A VARIATIONAL METHOD FOR THE SIMULATION OF SYSTEMS WITH DIVERSE FREQUENCIES

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CHAPTER I

INTRODUCTION

In recent years, an increased importance has been placed on the solution of dynamic system equations. This has resulted largely from the development of digital computing systems which has opened new fields in numerical engineering analysis and system optimization and control. The speed and versatility of the computer has made it feasible to perform dynamic analyses on systems which may be far too complex to study in any other way. The practicality of utilizing the digital computer in generating dynamic system responses has given rise to the development of new and varied numerical integration algorithms as well as the unveiling of older methods which previously found no application.

These algorithms have been implemented in many user oriented simulation programs and languages. The programs are usually very general in design allowing the engineer to easily transfer the system equations into the form required for simulation. The availability of such programs has made it possible to generate a solution for virtually any set of ordinary differential equations and to then have the solution displayed in both tabular and graphical form.

One practical consideration which must be made in almost any dynamic engineering analysis concerns the cost of system simulation. Before a dynamic analysis of a system can be of real value to the engineer, the results of the analysis must be of sufficient value to justify the

expense involved. One of the major contributors to the cost of such a computerized analysis is the actual expense incurred as computer time. Since the cost of obtaining a computer solution to any problem is directly dependent on the amount of computation time required, it is desirable to minimize this time to improve efficiency.

Many times physical systems are encountered which do not lead to efficient digital solutions. This is, of course, ambiguous without a definition to outline the characteristics of an "efficient" method for the solution of dynamic system equations. What is meant here is that the amount of useful information obtained from the solution of the equations must be sufficient to justify the computation effort required. This is still very general, but it will become more understandable as the purpose of this study is defined.

Consider a system as shown in Figure 1. This type of system is often encountered in engineering analyses, and leads to a definitely inefficient solution. The system is basically low-frequency as is demonstrated by the outputs only being explicitly dependent on the low-frequency components. Internally, there are also high-frequency components whose outputs are of only secondary importance in relation to the input-output characteristics of the total system. The high-frequency effects will be seen as a pertubation of the primary outputs and will often be of minor importance in determining the low-frequency responses.

This type of system will be considered to lead to an inefficient solution because the amount of computation effort required to generate the solution is not directly dependent on the primary outputs of the system. Although the importance of knowing the system response may justify the simulation expense, the solution itself is here considered

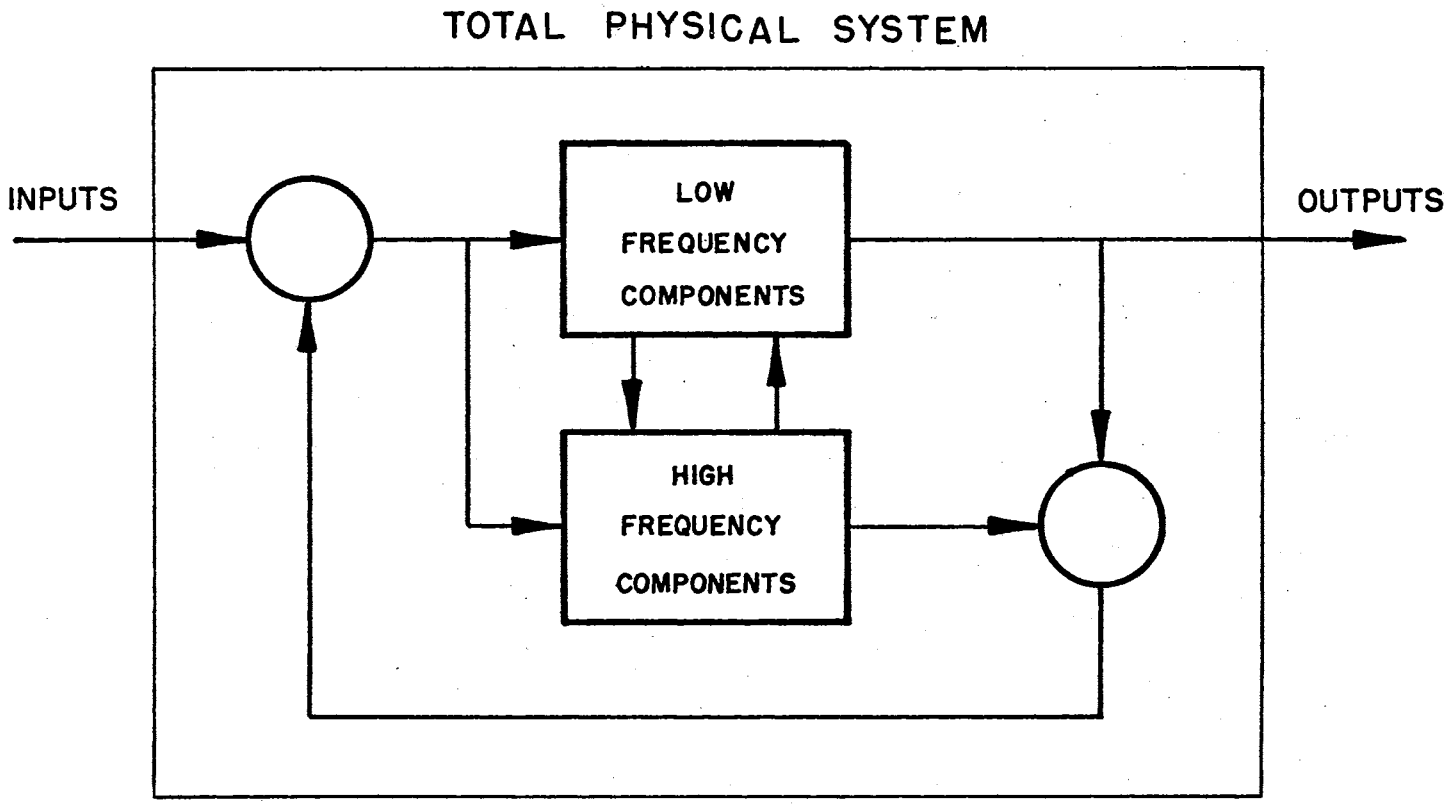

Figure 1. A System With Low-Frequency Outputs

to be inefficient due to the nature of the existing simulation methods.

Simulation programs and numerical integration methods are usually intended for application to a broad class of problems. Usually, the user is required to supply only two types of information which will completely define the problem in the form required by the simulation program. This information consists of the derivatives of the system states with respect to the independent variable, considered here to be time, plus certain other parameters which are used to control the program. Then, by evaluating the derivatives according to the requirements of the chosen numerical integration algorithm, the program generates a solution for the equations.

Critical in the generation of the solution is the integration time step or the change in the independent variable per step in the integration method. The time step is directly dependent on the highest frequency component present in the total dynamic system, and is usually defined as some fraction of the time constants or the period of oscillation associated with that component. This is discussed in more detail in Chapter II, but first the system in Figure 1 will be reconsidered.

The system shown in this figure is, as has been stated, basically low-frequency; and therefore it is desirable to base the integration step and subsequently the simulation cost on the low-frequency components. However, present methods require that the step size be selected by considering only the highest frequency component of the system which here is not even of primary interest as a system output. This discrepancy can lead to orders of magnitude increases in the required computataion time as will be shown in the following example.

The fluid power system shown in Figure 2 is typical of hundreds of

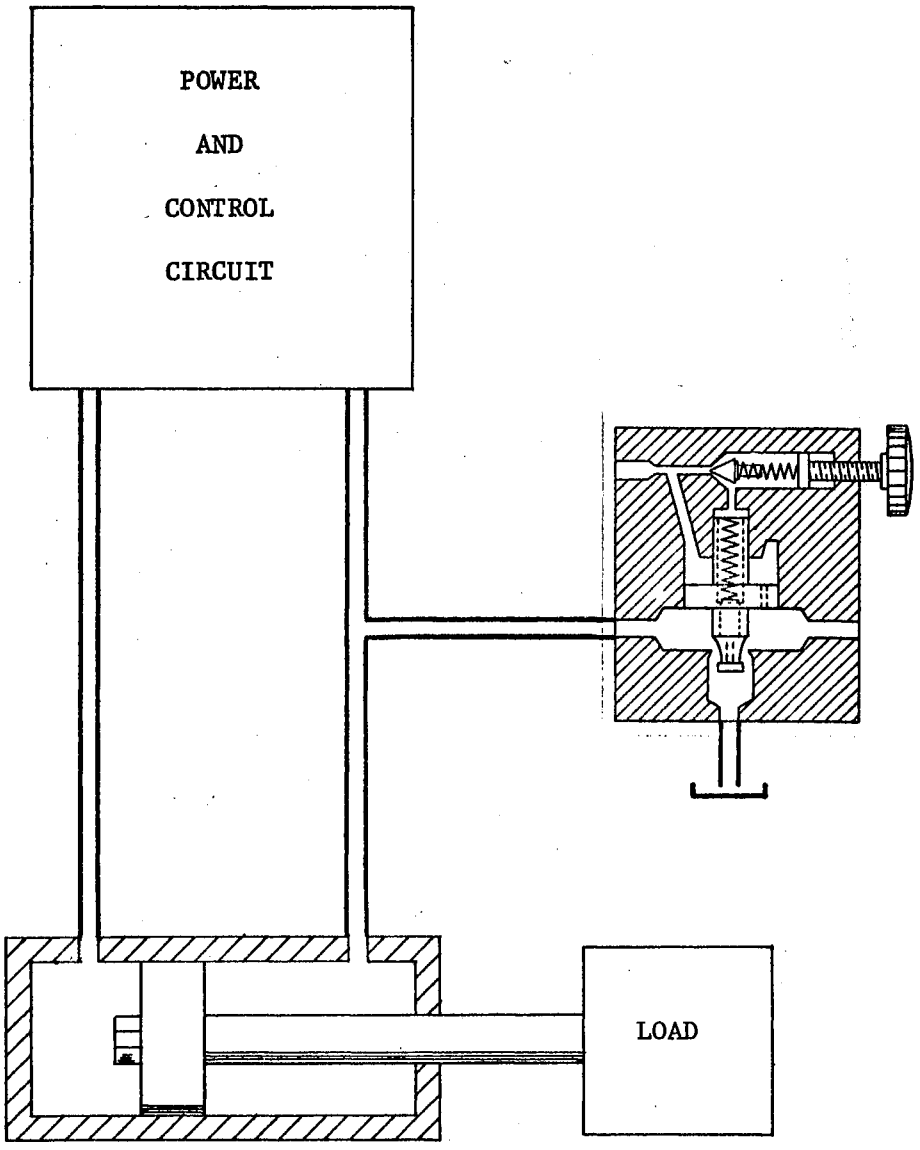

Figure 2. An Example of a "Low-Frequency" System

existing applications. For simulation purposes, the power and control circuit is often considered to have very little dynamics with the exception of the supply pressure, and the primary output is most often the response of the motor circuit. Here, the motor is shown as a simple linear actuator with an inertial load and is protected by a two-stage relief valve. It is desirable to simulate the response of the piston, but what "cost" will be involved in determining this response?

