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DEVELOPMENT OF SUITABLE APPROXIMATION ALGORITHMS TO BE
USED IN THE DESCRIPTION OF HETEROGENEOUS RESERVOIRS FOR
SECONDARY RECOVERY STUDIES

The University of Oklahoma   PH.D.   1981

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THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

DEVELOPMENT OF SUITABLE APPROXIMATION ALGORITHMS

TO BE USED IN THE DESCRIPTION OF HETEROGENEOUS

RESERVOIRS FOR SECONDARY RECOVERY STUDIES

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

EBERE PAULINUS OFOH

Norman, Oklahoma

1981

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APPROVED FOR THE DEPARTMENT OF

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by

DISSERTATION COMMITTEE

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ABSTRACT

Our current capability to accurately predict the performance of a reservoir given a detailed description of its heterogeneities, calls for an urgent need for an efficient method of describing these non-conformities at any given locations in a reservoir.

In this study, a suitable approximation algorithm was developed for use in the estimation of reservoir performance prior to waterflooding operations. This algorithm, a two dimensional cubic spline, constructs a smooth, and continuous function of the given data values. This smooth function with continuous first and second derivatives, removes the 'wiggly' and undulating characteristics often present in most polynomial approximations.

A number of tests using varying degrees of data contamination shows that the model has the potential to reduce bad data effects. It was also shown that the model can be used to easily determine the reservoir floodable volume, the water injection schedule for secondary recovery operations, and an overall waterflooding performance.

The prediction method makes use of flow capacity distributions in a heterogeneous reservoir.

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CHAPTER I

INTRODUCTION

Our present capability to accurately predict the per-
formance of a reservoir given a detailed description of its
heterogeneities, calls for an urgent need for an improved de-
scription of these non-conformities at given locations in a
field. Good waterflood engineering can be achieved by a pre-
cise and quantitative description of these reservoir hetero-
geneities. This need has become more critical particularly
now that the industry has turned its attention to the low
permeability regions of the field. Since it is now certain
that the probability of obtaining a homogeneous reservoir
structure is remote, experts are developing different mathe-
matical models to define reservoirs in terms of their inten-
sive properties. This is just part of the unending effort
by researchers to continuously evaluate factors affecting
waterflood performance in order to reduce operating costs
and increase ultimate recovery. In practice, a reservoir may
be non-uniform in all its intensive properties such as pore
size distribution, wettability, connate water saturations,
crude properties, permeability, porosity and thickness. This
study recognizes the last three parameters: permeability,

porosity and thickness as important tools needed in the plan-
ning of waterflood projects when we have available core
analysis data. Various techniques for describing a reser-
voir by means of these basic properties are well documented
in the literature.

Kruger [18] was the first to quantitatively describe
areal permeability distributions using observed differences
in well production history. Jacquard and Jain [13] used
field pressure data in their numerical technique model.
Jans [14] developed a regression analysis technique and used
pressure interference test data. Johnson et al published a
method [15], "Pulse testing," for describing reservoir flow
properties between wells. Their technique showed promise
for providing a measure of storage capacity (ϕh), and forma-
tion flow capacity (kh). Hutchinson [11] suggested that for-
mation outcrop be carefully examined to obtain information
on such factors as continuity of zones of specific perme-
ability, extent of shale breaks, and the degree of stratifi-
cation.

The use of core data in the study of reservoir proper-
ties has been controversial. Kruger [18] remarked that
the use of core analysis and/or well flow tests give sparse,
spotty, and doubtful information that are difficult to inter-
pret. However, Miller and Lenta [20] by use of a 'Posi-
tional Approach' method had disproved this belief. Using the
positional approach model, they were able to determine the

layering properties of Cotton-Valley reservoir project from core data obtained from different wells. Skov, et al[9] also reported success when they used a similar technique to match the performance of a number of fluid injection projects.

1.1 Review of Previous Waterflood Applications Using Core Data

One of the earliest attempts to use a two-dimensional approach of parameter distributions to predict waterflood performance in depleted or nearly depleted reservoirs was by Stiles [28]. He studied the lateral flow of fluids in formations of irregular permeability. He represented patterns of irregularities by a smoothed permeability distribution curve and a flow capacity distribution curve. The permeability values were arranged in order of decreasing values. Dimensionless values of permeability ratios were plotted against the cumulative thickness (ratio). Similarly he plotted dimensionless capacity ratio against cumulative thickness (ratio). He also calculated water cut values and recovery values using appropriate equations. Water Cut-Recovery Curve was then used to predict the behavior of a well. The prediction model, however did not consider the structural nature of the reservoir under study. To obtain better results, such other factors as the shape of field, the structural position of the individual wells, and well spacing should be considered by the model.

In his two dimensional approach Johnson [16] used a graphical representation of porosity and permeability

distribution to predict waterflood performance. His technique
was able to predict cumulative water injected and cumulative
oil produced in terms of percent water-cut by use of permea-
bility-porosity relationship. Classifying the reservoir per-
meability data (from core analysis) in order of decreasing
values, he made a plot of cumulative flow capacity versus the
logarithm of the corresponding cumulative volume. He observed
that the above plot is similar to the plot of logarithm of
cumulative oil recovery versus water cut. By this finding,
Johnson [16] is confirming the work of Stiles [28]. He also
observed that in highly stratified reservoirs, adjusting the
slope of the flow capacity-volume curves for mobility ratio
will produce a slope approximating the slope of the performance
curve (figure 1).

Here again, Johnson [16] did not consider the effect of
the well structure and pattern. The accuracy of his work may
be limited.

Schmalz and Rahme [25] studied the degree and the magni-
tude of variation in waterflood performance with variation in
permeability profile. Using several precise mathematical re-
gression models such as straight lines of varying slope and/or
various continuous curves, they calculated a composite group
of waterflood predictions under constant reservoir conditions.

By obtaining a plot of Lorenz coefficient versus per-
cent recovery (of recoverable oil), they made a preliminary
quantitative estimate of the expected performance of a flood,
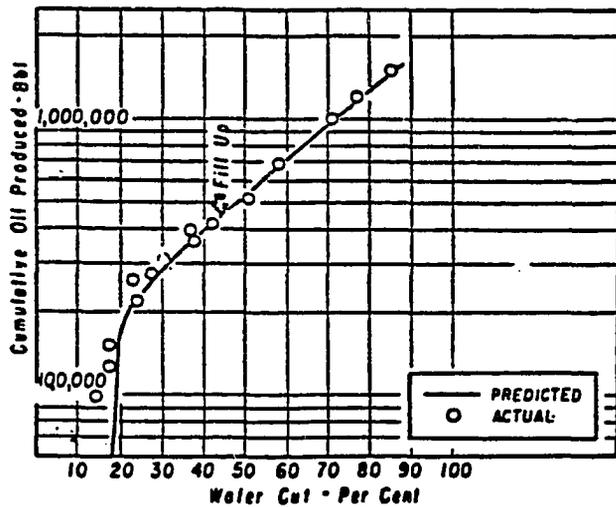
A —Project 1 performance.
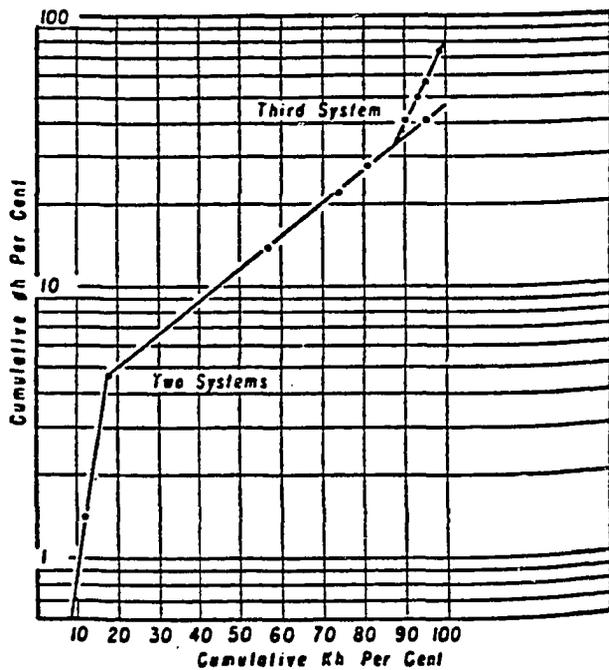
B—Project 1 classification composite.

Figure 1: Prediction of waterflood performance by use of reservoir parameter distribution (after Johnson [16]).

given only the permeability data. This general review shows the extent to which core data can be stretched to profit the petroleum industry. That oil wells perform in accordance with their joint parameter distributions has been demonstrated [14, 16, 20, 25, 28].

1.2 Review of Literature on Selective Plugging

In most oil reservoirs, the oil saturated formations have strata of varying permeabilities and thicknesses. In the waterflood process, the more permeable strata are depleted of oil first, and thereafter they continue to take large quantities of water even though they produce no more oil. This situation eventually results in large quantities of water circulating through depleted formations. This results in an unnecessary expense in the way of water costs and pumping. High back pressure in the vicinity of the producing well will interfere with the flow of oil from the less permeable strata.

Selective plugging is a practical method for preventing the wasteful circulation of water through most permeable strata. It generally involves the injection of dispersed solids and semi-solids of such a particle size that they will enter the pores of the most permeable strata (or factures). The dispersed solids will travel some distance and then stop. Particles keep building up behind this stop point until they form a plug which stops the flow of water into the strata. Wayne et al, [32] reported an unusually high degree of success

over a variety of field conditions using selective plugging
technique. Joseph [17] reported the case history of a number
of field tests as successful.

However, his procedure has some major drawbacks:

1. Where there is no bedding planes to isolate the
   plugged zone, the injection water may go round the
   surface plug and enter the zone of highest permea-
   bility. This means waste of a large quantity of
   water.

2. An injection well may develop a backflow which will
   wash away the surface plugging chemicals and thus
   reopen the offending zone to accept flood water
   again.

3. Field trials are limited since operators are reluc-
   tant to treat the formation selectively when they
   cannot determine quickly the location and effective-
   ness of such a treatment.

4. Selective plugging chemicals can get so costly that
   the application of the technique becomes uneconomic.

5. Irregular absorption of chemicals to the formation
   rock can cause the blocking of the wrong zones --
   hence defeating the purpose of the technique.

In view of these difficulties and the fact that the
first stage of selective plugging requires an accurate know-
ledge of waterflood profile, the method of this study proves
advantageous and superior. All we need to know in this

investigation is the core data, then from the interpolation algorithm and the resulting graphical display of the reservoir parameter, we can isolate zones of interest where special treatment may be necessary.

CHAPTER II

MAPPING AND VARIOUS INTERPOLATION PROCEDURES

2.1 Contour Mapping

Contouring is a way of representing pictorially values of a variable at various points in a study area. Contour maps are frequently used to obtain detailed pictorial representation of a function of two variables whose values are known at discrete points.

In geographical research, contour maps are used to represent terrain surfaces. In crystallography, contour maps are employed extensively to map electron density. Imaginary surfaces comprising of a continuum of all possible parameter values can be employed in reservoir engineering studies to map reservoir parameter distributions. Unlike the surface trend analysis employed in geographical studies, contour mapping in petroleum engineering studies is difficult to obtain due to the problem of limited data resources. Thus, we must therefore rely on some kind of an approximation theory to generate the much needed data matrix by interpolation procedure.

The approximation technique considered suitable for this sort of investigation is one that would represent the data information as a continuous function of two variables.

9

The continuity condition is desirable to eliminate the presence of maxima points at the nodes of given data points, and the existence of saddle points at the unknown data points -- both of which are present in other interpolation methods. The result is a curve, smooth and differentiable at all points. This mathematical approach is also considered noise free -- resulting in smooth contours.

Contour plotting involves the drawing of contour lines through equal values in a two-dimensional surface. A contour plotting algorithm [29] may be constructed by following contours from some starting points until they either close or intersect a boundary. Alternatively each cell of the grid can be examined in turn and then all contours found inside the cell are drawn.

Linear or non-linear interpolation procedures may be used in the execution of a contour. All contours intersecting the boundaries are drawn first and then all contours which do not intersect the boundaries are drawn later.

The contouring algorithm modified and used in this study is due to Synder [29]. It is presented under the Statistical Analysis System [24].

2.2 The Three Dimensional Projection

A conceptual surface is often obtained when a function whose points are related by some two-dimensional system of co-ordinates is associated, point to point, with the corresponding value of variable, z_i, say. The implication

of the above is that as many conceptual surfaces are possible
as there are variables.

In engineering, the determination and variation of
these conceptual surfaces can be related to the variation of
corresponding non-spatial functions from point to point.
This variation can be regarded as the flow on the conceptual
surface, and thus, the generation of this conceptual surface
can replace the physical system.

In petroleum engineering, this conceptual surface
can range from flow capacity (kh) to flow volume (ϕh). An
understanding of how the peaks, pits and beds associated
with these projections are related to one another, will give
the engineer the desired insight into the flow of the non-
spatial variable.

A sequence of points in three-space is usually con-
nected together by linear interpolation between adjacent
points. The three dimensional plot adopts a masking tech-
nique. Those lines or portions of lines which should be
hidden by previous lines are masked. This means that lines
in the foreground in the positive Z-direction are plotted
before lines in the background. A line or portion of a line
is hidden if it lies within the region bounded by previously
plotted lines.

The three dimensional algorithm modified and used in
this study is due to Watkins [31]. His algorithm was modi-
fied under the SAS [24]. The algorithm accepts three dimen-
sional data in various forms rotated in three dimensional

space. The projection of the resulting figure is plotted on to the x-y plane.

2.3 Available Interpolation Techniques

To be very effective, a good interpolation algorithm should provide some or all of the following conditions:

1. It should reduce estimation errors to a minimum and give exact values at the data points.

2. It should be simple and manageable so that the evaluation of data points is not tedious.

3. The interpolation function should be nice and continuously differentiable.

Two-dimensional approaches briefly discussed in this study include:

The multi-quadric approach

The trend surface model

The normal distribution model

The triangular model

Unfortunately most of the above models are inadequate for use.

2.4 Multiquadric Equations of Topography and Other Irregular Surfaces[10]

This is an analytical method that involves the summation of equations of quadric surfaces having unknown coefficients. The quadric surfaces are located at significant points throughout the region to be mapped. Contoured multiquadric surfaces are compared with topography and other

irregular surfaces from which the multiquadric equation was
derived.

Topography can be represented by various analytical,
numerical, and digital methods, in addition to the classical
contour map.

Earliest work in topography solved the problem stated
thusly: given continuous topographic information in a certain
region, reduce it to an equivalent set of discrete data example
spherical harmonic coefficients or digital terrain incre-
ments.

My study among others [10,23] is concerned with the
procedural inverse of the above statement, thus: given a
set of discrete data on a topographic surface (or reservoir
parameter), reduce it to a satisfactory continuous function
representing the topographic surface (or reservoir parameter).

2.5 The Multi-Quadric Model

The multi-quadric model is an approach introduced by
Hardy [10] for obtaining the equations of irregular surfaces.
This technique has been modified for the purpose of this re-
view to be applicable to cases of heterogeneous reservoirs.
The procedure requires a set of discrete reservoir data.
The algorithm reduces the data into a satisfactory continuous
function representing a quadric surface having unknown coef-
ficients. The summation of a number of quadric surfaces will
give the profile of the physical system.

Consider a function Z with co-ordinates X and Y from

a class of quadric surfaces Q,

$$Z = \sum_{i=1}^{n} A_i \; [Q(X_i, \; Y_i, \; X, \; Y]$$

2(1)

where A_i is a constant determining the algebraic sign or

flatness of the quadric term.

For a multi-quadric surface defined by circular

hyperboloids in two sheets: 2(1) becomes

$$Z_j = \sum_{i=1}^{n} A_i \; [(X_j - X_i)^2 + (Y_j - Y_i)^2 + C]^{\frac{1}{2}}$$

2(2)

For the case of cones and straight lines segments

C = 0 and 2(2) becomes

$$Z_j = \sum_{i=1}^{n} A_i \; [(X_j - X_i)^2 + (Y_i - Y_i)^2]^{\frac{1}{2}}$$

2(3)

For an ellipse (Ellipsiod), $0 < C < 1$, a good approximation is considered here to be C = .5

Then

$$Z_j = \sum_{i=1}^{n} A_i \; [(X_j - X_i)^2 + (Y_j - Y_i)^2 + 0.5]^{\frac{1}{2}}$$

2(4)

If we assume that reservoir parameter are normally distributed; (Gaussian distribution) we have: assumptions similar to this has been reported [10, 11, 14, 23].

$$\log_e Z_j = -\sum_{i=1}^{n} A_i \; [(X_j - X_i)^2 + (Y_j - Y_i)^2 + .5]^{\frac{1}{2}}$$

2(5)

$$Z_i = e^{-[A_i \Sigma \; [(X_j - X_i)^2 + (Y_j - Y_i)^2 + .5]^{\frac{1}{2}}]}$$

2(6)

$$Z_j = \sum_{i=1}^{n} C_i \; e^{-\frac{1}{2} \left[\frac{(X_j - X_i)^2 + (Y_j - Y_i)^2 + .5}{\lambda^2} \right]} \qquad 2(7)$$

or

$$Z_j = \sum_{i=1}^{n} C_i \; e^{\frac{[-(X_j - X_i)^2 - (Y_j - Y_i)^2 + 0.5]}{\lambda^2}} \qquad 2(8)$$

Where λ is the average spacing between data point positions ($\lambda_x = \lambda_y = \lambda$), equation 2(8) is similar to 2(4) but it has been modified to handle normally distributed random variables consisting of reservoir parameters for an elliptical system.

2(8) is the same as $f(t) = \Sigma c e^{-r^2/\lambda^2}$ $\qquad$ 2(8a)

Thus a circle of radius r is a theoretical contour containing essentially all the variables of the population.

The solution of 2(8) will yield

$$AX = B \qquad 2(9)$$

where

$$\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = X \qquad 2(10)$$

and

$$\begin{bmatrix} Z_1 \\ \vdots \\ \vdots \\ Z_n \end{bmatrix} = Z_i = B \qquad 2(11)$$

The matrix $[A_{ij}] = A$ (which is nxn) $\qquad$ 2(12)

from 2(8) - 2(12)

$$AX = B \qquad 2(13)$$

$$X = A^{-1}B \qquad 2(14)$$

When the known values of C_i are substituted into equation 2(8)

we have the required equation of the surface - which fits

the data points exactly and provides a logical interpolation

at intermediate points. However, the real problem in multi-

quadric approach is the placement of a saddle in a region

with no data information. Questions regarding the continuity

of the first and second derivatives of the function may be

raised.

