STARLIKE SETS, INVERSE STARLIKE SETS,
AND A GENERALIZATION OF CONVEXITY

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PREFACE

This paper deals with certain classes of subsets of a real linear space. The basic definitions and notation are given in Chapter 0. Chapter I deals with the basic properties of inverse starlike sets and it is demonstrated that the inverse starlike property is preserved under many of the operations and transformations in a linear space. The inverse star envelope and star envelope of a set are defined in Chapter II; it is proved that many of the properties of the inverse star envelope are determined by the given set. Certain inverse starlike (starlike) sets may be represented as the inverse star (star) envelope of a set of relative extreme points, an extension of the Krein-Milman Theorem. Chapter III contains a discussion of a metric space of starlike sets. Chapter IV deals with the class of all starlike subsets and it is shown that this class of sets satisfies all the requirements for a vector space except additive inverses; a restricted cancellation law is proved. Finally in Chapter IV it is shown that, for a particular order relation on the class of starlike sets, this class is a complete complemented lattice which is not distributive and not modular.

In Chapter V a generalization of convexity is given which includes convex, projectively convex, starlike, inverse starlike, property P_3, cone, and flat as special cases. Several properties of this generalization are then determined. Also a theorem due to Brunn [3] (numbers in square brackets refer to the bibliography at the end of the paper) is generalized.

Finally Chapter VI is a summary of the paper and lists several unsolved and partially solved problems that have been raised in the course of the investigation.

I wish to express my sincere thanks to the members of my advisory committee for their help in planning my program of study; to Dr. L. Wayne Johnson, Chairman of the Mathematics Department, for my graduate assistantship and for his wise counsel; and to Professor E. K. McLachlan for his kind assistance and guidance in the preparation of this thesis. Indebtedness is also acknowledged to the faculty at Louisiana Tech for their encouraging me to begin graduate study and for a good foundation in the subject. Finally I want to express my deepest appreciation to my parents for their support and encouragement, and especially to my wife, Jerrie, without whose help I could never have finished my work.

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CHAPTER 0

DEFINITIONS AND NOTATION

The setting for the results of this paper is a real linear space as defined by Day [6] ; several results are given in special linear spaces and these definitions also come from Day. The symbol L is used to denote the linear space, elements of L are denoted by lower case Latin letters, subsets of L are denoted by capital Latin letters, and real numbers are denoted by lower case Greek letters with the exception that in some instances subscripts are denoted by lower case Latin letters.

The following notation is used for portions of the line through elements u and v of L: $uv = \{\alpha u + (1 - \alpha)v : 0 \leq \alpha \leq 1\}$ and $\infty uv = \{\alpha u + (1 - \alpha)v : \alpha \geq 1\}$. The line through u and v is denoted by L(u,v) and $)uv(= uv \setminus \{u,v\}$ is the open segment between u and v where $\setminus$ denotes set difference.

A subset S of L is:

<u>flat</u> if for each x and y in S, L(x,y) is contained in S;

<u>convex</u> if for each x and y in S, xy is contained in S;

<u>starlike from a</u> if for each x in S, ax is contained in S;

a <u>cone with vertex at a</u> if for each x in S, $ax \cup \infty xa$ is contained in S;

<u>projectively convex</u> if for each x and y in S, either xy or $\infty xy \cup \infty yx$ is contained in S (see Hare [8]);

P_3 (or S has property P_3) if for each x, y, and z in S, at least one of the segments xy, yz, or xz is contained in S (see Valentine [14]).

The symbol E^n refers to n-dimensional Euclidean space with the usual topology.

For a subset S of L, C(S) denotes the set complement of S. The convex hull of S is denoted by k(S) and core(S) is the core of S. If L is a linear topological space (LTS) and S is a subset of L, then S^o denotes the interior of S and $\overline{S}$ is the closure of S. The boundary of S is symbolized by bdry(S).

It is assumed that all sets considered are contained in some real linear space L.

CHAPTER I

GENERAL PROPERTIES OF STARLIKE AND INVERSE STARLIKE SETS

Motzkin [11] has defined a subset S of L to be <u>inverse starlike</u> from an element a of L if for each x in S, ∞xa is contained in S.[1] In the plane the inversion of a starlike set in the unit circle has the property of being inverse starlike from the origin which accounts for the use of the term inverse starlike.

The notion of an inverse starlike set is not completely foreign or new to mathematical research. For example, distortion theorems in the theory of complex variables deal with simple mappings of the exterior of the unit circle which is an inverse starlike set (see Bieberbach [1]). Other results are obtained for sets which are the complements of starlike sets. It will be proved later that indeed the complement of a starlike set is inverse starlike. Recently, Lax, Morawetz, and Phillips [10] published results they obtained concerning solutions of the wave equation in the exterior of a starlike set. These are just a few of the examples of the use of inverse starlike and starlike sets in the literature.

The purpose of this chapter is to develop several basic properties of inverse starlike sets. Many of the theorems hold also for starlike sets and may be proved in the same fashion by merely changing the requirement on the scalars. The first result relates starlike sets to inverse starlike sets.

[1]The author had defined this notion and developed several properties of such sets prior to learning of Professor Motzkin's work at the Symposium on Convexity, Seattle, Washington, June, 1961.

Theorem 1: Let a subset S of L be such that a belongs to S. Then S is starlike from a if, and only if, C(S) is inverse starlike from a.

Proof: First assume that S is starlike from a and let x be an element of C(S). Then ∞xa is contained in C(S) since, if there is an element y of ∞xa which belongs to S, ay would be contained in S; but x belongs to ay contradicting that x is in C(S). Thus C(S) is inverse starlike from a.

Next suppose that C(S) is inverse starlike from a and let x belong to S. Then by assumption a belongs to S, and ax must be contained in S since if there is an element y of ax which belongs to C(S), ∞ya would be contained in C(S); but x is an element of ∞ya which contradicts that x is in S. Hence S is starlike from a.

A set is convex if, and only if, it is starlike from every point of the set, which proves the following corollary to Theorem 1.

Corollary: If K is inverse starlike from every point of C(K), then C(K) is convex.

The following group of theorems demonstrates that the property of being inverse starlike is quite a dominant property since it is preserved under many of the set operations.

Theorem 2: Let S_α be inverse starlike from a for each α in some index set Δ. Then $\bigcap_\alpha S_\alpha$ and $\bigcup_\alpha S_\alpha$ are inverse starlike from a.

Proof: First it is proved that $\bigcap_\alpha S_\alpha$ is inverse starlike from a. Let x be in $\bigcap_\alpha S_\alpha$. Then x is in S_α for each α in Δ and since each S_α

is inverse starlike from a, it follows that ∞xa is contained in S_α for each α in Δ. Thus ∞xa is contained in $\bigcap_\alpha S_\alpha$ which proves that $\bigcap_\alpha S_\alpha$ is inverse starlike from a.

Now consider $\bigcup_\alpha S_\alpha$. Let x be an element of $\bigcup_\alpha S_\alpha$. Then x is in S_β for some β in Δ, and since S_β is inverse starlike from a, it follows that ∞xa is contained in S_β. Hence ∞xa is contained in $\bigcup_\alpha S_\alpha$ which proves that $\bigcup_\alpha S_\alpha$ is inverse starlike from a.

<u>Corollary</u>: Let $\{S_n\}$ be a sequence of subsets of L. Then lim sup S_n and lim inf S_n are inverse starlike from a provided that each S_n is inverse starlike from a.

<u>Proof</u>: This corollary follows immediately from Theorem 2 and the fact that $\limsup S_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$ and $\liminf S_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n$.

<u>Theorem 3</u>: Let H and K be subsets of L and suppose that H is inverse starlike from a and K is inverse starlike from b. Let α be a real number. Then αH is inverse starlike from αa and H + K is inverse starlike from a + b.

<u>Proof</u>: It will first be proved that $\alpha H = \{\alpha x : x \in H\}$ is inverse starlike from αa. Let y be an element of αH and write $y = \alpha x$ for some x in H. Let $\lambda \geq 1$. Then $\lambda(\alpha x) + (1 - \lambda)(\alpha a) = \alpha(\lambda x + (1 - \lambda)a)$ which belongs to αH since $\lambda x + (1 - \lambda)a$ belongs to H. Hence αH is inverse starlike from αa.

Next it will be proved that H + K is inverse starlike from a + b. Let z be an element of H + K; then z = x + y for some x in H and y in K. Let $\lambda \geq 1$. Then $\lambda z + (1 - \lambda)(a + b) = \lambda(x + y) + (1 - \lambda)(a + b) = \lambda x + (1 - \lambda)a + \lambda y + (1 - \lambda)b$ which belongs to H + K since $\lambda x + (1 - \lambda)a$ is in H and $\lambda y + (1 - \lambda)b$ is in K. Thus H + K is inverse starlike from a + b.

It should be pointed out that the singleton $\{a\}$ is inverse starlike from a. Using this observation one may obtain the following corollary to Theorem 3.

Corollary: Let H be inverse starlike from a and let b belong to L. Then the translate H + b is inverse starlike from a + b.

Theorem 4: Let L and L' be linear spaces and let f be a linear transformation of L into L'. If a subset S of L is inverse starlike from a, then f(S) is inverse starlike from f(a) in L'.

Proof: Let y be an element of f(S). Then there is an element x of S so that y = f(x). Let $\lambda \geq 1$. Then $\lambda y + (1 - \lambda)f(a) = \lambda f(x) + (1 - \lambda)f(a) = f(\lambda x + (1 - \lambda)a)$ which belongs to f(S) since $\lambda x + (1 - \lambda)a$ is in S. Hence f(S) is inverse starlike from f(a).

Corollary 1: Let L_o be a linear subspace of L and suppose a subset S of L is inverse starlike from a. Then the subset S_o of L/L_o given by $S_o = \{x + L_o : x \in S\}$ is inverse starlike from $a + L_o$ in L/L_o.

An affine transformation f of the linear space L to the linear space L' is defined by f(x) = g(x) + c where g is a linear transformation of L into L' and c is an element of L'. Using the corollary to Theorem 3 one may obtain the following second corollary to Theorem 4.

Corollary 2: Let f be an affine transformation of L into L' and let a subset S of L be inverse starlike from a. Then the subset f(S) of L' is inverse starlike from $f(a)$.

If L_α is a real linear space for each α in some index set Δ, then $\prod_\alpha L_\alpha$ is defined to be the set of functions x on Δ so that $x_\alpha \in L_\alpha$ for all α in Δ and $\bigoplus_\alpha L_\alpha$ is the subset of $\prod_\alpha L_\alpha$ consisting of those functions x for which $x_\alpha = 0$ except for a finite number of α in Δ. Each of the sets $\prod_\alpha L_\alpha$ and $\bigoplus_\alpha L_\alpha$ is a linear space (see Day [6], page 5). Let $K_\alpha \subset L_\alpha$ for each α in Δ. Then $\prod_\alpha K_\alpha = \left\{(x_\alpha)_{\alpha\in\Delta} : x_\alpha \in K_\alpha\right\}$ and $\bigoplus_\alpha K_\alpha = \left\{(x_\alpha)_{\alpha\in\Delta} : x_\alpha \in K_\alpha \text{ and } x_\alpha = 0 \text{ except for a finite number of } \alpha \text{ in } \Delta\right\}$.

<u>Theorem 5</u>: Let L_α be a linear space for each α in some index set Δ, and for each α in Δ let K_α be a subset of L_α which is inverse starlike from 0, the origin of L_α. Then $\prod_\alpha K_\alpha$ is inverse starlike from 0 in the direct product space $\prod_\alpha L_\alpha$ and $\bigoplus_\alpha K_\alpha$ is inverse starlike from 0 in the direct sum space $\bigoplus_\alpha L_\alpha$.

<u>Proof</u>: Let $x = (x_\alpha)_{\alpha\in\Delta}$ belong to $\prod_\alpha K_\alpha$ and $\lambda \geq 1$. Then $\lambda x + (1 - \lambda)0 = (\lambda x_\alpha)_{\alpha\in\Delta}$ which belongs to $\prod_\alpha K_\alpha$ since λx_α is in K_α for each α in Δ. Thus $\prod_\alpha K_\alpha$ is inverse starlike from 0 in $\prod_\alpha L_\alpha$.

Next consider an element $y = (y_\alpha)_{\alpha\in\Delta}$ of $\bigoplus_\alpha K_\alpha$ and $\lambda \geq 1$. Then by definition $y_\alpha = 0$ for all but a finite number of the α in Δ and $\lambda y = (\lambda y_\alpha)_{\alpha\in\Delta}$ is in $\bigoplus_\alpha K_\alpha$ since λy_α is in K_α for each α in Δ. Therefore $\bigoplus_\alpha K_\alpha$ is inverse starlike from 0 in $\bigoplus_\alpha L_\alpha$.

<u>Theorem 6</u>: Let the subset S of L be inverse starlike from 0. Then the core of S is also inverse starlike from 0 or is empty.

<u>Proof</u>: If core(S) is not empty, then let x be an element of core(S)

and $\alpha \geq 1$. Now for each y in L there is a positive number $\epsilon(y)$ so that $x + \lambda y$ is in S for $|\lambda| < \epsilon(y)$. Now S is inverse starlike from 0 which implies that $\alpha x + \alpha\lambda y$ is in S. Also αx is in core(S) since for each y in L there is a positive number $\epsilon_1(y) = \epsilon(y)/\alpha$ so that $\alpha x + \lambda y$ is in S for $|\lambda| < \epsilon_1(y)$. Hence core(S) is inverse starlike from 0.

The corollary to Theorem 3 and the fact that the core of a set is preserved under translation prove the following corollary to Theorem 6.

Corollary: If S is inverse starlike from a, then core(S) is inverse starlike from a.

Theorem 7: Let K be a subset of the linear topological space L and suppose K is inverse starlike from a. Then K^o is inverse starlike from a or is empty and $\bar{K}$ is inverse starlike from a.

Proof: It is first proved that K^o is inverse starlike from a. Let x be an element of K^o and $\alpha \geq 1$. Then there is a neighborhood U of x so that U is contained in K and $\alpha U + (1 - \alpha)a \subset \alpha K + (1 - \alpha)a \subset K$. But $\alpha U + (1 - \alpha)a$ is open and $\alpha x + (1 - \alpha)a$ is an element of $\alpha U + (1 - \alpha)a$ which implies that $\alpha x + (1 - \alpha)a$ is in K^o. Therefore K^o is inverse starlike from a.

Next consider the closure of K and let x belong to $\bar{K}$ and $\alpha \geq 1$. Then x is either a point or a limit point of K. If x is in K, then ∞xa is contained in K. Hence let x be a limit point of K, so that there exists a sequence of points $\{x_n\}$ converging to x and where x_n is in K for each positive integer n. Also $\alpha x_n + (1 - \alpha)a$ is in K for each positive integer n and the sequence $\{\alpha x_n + (1 - \alpha)a\}$ converges to $\alpha x + (1 - \alpha)a$ which must be in $\overline{K}$. Hence $\overline{K}$ is inverse starlike from a.

The image of an inverse starlike set under a linear transformation has been considered. It seems natural to consider the image of an inverse starlike set under other types of transformations. Hare [8] has proved in his thesis that the image of a projectively convex set under a projective transformation is again projectively convex. A similar result may be proved for inverse starlike sets for a transformation with an order separation property.

Let a, b, x, and y be elements of L. Then the pair (a,b) order separates the pair (x,y) provided x is on ab and y is on ∞ba.

Theorem 8: Let T be a 1-1 map of L onto L which maps lines onto lines and is such that (a,b) order separates (x,y) if, and only if, (T(a),T(b)) order separates (T(x),T(y)) and T(0) = 0. If the subset K of L is inverse starlike from 0, then T(K) is also inverse starlike from 0.

Proof: Let T(x) belong to T(K) and $\alpha > 1$. It must be shown that αT(x) is in T(K). The mapping T is onto and hence there is a y in L so that T(y) = αT(x). Assume by way of contradiction that T(y) is not in T(K). Then certainly y is not in K. Since T maps lines onto lines and is 1-1, y is on L(0,x) because T(y) is on L(0,T(x)). Furthermore either y is on 0x or on ∞0x; for if y is on ∞x0, T(y) would be in T(K) contradicting the assumption that T(y) is not in T(K). Now αx is in K which implies that T(αx) is in T(K) and since T maps lines onto lines T(αx) must be on L(0,T(x)). Indeed it can be shown that T(αx) is on ∞T(x)0. Let $\beta > \alpha$. Then (0,αx) order separates (x,βx) which implies that (0,T(αx)) order separates

$(T(x),T(\beta x))$ and thus $T(\alpha x)$ is on $\infty T(x)0$. There are two cases to consider:

Case I: Assume that y is on 0x. Then $(0,x)$ order separates $(y,\alpha x)$ which implies that $(0,T(x))$ order separates $(T(y),T(\alpha x))$ and hence that $T(y)$ is on $0T(x)$ which contradicts that $T(y) = \alpha T(x)$ where $\alpha > 1$.

Case II: Assume that y is on $\infty 0x$. Then (y,x) order separates $(0,\alpha x)$ which implies that $(T(y),T(x))$ order separates $(0,T(\alpha x))$ and hence that 0 is on $T(y)T(x)$ which again contradicts that $T(y) = \alpha T(x)$ with $\alpha > 1$.

In each possible case a contradiction is reached and hence the hypothesis that $T(y)$ is not in $T(K)$ is untenable which implies that $T(y) = \alpha T(x)$ is in $T(K)$. Therefore $T(K)$ is inverse starlike from 0.

For a given linear space L, let $L^{\#}$ denote the linear space of all linear functionals on L. If K is a subset of L, then the subset K^{π} of $L^{\#}$ defined by $K^{\pi} = \{ f \in L^{\#} : f(x) \geq -1 \text{ for each } x \text{ in } K \}$ is called the polar set of K. Similarly if H is contained in $L^{\#}$, $H_{\pi} = \{ x \in L : f(x) \geq -1 \text{ for each } f \text{ in } H \}$.

<u>Theorem 9</u>: Let the subset K of a linear space L be inverse starlike from 0. Then K^{π} is inverse starlike from 0 in $L^{\#}$. Likewise if H is inverse starlike from 0 in $L^{\#}$, then H_{π} is inverse starlike from 0 in L.

<u>Proof</u>: It is proved that K^{π} is inverse starlike from 0. It is known already that 0 is in K^{π} and that K^{π} is convex (see Day [6], page 17). Let f belong to K^{π} and $\alpha \geq 1$. Then $(\alpha f)(x) = \alpha f(x) = f(\alpha x) \geq -1$ for each x in K since αx is in K. Therefore K^{π} is inverse starlike from 0.

A similar argument may be used to show that H_{π} is inverse starlike from 0.

A set which is convex, inverse starlike from 0, and contains 0 is clearly a convex cone. This establishes the following corollary.

Corollary 1: The sets K^{π} and H_{π} are convex cones with vertices at 0 in the spaces $L^{\#}$ and L, respectively.

A theorem of Day [6], page 20, may be used to obtain a second corollary.

Corollary 2: Let K be inverse starlike from 0 in the locally convex linear topological space L. Then $(K^{\pi})_{\pi}$ is the smallest weakly-closed cone in L containing K.

The term smallest is used here in the sense that if C is any weakly-closed cone in L containing K, then C also contains $(K^{\pi})_{\pi}$. A similar result for $(H_{\pi})^{\pi}$ follows also for an inverse starlike subset H of $L^{\#}$.

The next three theorems relate inverse starlike sets and flats. The first result characterizes flats in terms of inverse starlike sets, and is motivated by the theorem of Hare [8] in which he proved that a set S is flat if, and only if, $S \cap K$ is projectively convex for every projectively convex set K.

Theorem 10: Let S be contained in L and suppose that if K is inverse starlike from a, then $S \cap K$ is also inverse starlike from a (K and a not fixed). Then S is flat. Conversely, if S is flat and a belongs to S, then $S \cap K$ is inverse starlike from a.

Proof: First assume that $S \cap K$ is inverse starlike for each inverse starlike set K in L and show that S is flat. Let x and y belong to S; then it must be shown that $\infty xy \cup \infty yx \cup xy$ is contained in S. Now ∞xy is inverse starlike from y which implies that $\infty xy \cap S$ is inverse starlike from y. Also x is in $\infty xy \cap S$ so that ∞xy is contained in $\infty xy \cap S$ and hence in S. Likewise it may be shown that ∞yx is contained in S. It remains to be proved that xy is contained in S. Let w be a point on ∞xy. Then ∞xw is inverse starlike from w and hence by hypothesis $\infty xw \cap S$ is inverse starlike from w; but x is in $\infty xw \cap S$ so that ∞xw is contained in S. Furthermore xy is contained in ∞xw since if $z = \alpha x + (1 - \alpha)y$ is on xy with $0 < \alpha < 1$ and $w = \sigma x + (1 - \sigma)y$ with $\sigma > 1$, then $z = \beta x + (1 - \beta)w$ where $\beta = (\sigma - \alpha)/(\sigma - 1) > 1$, which implies that z is on ∞xw. Thus xy is contained in S and it has been proved that S is flat.