If the hydraulic cylinder was considered to have some known input pressure and the relief valve could be ignored, then experience shows that the integration step size will nominally range from 0.0001 to 0.001 seconds. Add the dual-stage relief valve, and the step size must be reduced to 0.0000001 to 0.000001 seconds. Thus, by including the effects of a component which does not explicitly determine the system output, the required computation "cost" has been increased by 100-10000. This is certainly an undesirable result, and it must be eliminated if the dynamic analysis of complex systems containing secondary high-frequency components is to become more attractive.

The purpose of this thesis is to present the results of an investigation of an approximate integration method based on the variational principle of mechanics. The method is intended for the simulation of systems such as those shown in Figures 1 and 2 in which the primary outputs of the system are basically low-frequency, and high-frequency components are present in only an indirect manner. The investigation has been limited to considering only one possible variational method and was not concerned with the development of a user-oriented simulation program. The study includes a review of the existing numerical integration methods, and a survey of the related topics from the areas of

linear and non-linear system analysis. The method is developed in Chapter III and applied to two detailed examples in Chapter IV. Chapter V presents the conclusions and recommendations for further work with an accuracy study being discussed in Appendix A.

The results of the example problems and the accuracy study show that the variational method of simulation is a valid and promising technique. The example problems demonstrated that the method gives accurate solutions while allowing the integration step size to be dependent only on the low-frequency system components. For the example problems, this resulted in a 75-94% reduction in computation time over the time required by a conventional method.

CHAPTER II

SURVEY OF RELATED TOPICS

It is not difficult to identify a large number of physical systems which lead to the type of dynamic representations discussed in Chapter I. This type of model occurs frequently in the analysis of mechanical and fluid power systems, and numerous other examples can be found in areas both in and out of the field of engineering.

Despite the common occurrence of systems having low-frequency outputs, there is a definite lack of literature pertaining directly to the simulation of such systems. Seemingly, it is assumed that once a dynamic model has been developed for a system the available simulation programs will be adequate to generate the dynamic response. This implies that there are no restrictions placed on the allowable cost of the simulation, but it is probably a reasonable assumption since there is no other alternative. There does not appear to be any method which has been developed specifically for the solution of sets of differential equations containing widely varying frequencies. This does not imply that there is no need for such an algorithm, but rather that the previous work has been directed mainly toward the solution of general sets of differential equations.

The absence of previous work in this area makes it necessary to deviate from the usual practice of presenting the pertinent results of a literature survey per se. There are, however, several related topics

which will serve as a background for the development of the algorithm in Chapter III. These topics are considered in the following discussion as they apply to the algorithm.

Numerical Integration Methods

The concept of calculating numerical solutions to differential equations is certainly not a recently proposed topic, since references show that one algorithm, the crude Euler method, was proposed two hundred years ago (3). Most of the methods which are in wide usage today were developed at a time when the absence of the digital computer made the application of the algorithms impractical. The advent of the computer has made it feasible to implement these algorithms while also providing an impetus for the derivation of new techniques.

The fundamental problem in numerical integration is to find the solution to the first-order equation

$$\frac{dX}{dt} = f(X,t),$$

which satisfies the initial condition on the X variable. More generally, X may be an n-dimension vector with n given initial conditions. The method then proceeds by increments h in the independent variable t and generates a pointwise solution for the equation or equations.

There are numerous ways in which one might attempt to classify numerical integration methods, depending on the comparison which is being made. One common criterion is to consider the methods on the basis of the accuracy which can be expected per step in the solution.

The accuracy of integration methods is considered to be a function of the time step plus other parameters which are not free to the user.

This is usually expressed by saying that the error is of the order h^p, meaning that the terms which have been ignored in the derivation of the method are of the order h^{p+1}. This estimate of the accuracy of an integration method is a result of the derivation of the method as opposed to being experimentally determined.

Since the accuracy of a numerical solution to a differential equation is directly dependent on the time step, it is necessary to have some guidelines to use in determining the step size for any given problem. The problem of fitting the step size to the differential equation is a critical area in nearly any solution, and it can have a profound influence on the resulting trajectory.

There are two topics to be considered in the selection of an integration step. The first of these is the accuracy of the digital computer. Even though theoretically the most accurate solution to a differential equation can be found by letting the step size approach zero, the accuracy of the digital computer does not allow this extreme. Since the computer can only operate with a finite number of significant digits in any calculation, there is a lower limit to the allowable step size. If the step size is chosen below this limit, the truncation error encountered in the solution of the problem will tend to negate the effect of time, and cause the solution to stay basically constant. The selection of a step size which is much too large often has the effect of driving the solution unstable. This can be thought of as applying a Taylor's series expansion of a function outside of the region of convergence and then expecting the results to be correct. If the function was then re-expanded about the incorrectly predicted point, the next prediction could diverge even further from the true function. This

provides a reasonable analogy to what occurs when too large a step size is selected. The numerical solution deviates from the exact solution, and the results become meaningless with respect to the original problem. The one redeeming characteristic of an excessively large step size is that it usually becomes quite evident that the solution is incorrect. This is not so obvious in the case of small step sizes.

A guideline for selecting the integration step can most easily be stated by considering a homogeneous, linear example. This gives a vector equation of the form

$$\dot{X}(t) = A\ X(t),$$

where A is a time invariant matrix. Denoting the eigenvalues of A as the vector D, the time step associated with the system can be selected as

$$h = B\ \mathrm{Min}(|1/\mathrm{Re}(D)|,\ |2\pi/\mathrm{Im}(D)|),$$

where the scalar B ranges from 1/10 to 1/100. This is surely a very general specification for a critical parameter, and a more specific rule could probably be derived if only linear systems were to be simulated. Since most accurate models contain non-linear terms and, thus, make the above guideline unusable per se, there is no need to be more specific. It is only necessary to realize that the time step must be chosen as some fraction, generally 1/10 to 1/100, of the smallest "time constant" or period of oscillation present in the system. (The quotation marks are used to indicate that whereas non-linear systems can have truly periodic solutions, the exponential responses may be quite perturbed.) The selection of a time step which is too small leads to truncation

errors and excessive computation time, while too large a step tends to give unstable solutions.

In addition to being classified as to their accuracy, the finite-difference integration methods can also be classified according to general types of algorithms. There are two general types, predictor-corrector and predictor. The predictor methods are based on the fundamental theorem of integral calculus and result in one formula which is used to predict a new solution point based on presently available information. The predictor-corrector methods extend this process through a backward-difference approximation for the function to yield a corrector formula. This formula then utilizes the original information plus the predicted results to give a corrected prediction. Reference (2) contains a thorough discussion of the details of several methods.

It is more important here to consider the philosophy of finite difference integration algorithms than to consider the details of any one method. Regardless of the method, the basic result is a formula of the form

$$X(t+h) = x(t) + h\,F(X,t),$$

where the function $F(X,t)$ may be a weighted sum of several derivative evaluations plus a linear combination of past values for X. An important result is that while many methods give solutions which are quite accurate, the methods must be restricted to a step size h which will allow the numerical solution to follow the exact solution. This leads to the type of problem posed in Chapter I. It also gives some information concerning a possible alternative.

Solutions of the kind generated by the equation shown above are

based on a finite amount of information. The derivatives are evaluated at pre-determined, discrete points in time. However, the derivative function at these points may not be indicative of the fundamental portion of the solution which is often as important for engineering considerations as the exact solution. This is demonstrated by the equation

$$\dot{X} = -X + \sin(20\pi t), \qquad X(0) = 1.$$

It is easily recognized that the solution will closely follow $X(t) = e^{-t}$, but this cannot be realized by considering the values of the derivative function for various times. Evaluation of the derivative along the exact solution will give values in the range $-2 < \dot{X} < 1$. This is quite different from the fundamental solution which would only give $-1 \leq \dot{X} < 0$.

If an algorithm is required to generate the outputs of a system having high-frequency components using a step size based on the low-frequencies, then that algorithm must be less dependent on discrete values of the derivatives. It must be able to follow the "trend" of the exact response while perhaps sacrificing some accuracy. It seems reasonable to consider a method which would generate a solution based on an infinite amount of information as opposed to discrete quantities.

This suggests that a completely new approach must be taken in order to develop a suitable algorithm. Some of the requirements which must be placed on the method are that it be "usable" as are the majority of the existing methods. The solutions generated must be of a reasonable accuracy, and, above all, the computation time must be reduced in comparison to present methods. Failure to meet the last criterion is a failure to solve the problem.

Linear Analysis Considerations

The concepts and methods available within the field of linear system theory undoubtedly provide some of the most rigorous analysis tools available to the engineer. The theories apply equally well to any set of linear dynamical equations, and the major problems encountered can almost be summarized as complications arising from the algebra involved. Unfortunately, the dynamic representations of most physical systems are non-linear, and it becomes impossible to apply the concepts of linear analysis directly.

Even though the theory often cannot be applied in its entirety, there are at least two concepts of linear analysis of interest here. These concepts do not lend directly to the development of the algorithm, but rather help to form a basis for a later assumption.

The general form of the system representation for a linear system is

$$\dot{X}(t) = A\,x(t) + B\,U(t), \qquad X(0) = X_0.$$

Here, the system is assumed to be time invariant and the control $U(t)$ is taken to be a piece-wise continuous function with a finite number of discontinuities. The general solution is well known as

$$X(t) = \varphi(t,0)\,X_0 + \int_0^t \varphi(t,\tau)\,B\,u(\tau)\,d\tau,$$

where φ is the state transition matrix. This solution is termed the "zero-input response" plus the "zero-state response" as can be easily understood by inspecting the two terms. A question now arises. Since the system response can explicitly be divided into the effects due to initial conditions and the effects due to the control, can the response

of the individual states be similarly divided? That is, does each state have a total response which can explicitly be divided into two parts which are due to the initial condition of only that state plus all other effects? One answer can be obtained by reformulating the general solution to linear differential equations.