One important aspect of the multivariate approach

is that it can handle cases where the data points are <u>irregu-</u>

<u>larly distributed.</u>

2.6 The Triangular Approach

Bengtsson and Nordbeck (2) recommended partitioning

the domain under study into triangles having data points as

vertices. A plane is then passed through the values at each

of the data points. The approach calls for ordered or ran-

dom data with series of triangles having data point at each

vertex (see figure 2.0). Each side of the triangle is subdi-

vided into "J" equal parts. Hence the number of vertices

for any triangle is given by

$$N = (J + 1)(J + 2)/2 \text{ vertices} \qquad 2(15)$$

2.7 Trend Surface Model

A Trend Surface is a statistically derived equation

to explain variations in given data values distributed regu-

larly or irregularly in X-Y space. It displays data by fit-

ting a continuous surface which can be described by a poly-

nomial equation. The parameters for an equation representing
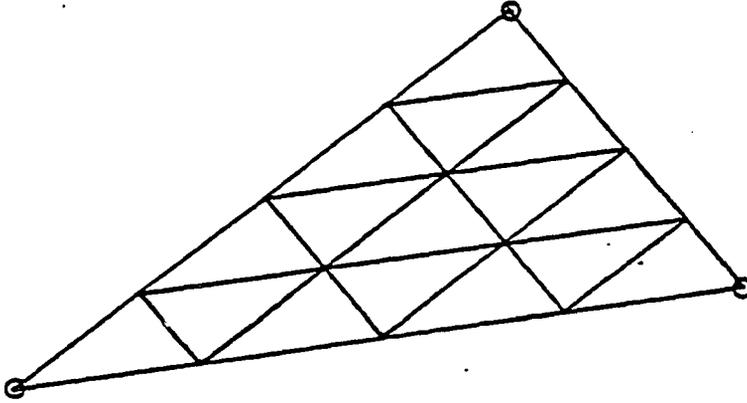
⊙ Triangle Vertex – Surface value known (data point)
• Triangle Vertex – Surface value unknown

Fig 2·0 – Triangular Approach

a surface is estimated by use of least squares fit. Hence the surface is fitted to data values in such a way that the sum of the squared deviations between the given values at data points position and the value of the computed surface at same points are minimized. A contour map displays data as a continuous surface interpolated from discrete data as a function of the distance of neighboring data points and their associated values. The difference between a contour map and a Trend Surface map using same data is the Residual map. A perfect fit is unlikely using this model.

Equations describing Trend Surfaces can be linear, quadratic, cubic--et cetera (figure 2.1).

The general form of the above surfaces using polynomial representation has been noted by Krumbein [19].

$$f(Z) = B_1 + B_2U + B_3V + B_4U^2 + B_5UV + B_6V^2 +$$
$$B_7U^3 + B_8U^2V + B_8V^2U + B_9V^3 \qquad\qquad 2(16)$$

Figure 2.1 gives the relationship of four orders of two-dimensional polynomial curves to their three dimensional counterparts [19].

It was pointed out [8] that the lower order surfaces are very effective in isolating important local trends from those that exist over a larger area while the higher order surfaces will reflect the Z-values very accurately. Trend Surface analysis is therefore considered as a filter which filters an input signal. The surface then represents the
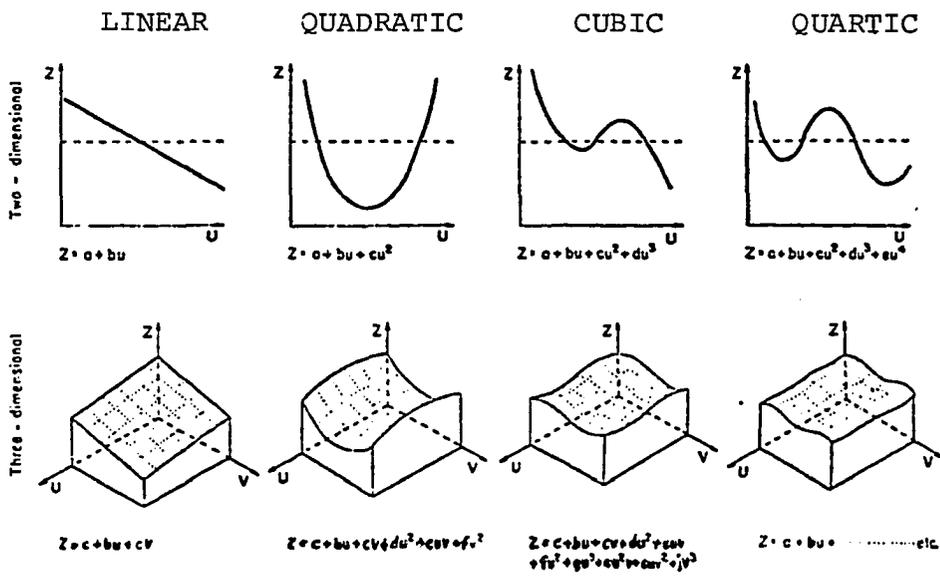
Fig. 2.1 Two and three dimensional representative of surface trend analysis (19).

result of this filtering process. As the order of the sur-
faces gets higher, the quantity of the input data that passes
through the filter gets bigger.

Figure 2.2 shows a trend surface map of Woods and Wood-
ward counties in Oklahoma. Raw data for this program was
obtained from Dwight's Natural Gas Data[7] and modified appro-
priately for Symap Program.[8] For the interest of this review,
figure 2.2 shows what happens to the data distribution when
the order of surfaces is varied from one to six. When the
order of the surfaces is one, very few data enter through
the filter but when the order of the surfaces is increased
to six, more data enter the system for analysis. Hence more
description of the data is reflected in the latter.

Figure 2.3 is a contour map of a theoretical surface
using Symap Method.[8] The contour plot shows how the relative
heights of the theoretical can be displayed for informational
purposes. Figure 2.4 is a trend surface analysis applied to
an oil bearing structure in Lost Springs area of Kansas.[22]
It is important to note that using a trend surface model, one
can identify oil bearing structure. This being the case, the
study of trend surface capabilities seems to be a cause worth
taking. But the trend surface procedure has a big handicap.
It is an inexact method and is only fairly acceptable when
the number of data points is large. But since in Petroleum
engineering we are faced with the problem of limited data
information, trend surface analysis is not very suitable.

A: order one   **B: order three**   **C: order six**

Fig. 2.2: Trend surface analysis for original gas in place in

Woods and Woodward counties using symap program[8]
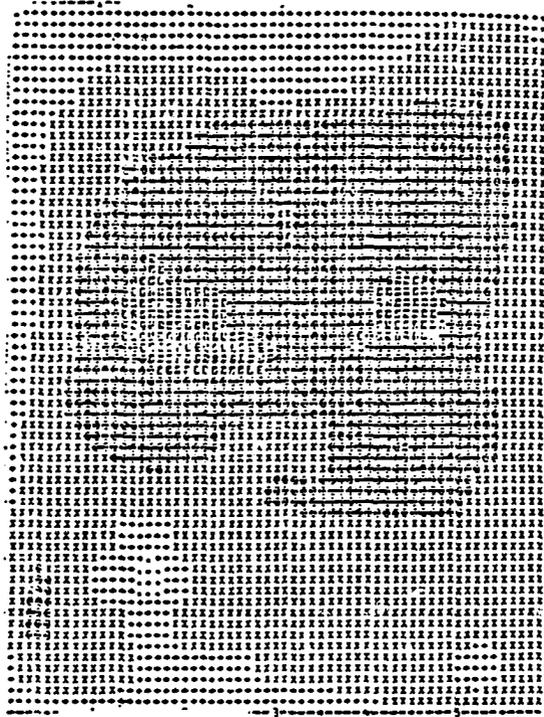
Fig. 2.3: Contour map of a theoretical surface using reference (8).
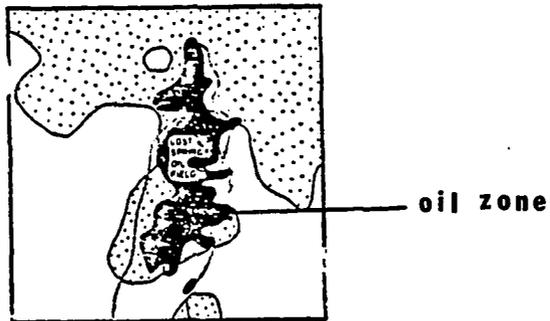
Fig. 2.4: Trend surface analysis applied to oil bearing structure in Lost Spring area of Kansas. (ref. 22).

Exact interpolation methods give exact values at data point positions. Two types of exact interpolation methods exist. One employs a single global function in describing a surface. The other uses a piecewise procedure. The approach of the former is generally unacceptable because of problems of "unmanageable complexity" associated with it, when data values get very large.[27] The latter is popular for its simplicity and ease of use. Many problems mask the results of most exact interpolation procedures. If the interpolating algorithm is one that does not have continuous derivatives, the interpolated points can develop saddle points, and unnecessary minima points.

The weighted average approach introduced in Chapter III was designed to correct most of the defects in exact interpolation procedure. It is adequately suited for heterogeneous reservoir application since it can comfortably handle irregularly spaced data. Similar procedure has been followed elsewhere,[27] but Shepard worked on topographic surfaces instead of reservoir parameters distribution.

CHAPTER III

Theory of the Weighted Averages

The weighted average procedure is an exact interpo-
lation method which puts into consideration a number of fac-
tors which affects interpolated parameters in a reservoir.
Such factors as distance between data points, the direction
of a desired interpolation value from the given data points,
the length and effect of discontinuities as they affect the
interpolation point, and the direction of the desired unknown
parameter value from the given data point locations, have
been considered in the formulation of the governing equation.
The governing equation is simply stated as:

$$f(z) = \sum_{i=1}^{N} (W_i Z_i) / \sum_{i=1}^{N} W_i \qquad\qquad 3(1a)$$

The result of this formulation is a curve that passes
through all the given data points. The curve is continuous
and so is its first and second derivatives. These properties
make an investigation of this algorithm worthwhile especially
when considering the estimation of reservoir parameters using
data points that are irregularly positioned.

3.1 The weighting technique [27]

Define a basic interpolation function as

$$f(P) = [(\sum_{i=1}^{N} (d_i)^{-2} z_i)/(\sum_{i=1}^{N} d_i^{-2})] \qquad 3(1)$$

where:

d_i = distance from point of interpolation to data
point position

P_i = interpolation point

z_i = data point value.

Implication of the inverse square distance expression used in 3(1):

higher weighting is placed on nearby data points with the result that the effect of far away data points is negligible. What this means is that depending on the point of interpolation, only nearby data points influence our results.

3.2 Conceptual procedure of the method

Define a standard radius as:

$$r_s = \left[\frac{n_p}{\pi} (\frac{A}{N})\right]^{0.5}$$

where

$A = \pi R^2$ = area of the largest circle enclosing
the reservoir

n_p = Number of data points to be used in the
interpolation

N = Total number of data points

To successfully interpolate a value for P_i we do the following:

(1) Specified minimum number of points required for interpolation (n_{min})

(2) Maximum number of points required for interpolation (n_{max})

(3) if $n_p < n_{min} \rightarrow n_p = n_{min}$, $d_i = d_{min}$

if $n_p > n_{max} \rightarrow n_p = n_{max}$, $d_i = d_{max}$

if $n_{min} < n_p < n_{max} \rightarrow n_p = \dfrac{n_{min} + n_{max}}{2}$

From figure 3.1,

we shall find an interpolated value of p by weighting the values at the data point positions while accounting for the following parameters.

(a) distance

(b) direction

(c) slope

(d) presence of discontinuity.

The overall weightings will give an estimate for P_i.

Boundary conditions: The boundary can be completely defined by assuming an impermeable boundary layer. This means that flow capacity, flow volume or rock permeability will be treated as zero at the boundary.

If however, a flow boundary is assumed then some values of permeability defining the boundary should be provided.
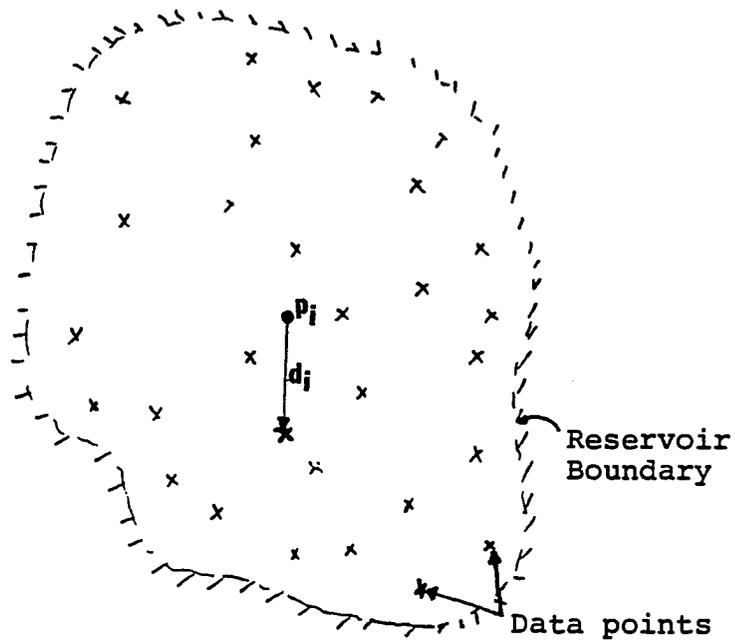
Figure 3.1: Conceptual procedure for the interpolation of reservoir data.

If l_i is a weighting function defined as :

$$l_i = 1/d_i, \quad d_i \neq 0 \qquad 3(2a)$$

Then 3(1) becomes

$$f(p) = \left| \Sigma(l_i^2 \, z_i) / \Sigma l_i^2 \right| \qquad 3(2)$$

Restrictions :

$$f(p) = \begin{bmatrix} 0 & \text{if} & d_i > r_p \\ f(.8r_p) & \text{if} & d_i > 0.5 \, r_p \\ z_i & \text{if} & d_i = 0 \end{bmatrix} \qquad 3(3)$$

3.2 Vectorial weighting:

let S_i be a directional weighting term

where $\quad 0 < S_i < 1$

$$g_i = S_i / \sum_{\substack{i=1 \\ j \neq i}}^{N_p} d_i \qquad\qquad 3(4)$$

where $\quad S_i = \frac{1}{2}(1-\cos\theta)$, and is such that

$S_i = 0$ for points in same direction

$S_i = 1$ for weighting in opposite direction.

$$S_i = \frac{1}{2}(1-\cos\theta) \qquad\qquad 3(5)$$

$\cos\theta = $ Angle $D_i\ P\ D_j$

$D_i \equiv x_i,\ y_i$ = data point Coordinate i

$D_j \equiv x_j,\ y_j$ = data point Coordinates j

$P \equiv P(x,\ y)$ = interpolation point $x,\ y$

$\cos\theta = D_i \vec{P} \cdot D_j \vec{} \cdot P$

$$\cos\theta = \frac{1}{2d_i d_j}\left[\Delta x_i \cdot \Delta x_j + \Delta y_i \cdot \Delta y_j\right] \qquad\qquad 3(6)$$

where: $d_i,\ d_j$ are distances of P from data point

positions $D_i,\ D_j$

$\Delta x_i,\ \Delta x_j$ are coordinate differences from P.

Thus $\Delta x_i = (x-x_i)$.

Let the new weighting function be W_i

$$W_i = \frac{1}{d_i^2}(1 + g_i) = L_i^2(1 + g^2) \qquad\qquad 3(7)$$

Then

$$
f(P) = \begin{bmatrix}
\sum_{i=1}^{N_p} (W_i Z_i) / \sum_{i=1} W_i & \text{if } d_i \neq 0 \\[2em]
Z_i & \text{if } d_i \neq 0
\end{bmatrix}
\qquad 3(8)
$$

3.3 Continuity Weighting

let

$$
p^i = \sum_{\substack{k=o \\ i=o}}^{N_p} \frac{\Delta Z_i}{d_k^2} \circ dx_i \qquad 3(9)
$$

$$
q^j = \sum_{\substack{k=o \\ i=o}}^{N_p} \frac{\Delta Z_i}{d_k^2} \cdot dy_i \qquad 3(10)
$$

where: p^i = weighting of derivative of Z with respect

x dir$\underline{n}$

q^j = weighting derivative of Z in y - dir$\underline{n}$

d_k = distance between any two data points con-

tained in P_n.

Let $M = (p^{i2} + q^{j2})^{0.5}$

define an increase in data point value:

$$
\Delta Z_i = (p^i \Delta x_i + q^j \Delta y_i) (\lambda_i) \qquad 3(12)
$$

where $\lambda_i = \dfrac{\alpha}{c_i}$ \qquad 3(13)

$c_i = \alpha + |d_i|$

$$
\alpha = \frac{\Delta Z_{max}}{M \cdot N_h} \qquad 3(14)
$$

N_h = Maximum number of data points allowable

for each point interpolation

The final equation is

$$f(Z) = \begin{cases} \sum_{i=1}^{N_p} W_i Z_i' \,/\, \sum_{i=1}^{N_p} W_i & \text{if } d_i \neq 0 \\ \\ Z_i & \text{if } d_i = 0 \end{cases} \qquad 3(15)$$

where $Z_i' = Z_i + \Delta Z_i$ 3(16)

3.4 Influence of Discontinuities

If d_c is the length of discontinuity and d_i is the length from P to data point position

Then

$$d_i' = (d_i^2 + d_c^2)^{0.5} \qquad 3(17)$$

This value of distance in 3(17) is then used in 3(2), 3(3) and 3(4).

3.6 Practical Example Problem

Find an interpolated value of flow capacity at a point P with co-ordinates at the origin, given the values at two other points A(2,0) with k_h value 300 md-ft, and B(4,0) with k_h value 500 md-ft.

