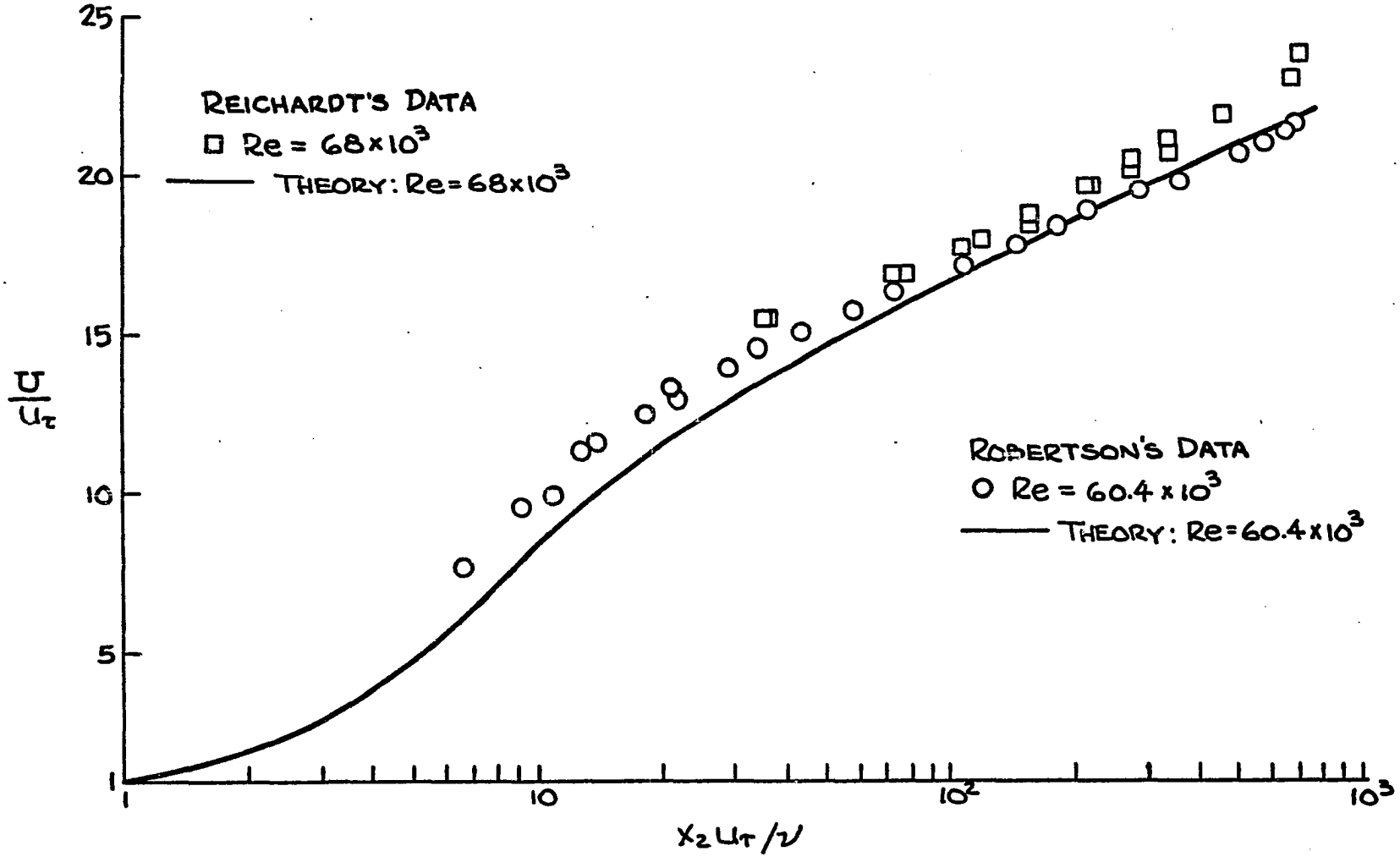
The values of the constants appearing in equation (4.1) were empirically determined by early (1920's and 1930's) investigators of boundary layer and channel flows, and as noted by Hinze [1959], p. 477, various investigators have found other values for these constants. Thus attempting to obtain u_τ from equation (4.1) is unsatisfactory.

The low speed water table experiments of Runstadler, et. al. [1963] are perhaps most similar to the Couette flow experiments of Reichardt. For a turbulent water boundary layer with zero pressure gradient they found that Clauser's form of the law of the wall correlated the logarithmic portion of their data quite well, e.g., Figure 3.20 of Runstadler [1963]. The constants chosen by Clauser [1956] in the law of the wall are

$$\frac{U}{u_\tau} = 2.44 \ln \frac{x_2 u_\tau}{\nu} + 4.9 \qquad (4.2)$$

Comparing equations (4.1) and (4.2) we see that it appears quite possible that Reichardt may have underestimated u_τ. Figure 6 shows Reichardt's data normalized with respect to the u_τ determined from equation (4.1). The theoretical mean velocity distribution is lower owing to a

Fig. 6. COMPARISON OF THEORETICAL AND SEMI-EMPIRICAL VALUES OF THE LAW OF THE WALL



higher theoretical value of u_τ. Figure 7 compares some values of the skin friction coefficient obtained from equation (4.1) with theory. It may be seen that the theory predicts about 15% larger values of C_f.

Robertson [1959], p. 33, used the Ross-Clauser method to obtain a value for the skin friction coefficient. Basically the Clauser Method [1954] consists of writing the law of the wall in the form

$$\frac{U}{U_1} = \sqrt{\frac{C_f}{2}} \left\{ \frac{1}{K} \ln \left(\frac{X_2 U_1}{\nu} \sqrt{\frac{C_f}{2}} \right) + B \right\} \tag{4.3}$$

where for Couette flow

$$U_1 = U_c \tag{4.4a}$$

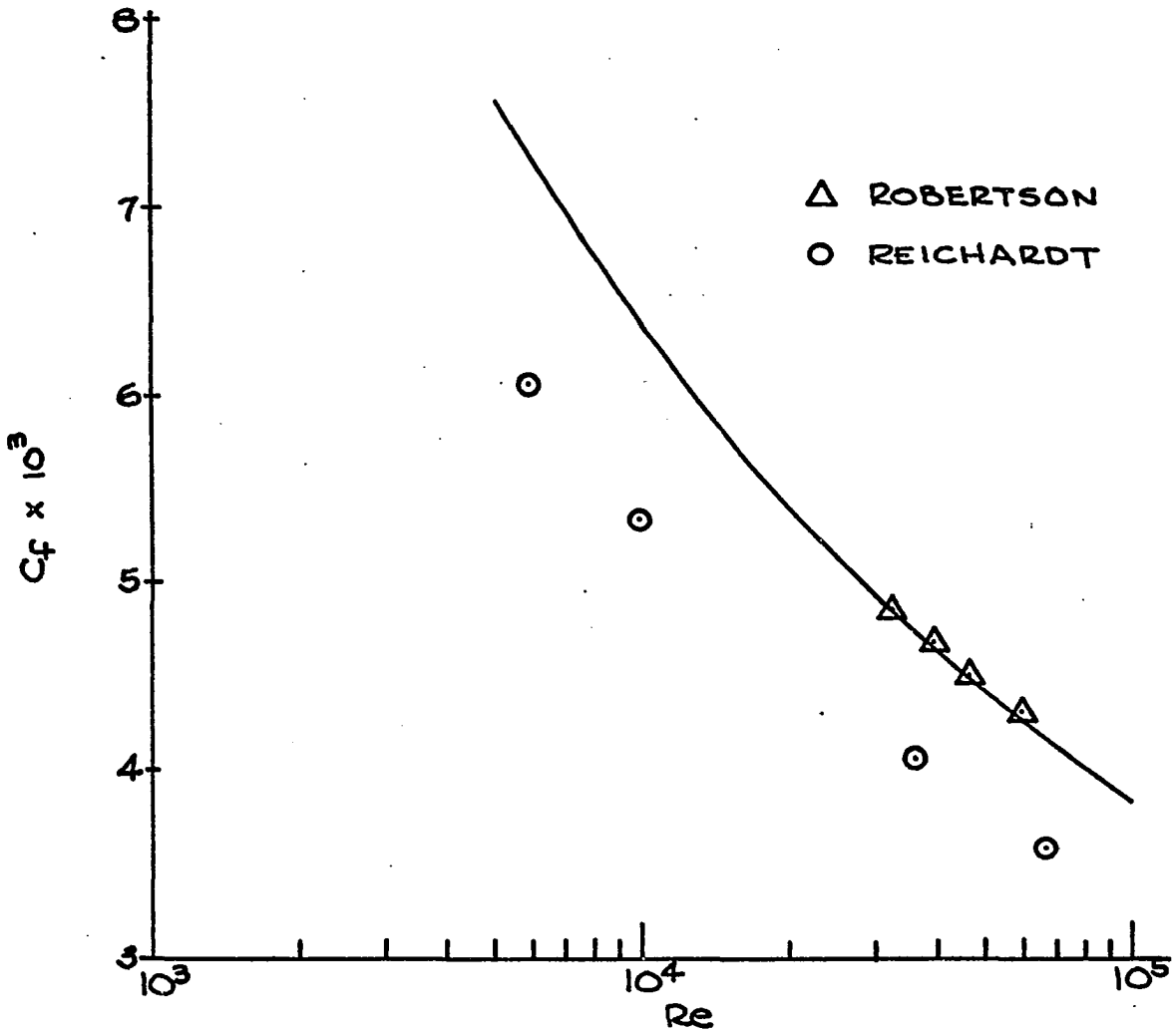
$$C_f = \tau_w / (\tfrac{1}{2} \rho U_c^2) \tag{4.4b}$$

A semi-logarithmic graph (similar to Figure 6) can be constructed from equation (4.3) with C_f as a parameter, and when a given mean velocity profile is plotted on the graph, the straight line (logarithmic) portion of the curve is discernible, thus allowing one to determine C_f by interpolation. It is assumed that Robertson followed such a procedure and used values of the constants found by Ross based on his analyses of data for pipe and boundary layer flows. According to Robertson, the values are

$$\frac{1}{K} = \frac{5.6}{\ln 10} \approx 2.43 \tag{4.5a}$$

$$B = 5.6 \tag{4.5b}$$

Fig. 7. TURBULENT SKIN FRICTION COEFFICIENT



Although these constants are quite close to those appearing in eq. (4.1), Robertson obtained values for C_f about 15% larger than those calculated from eq. (4.1). These values of C_f, determined by Robertson from his mean velocity profile data, are shown in Figure 7 and are found to be in excellent agreement with the theory.

Using the values of C_f presented by Robertson, his mean velocity data have been cast in the form of the law of the wall and are also presented in Figure 6. The slope and general shape of these data agree with the theory as expected owing to the close agreement shown in Figures 5 and 7. The difference in level between the Robertson data and the theory for the smaller values of $X_2 u_\tau/\nu$ is entirely due to the small differences in U shown in Figure 5.

It is of interest to place a straight edge along the line between $U/u_\tau = 5.5$ (since B = 5.5 in equation 4.1) and the last point of Reichardt's data shown in Figure 6 which corresponds to the midstream. It may be seen that it is questionable whether Reichardt's data obey the law of the wall. Indeed, Squire [1960] has observed that Reichardt's data are in better agreement with a square root law given by

$$\frac{U}{u_\tau} = 14 + \frac{3}{8}\left(\frac{X_2 u_\tau}{\nu} - 14\right)^{1/2} \tag{4.6}$$

Squire also examined Robertson's data and found that they appeared to be compatible with a law of the wall

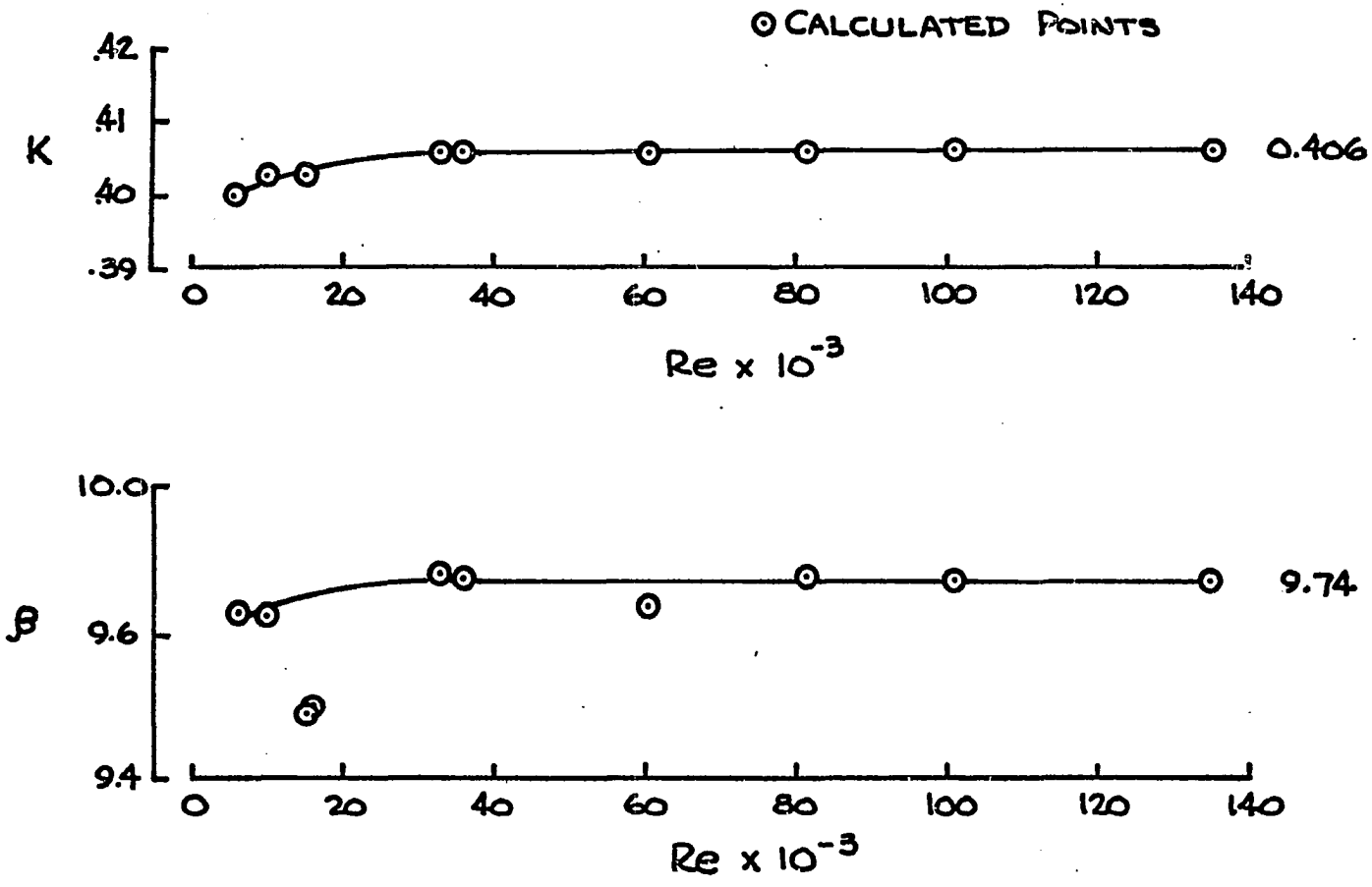
description. One cannot conclude, based on this limited experimental data, which, if either, set of data is correct; both experimental set-ups had sources of error and any conclusions must await further experimental results from a third source. However, as mentioned at the beginning of this chapter, the present theory indicates that the law of the wall is a valid representation for turbulent Couette flow.