Conversely, let S be flat, a belong to S, K be inverse starlike from a, and show that $S \cap K$ is inverse starlike from a. Let x be in $S \cap K$. Then ∞xa is contained in S since x and a are in S and L(x,a) is contained in S. Also ∞xa is contained in K since K is inverse starlike from a. Hence ∞xa is contained in $S \cap K$ which proves that $S \cap K$ is inverse starlike from a.

Theorem 11: Let the subset K of L be inverse starlike from 0. Then 0 is a core point of K if, and only if, K = L.

Proof: Clearly if K = L, then 0 is a core point of K. Hence assume that 0 is a core point of K and show K = L. Already K is contained in L so that it must be shown that L is contained in K. Let y belong to L. Then there is a positive number $\epsilon(y)$ so that $0 + \lambda y$ is in K for $|\lambda| < \epsilon(y)$. Indeed one may choose $0 < \lambda < 1$ and have $1/\lambda > 1$. Then K is inverse starlike from 0 implies that $(1/\lambda)(\lambda y) = y$ is in K which proves that L is a subset of K. Therefore K = L.

The core and the interior of a set are generally different. The following example in the plane (see Figure 1) shows that this is true also for inverse starlike sets. The set S is indicated by the shaded portion of the plane and S is inverse starlike from (0,0). The point (4,4), which is the point at which the circle is tangent to y = x, is a core point but not an interior point of S.

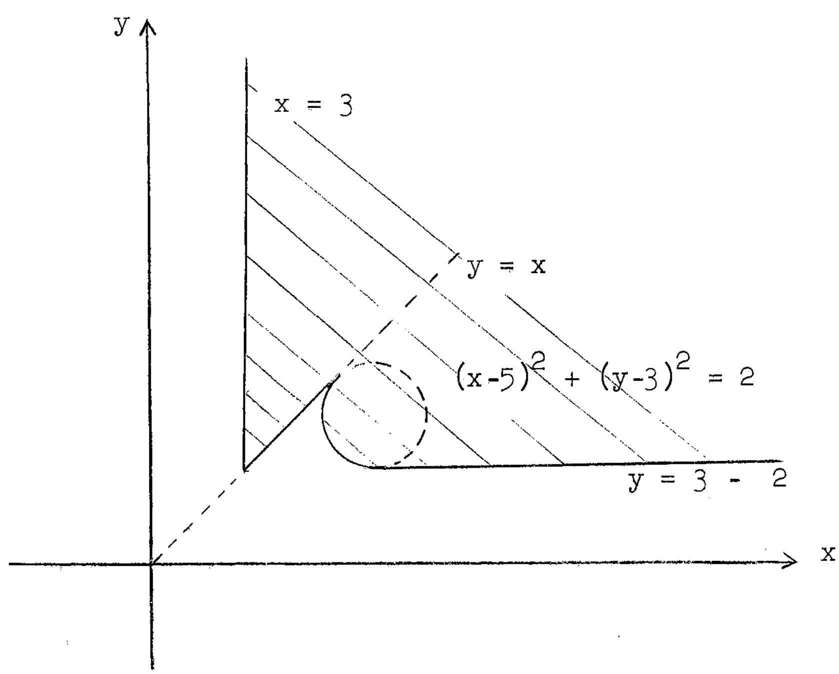

Figure 1

The corollary to Theorem 3 and the fact that the core is preserved under translation prove the following more general result.

Corollary: Let K be inverse starlike from a in L. Then a is a core point of K if, and only if, K = L.

Theorem 12: Let the subset K of L be inverse starlike from 0 and be symmetric. Then k(K) is a subspace of L.

Proof: If x is in K, then -x is in K and $(1/2)x + (1/2)(-x) = 0$ must belong to k(K). As has been observed before for a set which is convex, inverse starlike, and contains 0, it must follow that k(K) is a convex cone. Thus $k(K) + k(K)$ and $\lambda k(K)$ are contained in k(K) for $\lambda \geq 0$. But K is symmetric; hence if x is in k(K) and $\lambda < 0$, then -x is in k(K) and $-\lambda > 0$ so that $\lambda x = (-\lambda)(-x)$ is in k(K). It has been proved that k(K) is closed under scalar multiplication and sums and is therefore a subspace of L.

A subset S of L is called mid-point convex if for each pair of elements x and y of S, then $(1/2)x + (1/2)y$ is also in S. Obviously a set which is convex is also mid-point convex; a set which is convex is also starlike from each point of the set. Hence the notions of mid-point convex and starlike are generalizations of convexity. A subset S of L satisfies condition X from the point a of L provided that for each x in S there is a number $\alpha > 1$ so that $\alpha x + (1 - \alpha)a$ is in S. Again it is clear that if S is inverse starlike from a, then S satisfies condition X from a, so that condition X is a generalization of the inverse starlike property. The next group of theorems relates the properties of convex, mid-point convex, starlike, condition X, and inverse starlike.

Theorem 13: Let the subset S of L be starlike from 0 and assume that for each line P through 0, $P \cap S$ is an open segment. Then S satisfies condition X from 0.

Proof: Let x belong to S and let P be the line through 0 and x. Then $P \cap S$ is an open segment containing 0 and x. Therefore there is a number $\lambda > 1$ so that λx is in S. Hence S satisfies condition X from 0.

A set S is called linearly closed if for every line P, $P \cap S$ is closed in the line topology for P.

Theorem 14: Let the subset S of L be linearly closed and mid-point convex. Then S is convex.

Proof: Using the hypothesis that S is mid-point convex, one can prove that for each x and y in S, $(r/2^n)x + (1 - r/2^n)y$ is in S for all positive integers r and n and for which $0 \leq r \leq 2^n$. Since the set of such numbers $r/2^n$ is dense in the interval $[0,1]$ and since the line topology for L(x,y) is equivalent to the real line topology, it follows that the set of points $(r/2^n)x + (1 - r/2^n)y$ is dense in the interval xy. Thus since S is linearly closed, $xy \subset S$ and hence S is convex.

The following modification of Theorem 14 can be proved by using the same argument as above.

Theorem 15: Let the subset S of L contain 0, be linearly closed for lines passing through 0, and assume that for each line P through 0, it is true that $P \cap S$ is mid-point convex. Then S is starlike from 0.

Theorem 16: Let the subset S of L be such that for every line P through 0, $P \cap S$ is linearly closed and mid-point convex. Assume further that S satisfies condition X from 0. Then S is inverse starlike from 0.

Proof: Let x belong to S and $\alpha > 1$; then it must be shown that αx is in S. Let P be the line through 0 and x. Since S satisfies condition X, there is a number $\lambda > 1$ so that λx is in S. If $\lambda = \alpha$, then αx is in S. If $\lambda > \alpha$, then as in Theorem 14 (since $P \cap S$ is linearly closed and mid-point convex), $x(\lambda x)$ must be contained in S and hence αx in S since αx is on $x(\lambda x)$. Now assume that $\lambda < \alpha$ and that for each $\beta \geq \alpha$, βx is not in S. Then $N = \{\lambda : \lambda > 1, \lambda x \in S\}$ is not empty and thus $\lambda' = \sup \{\lambda : \lambda > 1, \lambda x \in S\}$ is finite since N is bounded above by α. But $P \cap S$ is linearly closed so that $\lambda' x$ is in S and thus there is a number $\lambda'' > 1$ so that $\lambda''(\lambda' x)$ is in S and $\lambda''\lambda' > \lambda'$ contradicting that $\lambda' = \sup N$. Therefore there is a number $\beta \geq \alpha$ so that βx is in S and the case for $\lambda > \alpha$ occurs again. In any case it has been proved that αx is in S and thus S is inverse starlike from 0.

Corollary: If in addition to the hypothesis of Theorem 16, it is assumed that 0 is in S, then S is a cone.

Theorem 17: Let the subset K of L be convex. Then $K + K \subset K$ if, and only if, K is inverse starlike from 0.

Proof: Assume that $K + K \subset K$ and show that K is inverse starlike from 0. Let x belong to K and $\alpha \geq 1$. Since $K + K \subset K$, nx is in K for $n = 1, 2, \cdots$. Let $n_1 < \alpha < n_2$ where n_1 and n_2 are positive integers. Then $n_1 x$ and $n_2 x$ are in K and K convex implies that $\frac{\alpha - n_1}{n_2 - n_1} n_2 x + (1 - \frac{\alpha - n_1}{n_2 - n_1}) n_1 x = \alpha x$ is in K. Thus K is inverse starlike from 0.

Next assume that K is inverse starlike and show $K + K \subset K$. Let x and y be in K. Then $(1/2)x + (1/2)y$ is in K since K is convex and $2\left[(1/2)x + (1/2)y\right] = x + y$ is in K since K is inverse starlike. Therefore $K + K \subset K$.

Corollary: If K is starlike from 0 and $K + K \subset K$, then K is a convex cone with vertex at 0.

Theorem 18: Let the subset K of L containing 0 be inverse starlike from 0 and mid-point convex. Then K is a convex cone.

Proof: First it is proved that K + K is contained in K. Let x and y belong to K. Then $(1/2)K + (1/2)K \subset K$ implies that $(1/2)x + (1/2)y$ is in K. Hence $2\left[(1/2)x + (1/2)y\right] = x + y$ is in K since K is inverse starlike. Therefore K + K is contained in K.

Next it is proved that λK is contained in K for $\lambda \geq 0$. If $\lambda \geq 1$, then $\lambda K \subset K$ since K is inverse starlike from 0. Hence consider $0 < \lambda < 1$. There is an integer n so that $1/2^n \leq \lambda$. If x is in K, then $(1/2^n)x$ is in K since K is mid-point convex and 0 is in K. Also there exists a number $\alpha \geq 1$ so that $\alpha(1/2^n) = \lambda$ and hence $\alpha(1/2^n)x = \lambda x$ is in K since K is inverse starlike from 0. Therefore K is a convex cone.

A real bilinear transformation (see Taylor [13], page 322) on $L \times L$ is a mapping B of $L \times L$ into the real numbers so that $B(\alpha x + \beta y, z) = \alpha B(x,z) + \beta B(y,z)$ and $B(x, \alpha y + \beta z) = \alpha B(x,y) + \beta B(x,z)$ for all real numbers α and β and all x, y, and z in L. Associated with each bilinear form B is the quadratic form Q defined by $Q(x) = B(x,x)$ for each x in L. Hare [8] has proved that the set of points at which a quadratic form is positive (non-negative, non-positive, negative) is a projectively convex set. Certain subsets of L defined by a quadratic form are inverse starlike.

Theorem 19: Let Q be the real quadratic form associated with the bilinear form B. The set $K = \{x \in L : Q(x) \geq 0\}$ is inverse starlike from each point a of L for which $Q(a) \geq 0$ and $B(x,a) \leq 0$ for each x in K.

Proof: First observe that a real bilinear form is symmetric so that $B(x,a) = B(a,x)$. Let x be in K and $\alpha \geq 1$; let a have the property that $Q(a) \geq 0$ and $B(x,a) \leq 0$ for all x in K. Then $Q(\alpha x + (1 - \alpha)a) = B(\alpha x + (1 - \alpha)a, \alpha x + (1 - \alpha)a) = \alpha^2 B(x,x) + (1 - \alpha)^2 B(a,a) + 2\alpha(1 - \alpha)B(x,a) = \alpha^2 Q(x) + (1 - \alpha)^2 Q(a) + 2\alpha(1 - \alpha)B(x,a)$ which is greater than or equal to 0. Thus K is inverse starlike from a.

Hare's result and Theorem 19 naturally raise the question that if a set is projectively convex and not convex, is it inverse starlike from some point? The following figure gives an example in the plane of a projectively convex set which is not inverse starlike from any point.

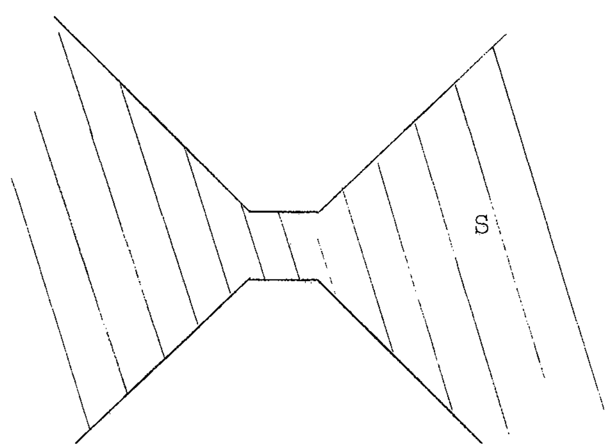

Figure 2

The quadratic form Q is called a positive quadratic form provided that $Q(x) \geq 0$ for all x in L. One is led then to investigate the set $S = \{x \in L : Q(x) = 0\}$ and obtain the following result.

<u>Theorem 20</u>: Let Q be the quadratic form associated with the bilinear form B and assume that Q is positive. Then $S = \{x \in L : Q(x) = 0\}$ is a subspace of L.

<u>Proof</u>: Observe that $Q(0) = 0$ so that 0 is in S. Let x and y be in S and let α and β be any real numbers. Then $Q(\alpha x + \beta y) = 2\alpha\beta B(x,y)$ since $Q(x) = Q(y) = 0$. Now since Q is positive, $Q(\alpha x + \beta y) \geq 0$ for all choices of x, y, α, and β, but $2\alpha\beta B(x,y)$ may be positive or negative depending on the choice of α and β which implies that $B(x,y) = 0$ for all x and y in S. Thus $Q(\alpha x + \beta y) = 0$ and hence S is a subspace of L.

A positive definite quadratic form is a positive quadratic form Q for which $Q(x) = 0$ implies that $x = 0$. One may consider certain sets associated with a positive definite quadratic form. For example, $\{x \in L : Q(x) \leq \alpha\}$ or $\{x \in L : Q(x) \geq \alpha\}$ or $\{x \in L : \alpha \leq Q(x) \leq \beta\}$ are sets associated with Q. So far as the author is able to determine, the properties that these sets have is unsolved.

CHAPTER II

THE INVERSE STAR ENVELOPE OF A SET

The convex hull of a subset S of the linear space L is defined to be the intersection of all convex subsets of L containing S; it is the smallest convex set containing S. Stoker [12] uses the notion of characteristic cone of a set, which is the smallest cone with a given vertex which contains the set. The purpose of this chapter is to define the inverse star envelope of a set, that is, the smallest inverse starlike set (from a given point) which contains the set, and to develop the properties of this envelope. It will be observed that many of the properties of the inverse star envelope of a set S are determined by properties of S. Many analogous results may be obtained for the star envelope of a given set, the star envelope being the smallest starlike set (from some given point) which contains the set.

The following definitions for the star envelope and the inverse star envelope will be used. The <u>star envelope</u> of a set S from the point a is denoted by Sa and is defined by $Sa = \bigcup_{x \in S} xa$. (This notation will not be confusing if one remembers that lower case Greek letters are used for scalars and lower case Latin letters for elements of L.) The <u>inverse star envelope</u> of a set S from the point a is denoted by ∞Sa and is defined by $\infty Sa = \bigcup_{x \in S} \infty xa$. Clearly the star envelope of S is starlike from a and the inverse star envelope of S is inverse starlike from a; also these are the smallest (in the sense of set inclusion) such sets containing S.

The first group of theorems gives results concerning set operations in L for the inverse star envelope.

Theorem 21: If $A \subset B$, then $\infty Aa \subset \infty Ba$.

Proof: Let x be in ∞Aa. Then there is an element y of A so that x is on ∞ya. But $A \subset B$ implies that y is in B and hence $\infty ya \subset \infty Ba$. Thus x belongs to ∞Ba and it follows that $\infty Aa \subset \infty Ba$.

Theorem 22: Let S_α be a subset of L for each α in some index set Δ. Then $\infty(\bigcup_\alpha S_\alpha)a = \bigcup_\alpha \infty S_\alpha a$ and $\infty(\bigcap_\alpha S_\alpha)a \subset \bigcap_\alpha(\infty S_\alpha a)$.

Proof: First consider the union of the sets S_α and let x belong to $\infty(\bigcup_\alpha S_\alpha)a$. Hence x is on ∞sa for some element s of $\bigcup_\alpha S_\alpha$. But then s must belong to S_β for some β in Δ and ∞sa is contained in $\infty S_\beta a$ which implies that x is in $\infty S_\beta a$. Thus x belongs to $\bigcup_\alpha \infty S_\alpha a$ and it follows that $\infty(\bigcup_\alpha S_\alpha)a \subset \bigcup_\alpha (\infty S_\alpha a)$. Next show inclusion the other way. Let y belong to $\bigcup_\alpha (\infty S_\alpha a)$. Then y belongs to $\infty S_\beta a$ for some β in Δ and hence there is an element z of S_β so that y is on ∞za. Now z is in $\bigcup_\alpha S_\alpha$ which implies that $\infty za \subset \infty(\bigcup_\alpha S_\alpha)a$ and thus that y belongs to $\infty(\bigcup_\alpha S_\alpha)a$. Therefore it has been proved that $\infty(\bigcup_\alpha S_\alpha)a = \bigcup_\alpha (\infty S_\alpha a)$.

Next consider the intersection of the sets S_α. Clearly $\bigcap_\alpha S_\alpha \subset S_\alpha \subset \infty S_\alpha a$ for all α in Δ. Hence by Theorem 21, $\infty(\bigcap_\alpha S_\alpha)a \subset \infty S_\alpha a$ for all α in Δ and it must then follow that $\infty(\bigcap_\alpha S_\alpha)a \subset \bigcap_\alpha \infty S_\alpha a$.

An example in E^1 which shows in the latter case that inclusion may be proper is the following. Let $S_n = [0, 1/n]$ for $n = 1, 2, \cdots$ and let $a = 0$. Then $\bigcap_{n=1}^{\infty} S_n = \{0\}$ so that $\infty(\bigcap_{n=1}^{\infty} S_n)0 = \{0\}$ but $\infty S_n 0 = [0, \infty)$

for each positive integer n and hence $\bigcap_{n=1}^{\infty} \infty S_n 0 = [0, \infty)$.

Theorem 23: Let S be a subset of L. Then $C(\infty Sa) \subset \infty C(S)a$.

Proof: Let x belong to $C(\infty Sa)$. Then $S \subset \infty Sa$ implies that x is in $C(S)$. But $C(S)$ is contained in $\infty C(S)a$ so that x must belong to $\infty C(S)a$. Thus $C(\infty Sa) \subset \infty C(S)a$.

Set inclusion in Theorem 23 may be proper as is demonstrated by the following example in the plane. Let $S = \{(x,y) : 0 \leq x \leq 1,\ y = 1 - x\}$ and $a = (0,0)$. Then $\infty Sa = \{(x,y) : x \geq 0,\ y \geq 0,\ y \geq 1 - x\}$ so that $C(\infty Sa)$ is not even inverse starlike from a. However, $\infty C(S)a = E^2$.

Theorem 24: Let S be a subset of L. Then $\infty k(S)a = k(\infty Sa)$ and $k(S)a = k(Sa)$.

Proof: Let x belong to $\infty k(S)a$. Then there is an element y of $k(S)$ so that x is on ∞ya and hence $x = \lambda y + (1 - \lambda)a$ for some $\lambda \geq 1$. Furthermore since y is in $k(S)$, $y = \sum_{i=1}^{n} \alpha_i y_i$ where y_i is in S and $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geq 0$ for $i = 1, 2, \cdots, n$. Then $x = \lambda \sum_{i=1}^{n} \alpha_i y_i + (1 - \lambda)a = \sum_{i=1}^{n} \alpha_i (\lambda y_i + (1 - \lambda)a)$

which must belong to $k(\infty Sa)$ since $\lambda y_i + (1 - \lambda)a$ is in ∞Sa for each $i = 1, 2, \cdots, n$. Thus $\infty k(S)a \subset k(\infty Sa)$.