Solution:

cosine Θ = (0-4)(0+2)/(4)(2) = -1 or Θ = 180°

S_i = ½)2) = 1.0

$g_i = S_i/d_i$ = ¼

g_2 = -½

$W_1 = \dfrac{1}{(4)^2}$ (1+¼) = $\dfrac{5}{64}$

$W_2 = \dfrac{1}{8}$

$P_1 = \dfrac{(500-300)}{(4+2)^2}$ (4) = $\underline{\underline{22.22}}$

$P_2 = \dfrac{(500-300)(2)}{6^2}$ = $\underline{\underline{11.11}}$

$p^i = {}_1^2 P_i$ = $\underline{\underline{33.33}}$

$\mu = (33.33^2 + 0^2)^{0.5}$ = 33.33

$\alpha = \dfrac{200}{333.3}$ = 0.6

C_1 = 0.6 + 4 = 4.6

C_2 = 0.6 + 2 = 2.6

$\lambda_1 = \dfrac{0.6}{4.6}$ = 0.13043

$\lambda_2 = \dfrac{0.6}{2.6}$ = 0.23077

$$\Delta Z_1 = ((22.22)(4) + 0)(0.13032) = 11.59262$$

$$\Delta Z_2 = ((11.11)(2) + 0)(0.23077)) = 5.1277$$

$$Z'_1 = Z_1 + \Delta Z_1 = 500 + 11.59262 = 511.59262$$

$$Z'_2 = Z_2 + \Delta Z_2 = 300 + 5.1277 = 305.1277$$

$$f(P) = \frac{W_1 Z'_1 + W_2 Z'_2}{W_1 + W_2} = \frac{\frac{5}{64}(511.59262) + (\frac{1}{8})(305.1277)}{\frac{5}{64} + \frac{1}{8}}$$

$$f(P) = \frac{39.97 + 38.141}{\frac{5}{64} + \frac{1}{8}} = \underline{\underline{384.54}}$$

An example of a case where a discontinuity exists:
(d_c = 3 units)

$$d_i^1 = \left(d_i^2 + 3^2\right)^{0.5}$$

$$d_1^1 = \left(16 + 9\right)^{0.5} = 5$$

$$d_2^1 = \left(4 + 9\right)^{0.5} = 3.61$$

$$L_1 = 1/5 = 0.2$$

$$l_2 = \left(1/3.61\right) = 0.277$$

with these new values of L_1 and L_2 we can calculate the required interpolation function.

NOMENCLATURE—WEIGHTED AVERAGE

d_i = distance from interpolation point to data point position

P_i = An interpolation point

Z_i = data point value

A = Area of the largest circle enclosed by the data points

N_p = Number of data points to be used in the interpolation process

N = Total number of data points

N_{max} = Maximum number of points needed for interpolation

N_m = Minimum number of points required for interpolation

L_i = weighting function associated with distance

S_i = directional weighting symbol

D_i, D_j = Data point position

d_j = distance of P from data point positions

W_i = a weighting function

Θ = Angle made by D_i P D_j

λ_i = Constant

N_h = Max. number of data points allowable for each point interpolation.

P^i, q^j = Partial derivatives of Z with respect to X,Y respectively

d_i' = equivalent distance used to account for discontinuity effect

M = a function of P^i, q^j.

CHAPTER IV

THEORY OF THE CUBIC SPLINE

Any third degree polynomial function which is continuous on an interval $a \leq x \leq b$, and has continuous first and second derivatives is referred to as a cubic spline.

For many years draftsmen used thin splines to smoothly connect points of interest in a given surface. These connections were made possible by means of weights or 'ducks' attached at specified points. Figure 4.1 is an example of what happens to a spline under various load conditions.

The forces tending to bend the spline can be accounted for by the Bernoulli-Euler Law:

$$Y''(x) = \delta M(x) \qquad\qquad 4(1)$$

$$Y''(x) = M_{(x)} \quad 1/EI \qquad\qquad 4(1a)$$

$$EI\ Y''(x) = M_{(x)}$$

where:

$M_{(x)}$ = Bending moment

$Y''(x)$ = Function Deflection

E = Young's modulus of Elasticity

I = Geometric moment of Inertia

α = Proportionality symbol
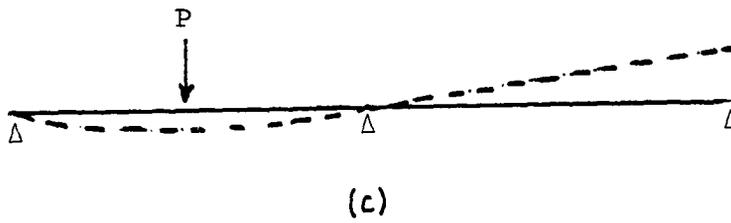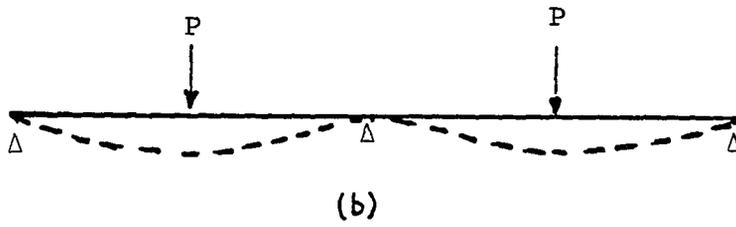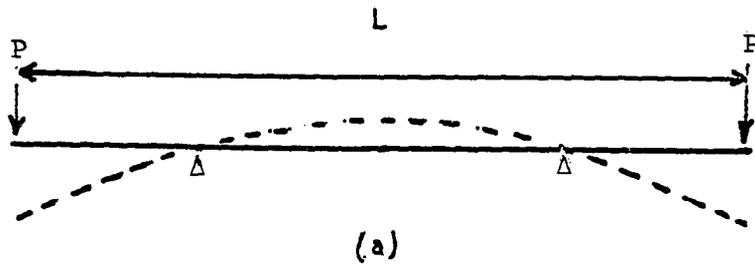
Figure. 4.1: Deflection of a continuous beam under various load conditions.

4.1 Reasons for the Cubic Spline Study

Since conventional polynomial approximations produce results that are highly inflected, the cubic spline approach is desirable for more reliable results. Whereas conventional polynomial approximations give simple continuous function, the cubic spline approach gives a continuous function whose first and second derivatives are continuous. These additional properties make the cubic spline approach capable of handling derivative dependent functions such as velocity in potential flow, and slopes to streamline curvature. The cubic spline approach can smooth the surface and also can represent curved sections with very few nodes.

An improved reservoir description is possible by assuming that the reservoir parameter forms smooth spline surfaces which are continuous. The nature of these surfaces may represent the degree of communication between different rock layers.

The two dimensional spline interpolator is a function of co-ordinate axes. Since reservoir heterogeneity can vary entirely with the co-ordinate axes, the spline approach is a good asset to reservoir engineers.

Finally interpolation using the cubic spline properties is currently very popular, particularly for interpolating relatively noise-free tables of physical data.

4.2 : Interpolation Using Cubic Spline

Due to the inflected or "Wiggly" character often de-
veloped by the conventional polynomial interpolation, the
cubic spline function is developed to produce smoother con-
tours. In view of this, conditions for an interpolating cu-
bic spline have been given as follows:

a) $F_i(X_i) = f(X_i)$ ⎤ Function $f(X_i)$ is continuous
→
b) $F_i(X_{i+1}) = f(X_{i+1})$ ⎦ at data point Value, $F_i(X_i)$.

c) $F_i'(X_i) = F'(X_{i-1})$ ⎤ Derivatives of X_i same when
→
d) $F_i''(X_i) = F''(X_{i-1})$ ⎦ X_i is approached from both
directions.

All four conditions indicate that both the function,
and its first and second derivatives are continuous in the
interval $X_0 < X < X_n$.

Figure 4.2 shows a linear function of the second deri-
vative $M_{(x)}$.
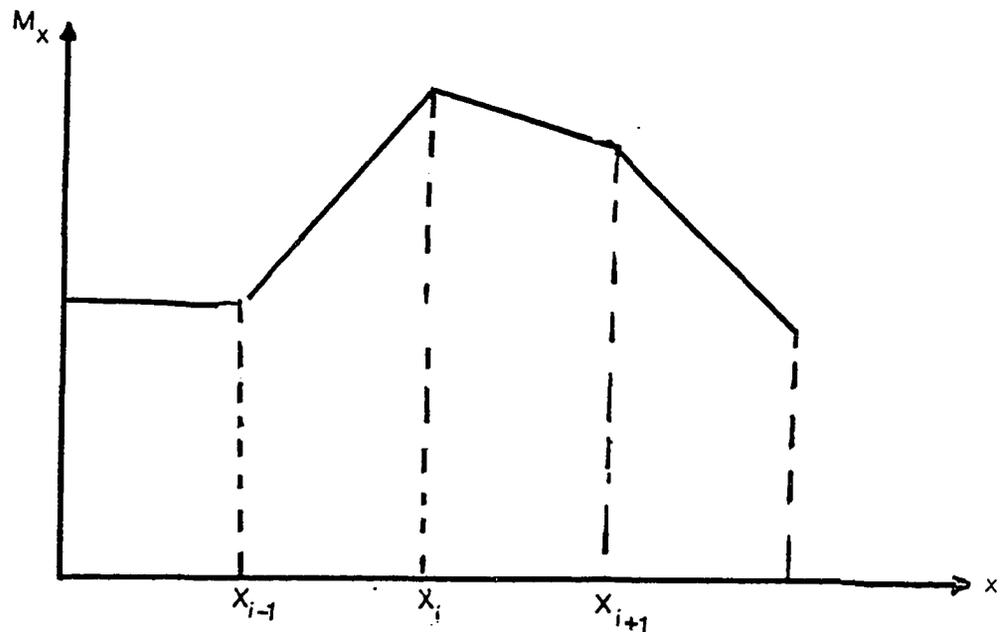
Figure 4.2: Linear relation of $M_{(x)}$ vs X

By linear interpolation, define the second derivative of the interpolation function at any point X as

$$S''(x) = M_{j-1} + \frac{x-x_{j-1}}{x_j-x_{j-1}} [M_j - M_{j-1}] \qquad 4(2)$$

$$S''(x) = M_{j-1} \frac{(X_j-X)}{h_j} + M_j \frac{x-x_{j-1}}{h_j} \qquad 4(3)$$

where :

$$h_j = x_j-x_{j-1}$$

$$S''(x) = M_{j-1} \frac{(x_j-x)}{h_j} + M_j \frac{(x-x_{j-1})}{h_j .} \qquad 4(3)$$

Integrate with respect to $(x-x_{j-1})$ or (x_j-x)

$$S'(x) = -M_{j-1} \frac{(x_j-x)^2}{2h_j} + \frac{m_j (x-x_{j-1})^2}{2h_j} + C_1 \qquad 4(4)$$

integrating again we have

$$S(x) = \frac{M_{j-1}(x_j-x)^3}{6h_j} + \frac{m_j (x-x_{j-1})^3}{6h_j} + C_1(x-x_{j-1}) + C_2 \qquad 4(B)$$

at $x = x_{j-1}$, $S(x) = Y_{j-1}$; 4(B) becomes

$$Y_{j-1} = \frac{M_{j-1}(x_j-x_{j-1})^3}{6h_j} + m_j(0) + C_1(0) + C_2 \qquad 4[C]$$

$$C_2 = Y_{j-1} - M_{j-1} \frac{(x_j-x_{j-1})^2}{6} \qquad 4[D]$$

when $x=x_i$, $S(x) = Y_j$

$$Y_j = M_{j-1}(0) + \frac{M_j h^2}{6} + C_1(h_j) + C_2$$

$$C_1 h_j = Y_j - \frac{M_j h^2}{6} - Y_{j-1} + M_{j-1} \frac{h^2}{6}$$

$$C_1 = \frac{Y_j - Y_{j-1}}{h_j} - \frac{(M_j - M_{j-1})h_j}{6} \qquad 4[E]$$

Putting in values of C_1 and C_2 in 4.(B) we have:

$$S(x) = \frac{M_{j-1}(x_j-x)^3}{6h_j} + M_j\frac{(x-x_{j-1})^3}{6h_j} + (Y_{j-1} - \frac{M_{j-1}h_j^2}{6})$$

$$\left(\frac{x_j-x}{h_j}\right) + (Y_j - \frac{M_jh_j^2}{6})\frac{x-x_{j-1}}{h_j} \quad \text{-------} \quad 4[F]$$

Now we can list the equations that obtain

$$S''(x) = M_{j-1} + \frac{x-x_{j-1}}{x_j-x_{j-1}}[M_j - M_{j-1}] \quad \text{------} \quad 4(2)$$

$$S''(x) = M_{j-1}\frac{(x_j-x)}{h_j} + M_j\frac{x-x_{j-1}}{h_j} \quad\quad 4(3)$$

$$S'(x) = -M_{j-1}\frac{(x_j-x)^2}{2h_j} + M_j\frac{(x-x_{j-1})^2}{2h_j} + \frac{Y_j-Y_{j-1}}{h_j} -$$

$$\frac{(M_j-M_{j-1})}{6}h_j \quad \text{- - --} \quad 4(4)$$

$$S(x) = M_{j-1}\frac{(x_j-x)^3}{6h_j} + M_j\frac{(x-x_{j-1})^3}{6h_j} + (Y_{j-1} -$$

$$\frac{M_{j-1}h_j^2}{6})\frac{(x_j-x)}{h_j} + (\frac{y_j-M_jh_j^2}{6})(\frac{x-x_{j-1}}{h_j}) \text{ ---} 4(5)$$

or

$$S(x) = \frac{M_{j-1}}{6}[\frac{(x_j-x)^3}{h_j} - (x_j-x)h_j] + \frac{M_j}{6}[\frac{(x-x_{j-1})^3}{h_j} -$$

$$(x-x_{j-1})h_j] + Y_{j-1}(\frac{x_j-x}{h_j}) + Y_j(\frac{x-x_{j-1}}{h_j}) \text{ -- } 4(6)$$

Putting the condition $S'_{j-1} = S'_j(x_j)$ in 4(4) we have

$$S'_{j-1}(x_j) = \frac{h_j}{6}M_{j-1} + \frac{h_j}{3}M_j + \frac{Y_j-Y_{j-1}}{h_j} \quad \text{- --}4(7)$$

$$S'_j(x_j) = \frac{-M_j}{3}h_{j+1} + 0 + \frac{Y_{j+1}-Y_j}{h_{j+1}} - \frac{(M_{j+1}-M_j)}{6}h_{j+1}$$

$$S'_j(x_j) = -M_j(\frac{h_{j+1}}{3}) - M_{j+1}(\frac{h_{j+1}}{6}) + \frac{Y_{j+1}-Y_j}{h_{j+1}} \text{ ---}4(8)$$

Equating 4[F] and 4(8) we have:

$$h_j M_{j-1} + 2(h_j + h_{j+1})M_j + h_{j+1}M_{j+1} = 6\left[\frac{Y_{j+1}-Y_j}{h_{j+1}} - \right.$$

$$\left.\frac{Y_j - Y_{j-1}}{h_j}\right] \qquad\qquad 4(9)$$

$$\frac{h_j}{h_j + h_{j+1}}M_{j-1} + 2M_j + \frac{h_{j+1}}{h_j + h_{j+1}}M_{j+1} = \left[6\left[\frac{Y_{j+1}-Y_j}{h_{j+1}(h_{j+1}+h_j)}\right.\right. -$$

$$\left.\left.\frac{Y_j - Y_{j-1}}{h_j(h_{j+1}+h_j)}\right]\right] \qquad\qquad 4(10)$$

or $\quad \mu_j M_{j-1} + 2M_j + \Lambda_j M_{j+1} = d_j \qquad\qquad 4(11)$

where:

$$\mu_j = \frac{h_j}{h_j + h_{j+1}} = 1 - \Lambda_j \qquad\qquad 4(12)$$

$$\Lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}} \qquad\qquad 4(13)$$

$$d_j = 6\left[\frac{(Y_{j+1}-Y_j)/h_{j+1}}{h_{j+1}+h_j} - \frac{(Y_j - Y_{j-1})/h_j}{h_{j+1}+h_j}\right] \qquad\qquad 4(14)$$

Applying boundary conditions to 4(11) at boundaries j=0, and j=N:

$$(\mu_0)M_{-1} + 2M_0 + \Lambda_0 M_1 = d_0 \qquad\qquad 4(15)$$

$$_N M_{N-1} + 2M_N + \Lambda_N M_{N+1} = d_N \qquad\qquad 4(16)$$

Solve for M_0, M_1, M_2, ..., M_{N-1} by matrix using Gauss-Elimination

$$
\begin{bmatrix}
2 & \lambda_0 & 0 & \cdots & 0 & 0 & 0 \\
\mu_1 & 2 & \lambda_1 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 2 & \lambda_{n-2} & 0 \\
0 & 0 & 0 & \cdots & \mu_{n-1} & 2 & \lambda_{n-2} \\
0 & 0 & 0 & & 0 & n & \mu 2
\end{bmatrix}
\begin{bmatrix}
M_0 \\ M_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_{n-2} \\ M_{n-1} \\ M_n
\end{bmatrix}
=
\begin{bmatrix}
d_0 \\ d_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_{n-2} \\ d_{n-1} \\ d_n
\end{bmatrix}
$$

Knowing M_j $(j = 0,1,\ldots,N)$, we

can now apply our interpolation function:

$$
g(x) = \frac{M_{j-1}}{6}\left[\frac{(x_j - x)^3}{h_j} - h_j(x_j - x)\right]
$$

$$
+ \frac{M_j}{6}\left[\frac{(x - x_{j-1})^3}{h_j} - h_j(x - x_{j-1})\right]
$$

$$
+ Y_{j-1}\left[\frac{x_j - x}{h_j}\right] + y_j\left[\frac{x - x_{j-1}}{h_j}\right] \qquad 4(18)
$$

We can now interpolate values of interest.

4.3 Sample Problem for one dimensional spline
         (Irregularly distributed data points)

Obtain an approximate value for $g(7)$, and $g(10)$ by

interpolation (Table 4.1). Given a natural spline boundary

conditions $(M_1 = M_6 = 0)$.