Instead of considering the most simple form of a system representation, attention will be given to Figure 3. This block diagram is admittedly redundant, but it lends an interesting insight into the response of coupled states in a system. The system has been divided into its high and low-frequency components which are denoted X_h and X_ℓ, respectively. This is the equivalent of partitioning the state vector into

$$[X] = \left[\frac{X_h}{X_\ell}\right].$$

Since X_h is independent of X_ℓ, it is possible to find the solution for X_h as

$$X_h(t) = \varphi_h(t,0)X_{h0} + \int_0^t \varphi_h(t,\tau)\ B_h\ u(\tau)\ d\tau.$$

Define an augmented control vector as

$$[U_A(t)] = \left[\frac{U(t)}{X_h(t)}\right].$$

The calculation of the augmented A and B matrices make possible the solution

$$X_\ell(t) = \varphi_\ell(t,0)\ X_{\ell 0} + \int_0^t \varphi_{\ell A}(t,\tau)\ B_{\ell A}\ u_A(\tau)\ d\tau.$$

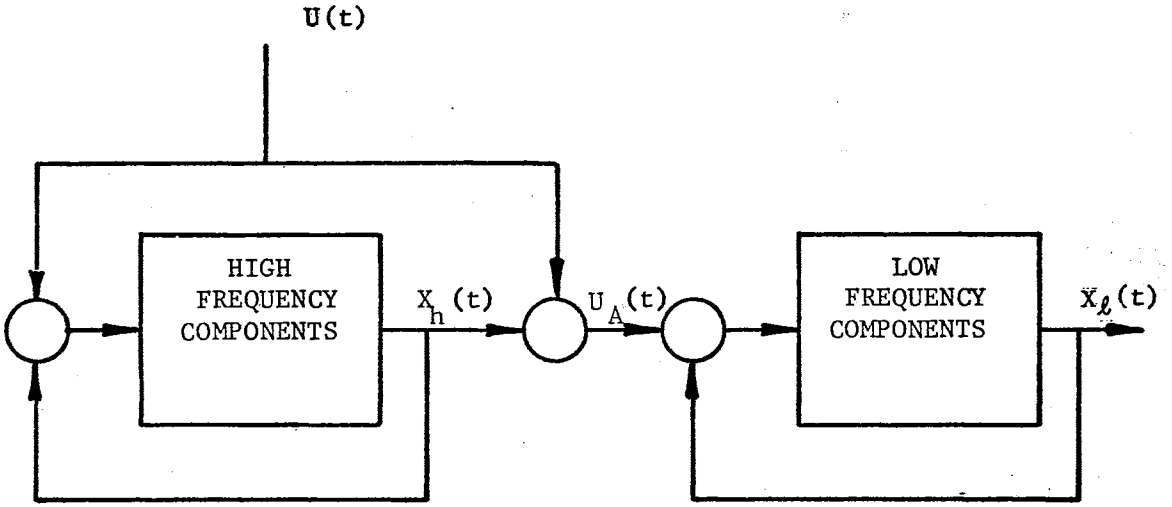

Figure 3. An Unsimplified Block Diagram Representation

Thus, the response of one portion of the system has been shown to be composed of effects due to its own initial conditions plus a response to the input and the other states of the system. It is interesting to note that the effect due to the remaining states can be re-defined as an input, and that this input appears within the convolution integral.

At this point, it is not significant that the system was partitioned into high and low-frequency components. It is also insignificant that X_h was independent of X_ℓ. This may not be obvious, but it can be realized by redesigning the system. Suppose that X_ℓ consists of only one state and that it does feed back to X_h. Generate the total system solution and then augment the input by a function equal to $X_\ell(t)$. The removal of the feedback to X_h from X_ℓ will give an equivalent system in which the response of X_ℓ is only dependent on its own initial condition plus a twice-augmented input vector. Similarly, the response of any state could be decomposed to show that the individual states have responses made up of a homogeneous portion plus the effects of some equivalent input. This becomes more significant in the following discussion which reviews frequency response analysis.

Frequency response analysis or Bode response magnitude analysis provides an important argument in the justification of the algorithm presented in this thesis. The results of this type of analysis reveal an important characteristic of the response of systems having high-frequency effects contributing to low-frequency outputs, and this can be used to good advantage in an integration algorithm.

The analysis consists of determining the magnitude of the ratio of a system output to its input over a range of input frequencies. The input is assumed to be sinusoidal, thereby making it convenient to

simply evaluate the transfer function for the system as the frequency is varied. The results are usually presented on some type of logarithmic scales making it convenient to determine the slope of the asymptotic response curves in terms of either decades or decibels.

The response of both first and second-order systems is shown for reference in Figure 4. The graph shows that as the input frequency increases, the response of the system not only decreases but becomes insignificant with respect to the input magnitude. This might be interpreted by saying that the system is acting as a low-pass filter and does not transmit high-frequency effects.

Response analysis is normally concerned with the over-all input-output characteristics of a system, but this is not necessary. It was shown above that each individual component of a system could be isolated and considered to be a sub-system in itself. The frequency response of each of these sub-systems could then be determined, and the results would usually be represented by Figure 4 since few system components are described by more than second-order models. The obvious results would be that the low-frequency components of any system tend to filter high-frequency effects. The exact response for any low-frequency will consist of some fundamental response with small high-frequency effects superimposed upon it.

Since it can be reasonably assumed that high-frequency effects become insignificant for the type of system of interest here, a question arises as to the importance of the solution for the high-frequency states. If the low-frequency states tend to filter the high-frequency contributions, it may not be necessary to generate "exact" high-frequency solutions. If the "exact" solutions could be approximated by

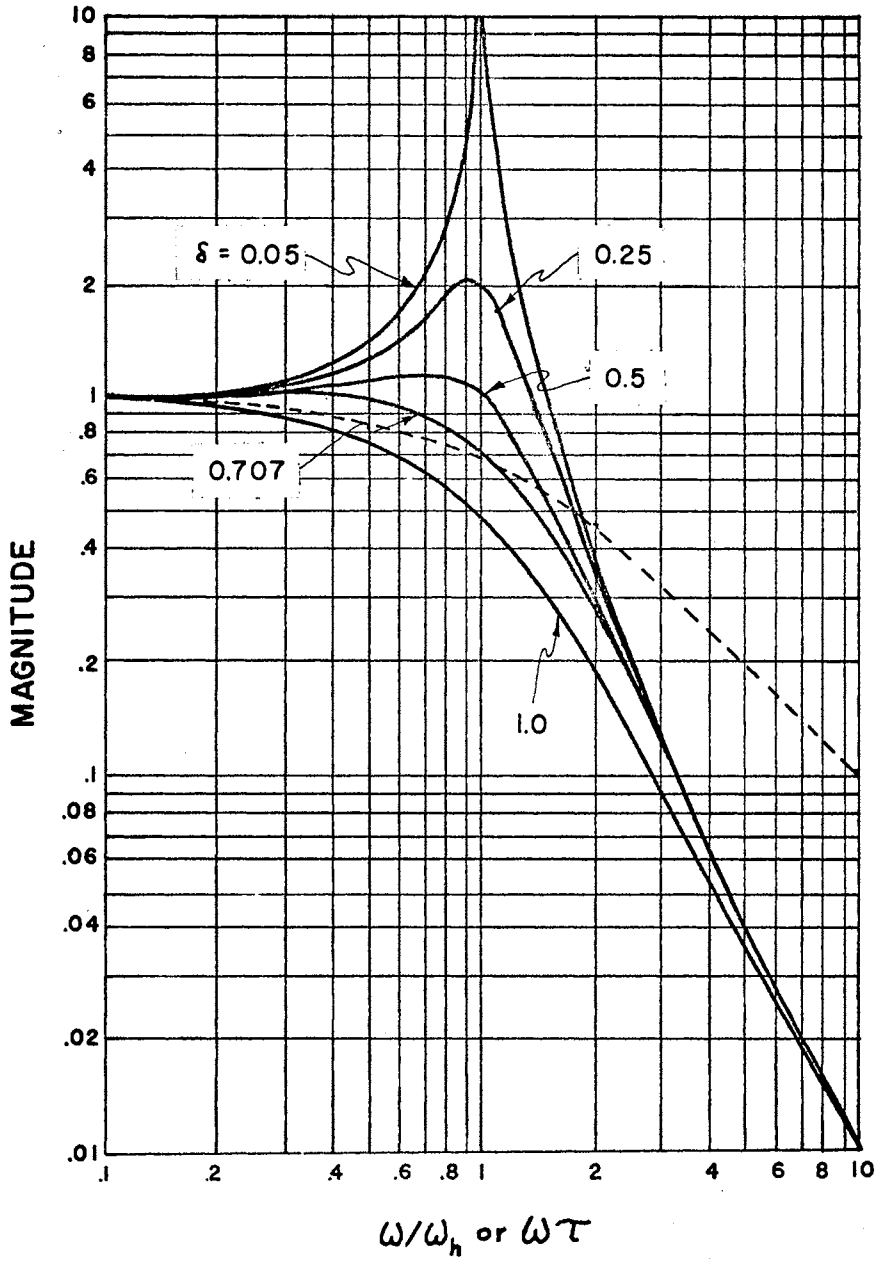

Figure 4. Response of First and Second-Order Systems

a trajectory which itself filters the oscillatory effects, then this approximation could be assumed to represent a nearly equivalent input to the low-frequency states. The use of this approximation concept would make it possible for a numerical integration method to span several periods of a high-frequency response in a single integration step. It would require that these high-frequencies be approximated by a trajectory which followed the fundamental lower frequency portion of their responses, thus making it impossible to investigate the "exact" state of the faster-responding components.

Neither linear analysis nor present simulation methods lend any information as to how such a solution might be generated. Also, the frequency response argument has only been presented for linear systems which are quite rare in practical applications. Therefore, it is necessary to also consider non-linear system theory and to derive from it the basis for the integration algorithm.

Non-Linear Analysis Considerations

Non-linear system analysis can be divided into two very different types of problems and techniques. One area is concerned with the study of the stability, existence, periodicity, and other characteristics of the solution of the system equations. The other area is concerned with determining approximate solutions to the equations. This generally excludes the use of the digital computer by dealing with graphical and approximate analytical solutions. The solution methods and specific results are of interest here.

There is no direct non-linear analogy to Bode frequency response analysis. At best, the response characteristics may be determined for a

particular problem or class of problems, with much being left to the experience of the engineer.

An example of the type of frequency response analysis which is involved with non-linear systems is shown by Arnold (1). The problems which he solved dealt with determining the response of systems with non-linear dynamic vibration absorbers. The non-linearity was present as either a hardening or softening in the coupling spring for the dynamic absorber. The results of this work indicate that at least one particular class of non-linear systems tends to filter high-frequency effects. (For more references in this area, the reader is referred to the Bibliography contained in Arnold's paper (1).)

It would be desirable to conclusively state that all non-linear systems respond as those studied by Arnold. This is not possible, and it is necessary to rely on a rather intuitive discussion of the expected response.

Many times, systems are referred to as being "nearly linear". This tends to imply that the response is a pertubation of some fundamental linear response, and that the system demonstrates basically linear characteristics. The continuation of this type of argument leads to the assumption that nearly linear systems demonstrate frequency response characteristics which follow linear response curves.

Systems which are very non-linear cannot be intuitively considered with the same ease. However, if the first assumption is that very non-linear systems do not respond to high-frequency inputs, then a type of argument can be constructed.

If a non-linear systems is being perturbed about some steady-state type value, then the effect of the non-linearity will become minimal.

The response of the system will be the same as that which would be predicted by linearization about the operating point, thereby effectively removing the non-linearity. However, this is totally dependent on the assumption that the system is not responding to its input. Without actually analyzing all of the possible non-linear system configurations, this assumption is probably best justified by engineering judgment. Experience shows that systems can withstand high-frequency inputs without being appreciably disturbed, and since almost all physical systems are non-linear it does not seem unreasonable to assume that non-linear systems will in general damp the effects of high-frequency inputs.

Although non-linear analysis does not represent a well-defined science applicable to all types of problems, it does provide a host of very useful techniques for determining approximate solutions to differential equations. These methods can be classed as either graphical or mathematical approximations of the true solution. The mathematical approaches are of particular interest in this discussion.

The approximate analytical solution methods used in non-linear analysis include pertubation techniques, equivalent linearization, the method of slowly varying amplitude and phase, and the Ritz-Galerkin averaging techniques. Each of these types of methods provides a procedure for determining an approximate solution to a differential equation by fitting an assumed form in some best sense. The definition for best is generally left to the user, and there is no assurance as to which type of method will give the most accurate approximation for any given equation and assumed solution. Although the mechanics of the methods differ greatly, the results can be summarized by stating that the algorithms make it possible to determine numerical values for free

parameters which appear in assumed solutions for the equations (4) (5).