Next let z belong to $k(\infty Sa)$. Then $z = \sum_{i=1}^{n} \alpha_i y_i$ where $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geq 0$,

and y_i is in ∞Sa for $i = 1, 2, \cdots, n$. Furthermore $y_i = \lambda_i w_i + (1 - \lambda_i)a$ where $\lambda_i \geq 1$ and w_i is in S for each $i = 1, 2, \cdots, n$. Then $z =$

$\sum_{i=1}^{n} \alpha_i (\lambda_i w_i + (1 - \lambda_i)a) = \sum_{i=1}^{n} (\alpha_i \lambda_i w_i + \alpha_i (1 - \lambda_i)a)$. Let $\alpha = \sum_{i=1}^{n} \alpha_i \lambda_i$; then

$1 - \alpha = \sum_{i=1}^{n} \alpha_i(1 - \lambda_i)$ and hence $z = \alpha \sum_{i=1}^{n} (\alpha_i\lambda_i/\alpha)w_i + (1 - \alpha)a$ which belongs to $\infty k(S)a$ since $\alpha \geq 1$ and $\sum_{i=1}^{n} (\alpha_i\lambda_i/\alpha)w_i$ is in $k(S)$. Therefore $\infty k(S)a = k(\infty Sa)$.

A similar argument will show that $k(S)a = k(Sa)$.

The fact that the convex hull of a set is convex is used to prove the following corollaries to Theorem 24.

Corollary 1: If S is convex, then ∞Sa is also convex.

Corollary 2: If S is inverse starlike (starlike) from a, then k(S) is inverse starlike (starlike) from a.

The next group of theorems concerns the algebraic operations on sets in L.

Theorem 25: Let S be a subset of L and let α be any non-zero real number. Then $\infty(\alpha S)a = \alpha(\infty Sb)$ where $\alpha b = a$.

Proof: Let x belong to $\infty(\alpha S)a$. Then $x = \lambda(\alpha y) + (1 - \lambda)a$ where $\lambda \geq 1$ and y is in S. Then $\lambda(\alpha y) + (1 - \lambda)a = \alpha(\lambda y) + (1 - \lambda)\alpha b = \alpha(\lambda y + (1 - \lambda)b)$ which belongs to $\alpha(\infty Sb)$ since $\lambda y + (1 - \lambda)b$ is in ∞Sb. Thus $\infty(\alpha S)a \subset \alpha(\infty Sb)$.

Next let z belong to $\alpha(\infty Sb)$. Then $z = \alpha(\lambda w + (1 - \lambda)b)$ where w belongs to S and $\lambda \geq 1$. Furthermore $z = \lambda(\alpha w) + (1 - \lambda)\alpha b = \lambda(\alpha w) + (1 - \lambda)a$ which belongs to $\infty(\alpha S)a$. Therefore it has been proved that $\infty(\alpha S)a = \alpha(\infty Sb)$.

Theorem 26: Let S and T be subsets of L. Then $\infty(S + T)a \subset \infty Sb + \infty Tc$ where $b + c = a$.

Proof: Let x belong to $\infty(S + T)a$. Then $x = \lambda(s + t) + (1 - \lambda)a$ where s belongs to S, t belongs to T, and $\lambda \geq 1$. Then $x = \lambda s + \lambda t + (1 - \lambda)(b + c) = (\lambda s + (1 - \lambda)b) + (\lambda t + (1 - \lambda)c)$ which belongs to $\infty Sb + \infty Tc$. Therefore $\infty(S + T)a \subset \infty Sb + \infty Tc$.

That inclusion may indeed be proper is demonstrated by the following examples in the plane. Let $S = \{(1,0)\}$ and $T = \{(0,1)\}$ and $a = (0,0)$. Then $\infty(S + T)a$ is a half-line but $\infty Sa + \infty Ta = \{(x,y) : x \geq 1, y \geq 1\}$.

One naturally seeks conditions under which equality would hold in Theorem 26. A partial solution to this problem is given in the next theorem.

Theorem 27: Let S and T be cones with vertices at b and c, respectively. Then $\infty(S + T)(b + c) = \infty Sb + \infty Tc$.

Proof: Using the result of Theorem 26, one needs only to prove that $\infty Sb + \infty Tc \subset \infty(S + T)(b + c)$. Since S and T are cones with vertices at b and c, it follows that $\infty Sb = S$ and $\infty Tc = T$ so that $\infty Sb + \infty Tc = S + T$. Furthermore $S + T \subset \infty(S + T)(b + c)$ which implies that $\infty Sb + \infty Tc \subset \infty(S + T)(b + c)$. Therefore $\infty Sb + \infty Tc = \infty(S + T)(b + c)$.

The converse of the above theorem is not true. The following example in E^1 gives two sets S and T which are not cones but for which $\infty(S + T)2b = \infty Sb + \infty Tb$. Let $S = [0,1]$ and $T = [0,2]$. Then $S + T = [0,3]$ and $\infty(S + T)0 = \infty S0 + \infty T0 = [0, \infty)$ but S and T are not cones.

The setting for this next group of theorems is a linear topological space and it will again be observed that many properties of the set carry over to the inverse star envelope of the set.

Theorem 28: Let S be a subset of a LTS. Then $\infty S^{o}a \subset (\infty Sa)^{o}$.

Proof: Let x belong to $\infty S^{o}a$. Then $x = \lambda y + (1 - \lambda)a$ where $\lambda \geq 1$ and y is in S^{o}. Furthermore there exists a neighborhood U of y so that $U \subset S$. Also $\lambda y + (1 - \lambda)a$ belongs to $\lambda U + (1 - \lambda)a$ which is a neighborhood of $\lambda y + (1 - \lambda)a$ and is contained in ∞Sa. Therefore x belongs to $(\infty Sa)^{o}$ which proves that $\infty S^{o}a \subset (\infty Sa)^{o}$.

The fact that $S = S^{o}$ if S is an open subset of L is used to obtain the following corollary to Theorem 28.

Corollary: If S is an open subset of L, then ∞Sa is also open.

That set inclusion in Theorem 28 may indeed be proper is demonstrated by the following example in E^{1}. Let $S = \{1\}$ and $a = 0$. Then $S^{o} = \emptyset$ so that $\infty S^{o}0 = \emptyset$ but $(\infty S0)^{o} = (1, \infty)$. A sharper result, however, can be proved for the closure operation.

Theorem 29: Let S be a subset of a LTS. Then $\infty \bar{S}a = \overline{\infty Sa}$.

Proof: Let x belong to $\infty \bar{S}a$. Then $x = \lambda y + (1 - \lambda)a$ where $\lambda \geq 1$ and y is some element of $\bar{S}$. Now y in $\bar{S}$ implies that there is a sequence of points y_n $(n = 1, 2, \cdots)$ of S converging to y. Then the points $\lambda y_n + (1 - \lambda)a$ form a sequence of points of ∞Sa which converges to $\lambda y + (1 - \lambda)a = x$ and which must therefore belong to $\overline{\infty Sa}$. Thus $\infty \bar{S}a \subset \overline{\infty Sa}$.

Next let u belong to $\overline{\infty Sa}$. Then there exists a sequence of points u_n $(n = 1, 2, \cdots)$ of ∞Sa converging to u. Also $u_n = \lambda_n x_n + (1 - \lambda_n)a$ where $\lambda_n \geq 1$ and x_n belongs to S. Solving for x_n one obtains $x_n = (1/\lambda_n)u_n + (1 - 1/\lambda_n)a$. Furthermore $0 < 1/\lambda_n \leq 1$ for each positive integer n so that by the Bolzano- Weierstrass theorem there exists a subsequence, say $\{1/\lambda_{n'}\}$, which converges to some number α with $0 \leq \alpha \leq 1$. Therefore the sequence of points $x_{n'} = (1/\lambda_{n'})u_{n'} + (1 - 1/\lambda_{n'})a$ must converge to $\alpha u + (1 - \alpha)a$ which is in $\overline{S}$ since $x_{n'}$ is in S for each n'. Now then if $\alpha \neq 0$, $u = (1/\alpha)(\alpha u + (1 - \alpha)a) + (1 - 1/\alpha)a$ which implies that u belongs to $\infty \overline{S}a$ since $1/\alpha \geq 1$. If $\alpha = 0$, then the sequence $\{x_{n'}\}$ converges to a which implies that the sequence of points $u_{n'} = \lambda_{n'}x_{n'} + (1 - \lambda_{n'})a$ converges to a. Since limits of sequences in L are unique, u = a. Also the sequence $\{x_{n'}\}$ converges to a so that a is in $\overline{S}$ which implies that a = u belongs to $\infty \overline{S}a$. Therefore it has been proved that $\infty \overline{S}a = \overline{\infty Sa}$.

Corollary: If the subset S of a linear topological space is closed, then ∞Sa is also closed.

Theorem 30: Let the subset S of the LTS L be connected. Then ∞Sa is also connected.

Proof: Assume by way of contradiction that ∞Sa is not connected. Then $\infty Sa = A \cup B$ where A and B are mutually separated and non-empty. Now $\infty Sa = \bigcup_{x \in S} \infty xa$ and each ∞xa is connected so that either $\infty xa \subset A$ or $\infty xa \subset B$. Thus $A = \bigcup \infty x'a$ and $B = \bigcup \infty x''a$ where x' runs through all x' in S for which $\infty x'a \subset A$ and likewise for x". Then $S \cap A \neq \emptyset$ and $S \cap B \neq \emptyset$. Since $A \neq \emptyset$, there exists an x' in S so that $\infty x'a \subset A$. Likewise it can be proved that $S \cap B \neq \emptyset$. But now $S \subset \infty Sa$ implies that $S = (S \cap A) \cup (S \cap B)$ which are mutually separated and non-empty, contradicting that S is connected. Therefore ∞Sa is connected.

Theorem 31: Let A be a dense subset of S in the LTS L. Then ∞Aa is dense in ∞Sa.

Proof: It must be shown that every point of ∞Sa is either a point or limit point of ∞Aa. Hence let y belong to ∞Sa. Then $y = \alpha x + (1 - \alpha)a$ where x is in S and $\alpha \geq 1$. Now x in S implies that x is either a point or limit point of A. If x is in A, then y is in ∞Aa. Hence assume that x is a limit point of A and let $\{x_n\}$ be a sequence of points of A converging to x. Then the sequence $\{\alpha x_n + (1 - \alpha)a\}$ is a sequence of points of ∞Aa converging to $\alpha x + (1 - \alpha)a = y$. Therefore ∞Aa is a dense subset of ∞Sa.

It seems that perhaps a closed inverse starlike set should be the inverse star envelope of its boundary. The next theorem shows that this is true in certain cases.

Theorem 32: Let K be a closed subset of the LTS L which is inverse starlike from a and so that a is not in K. If B is the boundary of K, then $K = \infty Ba$.

Proof: Since K is closed, $B \subset K$ and by Theorem 21 it follows that $\infty Ba \subset \infty Ka = K$. Hence let x belong to K. The point a is not in K implies there exists a neighborhood U of a so that $U \cap K = \emptyset$. Consequently, since x is in K and a is not in K, ax must intersect the boundary B in at least one point p. Then $p = \alpha x + (1 - \alpha)a$ for some number $0 < \alpha \leq 1$. But

$x = (1/\alpha)p + (1 - 1/\alpha)a$ and $1/\alpha \geq 1$ implies that x belongs to ∞Ba. Therefore $K = \infty Ba$.

It is not true in general that if a set is connected, then its boundary is also connected. This statement is true for certain inverse starlike sets.

Theorem 33: Let K be a connected set which is inverse starlike from a in the LTS L and assume that a is not in K. Then the boundary B of K is connected.

Proof: Assume by way of contradiction that B is not connected and write $B = M \cup N$ where M and N are mutually separated, non-empty, and both closed since B is closed. By Theorems 32 and 22, $\bar{K} = \infty Ba = \infty Ma \cup \infty Na$. Furthermore ∞Ma and ∞Na are both closed by the corollary to Theorem 29. It will be shown next that indeed $\infty Ma \cap \infty Na = \emptyset$. Assume the contrary and let z belong to $\infty Ma \cap \infty Na$. Then $z = \alpha m + (1 - \alpha)a$ and $z = \beta n + (1 - \beta)a$ where m belongs to M, n belongs to N, $\alpha \geq 1$, and $\beta \geq 1$. It may be assumed without loss of generality that $\beta \geq \alpha$ (see Figure 3). Thus $az \cap M \neq \emptyset$ and $az \cap N \neq \emptyset$ since m is in $az \cap M$ and n is in $az \cap N$. Now either the open segment $)mn($ is contained in K^o or mn is contained in B since n is in B and $m = (\beta/\alpha)n + (1 - \beta/\alpha)a$ with $\beta/\alpha \geq 1$ which implies that $mn \subset \bar{K}$. If $)mn($ is contained in K^o and p is on $)mn($, then there exists a neighborhood U of p so that $U \subset K$. If $p = \sigma n + (1 - \sigma)a$ and $m = \epsilon n + (1 - \epsilon)a$, $\sigma > 1$ and $\epsilon > 1$, then m belongs to $(\epsilon/\sigma)U + (1 - \epsilon/\sigma)a \subset K^o$ which contradicts that m belongs to B. Also if $mn \subset B$ then M and N must have common points since they are both closed and mn is a connected subset of B containing points of the separated sets M and N. This is a contradiction that $M \cap N = \emptyset$. In either case a contradiction has been reached so that $\infty Ma \cap \infty Na = \emptyset$. But this implies that $\overline{K}$, and hence K, is not connected. But the hypothesis was that K is connected. Therefore the assumption that B is not connected is untenable and it must follow that B is connected.

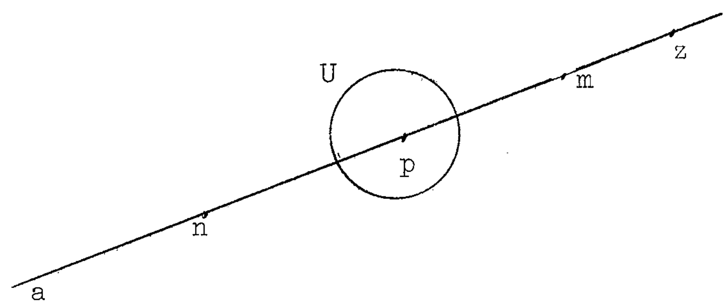

Figure 3

An inverse starlike subset of a normed linear space is unbounded. Stoker [12] made a study of unbounded convex sets in E^3 and proved that the boundary of an unbounded convex set with interior points in E^3 is empty or is homeomorphic to either two parallel planes, the surface of an infinite right circular cylinder, or a plane. If in addition it is assumed that the set is inverse starlike, the nature of the boundary may be restricted even further.

Theorem 34: Let K be a closed convex set in E^n which has interior points and is inverse starlike from the origin but does not contain the origin. Then the boundary B of K is homeomorphic to a hyperplane.

Proof: Let $(x_1, \cdots, x_n)$ be the representation for points in E^n and let $\overline{0} = (0, 0, \cdots, 0)$. It may be assumed without loss of generality that the coordinate system for E^n has been chosen so that the point $(0, \cdots, 0, \lambda)$ is in K^o with $\lambda > 0$ (see Figure 4). Let H be the hyperplane defined by $x_n = \sigma$ with $0 < \sigma < \lambda$. Let $(b_1, b_2, \cdots, b_n)$ belong to bdry(K). Define the mapping f of bdry(K) onto H by $f(b_1, b_2, \cdots, b_n) = (b_1, b_2, \cdots, b_{n-1}, \sigma)$. This mapping f and its inverse are continuous since a sphere of radius ϵ about $(b_1, b_2, \cdots, b_n)$ will project into a sphere of radius ϵ about $(b_1, \cdots, b_{n-1}, \sigma)$.

It is proved next that f is 1-1. Let $(b_1, b_2, \cdots, b_n)$ belong to bdry(K). Then since $(0, 0, \cdots, 0, \lambda)$ is in K^o, there exists a sphere U about

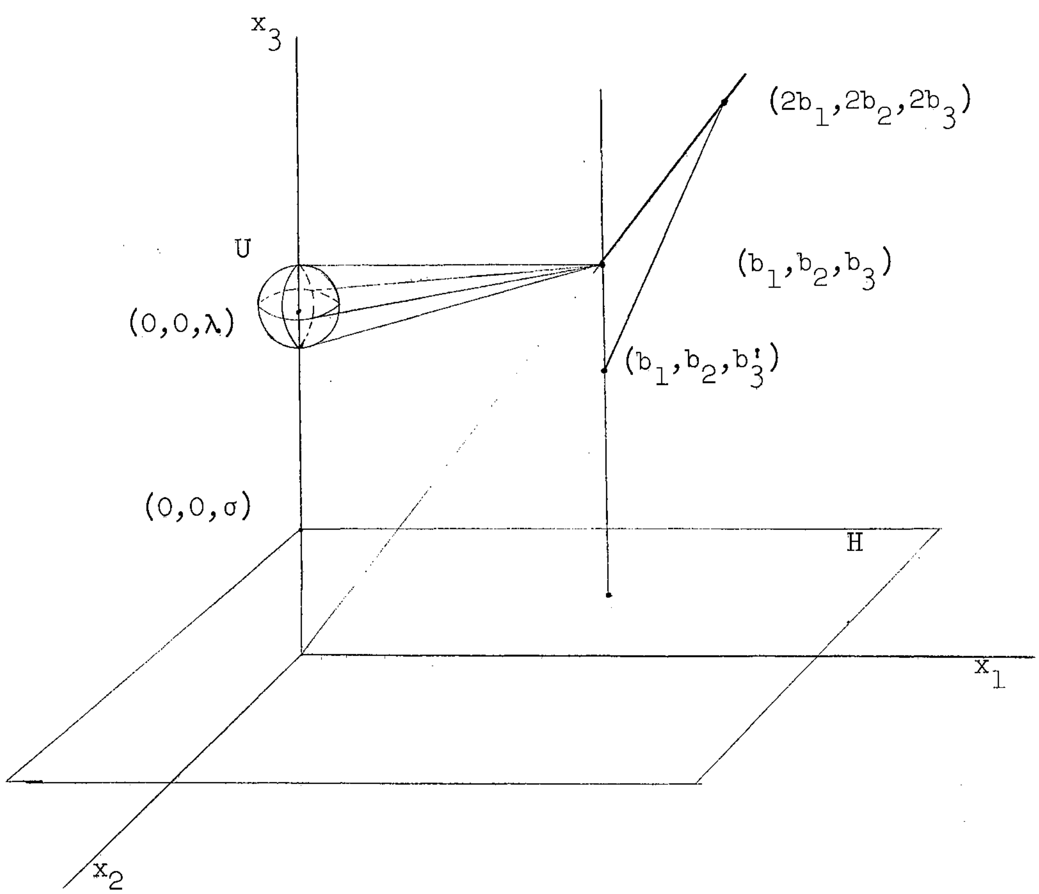

Figure 4

$(0, 0, \cdots, 0, \lambda)$ so that $U \subset K^o$. Also $S = \mathrm{co}\left[k(U \bigcup (b_1, b_2, \cdots, b_n))^o\right] \bar{0} \subset K^o$ since K is convex. Furthermore $(b_1, b_2, \cdots, b_{n-1}, \alpha b_n)$ belongs to S for all $\alpha > 1$ since the segment joining the origin to $(b_1, b_2, \cdots, b_{n-1}, \alpha b_n)$ intersects the set $k(U \bigcup (b_1, b_2, \cdots, b_n))^o \subset K^o$. Hence assume there exists a point $(b_1, b_2, \cdots, b_{n-1}, b_n')$ of bdry(K) different from $(b_1, b_2, \cdots, b_n)$ with $b_n' < b_n$. Then $T = k(U \bigcup (b_1, b_2, \cdots, b_{n-1}, b_n') \bigcup (2b_1, 2b_2, \cdots, 2b_n))^o \subset K^o$ since K is convex and furthermore $(b_1, b_2, \cdots, b_n)$ belongs to $\mathrm{co}T\bar{0}$ contradicting that $(b_1, b_2, \cdots, b_n)$ is in bdry(K). Therefore the line through $(b_1, b_2, \cdots, b_n)$ and $(b_1, b_2, \cdots, b_{n-1}, \sigma)$ intersects the bdry(K) only at one point which proves that f is 1-1.