Divide equation 4(9) by h_{j+1} and we get

$$\frac{h_j}{h_{j+1}} \left[M_{j-1}\right] + \frac{2(h_j + h_{j+1})}{h_{j+1}} \left[M_j\right] + M_{j+1}$$

$$= 6 \left[\frac{y_{j+1}-y_j}{(h_{j+1})^2} - \frac{y_j - y_{j-1}}{h_j(h_{j+1})}\right] \qquad 4(19)$$

For:

$j=2;\ h_j=4;\ h_{j+1}=3;\ y_{j-1}=4;\ y_j=10;\ y_{j+1}=15$

$$\frac{4}{3}\left[M_{j-1}\right] + \frac{7}{2}\left[M_j\right] + M_{j+1} = 6\left[\frac{5}{9} - \frac{1}{2}\right] = \frac{1}{3} \qquad 4(20)$$

Since $M_1 = 0$ 4(20) becomes:

$$21\ M_2 + 6\ M_3 = 2 \qquad\qquad 4(21)$$

For:

$j=3;\ h_j=3;\ h_{j+1}=3;\ y_{j-1}=10;\ y_j=15;\ y_{j+1}=8$

$$M_2 + 2\ M_3 + M_4 = -8 \qquad 4(22)$$

For:

$j=4;\ h_j=3;\ h_{j+1}=4;\ y_{j-1}=15;\ y_j=8;\ y_{j+1}=3$

$$\frac{3}{4}\ M_3 + \frac{7}{2}\ M_4 + M_5 = \frac{13}{8} \qquad 4(23)$$

$$18\ M_3 + 84\ M_4 + 24\ M_5 = 39 \qquad 4(24)$$

But $M_1 = M_5 = 0$

$$18\ M_3 + 84\ M_4 = 39 \qquad 4(25)$$

from 4(21) and 4(22) we get

$$-36\ M_3 - 21\ M_4 = 168 \qquad 4(26)$$

$4 \times (4(25)) + (4(26))$

$$-126 \ M_3 = 711$$

$$M_3 = -5.643$$

$$M_4 = 1.673$$
4(27)

$$M_2 = 1.612$$

$$M_1 = M_5 = 0.0$$

Since we want to interpolate the function value when $X_i = 7.0$, we must use the cubic approximation to the interval between $j=2$, and $j=3$ (or $6 \leq x \leq 9$)

From equation 4(18), we can calculate $g(7)$ using node $j=2$

Thus:

$$h_j = 4; \ h_{j+1} = 3; \qquad x_{j-1} = 2; \ x_j = 6; \ x_{j+1} = 9$$

$$y_j = 10; \ y_{j+1} = 15$$

$$g(7) = \frac{M_1}{6} \left[\frac{(6-7)^3}{4} - 4 \ (6-7) \right]$$

$$+ \frac{M_2}{6} \left[\frac{(7-2)^3}{4} - 4(7-2) \right]$$

$$+ 4 \left[\frac{6-7}{4} \right] + 10 \left[\frac{7-2}{4} \right]$$

$$g(7) = 0 + 3.02 + 11.5 = 14.52$$

Using linear interpolation $g(7) = 11.86$

Exact value is 14.0

Similarly $g(10) = \frac{M_2}{6} \left[\frac{(9-10)^3}{3} - 3(9-10) \right]$

$$+ \frac{M_3}{6} \left[\left(\frac{10-6}{3}\right)^3 - 3\left(\frac{10-6}{3}\right) \right]$$

$$+ 10 \left(\frac{9-10}{3}\right) + 15 \frac{(10-6)}{3}$$

$$= \left(\frac{1.612}{6}\right) (2\ 2/3) + \left(\frac{-5.64}{6}\right)\left(\frac{64}{3} - 12\right) - \frac{10}{3} + 20$$

$$g(10) = \qquad\qquad 8.61$$

By linear interpolation $g(10) = 12.7$

Exact value is $\qquad g(10) = 9.3$

Figure 4.3 shows how the one-dimensional method of this study approximates the original curve. The closeness of the values obtained to the exact values explains the quality of this approach as compared to other approaches such as linear interpolation method. The method of this study is recommendable to problems dealing with curvatures, etcetera.

Table 4.1. Sample Data.

j	x_j	y_i
1	2	4
2	6	10
3	9	15
4	12	8
5	16	3

$$
\begin{array}{cccccc}
M_1 & M_2 & M_3 & M_4 & M_5 & M_6
\end{array}
$$

$$
\begin{bmatrix}
0 & 2 & 3/7 & 0 & 0 \\
0 & 4/7 & 2 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} & 2 & 4/7
\end{bmatrix}
\begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
M_5
\end{bmatrix}
=
\begin{bmatrix}
1/7 \\
-4 \\
13/14
\end{bmatrix}
$$

$$ M \qquad d_j $$

Solutions:

$$ M_1 = 0.0 $$

$$ M_2 = 1.612 $$

$$ M_3 = 5.643 $$

$$ M_4 = 1.673 $$

$$ M_5 = 0.0 $$

$$ g(10) = 8.61 $$

Exact solution $g(10) = 9.3$

By Linear interpolation $g(10) = 12.7$
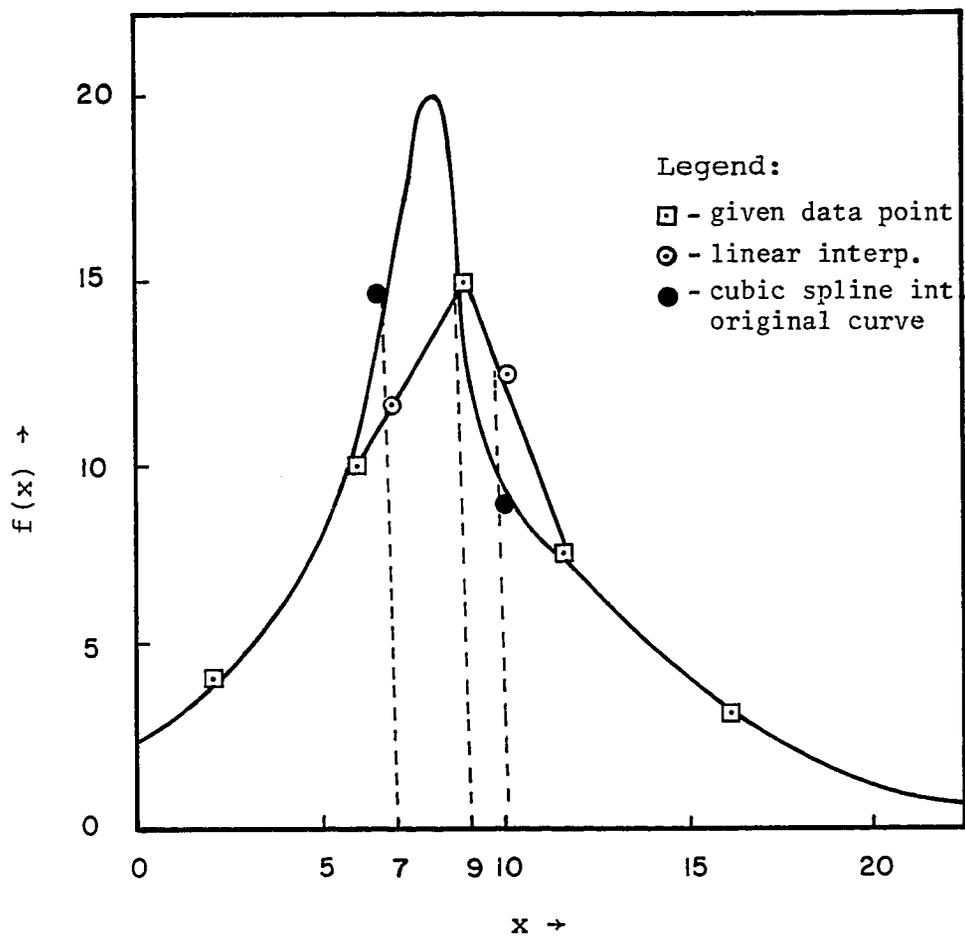
Figure 4.3 Comparison of the result of sample calc.
and linear interpolation with original
curve.

4.4 Two Dimensional Spline Approach

In the one dimensional spline procedure, we constructed an interpolating spline of the form g(x) as a function of a single variable x. In the two dimensional approach, our interpolating spline will be a function S(x,y) which is dependent on the x and y co-ordinate axes. However, the two dimensional approach is essentially an extension of the one dimensional case applied in two dimensions. The major difference between the one dimensional method and the two dimensional approach is that the latter is mesh dependent as can be seen later in this development.

The procedure for the two-dimensional development of the cubic spline interpolation is similar to that of the two dimensional linear interpolation. Both use the basic theory of Hilbert Space [1]. Because of its simplicity, one can present the development of the two-dimensional linear interpolation first for illustrative purposes.

4.4.(a) Two-Dimensional Linear Interpolation

The procedure for obtaining the two dimensional linear interpolation is represented in figure 4.4. First, carry out linear interpolation along the x-axis to obtain values for f_A and f_B. Then interpolate between f_A and f_B linearly along the y direction while keeping x constant to obtain the approximation value f(x,y) .
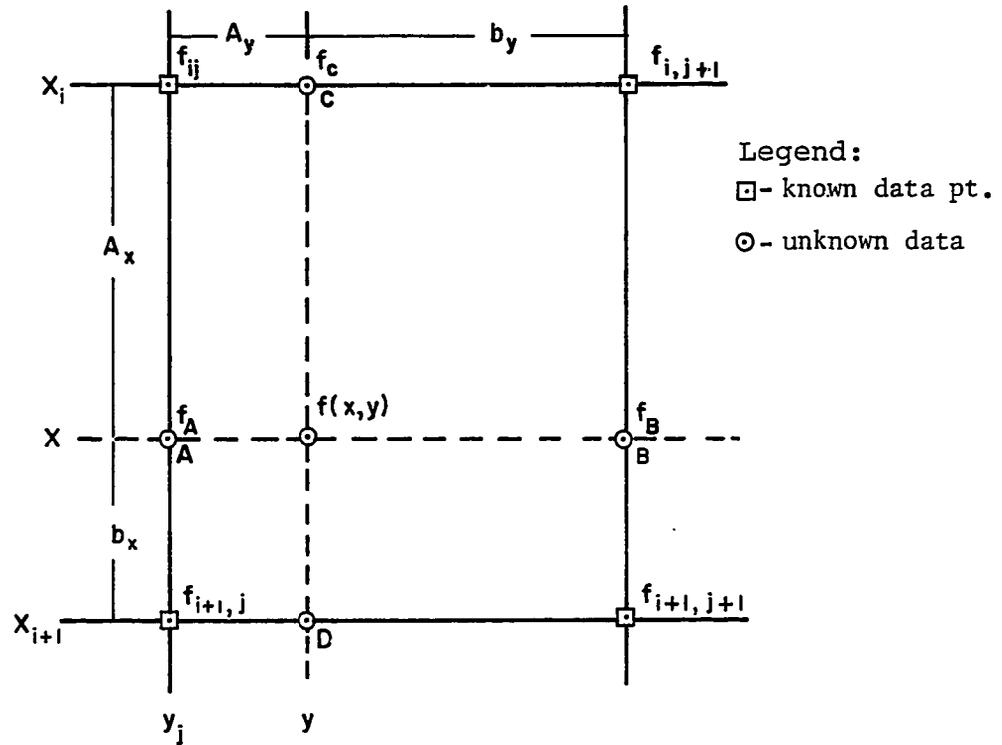
Figure 4.4   2-Dimensional linear interpolation
(rectangular mesh system).

<u>Derivations:</u> $\qquad x_i < x < x_{i+1}; \; y_i < y < y_{j+1}$

Let $A_x = x - x_i \; ; \; A_y = y - y_j$

$$b_x = x_{i+1} - x$$

$$b_y = y_{i+1,j+1} - y$$

$$f_A = f_{ij} + (f_{i+1,j} - f_{i,j})\left(\frac{A_x}{A_x + b_x}\right) \qquad\qquad 4(28)$$

$$f_A = \left(1 - \frac{A_x}{A_x + bx}\right) f_{ij} = \left(\frac{A_x}{A_x + b_x}\right) f_{i+1,j} \qquad 4(29)$$

Similarly

$$f_B = \left(-\frac{A_x}{A_x + b_x}\right) f_{i,j+1} + \left(\frac{A_x}{A_x + b_x}\right) f_{i+1,j+1} \qquad 4(30)$$

$$f(x,y) = f_A + (f_B - f_A)\left(\frac{A_y}{A_y + b_y}\right) \qquad\qquad 4(31)$$

$$f(x,y) = (1 - \beta) f_A + \beta f_B \qquad\qquad 4(32)$$

where:

$$\beta = \frac{A_y}{A_y + b_y} \; , \qquad \alpha = \frac{A_x}{A_x + b_x}$$

then

$$f_A = (1 - \alpha) f_{ij} + \alpha f_{i+1,j} \qquad\qquad 4(33)$$

$$f_B = (1 - \alpha) f_{i,j+1} + \alpha f_{i+1,j+1} \qquad\qquad 4(34)$$

$$f(x,y) = (1 - \beta)(1 - \alpha) f_{ij} + \beta(1 - \alpha) f_{i,j+1}$$

$$+ \; \alpha(1 - \beta) f_{i+1,j} + (\alpha\beta) f_{i+1,j+1} \qquad 4(35)$$

4.4.(b) Cubic Spline Interpolation in Two-dimensions:

The two dimensional spline interpolation like its two dimensional linear counterpart will first interpolate values of S_A, S_B, S_C and S_D in the x-direction while holding y axis constant. The one dimensional spline approach is further applied in the y-direction using known values of f_A, f_B, f_C and f_D to obtain an estimate of $S(x,y)$ at the desired points.

The interpolating function is as given in 4(6) for x axis:

$$S(x) = \frac{M_{i-1}}{6} \left[\frac{(x_i-x)^3}{h_i} - (x_i - x) h_i \right]$$

$$+ \frac{M_i}{6} \left[\frac{(x-x_{i-1})^3}{h_i} - (x-x_{i-1})h_i \right]$$

$$+ f_{i-1} \left(\frac{x_i - x}{h_i} \right) + f_i \left(\frac{x - x_{i-1}}{h_i} \right) \qquad \qquad 4(36)$$

where f_i are the function values (Figure 4.5).

By putting appropriate values of distances A, B, C, D, we can determine f_A, f_B, f_C, f_D where x is such that $x_{i-1} < x < x_i$

NOTE: $f_A \equiv f_{j-1}$, $f_B \equiv f_j$, $f_C \equiv f_{j+1}$, $f_D \equiv f_{j+2}$

These values thus determined form new data points for a one-dimensional interpolation along the y-axis.
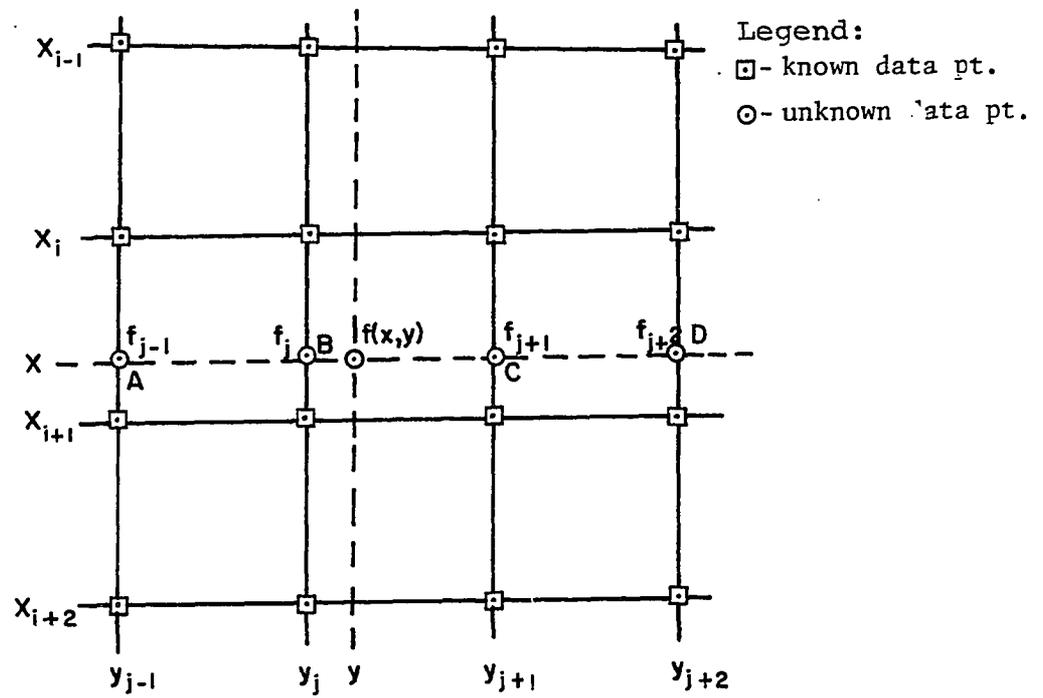
Figure 4.5   Two Dimensional Spline
Interpolation (mesh system).

Let S*(y) represent an independent interpolation along the y-axis.

$$S^*(y) = \frac{M_{j-1}}{6} \left[\frac{(y_j - y)^3}{h_j} - (y_j - y)h_j \right]$$

$$+ \frac{M_j}{6} \left[\frac{(y - y_{j-1})^3}{h_j} - (y - y_{j-1})h_j \right]$$

$$+ f_{j-1}\left(\frac{y_j - y}{h_j}\right) + f_j \left(\frac{y - y_{j-1}}{h_j}\right) \qquad 4(37)$$

Where f_j are the function values represented by f_A, f_B, f_C and f_D

y is such that $y_{j-1} < y < y_j$

Since f_A, f_B, f_C and f_D are functions of x, then S*(y) is a function also of x

$$S^*(y) = S(x,y).$$

Theoretically we can look at the overall relationship as a product of two one dimensional splines given by the relation (from 4(6)).

$$S^*(y) = S(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} K_{ij} (x-x_i)^i (y-y_j)^j \qquad 4(38)$$

where K_{ij} is the coefficient to be determined. Alternatively we can apply a Hilbert Space theory to the problem.

4.5 Theoretical Validation of the Procedure

Outlined by Use of Hilbert Space Theory [1]

Let L_x, L_y represent two differential operators in a rectangular mesh, R_{ij}, operating in the x and y directions respectively.

$$L_x = A_n(x)\ D_x^m + A_{m-1}(x)\ D_x^{m-1} + \ldots + A\ D(x) \qquad 4(39)$$

$$L_y = B_n(y)\ D_y^n + B_{n-1}(y)\ D_y^{n-1} + \ldots + B\ D(y) \qquad 4(40)$$

where:

$A_m(x)$, $B_n(y)$ posses continuous m^{th}, n^{th} derivatives respectively.