4.2 Theoretical Variation of the Constants in the Law of the Wall

The constants K and β , which appear in equation (3.34), were found to be a function of Reynolds number and are shown in Figure 8. According to Kline, et al [1967], Comte-Bellot has also shown that the coefficients in the logarithmic law of the wall depend on Reynolds number. In addition, it has been found experimentally, e.g., Gill and Scher [1961], in turbulent flow through tubes and between parallel, flat plates (channel flow) that the coefficients show a definite Reynolds number effect up to Reynolds numbers of approximately thirty thousand beyond which they appear to be genuine constants. This trend has also been predicted by the present theory as shown in Figure 8.

It may be noted that the calculations show a small amplitude oscillation in K and β as the Reynolds number decreases. The exact cause of this behavior has not been found, but the instability may be caused by inaccuracies

Fig. 8. VARIATION OF CONSTANTS IN THE LAW OF THE WALL WITH REYNOLDS NUMBER



in the numerical calculations and/or inadequacy of the law of the wall to correctly describe the mean velocity profile at lower Reynolds numbers.

It is of interest to note the limiting values predicted by the present theory for Couette flow; for this purpose, we note for large values of U/u_τ that equation (3.34) may be written as

$$\frac{x_2 u_\tau}{\nu} \approx \frac{1}{\beta} e^{\kappa U/u_\tau} \tag{4.7}$$

Upon taking the natural logarithm, we obtain

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln \frac{x_2 u_\tau}{\nu} + \frac{\ln \beta}{\kappa} \tag{4.8}$$

Thus, comparing equation (4.7) with equation (3.33b), we see that

$$B = \frac{\ln \beta}{\kappa} \tag{4.9}$$

Now, if we estimate the asymptotic values of β and κ from Figure 8 as being 9.74 and 0.406, respectively, we find the following values for the coefficients in the law of the wall valid for Couette flow as $Re \to \infty$.

$$\frac{1}{\kappa} \approx 2.46 \quad , \quad B \approx 5.6 \tag{4.10}$$

4.3 Additional Theoretical Results

Figure 9 presents the predicted distribution of Reynolds stress from the wall to midstream. It is important to keep in mind that the large eddies are primarily responsible for turbulent transport phenomena, and in particular the momentum transport caused by the large eddies produce the Reynolds stress. Thus, Figure 9, simply shows that the largest eddies exist in the middle of the flow, as one might expect intuitively, and are rapidly reduced in size very near the boundaries where $y < 30$ or X_2/h less than about 0.03.

The theoretical distribution of the average squared-velocity associated with the isotropic, small-scale motions is presented in Figure 10. It is pertinent to notice that these curves do not show a large maximum within the flow as Robertson [1959] found for $\overline{u_1^2}$ at $Re \approx 60 \times 10^4$, see Figure 3. Presumably, this is due to the fact that $\overline{(u')^2}$ is calculated in the theory in such a way as to give the best solution for isotropic dissipation, eq. (3.29). Thus $\overline{(u')^2}$ is not a measure of the actual turbulent kinetic energy per unit mass, but rather is a measure of the <u>average</u> of the three correlations $\sum_i \overline{u_i u_i}$ associated with the <u>small-scale</u> motions.

The predictions for Taylor's dissipation scale are shown in Figure 11 which show that the dissipation scale

Fig. 9. THEORETICAL DISTRIBUTION OF REYNOLDS STRESS

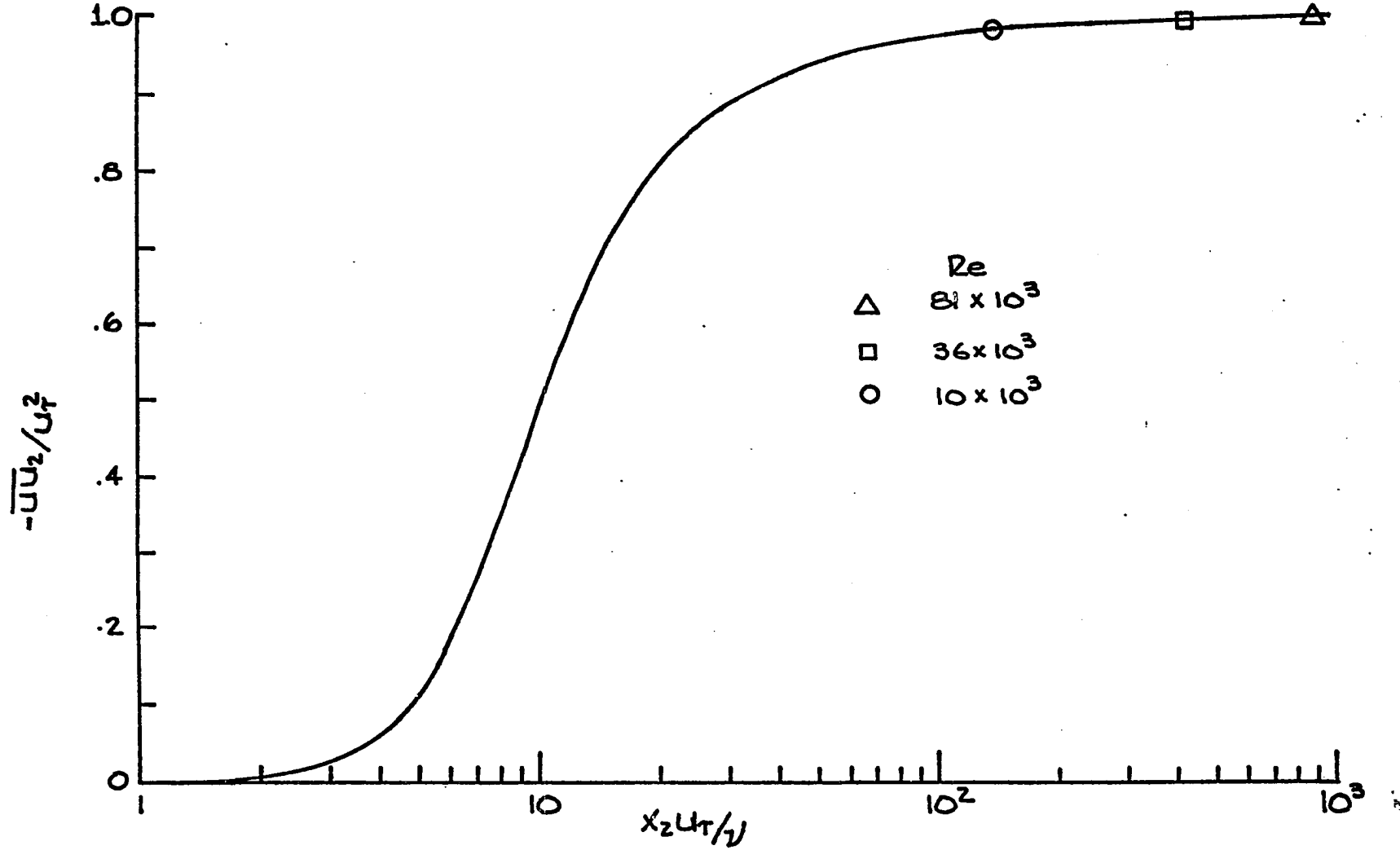


Fig. 10. THEORETICAL DISTRIBUTIONS OF MEAN-SQUARED VELOCITY ASSOCIATED WITH THE ISOTROPIC, SMALL-SCALE MOTIONS

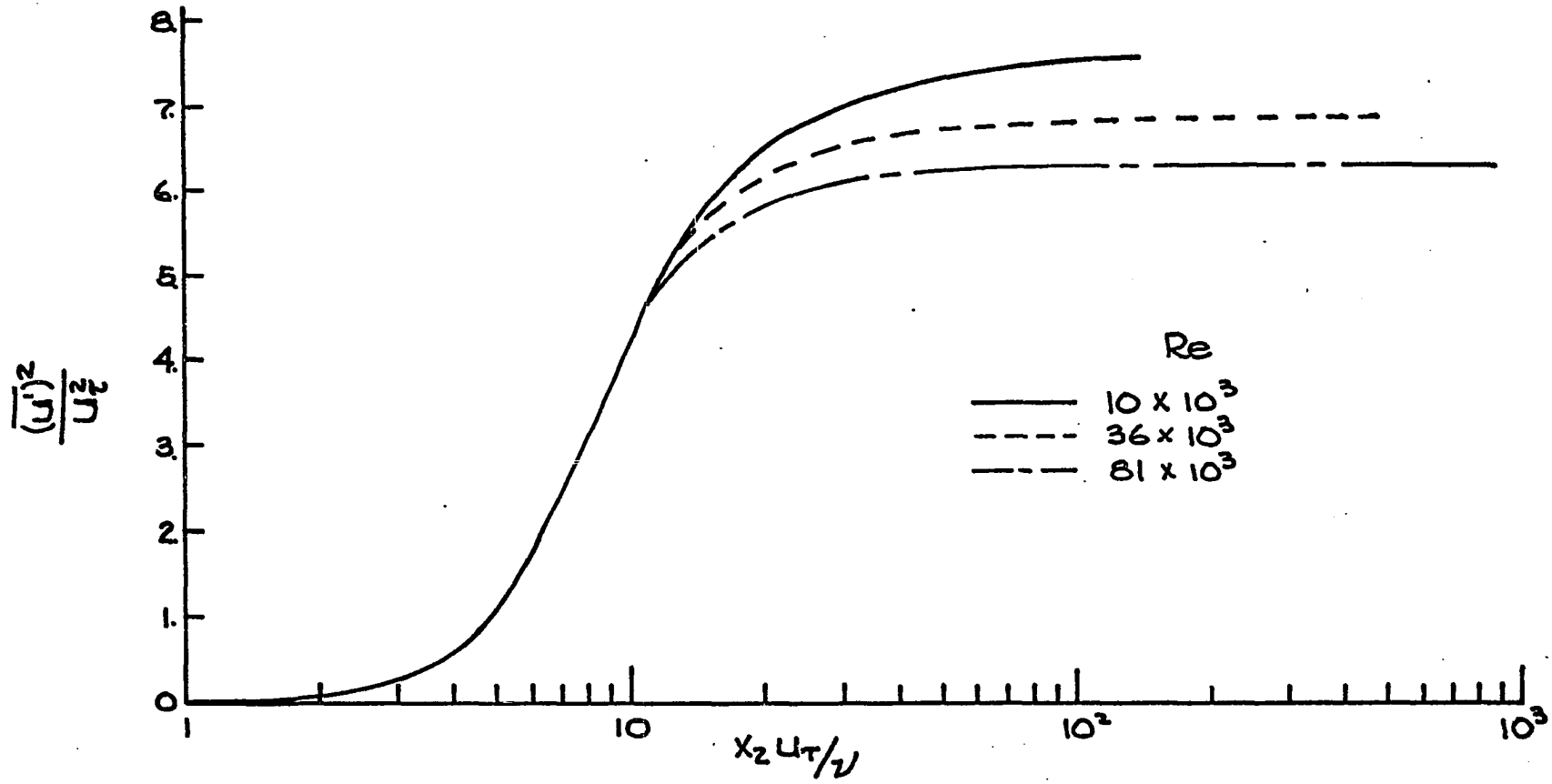
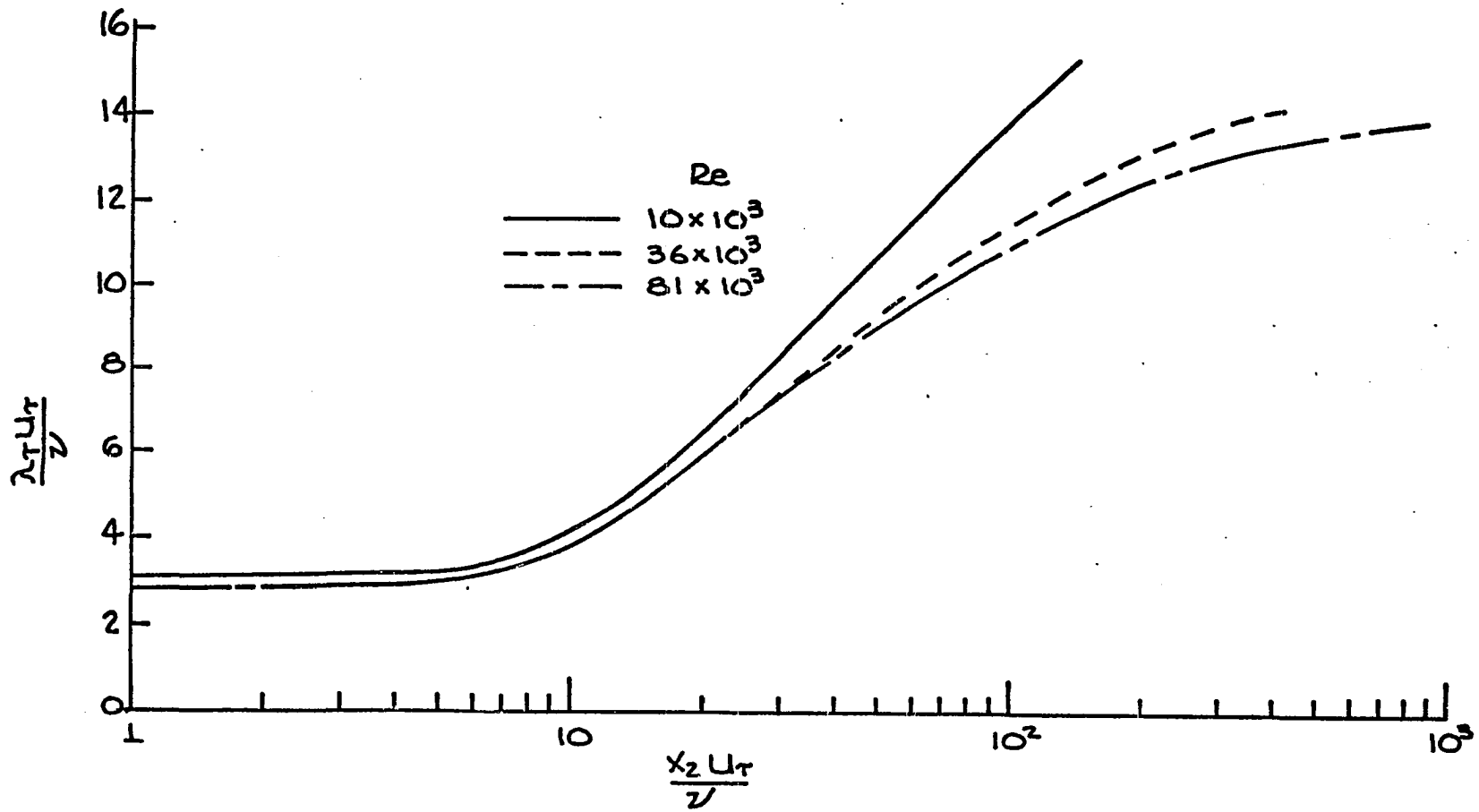


Fig. 11. THEORETICAL DISTRIBUTIONS OF TAYLOR'S DISSIPATION SCALE



decreases slightly with increasing Reynolds number. The nearly constant value for λ_T when $y < 5$ is indicative of a nearly constant value of dU/dX_2 very close to the wall, eq. (3.38).