It is proved next that f is onto H. Let $(h_1, h_2, \cdots, h_{n-1}, \sigma)$ belong to H and let $(\alpha h_1, \alpha h_2, \cdots, \alpha h_{n-1}, \alpha\sigma + (1 - \alpha)\lambda)$ be a point of the segment joining $(h_1, h_2, \cdots, h_{n-1}, \sigma)$ to $(0, 0, \cdots, 0, \lambda)$ with $0 < \alpha < 1$ and inside the sphere U. Then $(1/\alpha)(\alpha h_1, \alpha h_2, \cdots, \alpha h_{n-1}, \alpha\sigma + (1 - \alpha)\lambda) = (h_1, h_2, \cdots, h_{n-1}, \sigma + (-1 + 1/\alpha)\lambda)$ belongs to K. If the line M through the points $(h_1, h_2, \cdots, h_{n-1}, \sigma)$ and $(h_1, h_2, \cdots, h_{n-1}, \sigma + (-1 + 1/\alpha)\lambda)$ is not contained in K, then it intersects the bdry(K) at some point. Hence assume that the line M is contained in K. Then $k(M \bigcup (0, 0, \cdots, 0, \lambda)) \subset K$ since K is convex. But this implies that $\bar{0}$ is in K since $\bar{0}$ belongs to $\overline{k(M \bigcup (0, 0, \cdots, 0, \lambda))} \subset K$ contradicting that $\bar{0}$ does not belong to K. Thus the line M must intersect bdry(K) at some point which proves that f is onto H.

Therefore H is homeomorphic with bdry(K).

There are many sufficient conditions that may be imposed on a set S to insure that $\bigcup_{a \in S} \mathrm{co}Sa = L$. For example, if S has an interior point or a core point, then $\bigcup_{a \in S} \mathrm{co}Sa = L$. Also if S contains at least three points of every line through the origin, then the above equality is again satisfied since in the segment ∞xa, the three points may be varied so as to include all of the line through the collinear points. An unsolved problem is to characterize in some way, or to find a necessary and sufficient condition on, the set S so that $\bigcup_{a \in S} \infty Sa = L$. The following theorem is a result in this direction.

<u>Theorem 35</u>: The subset S of L is flat if, and only if, $\bigcup_{a \in S} \infty Sa = S$.

<u>Proof</u>: Assume first that S if flat. Then for every x in S and a in S, $L(a,x) \subset S$ and since $\infty xa \subset L(a,x)$, $\infty xa \subset S$ which implies that $\bigcup_{a \in S} \infty Sa \subset S$. Also $S \subset \infty Sa$ for each a in S so that $S \subset \bigcup_{a \in S} \infty Sa$. Therefore $\bigcup_{a \in S} \infty Sa = S$.

Next suppose that $S = \bigcup_{a \in S} \infty Sa$ and show that S is flat. Hence let x and y belong to S and prove that $L(x,y) = \infty xy \cup \infty yx \cup xy \subset S$. Since $S = \bigcup_{a \in S} \infty Sa$, $\infty xy \cup \infty yx \subset S$. It remains to be proved that xy is contained in S. Let u be on ∞xy; then u is in S so that $\infty xu \subset \bigcup_{a \in S} \infty Sa = S$. But $xy \subset \infty xu$ so that $xy \subset S$. Therefore $L(x,y) \subset S$ which implies that S is flat.

A point p of a set S is called an extreme point of S if p is contained in no open segment contained in S. The Krein-Milman theorem (see Day [6], page 78) states that a compact convex set is equal to the convex hull of its extreme points. A similar result can be proved for starlike and inverse starlike sets in terms of the star envelope and inverse star envelope using the notion of relative extreme points defined by Klee [9]. Let L denote a linear topological space and let X and Z be subsets of L. A point z in Z is said to be extreme in Z relative to X provided z does not lie in any open segment $)xz'($ determined by distinct points x of X and z' of Z. The set of all such relative extreme points will be denoted by $ex_X Z$. When Z is convex, a point z of Z is an extreme point of Z in the usual sense if, and only if, z belongs to $ex_Z Z$.

<u>Theorem 36</u>: Let S be a closed subset of the LTS L which is starlike from a and suppose S contains no ray emanating from a. Then $S = (ex_a S)a$.

<u>Proof</u>: Certainly $(ex_a S)a \subset S$ since S is starlike from a. Hence let x be in S and show x is in $(ex_a S)a$. Let $P = ax \bigcup \infty xa$ be the ray through x emanating from a. Then $P \bigcap C(S) \neq \emptyset$ since S contains no ray emanating from a. Let Q be the closure of $P \bigcap C(S)$. The set $Q \bigcap S \neq \emptyset$ since if $Q \bigcap S = \emptyset$, then $P = (S \bigcap P) \bigcup Q$ and $S \bigcap P$ is closed and Q is closed so that P is the union of separated sets contradicting that P is connected. Thus let p belong to $Q \bigcap S$. Then p is in $ex_a S$ because if p does not belong to $ex_a S$, then there is an element y of S so that p is on the open segment $)ay($ which implies that p is in the interior (with respect to the line topology) of $S \bigcap P$. But then p cannot belong to Q which contradicts that p is in $Q \bigcap S$. Now $ap \subset S$ since p is in S and S is starlike from a. Also x is on ap since if x is not on ap, then there is a number $\alpha > 1$ so that $x = \alpha p + (1 - \alpha)a$ which implies that p is on the open segment $)ax($ again contradicting that p is in Q. Hence x is on ap which proves that $S = (ex_a S)a$.

<u>Theorem 37</u>: Let S be a closed subset of the LTS L which is inverse starlike from a and suppose a belongs to C(S). Then $S = \infty(ex_a E)a$ where $E = \bigcup \{cl(ax \bigcap C(S)) : x \in S\}$ and $cl(ax \bigcap C(S))$ denotes the closure of $ax \bigcap C(S)$ in the line topology for L(a,x).

Proof: Let p be different from a and belong to $ex_a E$. Then p is not in E since if p belongs to E, p must belong to $)ax(\cap C(S)$ for some element x of S. But $)ax(\cap C(S)$ is an open segment, since S is closed and $)ax(\cap C(S)$ is non-empty, and p on this open segment contradicts that p is in $ex_a E$. Hence p is a limit point (in the line topology) of $ax \cap C(S)$ for some x in S. But then p must belong to S since S is closed so that $\infty pa \subset S$. Therefore $\infty(ex_a E)a \subset S$.

Next let x belong to S. Now $ax \cap S$ is a closed segment since S is closed. Thus $ax \cap S = px$ and it must be proved that p is in $ex_a E$. If p does not belong to $ex_a E$, then there exists an element w of E so that p is on $)aw(\subset E$. Since p is in S, w is also in S. Thus $ap \subset E$ and $aw \subset E$ with $ap = cl(ax \cap C(S))$ and $aw = cl(ax \cap C(S))$ implies that $p = w$, a contradiction. Therefore p belongs to $ex_a E$ which proves that $S = \infty(ex_a E)a$.

Theorem 37 is used to prove the following theorem for the inverse star envelope of the intersection of sets.

Theorem 38: Let A and B be non-empty closed subsets of a LTS so that $\infty Aa = \infty Ba$ and a does not belong to $A \cap B$. Then $A \cap B \neq \emptyset$ and $\infty Aa = \infty Ba = \infty(A \cap B)a$.

Proof: Since $A \cup B$ is closed and a does not belong to $A \cup B$, there exists a neighborhood U of a so that $U \cap (A \cup B) = \emptyset$. Hence $\infty Aa = \infty Ba$ is closed by the corollary to Theorem 29 and contains no ray emanating from a. Thus by Theorem 37 $\infty Aa = \bigcup \{\infty pa : p \in ex_a E\}$ where $E = \bigcup \{cl(ax \cap C(S)) : x \in S\}$.

Now $A \cap B \subset A$ implies that $\infty(A \cap B)a \subset \infty Aa$. It remains to be proved that $\mathcal{O}Aa \subset \mathcal{O}(A \cap B)a$. Hence let x belong to $\mathcal{O}Aa$. Then there exists an element p of ex_aE so that x is on $\mathcal{O}pa$. Also p belongs to $A \cap B$ since if p does not belong to $A \cap B$, then either p does not belong to A or p does not belong to B. If p does not belong to A, then since p is in $\mathcal{O}Aa$ there exists a number $0 < \alpha < 1$ so that $\alpha p + (1 - \alpha)a$ is in $A \subset \mathcal{O}Aa$ which contradicts that p is in ex_aE. A similar argument will hold if p does not belong to B. Hence p is in $A \cap B$ which implies that $A \cap B \neq \emptyset$ and further that x belongs to $\mathcal{O}(A \cap B)a$ since x is on $\mathcal{O}pa$. This completes the proof that $\mathcal{O}Aa = \mathcal{O}(A \cap B)a$.

This theorem is not true for sets A and B which are not closed and for which a belongs to $A \cup B$ as is demonstrated by the following sets in the plane. Let $A = \{(x,y) : x^2 + y^2 = r^2$ and r is a rational number with $0 \leq r \leq 1\}$ and $B = \{(x,y) : x^2 + y^2 = r^2$ and r is an irrational number with $0 < r < 1$ or $r = 0\}$. Then $\mathcal{O}Aa = \mathcal{O}Ba = E^2$ where $a = (0,0)$ but $A \cap B = \{(0,0)\}$ and clearly $\mathcal{O}(A \cap B)a \neq E^2$.

Convex sets may be characterized in the following manner: A set S is convex if, and only if, $S = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1, x_i \in S \right\}$; the index n is not fixed. Hence the convex hull of any set K may be represented by $k(K) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1, x_i \in K \right\}$ where again n may be any positive integer. The final result of this chapter gives a similar characterization for the star envelope and the inverse star envelope of a convex set.

<u>Theorem 39</u>: Let S be a convex subset of L. Then $Sa = \left\{ \sum_{i=1}^{n} \alpha_i x_i + (1 - \alpha)a : \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = \alpha \leq 1, x_i \in S \right\}$ and $\mathcal{O}Sa = \left\{ \sum_{i=1}^{n} \alpha_i x_i + (1 - \alpha)a : \right.$

$\alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = \alpha \geq 1, \ x_i \in S \}$.

Proof: Only the proof for ∞Sa is given; the proof for Sa is very similar and will be omitted. The first argument demonstrates that the set K = $\{ \sum_{i=1}^{n} \alpha_i x_i + (1 - \alpha)a : \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = \alpha \geq 1, \ x_i \in S \}$ is inverse starlike from a. Let $y = \sum_{i=1}^{n} \alpha_i x_i + (1 - \alpha)a$ belong to K and let $\beta \geq 1$. Then $\beta y + (1 - \beta)a = \sum_{i=1}^{n} \beta\alpha_i x_i + \beta(1 - \alpha)a + (1 - \beta)a = \sum_{i=1}^{n} \beta\alpha_i x_i + (1 - \beta\alpha)a$ which belongs to K. Hence K is inverse starlike from a.

Since ∞Sa is the smallest inverse starlike set containing S and since K is an inverse starlike set containing S, it follows that $\infty Sa \subset K$. The next argument demonstrates that $K \subset \infty Sa$. Let $y = \sum_{i=1}^{n} \alpha_i x_i + (1 - \alpha)a$ belong to K. Then $y = (\alpha y)/\alpha = \alpha \sum_{i=1}^{n} (\alpha_i/\alpha)x_i + (1 - \alpha)a$ belongs to ∞Sa since $\sum_{i=1}^{n} (\alpha_i/\alpha)x_i$ belongs to S and S is convex. Thus K = ∞Sa.

CHAPTER III

A METRIC SPACE OF STARLIKE SETS

In order to prove the Blaschke selection theorem, which asserts that the class of closed convex subsets of a closed bounded convex set of E^n can be made into a compact metric space, Eggleston [7] (page 59) develops a metric space of bounded convex sets. The purpose of this chapter is to develop in a similar fashion a metric space of closed bounded starlike sets, which will, of course, include all closed bounded convex sets. A theorem for starlike sets analogous to the Blaschke selection theorem for convex sets cannot be proved in the same manner, however, since not all the theorems leading up to the Blaschke selection theorem carry over to starlike sets.

The setting for the results of this chapter is a normed linear space L where the norm of an element x of L is denoted by $\|x\|$ and the distance from x to y is $\|x - y\|$.

Let S(R) denote a sphere of radius R whose center is the origin and let $\mathfrak{A}$ denote the class of closed sets contained in S(R). The distance from a point x to a set Y is defined by $\rho(x,Y) = \inf \left\{ \|x - y\| : y \in Y \right\}$ and a set of the form $U(Y,\sigma) = \left\{ x : \rho(x,Y) < \sigma \right\}$ is called a σ-neighborhood of Y.

The metric for $\mathfrak{A}$ is defined as follows. Let X_1 and X_2 be elements of $\mathfrak{A}$ and let σ_1 be the greatest lower bound of positive numbers σ such that $U(X_1,\sigma) \supset X_2$ and let σ_2 be the greatest lower bound of positive numbers σ such that $U(X_2,\sigma) \supset X_1$. The distance between X_1 and X_2 is $\Delta(X_1,X_2) = \sigma_1 + \sigma_2$. It can be proved that this distance function satisfies all the conditions for a metric.

A sequence $\{X_i\}$ of members of $\mathfrak{a}$ is said to converge to a member X of $\mathfrak{a}$ (or X_i tends to X) provided that $\Delta(X_i,X)$ tends to 0 as i tends to ∞.

The results of this chapter are concerned with the subclass $\mathcal{S}$ of $\mathfrak{a}$ of all closed subsets of S(R) which are starlike from some point of S(R).

<u>Theorem 40</u>: In the metric space $\mathfrak{a}$ if X_i tends to X, X_i is starlike from a_i, and a_i tends to a, then X is starlike from a.

<u>Proof</u>: If X is not starlike from a, then there exists an element x of X so that ax is not contained in X. Hence there exists an element x_o of ax which is not in X. The set X is closed which implies that there exists a neighborhood $U(x_o,\sigma)$ so that $U(x_o,\sigma)\cap X = \emptyset$. Since X_i tends to X and a_i tends to a, it is possible to choose i large enough so that $\|a_i - a\| < \sigma/2$ and $\Delta(X_i,X) < \sigma/2$. Then there is an element x' of X_i so that $\|x' - x\| < \sigma/2$. Hence there exists an element x'_o of $a_i x'$ so that $\|x_o - x'_o\| < \sigma/2$. But then also for each y of X, $\|y - x'_o\| > \sigma/2$ since x'_o is in $U(x_o,\sigma)$ and this contradicts that $\Delta(X_i,X) < \sigma/2$. Therefore the assumption that X is not starlike from a leads to a contradiction which proves that X is starlike from a.

Let $\mathcal{S}'$ be the class of all elements S of $\mathcal{S}$ for which $S^o \neq \emptyset$.

<u>Theorem 41</u>: Let X_i tend to X where X_i and X belong to $\mathcal{S}'$, with X_i starlike from a_i which tends to a. Let D be a member of $\mathcal{S}'$ which is also starlike from a and assume a belongs to $(X\cap D)^o$. Then $X_i\cap D$ tends to $X\cap D$.

Proof: It is proved that for every positive number ϵ there exist integers M and N so that

(1) $U(X\cap D,\epsilon)\supset X_i\cap D$ for each integer $i\geq M$ and

(2) $U(X_i\cap D,\epsilon)\supset X\cap D$ for each $i\geq N$.

Suppose by way of contradiction that (1) is false. Then there exist an $\epsilon>0$, a sequence of positive integers i_j tending to ∞, and a sequence $\{p_j\}$ so that p_j belongs to $X_{i_j}\cap D$ and for every j and for every x in $X\cap D$, $\| x - p_j \| > \epsilon$. There exists a subsequence of $\{p_j\}$ which converges and it may be assumed that p_j tends to p without loss of generality. Furthermore p is in D since D is closed and each p_j belongs to D. Since X_i tends to X and p_j belongs to X_{i_j}, it follows that p belongs to X and hence that p belongs to $X\cap D$. But for the given ϵ there exists a sufficiently large integer j so that $\| p_j - p \| < \epsilon$ which is a contradiction since for every x in $X\cap D$, $\| x - p_j \| > \epsilon$. Therefore (1) is true.

Suppose by way of contradiction that (2) is false. Then there exist a sequence of positive integers i_j tending to ∞ and a sequence of points p_j of $X\cap D$ so that for every x in $X_{i_j}\cap D$, $\| x - p_j \| > \epsilon$ for $j = 1, 2, \cdots$. Again it may be assumed without loss of generality that p_j tends to p which belongs to $X\cap D$, since $X\cap D$ is compact. By hypothesis a belongs to $(X\cap D)^{o}$. If p = a, then one may choose another point q of $(X\cap D)^{o}$ and proceed with the same argument. Now let r be on ap and be such that $\| p - r \| < \epsilon/2$ (see Figure 5). The point r may be chosen so that r belongs to D^{o} since a belongs to D^{o}; also r belongs to X. Then there exists a positive number σ so that $U(r,\sigma)\subset D$. There exists a positive integer N_1 so that for every integer $i\geq N_1$, $\Delta(X_i,X)<\sigma$ and thus if $i_j\geq N_1$, there exists an element r_j of X_{i_j} so that $\| r_j - r \| < \sigma$. Now r_j belongs to $X_{i_j}\cap D$ since $\| r_j - r \| < \sigma$ and

$U(r,\sigma) \subset D$. Furthermore $\| p - r_j \| < \sigma + \epsilon/2 < 3\epsilon/4$ since $U(r,\sigma) \subset U(p,\epsilon)$. A positive integer M may be chosen large enough so that for every $j \geq M$, $\| p_j - p \| < \epsilon/4$. Then for an integer j satisfying $j \geq M$ and $i_j \geq N_1$, $\| p_j - r_j \| < \epsilon$ since $\| p_j - r_j \| = \| p_j - p + p - r_j \| \leq \| p_j - p \| + \| p - r_j \| < \epsilon$. But this contradicts that for every x in X_{i_j}, $\| x - p_j \| > \epsilon$. Therefore (2) must be true.

The theorem now follows from the definition of the metric for $\mathcal{A}$.

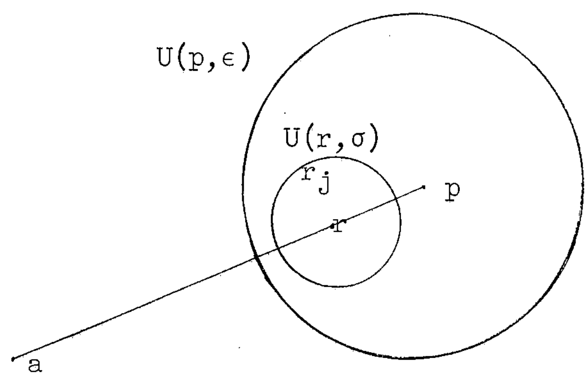

Figure 5

The σ-neighborhood of any convex subset of a normed linear space is again convex. The final theorem of this chapter shows that this is also true for starlike sets.

Theorem 42: If the subset S of a normed linear space is starlike from a and $\sigma > 0$, then $U(S,\sigma)$ is starlike from a.

Proof: Let x belong to $U(S,\sigma)$; it must be proved that ax is contained in $U(S,\sigma)$. Since x is in $U(S,\sigma)$, there exists an element y of S so that $\| x - y \| < \sigma$, and y in S implies that ay is contained in S. Let x' =

$\alpha x + (1 - \alpha)a$ be on ax $(0 \leq \alpha \leq 1)$ and for the same value of α let $y' = \alpha y + (1 - \alpha)a$. Then $\| x' - y' \| = \| \alpha x + (1 - \alpha)a - \alpha y - (1 - \alpha)a \| = \| \alpha(x - y) \| = \alpha \|x - y\| < \alpha\sigma < \sigma$ which implies that x' belongs to $U(S,\sigma)$. Therefore $U(S,\sigma)$ is starlike from a.

The σ-neighborhood of an inverse starlike set need not be inverse starlike and the same is true for projectively convex sets. The following example in the plane is projectively convex and inverse starlike from the origin but no σ-neighborhood of it has either of these properties. Let $S = \left\{ (x,0) \; ; \; |x| \geq 1 \right\}$.

CHAPTER IV

A SEMIGROUP AND A LATTICE OF STARLIKE SETS

In Theorem 3 it was proved that if H is inverse starlike from a and K is inverse starlike from b, then H + K is inverse starlike from a + b. A similar result holds for starlike sets. The first purpose of this chapter is to develop a commutative semigroup of starlike sets with an identity, a non-trivial set of units, a separation theorem, and a restricted cancellation law. Finally a complete complemented lattice of starlike sets is obtained. Most of the results of this chapter hold also for inverse starlike sets; this is not true, however, for the separation theorem and the cancellation law.