D_x, D_y = partial derivatives with respect to x, y respectively.

Then

$$L_x \cdot L_x\ S(x,y) = 0 \qquad 4(41)$$

$$L_y \cdot L_y\ S(x,y) = 0 \qquad 4(42)$$

If the fundamental solutions of the above are given by $U_i(x)$, $U_j(y)$ respectively

Then we can solve for S(x,y) from the $U_i(x)$ solution.

$$S(x,y) = A_i(x)\ U_i(x) \qquad 4(43)$$

putting this value of S(x,y) into the second equation will give the relation for the y-direction [cross product relation]

$$S(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} A_i\ U_i(x)\ B_j\ U_j(y) \qquad 4(44)$$

where:

A$_i$, B$_j$ are functions of x,y respectively with the property of continuous nth derivatives. m,n are no. of terms in x,y respectively

U$_i$(x), U$_j$(y) are functions of x, and ý respectively

However looking at equation 4(6) one can rightly say this:

$$S(x,y) = \sum_{i=o}^{3} \sum_{j=o}^{3} K_{ij} (x-x_i)^i (y-y_j)^j \qquad 4(45)$$

where:

$$K_{i,j} = f(f(x,y), P_{ij}, q_{ij}, e_{ij}) \qquad 4(46)$$

f(x,y) is the functional cubic equation in x and y

P$_{ij}$ = partial derivation of the interpolation equation with respect to x

q$_{ij}$ = partial derivative of the interpolation equation with respect to y

e$_{ij}$ = Second derivative of the interpolation equation with respect to x,y.

Finally we can define the coefficient k$_{ij}$ thusly:

$$k_{ij} = \sum_{i=o}^{3} \sum_{j=o}^{3} C_{ij} \cdot 1/H_{ij} \qquad 4(47)$$

where:

$$C_{ij} = \begin{vmatrix} f(x,y) & P_{ij} \\ q_{i,j} & e_{ij} \end{vmatrix} \qquad 4(48)$$

$$H_{ij} = \sum_{i=o}^{3} \sum_{j=o}^{3} \left[h_i\right]^i \left[h_j\right]^j \qquad 4(49)$$

where:

$$h_i = x_i - x_{i-1} \qquad 4(50)$$

$$h_j = y_j - y_{j-1} \qquad 4(51)$$

The computation of $P_{i,j}$, $q_{i,j}$ and $e_{i,j}$ are presented in the succeeding pages.

4.6 Computation of Derivatives

The partial derivatives needed at each of the four corners of the subrectangle are defined thus:

Let

$$P_{ij} = \frac{\delta S}{\delta x}$$

$$q_{ij} = \frac{\delta S}{\delta y} \qquad \cdot 4(52)$$

$$e_{ij} = \frac{\delta^2 S}{\delta y \delta x}$$

Then as in 4(10) for P_{ij}:

$$\Delta X_{i-1} P_{i-1,j} + 2(\Delta X_{i-1} + \Delta X_i) P_{ij} + \Delta X_i P_{i+1,j}$$

$$= 6\left[\frac{1}{\Delta X_i} (U_{i+1,j} - U_{i,j}) - \frac{1}{\Delta X_{i-1}} (U_{i,j} - U_{i-1,j})\right] \qquad 4(53)$$

In y-direction : q_{ij}

$$\Delta Y_{j-1} q_{i,j-1} + 2(\Delta Y_{j-1} + \Delta Y_j) q_{i,j} + \Delta Y_j q_{i,j+1}$$

$$= 6 \left[\frac{1}{\Delta Y_j} (U_{i,j+1} - U_{i,j}) - \frac{1}{\Delta Y_{j-1}} (U_{i,j} - U_{i,j-1}) \right] \qquad 4(54)$$

for second derivative e_{ij}:

$$\Delta X_{i-1} e_{i-1,j} + 2(\Delta X_{i-1} + \Delta X_i) e_{ij} + \Delta X_i e_{i+1,j}$$

$$= 6 \left[\frac{1}{\Delta X_i} (q_{i+1,j} - q_{ij}) - \frac{1}{\Delta X_{i-1}} (q_{ij} - q_{i-1,j}) \right] \qquad 4(55)$$

Equations 4(52) through 4(55) can be solved independently using a tridiagonal matrix as in 4(17) to calculate values for P_{ij}, q_{ij} and e_{ij} respectively.

Thus:

$$
\begin{bmatrix}
2 & \lambda_0 & 0 & \cdots & 0 & 0 & 0 \\
\mu & 2 & \lambda_1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots 2 & \lambda_{n-2} & 0 \\
0 & 0 & 0 & \cdots \mu_{n-1} & 2 & \lambda_{n-1} \\
0 & 0 & 0 & \cdots 0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
P_0 \\
P_1 \\
\vdots \\
\vdots \\
\vdots \\
P_{m-2} \\
P_{m-1} \\
P_m
\end{bmatrix}
=
\begin{bmatrix}
d_0 \\
d_1 \\
\vdots \\
\vdots \\
\vdots \\
d_{m-2} \\
d_{m-1} \\
d_m
\end{bmatrix}
\qquad 4(56)
$$

Similar approach can be used to obtain q_{ij} and e_{ij}

The above is equivalent to

$$AX = D \qquad\qquad 4(57)$$

$$X = A^{-1}D \qquad\qquad 4(58)$$

where:

$$X = P_1, P_2, \ldots, P_{m-1} \qquad\qquad 4(59)$$

$$D = d_1, d_2, \ldots, d_{m-1} \qquad\qquad 4(60)$$

The divided difference version for calculating the partial derivatives can also be used to simplify calculations.

$$\frac{\delta S}{\delta x} = P_{ij} = [S_{i+1,j} - S_{i,j}] / (X_{i+1} - X_i) \qquad\qquad 4(61)$$

$$\frac{\delta S}{\delta y} = q_{ij} = [S_{i,j+1} - S_{ij}] / (Y_{j+1} - Y_j) \qquad\qquad 4(62)$$

$$\frac{\delta^2 S}{\delta y \delta x} = e_{ij} = [P_{i,j+1} - P_{ij}] / (Y_{j+1} - Y_j) \qquad\qquad 4(63)$$

$$= [q_{i+1,j} - q_{ij}] / (X_{i+1} - X_i) \qquad\qquad 4(64)$$

4.7 Sample Problem for two-dimensional Spline Interpolation (regularly distributed Data Points)

Obtain an approximate value at $f(1.5,1.5)$ using a two-dimensional cubic spline interpolation. Data values are tabulated in table 4.2.

Assume a natural spline condition.

Solution:

Start the solution by noting that

$$\lambda = \mu = \tfrac{1}{2}$$

$$d_j = 3(f_{j-1} - 2f_j + f_{j+1})$$

$$h_j = h_{j+1} = 1.0$$

a. Interpolate x first, and then y.

(see figure 4.6 and table 4.3)

b. Y first then x (figure 4.7 and table 4.4)

The two values obtained by reversing the axes are same.

Table 4.2: Two Dimensional Interpolation Method

X_i	Y_j 0	1	2	3
0	4	6	8	12
1	9	10	18	20
2	15	16	30	32
3	7	8	14	16

Table 4.3: Solution values--interpolation of X, first.

Y_j	M_1	M_2	M_3	M_4	$g_x(1.5)$
0	0.0	7.2	-11.8	0.0	12.98
1	0.0	8.8	-23.2	0.0	13.9
2	0.0	5.433	-15.73	0.0	24.644
3	0.0	17.6	-46.4	0.0	27.8

Interpolate in the x direction:
from $g_x(1.5)$, $M_2 = 18.743$, $M_3 = -16.065$, $f(x,y)= [19.6]$

Table 4.4: Solution values--interpolating y first, then x.

j	x_j	m_1	m_2	m_3	m_4	$g_y(1.5)$
1	0	0.0	0.0	0.0	0.0	7
2	1	0.0	13.6	-12.4	0.0	13.925
3	2	0.0	25.6	-24.4	0.0	22.425
4	3	0.0	9.6	-8.4	0.0	10.925

Interpolate in the x direction using value of
$g_y(1.5)$: $M_2 = 11.72$, $M_3 = -34.43$, $f(y,x) = [19.6]$
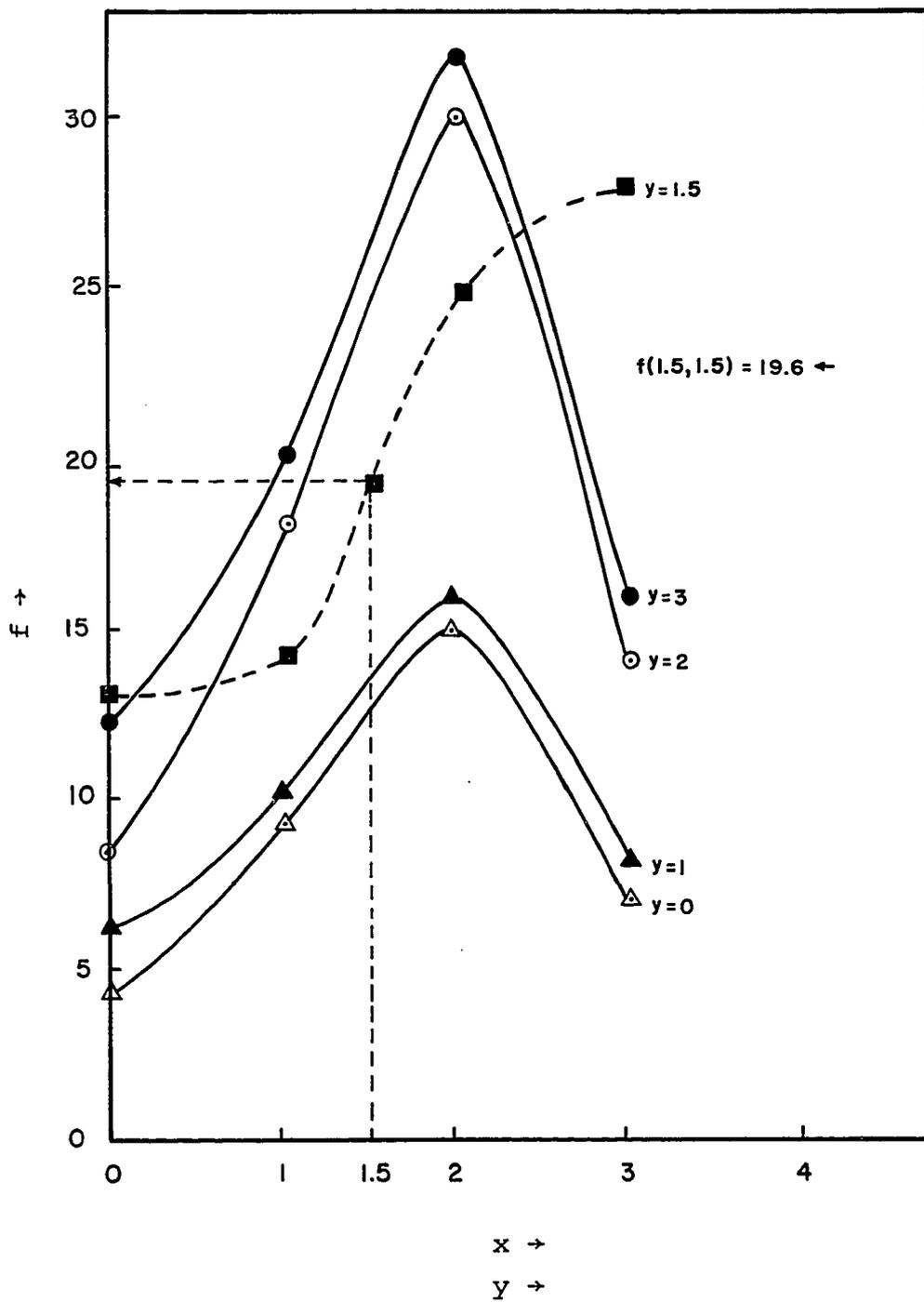
Figure 4.6   Interpolation along x axis, first
and then along y axis.
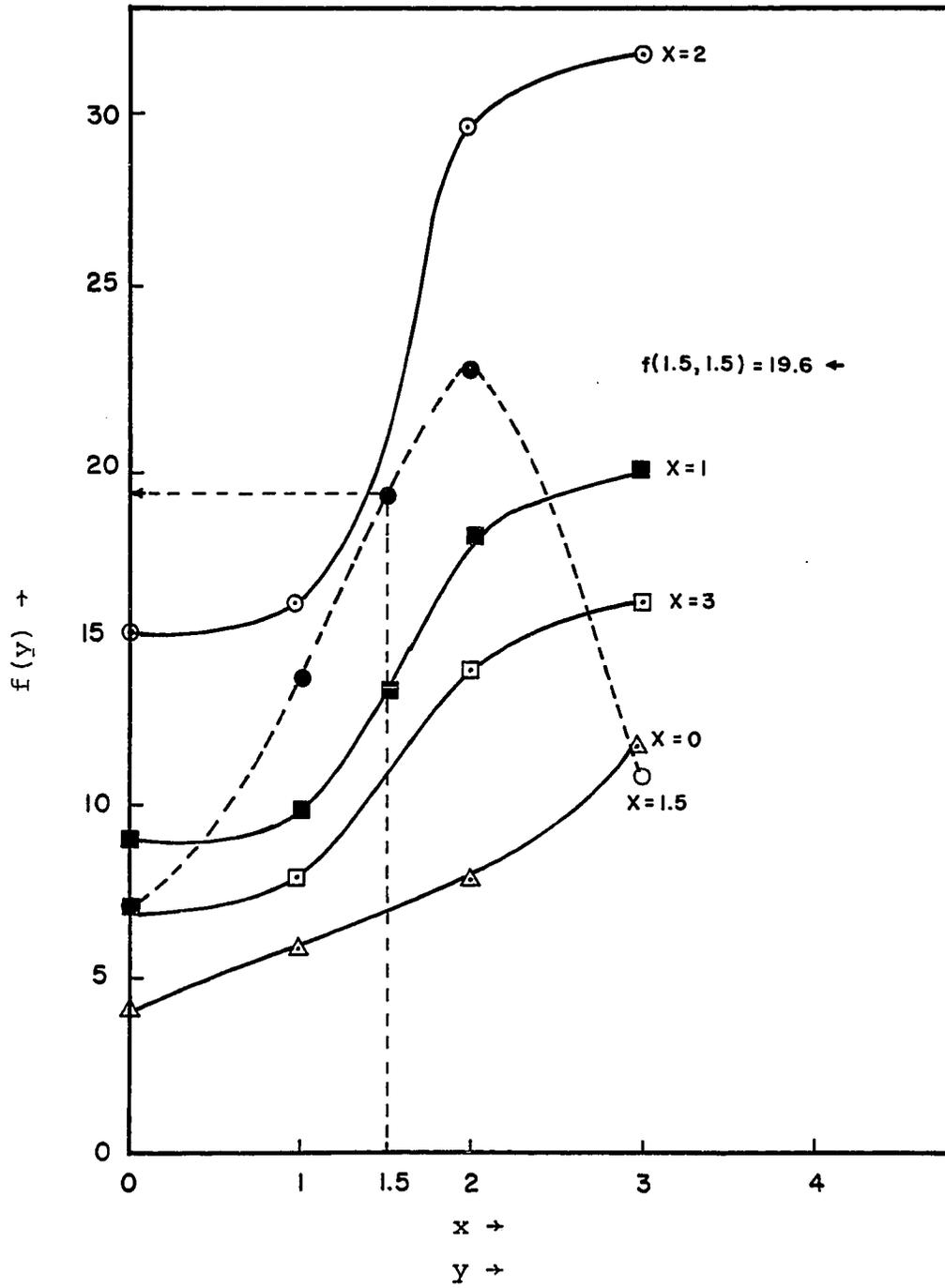
Figure 4.7   Interpolation along y axis
first and then x axis.

NOMENCLATURE FOR THE CUBIC SPLINE FUNCTIONS

$y(x)$ = functional value of x

$y''(x)$ = Second derivative of y with respect to x

$y'(x)$ = first derivative of function y with respect to X

$M(x)$ = 'Bending moment' function

E = Young's modulus of Elasticity

I = Geometric moment of Inertia

α = Proportionality symbol

C_1, C_2 = constants of integragion

μ λ = constants

d_j = n x 1 matrix

M_j = "Bending moment" at node j →• a constant

y_j = functional value at node j

h_j = interval constant between node at j and node at $j-1$

$g(x)$ = One-dimensional interpolation function

$S(x,y)$ = Two-dimensional interpolation function

f_A, f_B, f_C, f_D = Interpolation values obtained by interpolation along the χ-axis.

α, β = Proportional constants in the two dimensional linear interpolation

$f_{i,j}$ = Two-dimensional functional values at coordinate nodes i, j.

$S^*(y)$ = An interpolating function along the y-axis.

$K_{i,j}$ = Coefficient matrix

L_x, L_y = Differential operators in x and y

A_m, B_n = functions in x, and y possessing continuous m^{th} and n^{th} derivatives respectively

U_i, U_j = fundamental solutions of sets of equations in the x, and y directions.

$P_{i,j}$ = Partial derivative of the interpolation equation with respect to x

q_{ij} = Partial derivative of the interpolation equation with respect to y.

$\rho_{i,j}$ = Partial derivative of the interpolation equation with respect to x, y

$H_{i,j}$ = Coefficient Matrix

CHAPTER V

MODEL TESTING

In order to show the superiority of this study model
over some of the conventional approaches, a few tests were
run. The first test, figure 5.1, is a plot that compares
the quality of the Lagrange interpolator with the one dimen-
sional approach of this study. The results are tabulated in
table 5.1, and figure 5.1 is the resulting plot. A compari-
son of both plots with the original curve shows that the
one-dimensional method of this study is a much closer approx-
imator than the Lagrange polynomial interpolator. One of
the possible reasons for this outcome is the fact that the
first and the second derivatives of the method of this study
are continuous. These properties make it easier for the Spline
model to closely approximate curvatures than the conventional
polynomial approximations.