The averaging techniques which are often referred to as the Ritz-Galerkin method can be generalized to many types of problems other than non-linear analysis. The basic result is that an integral is being minimized in some sense, and that the integrand is chosen such that it represents the error between the exact and assumed solution. This type of method was first proposed by Ritz in 1908, and suprisingly enough was not first applied to the solution of non-linear equations. Rather, the method was applied to the minimization of a integral associated with a boundary-value problem involving partial differential equations. The method was later applied to problems in solid mechanics and non-linear analysis. In 1915, Galerkin presented a method which was somewhat simpler to apply, and this method is usually used in reference to Ritz-Galerkin averaging technique (5).

Moneymaker (7) has proposed three methods which are based on the variational principle of mechanics and are shown to be applicable to very broad classes of problems. The derivations rely heavily on a physical insight into the variational principle and its application to the response of dynamic systems. It is shown that the results contain the Ritz method, the Galerkin method, and the method of slowly varying amplitude and phase, thereby proving to be a desirable basis for determining non-linear responses.

The derivation of the virtual work method of variational analysis shown in Chapter III gives the first method proposed by Moneymaker. The essential difference occurs in the selection of the approximating solution which is generally assumed to be of the same form as the exact solution. This assumption is not necessary, but it is usually presented

as being vitally important in determining the "goodness of fit" which is obtained in the analysis. However, if the technique which is used to determine the free parameters in the approximating solution truly gives a best fit in some sense, then it seems reasonable that even a very poorly fitting solution could be assumed. The fit would be less accurate, but the method would remain valid. This is discussed further in Chapter III after the virtual work principle is developed.

CHAPTER III

DEVELOPMENT OF THE INTEGRATION ALGORITHM

The discussion in Chapter II considered the general response characteristics of systems with low-frequency outputs. Several techniques used in non-linear analysis were mentioned briefly, and in this chapter the variational method based on the virtual work principle will be developed. The necessary considerations in selecting an approximating solution are then presented, and the chapter ends with a statement of the algorithm.

Hamilton's Modified Principle

One method for determining free parameters in assumed solutions for differential equations can be derived by considering the Lagrangian formulation of the equations of motion. This classical method results in

$$\left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{X}_1}+\frac{\partial L}{\partial X_1}+F_1\right)\delta X_1+\left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{X}_2}+\frac{\partial L}{\partial X_2}+F_2\right)\delta X_2+\cdots$$

$$+\left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{X}_n}+\frac{\partial L}{\partial X_n}+F_n\right)\delta X_n=0,$$

where:

L = the Lagrangian

= Kinetic energy - Potential energy

X_i = the i^{th} generalized coordinate

F_i = the i^{th} generalized force

δ is the variation.

This is a statement that the virtual work of a system at each point in time is zero. Define E as an operator which operates on the vector X(t) as:

$$E_i = - \frac{d}{dt} \frac{\partial L}{\partial \dot{X}_i} + \frac{\partial L}{\partial X_i} + F_i$$

$$= E_i(X,t).$$

This leads to the condensed notation:

$$\sum_{i=1}^{n} E_i \, \delta X_i = 0.$$

Since the system is assumed holonomic, the generalized coordinates are independent. The δX_i are in general not zero which leads to the set of equations:

$$E_i = 0, \qquad i = 1, 2, \ldots, n,$$

which are the Euler-Lagrange equations for a non-conservative system.

It is possible to select n approximating solutions each containing one free parameter and to then determine the parameters by substitution into the above equations. This requires that the time be specified, and it gives a solution only applicable at this discrete time. This does not lead to a valuable solution method, and it is usually disregarded as an approximate solution technique.

The Euler-Lagrange equations can also be derived from Hamilton's

Principle which states that for conservative systems the line integral:

$$A = \int_{t_1}^{t_2} L \, dt,$$

evaluated along the path of motion is an extremum. For an extremum to occur, the calculus of variations requires that:

$$\delta A = 0$$

$$= \delta \int_{t_1}^{t_2} L \, dt$$

$$= \int_{t_1}^{t_2} \delta L \, dt$$

where t_1 and t_2 are fixed points in time. This can be extended to Hamilton's modified Principle for non-conservative systems (11):

$$0 = \int_{t_1}^{t_2} \left(\delta L + \sum_{i=1}^{n} F_i \, \delta X_i \right) dt$$

$$= \int_{t_1}^{t_2} E_i \, \delta X_i \, dt \qquad i = 1, 2, \ldots, n. \qquad (1)$$

This is a statement that the integral of the virtual work over any definite period is zero.

It is possible to assume solutions for the independent coordinates of the form:

$$\overline{X}_i = \overline{X}_i(f_{ij}, a_{ij}), \qquad i = 1, 2, \ldots, n$$

$$j = 1, 2, \ldots, m,$$

where the f's are functions of time and the a's are assumed to be independent parameters. Substituting the $\overline{X}_i$ into E_i gives $\overline{E}_i$ and also:

$$\delta X_i = \sum_{j=1}^{m} \frac{\partial X_i}{\partial a_{ij}} \delta a_{ij}, \qquad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m. \end{matrix}$$

Subsequent substitution into (1) gives:

$$0 = \int_{t_1}^{t_2} \overline{E}_i \sum_{j=1}^{m} \frac{\partial X_i}{\partial a_{ij}} \delta a_{ij} \, dt, \qquad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m. \end{matrix}$$

Since the a's are assumed to be independent arbitrary parameters and δa is in general not zero the integral becomes:

$$0 = \int_{t_1}^{t_2} \overline{E}_i \frac{\partial X_i}{\partial a_{ij}} \, dt, \qquad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m. \end{matrix} \qquad (2)$$

The completion of the integration gives $n \cdot m$ independent equations which can be solved for the $n \cdot m$ parameters a_{ij}.

This is the result which Moneymaker (7) presents as the virtual work method. It is an extremely versatile technique and can be shown to contain several other methods of non-linear analysis as special cases.

In order to apply the method, it is necessary to select the approximating solutions $\overline{X}$. It is also necessary to set certain guidelines which make it possible to implement the method in a simulation program.

The Approximating Solution

The approximate solutions are usually selected such that they are

of the same form as the expected exact solution. This implies that periodic solutions are approximated as sinusoids, transients as exponentials, etc. This type of selection process is not practical for an algorithm which must be applicable to a broad class of problems.

Additional restraints are placed on the approximate solution by the intended utilization of the digital computer. It is desirable to have the equations which result from the analytical integrations of a form which can be easily solved by a numerical algorithm. The solutions must contain an adequate number of free parameters to allow a good fit to the exact response, and it is necessary that the solutions be differentiable and easily integrable. This will facilitate the substitutions and integrations implied by Equation (2).

It has been implied that each solution must contain m parameters and m functions of time. This constraint is purely for convenience, and there is no general restraint on either the number of parameters or the number of functions. Two of the following solutions demonstrate how these numbers might vary.

One candidate for a general solution is the exponential function

$$X_i(t) = \sum_{j=1}^{m} a_{ij} e^{z_{ij}t},$$

where the a_{ij} is an undetermined real number and the z_{ij} could at least theoretically be either a real or complex number. This form seems reasonable since it is a general solution to linear systems. However, it does not lend to ease of integration when combined with other functions, and the resulting algebraic equations would be difficult to solve.

Another possible solution is

$$\bar{X}_i = \sum_{j=1}^{m} a_{ij} \sin (\omega_{ij} t + b_{ij}),$$

where ω, b, and a are all free parameters. This is the solution assumed by the method of slowly varying amplitude and phase ($m = 1$ and ω specified). Extending it to include other than the fundamental response to give a Fourier type of approximate solution leads to problems in the integration. The problems arise both due to the difficulties involved in actually performing the integrations and due to the orthogonality characteristics of sinusoids.

Probably the most simple approximation for any continuous function can be found using a Taylor's series expansion. Since the expansion simply results in a polynomial in time, it is reasonable to assume a solution of the form

$$\bar{X}_i = \sum_{j=1}^{m} a_{ij} t^{j-1}.$$

In addition to being easily manipulated, this type of solution also has other desirable characteristics. If it is assumed that the solution method will determine the a_{ij} such that they are nearly equal to the corresponding derivatives of X, then it can be shown that the series will converge for some t. Also, if one series converges and a second converges absolutely, then the sum and product will converge as well as the derivative and integral of each (8). Thus, within the limits of one assumption, this solution can be considered valid for some t.

The substitution of the power series into (2) requires:

$$\frac{\partial \bar{X}_i}{\partial a_{ij}} = t^{j-1} \qquad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m. \end{matrix}$$

This reduces the Equation (2) to:

$$0 = \int_{t_1}^{t_2} \overline{E}_i(\overline{X},\ t)t^{j-1}\ dt, \qquad \begin{matrix} i = 1,\ 2,\ \ldots,\ n \\ j = 1,\ 2,\ \ldots,\ m. \end{matrix} \tag{3}$$

(This is a Galerkin integral since the parameters a_{ij} appear only as coefficients of the selected functions.) The integral can now be changed to give a form which will be easier to apply.

Assume the solution is desired over some interval from t_0 to t_f and

$$t_0 \leq t_1 < t_2 \leq t_f.$$

Define the numerical integration step size h as

$$h = t_2 - t_1.$$

Also define

$$z = t - t_1$$

$$dz = dt.$$

Substitution into (3) and changing the limits of integration gives:

$$0 = \int_0^h \overline{E}_i(\overline{X}(t+z),t+z)\ z^{j-1}dz, \qquad \begin{matrix} i = 1,\ 2,\ \ldots,\ h \\ j = 1,\ 2,\ \ldots,\ m. \end{matrix} \tag{4}$$

This is the form of the integral which will be used in the simulation algorithm. It can be applied to any set of differential equations which result in integrable functions $\overline{E}_i z^{j-1}$, but no provision has yet been made for including the influence of initial conditions.

The Initial Conditions

Thus far no consideration has been given to including the initial conditions of a system into the solution. Also, no distinction has been made between the solution for the low-frequency states of a system as compared to the high-frequency responses. Another topic which must be considered is the total number of terms to be included in the power series solutions, since obviously some finite number must be used. All of these areas can be considered by example.

Consider a system of equations:

$$\dot{X}_1 = X_2 \qquad X_1(0) = X_{10}$$

$$\dot{X}_2 = -X_1 \qquad X_2(0) = X_{20},$$

which will be assumed to be low-frequency. This requires that the approximate solution must <u>follow</u> the exact solution. Assume the solutions:

$$\overline{X}_1 = a_{11} + a_{12}z + a_{13}z^2 + a_{14}z^3$$

$$\overline{X}_2 = a_{21} + a_{22}z + a_{23}z^2 + a_{24}z^3$$

with derivatives

$$\dot{\overline{X}}_1 = a_{12} + 2\,a_{13}z + 3\,a_{14}z^2$$

$$\dot{\overline{X}}_2 = a_{22} + 2\,a_{23}z + 3\,a_{24}z^2.$$

In order for the solution to fit the initial conditions, fix a_{11} and a_{21} as:

$$a_{11} = X_{10}$$

$$a_{21} = X_{20}.$$

The differential equation also requires that

$$\dot{\overline{X}}_1(0) = \overline{X}_2(0),$$

which requires

$$a_{12} = X_{20}.$$

The power series then become:

$$\overline{X}_1 = X_{10} + X_{20}z + a_{13}z^2 + a_{14}z^3$$

$$\overline{X}_2 = X_{20} + a_{22}z + a_{23}z^2 + a_{23}z^3.$$

The derivative of $\overline{X}_1$ is:

$$\dot{\overline{X}}_1 = X_{20} + 2\,a_{13}z + 3\,a_{14}z^2.$$

This requires that the power series include a minimum of three terms if $\dot{\overline{X}}_1$ is to demonstrate any dependence on time.