Let $\bar{\mathcal{S}}$ be the class of all starlike subsets of a linear space L and let $\mathcal{S}$ be the subclass of $\bar{\mathcal{S}} \times L$ consisting of all ordered pairs (S,a) for which S is starlike from a. Define $(S,a) \sim (S,b)$ if S is starlike from both a and b. This $\sim$ is an equivalence relation since it is easily verified that it is reflexive, symmetric, and transitive. Hence $\sim$ partitions $\mathcal{S}$ into equivalence classes which are denoted by $(S,a)^*$, and the collection of all such equivalence classes is denoted by $\mathcal{S}^*$.

A sum operation + is defined in $\mathcal{S}^*$ by $(S_1,a)^* + (S_2,b)^* = (S_1 + S_2, a + b)^*$. The addition operation is well-defined since if (S_1,a) and (S_1,a') are any two representatives of $(S_1,a)^*$ and if (S_2,b) and (S_2,b') are any two representatives of $(S_2,b)^*$, then $(S_1 + S_2, a + b)^* = (S_1 + S_2, a' + b')^*$ since $S_1 + S_2$ is starlike from both a + b and a' + b'. The collection $\mathcal{S}^*$ is closed under the addition operation and furthermore it is easily shown to be associative and commutative, and that $(0,0)^*$ is the additive identity.

The group of units for $\mathcal{S}^*$ is the set of singletons $(a,a)^*$ and $(a,a)^* + (-a,-a)^* = (0,0)^*$. The linear space L is hence isomorphic to the group of units of $\mathcal{S}^*$.

If S is starlike from a and λ is real, then λS is starlike from λa, which may be proved similarly to Theorem 3. This result suggests the following definition for a scalar product in $\mathcal{S}^*$. Let $(S,a)^*$ belong to $\mathcal{S}^*$ and λ be real; define $\lambda(S,a)^* = (\lambda S,\lambda a)^*$. This operation is well-defined since if (S,a) and (S,a') are two representatives from $(S,a)^*$, then $(\lambda S,\lambda a)^* = (\lambda S,\lambda a')^*$ since λS is starlike from both λa and $\lambda a'$. This operation also enjoys the usual properties for a scalar product in a vector space:

$$\alpha\left[(S,a)^* + (T,b)^*\right] = \alpha(S,a)^* + \alpha(T,b)^*$$

$$(\alpha + \beta)(S,a)^* = \alpha(S,a)^* + \beta(S,a)^*$$

$$(\alpha\beta)(S,a)^* = \alpha\left[\beta(S,a)^*\right]$$

$$1(S,a)^* = (S,a)^*.$$

It seems desirable to make the convention that the empty set $\emptyset$ is starlike from no point and to define $(\emptyset,\emptyset)^* = \left\{(\emptyset,\emptyset)\right\}$, $(\emptyset,\emptyset)^* + (S,b)^* = (\emptyset,\emptyset)^*$, and $\lambda(\emptyset,\emptyset)^* = (\emptyset,\emptyset)^*$. These definitions are consistent with the above properties of addition and scalar multiplication.

Hence $\mathcal{S}^*$ is a commutative semigroup under addition with an identity, a non-trivial set of units, and $\mathcal{S}^*$ is furnished with a scalar multiplication. With the operations of addition and scalar multiplication thus defined, $\mathcal{S}^*$ satisfies all the requirements for a vector space except for additive inverses.

<u>Theorem 43</u>: The space $\mathcal{S}^*$ is homomorphic to the subclass $\mathcal{S}_0^* = \left\{(S,0)\right\}$.

Proof: The homomorphism h of $\mathcal{A}^*$ to $\mathcal{A}_0^*$ is defined by $h\left[(S,a)^*\right] = (S-a,0)^*$. It must be proved that the function thus defined is indeed a homomorphism. If $(S,a)^*$ and $(T,b)^*$ are in $\mathcal{A}^*$, then $h\left[(S,a)^* + (T,b)^*\right] = h\left[(S+T,a+b)^*\right] = (S+T-a-b,0)^* = (S-a,0)^* + (T-b,0)^* = h\left[(S,a)^*\right] + h\left[(T,b)^*\right]$; and if α is real, then $h\left[\alpha(S,a)^*\right] = h\left[(\alpha S,\alpha a)^*\right] = (\alpha S-\alpha a,0)^* = \alpha(S-a,0)^* = \alpha h\left[(S,a)^*\right]$. Therefore h is a homomorphism.

The class $\mathcal{A}_0^*$ has the same properties as $\mathcal{A}^*$ and furthermore is closed under the operations of addition and scalar multiplication. Hence $\mathcal{A}_0^*$ is a substructure of $\mathcal{A}^*$.

The cancellation law for addition does not hold even for very simple starlike sets as the following examples demonstrate. Let $S_1 = \{(0,y) : 0 \leq y < \infty\} \cup \{(x,0) : 0 \leq x < \infty\}$, $S_2 = \{(x,0) : 0 \leq x < \infty\}$, and $S_3 = \{(1,y) : 1 \leq y < \infty\} \cup \{(x,1) : 1 \leq x \leq 2\}$. Then $S_1 + S_3 = \{(x,y) : 1 \leq x < \infty, 1 \leq y < \infty\} = S_2 + S_3$, but $S_1 \neq S_2$. However, a restricted cancellation law can be proved using the following separation theorem. The setting for this theorem is E^n.

Theorem 44: (Separation Theorem) Let S be a closed subset of E^n which is starlike from a and let p belong to C(S). Then there exists a set H containing p which is homeomorphic to a hyperplane and so that $H \cap S = \emptyset$.

Proof: If p does not belong to k(S), then the separation theorem for closed convex sets (see Day [6], page 22) assures the existence of such a hyperplane.

If p belongs to k(S), then let U be a sphere of radius ϵ and center at p so that $\overline{U} \cap S = \emptyset$. It may be assumed without loss of generality that the coordinate system for E^n has been chosen so that $a = 0$ and $p = (0, \cdots, 0, \sigma)$, that is, p is on the x_n-axis with $\sigma > 0$. Let H' be the hyperplane defined by $x_n = \sigma$ (see Figure 6). Next consider the set $\infty(\bar{U} \cap H')0 = \left\{ (x_1, \cdots, x_n) : x_n \geq (\sigma/\epsilon)(x_1^2 + \cdots + x_{n-1}^2)^{1/2} \text{ if } x_1^2 + \cdots + x_{n-1}^2 \geq \epsilon^2 \text{ and } x_n \geq \sigma \text{ if } x_1^2 + \cdots + x_{n-1}^2 < \epsilon^2 \right\}$ which is closed and convex by Theorems 29 and 24. The $\text{bdry}(\infty(\bar{U} \cap H')0) = \left\{ (x_1, \cdots, x_n) : x_n = (\sigma/\epsilon)(x_1^2 + \cdots + x_{n-1}^2)^{1/2} \text{ if } x_1^2 + \cdots + x_{n-1}^2 \geq \epsilon^2 \text{ and } x_n = \sigma \text{ if } x_1^2 + \cdots + x_{n-1}^2 < \epsilon^2 \right\}$ and is the desired set H. Certainly if $H = \text{bdry}(\infty(\bar{U} \cap H')0)$, then $H \cap S = \emptyset$ since if there is an element x in $H \cap S$, then $0x \cap \bar{U}$ contains a point of S, contradicting that $\bar{U} \cap S = \emptyset$. The set H is homeomorphic with the hyperplane H' given by $x_n = \sigma$ and the homeomorphism is defined by $f(z) = z$ if z belongs to $\bar{U} \cap H'$ and $f(z) = w$ if z belongs to $H \setminus \bar{U} \cap H'$ where w is the perpendicular projection of z onto H'. More precisely the function f may be defined at each point $z = (x_1, \cdots, x_n)$ of H by $f(z) = (x_1, \cdots, x_{n-1}, \sigma)$.

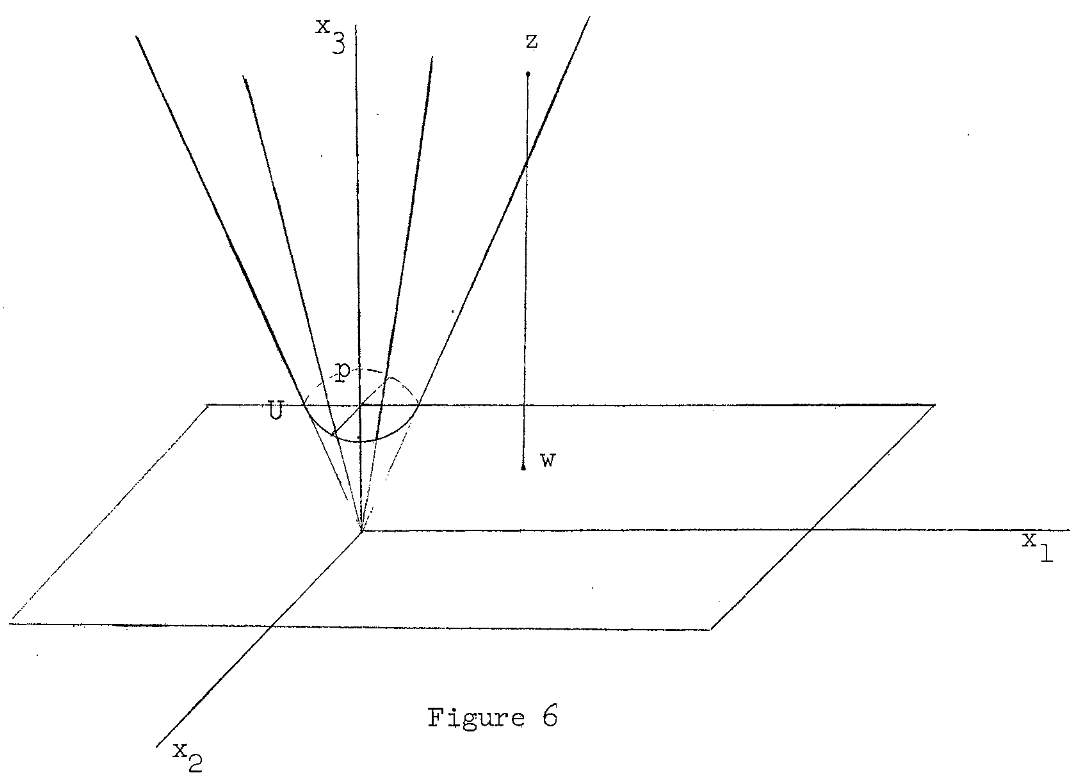

Figure 6

Theorem 45: Let A, B, and C be subsets of $L = E^n$ with A compact and starlike from a; assume B and C are closed and starlike from b and c, respectively. If $A + B = A + C$, then $B = C$.

Proof: There are two cases and in each case it is assumed that $B \neq C$ and shown that $A + B \neq A + C$.

Case I: Suppose there exists an element p of $B \setminus C$ such that p does not belong to $k(C)$. By translation take $p = 0$. There exists an element f of L^* so that $f(0) = 0$ and $f(z) > 0$ for every z in $k(C)$ (see Day $\left[6\right]$, page 22). Since f is continuous on the compact set A, there exists an element u of A so that $f(u) = \inf \left\{ f(z) : z \in A \right\}$. Suppose u belongs to $A + C$. Then there exist elements a' of A and c' of C so that $u = a' + c'$. Thus $f(u) = f(a' + c') = f(a') + f(c') > f(u) + 0 = f(u)$, a contradiction. Hence u does not belong to $A + C$. But $u = u + 0$ belongs to $A + B$. Therefore if $B \neq C$, then $A + B \neq A + C$.

Case II: Suppose for every p in $B \setminus C$, p belongs also to $k(C)$. Let p belong to $B \setminus C$. Again by translation take $p = 0$. By Theorem 44 there exists a set H containing 0 which is homeomorphic to a hyperplane H' and so that $k(H) \cap C = \emptyset$. The set H' may be chosen to contain 0 and not c since 0 does not belong to C and $c \neq 0$. Then there exists an element f of L^* so that $H' = \left\{ z \in L : f(z) = 0 \right\}$. Let g be the homeomorphism of H' to H and let h be a homeomorphism of L onto L so that $h(z) = g(z)$ for each z in H with $h(0) = 0$ and so that h transforms $\left\{ z : f(z) > 0 \right\}$ 1-1 onto $L \setminus k(H)$ and $\left\{ z : f(z) < 0 \right\}$ 1-1 onto $k(H) \setminus H$.

A new vector space structure is introduced on L by defining $x \oplus y = h\left[h^{-1}(x) + h^{-1}(y)\right]$ and $\lambda * x = h\left[\lambda h^{-1}(x)\right]$ where x and y are in L and λ is real. It is necessary to verify that L is indeed a vector space using these operations. The addition $\oplus$ is a function defined on $L \times L$ to L and $*$ is a function defined on $R \times L$ to L where R is the space of real numbers. Furthermore:

(1) $(x \oplus y) \oplus z = h\left[h^{-1}(x \oplus y) + h^{-1}(z)\right] = h\left[\left\{h^{-1}(h(h^{-1}(x) + h^{-1}(y)))\right\} + h^{-1}(z)\right] = h\left[(h^{-1}(x) + h^{-1}(y)) + h^{-1}(z)\right] = h\left[h^{-1}(x) + (h^{-1}(y) + h^{-1}(z))\right] = h\left[h^{-1}(x) + \left\{h^{-1}\left[h(h^{-1}(y) + h^{-1}(z))\right]\right\}\right] = h\left[h^{-1}(x) + h^{-1}(y \oplus z)\right] = x \oplus (y \oplus z)$;

(2) $x \oplus y = h\left[h^{-1}(x) + h^{-1}(y)\right] = h\left[h^{-1}(y) + h^{-1}(x)\right] = y \oplus x$;

(3) $x \oplus 0 = h\left[h^{-1}(x) + h^{-1}(0)\right] = h\left[h^{-1}(x) + 0\right] = h\left[h^{-1}(x)\right] = x$;

(4) $x \oplus (-1 * x) = h\left[h^{-1}(x) + h^{-1}(-1 * x)\right] = h\left[h^{-1}(x) - h^{-1}(x)\right] = h(0) = 0$;

(5) $\lambda * (x \oplus y) = h\left[\lambda h^{-1}(x \oplus y)\right] = h\left[\lambda(h^{-1}(x) + h^{-1}(y))\right] = h\left[\lambda h^{-1}(x) + \lambda h^{-1}(y)\right] = h\left[h^{-1}(\lambda * x) + h^{-1}(\lambda * y)\right] = \lambda * x \oplus \lambda * y$;

(6) $(\lambda + \mu) * x = h\left[(\lambda + \mu)h^{-1}(x)\right] = h\left[\lambda h^{-1}(x) + \mu h^{-1}(x)\right] = h\left[h^{-1}(\lambda * x) + h^{-1}(\mu * x)\right] = \lambda * x \oplus \mu * x$;

(7) $\lambda * (\mu * x) = \lambda * h(\mu h^{-1}(x)) = h\left[\lambda h^{-1}(h(\mu h^{-1}(x)))\right] = h\left[(\lambda\mu)h^{-1}(x)\right] = (\lambda\mu) * x$;

(8) $1 * x = h(1h^{-1}(x)) = h(h^{-1}(x)) = x$.

Consequently, the collection of elements of L furnished with the operations $\oplus$ and $*$ is a vector space which will hereafter be denoted by $h(L)$. The next argument shows that $h(L)$ can be furnished with a topology for which the operations $\oplus$ and $*$ are continuous.

There exists a neighborhood basis $\mathcal{U}$ of 0 for L which satisfies the conditions a-f of §4(2), page 11 of Day [6]. Let $h(\mathcal{U}) = \{h(U) : U \in \mathcal{U}\}$. It is next shown that $h(\mathcal{U})$ satisfies the conditions a-f and is hence a neighborhood basis of 0 for $h(L)$.

(a) If x belongs to the intersection of all sets $h(U)$, then x belongs to

$h(U)$ for each U in $\mathcal{U}$ which implies that $h^{-1}(x)$ belongs to U for every U in $\mathcal{U}$ and hence $h^{-1}(x) = 0$ so that $x = 0$.

(b) If U and V are in $\mathcal{U}$, then there exists an element W of $\mathcal{U}$ so that $W \subset U \cap V$. Hence $h(W) \subset h(U \cap V) \subset h(U) \cap h(V)$.

(c) If U belongs to $\mathcal{U}$ and $|\alpha| < 1$, then $\alpha U \subset U$. Hence $\alpha * h(U) \subset h(U)$ since if $h(x)$ is in $h(U)$, then $\alpha * h(x) = h(\alpha x)$ is in $h(U)$ because αx is in U.

(d) If U belongs to $\mathcal{U}$, then there exists an element V of $\mathcal{U}$ so that $V + V \subset U$. Hence $h(V) \oplus h(V) \subset h(U)$ since if $h(x)$ and $h(y)$ belong to $h(V)$, it follows that $h(x) \oplus h(y) = h(x + y)$ belongs to $h(U)$ because $x + y$ is in $V + V \subset U$.

(e) The point 0 is a core point of each U in $\mathcal{U}$ which means that for each y in L there exists $\epsilon(y) > 0$ so that λy is in U for $|\lambda| < \epsilon(y)$. Hence for each $h(y)$ in $h(L)$ there exists $\epsilon(h(y)) = \epsilon(y)$ so that $\lambda * h(y)$ is in $h(U)$ for $|\lambda| < \epsilon(h(y))$ since $\lambda * h(y) = h(\lambda y)$ which belongs to $h(U)$ since λy belongs to U.

(f) Each U in $\mathcal{U}$ is convex in L and hence each $h(U)$ is convex in $h(L)$. Let $h(x)$ and $h(y)$ belong to $h(U)$ and $0 \leq \alpha \leq 1$. Then $\alpha * h(x) \oplus (1 - \alpha) * h(y) = h(\alpha x) \oplus h((1 - \alpha)y) = h(\alpha x + (1 - \alpha)y)$ which belongs to $h(U)$ since $\alpha x + (1 - \alpha)y$ belongs to U.

Consequently, it has been proved that $h(L)$ is a locally convex linear topological space with the operations $\oplus$ and $*$ and the neighborhood basis $h(\mathcal{U})$.

The set A is compact and B and C are closed in L so that $h(A)$ is compact and $h(B)$ and $h(C)$ are closed in $h(L)$ since h is a homeomorphism. Also $h(B)$ and $h(C)$ are starlike from $h(b)$ and $h(c)$, respectively. In order to show that $h(B)$ is starlike from $h(b)$, let x belong to $h(B)$ and $0 \leq \alpha \leq 1$. Then

$\alpha * x \oplus (1 - \alpha) * h(b) = h(\alpha h^{-1}(x)) \oplus h((1 - \alpha)b) = h\left[\alpha h^{-1}(x) + (1 - \alpha)b\right]$ which belongs to $h(B)$ since $h^{-1}(x)$ is in B and B is starlike from b. A similar argument will prove that $h(C)$ is starlike from $h(c)$.

Consider the function F of $h(L)$ into the real numbers defined by $F(x) = f(h^{-1}(x))$. The function F is linear on $h(L)$ since if x and y belong to $h(L)$, then $F(x \oplus y) = f(h^{-1}(x \oplus y)) = f\left[h^{-1}(h(h^{-1}(x) + h^{-1}(y)))\right] = f(h^{-1}(x) + h^{-1}(y)) = f(h^{-1}(x)) + f(h^{-1}(y)) = F(x) + F(y)$; also if α is real, then $F(\alpha * x) = f(h^{-1}(\alpha * x)) = f(h^{-1}(h(\alpha h^{-1}(x)))) = f(\alpha h^{-1}(x)) = \alpha f(h^{-1}(x)) = \alpha F(x)$. Furthermore F is continuous since f and h^{-1} are continuous so that F belongs to $h(L)^*$. It follows that $F(z) = 0$ for every z in H; also $F(z) > 0$ if, and only if, z belongs to $L \setminus k(H)$ and $F(z) < 0$ if, and only if, z belongs to $k(H)^o$.

The remainder of the argument is now similar to Case I. The set A is compact in $h(L)$ since h is a homeomorphism. The function F is continuous on the compact set A in $h(L)$ so that there exists an element u of A for which $F(u) = \inf \left\{F(z) : z \in A\right\}$. Assume that u belongs to $A \oplus C$; then there are elements a' of A and c' of C so that $u = a' \oplus c'$. Hence $F(u) = F(a' \oplus c') = F(a') + F(c') > F(u) + 0$ since $C \cap k(H) = \emptyset$ and so c' belongs to $L \setminus k(H)$. This is a contradiction and therefore u does not belong to $A \oplus C$. But $u = u \oplus 0$ belongs to $A \oplus B$ since 0 is in B. Hence $A \oplus B \neq A \oplus C$ which implies that $A + B \neq A + C$ since h is 1-1 and onto.