The two dimensional approach of this study was also
compared with the two-dimensional Hermite interpolation
method. Hermite interpolation method has a continuous first
derivative. Figure 5.2 A is the result of the Hermite inter-
polation method while figure 5.2 B is the result of interpo-
lation using the method of this study. Data for this test
is presented in table 5.2. A comparison of both methods shows

Table 5.1. Data for Comparison of Lagrange
interpolation method versus the
method of this study.

X	Y		
	Original curve	Lagrange method	This study
*2	4.0	4.0	4.0
*6	10.0	10.0	10.0
7	14.0	11.6	14.5
8	20.0	10.0	21.0
*9	15.0	15.0	15.0
10	9.5	9.3	8.8
11	--	--	--
*12	8.0	8.0	8.0
*16	3.0	3.0	3.0

*Supplied data points.
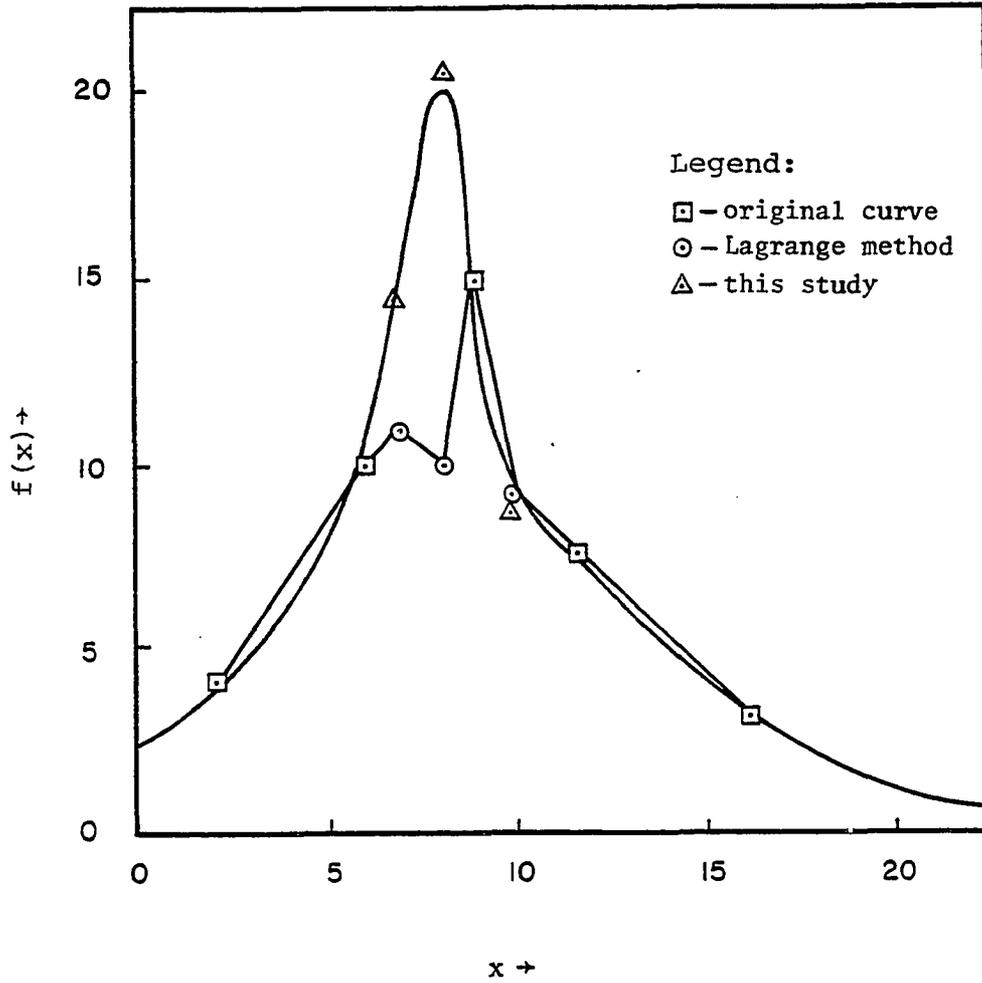
Figure 5.1. Comparison of Lagrange interpolation method with the method of this study.

Table 5.2: Reservoir data from Sitio Grand Field in New Mexico (data of Perez (23)).

X	Y	kh
2.95	12.3	0.4
4.	11	16.3
4	7	27
5	12.3	14.
5	8.8	19.4
5	5.3	6.5
6	11	4.2
6	7	10.0
7	12.3	0.31
7	8.8	11.2
7	5.3	6.82
8	11.	1.2
8	7	59.1
9	8.8	4.1
9	5.3	3.2
10	7	1.45
10	3.5	0.15
11	5.3	4.2

Fig 5.2 a: Hermite Interpolation.   Fig 5.2 b: this Study

Fig. 5.2:   Contour plots of the two-dimensional approach of
this study and the Hermite Interpolation method.
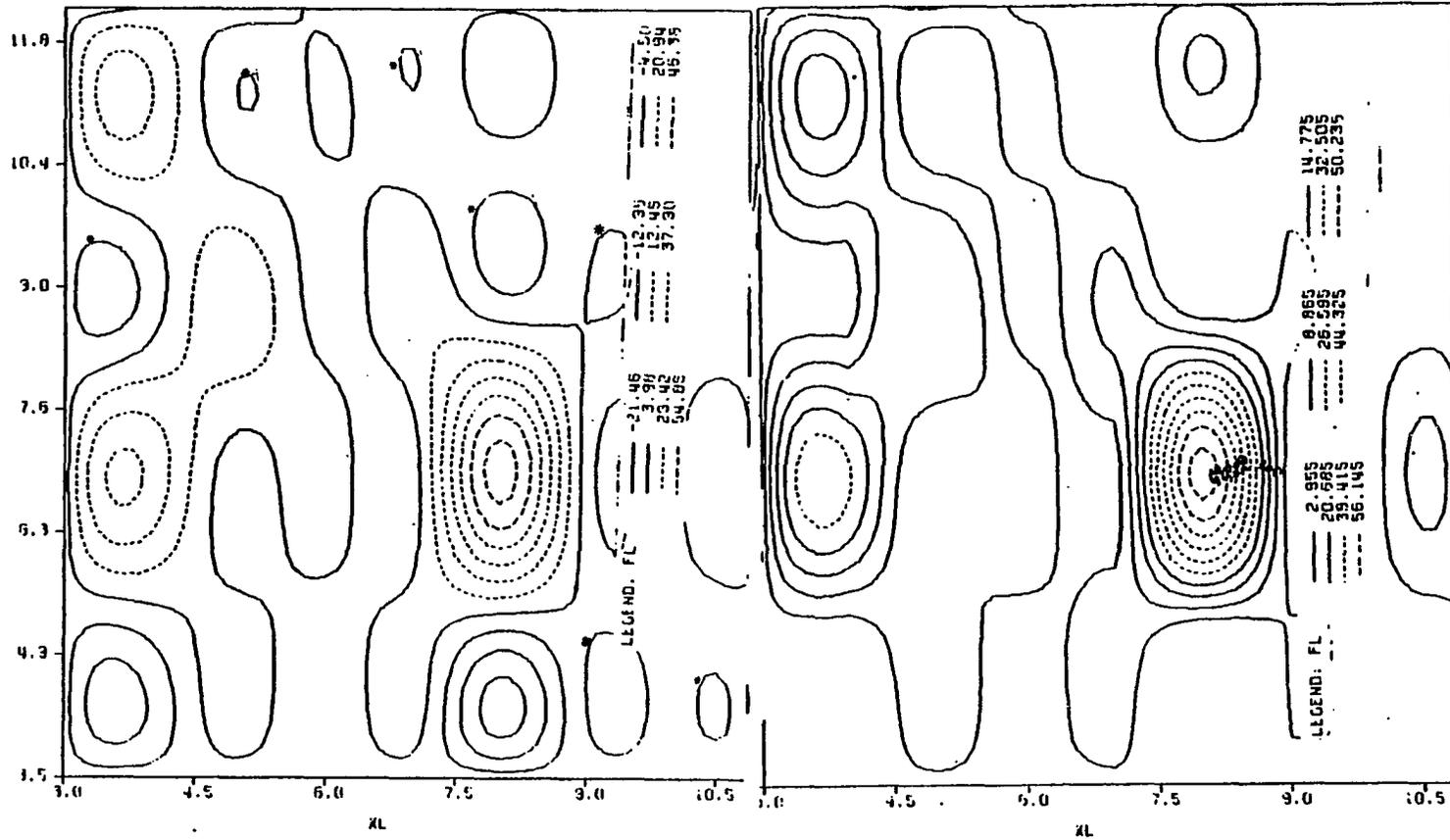
that the Hermite interpolation method exhibits some undersir-
able minima contour values which are absent in the contours
produced by the method of this study. The exhibition of
minima contour values (indicated by '*') is often referred .
to as undulations. One big advantage of the model of this
study over the Hermite interpolator is the elimination of
excessive undulations often present in the latter. It can
be easily said that most polynomial approximations exhibit
excessive undulations which are caused by interpolated values
that are much smaller than the lowest available data values.

A test of how this model will behave in the presence
of real reservoir parameter was carried out using data given
in table 5.2. The data was obtained from Sitio Grand Field[23]
which is producing from limestone of Cretaceous age. It is
located in southeast Mexico. The results are shown in
figure 5.3. Figure 5.3 is a contour map showing the flow
capacity values at different positions of the field. This
sort of distribution is desirable especially for a researcher
who may need some flow capacity information at different
points in the field. A three dimensional plot is presented
in figure 5.4. The three dimensional plot shows zones of the
field with peaks and pits. These phenomena elicit informa-
tion as to those areas of the field or reservoir where high
or low values of flow potential are expected.

Further tests were carried out using data from table 5.3.
The data is a waveform studies information given in ref. (4).

Fig. 5.3: Flow-capacity distribution in the Sitio Grand
field.

Table 5.3: Wave form study data obtained from (≉).

x	Y								
	0.00	5.00	10.00	15.00	20.00	25.00	30.00	35.00	40.00
0.	58.20	61.50	47.90	62.30	34.60	45.50	38.20	41.20	41.70
5.	37.20	40.00	0.00	41.30	14.10	24.50	17.30	20.20	20.80
10.	22.40	22.50	14.60	22.50	4.70	7.20	1.60	2.10	2.10
15.	21.80	20.50	12.80	17.60	5.80	7.60	0.80	0.60	0.60
20.	16.80	14.40	8.10	6.90	6.20	0.60	0.10	0.00	0.00
25.	12.00	8.00	5.30	2.90	0.60	0.00	0.00	0.00	0.00
30.	7.40	4.80	1.40	0.10	0.30	0.00	0.00	0.00	0.00
35.	3.20	0.70	0.00	0.00	0.00	0.00	0.00	0.00	0.00
40.	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
45.	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
50.	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Fig. 5.4:   3-D plot of Flow-Capacity distribution--Sitio
Grand field

Here again, high peaks indicating high wave conditions were
observed. Similarly, low pits indicating low wave condi-
tions were observed (figure 5.5). Since wave condition may
affect the way debris for off-shore hydrocarbon accumulation
are distributed in the ocean the approach of this study can
be used to influence the future of various off-shore prospect-
ing.

In an attempt to see what happens to the ability of
this model to approximate an original surface and identify
parameter 'high points', varying percentages of known error
were introduced into the original data constituting the known
surface. Figure 5.6 is the original data as given in table
5.3. Figure 5.7 is the result obtained using the model of
this study after replacing 5 percent of the original data
values by zeros. Zero values were purposely chosen in order
to amplify the results in each situation. Figure 5.7 (A) shows
smooth contours of the resulting surface. A comparison of
the contours of figure 5.7 (A) and those of the original data
shows no marked differences, apart from the recognition that
the former are smoother contours.

Figure 5.7 (B) is a three dimensional projection of the
data. The three dimensional plot of figure 5.7 (B) carefully
identifies areas having very high and very low parameter
values which correspond to those indicated by the contours
of figure 5.7 (A). The importance of the three dimensional
plot is to afford an easy recognition of anomalous portions
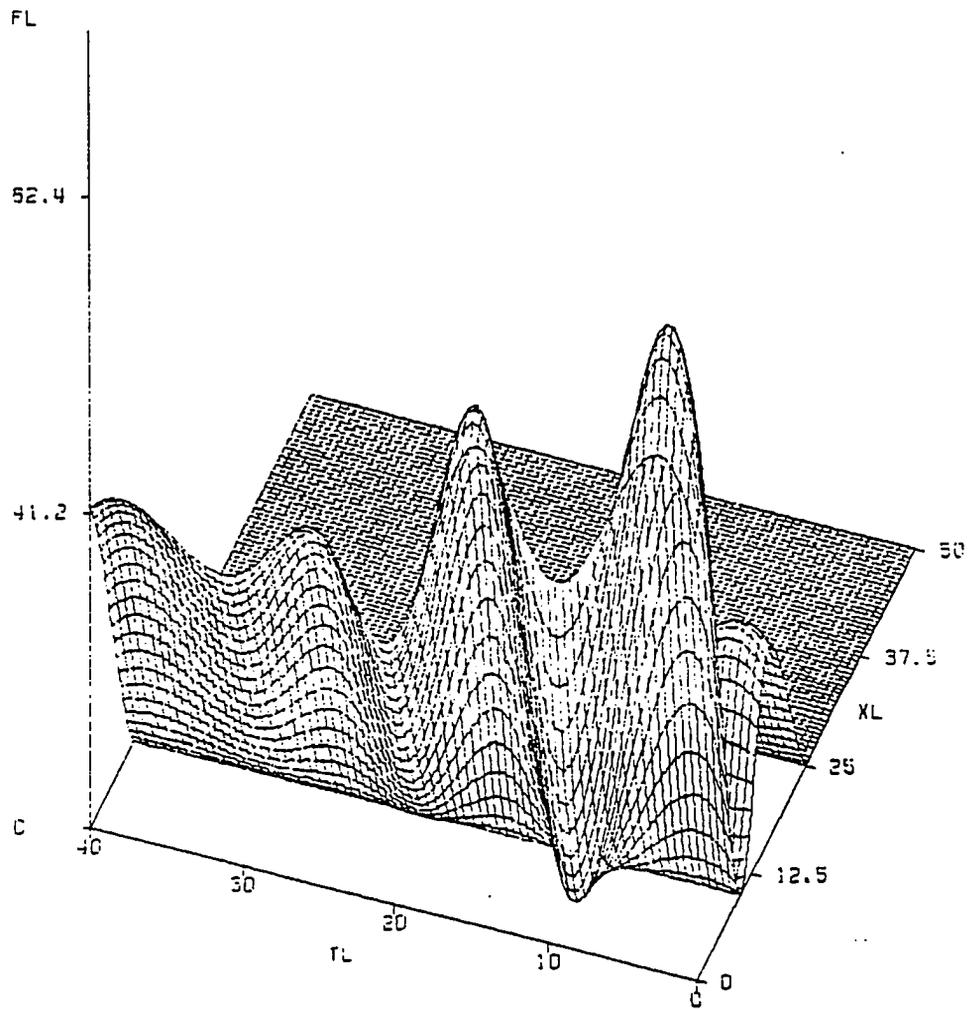
Fig. 5.5: 3-Dim. plot of waveform distribution