It is relevant to compare these values of λ_T with those obtained by Robertson [1959] who also assumed the isotropic relation, eq. (3.29). Using information from Figures 3 and 7, we can calculate a nondimensional value for Robertson's dissipation scale, i.e.

$$\frac{\lambda_T u_\tau}{\nu} = \frac{\lambda_T}{h}\left(\frac{U_c h}{\nu}\right)\frac{u_\tau}{U_c}$$

$$\approx \frac{0.01\ \text{ft.}}{1/6\ \text{ft.}}\left(3 \times 10^4\right)\sqrt{\frac{4.3 \times 10^{-3}}{2}}$$

$$\approx 83.5 \qquad (4.1)$$

which is about 6 times the midstream theoretical values of Figure 10. The values of λ_T shown in Figure 3 were determined by using the isotropic relation

$$\overline{\left(\frac{\partial u_1}{\partial X_1}\right)^2} = \frac{\overline{(u_1)^2}}{\lambda_T^2} \qquad (4.12)$$

where the two correlations were measured data. Since the turbulence is not isotropic, which means

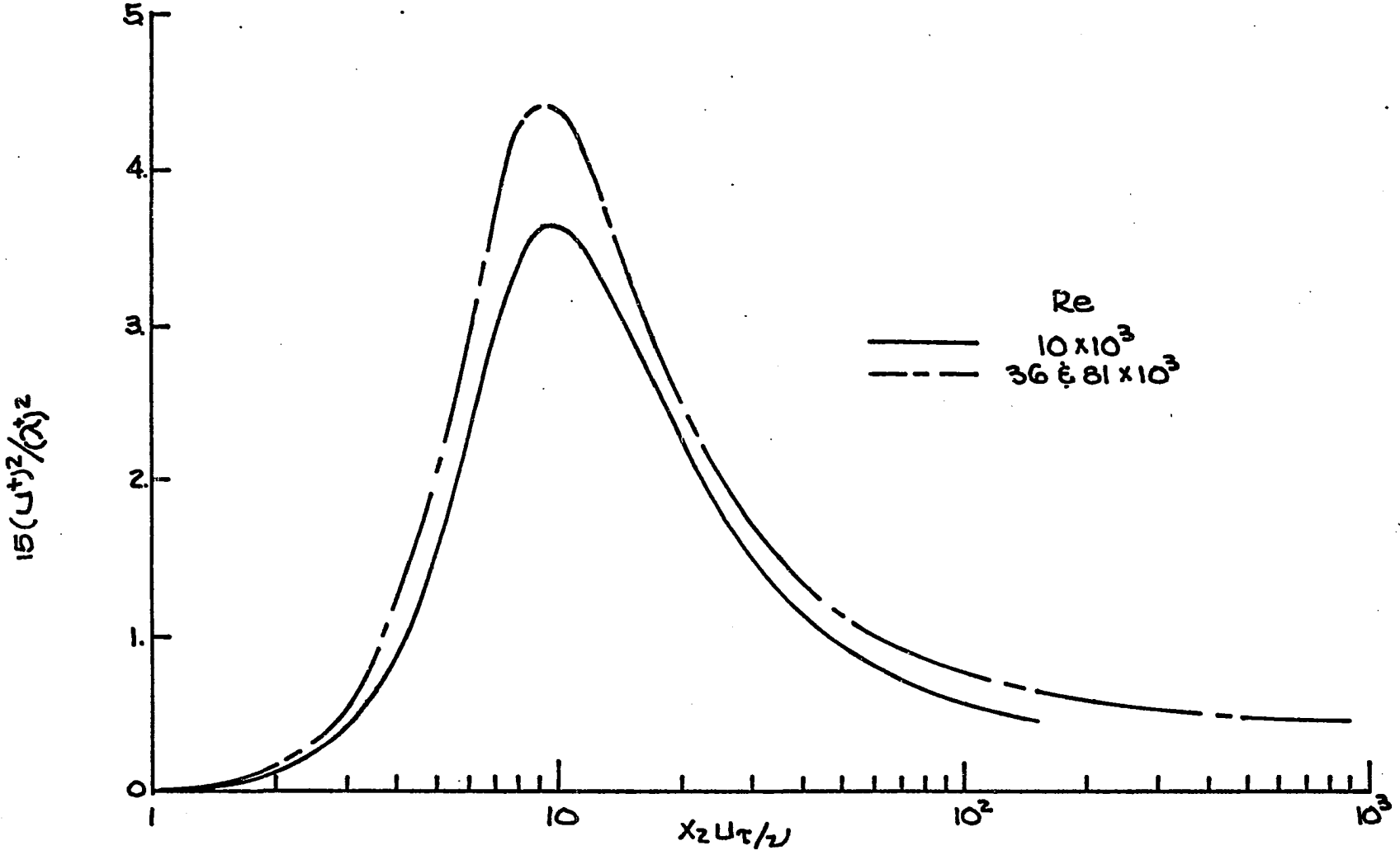
$$\overline{(u_1)^2} \neq \overline{(u_2)^2} \neq \overline{(u_3)^2} \neq \overline{(u')^2}$$

it is suspected that the resulting value of λ_T is <u>not</u> a

measure of the dissipation scale because $\overline{(u_1)^2}$ of the complete motion >> $\overline{(u')^2}$ associated with the small-scale, isotropic motions.

Theoretical distributions of the dissipation are shown in Figure 12. It is most interesting to compare the high Reynolds number distributions with the distribution obtained by Laufer in the wall region of a pipe flow, e.g., Hinze [1959], Figure 7-43. Laufer's data exhibit a peak in the dissipation curve near y = 8 which has a nondimensional value of about 0.25. Thus the distributions of Figure 12 have the correct shape, and the larger magnitudes can be explained by the lower skin friction (or u_τ) which exists in a Couette flow.

Fig. 12. THEORETICAL DISTRIBUTIONS OF ISOTROPIC DISSIPATION



CONCLUSIONS AND RECOMMENDATIONS

A new analytical method, based on the general evolution criterion of Glansdorff and Prigogine, has been applied to a simple, turbulent shear flow. The results for plane Couette flow consist of the following:

1. The concept of ergodic turbulence is a useful tool in the analysis of this planar flow.
2. The law of the wall appears to be a valid representation for fully-developed turbulent flow.
3. The relation between the constants in the law of the wall and Reynolds number is established.
4. The relation between skin friction coefficient and Reynolds number is established.
5. Additional experimental work needs to be done in order to establish with certainty the true nature of the flow and to compare with the present theoretical results. In particular, the work should include direct measurements of shear stress and distributions of dissipation.

In addition, it appears possible to extend the present work to analyze turbulent Couette flow in the case of:

1. a compliant boundary
2. a compressible fluid

However, before attempting such extensions, the convergence of the series representations for $(\lambda)^2$ and $(u^+)^2$, i.e.,

$$1/\lambda^2 = \sum_{i=0}^{N} A_i \left[dU^+/dy\right]^i$$

$$(u^+)^2 = \sum_{j=1}^{M} C_j \left[1 - dU^+/dy\right]^j$$

should be investigated. Specifically, it should be determined whether the Ritz method will converge to a particular solution with increasing N and M.

With regard to general applications, additional original work must be done before the basic method can be applied to more complex turbulent shear flows, such as the notoriously difficult problem of the turbulent boundary layer with pressure gradient*. But the method does offer a promising, new, analytical procedure for obtaining approximate solutions without the use of empirical constants so prevalent in most existing theories of turbulent shear flows.

As a final note, it is strongly recommended that the very limited and inefficient computer program used herein be replaced by one of the more sophisticated routines for solving sets of nonlinear algebraic equations. For example, the Lockheed program written by Remmler, et al [1966] (available through COSMIC at the University of Georgia) appears to be worthy of consideration.

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*For instance, the applicability of the ergodic hypothesis must be re-examined since the turbulence would would vary along the flow. Also the necessary constraints on the variations of the local potential will be more complicated.

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Appendix I

RELATION BETWEEN VARIATIONAL PRINCIPLES AND SELF-ADJOINT DIFFERENTIAL OPERATORS

The problem of finding a curve $y(x)$ that minimizes a functional $I(y)$ given by

$$I(y) = \int_{x_1}^{x_2} F(x, y, y') \, dx \qquad (I\text{-}1)$$

where the integrand is a specified function of $y(x)$, the derivative $y'(x)$, and the independent variable x, is called the simplest problem in the calculus of variations. This basic problem is used here to illustrate the relation between variational principles and self-adjointness of linear differential operators.

The functional $I(y)$ is defined to be stationary if and only if its first variation vanishes for every permissible variation $\delta y(x) = y(x) - \overline{y}(x)$, where $\overline{y}(x)$ is the desired function or curve which gives $I(y)$ its smallest value relative to the class of admissible functions $y(x)$. The variation $\delta y(x)$ is understood to represent an infinitesimal change in a function $y = f(x)$ at the point x and is virtual in the sense that it is arbitrary and is only a mathematical experiment to determine the properties of

I at the "point" $y(x)$*.

Since the variation operator commutes with both the operations of differentiation and integration with respect to an independent variable (which is never varied), we may write for the first variation of the integral

$$\delta I = \delta\int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \delta F\,dx$$

or

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]\,dx \quad \dagger$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta\frac{dy}{dx}\right]\,dx$$

Now integrating by parts, we obtain

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,dx + \left[\frac{\partial F}{\partial y'}\,\delta y\right]_{x_1}^{x_2} \qquad \text{(I-2)}$$

Note that $dy(x)$ and $\delta y(x)$ are fundamentally different. Both represent infinitesimal changes in the value of y; however, dy denotes the change in value of the <u>given</u> function $y(x)$ caused by the infinitesimal change dx of the independent variable, while δy is an infinitesimal change in the function $y(x)$ which produces a <u>new</u> function $y(x) + \delta y(x)$ which, of course, has a different value at the <u>point</u> x.

† From ordinary calculus one would expect

$$\delta F(x, y, y') = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

but since x is not varied, $\delta x = 0$.

If the integral is to have a stationary value for arbitrary, but small, variations $\delta y(x)$, then the following must be true.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{I-3a}$$

and

$$\frac{\partial F}{\partial y'}\,\delta y\Big]_{x_1}^{x_2} = 0 \tag{I-3b}$$

Eq. (I-3a) is known as the Euler-Lagrange equation.

When the value of the sought function $\overline{y}(x)$ is not preassigned at one or both of the end points $x = x_1, x_2$, the difference $\delta y(x)$ between $\overline{y}(x)$ and the varied function $y(x) = \overline{y}(x) + \delta y(x)$ need not vanish. However, eq. (I-3b) is a necessary condition which must be satisfied when $y(x)$ is identified with the minimizing (or maximizing) function and must hold for all permissible variations $\delta y(x)$. Therefore, if $\delta y(x)$ is not zero at the end points, we have the so-called natural boundary conditions, i.e.,

$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_2} = 0 \quad , \quad \left.\frac{\partial F}{\partial y'}\right|_{x=x_1} = 0 \tag{I-4}$$

which must be satisfied if $I(y)$ is to be stationary.

At this point, we are ready to consider the definition of adjoint of a linear differential operator. The following discussion of adjointness is taken from Ince [1956].

Apparently, the term adjoint was introduced by the German mathematician Fuchs in 1873 in connection with work concerning integrating factors for linear differential equations of order two or higher. The elementary concept of the integrating factor for linear, first order, differential equations is one of the first methods introduced in beginning courses on differential equations and is familiar to most sophmore science and engineering students. However, the student often hears nothing about integrating factors for higher order differential equations and is introduced to the term self-adjoint in the study of the classical Sturm-Liouville problem, where it is often shown, without explanation, that the differential operator associated with the problem can be written in the so-called self-adjoint form. The confusion which such a presentation can lead to is obvious.

The adjoint originates from consideration of the following linear differential operator.

$$L(y) = P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + - - - + P_{n-1} \frac{dy}{dx} + P_n y \qquad \text{(I-5)}$$

Now suppose that a function $z(x)$ exists such that $zL(y)dx$ is an exact differential. In addition, one needs the following formula which can be shown to be true by using the elementary calculus rule for differentiation of a product.

$$u^{(n)}v = \frac{d}{dx}\left\{u^{(n-1)}v - u^{(n-2)}v' + \cdots (-1)^{n-1}uv^{(n-1)}\right\} + (-1)^n uv^{(n)} \tag{I-6}$$

where the superscripts in parenthesis denote derivatives with respect to x.