This completes the proof of Theorem 45.

The kernel of a set is the collection of points from which the set is starlike. Brunn [3] proved that the kernel for closed sets in the plane is closed and convex. This result will be proved in Chapter V for a general

LTS. The kernel of a set S is denoted by ker(S).

Using the notion of the kernel, one is able to define an order relation $\leq$ on $\mathcal{S}^*$ by $(S_1,a)^* \leq (S_2,b)^*$ provided that $S_1 \subset S_2$ and $\ker(S_1) \subset \ker(S_2)$. It is easily verified that the order relation $\leq$ is reflexive, transitive, and antisymmetric.

With the order relation thus defined $\mathcal{S}^*$ is a lattice (see Birkhoff [2]). The notation $(S,a)^* \vee (T,b)^*$ is used for the least upper bound of $(S,a)^*$ and $(T,b)^*$, and $(S,a)^* \wedge (T,b)^*$ for their greatest lower bound.

The least upper bound of $(S,a)^*$ and $(T,b)^*$ is given by $(S,a)^* \vee (T,b)^* = (K,c)^*$ where c belongs to $\ker(S) \cup \ker(T)$ and $K = \bigcup \{px : x \in S \cup T, p \in \ker(S) \cup \ker(T)\}$. Note that $\ker(S) \cup \ker(T) \subset \ker(K)$ and hence $k(\ker(S) \cup \ker(T)) \subset \ker(K)$ since the kernel is convex, but equality may not hold as the following example in the plane demonstrates:

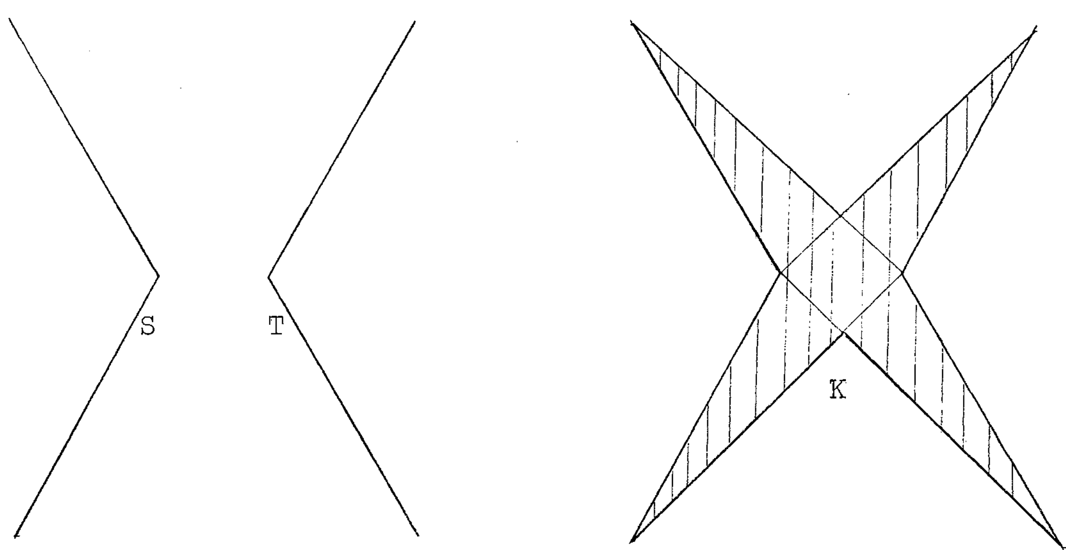

Figure 7

There are two cases for the greatest lower bound. If $\ker(S)\cap\ker(T) = \emptyset$, then $(S,a)^* \wedge (T,b)^* = (\emptyset,\emptyset)^*$. If $\ker(S)\cap\ker(T) \neq \emptyset$ and c belongs to $\ker(S)\cap\ker(T)$, then $(S,a)^* \wedge (T,b)^* = (S\cap T,c)^*$.

The lattice $\mathcal{S}^*$ has a least element $(\emptyset,\emptyset)^*$ and a greatest element $(L,a)^*$ since for every element $(S,b)^*$ of $\mathcal{S}^*$, $(\emptyset,\emptyset)^* \leq (S,b)^* \leq (L,a)^*$.

<u>Theorem 46</u>: The lattice $\mathcal{S}^*$ is complemented.

<u>Proof</u>: The complement of $(L,a)^*$ is $(\emptyset,\emptyset)^*$ and conversely. Hence let $(S,a)^*$ be a member of $\mathcal{S}^*$ and suppose $S \neq L$ and $S \neq \emptyset$. Let p be any point of $C(S)$. Since $C(S)$ is inverse starlike from a by Theorem 1, ∞pa is also contained in $C(S)$. There exists a ray M emanating from p so that $M \neq \infty pa$ and $M\cap\ker(S) = \emptyset$ since if for every ray M emanating from p, $M\cap\ker(S) \neq \emptyset$, every line through p would intersect $\ker(S)$ on each side so that p would belong to $\ker(S)$ since $\ker(S)$ is convex; but this contradicts that p belongs to $C(S)$. Let q be any point different from p on M and set $S' = C(\infty pq)$. Then $\ker(S') = M$ and it follows that $(S',p)^*$ is the complement of $(S,a)^*$. Certainly $(S,a)^* \wedge (S',p)^* = (\emptyset,\emptyset)^*$ since $\ker(S)\cap\ker(S') = \emptyset$. Then it must be proved that $(S,a)^* \vee (S',p)^* = (L,a)^*$. If x belongs to S', then x belongs to K (referring to the notation used above in describing the least upper bound). If x is in $C(S')$, then the line through a and x intersects $C(S')$ only at x (since a and M are not collinear) and consequently there exists an element z of S' so that x is on az which implies again that x belongs to K. Since $L = S'\cup C(S')$, it follows that $L = K$ and that $(S,a)^* \vee (S',p)^* = (L,a)^*$.

It should be observed that the complement constructed above is by no means unique since there are an infinite number of choices for the ray M.

In a distributive lattice complements are unique (see Birkhoff [2], page 75) and hence the lattice $\mathcal{S}^*$ is not distributive.

The lattice $\mathcal{S}^*$ is not modular as is shown by the following examples. Let $S_2 = \{(x,1) : -2 \leq x \leq 2\} \cup \{(x,y) : -1 \leq x \leq 1, |x| \leq y \leq 1\}$, $S_1 = \ker(S_2) = \{(x,1) : -1 \leq x \leq 1\}$, and $S_3 = \{(x,-1) : -1 \leq x \leq 1\}$. Then $(S_1,a)^* \leq (S_2,b)^*$ and $(S_1,a)^* \vee [(S_2,b)^* \wedge (S_3,c)^*] = (S_1,a)^* \vee (\emptyset,\emptyset)^* = (S_1,a)^*$ but $[(S_1,a)^* \vee (S_2,b)^*] \wedge (S_3,c)^* = (S_2,b)^* \wedge (S_3,c)^* = (\emptyset,\emptyset)^*$ (see Figure 8).

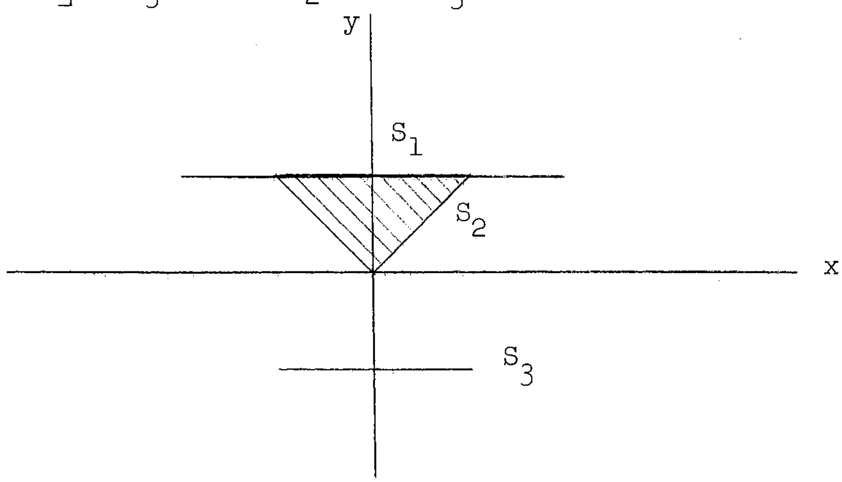

Figure 8

Consider now an infinite collection of elements of $\mathcal{S}^*$, say $\{(S_\alpha,a_\alpha)^*\}$ where α is in some index set Δ. Denote the least upper bound of this collection by $\bigvee_\alpha (S_\alpha,a_\alpha)^*$ and its greatest lower bound by $\bigwedge_\alpha (S_\alpha,a_\alpha)^*$. Then:

(1) $\bigvee_\alpha (S_\alpha,a_\alpha)^* = (K,c)^*$ where c belongs to $\bigcup_\alpha \ker(S_\alpha)$ and $K = \bigcup\{px : x \in \bigcup_\alpha S_\alpha,\ p \in \bigcup_\alpha \ker(S_\alpha)\}$;

(2) $\bigwedge_\alpha (S_\alpha,a_\alpha)^* = (\bigcap_\alpha S_\alpha,a)^*$ if $\bigcap_\alpha \ker(S_\alpha) \neq \emptyset$ and a belongs to $\bigcap_\alpha \ker(S_\alpha)$;

(3) $\bigwedge_\alpha (S_\alpha,a_\alpha)^* = (\emptyset,\emptyset)^*$ if $\bigcap_\alpha \ker(S_\alpha) = \emptyset$.

Therefore the lattice $\mathcal{S}^*$ is complete (see Birkhoff [2], page 17).

The properties of the lattice $\mathcal{S}^*$ may be summarized as follows: $\mathcal{S}^*$ is a complete complemented lattice which is not distributive and not modular.

CHAPTER V

A GENERALIZATION OF CONVEXITY

V-1: Introduction

All of the results so far have been for inverse starlike and starlike sets with a few references to convex sets, projectively convex sets, sets with property P_3, cones, and flats. The purpose of this chapter is to give a definition which will include all of the above definitions as special cases and then to develop several basic results for this more general notion.

In order to make such a definition, it is necessary to consider carefully the definitions of convex, starlike, inverse starlike, property P_3, projectively convex, cone, and flat. In all cases either two or three points are chosen; for some sets one of these points is fixed and others are arbitrary. Also a certain portion of the line or lines joining these points is required to be contained in the set. Hence the principle features of these sets are the choice of a finite number of points and the use of linear segments. Keeping these features in mind one is able to arrive at the definition given below.

Let $A^k = \{a_1, \cdots, a_k\}$ denote a set of k fixed points and let $X = X^{n-k} = \{x_1, \cdots, x_{n-k}\}$ $(n \geq k)$ denote any set of n-k points. Let T represent a collection of the following types of linear segments: uv, ∞uv, and unions of such segments where u and v are in L and may be any a_i $(i \leq k)$ or any x_i $(i \leq n-k)$.

Definition: A subset S of L has property $P_n^r(A^k;X;T)$ provided that for every subset $X = \{x_1, \cdots, x_{n-k}\}$ of $S \setminus A^k$, at least r distinct segments of types T are in S.

There are several variations that could be made in the definition. The finite character of A^k and X might be omitted and curves or continua or some other set used instead of linear segments. Such an alteration would certainly be more general and constitute a research problem within itself. Also instead of requiring that at least r distinct segments of types T be in S, one might require exactly r or at most r segments of types T to be in S and these alterations lead also to some interesting problems.

In their paper "Helly's Theorem and Its Relatives" (which has not yet been published and which the author learned of only recently) Danzer, Grunbaum, and Klee [5] outline the following scheme for generalizing convexity: In a set X a family $\underset{\sim}{F}$ of sets is given together with a function η which assigns to each element F of $\underset{\sim}{F}$ a family ηF of subsets of X. A subset K of X is called η-convex provided that K contains at least one member of ηF whenever F is contained in K and F belongs to $\underset{\sim}{F}$.

The generalization of convexity defined above is a special case of η-convex. In this case X is the linear space L. If for example the subset K of L has property $P_n^1(A^k;X;T)$, then the family $\underset{\sim}{F} = \{X = \{x_1, \cdots, x_{n-k}\} : X \subset K \setminus A^k\}$ and the function η assigns to each X in $\underset{\sim}{F}$ the family of all possible segments of types T. The set K has property $P_n^1(A^k;X;T)$ provided that K contains at least one segment of type T.

It should be observed that if a subset S has property $P_n^r(A^k;X;T)$, then S also has property $P_n^s(A^k;X;T)$ if $1 \leq s \leq r$. Consequently all of the results are proved for sets which have property $P_n^1(A^k;X;T)$ and for convenience the notation will be modified to $P_n(A^k;X;T)$.

Now then it is possible to express the previous notions of convexity, etc., in terms of property $P_n(A^k;X;T)$:

(1) Convex: $P_2(\emptyset;X;x_1x_2)$

(2) Projectively convex: $P_2(\emptyset;X;x_1x_2, \infty x_1x_2 \bigcup \infty x_2x_1)$

(3) Property P_3: $P_3(\emptyset;X;x_ix_j)$

(4) Starlike from a: $P_2(a;x;ax)$

(5) Inverse starlike from a: $P_2(a;x; \infty xa)$

(6) Cone with vertex at a: $P_2(a;x;ax \bigcup \infty xa)$

(7) Flat: $P_2(\emptyset;X;x_1x_2 \bigcup \infty x_1x_2 \bigcup \infty x_2x_1)$

V-2: Sets With Property $P_n(A^k;X;T)$

The first group of theorems develops the basic properties of sets with property $P_n(A^k;X;T)$. Then several special cases for the general property will be considered. The proofs are given only for the segments of types uv and ∞uv since all other types are unions of these and hence uv and ∞uv are the basic ingredients that must be considered.

Theorem 47: Let the subset S of L have property $P_n(A^k;X;T)$ and α be any real number. Then αS has property $P_n(\alpha A^k;\alpha X;T)$.

Proof: Let $y_1, \cdots, y_{n-k}$ be elements of $\alpha S \setminus \alpha A^k$. Then there exist elements $x_1, \cdots, x_{n-k}$ of $S \setminus A^k$ so that $y_1 = \alpha x_1, \cdots, y_{n-k} = \alpha x_{n-k}$. Also there exist elements u and v of $A^k \bigcup X$ so that either uv, ∞uv, or a union of such segments is in S. But $\alpha(uv) = (\alpha u)(\alpha v)$ and $\alpha(\infty uv) = \infty(\alpha u)(\alpha v)$ so that if $uv \subset S$, then $\alpha(uv) \subset \alpha S$ and if $\infty uv \subset S$, then $\alpha(\infty uv) \subset \alpha S$ and likewise for unions. Hence αS has property $P_n(\alpha A^k;\alpha X;T)$.

Theorem 48: Let the subset S of L have property $P_n(A^k;X;T)$ and c belong to L. Then c + S has property $P_n(c+A^k;c+X;T)$.

Proof: Let $y_1, \cdots, y_{n-k}$ be elements of $(c+S)\setminus(c+A^k)$. Then there exists a set $X = \{x_1, \cdots, x_{n-k}\} \subset S\setminus A^k$ so that $y_1 = c + x_1, \cdots, y_{n-k} = c + x_{n-k}$. Also there exist elements u and v of $A^k \cup X$ so that either uv, ∞uv, or a union of such segments is in S. But $(c + u)(c + v) = c + uv$ and $\infty(c + u)(c + v) = c + \infty uv$ so that if $uv \subset S$, then $(c + u)(c + v) \subset c + S$ and if $\infty uv \subset S$, then $\infty(c + u)(c + v) \subset c + S$ and likewise for unions. Hence c + S has the desired property.

Theorem 49: Let S be a subset of the linear space L_1 and let f be a linear transformation of L_1 into a linear space L_2. If S has property $P_n(A^k;X;T)$, then f(S) has property $P_n(f(A^k);f(X);T)$.

Proof: Let $y_1, \cdots, y_{n-k}$ be elements of $f(S)\setminus f(A^k)$. Then there exists a set $X = \{x_1, \cdots, x_{n-k}\} \subset S\setminus A^k$ so that $y_1 = f(x_1), \cdots, y_{n-k} = f(x_{n-k})$. Also there exist elements u and v of $A^k \cup X$ so that either uv or ∞uv (or a union of such segments) is contained in S. But $f(uv) = f(u)f(v)$ and $f(\infty uv) = \infty f(u)f(v)$ so that if $uv \subset S$ then $f(u)f(v) \subset f(S)$, and if $\infty uv \subset S$ then $\infty f(u)f(v) \subset f(S)$. Therefore f(S) has the desired property.

Theorem 50: Let the subset S have property $P_n(A^k;X;T)$. Then the intersection of S with a flat M has this same property provided $A^k \subset S \cap M$.

Proof: Let $x_1, \cdots, x_{n-k}$ be elements of $(S\cap M)\setminus A^k$. Then $x_1, \cdots, x_{n-k}$ belong to $S\setminus A^k$ so that there exist elements u and v of $A^k \cup X$ so that either uv or ∞uv is in S. Since M is flat, uv or ∞uv is also contained in M which proves that $S \cap M$ has the stated property.

Theorem 51: Let each of the sets $S_1, \cdots, S_m$ have property $P_n(A^k;X;T)$. Then $\bigcup_{i=1}^{m} S_i$ has property $P_{mn}(A^k;X;T)$.

Proof: Let $X = \{x_1, \cdots, x_{mn-k}\} \subset \bigcup_{i=1}^{m} S_i \setminus A^k$. There are mn - k points in X and $mn - k > m(n - k)$ which implies that at least n - k of the points of X are in some S_i. But S_i has property $P_n(A^k;X;T)$ so that at least one segment of type T is in S_i and hence in $\bigcup_{i=1}^{m} S_i$. Therefore $\bigcup_{i=1}^{m} S_i$ has property $P_{mn}(A^k;X;T)$.

Obviously the number of sets in Theorem 51 must be restricted to be finite. The question of intersections of sets with property $P_n(A^k;X;T)$ must be raised. However, it seems that no conclusions may be drawn here as is demonstrated by the following examples. Let S_1 be a set in E^2 consisting of three lines parallel to the x-axis and let S_2 be a set consisting of three lines parallel to the y-axis. Then $S_1 \cap S_2$ consists of nine isolated points but each of the sets has property $P_4(\emptyset;X;x_i x_j)$.

Theorem 52: Let the subset S have property $P_n(A^k;X;T)$. Then ∞SO also has property $P_n(A^k;X;T)$.

Proof: Let $\{y_1, \cdots, y_{n-k}\} = Y \subset \infty SO \setminus A^k$. Then there exists a set $\{x_1, \cdots, x_{n-k}\} = X \subset S \setminus A^k$ so that $y_i = \alpha_i x_i$, $\alpha_i \geq 1$, $i = 1, \cdots, n-k$. Then there are elements u and v of $X \cup A^k$ so that uv or ∞uv is contained in S. In order to complete the proof, it appears to be best to consider all of the possibilities for u and v.

Case I: Suppose $a_j x_i \subset S$. Then it will be proved that $a_j y_i \subset \infty SO$. Let $z = \beta a_j + (1 - \beta) y_i = \beta a_j + (1 - \beta)\alpha_i x_i$ be on $a_j y_i$ with $0 \leq \beta \leq 1$. Then $z = \sigma(\epsilon a_j + (1 - \epsilon) x_i)$ where $\sigma = (1 - \beta)\alpha_i + \beta$ and $\epsilon = \beta/\sigma$ with $\sigma \geq 1$ and $0 \leq \epsilon \leq 1$ which implies that z belongs to ∞SO. Thus $a_j y_i \subset \infty SO$.

Case II: Suppose $x_i x_j \subset S$. It will be proved that $y_i y_j \subset \infty SO$. Let $z = \beta y_i + (1 - \beta) y_j = \beta\alpha_i x_i + (1 - \beta)\alpha_j x_j$ with $0 \leq \beta \leq 1$. Then $z = \sigma(\epsilon x_i + (1 - \epsilon) x_j)$ where $\sigma = (1 - \beta)\alpha_j + \beta\alpha_i \geq 1$ and $\epsilon = \beta\alpha_i/\sigma$, $0 \leq \epsilon \leq 1$, which implies that z belongs to ∞SO. Hence $y_i y_j \subset \infty SO$. A similar argument will take care of the case $a_i a_j$.