Fig. 5.6: Original surface
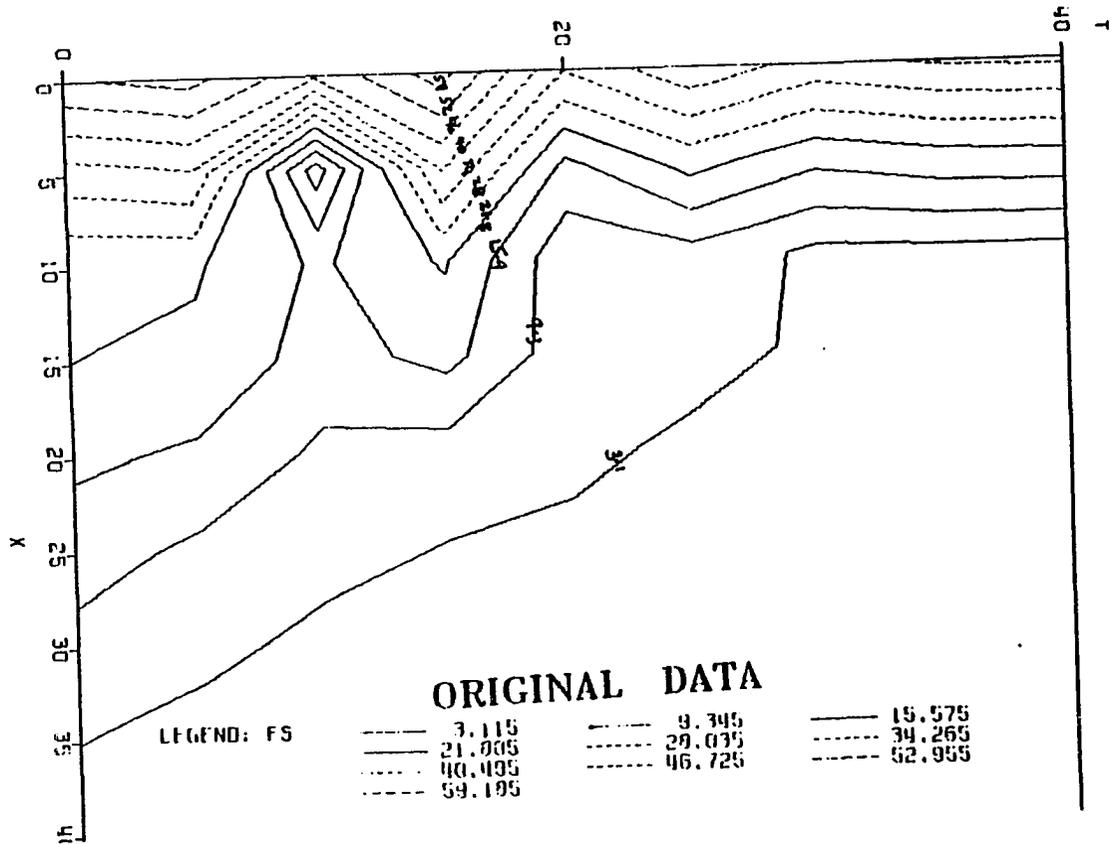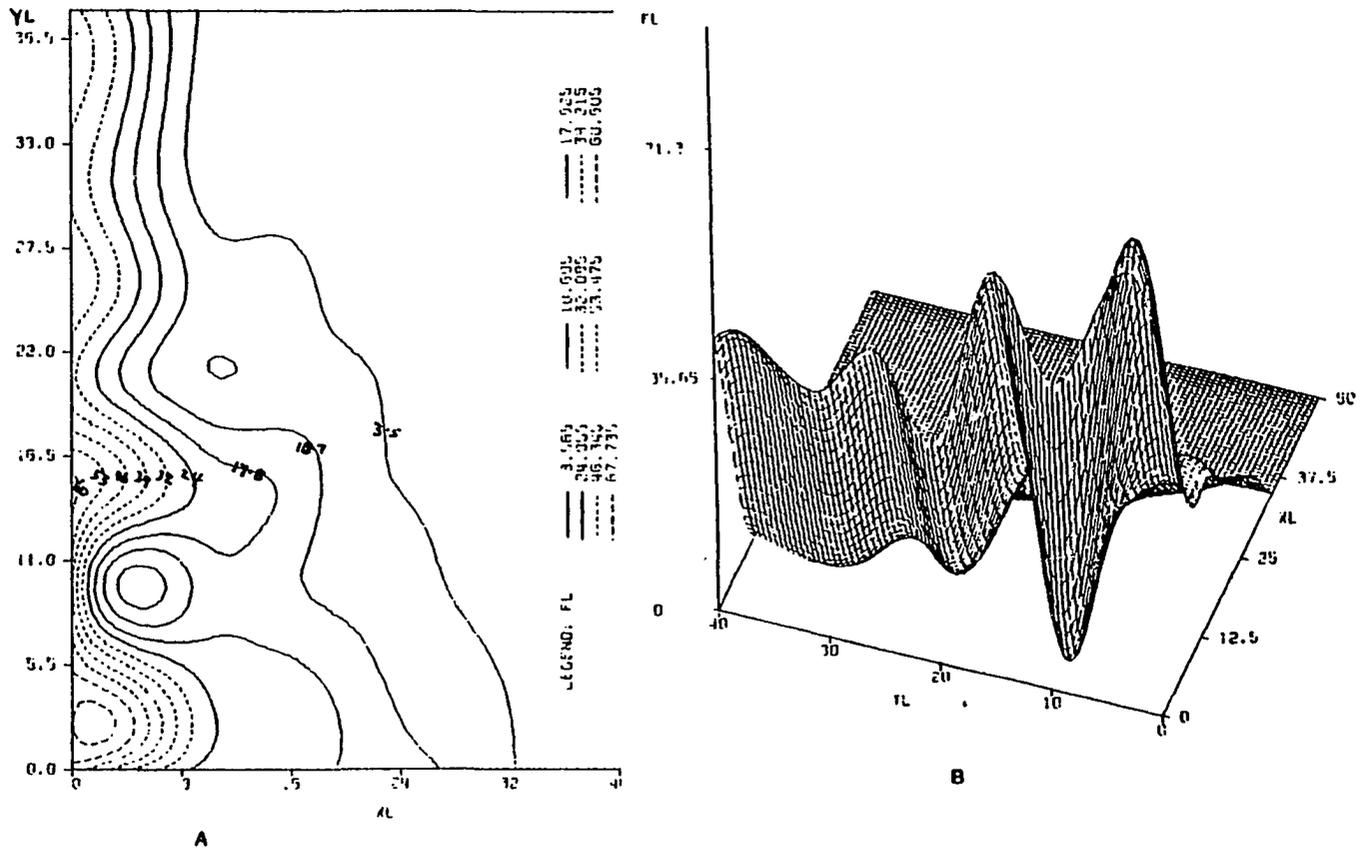
Fig. 5.7: Surface obtained by replacing 5% of data values
with zeros (extreme cases).

of a field or reservoir where special attention may be necessary.

Figure 5.8 shows the result of contaminating 10 percent of the original data. This was done as before by replacing 10 percent of the original data with zeros. Comparison of the produced contours with the contours obtained using the original data does not easily reveal any obvious differences. The resulting contours and the corresponding three dimensional display are very closely similar to those of figure 5.7. This means that the model of this study can closely approximate the original data distribution even when 10 percent of the data information are faulty.

A further test was carried out using 80 percent of the correct data information and 20 percent data contamination. The result of this operation is shown in figure 5.9. A comparison of this result with that of the original data shows some little differences. However the actual positions of the peaks and 'pits' were not basically changed. As the contamination was increased beyond 20 percent, marked differences were observed. Figure 5.10 is the result of introducing a 30 percent contamination into the original data. Two distinct contour maps are produced (figure 5.10A). The three dimensional display of figure 5.10B shows peaks that are much lower than the corresponding peaks in the previous plots, and also the pits are diminished.
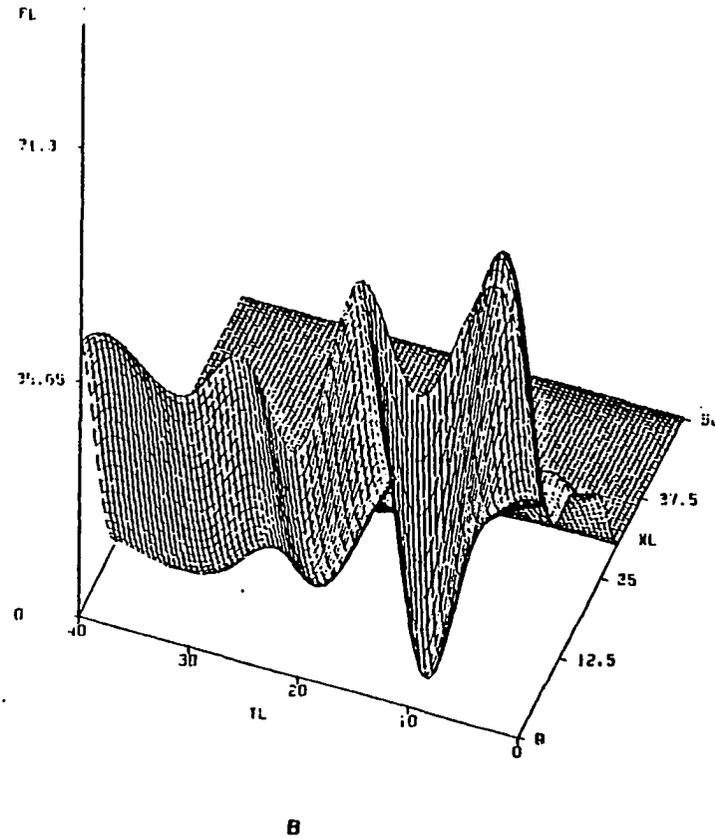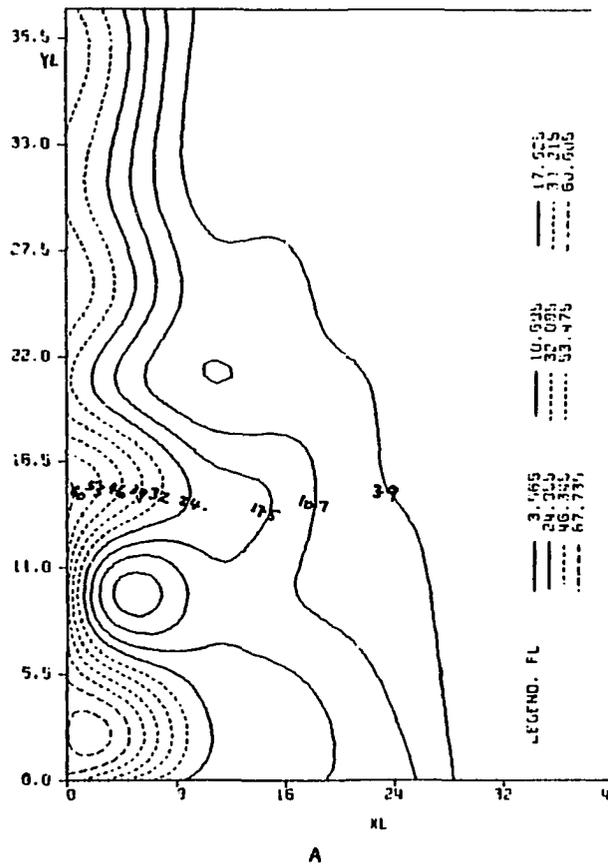
Fig. 5.8: Surface obtained by replacing 10% of data values with zeros (extreme cases).
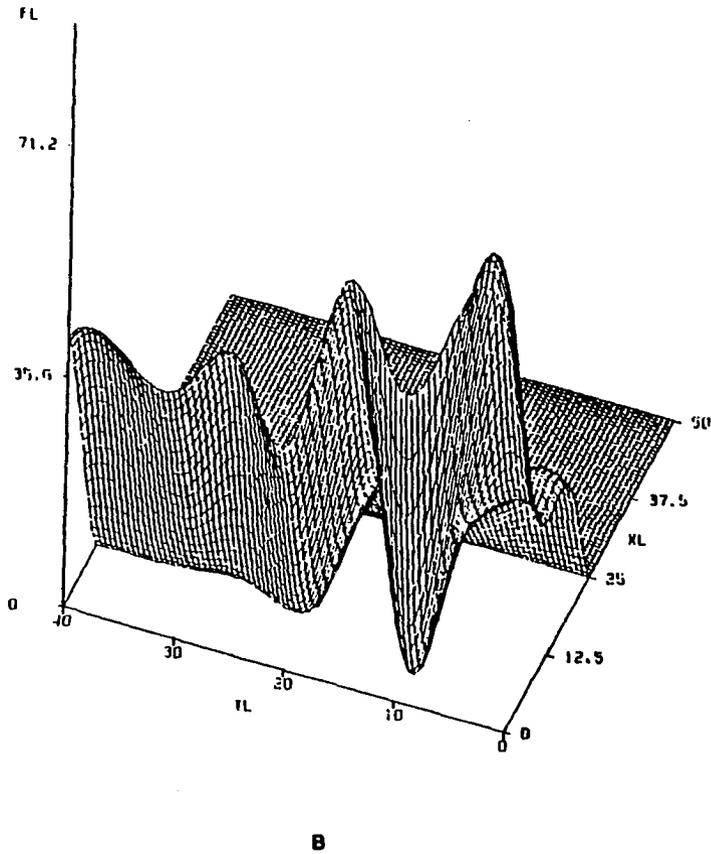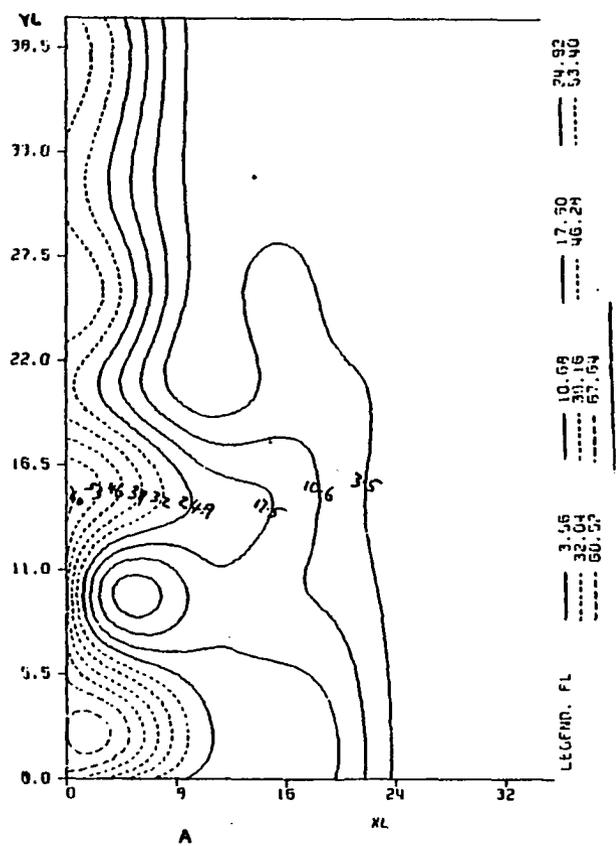
Fig. 5.9: Surface obtained by replacing 20% of data values
with zeros (extreme cases).
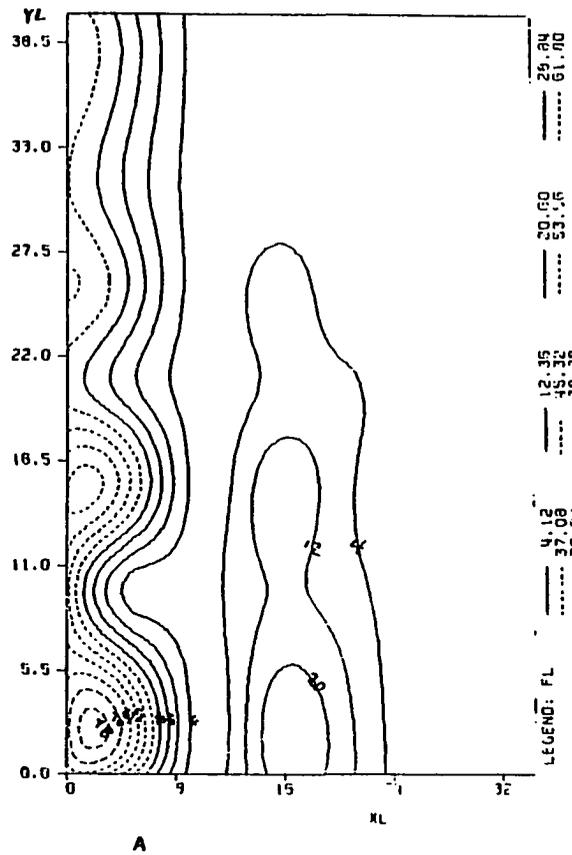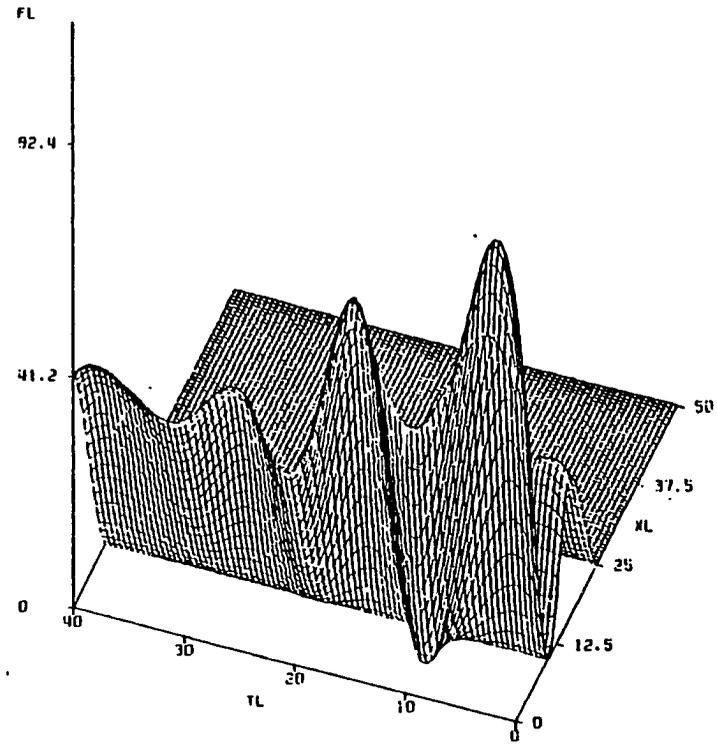
Fig. 5.10: Surface obtained by replacing 30% of data points
with zeros (extreme cases).

CHAPTER VI

DISCUSSION AND PRACTICAL PETROLEUM ENGINEERING

APPLICATION OF THE PROPOSED MODEL

One of the aims of this study is to develop a practical model that will aid petroleum engineers in the secondary oil recovery operations.

In the application scheme for this model, contour maps [29] have been used extensively to obtain detailed pictorial representation of a function of two variables whose values are known at discrete points. Detailed discussion on contour mapping has been covered in Chapter II, section 1. In reservoir engineering studies, imaginary surfaces comprising of a continuum of all possible reservoir parameter values can be mapped using the proposed model to display the distribution of variables of interest in the reservoir. The distribution of reservoir parameters when there is limited data information is often obscure and difficult to obtain in the absence of an appropriate approximation algorithm. The model of this study has been considered appropriate for use in the study of reservoir parameter distributions.

Similarly a three-dimensional model due to Watkins (31) and modified under the SAS (24) has been employed to obtain a conceptual surface representing appropriate reservoir

parameter. In reservoir engineering, the determination and variation of conceptual surfaces can be directly related to the variation of some corresponding non-spatial functions from point to point. This variation can be regarded as the flow on the conceptual surface, and thus, the generation of this conceptual surface can replace the physical system. An understanding of how the peaks, 'pits' and 'beds' associated with these projections are related to the physical system may relate to the reservoir engineer a. desired insight into better ways to optimize his project.

Possibly the area of greatest uncertainty in designing a waterflood project is the quantitative overall knowledge of the variation of rock properties within a reservoir. Though other reservoir parameters such as porosity and connate water saturation vary both areally and vertically within a reservoir, the reservoir parameter whose variation is the most important in influencing waterflood performance is permeability. Permeability is a measure of fluid transmissibility of a rock. Several sources of obtaining good information on permeability are available:

1. Direct measurements of permeability on cores removed from wells.

2. Formation tests during drilling and production.

3. Transmissibilities obtainable from carefully run injection profiles.

4. Inferential information from well Logs,

6.1 Procedure for estimation of flood performance using the results of this model:

a) From the flow capacity distribution shown in figures 6.1 and 6.2, obtain flow capacity values at fixed contour intervals (here a contour interval of 6 md-ft is convenient).

b) Using a planimeter method, obtain floodable reservoir volume for the successive contour levels selected.

c) Calculate water cut and oil recovery by using flow-capacity values (cumulative fraction of the total capacity), and a functional equation (simplifying assumptions were made where necessary).

Derivation of functional equation:

Simplifying assumptions made in this derivation are:

1. flow rate is proportional to flow capacity.

2. each flow contour level has same values of ϕ, S_{or}, k_{rw}, k_{ro}.