If eq. (I-6) is applied to each term in the product z times eq. (I-5), one obtains

$$\begin{aligned} zL(y) = {} & \frac{d}{dx}\left\{y^{(n-1)}p_0 z - y^{(n-2)}(p_0 z)' + \cdots (-1)^{n-1} y (p_0 z)^{(n-1)}\right\} \\ & + \frac{d}{dx}\left\{y^{(n-2)}p_1 z - y^{(n-3)}(p_1 z)' + \cdots (-1)^{n-2} y (p_1 z)^{(n-2)}\right\} \\ & + \cdots \\ & + \frac{d}{dx}\left\{y' p_{n-2} z - y (p_{n-2} z)'\right\} \\ & + \frac{d}{dx}(y p_{n-1} z) + y L^*(z) \end{aligned} \tag{I-7}$$

where

$$\begin{aligned} L^*(z) \equiv {} & (-1)^n \frac{d^n (p_0 z)}{dx^n} + (-1)^{n-1} \frac{d^{n-1}(p_1 z)}{dx^{n-1}} + \cdots \\ & - \frac{d(p_{n-1} z)}{dx} + p_n z \end{aligned} \tag{I-8}$$

$\equiv$ the <u>adjoint</u> to L(y)

and

$$L^*(z) = 0 \tag{I-9}$$

is called the <u>adjoint equation</u> corresponding to

$$L(y) = 0.$$

Eq. (I-7) can be rewritten in the form

$$zL(y) - yL^*(z) = \frac{d}{dx}\{B(y,z)\} \tag{I-10}$$

which is known as the Lagrange identity. The term B(y,z) is linear and homogeneous in both y and z and their derivatives and is labeled the bilinear concomitant. Thus we see that the product zL(y)dx is an exact differential, or z(x) is an integrating factor, if and only if z satisfies the adjoint equation, eq. (I-9).

When an equation is identical with its adjoint it is said to be self-adjoint. For example, the Sturm-Liouville differential operator is

$$\begin{aligned} A(y) &= (Py')' + qy \\ &= Py'' + P'y' + qy \end{aligned} \tag{I-11}$$

Now using eq. (I-8) with n = 2 as the defining relation for the adjoint, we have the following adjoint for A(y).

$$\begin{aligned} A^*(y) &= (Py)'' - (P'y)' + qy \\ &= A(y) \end{aligned} \tag{I-12}$$

which proves the Sturm-Liouville operator is indeed self-adjoint.

Comparing eqs. (I-11) and (I-3a) we see that they have the same differential form. Thus, processes which obey such self-adjoint differential operators admit the possibility of defining a variational principle such that

some integral I has as its Euler-Lagrange equation the governing differential equation. Note also that the boundary conditions appropriate to the particular problem must be compatible with a condition such as eq. (I-3b) in order for I to be stationary.

This basic relation between variational principles and linear differential operators is also valid for more complex cases. For example, a self-adjoint, fourth order, linear differential operator must have the form

$$L(y) = (Sy'')'' + (Py')' + qy = L^*(y) \tag{I-13}$$

Eq. (I-13) has the same form as the Euler-Lagrange equation for the following functional

$$I = \int_{x_1}^{x_2} F(x,y,y',y'')dx \tag{I-14}$$

Requiring I to be stationary leads to

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \tag{I-15a}$$

with boundary conditions

$$\left(\frac{\partial F}{\partial y'} - \frac{d}{dx}\frac{\partial F}{\partial y''}\right)\delta y\Big]_{x_1}^{x_2} = 0 \tag{I-15b}$$

and

$$\left.\frac{\partial F}{\partial y'''}\ \delta y'\right]_{x_1}^{x_2} = 0 \qquad \text{(I-15c)}$$

The boundary conditions vanish identically if y and y' are specified at $x = x_1, x_2$. Thus, a variational principle can be formulated for any process which is governed by eq. (I-13).

In the case of non-self-adjoint problems such self-contained variational formulations do not exist. For additional related discussions the interested reader is referred to Hildebrand [1965] and Finlayson and Scriven [1967].

Appendix II

COMPUTER PROGRAM

The integral over y appearing in eq. (3.41) may be reduced to twice an integral over $0 \leq y \leq h^+/2$ by noting that the integral is an even function of y since dU/dy is symmetric about $y = h^+/2$. This may be seen by observing that the mean velocity profile between h/2 and h is the reversed mirror image of eq. (3.34), see Figure 2. Thus, the complete mean velocity profile is given by

$$y = U^+ + \frac{2}{\beta}\left[\cosh(\kappa U^+) - \frac{(\kappa U^+)^2}{2} - 1\right]$$

$$\text{for } 0 \leq X_2 \leq h/2$$

(II.1a)

and

$$(h^+ - y) = (U_m^+ - U^+) + \frac{2}{\beta}\left[\cosh \kappa(U_m^+ - U^+) - \frac{\kappa^2(U_m^+ - U^+)^2}{2} - 1\right]$$

$$\text{for } h/2 \leq X_2 \leq h$$

(II.1b)

Upon taking the derivative, one finds that $d\bar{U}/dy$ is indeed symmetric about the middle of the flow as required.

If this information is combined with the fact that only the integral of the dissipation is a function of the constants A_1, A_2, C_1, C_2, β and K, one can define a simplified functional I' whose derivatives with respect to the constants lead to the same set of homogeneous equations as would be obtained by evaluating the corresponding derivatives of $T\bar{\Phi}$, i.e.,

$$\frac{\partial(T\bar{\Phi})}{\partial \alpha_i} = 0 = \frac{\partial I'}{\partial \alpha_i}$$

where

$$I' \equiv 2\int_0^{h^+/2} \left\{ \left[A_1 + A_2 \frac{d\bar{U}}{dy} \right] \left[C_1\left(1 - \frac{d\bar{U}}{dy}\right) + C_2\left(1 - \frac{d\bar{U}}{dy}\right)^2 \right] \right\} dy \tag{II.2}$$

Expanding and integrating where possible, this equation becomes

$$\begin{aligned} I' = {} & A_1(C_1 + C_2)h^+ + A_2(C_1 + C_2)U_m^+ - A_1(C_1 + 2C_2)U_m^+ \\ & - 2A_2(C_1 + 2C_2)\int_0^{\frac{U_m^+}{2}} \frac{dU^+}{dy}\, dU^+ + 2A_1C_2\int_0^{\frac{U_m^+}{2}} \frac{dU^+}{dy}\, dU^+ \\ & + 2A_2C_2\int_0^{\frac{U_m^+}{2}} \left(\frac{dU^+}{dy}\right)^2 dU^+ \end{aligned} \tag{II.3}$$

Collecting like terms, the results are

$$I' = [A_1 h^+ + A_2 U_m^+](C_1 + C_2) - A_1(C_1 + 2C_2)U_m^+ + 2[A_1 C_2 - A_2(C_1 + 2C_2)]T1 + 2A_2 C_2 T3$$

(II.4a)

where

$$T1 \equiv \int_0^{\frac{U_m^+}{2}} \frac{dU^+}{dy} dU^+ = \int_0^{\frac{U_m^+}{2}} \frac{dU^+}{1 + \frac{2K}{\beta}(\sinh KU^+ - KU^+)}$$

(II.4b)

$$T3 \equiv \int_0^{\frac{U_m^+}{2}} \left[\frac{dU^+}{dy}\right]^2 dU^+ = \int_0^{\frac{U_m^+}{2}} \frac{dU^+}{\left[1 + \frac{2K}{\beta}(\sinh KU^+ - KU^+)\right]}$$

(II.4c)

Now making the substitutions

$$A_2 \longrightarrow A_2 C_1$$

$$C_2 \longrightarrow A_1 C_2$$

we obtain

$$I' = [A_1 h^+ + A_2 C_1 U_m^+](C_1 + A_1 C_2) - A_1(C_1 + 2A_1 C_2)U_m^+ + 2[A_1^2 C_2 - A_2 C_1(C_1 + 2A_1 C_2)]T1 + 2A_1 A_2 C_1 C_2 T3 \quad \text{(II.5)}$$

Before taking the derivative of I' with respect to the constants, we must apply two constraints on the admissible values of the constants, viz.,

1. the constraint J_1, defined by eq. (3.48), and
2. a constraint on β and K such that the mean velocity $= U_m/2$ at $x_2 = h/$, i.e.,

$$0 = J_2 \equiv -\frac{h^+}{2} + \frac{U_m^+}{2} + \frac{2}{\beta}\left[\cosh\left(\frac{\kappa U_m^+}{2}\right) - \frac{(\kappa U_m^+)^2}{8} - 1\right] \tag{II.6}$$

These constraints are applied by using the Method of Lagrange Multipliers, i.e., the set of equations for the constants are determined by the conditions:

$$\frac{\partial I}{\partial \alpha_i} \equiv \frac{\partial (I' + \eta_1 J_1 + \eta_2 J_2)}{\partial \alpha_i} = 0$$

$$J_1 = 0$$

$$J_2 = 0 \tag{II.7}$$

Where η_1 and η_2 are the Lagrange multipliers.

The corresponding set of equations, which follow from eqs. (II.5) and (II.7), is

$$\frac{\partial I}{\partial A_1} = h^+(C_1 + 2A_1C_2) + U_m^+ A_2 C_1 C_2 - U_m^+(C_1 + 4A_1C_2) + 4[A_1C_2 - A_2C_1C_2]T1 + 2A_2C_1C_2T3$$

$$\frac{\partial I}{\partial A_2} = U_m^+[C_1 + A_1C_2] - 2(C_1 + 2A_1C_2)T1 + 2A_1C_2T3$$

$$\frac{\partial I}{\partial C_1} = A_1h^+ + 2A_2C_1U_m^+ - A_1U_m^+ - 4A_2(C_1 + A_1C_2)T1 + 2A_1A_2C_3T3$$

$$\frac{\partial I}{\partial C_2} = A_1h^+ + A_2C_1U_m^+ - 2A_1U_m^+ + 2(A_1 - 2A_2C_1)T1 + 2A_2C_1T3$$

$$J_1 = U_m^+ - 2T1 - 15I' \qquad \text{(II.8)}$$

$$J_2 = -\frac{h^+}{2} + \frac{U_m^+}{2} + \frac{2}{\beta}\left[\cosh\left(\frac{KU_m^+}{2}\right) - \frac{(KU_m^+)^2}{8} - 1\right]$$

$$\frac{\partial I}{\partial K} = \eta_1(T2 + KT5) + (\eta_1 - 1)\left\{[A_1^2C_2 - A_2C_1(C_1 + 2A_1C_2)](T2 + KT5) + 2A_1A_2C_1C_2(T4 + KT6)\right\} + \frac{\eta_2}{2}\left[\frac{U_m^+}{2}\,\text{SINh}\left(\frac{KU_m^+}{2}\right) - K\left(\frac{U_m^+}{2}\right)^2\right]$$

$$\frac{\partial I}{\partial \beta} = \eta_1 T2 + (\eta_1 - 1)\left\{T2[A_1^2C_2 - A_2C_1(C_1 + 2A_1C_2)] + 2A_1A_2C_1C_2T4\right\} + \frac{\eta_2}{2K}\left[\cosh\left(\frac{KU_m^+}{2}\right) - \frac{(KU_m^+)^2}{8} - 1\right]$$

Where $T2 = \int_0^{\frac{U_m^+}{2}} \frac{(\sinh KU^+ - KU^+)\, dU^+}{\left[1 + \frac{2K}{\beta}(\sinh KU^+ - KU^+)\right]^2}$

$T4 = \int_0^{\frac{U_m^+}{2}} \frac{(\sinh KU^+ - KU^+)\, dU^+}{\left[1 + \frac{2K}{\beta}(\sinh KU^+ - KU^+)\right]^3}$

$T5 = \int_0^{\frac{U_m^+}{2}} \frac{U^+(\cosh KU^+ - 1)\, dU^+}{\left[1 + \frac{2K}{\beta}(\sinh KU^+ - KU^+)\right]^2}$

$T6 = \int_0^{\frac{U_m^+}{2}} \frac{U^+(\cosh KU^+ - 1)\, dU^+}{\left[1 + \frac{2K}{\beta}(\sinh KU^+ - KU^+)\right]^3}$

A computer program was necessary to numerically integrate and solve the above set of equations. The integrals are evaluated via Simpson's rule, and approximate solutions to the set of nonlinear, algebraic equations are obtained by using what might be called a sequential minimization procedure which is analogous to the method of least squares. The procedure consists of three basic steps:

1. begin with a set of initial guesses,
2. consecutively vary the value of each constant (α_i, η_1 and η_2) until the sum of the squares of the right-hand sides of eq. (II.8) becomes a minimum,
3. the iteration is continued until the sum is less than some allowable error.

Since the approximate values of K and β were known a priori, it was decided that the method would converge

faster if these two variables were placed last in the set of eight constants, α_i, η_1, η_2. Thus, in the computer program the following identifications are made.

A_1 = X (1) —— F(1) = $\partial I/\partial A_1$

A_2 = X (2) —— F(2) = $\partial I/\partial A_2$

C_1 = X (3) —— F(3) = $\partial I/\partial C_1$

C_2 = X (4) —— F(4) = $\partial I/\partial C_2$

η_1 = X (5) —— F(5) = J_1

η_2 = X (6) —— F(6) = J_2

K = X (7) —— F(7) = $\partial I/\partial K$

β = X (8) —— F(8) = $\partial I/\partial \beta$

The input to the program consists of the following.

1. Three items must be specified on card #1 of the data:

   a. the total number of unknown constants (≤ 10) in columns 1 and 2,

   b. a print parameter (an integer) in column 10 which if non-zero will cause the sum of the squares to be printed for <u>each</u> value of the constant being incremented, and if zero will only print out the sum and the resultant values of the constants at the end of one complete cycle of incrementation.

   c. the total number of cases to be run in columns 19 and 20.

2. Each additional card specifies the maximum velocity U_m in columns 10-22 and the width of the flow h divided by the kinematic viscosity ν in columns 30-42. The number of

these cards is equal to the total number of cases.