Case III: Suppose $\infty x_i a_j \subset S$. It will be proved that $\infty y_i a_j \subset \infty SO$. Let $z = \beta y_i + (1 - \beta) a_j = \beta\alpha_i x_i + (1 - \beta) a_j$ be on $\infty y_i a_j$ with $\beta \geq 1$. Then $z = \sigma(\epsilon x_i + (1 - \epsilon) a_j)$ where $\sigma = \beta\alpha_i + 1 - \beta \geq 1$ and $\epsilon = 1 - \sigma(1 - \beta) \geq 1$ which implies that z belongs to ∞SO. Hence $\infty y_i a_j \subset \infty SO$.

Case IV: Suppose $\infty a_i x_j \subset S$. It will be proved that $\infty a_i y_j \subset \infty SO$. Let $z = \beta a_i + (1 - \beta) y_j = \beta a_i + (1 - \beta)\alpha_j x_j$ be on $\infty a_i y_j$ with $\beta \geq 1$. Then $z = \sigma(\epsilon a_i + (1 - \epsilon) x_j)$ where $\sigma = (1 - \beta)\alpha_j + \beta \geq 1$ and $\epsilon = \beta/\sigma \geq 1$ which implies that z belongs to ∞SO. Hence $\infty a_i y_j \subset \infty SO$.

Case V: Suppose $\infty x_i x_j \subset S$. It will be proved that $\infty y_i y_j \subset \infty SO$. Let $z = \beta y_i + (1 - \beta) y_j = \beta\alpha_i x_i + (1 - \beta)\alpha_j x_j$ be on $\infty y_i y_j$ with $\beta \geq 1$. Then $z = \sigma(\epsilon x_i + (1 - \epsilon) x_j)$ where $\sigma = (1 - \beta)\alpha_j + \beta\alpha_i \geq 1$ and $\epsilon = \beta\alpha_i/\sigma \geq 1$ which implies that z belongs to ∞SO. Hence $\infty y_i y_j \subset \infty SO$.

These five cases represent all possibilities for the basic segments uv and ∞uv. In each case it has been proved that ∞SO has property $P_n(A^k;X;T)$.

<u>Corollary</u>: Let S have property $P_n(A^k;X;T)$. Then ∞Sa also has

property $P_n(A^k;X;T)$.

Proof: It is first proved that $\infty Sa = \infty(S - a)0 + a$. Let $z = \alpha s + (1 - \alpha)a$ belong to ∞Sa where s is in S and $\alpha \geq 1$. Then $z = \alpha(s - a) + a$ which belongs to $\infty(S - a)0 + a$ so that $\infty Sa \subset \infty(S - a)0 + a$. Let $u = \alpha(s - a) + a$ belong to $\infty(S - a)0 + a$ where s is in S and $\alpha \geq 1$. Then $u = \alpha s + (1 - \alpha)a$ which belongs to ∞Sa. Therefore $\infty Sa = \infty(S - a)0 + a$.

The set S has property $P_n(A^k;X;T)$ so that S - a has property $P_n(A^k-a;X-a;T)$ by Theorem 48. By Theorem 52 $\infty(S - a)0$ has property $P_n(A^k-a;X-a;T)$. Again by Theorem 48 it follows that $\infty(S - a)0 + a = \infty Sa$ has property $P_n(A^k;X;T)$.

V-3: Some Special Cases of Sets with Property $P_n(A^k;X;T)$

The remainder of the results in this chapter deal with sets with property $P_n(A^k;X;T)$ in which the T is specifically described. The next group of theorems concerns sets with property $P_n(\emptyset;X;x_ix_j)$ and the notation will be shortened to P_n since this property generalizes property P_3.

Theorem 53: Let the subset S of L have property P_n. Then S is the union of n - 1 or fewer sets A_i with A_i starlike from a_i.

Proof: Let a_1 belong to S and define $A_1 = \bigcup\left\{a_1x : x \in S, a_1x \subset S\right\}$. Assume that a_{i-1} and A_{i-1} have been defined. Let a_i belong to $S \setminus \bigcup_{k=1}^{i-1} A_k$ and define $A_i = \bigcup\left\{a_ix : x \in S, a_ix \subset S\right\}$. Clearly each of the sets A_i is starlike from a_i.

It must be proved that $S = \bigcup_{i=1}^{r} A_i$ where $r \leq n - 1$. Assume by way of contradiction that there exists an element x of $S \setminus \bigcup_{i=1}^{n-1} A_i$ and consider the n points $a_1, \cdots, a_{n-1}, x$ of S. If $a_ix \subset S$, then x belongs to A_i contradicting that x is in $S \setminus \bigcup_{i=1}^{n-1} A_i$. If $a_i a_j \subset S$, then (if $j > i$) a_j belongs to A_i contradicting that a_j belongs to $S \setminus \bigcup_{k=1}^{j-1} A_k$. But this contradicts that S has property P_n. Therefore $S = \bigcup_{i=1}^{r} A_i$ where $r \leq n - 1$.

If each of the sets A_i of Theorem 53 contains no ray emanating from a_i and if S is closed, then each set A_i is the star envelope of its relative extreme points by Theorem 36.

Corollary: Let the subset S of a LTS be closed, have property P_n, and assume that each of the sets A_i of Theorem 53 contains no ray emanating from a_i. Then $S = \bigcup_{i=1}^{r} (\mathrm{ex}_{a_i} A_i) a_i$.

Valentine [14] has proved that under certain conditions a set with property P_3 is the union of three or fewer convex sets. It seems likely that a set with property P_n is the union of n or fewer convex sets, although this has not yet been proved. Theorem 53 is a result in this direction.

A set which is convex also has property P_n. Furthermore a set is convex if, and only if, the intersection of every line with the set is either empty or connected. A similar property can be proved for sets with property P_n.

Theorem 54: Let the subset S of L have property P_n and let M be a line in L. Then $M \cap S$ consists of at most n - 1 segments.

Proof: If $M \cap S$ contains at least n components, then a point may be chosen from each of these components and thus contradict that S has property P_n.

A closed bounded set is convex if, and only if, for every two boundary points x and y, xy is contained in the set. A problem yet to be solved is to prove that a closed bounded set S has property P_n if, and only if, for every n boundary points $x_1, \cdots, x_n$, at least one segment $x_i x_j$ is contained in S.

Theorem 55: Let the subset S of L have property P_n and let M be convex. Then S + M also has property P_n.

Proof: Let $y_1, \cdots, y_n$ be elements of S + M. Then $y_i = s_i + m_i$ where s_i is in S and m_i is in M for i = 1, $\cdots$, n. There exist elements s_i and s_j so that $s_i s_j \subset S$ since S has property P_n. For the same i and j, $m_i m_j \subset M$ since M is convex. Furthermore $y_i y_j = (s_i + m_i)(s_j + m_j) \subset S + M$ since $(s_i + m_i)(s_j + m_j) \subset s_i s_j + m_i m_j \subset S + M$. Therefore S + M has property P_n.

The next few results are for sets with property P_n contained in a LTS and describe the nature of the components of such a set.

Theorem 56: Let the subset S of a LTS have property P_n. Then S has at most n - 1 components which have property P_n. If S has exactly n - 1 components, each component is convex.

Proof: Suppose by way of contradiction that S has more than n - 1 components and write $S = B_1 \cup B_2 \cup \cdots \cup B_n \cup (S \setminus \bigcup_{i=1}^{n} B_i)$ where each B_i, i = 1, $\cdots$, n, is a component of S. Let x_i belong to B_i. Then for some i and j, $x_i x_j \subset S$ since S has property P_n. But then $B_i \cup B_j \cup x_i x_j$ is a connected subset of S contradicting that B_i and B_j are components of S. Therefore S has at most n - 1 components. Obviously each component has property P_n.

Assume that S has exactly n - 1 components $B_1, \cdots, B_{n-1}$. Let x_i and y_i belong to B_i and let x_j belong to B_j for j = 1, $\cdots$, n-1 and $j \neq i$. Then x_ix_j and y_ix_j are not contained in S since $B_1, \cdots, B_{n-1}$ are components of S. Therefore $x_iy_i \subset S$ and hence is in B_i which proves that B_i is convex.

Theorem 57: Let the set S in a LTS have property P_n. If S has exactly n - r components, then each component has property P_{r+1} and at least one of the components has property P_r.

Proof: Let $S = B_1 \bigcup B_2 \bigcup \cdots \bigcup B_{n-r}$ where $B_1, \cdots, B_{n-r}$ are the components of S. Let $x_1, \cdots, x_{r+1}$ be elements of B_k and let y_i belong to B_i for i = r+2, $\cdots$, n. Then if $x_iy_j \subset S$ for some i and j, $B_k \bigcup B_j \bigcup x_iy_j$ is a connected subset of S contradicting that B_k and B_j are components of S. Also if $y_iy_j \subset S$ for some i and j, then $B_i \bigcup B_j \bigcup y_iy_j$ is a connected subset of S contradicting that B_i and B_j are components of S. Therefore $x_ix_j \subset S$ for some i and j and hence $x_ix_j \subset B_k$ since B_k is a component of S. Hence B_k has property P_{r+1}.

Now suppose that none of the components B_i has property P_r. Then for each B_i, there exist points x_j^i of B_i, j = 1, $\cdots$, r, so that $x_j^ix_k^i$ is not contained in B_i for any j and k. Consider the set $X = \left\{ x_1^1, x_2^1, \cdots, x_r^1, x_1^2, x_2^2, x_1^3, x_1^4, \cdots, x_1^{n-r} \right\}$ of n points of S. Also none of the segments with end-points in X is contained in S which contradicts that S has property P_n. Therefore at least one of the n - r components has property P_r.

Corollary: Let S have property P_n $(n > 3)$ and suppose S has exactly n - 2 components. Then at least one of the components is convex.

The σ-neighborhood of a convex set in a normed linear space is again convex. This property carries over to sets with property P_n.

Theorem 58: Let the subset S of a normed linear space L have property P_n and $\sigma > 0$. Then $U(S,\sigma)$ also has property P_n.

Proof: Let $x_1, \cdots, x_n$ be elements of $U(S,\sigma)$. Then there exist points $y_1, \cdots, y_n$ of S so that $\| x_i - y_i \| < \sigma$ for $i = 1, \cdots, n$.

Case I: Assume that all of $y_1, \cdots, y_n$ are distinct. Then since S has property P_n there exist i and j so that $y_i y_j \subset S$. For the same i and j, it will be proved that $x_i x_j \subset U(S,\sigma)$. Let $x = \alpha x_i + (1 - \alpha)x_j$ be on $x_i x_j$ with $0 \leq \alpha \leq 1$. The point $y = \alpha y_i + (1 - \alpha)y_j$ (for the same α) belongs to S. Furthermore $\| x - y \| = \| \alpha x_i + (1 - \alpha)x_j - \alpha y_i - (1 - \alpha)y_j \| =$ $\| \alpha(x_i - y_i) + (1 - \alpha)(x_j - y_j) \| < \alpha\sigma + (1 - \alpha)\sigma = \sigma$ which implies that x belongs to $U(S,\sigma)$. Therefore $U(S,\sigma)$ has property P_n.

Case II: Suppose that $y_1, \cdots, y_n$ are not all distinct and assume that $y_i = y_j$ for some $i \neq j$. Assume further that it is not possible to choose an element y_j' of S so that $y_i \neq y_j'$ and $\| x_j - y_j' \| < \sigma$; if this is possible for each pair that are equal, then Case I occurs again. Thus $U(y_i,\sigma) \cap S = \{y_i\}$. Also x_i and x_j belong to $U(y_i,\sigma)$ which is convex so that $x_i x_j \subset U(y_i,\sigma) \subset U(S,\sigma)$. Therefore $U(S,\sigma)$ has property P_n.

The complement of a convex set is inverse starlike from each point of the convex set. The question of the nature of the complement of a set with property P_n has not been answered. The following theorem, however, does give an answer for $n = 3$.

Theorem 59: Let the subset S of L have property P_3. Then C(S) has property $P_2(\emptyset;X;x_1x_2, \infty x_ix_j)$.

Proof: Let x_1 and x_2 belong to C(S) and assume that neither x_1x_2, ∞x_1x_2, nor ∞x_2x_1 is in C(S). Then there is a number α with $0<\alpha<1$ so that $\alpha x_1 + (1 - \alpha)x_2 = y$ is in S. Also there exist numbers β and β' both greater than one so that $\beta x_1 + (1 - \beta)x_2 = z$ is in S and $\beta' x_2 + (1 - \beta')x_1 = z'$ is in S. The three points y, z, and z' are in S so that either yz, yz', or zz' is contained in S. But this cannot be since x_1 is on yz, x_2 is on yz', and x_1 is on zz' and x_1 and x_2 are in C(S). This is a contradiction and thus C(S) has the desired property.

The converse of Theorem 59 is not true since if C(S) is the unit disc in the plane, then C(S) has property $P_2(\emptyset;X;x_1x_2, \infty x_ix_j)$. However S does not have property P_3.

This completes all results concerning property P_n. The next few theorems deal with property $P_n(A^k;X;a_ix_j)$. The first result characterizes such sets in terms of starlike sets.

Theorem 60: Let the subset S of L have property $P_n(A^k;X;a_ix_j)$. Then S is the union of k starlike sets and a finite set of n - k - 1 or fewer points.

Proof: Let $A_i = \left\{x \in S : a_ix \subset S\right\}$ for i = 1, $\cdots$, k. Clearly each set A_i is starlike from a_i. Assume by way of contradiction that $S \setminus \bigcup_{i=1}^{k} A_i$ contains at least n - k points $x_1, \cdots, x_{n-k}$. If $a_ix_j \subset S$ for some i and j, then x_j belongs to A_i contradicting that x_j is in $S \setminus \bigcup_{i=1}^{k} A_i$. But this contradicts that S has property $P_n(A^k;X;a_ix_j)$. Therefore $S \setminus \bigcup_{i=1}^{k} A_i$ contains at most n - k - 1 points.

Theorem 61: Let the subset S of a LTS have property $P_n(A^k;X;a_ix_j)$. Then S has at most k non-degenerate components.

The proof of Theorem 61 is very similar to that for Theorem 56 and will be omitted.

A subset S of E^n is said to have the n-point property if for every element p of k(S), there exist n points $q_1, \cdots, q_n$ of S and n non-negative numbers $\alpha_1, \cdots, \alpha_n$ for which $\sum_{i=1}^{n} \alpha_i = 1$ so that $p = \sum_{i=1}^{n} \alpha_i q_i$. Bunt [4] has proved that a subset S of E^n which has at most n components has the n-point property. Theorems 56 and 61 state that sets which have property P_n or property $P_n(A^k;X;a_ix_j)$ have a finite number of components. Consequently a subset S of E^{n-1} which has property P_n or property $P_n(A^k;X;a_ix_j)$ has the n-point property.

Theorem 62: Let the subset S be either open or closed in a linear topological space and have property $P_n(A^k;X;a_ix_j)$. Assume S has exactly k components $B_1, \cdots, B_k$ where a_m belongs to B_m, $m = 1, \cdots, k$. Then B_m is starlike from a_m.

Proof: Let x belong to B_m and assume that a_mx is not contained in B_m. Then there is an element $z = \alpha a_m + (1 - \alpha)x$ of a_mx with $0 < \alpha < 1$ so that z does not belong to B_m. If S is open, then there is a neighborhood U of x so that z does not belong to U and $a_mx \cap U$ is a segment since B_m is open and U may be chosen in B_m. Let $x_1, \cdots, x_{n-k-1}$ be points on $a_mx \cap U$. Then neither $a_jx \subset S$ $(j = 1, \cdots, k)$, $a_jx_i \subset S$ $(j = 1, \cdots, k$ and $i = 1, \cdots, n\text{-}k\text{-}1)$,

$a_m x \subset S$, nor $a_m x_j \subset S$ ($j = 1, \cdots, n-k-1$) which contradicts that S has property $P_n(A^k;X;a_i x_j)$. Thus B_m is starlike from a_m for $m = 1, \cdots, k$.

Next suppose that S is closed. Hence B_m is closed for $m = 1, \cdots, k$. Again suppose that $a_m x$ is not contained in B_m and let z be defined as above. Since B_m is closed, there exists a neighborhood U of z so that $U \cap B_m = \emptyset$. The point x is a limit point of B_m and, if B_m is non-degenerate, there exists a sequence of distinct points s_r of B_m which converges to x. The segments $a_m s_j$ converge to $a_m x$. Consequently there exists a sequence of points z_j where z_j belongs to $a_m s_j$ and so that $\{z_j\}$ converges to z. Also there exists a positive integer N so that if $r > N$, then z_r belongs to U. Consider the n - k points $s_{N+1}, \cdots, s_{N+n-k}$ of B_m. Then $a_j s_r$ is not contained in S for $j = 1, \cdots, k$ and $r = N+1, \cdots, N+n-k$ which contradicts that S has property $P_n(A^k;X;a_i x_j)$. Hence again B_m is starlike from a_m for $m = 1, \cdots, k$.

The following example in the plane demonstrates that it is necessary to require the set in Theorem 62 to be either open or closed. Let $S = \{(x,y) : 0 \leq x \leq 1, 0 < y < 1\} \cup \{(x,y) : 2 \leq x \leq 3, 0 < y \leq 1\} \cup \{(0,0), (1,0), (2,0), (3,0)\}$. Let $a_1 = (0,0)$ and $a_2 = (2,0)$. Then S has property $P_5(a_1 a_2;X;a_i x_j)$ and has exactly two components but neither component is starlike from a_i.

The σ-neighborhood of a starlike set is starlike. It is not possible to prove that the σ-neighborhood of a set with property $P_n(A^k;X;a_i x_j)$ also has this property as the following example in the plane demonstrates. Let $S = \{(x,0) : 0 \leq x \leq 1\} \cup \{(0,1)\}$ and $a = (0,0)$. Then S has property $P_3(a;X;ax_i)$ but the σ-neighborhood of S for $\sigma = 1/3$ does not have this property. It does follow, however, that if S has property $P_n(A^k;X;a_i x_j)$, then the σ-neighborhood of S is the union of n - 1 or fewer starlike sets by Theorem 60.

The next several theorems prove that sets which have property $P_n(A^k;X;T)$ for some other special choices of T can be decomposed into sets which are starlike or inverse starlike and extend further the results established in Theorems 53 and 60.

Theorem 63: Let the subset S of L have property $P_n(\emptyset;X;x_ix_j \bigcup \infty x_ix_j)$. Then S is the union of n - 1 or fewer starlike sets.

Proof: Obviously a set which has property $P_n(\emptyset;X;x_ix_j \bigcup \infty x_ix_j)$ also has property P_n. Therefore the result follows from Theorem 53.

Theorem 64: Let the subset S of L have property $P_n(A^k;X;\infty x_ia_j)$. Then S is the union of k inverse starlike sets and a finite set of n - k - 1 or fewer points.

Proof: Let $A_i = \{x \in S : \infty xa_i \subset S\}$ where $A^k = \{a_1, \cdots, a_k\}$ and $i = 1, \cdots, k$. Clearly each set A_i is inverse starlike from a_i. Suppose by way of contradiction that $S \setminus \bigcup_{i=1}^{k} A_i$ consists of at least n - k points $x_1, \cdots, x_{n-k}$. If $\infty x_ia_j \subset S$, then x_i belongs to A_j contradicting that x_i is in $S \setminus \bigcup_{i=1}^{k} A_i$. Thus ∞x_ia_j is not contained in S for $i = 1, \cdots, n - k$ and $j = 1, \cdots, k$. But this contradicts that S has property $P_n(A^k;X;\infty x_ia_j)$. Therefore $S \setminus \bigcup_{i=1}^{k} A_i$ contains at most n - k - 1 points.

Theorem 65: Let the subset S of L have property $P_n(a;X;\infty ax_i)$. Then S is the union of a cone with vertex at a and a set of at most n - 2 points.

Proof: It is first proved that S has property $P_n(a;X;ax_i \bigcup \infty x_ia \bigcup \infty ax_i)$.

Hence let $x_1, \cdots, x_{n-1}$ be elements of S. Then $\infty ax_i \subset S$ for some i. Let $y_1, \cdots, y_{n-1}$ be points on ∞ax_i. Then for some j, $\infty ay_j \subset S$. But $ax_i \bigcup \infty x_i a \subset \infty ay_j \subset S$ which proves that S has the stated property.