The specification of certain of the parameters to include the initial conditions for low-frequency states then has at least two effects. The number of free parameters is reduced by at least one per state. Also, it requires that m must be chosen large enough to insure that each derivative has at least one degree of freedom.

Since the time step used in this method is to be based on the low-frequency responses, special consideration must be given to the high-frequency initial conditions. If the large step size is used, then one single solution step will span at least one complete high-frequency oscillation. However, the effect of the power series approximation of a sinusoid will tend to filter the oscillations and should predict only

the "steady state" portion of the high-frequency response. This can only occur if the high-frequency initial conditions are relaxed, and the method is allowed to determine the best fit initial value. If such a relaxation is not made, the effect is the equivalent of weighting one point in a curve-fitting problem more than any other point. The result will be a skewed fit, and experimentation has shown that the high-frequency solutions diverge if the initial conditions are not relaxed.

The relaxation of the high-frequency initial conditions is one concept which makes this numerical integration technique different than other methods. It allows the determination of a truly approximate high-frequency solution thereby making it possible to simulate complex systems without altering the system model.

The total algorithm has been developed in this chapter including the specification of the approximating solutions. It is now possible to present the complete numerical integration algorithm in a step-by-step fashion.

Outline of the Algorithm

1. Formulate the system equations in first-order form.
2. Partition the state vector into the low-frequency and high-frequency states. Determine the smallest time constant or period of oscillation in each portion of the vector.
3. Select a step size h as $^1/_5$ to $^1/_{20}$ of the low-frequency value from 2.
4. If the high-frequency value from 2 is greater than the step size, use a conventional method.

5. Determine which a_{ij}'s in

$$\overline{X}_i = \sum_{j=1}^{m} a_{ij} z^{j-1}$$

are fixed by initial conditions on the low-frequency states. Select m.

6. Substitute $\overline{X}_i$ into the differential equations and perform the integrations

$$0 = \int_0^h E_i(\overline{X}(t+z),\ t+z) z^{j-1}\ dz$$

for each z^{j-1} not preceded by a specified a_{ij}.

7. Solve the algebraic equation from 6 for a_{ij}.

8. Substitute all a_{ij} into the power series in 5 to determine $\overline{X}(t+h)$.

9. Repeat 7 through 9 with $t = t+h$ until $t+h$ equals the final time.

CHAPTER IV

APPLICATIONS

The versatility of the algorithm presented in Chapter III cannot be realized except by example. The derivation does not give any information concerning the amount of computation time that might be required in comparison to other methods, and it is also necessary to investigate the accuracy of the method.

Two example problems have been selected to demonstrate the application and capabilities of the variational method. Example problem one demonstrates the handling of a non-linearity and an analytical input function, and the second is an application to a system with an internal high-frequency component. A study was also made which compares the accuracy of the variational method with a Runge-Kutta fourth-order (Kutta-Simpson) algorithm. The results are shown in Appendix A.

Example Problem One

Fluid power systems provide many examples of low-frequency system outputs coupled to other high-frequency components. One such system was shown in Chapter I, Figure 1, in which the two-stage relief valve was assumed to cause oscillations in the pressure applied to the cylinder. This problem can be simplified by assuming the instantaneous pressure to be a known function of time. This simplification leads to the following problem.

The cylinder in Figure 5 has a known pressure applied across the ports. The damper provides both viscous and velocity squared damping, and it is desired to determine the velocity response of the piston. Assume the following information:

$X(0) = \dot{X}(0) = 0$

$P = 1000 + 50 \sin(\omega t)$

Damping Force $= B\dot{X} + C\dot{X}^2$

Piston Diameter = 4 inches

$M = 1158\ lb_m$

$B = 90\ lb_f$ sec./in.

$C = 180\ lb_f$ $sec.^2/in.^2$

$\omega = 2000\ \pi$/sec.

$A = \pi d^2/4 = 12.57\ in.^2$

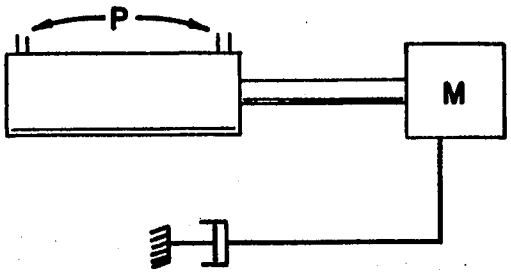

Figure 5. System for Example Problem One

Since the velocity response is desired, define a variable

$$V = \dot{X}.$$

The equation of motion is then:

$$\frac{M}{g}\dot{V} + BV + CV^2 = PA, \qquad V(0) = 0.$$

The form required for simulation is:

$$\dot{V} + \frac{g}{M}(BV + CV^2 - PA) = 0.$$

To determine the response, assume the solution:

$$\bar{V}(t+z) = a_0 + a_1 z + a_2 z^2.$$

The initial conditions require:

$$\bar{V}(0) = V_0.$$

Then:

$$\bar{V}(t+z) = V(t) + a_1 z + a_2 z^2$$

$$\dot{\bar{V}}(t+z) = a_1 + 2a_2 z.$$

The functions which have undetermined coefficients are z and z^2. To determine the parameters, a_1 and a_2 perform the substitutions to give the integral equations:

$$0 = \int_0^h (\dot{\bar{V}} + \frac{g}{M}(B\bar{V} + C\bar{V}^2 - PA)) \begin{bmatrix} z \\ z^2 \end{bmatrix} dz$$

$$0 = \int_0^h (a_1 + 2a_2 z + \frac{g}{M}(B(V_0 + a_1 z + a_2 z^2) + C(V_0 + a_1 z + a_2 z^2)^2 -$$

$$A(1000 + 50\sin(\omega(t+z)))) \begin{bmatrix} z \\ z^2 \end{bmatrix} dz.$$

Rearrangement after performing the integrations gives the two equations:

$$\left(\frac{1}{2}h^2 + \frac{1}{3}\left(\frac{Bg}{M} + \frac{2V_oCg}{M}\right)h^3\right)a_1 + \frac{Cg}{4M}h^4a_1^2 + \frac{2Cg}{5M}h^5\,a_1a_2$$

$$+\left(h^2 + \frac{1}{3}\left(\frac{Bg}{M} + \frac{2V_oCg}{M}\right)h^3\right)a_2 + \frac{Cg}{6M}h^6\,a_2^2 =$$

$$\frac{g}{M}\left[-\frac{1}{2}BV_oh^2 - \frac{1}{2}CV_o^2h^2 + 500Ah^2 + A\left(\frac{50}{\omega^2}\sin(\omega(t+z)) - \frac{50z}{\omega}\right.\right.$$

$$\left.\left.\cos(\omega(t+z))\right)_{z=0}^{z=h}\right]$$

$$\left(\frac{1}{3}h^3 + \frac{1}{4}\left(\frac{Bg}{M} + \frac{2V_oCg}{M}\right)h^4\right)a_1 + \frac{Cg}{5M}h^5a_1^2 + \frac{Cg}{3M}h^6\,a_1a_2$$

$$+\left(\frac{2}{3}h^3 + \frac{1}{4}\left(\frac{Bg}{M} + \frac{2V_oCg}{M}\right)h^4\right)a_2 + \frac{Cg}{7M}h^7a_2^2 =$$

$$\frac{g}{M}\left[-\frac{1}{3}BV_oh^3 - \frac{1}{3}CV_o^2h^3 + \frac{1000Ah^3}{3} + A\left(\frac{100z}{\omega^2}\sin(\omega(t+z))\right.\right.$$

$$\left.\left.-\frac{50(\omega^2z^2 - 2)}{\omega^3}\cos(\omega(t+z))\right)_{z=0}^{z=h}\right].$$

These equations are evaluated from z = 0 to z = h at time t such that:

$$V(t) = V_o.$$

The time step for this problem can be chosen by assuming that the response will be similar to an exponential and that the time constant can be crudely approximated as:

$$\tau = \frac{1}{\frac{g}{M}(B+C)} \approx 0.01 \text{ sec.}$$

Select the time step as:

$$h = 0.1\ \tau = 0.001 \text{ sec.}$$

Conventional integration methods would require:

$$h_c = 0.1(2\pi/\omega) = 0.0001 \text{ sec.}$$

This implies that the number of necessary integration steps will be reduced to only 0.1 of the steps required by conventional methods.

This problem was solved for $0 \leq t \leq 0.2$ seconds using an iterative method for solving the non-linear equations. The response of the piston is shown in Figure 6. The variational method indicated that the piston reached a constant velocity of 8.07 in./sec. in approximately 0.1 seconds.

The problem was also solved using a fourth-order Runge-Kutta method. This solution indicated that the steady-state oscillation due to the pressure fluctuation caused the final velocity to vary within $8.078 \leq V_{ss} < 8.114$. This certainly demonstrates that the method is sufficiently accurate for many engineering analyses. More information concerning the actual computer time required to obtain the solutions can be found at the end of the chapter.