The basic output of the program is the sum of the squares of the homogeneous equations, eqs. (II.7), the corresponding values of the constants (which are incremented up to 50 times*), and the distributions across the flow of pertinent flow quantities, viz.,

x_2/h = X OVER H

$\frac{x_2 u_\tau}{\nu}$ = X2 - STAR

U/U_m = U OVER UMAX

$(u^+)^2 \equiv \overline{(u')^2}/u_\tau^2$ = USQ/UTAU SQ.

$\frac{\lambda_T u_\tau}{\nu}$ = DSCALE * UTAU/NU

$15\,(u^+)^2/(\lambda)^2$ = DISSIPATION (ND)

$-\overline{u_1 u_2}/u_\tau^2$ = REY. STRESS(ND)

$\frac{U}{u_\tau}$ = U-STAR

$\frac{\overline{(u')^2}}{U_m^2}$ = TISQ

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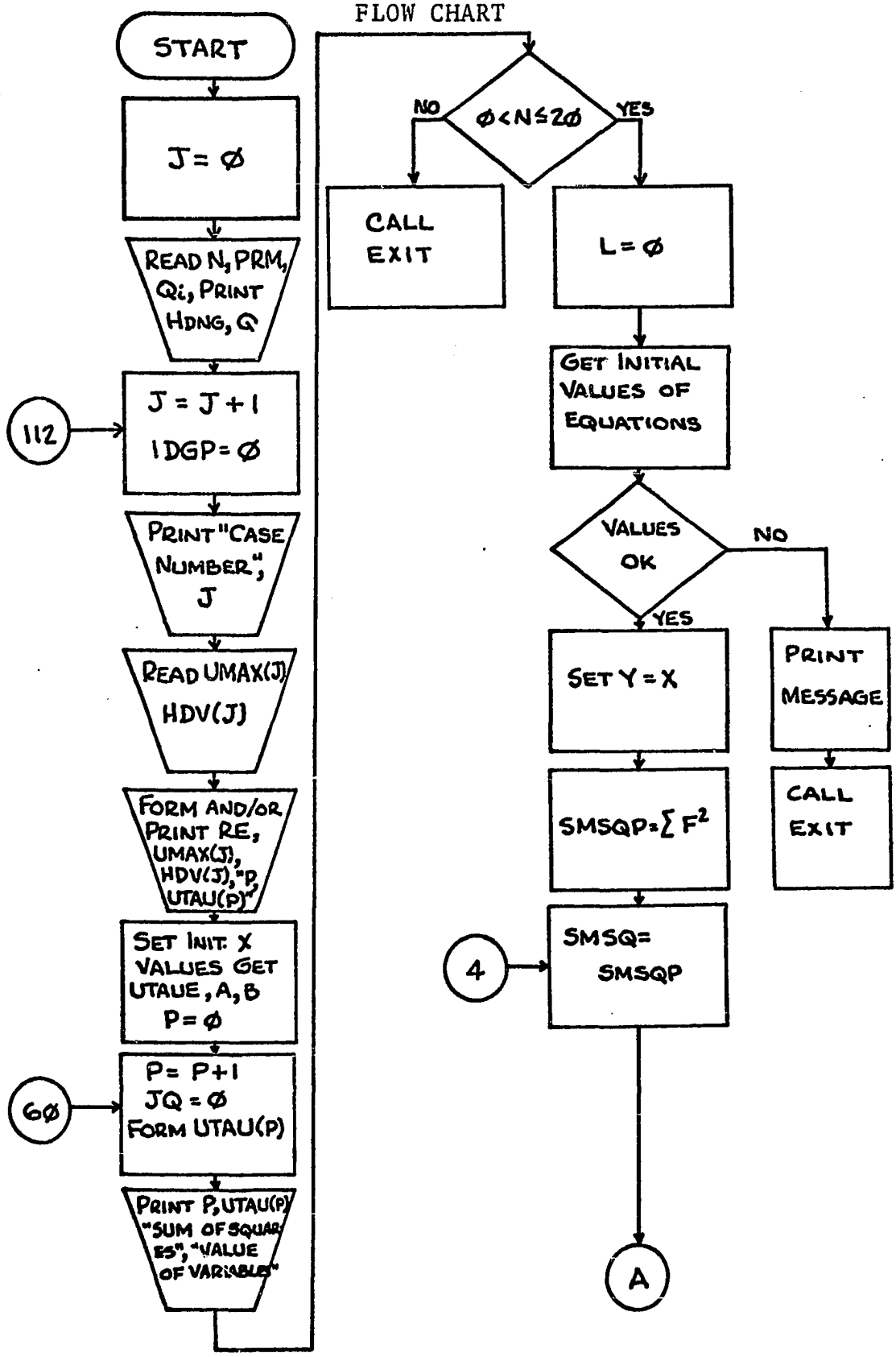
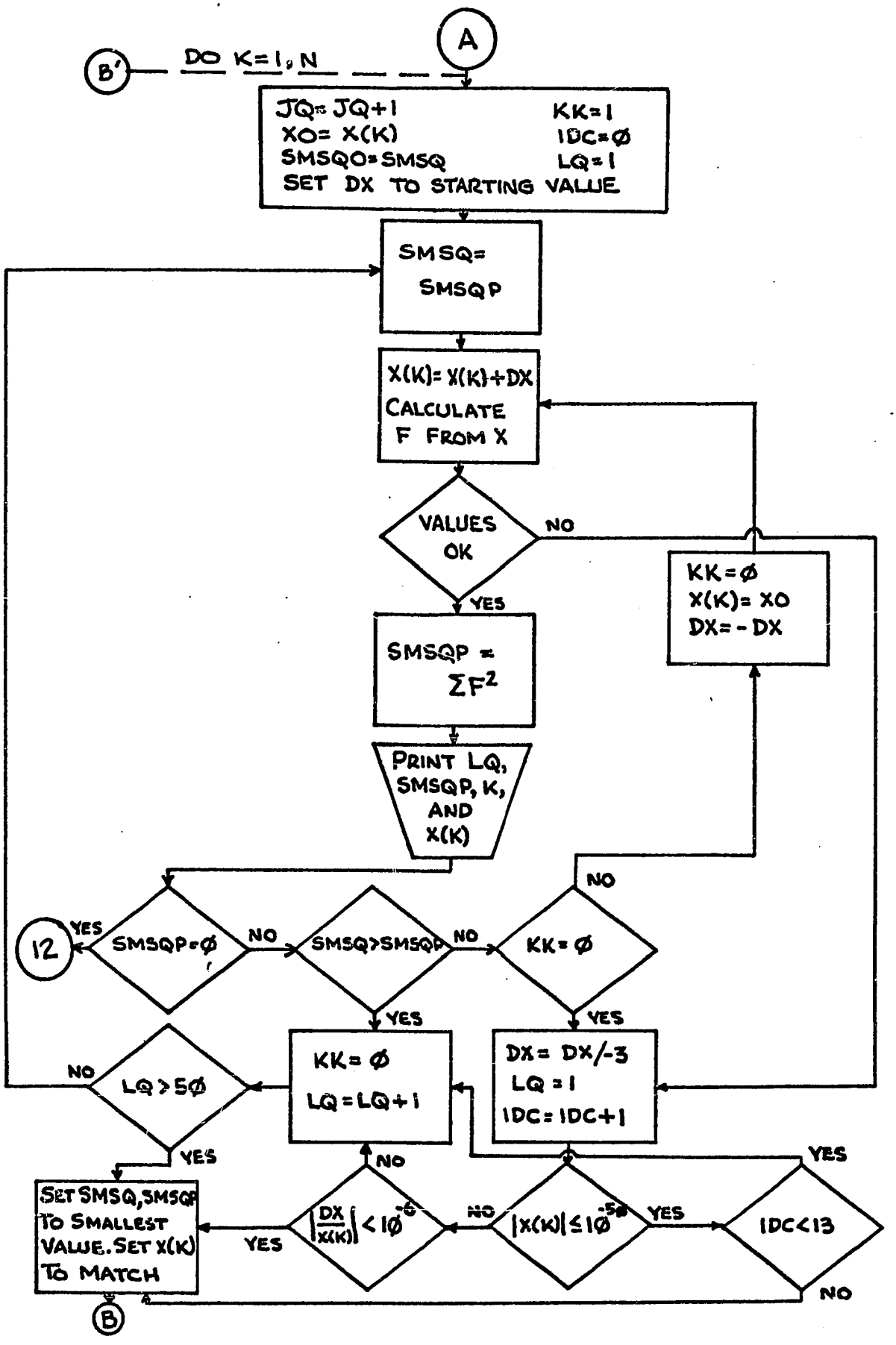
*When a constant has been incremented 50 times without finding a minimum in the sum, the sum is printed out with a minus sign.

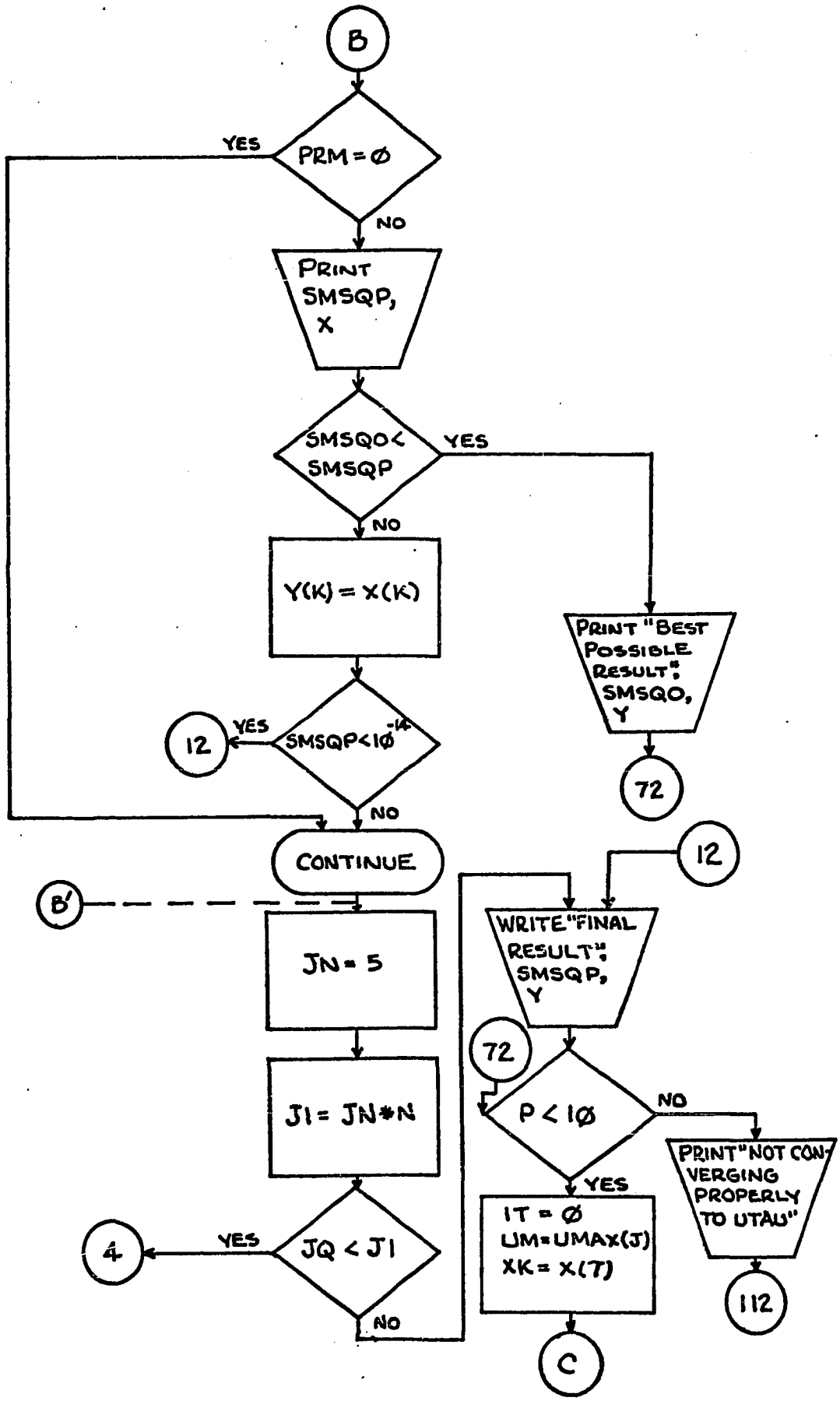
(ND) denotes nondimensional quantity.

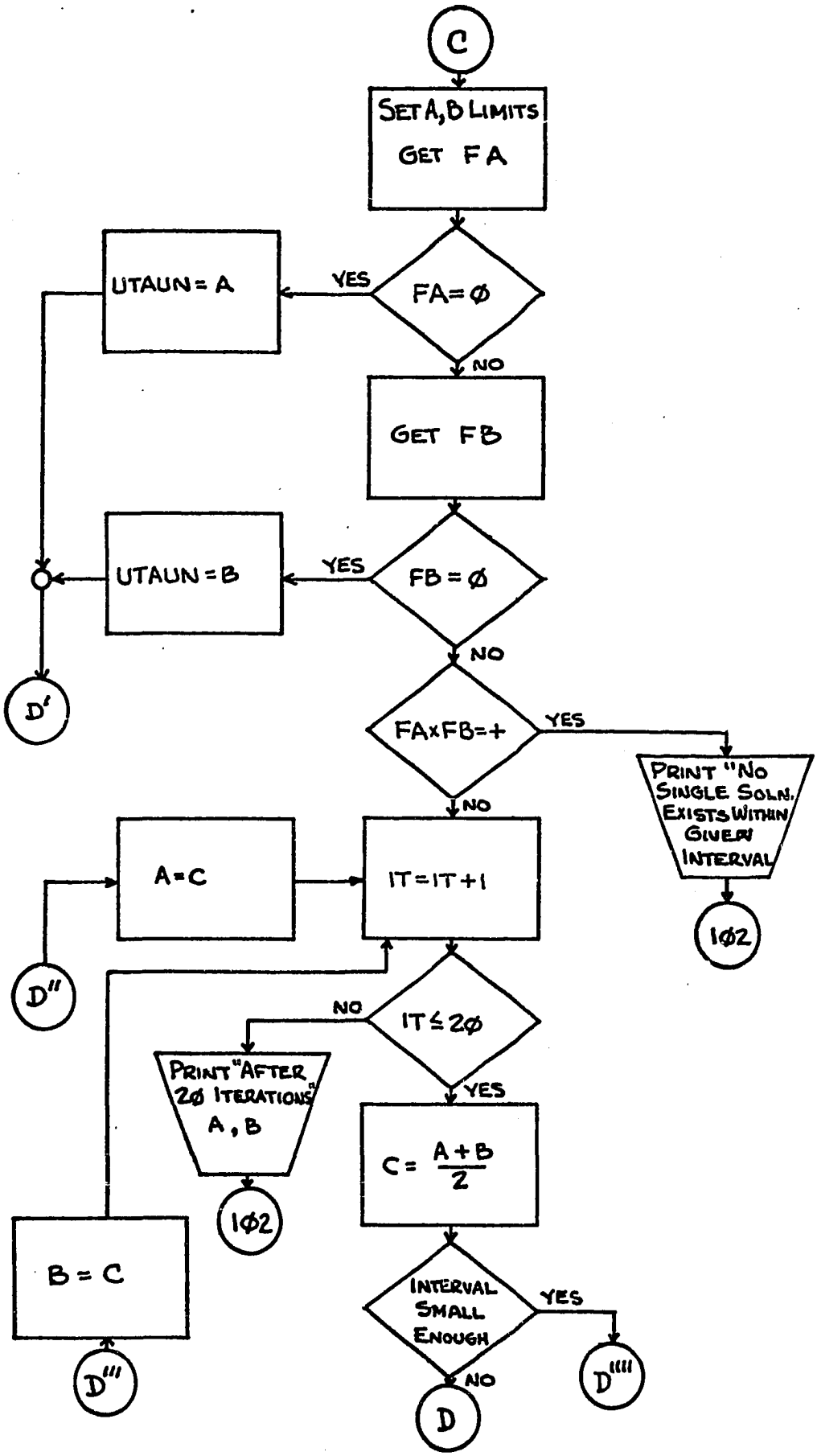
The following pages show a flow chart, a listing of the Fortran IV statements with example inputs, and a typical table of output for the various profiles.