Let $A = \left\{ x \in S : ax \bigcup \infty xa \subset S \right\}$ which is clearly a cone with vertex at a. Assume by way of contradiction that $S \setminus A$ contains at least n - 1 points $z_1, \cdots, z_{n-1}$. If $\infty az_i \subset S$, then as was proved above $az_i \bigcup \infty z_i a \subset S$ which implies that z_i is in A, a contradiction. But this contradicts that S has property $P_n(a;X; \infty ax_i)$. Therefore $S \setminus A$ contains at most n - 2 points.

It is not possible to decompose a set which has property $P_n(A^k;X; \infty a_i x_j)$ for $k \geq 2$ into a finite number of starlike and inverse starlike sets as is proved by the following example in the plane. Let $A = \left\{ (x,y) : x \leq -1, x + 1 \leq y \leq -x - 1 \right\}$, $B = \left\{ (x,y) : x \geq 1, -x + 1 \leq y \leq x - 1 \right\}$, $C_m = \left\{ (1/m,y) : (1/m) - 1 \leq y \leq (-1/m) + 1 \right\}$, and $D_m = \left\{ (-1/m,y) : (1/m) - 1 \leq y \leq (-1/m) + 1 \right\}$ with m = 2, 3, $\cdots$. Then define $S = A \bigcup B \bigcup (\bigcup_{m=2}^{\infty} C_m) \bigcup (\bigcup_{m=2}^{\infty} D_m)$ and $a_1 = (-1,0)$ and $a_2 = (1,0)$. Then S has property $P_n(A^2;X; \infty a_i x_j)$ (see Figure 9).

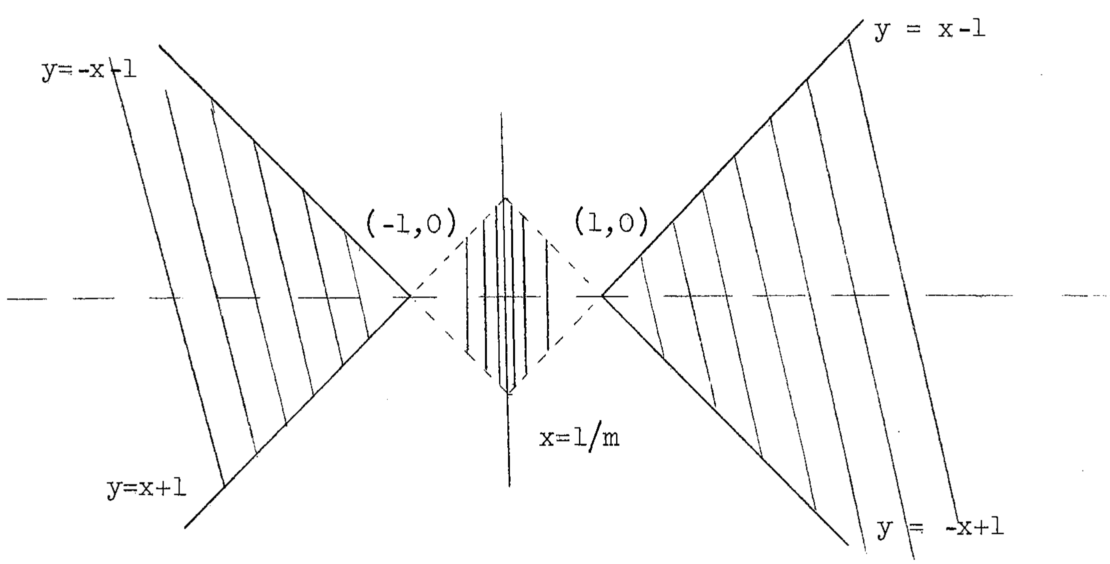

Figure 9

Theorem 66: Let S have property $P_n(A^k;X;a_ix_j \bigcup \infty x_ja_i)$. Then S is the union of k cones and a finite set of n - k - 1 or fewer points.

Proof: Let $A_i = \left\{x \in S : a_ix \bigcup \infty xa_i \subset S\right\}$ for i = 1, $\cdots$, k and where $A^k = \left\{a_1, \cdots, a_k\right\}$. Clearly each set A_i is a cone with vertex at a_i. Assume by way of contradiction that $S \setminus \bigcup_{i=1}^{k} A_i$ contains at least n - k points $x_1, \cdots, x_{n-k}$. If $a_ix_j \bigcup \infty x_ja_i \subset S$ for some i and j, then x_j belongs to A_i contradicting that x_j is in $S \setminus \bigcup_{i=1}^{k} A_i$. But this contradicts that S has property $P_n(A^k;X;a_ix_j \bigcup \infty x_ja_i)$. Therefore $S \setminus \bigcup_{i=1}^{k} A_i$ contains at most n - k - 1 points.

V-4: A Generalization of a Theorem Due to Brunn

It has already been pointed out that the set of points from which a closed set in E^2 is starlike is closed and convex (see Brunn [3]). The following theorem, which will be used to generalize this result, proves that Brunn's result is true for any linear topological space.

Theorem 67: Let A be a closed subset of a linear topological space. Then ker(A) is closed and convex.

Proof: Let K = ker(A). It is first proved that K is convex. Hence let a and b belong to K and let $c = \alpha a + (1 - \alpha)b$ be on ab with $0 < \alpha < 1$. It must be proved that A is starlike from c. Let x belong to A and prove $cx \subset A$. Since a and b belong to K, $ax \subset A$, $bx \subset A$, and $ab \subset A$ so that c belongs to A. Let $d = \sigma c + (1 - \sigma)x$ be on cx with $0 < \sigma < 1$. The segment $bx \subset A$ implies that $y = \rho b + (1 - \rho)x$ belongs to A for $0 < \rho < 1$. Thus $ay \subset A$ for every y on bx since A is starlike from a. Let $\rho = \sigma(1 - \alpha)/(1 - \sigma\alpha)$ with $1 - \sigma\alpha \neq 0$ since $0 < \alpha < 1$ and $0 < \sigma < 1$; also this value of ρ satisfies $0 < \rho < 1$. Then

$y = \frac{\sigma(1 - \alpha)}{1 - \sigma\alpha} b + (1 - \frac{\sigma(1 - \alpha)}{1 - \sigma\alpha})x$ and $d = \sigma\alpha a + (1 - \sigma\alpha)y$. Therefore d is on ay since $0 < \sigma\alpha < 1$ so that d is in A and hence $cx \subset A$. But this implies that A is starlike from c and that c belongs to K. Therefore K is convex.

Next it is proved that K is closed. Let p be a limit point of K and show that p belongs to K, that is, show A is starlike from p. Let x belong to A. The space L is Hausdorff so there exists an infinite sequence of distinct points of K, say $\{p_i\}$, which converges to p. Then p_i in K implies that $p_i x \subset A$ since p_i is in K for $i = 1, 2, \cdots$. Thus the sequence of segments $p_i x$ converges to px which implies that $px \subset A$ since A is closed. Thus A is starlike from p so that p belongs to K which proves that K is closed.

It should be noted that in Theorem 67 the proofs that K is convex and K is closed are independent.

The kernel of a set S may be described as the set of points a of S so that S has property $P_2(a;x;ax)$. A natural generalization of the notion of the kernel could be described as the set of points a of S (or of L) so that S has property $P_n(a;X;T)$. The final two theorems of this chapter show that for special choices of T this generalized kernel for closed sets is also closed and convex.

Theorem 68: Let S be a closed subset of a LTS. Then the subset $K = \{a \in S : S \text{ has property } P_n(a;X;ax_i)\}$ is closed and convex.

Proof: It is first proved that if $K \neq \emptyset$, then S has exactly one non-degenerate component. Assume that S has two non-degenerate components A_1 and A_2 and assume $A_1 \cap K \neq \emptyset$. Let a belong to $A_1 \cap K$ and $x_1, \cdots, x_{n-1}$ belong to A_2. Then clearly ax_i is not contained in S for any i and j which contradicts that S has property $P_n(a;X;ax_i)$. Furthermore, if $K \neq \emptyset$, then S has exactly one non-degenerate component since for an element a of K there exists an element x of S so that $x \neq a$ and $ax \subset S$. Also it is clear that S can have no more than n - 2 degenerate components since the contrary would contradict that S has property $P_n(a;X;ax_i)$ for some a in K. Therefore $S = A \cup \{b_1, \cdots, b_m\}$, $m \leq n - 2$, where A is the non-degenerate component of S. Obviously $K \subset A$.

It is next proved that K = ker(A), and since A is closed, the conclusion will follow from Theorem 67. Let a belong to K, x belong to A, and prove $ax \subset A$. Let $\{U_k\}$ be a sequence of neighborhoods of x closing down on x. Assume that all of the sets U_k are distinct and for each positive integer k let $x_1^k, \cdots, x_{n-1}^k$ be elements of $U_k \cap A$. Then for each k there exists an element x_i^k so that $ax_i^k \subset A$ since A has property $P_n(a;X;ax_i)$. Also the sequence of such points x_i^k converges to x as k tends to ∞. Since A is closed, and since the segments ax_i^k converge to ax, $ax \subset A$. Therefore A is starlike from a which proves that $K \subset \ker(A)$. Obviously if a belongs to ker(A), then a belongs to K. Therefore K = ker(A) and by Theorem 67 it follows that K is closed and convex.

It is necessary in Theorem 68 to assume that S is closed as the following example in the plane proves. Let $S = \{(x,y) : 0 < x < 1, 0 < y < 1\} \cup \{(1/2,0), (1/4,0)\}$. Then for $n > 2$, K = S which is neither closed nor convex.

<u>Theorem 69</u>: Let S be a closed subset of a LTS. Then the subset $H = \{a \in L : S \text{ has property } P_n(a;X; \infty x_i a)\}$ is closed and convex.

<u>Proof</u>: It is first proved that H is convex. Assume that H contains

more than one point. Let a and b be elements of H and let $c = \lambda a + (1 - \lambda)b$ with $0 < \lambda < 1$. Let $x_1, \cdots, x_{n-1}$ be elements of S. Then it must be proved that there is a positive integer $i \leq n-1$ so that $\infty x_i c \subset S$ or that $y = \sigma x_i + (1 - \sigma)c$ belongs to S for $\sigma \geq 1$. There exists an x_k $(1 \leq k \leq n-1)$ so that $\infty x_k a \subset S$; that is, $z = \alpha x_k + (1 - \alpha)a$ belongs to S for $\alpha \geq 1$. It will be proved that $\infty x_k c \subset S$.

The next argument demonstrates that for each z on $\infty x_k a$, $\infty zb \subset S$. Certainly z is a limit point of $\infty x_k a$ in the line topology on $\infty x_k a$. Hence let the sequence of points p_i of $\infty x_k a$ converge to z. Next partition the set of positive integers as follows: $I_1 = \{1, \cdots, n-1\}$, $I_2 = \{n+1, \cdots, 2(n-1)\}$, $\cdots$, $I_m = \{n+(m-1), \cdots, m(n-1)\}$, $\cdots$. Then for every positive integer m, there is a positive integer i_m in I_m so that $\infty p_{i_m} b \subset S$. Also since $\{p_{i_m}\}$ converges to z, $\{\epsilon p_{i_m} + (1 - \epsilon)b\}$ converges to $\epsilon z + (1 - \epsilon)b$ for $\epsilon \geq 1$ which must belong to S since S is closed. Thus $\infty zb \subset S$ for each z on $\infty x_k a$.

If $\alpha \geq 1$, $z = \alpha x_k + (1 - \alpha)a$ belongs to S and if $\theta \geq 1$, $\theta z + (1 - \theta)b$ belongs to S as was proved above. In particular let $\alpha = \sigma/(\lambda + (1 - \lambda)\sigma) \geq 1$ and $\theta = \lambda + (1 - \lambda)\sigma \geq 1$. Then $\theta z + (1 - \theta)b = \sigma x_k + (1 - \sigma)c$ which belongs to S. Therefore H is convex.

It is next proved that H is closed. Let q be a limit point of H with $\{q_i\}$ a sequence of points of H converging to q. Again let $\{x_1, \cdots, x_{n-1}\} = X$ be elements of S and show there is some x_k so that $\infty x_k q \subset S$. For each q_m there is an x_{i_m} of X so that $\infty x_{i_m} q_m \subset S$ for $m = 1, 2, \cdots$. Since X is finite there exists some x_k of X so that $\infty x_k q_m \subset S$ for infinitely many of the points of $\{q_m\}$, say for $\{q_{m_i}\}$. But $\{q_{m_i}\}$ also converges to q and hence $\{\alpha x_k + (1 - \alpha)q_{m_i}\}$ converges to $\alpha x_k + (1 - \alpha)q$, $\alpha \geq 1$, which must belong to S since S is closed. Therefore q belongs to H which proves that H is closed.

V-5: An Ordering for the Classes of Subsets With Property $P_n(A^k;X;T)$

Property $P_n(A^k;X;T)$ determines a class or collection of subsets of L, namely the class of subsets of L which have this particular property. Hence with each property $P_n(A^k;X;T)$ is associated a class of subsets of L which will be denoted by $\overline{P}_n(A^k;X;T)$. The collection of all such classes may be ordered by inclusion. For example, $\overline{P}_3(\emptyset;X;x_ix_j) \subset \overline{P}_4(\emptyset;X;x_ix_j)$. The problem naturally arises as to the precise nature of this order relation--can it be described in terms of the n, k, A^k, and T? If $\overline{P}_n(A^k;X;T)$ and $\overline{P}_m(B^p;X;T')$ are two such classes of sets, and if $B^p \subset A^k$, $n-k \geq m-p$, and either $T' \subset T$ or each segment of type T' includes a segment of type T, then $\overline{P}_m(B^p;X;T') \subset \overline{P}_n(A^k;X;T)$. The converse of this statement seems also to be true although the problem has not yet been fully investigated. All such classes of sets for which n = 2 have been determined and the remainder of this chapter is devoted to the description and ordering of these classes.

Consider first the classes of sets with property $P_2(\emptyset;X;T)$. There are 31 possible choices for the types of segments T. However, only 7 of these yield distinct classes of sets and these 7 possibilities are:

(1) x_1x_2

(2) ∞x_ix_j

(3) $x_1x_2 \cup \infty x_ix_j$

(4) $x_1x_2 \cup \infty x_1x_2 \cup \infty x_2x_1$

(5) x_1x_2, ∞x_ix_j

(6) x_1x_2, $\infty x_1x_2 \cup \infty x_2x_1$

(7) $x_1x_2 \cup \infty x_ix_j$, $\infty x_1x_2 \cup \infty x_2x_1$.

The lattice structure for these seven classes using the inclusion order relation is given in Figure 10.

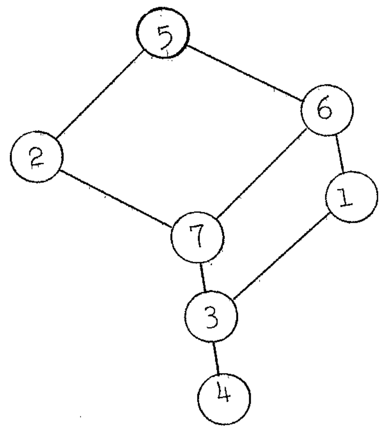

Figure 10

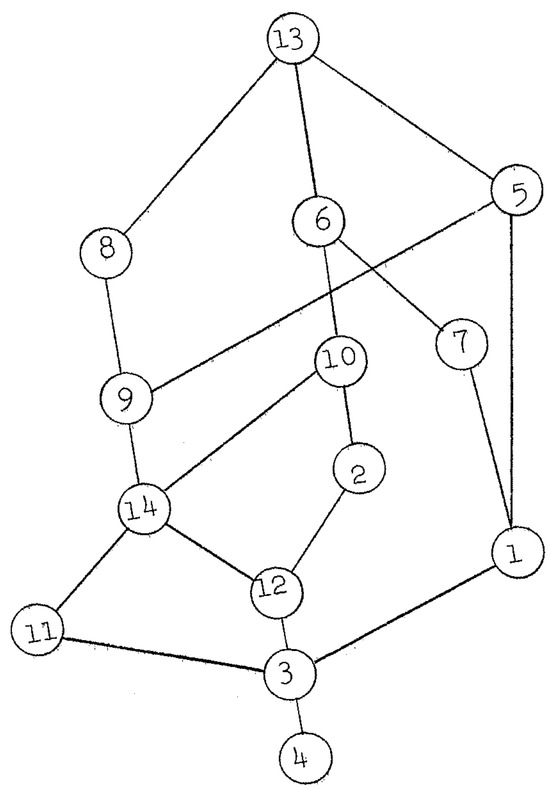

Figure 11

Finally consider the other possibility for n = 2, namely $P_2(a;x;T)$. There are 127 possible choices for T in this case. However, only 14 of these yield distinct classes of sets and they are:

(1) ax	(8) co ax, co xa
(2) co xa	(9) co ax, ax $\bigcup$ co xa
(3) ax $\bigcup$ co xa	(10) co xa, ax $\bigcup$ co ax
(4) ax $\bigcup$ co ax $\bigcup$ co xa	(11) ax $\bigcup$ co ax, ax $\bigcup$ co xa
(5) ax, co ax	(12) ax $\bigcup$ co xa, co ax $\bigcup$ co xa
(6) ax, co xa	(13) ax, co ax, co xa
(7) ax, co xa $\bigcup$ co ax	(14) ax $\bigcup$ co ax, ax $\bigcup$ co xa, co xa $\bigcup$ co ax.

The lattice structure for these 14 classes using the inclusion order relation is given in Figure 11.

For property $P_3(\emptyset;X;T)$ there are $2^{511} - 1$ possible choices for T and, of course, not all of these yield distinct classes of sets. A more economical technique has yet to be found for determining the distinct possible classes for $n > 2$.

CHAPTER VI

SUMMARY AND UNSOLVED PROBLEMS

The primary purpose of this paper has been to develop, and in some cases extend, properties of certain classes of subsets of a linear space. The class of inverse starlike sets has been rather thoroughly investigated. A metric space of starlike sets was discussed. The class of all starlike subsets of a linear space was shown to have an addition and scalar multiplication operation which have all the properties of a vector space addition and scalar multiplication except for additive inverses and even a restricted cancellation law. An order relation was defined on the class of all starlike subsets of L and with this ordering, the class was shown to be a complete complemented lattice which is not distributive and not modular. Finally a notion of generalized convexity was defined which includes convex, projectively convex, starlike, inverse starlike, property P_3, cone, and flat all as special cases. Basic theorems were proved for this generalization and it was shown that many of these classes of sets consist of sets which are the union of starlike and inverse starlike sets. A theorem of Brunn was generalized to some special classes of sets determined by the generalized convex property.

Throughout the paper several references have been made to problems that were unsolved or only partially solved. The purpose of this chapter is to bring together and summarize these questions and others which have been raised in connection with this research.

The properties of certain subsets determined by a quadratic form are discussed at the end of Chapter II. The question is raised concerning the properties of certain other subsets determined by a positive definite quadratic form.

A partial solution is given to the problem of conditions under which $\infty(S + T)a = \infty Sb + \infty Tc$ where $b + c = a$ in Theorem 27. A necessary and sufficient condition on S to insure that $\bigcup_{a \in S} \infty Sa = L$ remains to be determined. Similar problems arise in connection with Theorems 22, 23, and 28.

Variations in the definition of property $P_n^r(A^k;X;T)$ such as omitting the finite character of A^k and X and using other than linear segments poses a problem for further research.

The intersection and sum of sets with property $P_n(A^k;X;T)$ remain to be characterized. The topological properties of sets with property $P_n(A^k;X;T)$ such as the closure and interior have not been determined. Can a set which has property P_n be expressed as the union of n or fewer convex sets? If a closed set S in a LTS has the property that for every n points $x_1, \cdots, x_n$ on the boundary, then at least one segment $x_i x_j \subset S$, does S have property P_n? If S has property $P_n(A^k;X;T)$, then what property, if any, does C(S) have? Can a set which has property $P_n(\emptyset;X; \infty x_i x_j)$ be decomposed into a union of a finite number of starlike or inverse starlike sets?

The nature of the ordering of the classes $\overline{P}_n(A^k;X;T)$ has not been fully investigated and a technique has not been determined for finding all the distinct classes for property $P_n(A^k;X;T)$ for a given value of n.

Property $P_n(A^k;X;T)$ generalizes convexity. A similar generalization for convex functions can be made and the properties of such functions investigated.

If f is a function defined on the reals to the reals, and if $K = \{(x,y) : y \geq f(x),\ x > 0\}$ is inverse starlike from (0,0), then f is sub-additive. The properties of functions for which K has other properties $P_n(A^k;X;T)$ are not known.

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