Example Problem Two

The second problem involves determining the response of the system represented in Figure 7. This system could be thought of as a second-order response with high-frequency noise in the feedback, where the noise is being modeled as a constant amplitude oscillation. The equations of motion for the response are:

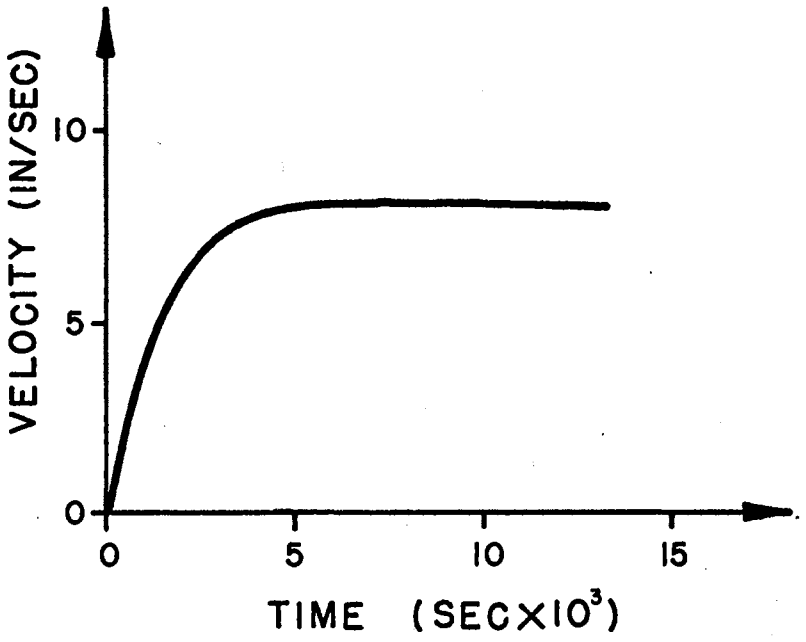

Figure 6. Piston Response From Example One

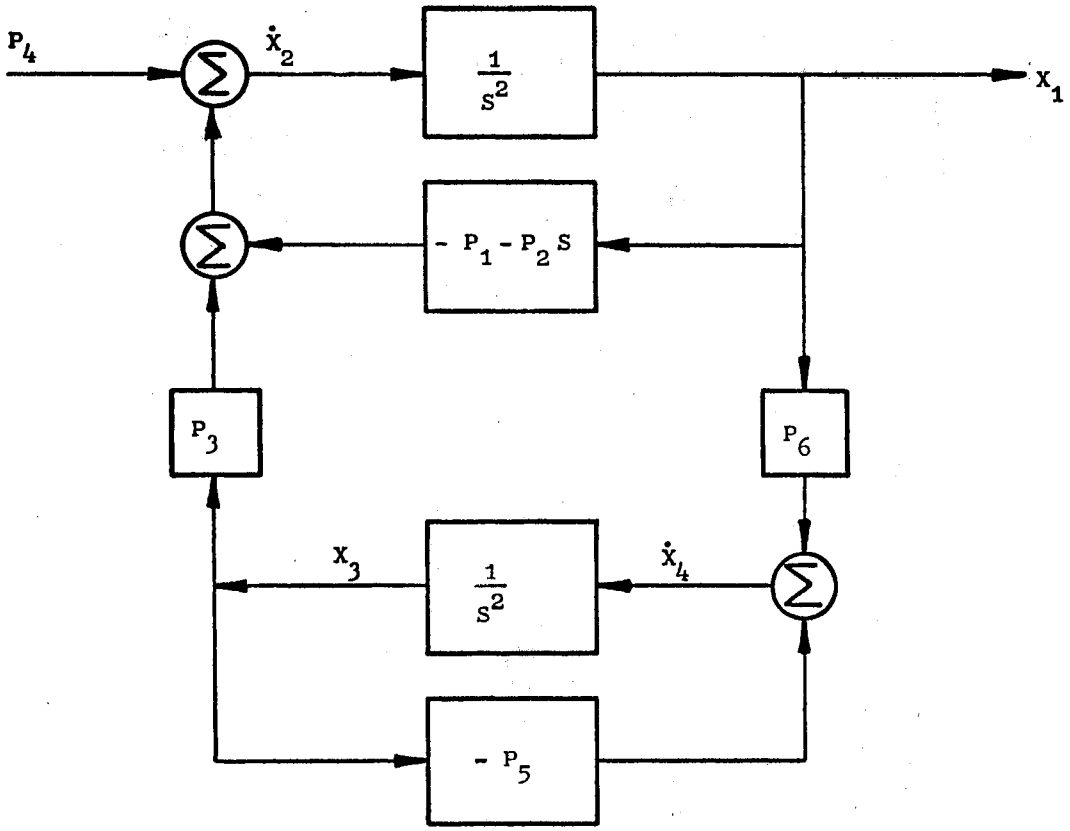

Figure 7. Block Diagram for Example Problem Two

$$\dot{X}_1 = X_2 \qquad X_1(0) = 0$$

$$\dot{X}_2 = -P_1X_1 - P_2X_2 + P_3X_3 + P_4 \qquad X_2(0) = 0$$

$$\dot{X}_3 = X_4 \qquad X_3(0) = 1$$

$$\dot{X}_4 = - P_5X_3 + P_6X_1 \qquad X_4(0) = 0.$$

The response is to be determined for two cases.

	Case 1	Case 2
P_1	25	25
P_2	2.5	2.5
P_3	200	200
P_4	20	20
P_5	562500	22500
P_6	56250	2250

To determine the solution using the variational method, assume:

$$\bar{X}_1(t+z) = a_{11} + a_{12}z + a_{13}z^2 + a_{14}z^3$$
$$\bar{X}_2(t+z) = a_{21} + a_{22}z + a_{23}z^2 + a_{24}z^3$$
$$\bar{X}_3(t+z) = a_{31} + a_{32}z + a_{33}z^2$$
$$\bar{X}_4(t+z) = a_{41} + a_{42}z + a_{43}z^2.$$

The responses $X_3(t)$ and $X_4(t)$ are assumed to be high-frequency and do not appear as outputs of the system. According to the algorithm, the initial conditions for these states must be relaxed making it necessary to consider only the initial conditions for $X_1(t)$ and $X_2(t)$. This is done by letting $z = 0$ so that

$$X_1(t) = \overline{X}_1(t) = a_{11}$$

$$a_{11} = X_1(0)$$

$$X_2(t) = \overline{X}_2(t) = a_{21}$$

$$a_{21} = X_2(0).$$

The state equations also require

$$\dot{X}_1(0) = X_2(0).$$

This is satisfied by letting:

$$a_{12} = X_2(0).$$

The first two approximating solutions then become:

$$\overline{X}_1(t+z) = X_1(t) + X_2(t)z + a_{13}z^2 + a_{14}z^3$$

$$\overline{X}_2(t+z) = X_2(t) + a_{22}z + a_{23}z^2 + a_{24}z^3,$$

where t corresponds to the time for which the state vector is known.

In order to perform the integrations for the variational method, the state equations must be re-arranged as:

$$\dot{X}_1 - X_2 = 0$$

$$\dot{X}_2 + P_1X_1 + P_2X_2 - P_3X_3 - P_4 = 0$$

$$\dot{X}_3 - X_4 = 0$$

$$\dot{X}_4 + P_5X_3 - P_6X_1 = 0.$$

The substituion of the approximate solutions gives the integral equations:

$$0 = \int_0^h \left(2z\, a_{13} + 3z^2 a_{14} - z\, a_{22} - z^2 a_{23} - z^3 a_{24}\right)\begin{bmatrix} z^2 \\ z^3 \end{bmatrix} dz$$

$$0 = \int_0^h \Big(P_1 z^2 a_{13} + P_1 z^3 a_{14} + (1 + P_2 z)a_{22} + (2z + P_2 z^2)a_{23} +$$

$$\left(3z^2 + P_2 z^3\right)a_{24} - P_3 a_{31} - P_3 z\, a_{32} - P_3 z^2 a_{33} + P_1\, X_{10} + P_2 X_{20} z$$

$$+ P_1 X_{20} z^2 + P_4\Big)\begin{bmatrix} z \\ z^2 \\ z^3 \end{bmatrix} dt$$

$$0 = \int_0^h \left(a_{32} + z\, a_{33} - a_{41} - z\, a_{42} - z^2 a_{43}\right)\begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix} dz$$

$$0 = \int_0^h \Big(- P_6 z^2 a_{23} - P_6 z^3 a_{24} + P_5 a_{31} + P_5 z\, a_{32} + P_5 z^2 a_{33} + a_{42}$$

$$+ z\, a_{43} - P_6\, X_{10} - P_6\, X_{20} z\Big)\begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix} dz.$$

The integrations give a set of eleven linear algebraic equations of the form:

$$[B]\,[A] = [C],$$

where B is a constant matrix and C is a vector. An expansion of this notation gives:

$$\begin{bmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,11} \\ B_{2,1} & & & \vdots \\ \vdots & \ddots & & \vdots \\ & & \ddots & \\ B_{11,1} & \cdots & \cdots & B_{11,11} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{14} \\ a_{22} \\ a_{23} \\ a_{24} \\ a_{31} \\ a_{32} \\ a_{33} \\ a_{41} \\ a_{42} \\ a_{43} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{11} \end{bmatrix}$$

The complete result of the integrations is shown in Appendix B where each term in the B matrix and C vector is defined.

The time step for this problem must be based on states X_1 and X_2 since together they form a second-order output response. From the characteristic equation, the eigenvalues can be found as:

$$\lambda = -1.25 \pm 4.8\ j.$$

Thus, the time constant and period of oscillation are:

$$\tau = \frac{1}{1.25} = 0.8 \text{ sec.}$$

$$p = 2\pi/4.8 \approx 1.3 \text{ sec.}$$

Experience in solving the problem showed that it was possible to select the time step as:

$$h = \frac{1}{4}\tau = 0.2 \text{ sec.}$$

Conventional methods would require

$$h = \frac{1}{10}\left(2\pi/\sqrt{P_5}\right),$$

which gives 0.0008 seconds for Case 1 and 0.004 seconds for Case 2.

This problem was solved by both the variational method and Runge-Kutta 4. The responses for Case 2 are shown in Figure 8. The variational solution is shown in its entirety; every point generated is shown in the plot. The accuracy of the method is well demonstrated by the responses of X_1 and X_3. The low-frequency response X_1 follows the exact solution extremely well for an approximation method. It can also be seen that the high-frequency response, X_3, was determined as expected. The "steady state" portion of the response was predicted by the variational method.

Discussion of Results

The examples which have been presented show that the variational method can predict accurate responses for low-frequency components using a large time step, since the time step required by the variational method was in all cases an order magnitude larger than the requirements for other methods. It is also important to compare the actual computation time which was required for the solutions.

(The example problems presented in this work were solved using an IBM 360 Model 65 computer with a Fortran compiler. Computation time was determined as the time required for the central processing unit, cpu, to calculate the solution exclusive of all input and output. A subroutine was used to query the computer clock and determine the cpu time in milliseconds.)