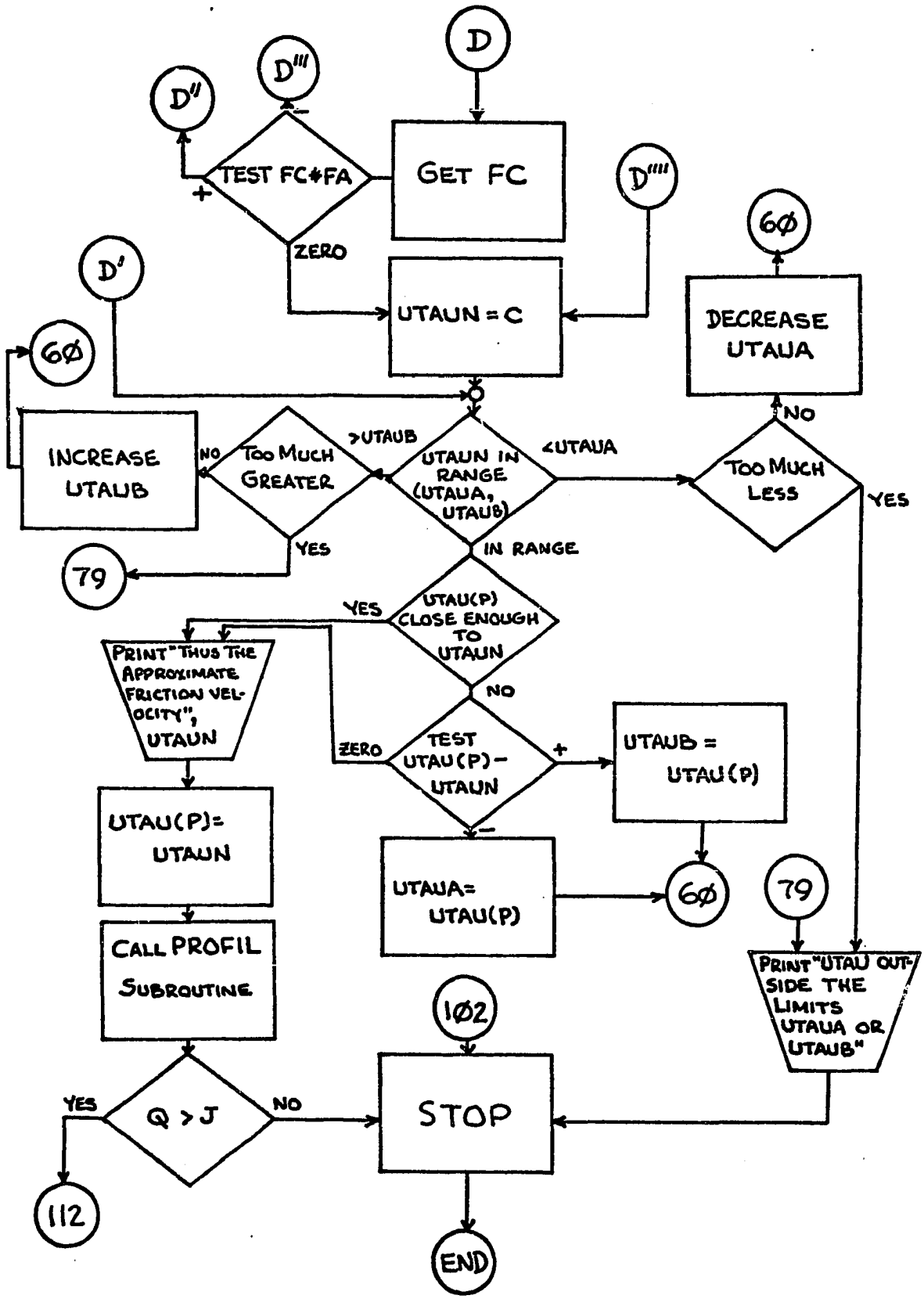
FLOW CHART

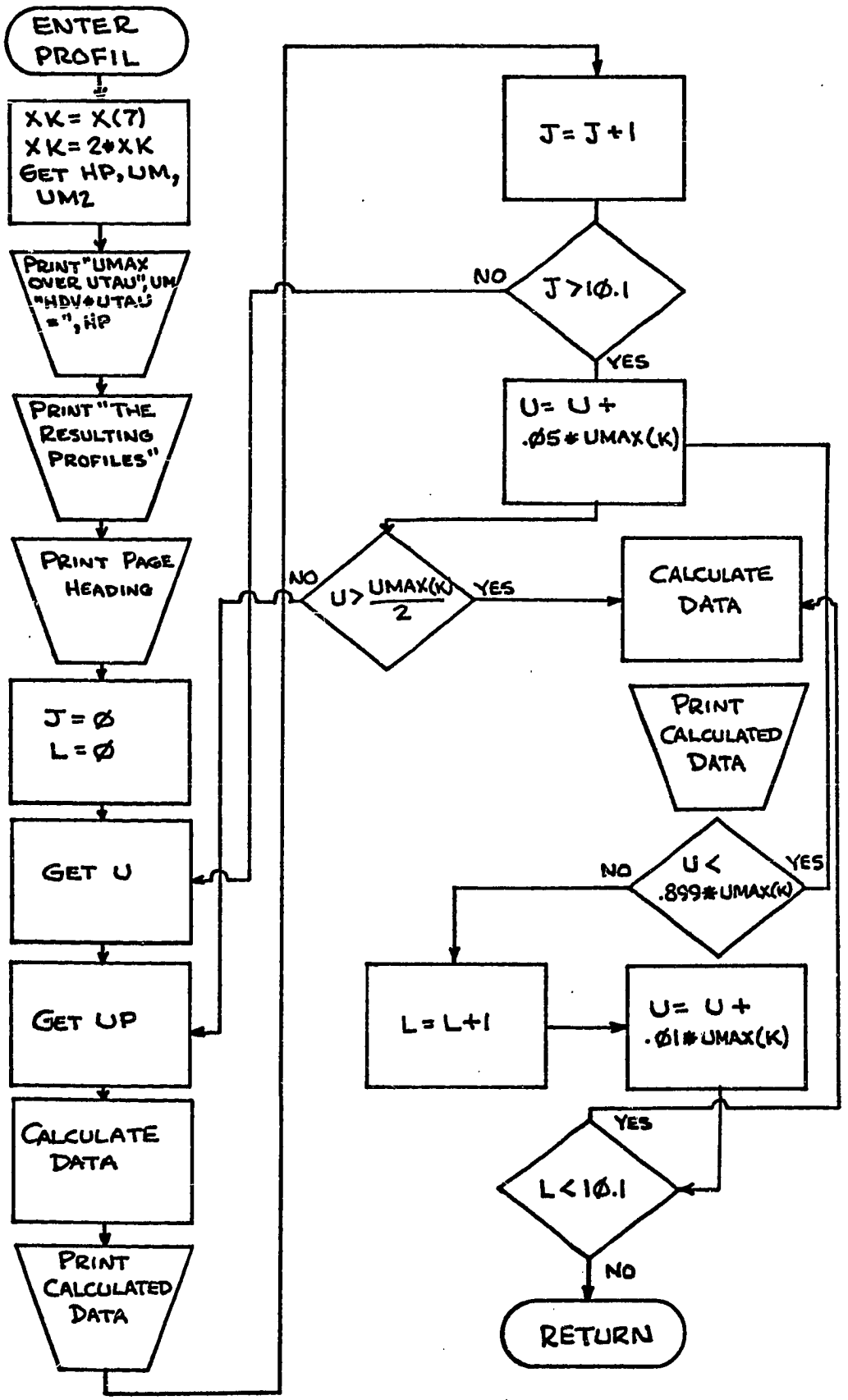

START
J = Ø
READ N, PRM, Qi, PRINT HDNG, Q
112
J = J + 1
IDGP = Ø
PRINT "CASE NUMBER", J
READ UMAX(J) HDV(J)
FORM AND/OR PRINT RE, UMAX(J), HDV(J), "P, UTAU(P)"
SET INIT. X VALUES GET UTAUE, A, B
P = Ø
60
P = P + 1
JQ = Ø
FORM UTAU(P)
PRINT P, UTAU(P) "SUM OF SQUARES", "VALUE OF VARIABLES"
Ø < N ≤ 2Ø
NO
YES
CALL EXIT
L = Ø
GET INITIAL VALUES OF EQUATIONS
VALUES OK
NO
YES
SET Y = X
PRINT MESSAGE
CALL EXIT
SMSQP = Σ F²
4
SMSQ = SMSQP
A


B'
DO K=1, N
A
JQ=JQ+1
XO=X(K)
SMSQO=SMSQ
SET DX TO STARTING VALUE
KK=1
IDC=Ø
LQ=1
SMSQ=
SMSQP
X(K)=X(K)+DX
CALCULATE
F FROM X
VALUES
OK
NO
YES
KK=Ø
X(K)=XO
DX=-DX
SMSQP=
ΣF²
PRINT LQ,
SMSQP, K,
AND
X(K)
12
YES
SMSQP=Ø
NO
SMSQ>SMSQP
NO
KK=Ø
NO
YES
YES
KK=Ø
LQ=LQ+1
DX=DX/-3
LQ=1
IDC=IDC+1
NO
LQ>5Ø
YES
NO
YES
SET SMSQ, SMSQP
TO SMALLEST
VALUE. SET X(K)
TO MATCH
|DX/X(K)|<10⁻⁶
YES
NO
|X(K)|≤10⁻⁵Ø
YES
IDC<13
YES
NO
B


B
PRM = Ø
YES
NO
PRINT SMSQP, X
SMSQO < SMSQP
YES
NO
Y(K) = X(K)
PRINT "BEST POSSIBLE RESULT"; SMSQO, Y
SMSQP < 10^-14
YES
12
NO
72
CONTINUE
12
B'
JN = 5
WRITE "FINAL RESULT"; SMSQP, Y
72
JI = JN*N
P < 10
NO
YES
PRINT "NOT CONVERGING PROPERLY TO UTAU"
IT = Ø
UM = UMAX(J)
XK = X(I)
JQ < JI
4
YES
NO
112
C


C
SET A,B LIMITS
GET FA
FA=Ø
YES
UTAUN = A
NO
GET FB
FB = Ø
YES
UTAUN = B
NO
D'
FAxFB=+
YES
PRINT "NO SINGLE SOLN. EXISTS WITHIN GIVEN INTERVAL
1Ø2
NO
A=C
IT=IT+1
D''
IT≤2Ø
NO
PRINT "AFTER 2Ø ITERATIONS" A, B
1Ø2
YES
C = A+B/2
B = C
D'''
INTERVAL SMALL ENOUGH
YES
D''''
NO
D


D
D'''
D''
TEST FC#FA
+
-
GET FC
D''''
ZERO
UTAUN = C
60
DECREASE UTAUA
D'
60
INCREASE UTAUB
NO
TOO MUCH GREATER
>UTAUB
UTAUN IN RANGE (UTAUA, UTAUB)
<UTAUA
NO
TOO MUCH LESS
YES
YES
79
IN RANGE
YES
UTAU(P) CLOSE ENOUGH TO UTAUN
NO
PRINT "THUS THE APPROXIMATE FRICTION VELOCITY", UTAUN
ZERO
TEST UTAU(P) - UTAUN
+
-
UTAUB = UTAU(P)
UTAU(P) = UTAUN
UTAUA = UTAU(P)
60
79
CALL PROFIL SUBROUTINE
PRINT "UTAU OUTSIDE THE LIMITS UTAUA OR UTAUB"
102
YES
Q > J
NO
STOP
112
END


ENTER PROFIL
XK = X(7)
XK = 2*XK
GET HP, UM, UM2
PRINT "UMAX OVER UTAU", UM "HDV*UTAU =", HP
PRINT "THE RESULTING PROFILES"
PRINT PAGE HEADING
J = Ø
L = Ø
GET U
GET UP
CALCULATE DATA
PRINT CALCULATED DATA
J = J + 1
NO
J > 1Ø.1
YES
U = U + .Ø5 * UMAX(K)
NO
U > UMAX(K) / 2
YES
CALCULATE DATA
PRINT CALCULATED DATA
NO
U < .899 * UMAX(K)
YES
L = L + 1
U = U + .Ø1 * UMAX(K)
YES
L < 1Ø.1
NO
RETURN

```
C  ROUTINE USED TO OBTAIN AN APPROXIMATE SOLUTION
C  TO A SYSTEM OF NONLINEAR ALGEBRAIC EQUATIONS
      DIMENSION X%10<,F%10<,Y%10<,UMAX%10<,HDV%10<,UTAU%10<
      INTEGER  P,Q
      J # 0
C  READ IN THE NUMBER%N< OF UNKNOWNS , PRINT PARAMETER, TOTAL NUMBER OF CASES
      READ%1,1< N,PRM,Q
    1 FORMAT %I2,7X,I1,8X,I2<
      WRITE%3,110< Q
  110 FORMAT%//9X,25HTOTAL NUMBER OF CASES IS ,I2//<
  112 J # J & 1
      IDGP # 0
      WRITE%3,170< J
  170 FORMAT%//9X,12HCASE NUMBER ,I2//<
C  READ IN U-MAX AND SEPARATION DISTANCE/KINEMATIC VISCOSITY
   17 READ%1,2<UMAX%J<,HDV%J<
    2 FORMAT%9X,E13.7,7X,E13.7 <
      RE #UMAX%J<*HDV%J<
      WRITE%3,114< RE,UMAX%J<,HDV%J<
  114 FORMAT %9X,11HREY. NO. = ,1PE13.6,5X,7HUMAX = ,1PE13.6,5X,12HH OVE
     1R NU = ,1PE13.6//<
      WRITE %3,117<
  117 FORMAT %12X, 1HP,7X,7HUTAU(P) <
C  INITIAL ESTIMATES OF THE N UNKNOWNS
      X%1< # 0.244
      X%2< # 0.011
      X%3< # 11.2
      X%4< # -1000.
      X%5< # -1.1
      X%6< # 1.0
      X%7< # 0.41
      X%8< # 9.1
      UTAUE # UMAX%J<*0.5*0.191/ALOG10%RE*0.25<
      UTAUA # 0.75*UTAUE
      UTAUB # 1.25*UTAUE
      P # 0
   60 P # P & 1
      JQ#0
      UTAU%P< # %UTAUA & UTAUB</2.0
   51 WRITE %3,116< P, UTAU%P<
```

```
115 FORMAT%/ 10X,I3,5X,E13.7//<
    WRITE%3,23<
 23 FORMAT%1HO,14HSUM OF SQUARES,29H        VALUES  OF  VARIABLES<
    IF%N<15,15,16
 15 CALL EXIT
 16 IF%N-20<71,71,15
 71 L#0
    CALL EQSNL%L,X,F,UMAX,HDV,UTAU,J, P,IDGP<
    IF %IDGP< 33,32,33
 33 WRITE %3,34<
 34 FORMAT %32H INITIAL VALUES NOT GOOD ENOUGH <
    GO TO 15
 32 DO 40 MN#1,N
 40 Y%MN< # X%MN<
    M#1
    GO TO 3
  4 SMSC#SMSQP
    DO 5 K#1,N
    JQ#JQ&1
    XO # X%K<
    SMSCO # SMSC
    KK#1
    IDC#0
    LQ#1
    IF% ABS%X%K<<-1.E-8< 7,7,6
  6 DX# X%K</3.0
    GO TO 29
 42 DX#1.0
    GO TO 29
  7 DX# 0.33E-8
 29 SMSC # SMSQP
 26 X%K<#X%K<&DX
    CALL EQSNL%L,X,F,UMAX,HDV,UTAU,J, P,IDGP<
    IF%IDGP< 36,35,36
 36 X%K< # XO
    SMSQP # SMSCO
    GO TO 25
 35 M#2
  3 SMSQP # 0.0
    DO 18 I#1,N
```

```
 18 SMSQP # SMSQP & F%I<*F%1<
    GO TO %4,19<,M
 19 WRITE%3,38< LQ,SMSQP,K,X%K<
 38 FORMAT%4X,5HLQ = ,I2,5X,8HSMSQP = ,E14.7,5X,4HK = ,I2,5X,7HX%K< =
   1,E14.7<
    IF%SMSQP< 28,12,28
 28 IF %SMSQ - SMSQP< 9,9 ,8
  9 IF%KK<24,25,24
 24 KK#0
    X%K< # XO
    DX#-DX
    GO TO 26
 25 DX#DX/%-3.0<
    LQ#1
    IDC#IDC&1
    IF% ABS%X%K<<- 1.E-50< 11,11,10
 11 IF%IDC-13<8,22,22
 10 IF% ABS%DX/X%K<<-1.E-6<22,8,8
  8 KK#0
    LQ # LQ & 1
    IF %LQ-50< 29,29,22
 22 IF%SMSQ-SMSQP< 109,111,111
109 X%K< # X%K< & 3.0*DX
    SMSQP # SMSQ
    GO TO 113
111 SMSQ # SMSQP
113 IF%PRM<21,5,21
 21 IF%LQ-50<30,30,31
 31 SMSQP #-SMSQP
 30 WRITE%3,14< SMSQP,%I,X%I<,I#1,N<
    SMSQP # ABS%SMSQP<
    IF%SMSQ -SMSQO< 27,27,73
 27 Y%K< # X%K<
    IF%SMSQP .LT. 1.E-14< GO TO 12
  5 CONTINUE
    JN # 5
    JI # JN*N
    IF%JQ - JI< 4,12,12
 73 WRITE%3,74<
 74 FORMAT%1HO,21H BEST POSSIBLE RESULT<
```

```
      WRITE(3,14)SMSQQ,(Y(I),I=1,N)
      GO TO 72
   12 WRITE(3,13)
   13 FORMAT(1HO,12HFINAL RESULT)
      WRITE (3,14) SMSQP,(Y(I),I=1,N)
   14 FORMAT(1X,E14.7,5(I5,2H)=,E14.7),/15X,5(I5,2H)=,E14.7),/15X,5(I5,2
     1H)=,E14.7)/15X,5(I5,2H)=,E14.7))
   72 IF(P - 10) 123,121,121
  121 WRITE(3,119)
  119 FORMAT (32H NOT CONVERGING PROPERLY TO UTAU)
      GO TO  112
  123 IT = 0
      UM = UMAX(J)
      XK = X(7)
      A = 0.75*UTAU(P)
      B = 1.25*UTAU(P)
      FA = -HDV(J)*A*0.5 * UMAX(J)*0.5/A *2.0/X(8)**(COSH(XK*UM      *.5/A
     2) - (XK*UMAX(J))**2/(8.0*A**2) -1.0)
      IF (FA) 91,99,91
   91 FB = -HDV(J)*0.5*B * UMAX(J)*0.5/B * 2.0/X(8)**(COSH(XK*UM      *0.5
     2/B) - (XK*UMAX(J))**2/(8.0*B**2) - 1.0)
      IF (FB) 92,100,92
   92 IF(FA*FB) 82,82,80
   82 IT = IT + 1
      IF(IT .GT. 20) GO TO 98
      C = (A+B)/2.0
      IF(.005*UTAU(P) - ABS(B-A)) 83,101,101
   83 FC = -HDV(J)*C*0.5 * UMAX(J)*0.5/C *2.0/X(8)**(COSH(XK*UM      *.5/C
     2) - (XK*UMAX(J))**2/(8.0*C**2) -1.0)
      IF(FC*FA) 85,101,86
   85 B = C
      GO TO 82
   86 A = C
      GO TO 82
   80 WRITE(3,81)
   81 FORMAT(// 52H NO SINGLE SOLUTION EXISTS WITHIN THE GIVEN INTERVAL)
      GO TO 102
   98 WRITE(3,97)A,B
   97 FORMAT(// 25H AFTER 20 ITERATIONS A = ,E13.7,5X,5H B = ,E13.7)
      GO TO 102
```

```
    99 UTAUN # A
       GO TO 62
   100 UTAUN # B
       GO TO 62
   101 UTAUN # C
    62 IF%UTAUN .LT. UTAUA< GO TO 59
       IF%UTAUN .GT. UTAUB< GO TO 57
       IF%.01*UTAUN - ABS%UTAU%P<-UTAUN<< 69,61,61
    61 IF%UTAU%P< - UTAUN< 70,61,76
    70 UTAUA # UTAU%P<
       GO TO 60
    76 UTAUB # UTAU%P<
       GO TO 60
    59 IF%UTAUA-UTAUN - .13/P*UTAU%P<< 56,56,79
    56 UTAUA # UTAUA - .13/P*UTAU%P<
       GO TO 60
    57 IF%UTAUN-UTAUB - .13/P*UTAU%P<<58,58,79
    58 UTAUN # UTAUB & .13/P*UTAU%P<
       GO TO 60
    79 WRITE%3,77<
    77 FORMAT%//40H UTAUN OUTSIDE THE LIMITS UTAUA OR UTAUB <
       GO TO 102
    60 WRITE %3,106< UTAUN
   106 FORMAT 6//5X,41HTHUS THE APPROXIMATE FRICTION VELOCITY = ,E13.7//<
       UTAU%P< # UTAUN
       CALL PROFILE %UMAX,HDV,UTAU,Y,J,P,,<
       IF %J - J< 102,102,112
   102 STOP
       END

C  SUBROUTINE USED TO INPUT THE SET OF N NONLINEAR ALGEBRAIC EQUATIONS
       SUBROUTINE EQSNL&L,X,F,UMAX,HDV,UTAU,K,P,IDGP<
       DIMENSION X%10<,F%10<,UMAX%10<,HDV%10<,UTAU%30<
       INTEGER  P
       IDGP # 0
       IF%L<2,1,2
     1 L#1
       UP # HDV%K<*UTAU%P<
       UM # UMAX%K</UTAU%P<
       UM2 # UM/2.0
```

```
      UMSQ# UM**2
      UM3 #       UMSQ*UM
      UM4 # UM3*UM
      XK # 0.41
      T1#0.0
      T2#0.0
      T3#0.0
      T4 # 0.0
      T5 # 0.0
      T6 # 0.0
      X1 #0.0
    2 IF(ABS(X(8)-X1)/X(8)) .LT. 0.001) GO TO 4
      X1 # X(8)
      CALL INTS(U**2,UTAU,P,X1,T1,T2,T3,T4,T5,T6,XK)
      WRITE (3,9) T1,T2,T3,T4,T5,T6
    9 FORMAT(1 14H INTEGRAL#1 # ,E13.7,14H INTEGRAL#2 # ,E13.7,14H INTEGR
     1AL#3 # ,E13.7,14H INTEGRAL#4 # ,E13.7,/1X,14H INTEGRAL#5 # ,E13.7,
     214H INTEGRAL#6 # ,E13.7)
    4 IF(ABS(X(7)-XK)/X(7)) .LT. 0.001) GO TO 5
      XK # X(7)
      CALL INTS(U**2,UTAU,P,X1,T1,T2,T3,T4,T5,T6,XK)
      WRITE (3,6) T1,T2,T3,T4,T5,T6
    6 F(1) = 4.*(HP-UM)*X(3) + (2.0*X(1)*(HP-2.0*UM)-2.0*T1)*X(4) + X(3)*
     1X(4)*X(2)*(1.0-4.0*T1 + 2.0*T3)                              1ST.EQN.
      F(2) = X(3)*(1.0-2.0*T1) + X(4)*X(1)*(1.0-4.0*T1 + 2.0*T3)    2ND.EQN.
      F(3) = X(1)*(HP-UM) + X(2)*X(3)*(2.0-4.0*T1) (2.0*X(1)*X(4)*T3-2.*3RD.EQN.
     1T1)*X(2)
      F(4) = X(1)*(HP-2.0*UM) 2.0*T1) + X(2)*X(3)*(1.0-4.0*T1 + 2.0*T3 4TH.EQN.
      F(5) = (UM-2.0*T1)/15.0 -(X(3)*(X(1)*X(4)*(HP + X(2)*X(3)) + (    5TH.EQN.
     1X(1)*UM*(X(3)(2.0*X(1)-X(4)) + 2.0*X(2)*X(3)*T1*(X(3)+ 2.0*X(1)*
     3X(4)) - 2.0*X(1)*X(4)*(X(1)*T1 + X(2)*X(3)*T3)
      F(6) = -HP/2.0 + UM2 * 2.0/(X(4)*(SQRT(XK*UM2)-XK*UM2)**2/2.0 - 1.0) 6TH.EQN.
      F(7) = X(5)*T2 + (X(5)*T5) + (X(5)-1.0)*(X(1)**2*X(4) - X(2)*X(3)) 7TH.EQN.
     2(X(3) + 2.*X(1)*X(4)*(T2 + XK*T5) + X(1)*X(2)*X(3)*X(4)*T4 + XK
     3*T6)*2.0) (X(6)*2.0/XK/UM2*SIN(XK*UM2) - XK*UM2**2)
      F(8) = X(5)*T2 + (X(5)-1.)*(X(1)**2*X(4)-X(2)*X(3))*(2.0* 8TH.EQN.
     2X(1)*X(4)) + 2.0*X(1)*X(2)*X(3)*X(4)*T4 ) -X(6)*2.0/X(8)**2*(COSH
     3(XK*UM2) - (XK*UM2)/2.0 - 1.0)
      RETURN
      END
```

```
C  NUMERICAL EVALUATION OF INTEGRALS VIA SIMPSON,S RULE
      SUBROUTINE INTS%UM2,UTAU,P,X1,T1,T2,T3,T4,T5,T6,XK<
      DIMENSION Y%20<,YY%20<,SU%20<,CU%20<,W%20<,Z%20<,YYY%20<,ZZ%20<
      DIMENSION ZZZ%20<
      INTEGER P
      XK2 # 2.0*XK
      H # UM2/20.0
      DO 2 I#1,20
      CH# COSH%XK*H*I<
      CU%I< # H*I*%CH - 1.0<
      SU%I< #SINH%XK*H*I< - XK*H*I
    2 Y%I< # 1.0/%1.0 & XK2/X1*SU%I< <
      YO #0.0
      DO 4 I#1,19,2
    4 YO # YO & Y%I<
      YE#0.0
      DO 6 I# 2,18,2
    6 YE#YE & Y%I<
      T1# H/3.0*%Y%20< &1.0 & 4.0*YO & 2.0*YE<
      DO 8 J#1,20
      YY%J< # Y%J<**2
    8 Z%J<# SU%J<*YY%J<
      ZO#0.0
      DO 10 J#1,19,2
   10 ZO # ZO & Z%J<
      ZE# 0.0
      DO 12 J#2,18,2
   12 ZE # ZE & Z%J<
      T2 # H/3.0*%Z%20< & 4.0*ZO & 2.0*ZE<
      WO # 0.0
      DO 14 J # 1,19,2
   14 WO # WO & YY%J<
      WE # 0.0
      DO 16 J # 2,18,2
   16 WE # WE & YY%J<
      T3 # H/3.0*%1.0 & 4.0*WO & 2.0*WE & YY%20<<
      DO 18 J # 1,20
   18 YYY%J< # Y%J<*YY%J<*SU%J<
      XO # 0.0
      DO 20 J # 1,19,2
```

```
   20 XO # XO & YYY%J<
      XE # 0.0
      DO 22 J # 2,18,2
   22 XE # XE & YYY%J<
      T4 # H/3.0*%4.0*XO & 2.0*XE & YYY%20<<
      DO 24 I # 1,20
   24 ZZ%I< # CU%I<*YY%I<
      ZO # 0.0
      DO 26 J#1,19,2
   26 ZO # ZO & ZZ%J<
      ZE # 0.0
      DO 28 J#2,18,2
   28 ZE # ZE & ZZ%J<
      T5 # H/3.0*%4.0*ZO & 2.0*ZE & ZZ%20<<
      DO 30 I # 1,20
   30 ZZZ%I< # ZZ%I<*Y%I<
      YO # 0.0
      DO 32 J#1,19,2
   32 YO # YO & ZZZ%J<
      YE # 0.0
      DO 34 J#1,18,2
   34 YE # YE & ZZZ%J<
      T6#H/3.0*%4.0*YO & 2.0*YE & ZZZ%20<<
      RETURN
      END

C  CALCULATION OF THE DISTRIBUTIONS OF VARIOUS FLOW QUANTITIES ACROSS THE FLOW
      SUBROUTINE PROFIL %UMAX,HDV,UTAU,X,K,P,Q<
      INTEGER  P,Q
      DIMENSION X%10<,F%10<,UMAX%10<,HDV%10<,UTAU%30<
      XK # X%7<
      XK2 # 2.0*XK
      HP # HDV%K<*UTAU%P<
      UM # UMAX%K</UTAU%P<
      UM2 # UM/2.0
      WRITE%3,17< UM, HP
    17 FORMAT%10X,18HUMAX OVER UTAU  = , E13.7/,10X,14HHDV * UTAU  = ,E13
     2.7//<
      WRITE%3,119<
   119 FORMAT%30X,23H THE RESULTING PROFILES//<
```

```
    WRITE %3,120<
120 FORMAT %11X,8HX OVER H,10X,8HX2-STAR    ,9X,11HU OVER UMAX,7X,13H U
   2SQ/UTAU SQ.,4X,14HDSCALE*UTAU/NU,4X,15HDISSIPATION(ND),4X,14HREY.S
   3TRESS(ND) <
    J # 0
    L # 0
124 U # J*0.01*UMAX%K<
 31 UP # U/UTAU%P<
    UDUMAX # U/UMAX%K<
    XDH # %UP & %2.*COSH%XK*UP<-%XK*UP<**2 - 2.0</X%8<<
    YS    # XDH*HP
    DUDX = 1.0/%1.0 &  XK2*%SINH%XK*UP< - XK*UP</X%8<<
    RES # 1.0 - DUDX
    USQ # X%3<*RES & X%1<*X%4<*RES**2
    DSCALE # %1.0/%X%1< & X%2<*X%3<*DUDX<<**0.5
    DISSIP #% USQ/DSCALE**2<*15.0
    TISQ # UTAU%P<**2*USQ/UMAX%K<**2
    WRITE %3,122< XDH,YS, UDUMAX, USQ,DSCALE,DISSIP,RES
122 FORMAT%10X,E13.7,6%5X,E13.7< <
    WRITE%3,123< UP,TISQ
123 FORMAT%5X,9H U-STAR = ,E13.7,5X,6H TISQ # ,E13.7/<
    J # J & 1
    IF%J - 10.1<124,124,125
125 U # U & 0.05*UMAX%K<
    IF%U - 0.5*UMAX%K<< 31,31,34
 34 UP # U/UTAU%P<
    UDUMAX # U/UMAX%K<
    XDHT # 1.0 - %UM-UP & %2.0*COSH%XK*%UM-UP<< - %XK*%UM-UP<<**2-2.0
   2</X%8<<
    YS # XDHT*HP
    DUDX # 1.0/%1.0 &  XK2*%SINH%XK*%UM-UP<< - XK*%UM-UP<</X%8<<
    RES # 1.0 - DUDX
    USQ # X%3<*RES & X%1<*X%4<*RES**2
    DSCALE # %1.0/%X%1< & X%2<*X%3<*DUDX<<**0.5
    DISSIP #% USQ/DSCALE**2<*15.0
    TISQ # UTAU%P<**2*USQ/UMAX%K<**2
    WRITE %3,122<XDHT,YS, UDUMAX, USQ,DSCALE,DISSIP,RES
    WRITE%3,123< UP,TISQ
     IF%U - 0.899*UMAX%K<< 125,33,33
 33 L # L & 1
```

```
      U = U &   0.01*UMAX&KK
      IF&L - 10.1< 34,35,35
   35 RETURN
      END

C  INITIAL SPECIFICATIONS: NO. OF EQUATIONS, PRINT PARAMETER, TOTAL NO. OF CASES
 8        1          2
C  TYPICAL DATA INPUT: U-MAX AND SEPARATION DISTANCE/KINEMATIC VISCOSITY
         0.1510CC0E&01          0.4500000E&05
         0.3500000E&00          0.4500000E&05
```