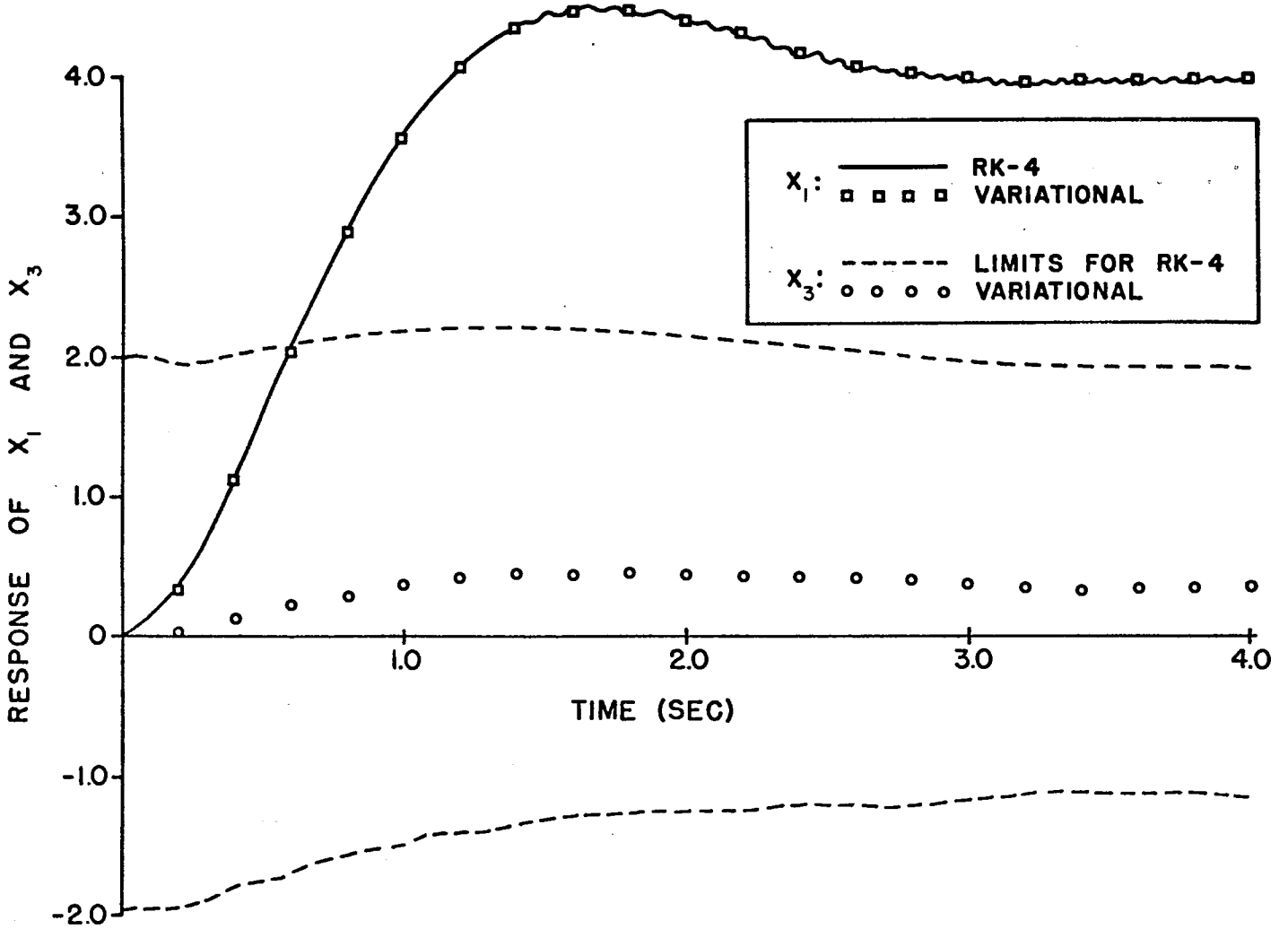

Figure 8. System Response From Example Two

The Runge-Kutta fourth-order simulation program used to obtain the comparisons was the DYSIMP Simulation Program, modified only to the extent necessary to allow the monitoring of cpu time (9).

The results for Example Problem One are shown in Table I. The cpu time per integration step is much greater for the variational method, but this is more than compensated for by the decrease in the number of steps required. The cost ratio, defined as the ratio of the RK-4 cpu time to the variational cpu time, shows that variational method reduced the cost by 75% over the RK-4 solution. This can surely be considered a significant reduction in the cost of the simulation.

Table II presents similar data for both cases of Problem Two. Here the difference in the required time step is more pronounced as is demonstrated by the cost ratios. For Case 2, where the ratio of the high-frequency to the low-frequency is 30, the variational method gave a 75% savings. For Case 1, where the frequency ratio is 150, the savings amounted to 94% of the RK-4 cost. This type of savings strongly support the validity of a variational simulation program.

The example problems also have implicit results which must be considered. In the second example, it was possible to use a time step equal to 0.25 of the time constant. Using a conventional method to simulate just the low-frequency portion of the system would have required a time step two and one-half times smaller. This implies a reduction in the total number of required integration steps; a result which is also supported by the comparisons in Appendix A. However, this does not represent a directly corresponding reduction in solution cost since each step in the variational program requires more computation time than one RK-4 step. By considering the results in Table II, it

TABLE I

DATA FROM EXAMPLE PROBLEM ONE

	Variational	RK-4
Real Time Simulated (sec.)	0.2	0.2
Time Step (sec.)	0.001	0.0001
CPU Time (msec.)	961	3840
CPU msec/step	4.8	1.9
Cost Ratio	1	3.96

TABLE II

DATA FROM EXAMPLE PROBLEM TWO

	Case 1		Case 2	
	Variational	RK-4	Variational	RK-4
Real Time Simulated (sec.)	4.0	4.0	4.0	4.0
Time Step (sec.)	0.2	0.0008	0.2	0.004
CPU Time (msec.)	705	11777	738	3027
CPU msec/step	35.3	2.36	36.9	3.03
Cost Ratio	1	16.7	1	4.09

seems that the break-even point occurs for a high-to-low frequency ratio of 7.5. If the frequency ratio falls below this limit, the variational method becomes more costly.

Another important result of using the variational method is that the time step does not decrease as the high-frequencies increase. This contrasts standard algorithms in which the time step linearly decreases with increasing frequency. Also, as the frequency increases, the accuracy of the variational method can be expected to improve.

The major difficulty encountered in using the method stemmed from the tedious substitutions and integrations required. This portion of the method proved to be time consuming, and represents an area for further study. Other problem areas are reflected by the recommendations in the following chapter.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

A numerical integration method for the solution of differential equations has been developed. The algorithm is based on the variational principle of mechanics, and represents a new approach to system simulation. The method is especially applicable to systems which have basically low-frequency outputs coupled to secondary high-frequency components.

The results of applying the algorithm to two example problems show the method to give accurate solutions while reducing the required computation time by as much as 94%. A study of the accuracy of the method showed it to be comparable with a fourth-order Runge-Kutta technique.

Based on these results, the variational method of simulation has been shown to be a valid algorithm for the simulation of systems with low-frequency outputs. The reduction in computation time which can be gained through this type of approach should encourage further work leading to the application of this type of method to a large number of problems in dynamic system analysis.

Recommendations for Further Study

The following points must be considered in further research into the topic:

1. Certain types of non-linearities lead to integrations

which may be impossible. For example, the equation

$$\dot{X} = -X^{1/2}$$

would result in the integral

$$0 = \int_0^h (a_1 + 2a_2 z + \sqrt{a_0 + a_1 z + a_2 z^2}) \begin{bmatrix} z \\ z^2 \end{bmatrix} dz.$$

This would be a very difficult integration, if it can be done. An alternative could be the use of

$$\dot{X}^2 = X,$$

which eliminates the radical. This and many other non-linearities must be investigated to determine whether they can be handled by the method.

2. An efficient method for the solution of non-linear algebraic equations is required in order to apply the variational method to general sets of differential equations. Unruh (10) presents a method which may be applicable.
3. The example problems demonstrated that the method is tedious in its application, and that there are many opportunities for blunders in setting up the problems. A computerized symbolic manipulation program is needed to perform the substitutions and integrations. Unruh (10) again presents work in a similar area.
4. The other variational methods presented by Moneymaker (7) should be compared to the virtual work method applied in this study.

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APPENDIX A

ERROR STUDY

The derivation of conventional numerical integration algorithms makes it possible to classify the method as to the expected error. However, the variational method does not give any such quantitative estimate of error, and, therefore, a comparison study was made to gain some insight into the matter. The study involved the solution of two linear differential equations by the variational method and Runge-Kutta 4.

The variational solutions were obtained for three series of the form

$$\overline{X} = a_0 + \sum_{i=1}^{m} a_i z^i,$$

where m is taken to be the number of terms in the series. This appears contradictory, but since a_0 is generally fixed by initial conditions, the method must specify m terms in the series. Solutions were determined for m = 1, 2, 3 for each differential equation.

The accuracy of a solution was defined as the per cent difference between the exact analytical solution and the simulated response. The time for which this error is reported is somewhat arbitrary, but is thought to be representative of what might be required in many simulation programs.

The first case considers the solution of

$$\dot{X} = -X \qquad X(0) = 1$$

$$X(t) = e^{-t}.$$

The results are reported in Figure 9 where the time step is normalized by the time constant. It was expected that the accuracy of the method would increase with the number of terms in the series. The results do not support this assumption, but do show that the method compares favorably with a fourth-order method for series with more than one free term.

Solutions were also compared for the equation

$$\ddot{X} = -X \qquad X(0) = 1$$

$$\dot{X}(0) = 0$$

$$X(t) = \cos(t).$$

The results in Figure 10 show a better correlation between the number of terms considered and the accuracy. A three-term series is shown to be as accurate as the RK-4 method while using a step size which is twice as large. This is particularly interesting since it implies a 50% reduction in the number of integration steps required in a solution. Again, the two-term solution gave accuracy comparable with RK-4, while the one-term series gave very inaccurate solutions. In Figure 10, the time step has been normalized by the period of the exact solution.