THUS THE APPROXIMATE FRICTION VELOCITY = 0.3096802E-01

REY. NO. = 6.038996E 04 UMAX = 1.341999E 00 H OVER NU = 4.500000E 04
UMAX OVER UTAU = 0.4333499E 02
HOV * UTAU = 0.1393561E 04

THE RESULTING PROFILES

X OVER H	X2-STAR	U OVER UMAX	USQ/UTAU SQ.	DSCALE*UTAU/NU	DISSIPATION(ND)	REY.STRESS(ND)
0.0	0.0	0.0	0.0	0.2807534E 01	0.0	0.0
U-STAR =0.0	TISQ = 0.0					
0.3109716E-03	0.4333578E 00	0.9999994E-02	0.8414206E-03	0.2807639E 01	0.1601112E-02	0.7534027E-04
U-STAR =0.4333498E 00	TISQ = 0.4480596E-06					
0.6220266E-03	0.8668319E 00	0.1999999E-01	0.6832350E-02	0.2808365E 01	0.1299434E-01	0.6119013E-03
U-STAR =0.8666997E 00	TISQ = 0.3639252E-05					
0.9333808E-03	0.1300722E 01	0.2999999E-01	0.2321551E-01	0.2810352E 01	0.4409080E-01	0.2080441E-02
U-STAR =0.1300049E 01	TISQ = 0.1236234E-04					
0.1245404E-02	0.1735545E 01	0.3999999E-01	0.5539925E-01	0.2814281E 01	0.1049204E 00	0.4970551E-02
U-STAR =0.1733399E 01	TISQ = 0.2950030E-04					
0.1558625E-02	0.2172038E 01	0.4999998E-01	0.1089668E 00	0.2820878E 01	0.2054078E 00	0.9796500E-02
U-STAR =0.2166748E 01	TISQ = 0.5802524E-04					
0.1873758E-02	0.2611195E 01	0.5999998E-01	0.1895360E 00	0.2830939E 01	0.3547495E 00	0.1709211E-01
U-STAF =0.2600099E 01	TISQ = 0.1009286E-03					
0.2191711E-02	0.3054282E 01	0.6999987E-01	0.3025433E 00	0.2845338E 01	0.5605459E 00	0.2740151E-01
U-STAR =0.3033446E 01	TISQ = 0.1611054E-03					
0.2513634E-02	0.3502901E 01	0.7999992E-01	0.4529231E 00	0.2865052E 01	0.8276575E 00	0.4126251E-01
U-STAR =0.3466798E 01	TISQ = 0.2411832E-03					
0.2840930E-02	0.3959007E 01	0.8999991E-01	0.6446707E 00	0.2891152E 01	0.1153677E 01	0.5918074E-01
U-STAR =0.3900147E 01	TISQ = 0.3432895E-03					
0.3175315E-02	0.4424994E 01	0.9999990E-01	0.8803089E 00	0.2924830E 01	0.1543565E 01	0.8159345E-01
U-STAR =0.4333495E 01	TISQ = 0.4687677E-03					
0.5043112E-02	0.7027883E 01	0.1499999E 00	0.2643772E 01	0.3255937E 01	0.3740709E 01	0.2662473E 00
U-STAR =0.6500264E 01	TISQ = 0.1407816E-02					
0.7656857E-02	0.1067030E 02	0.1999999E 00	0.4600108E 01	0.4005249E 01	0.4301308E 01	0.5280433E 00
U-STAR =0.8666993E 01	TISQ = 0.2449574E-02					
0.1222132E-01	0.1703114E 02	0.2499998E 00	0.5769777E 01	0.5352893E 01	0.3020460E 01	0.7525500E 00
U-STAR =0.1083374E 02	TISQ = 0.3072426E-02					
0.2164494E-01	0.3016353E 02	0.2999998E 00	0.6242947E 01	0.7335526E 01	0.1740275E 01	0.8860592E 00
U-STAR =0.1300049E 02	TISQ = 0.3324391E-02					
0.4293867E-01	0.5983763E 02	0.3499998E 00	0.6412481E 01	0.9675437E 01	0.1027486E 01	0.9507176E 00
U-STAR =0.1516724E 02	TISQ = 0.3414668E-02					
0.9299845E-01	0.1295990E 03	0.3999998E 00	0.6474877E 01	0.1178685E 02	0.6990803E 00	0.9792288E 00
U-STAR =0.1733398E 02	TISQ = 0.3447895E-02					

0.2125416E 00	0.2961895E 03	0.4499997E 00	0.6499078E 01	0.1322358E 02	0.5574999E 00	0.9913321E 00
U-STAR =0.1950073E 02	TISQ = 0.3460781E-02					
0.4996910E 00	0.6963496E 03	0.4999998E 00	0.6508799E 01	0.1400294E 02	0.4979135E 00	0.9963961E 00
U-STAR =0.2166748E 02	TISQ = 0.3465959E-02					
0.7874557E 00	0.1097367E 04	0.5499997E 00	0.6499078E 01	0.1322361E 02	0.5574978E 00	0.9913322E 00
U-STAR =0.2383423E 02	TISQ = 0.3460781E-02					
0.9070005E 00	0.1263960E 04	0.5999997E 00	0.6474877E 01	0.1178689E 02	0.6990763E 00	0.9792291E 00
U-STAR =0.2600098E 02	TISQ = 0.3447895E-02					
0.9570610E 00	0.1333723E 04	0.6499997E 00	0.6412482E 01	0.9675465E 01	0.1027480E 01	0.9507181E 00
U-STAR =0.2816772E 02	TISQ = 0.3414670E-02					
0.9783549E 00	0.1363397E 04	0.6999997E 00	0.6242949E 01	0.7335554E 01	0.1740263E 01	0.8860604E 00
U-STAR =0.3033447E 02	TISQ = 0.3324393E-02					
0.9877786E 00	0.1376529E 04	0.7499993E 00	0.5769793E 01	0.5352923E 01	0.3020434E 01	0.7525534E 00
U-STAR =0.3250121E 02	TISQ = 0.3072435E-02					
0.9923431E 00	0.1382890E 04	0.7999990E 00	0.4600153E 01	0.4005280E 01	0.4301284E 01	0.5280505E 00
U-STAR =0.3466794E 02	TISQ = 0.2449598E-02					
0.9949569E 00	0.1386533E 04	0.8499987E 00	0.2643829E 01	0.3255949E 01	0.3740843E 01	0.2662539E 00
U-STAR =0.3683469E 02	TISQ = 0.1407849E-02					
0.9968247E 00	0.1389136E 04	0.8999984E 00	0.8803511E 00	0.2924836E 01	0.1543633E 01	0.8159751E-01
U-STAR =0.3900142E 02	TISQ = 0.4687901E-03					
0.9971590E 00	0.1389602E 04	0.9099978E 00	0.6447245E 00	0.2891160E 01	0.1156967E 01	0.5918580E-01
U-STAR =0.3943474E 02	TISQ = 0.3433181E-03					
0.9974863E 00	0.1390058E 04	0.9199972E 00	0.4529707E 00	0.2865058E 01	0.8277412E 00	0.4126692E-01
U-STAR =0.3986807E 02	TISQ = 0.2412085E-03					
0.9978082E 00	0.1390506E 04	0.9299966E 00	0.3025922E 00	0.2845344E 01	0.5606341E 00	0.2740598E-01
U-STAR =0.4030139E 02	TISQ = 0.1611314E-03					
0.9981261E 00	0.1390949E 04	0.9399959E 00	0.1895766E 00	0.2830941E 01	0.3548250E 00	0.1709580E-01
U-STAR =0.4073471E 02	TISQ = 0.1009503E-03					
0.9984413E 00	0.1391398E 04	0.9499953E 00	0.1090084E 00	0.2820881E 01	0.2054858E 00	0.9800255E-02
U-STAR =0.4116803E 02	TISQ = 0.5804739E-04					
0.9987544E 00	0.1391825E 04	0.9599947E 00	0.5542046E-01	0.2814284E 01	0.1049603E 00	0.4972458E-02
U-STAR =0.4160136E 02	TISQ = 0.2951159E-04					
0.9989665E 00	0.1392260E 04	0.9699940E 00	0.2322615E-01	0.2810355E 01	0.4411090E-01	0.2081394E-02
U-STAR =0.4203468E 02	TISQ = 0.1236801E-04					
0.9993778E 00	0.1392694E 04	0.9799934E 00	0.6842993E-02	0.2808365E 01	0.1301458E-01	0.6128550E-03
U-STAR =0.4246800E 02	TISQ = 0.3643919E-05					
0.9996888E 00	0.1393127E 04	0.9899928E 00	0.8520710E-03	0.2807839E 01	0.1621378E-02	0.7629395E-04
U-STAR =0.4290132E 02	TISQ = 0.4537309E-06					
0.9999998E 00	0.1393560E 04	0.9999921E 00	0.0	0.2807534E 01	0.0	0.0
U-STAR =0.4333466E 02	TISQ = 0.0					