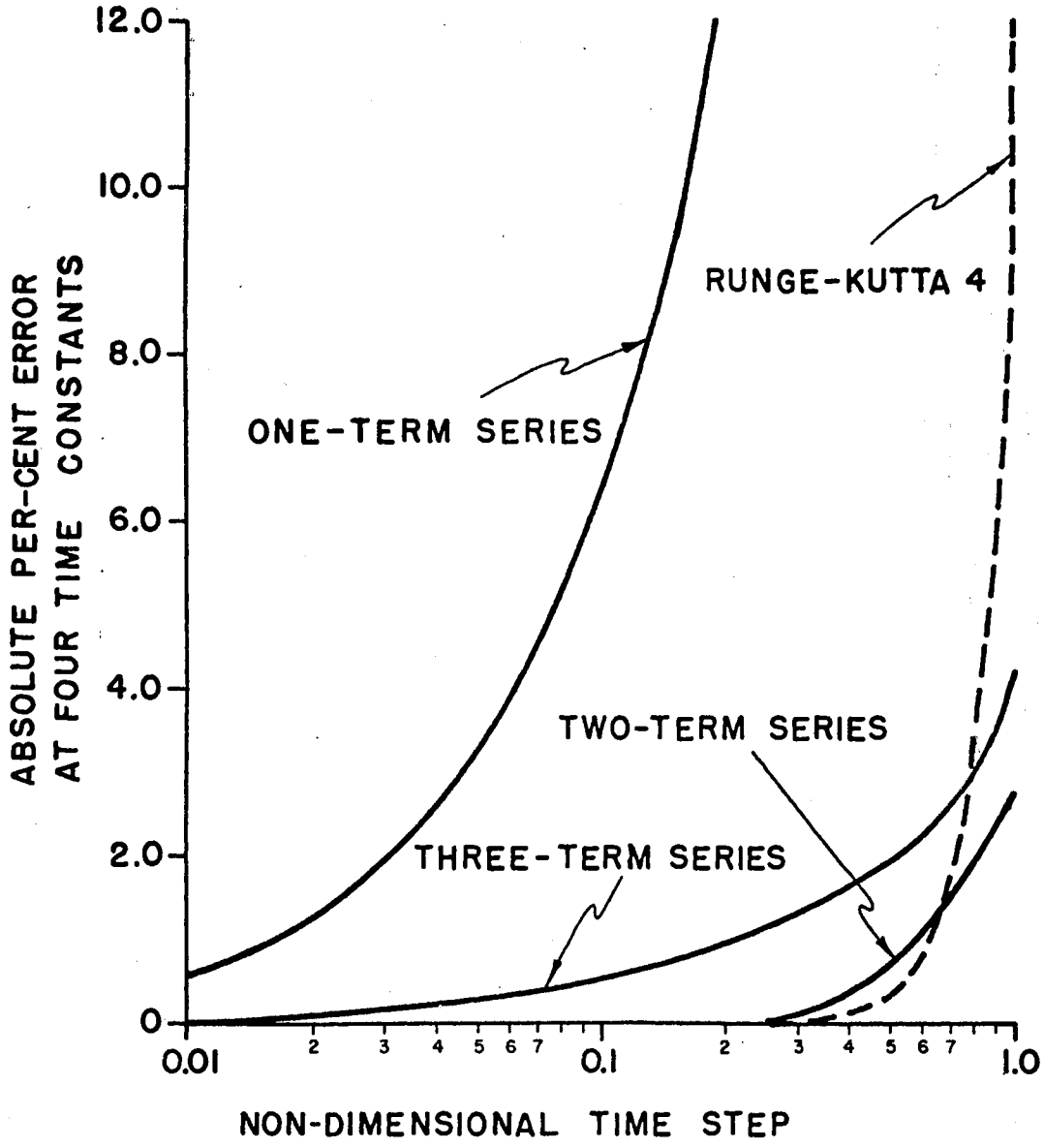

Figure 9. Accuracy of First-Order Solutions

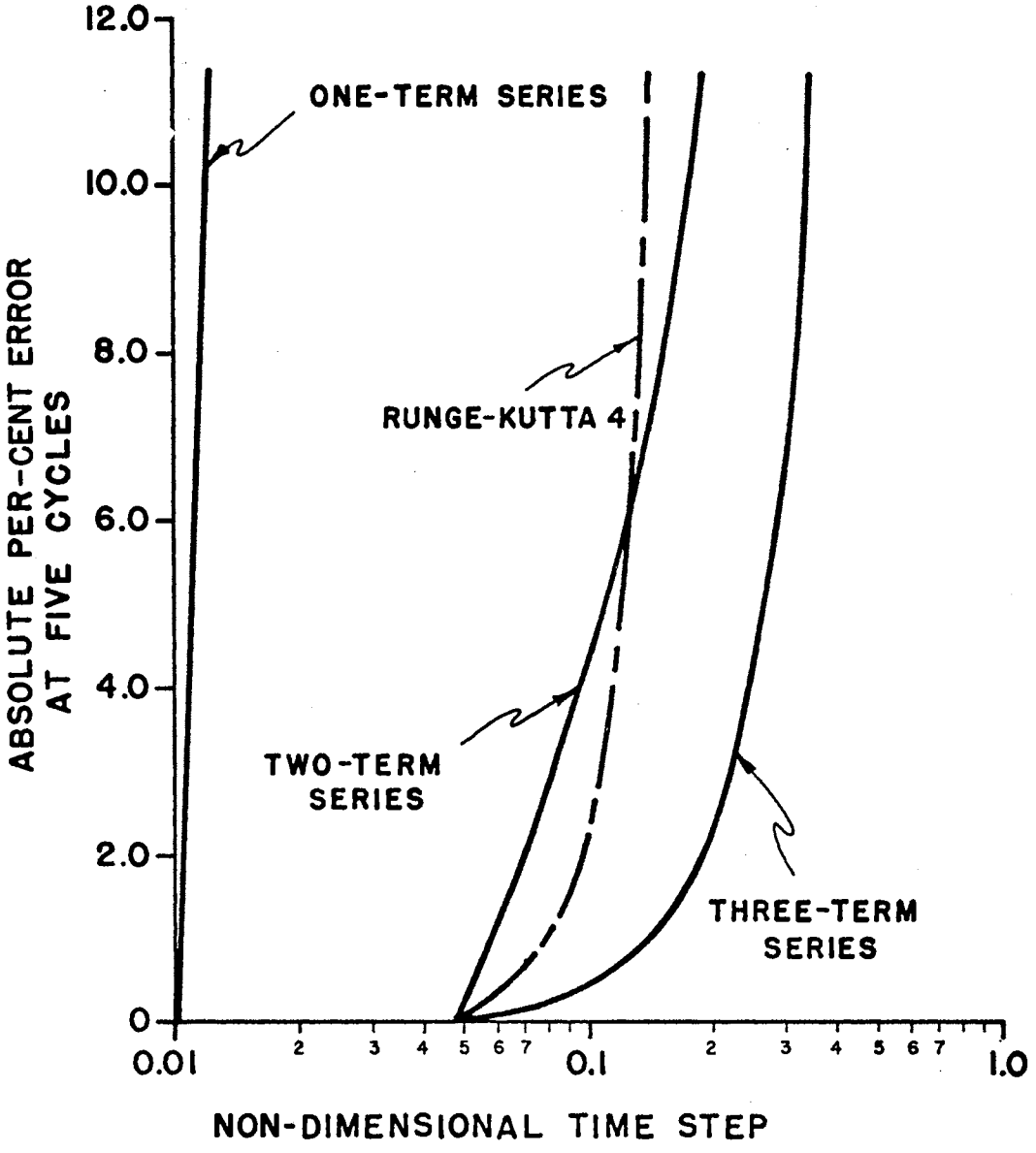

Figure 10. Accuracy of Second-Order Solutions

APPENDIX B

DEFINITIONS FOR EXAMPLE PROBLEM TWO

The following definitions refer to the B matrix and C vector in Example Problem 2. All entries which are not defined are zero.

$$C_3 = -\frac{1}{2} P_1 X_{10} h^2 - \frac{1}{2} P_2 X_{20} h^2 - \frac{1}{3} P_1 X_{20} h^3 + \frac{1}{2} P_4 h^2$$

$$C_4 = -\frac{1}{3} P_1 X_{10} h^3 - \frac{1}{3} P_2 X_{20} h^3 - \frac{1}{4} P_1 X_{20} h^4 + \frac{1}{3} P_4 h^3$$

$$C_5 = -\frac{1}{4} P_1 X_{10} h^4 - \frac{1}{4} P_2 X_{20} h^4 - \frac{1}{5} P_1 X_{20} h^5 + \frac{1}{4} P_4 h^4$$

$$C_9 = P_6 X_{10} h + \frac{1}{2} P_6 X_{20} h^2$$

$$C_{10} = \frac{1}{2} P_6 X_{10} h^2 + \frac{1}{3} P_6 X_{20} h^3$$

$$C_{11} = \frac{1}{3} P_6 X_{10} h^3 + \frac{1}{4} P_6 X_{20} h^4$$

$$B_{1,1} = \frac{1}{2} h^4 \qquad B_{2,1} = \frac{2}{5} h^5$$

$$B_{1,2} = \frac{3}{5} h^5 \qquad B_{2,2} = \frac{1}{2} h^6$$

$$B_{1,3} = -\frac{1}{4} h^4 \qquad B_{2,3} = -\frac{1}{5} h^5$$

$$B_{1,4} = -\frac{1}{5} h^5 \qquad B_{2,4} = -\frac{1}{6} h^6$$

$$B_{1,5} = -\frac{1}{6} h^6 \qquad B_{2,5} = -\frac{1}{7} h^7$$

$$B_{3,1} = \frac{1}{4} P_1 h^4 \qquad B_{4,1} = \frac{1}{5} P_1 h^5$$

$$B_{3,2} = \frac{1}{5} P_1 h^5$$

$$B_{3,3} = \frac{1}{2} h^2 + \frac{1}{3} P_2 h^3$$

$$B_{3,4} = \frac{2}{3} h^3 + \frac{1}{4} P_2 h^4$$

$$B_{3,5} = \frac{3}{4} h^4 + \frac{1}{5} P_2 h^5$$

$$B_{3,6} = -\frac{1}{2} P_3 h^2$$

$$B_{3,7} = -\frac{1}{3} P_3 h^3$$

$$B_{3,8} = -\frac{1}{4} P_3 h^4$$

$$B_{5,1} = \frac{1}{6} P_1 h^6$$

$$B_{5,2} = \frac{1}{7} P_1 h^7$$

$$B_{5,3} = \frac{1}{4} h^4 + \frac{1}{5} P_2 h^5$$

$$B_{5,4} = \frac{2}{5} h^5 + \frac{1}{6} P_2 h^6$$

$$B_{5,5} = \frac{1}{2} h^6 + \frac{1}{7} P_2 h^7$$

$$B_{5,6} = -\frac{1}{4} P_3 h^4$$

$$B_{5,7} = -\frac{1}{5} P_3 h^5$$

$$B_{5,8} = -\frac{1}{6} P_3 h^6$$

$$B_{8,7} = \frac{1}{3} h^3$$

$$B_{8,8} = \frac{1}{2} h^4$$

$$B_{8,9} = -\frac{1}{3} h^3$$

$$B_{4,2} = \frac{1}{6} P_1 h^6$$

$$B_{4,3} = \frac{1}{3} h^3 + \frac{1}{4} P_2 h^4$$

$$B_{4,4} = \frac{1}{2} h^4 + \frac{1}{5} P_2 h^5$$

$$B_{4,5} = \frac{3}{5} h^5 + \frac{1}{6} P_2 h^6$$

$$B_{4,6} = -\frac{1}{3} P_3 h^3$$

$$B_{4,7} = -\frac{1}{4} P_3 h^4$$

$$B_{4,8} = -\frac{1}{5} P_3 h^5$$

$$B_{6,7} = h$$

$$B_{6,8} = h^2$$

$$B_{6,9} = -h$$

$$B_{6,10} = -\frac{1}{2} h^2$$

$$B_{6,11} = -\frac{1}{3} h^3$$

$$B_{7,7} = \frac{1}{2} h^2$$

$$B_{7,8} = \frac{2}{3} h^3$$

$$B_{7,9} = -\frac{1}{2} h^2$$

$$B_{7,10} = -\frac{1}{3} h^3$$

$$B_{7,11} = -\frac{1}{4} h^4$$

$$B_{10,1} = -\frac{1}{4} P_6 h^4$$

$B_{8,10} = -\frac{1}{4} h^4$

$B_{8,11} = -\frac{1}{5} h^4$

$B_{9,1} = -\frac{1}{3} P_6 h^3$

$B_{9,2} = -\frac{1}{4} P_6 h^4$

$B_{9,6} = P_5 h$

$B_{9,7} = \frac{1}{2} P_5 h^2$

$B_{9,8} = \frac{1}{3} P_5 h^3$

$B_{9,10} = h$

$B_{9,11} = h^2$

$B_{11,1} = -\frac{1}{5} P_6 h^5$

$B_{11,2} = -\frac{1}{6} P_6 h^6$

$B_{10,2} = -\frac{1}{5} P_6 h^5$

$B_{10,6} = \frac{1}{2} P_5 h^2$

$B_{10,7} = \frac{1}{3} P_5 h^3$

$B_{10,8} = \frac{1}{4} P_5 h^4$

$B_{10,10} = \frac{1}{2} h^2$

$B_{10,11} = \frac{2}{3} h^3$

$B_{11,6} = \frac{1}{3} P_5 h^3$

$B_{11,7} = \frac{1}{4} P_5 h^4$

$B_{11,8} = \frac{1}{6} P_5 h^6$

$B_{11,10} = \frac{1}{3} h^3$

$B_{11,11} = \frac{1}{2} h^4$

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