3. Isothermal conditions exist throughout the reservoir.

The procedure of this mathematical analysis is similar to that reported by Stiles (28). The major difference is this model is using a volume weighted value of flow capacity values to calculate the water cut, while Stiles used randomly varying permeability values to obtain his water cut.
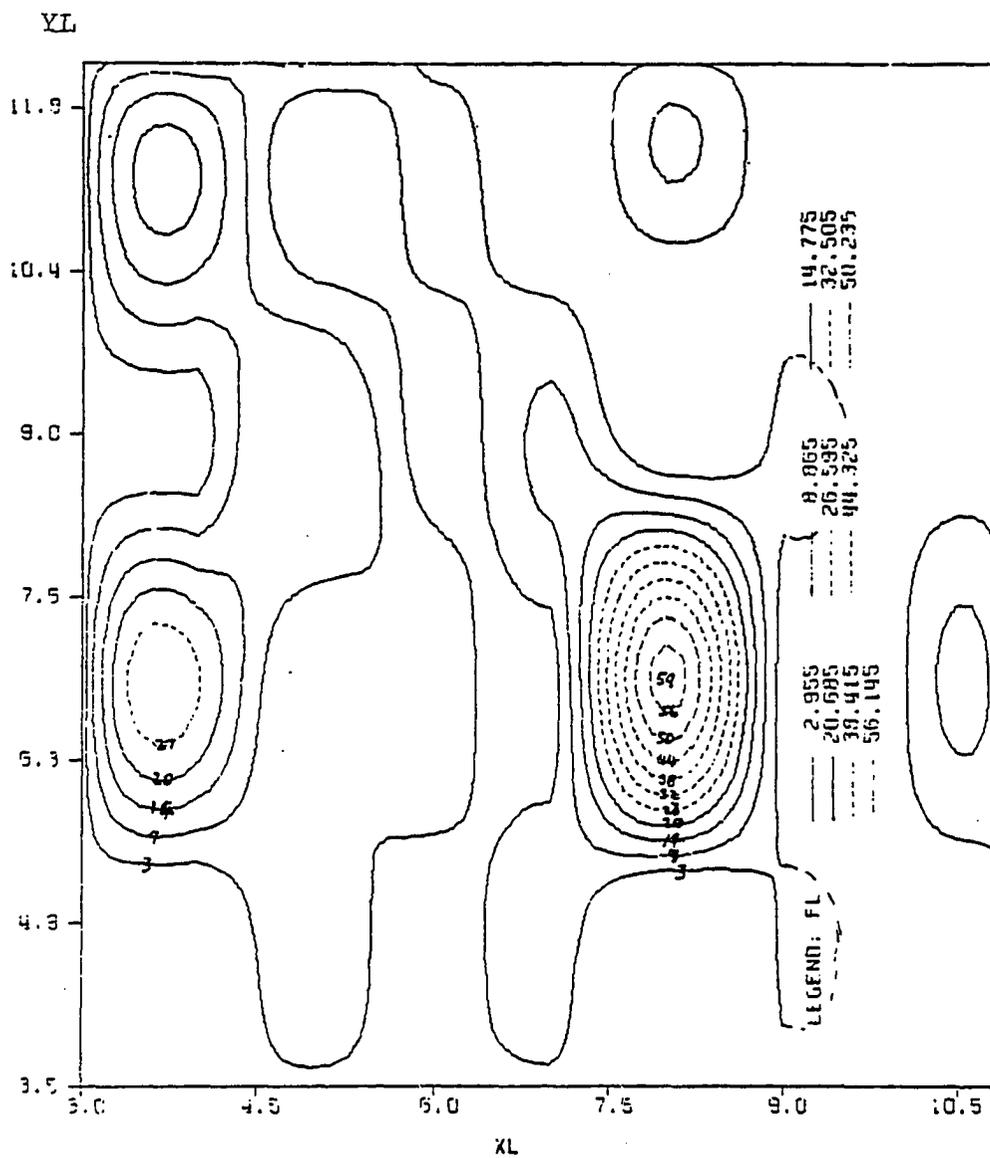
Fig. 6.1 : Flow-capacity distribution in the Sitio Grand
field (data of ref. 23)
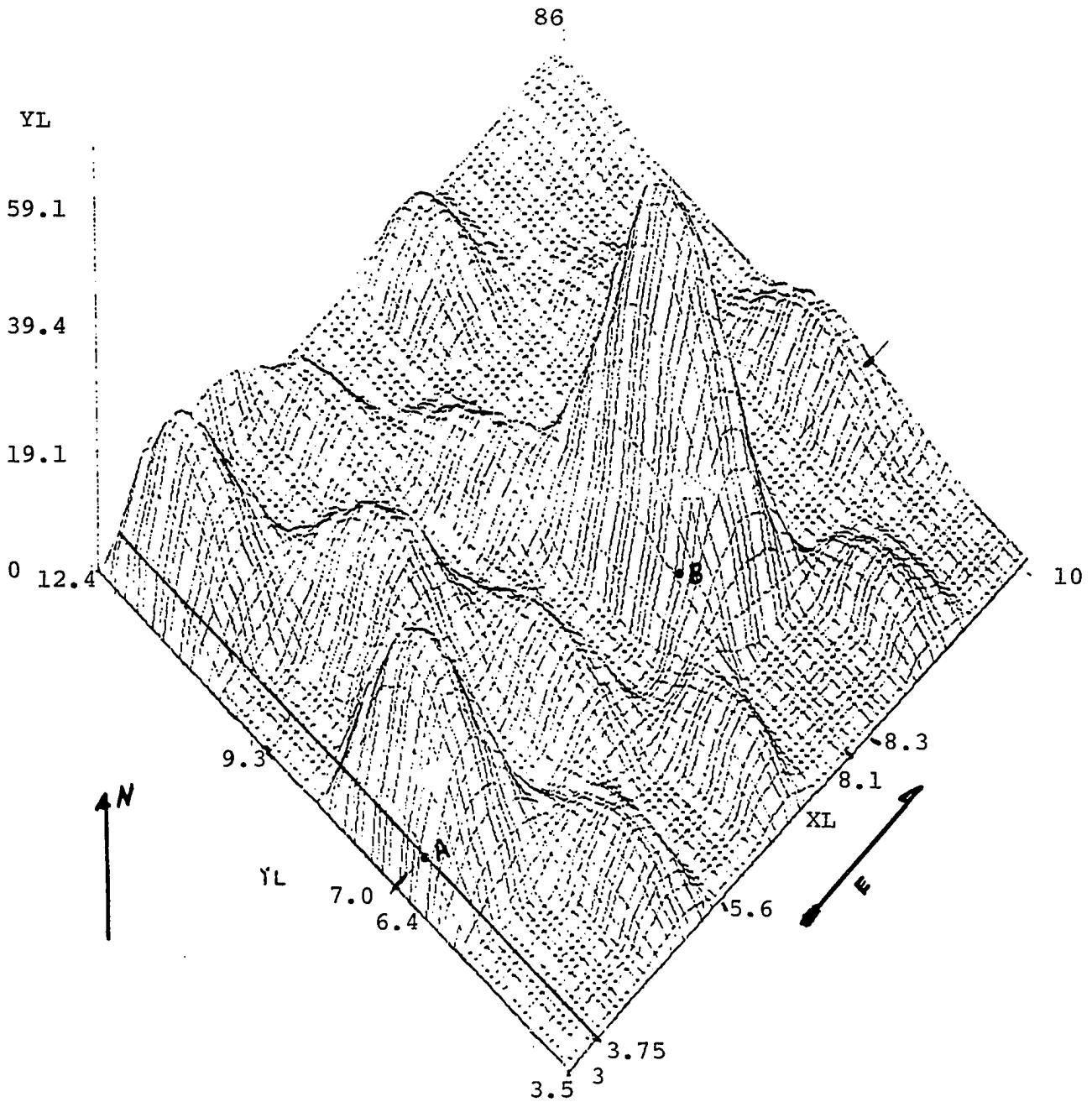
Fig. 6.2: 3-Dimensional plot of flow-capacity distribution in Sitio Grand field (data of ref. 23)

Define water flow rate

$$q_w = \rho_i^* \left(\frac{k_{rw}}{\mu\omega}\right)$$ 6.1

$$q_o = (i-\rho_i)\left(\frac{k_{ro}}{\mu o}\right)\left(\frac{1}{\beta o}\right)$$ 6.2

water cut:

$$f_w = \frac{q_w}{q_w+q_o} = \frac{\rho_i \left(\frac{k_{rw}}{\mu\omega}\right)}{\rho_i \left(\frac{k_{rw}}{\mu\omega}\right) + (1-\rho_i)\left(\frac{k_{ro}}{\mu_o}\right)\frac{1}{\beta_o}}$$ 6.3

$$f_w = \left[1 + \lambda (1 - \rho_i)\right]^{-1}$$ 6.4

where:

$$\lambda = \frac{k_{ro}}{k_{rw}} \cdot \frac{\mu_\omega}{\mu_o} \cdot \frac{1}{\beta_o}$$

$$^*\rho_i = (\Sigma\rho_i V_i)/\Sigma V_i$$

ρ_i = flow capacity (weighted value) at point, i

$\mu_o \mu_\omega$ = oil and water viscosities respectively

k_{rw}, k_{ro} = relative permeabilities to water and
oil respectively.

V_i = floodable volume for flow capacity level, i.

Similarly

$$f_O = \frac{(1-\rho_i)\ (\frac{k_{ro}}{\mu_o})\ (\frac{1}{\beta_o})}{\rho_i\ (\frac{k_{rw}}{\mu\omega})+ (1-\rho_i)\ (\frac{k_{ro}}{\mu_o})\ \frac{1}{\beta_o}} \qquad 6.5$$

Sample Calculation: Contoured data from figure 6.1 is being used to show how the model of this study can aid the reservoir engineer in the estimation of reservoir performance prior to the start of water flooding operations. The necessary data needed in this analysis are given as follows:

k_{rw} = 0.3, k_{ro} = 0.7, ϕ = 0.2, S_{or} = 0.25, S_w = 0.3 and β_O = 1.1

By using the flow capacity values given in table 6.1, and the flow equation of 6.4 and 6.5, · obtain water cut, and recovery values shown in table 6.1. Unit recovery for this reservoir has been estimated to be 458.4 bbls/acre-ft (see equation 6.6).

Figure 6.4 shows the plot of oil recovery versus water cut for the above field. The characteristics of the curves are similar to those reported by Stiles (28) and Johnson (16). A plot of flow-capacity versus oil recovery is shown in figure 6.5. A plot of flow capacity versus water cut is presented in figure 6.6. It is interesting to note that the three plots depicted in the above figures have similar characteristics.

Where V_i = floodable volume for flow capacity level, i.

Table 6.1. Calculation of Reservoir Performance

flow cap. cap. (cum.)	(fraction of total)	water cut (fract.)	Recovery (frac. of total)	Recovery fraction (cum)
--	0.0	0.00	0.21	0.21
59	0.17	0.33	0.143	0.35
115.1	0.33	0.37	0.135	0.48
165.73	0.47	0.43	0.121	0.60
209.66	0.59	0.49	0.109	0.71
248.07	0.7	0.55	0.096	0.81
280.57	0.79	0.65	0.075	0.88
307.87	0.87	0.75	0.05	0.93
327.87	0.93	0.84	0.03	0.96
342.67	0.97	0.93	0.015	0.97
351.57	0.99	0.98	0.0043	0.98
354.57	1.00	1.00	0.0	1.00

$$\text{Unit Recovery} = 7758 \left(\frac{S_{oi} - R_{so} \beta_o}{\beta_o} \right) \phi \qquad 6.6$$

$$= (7758) \left(\frac{0.6 - (0.25)(1.1)}{1.1} \right)(0.2)$$

$$= 458.4 \quad \text{bbls/acre-ft}$$

DATA: $K_{rw} = 0.3$ $\qquad S_w = 0.3$

$\qquad\qquad K_{ro} = 0.7$ $\qquad R_{so} = 0.25$

$\qquad\qquad \phi = 0.2$ $\qquad \beta_o = 1.1$

$\qquad\qquad$ Sweep eff = 0.95
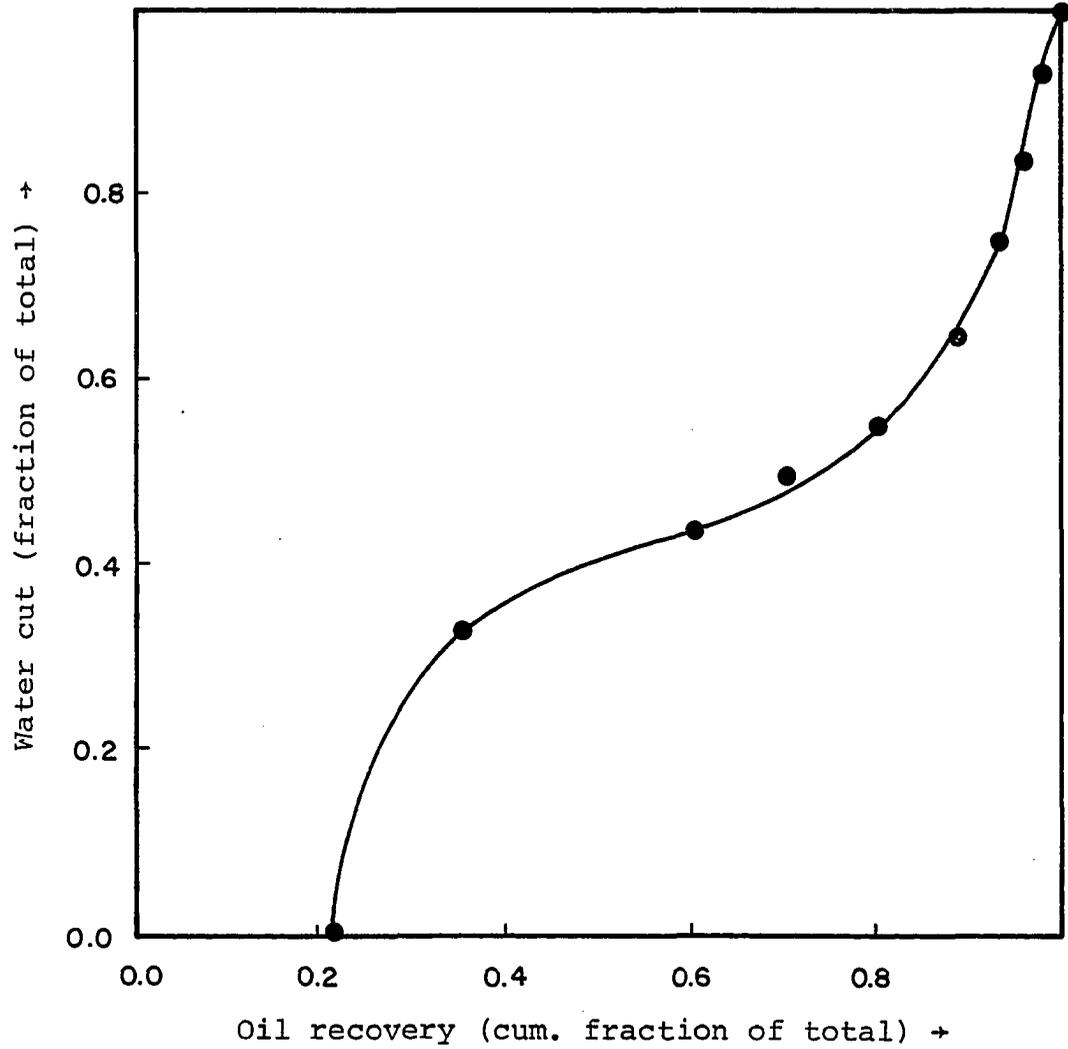
Figure 6.4   Plot of water cut versus oil recovery.
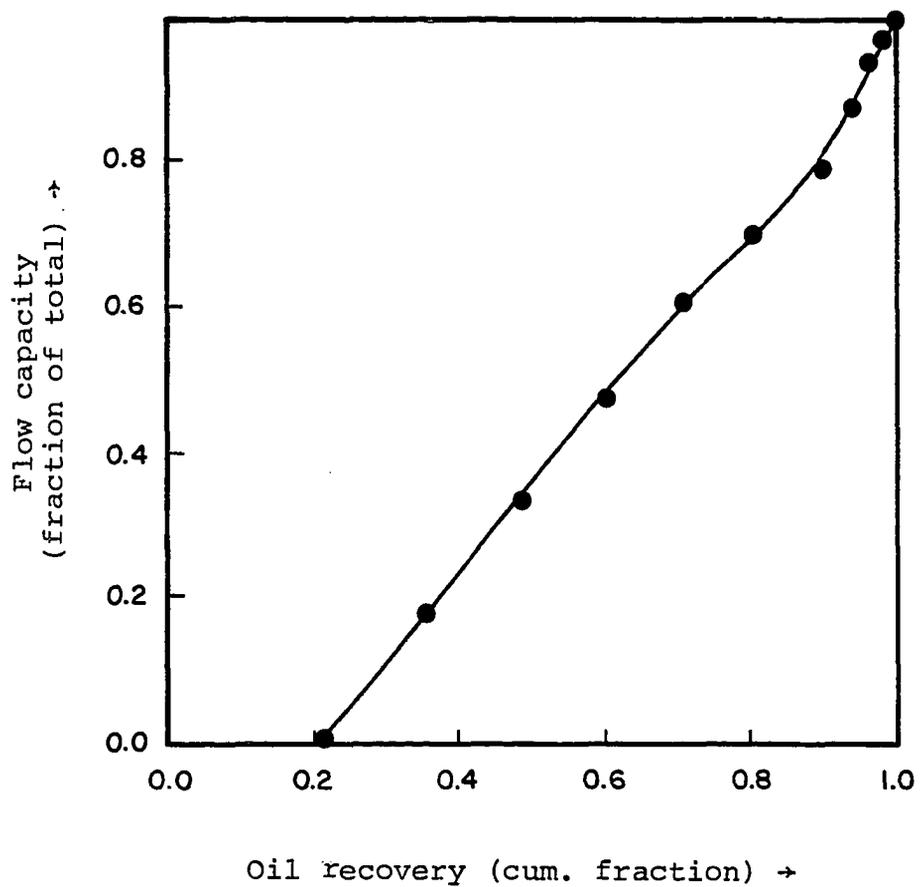
Figure 6.5 Plot of flow capacity
versus oil recovery.
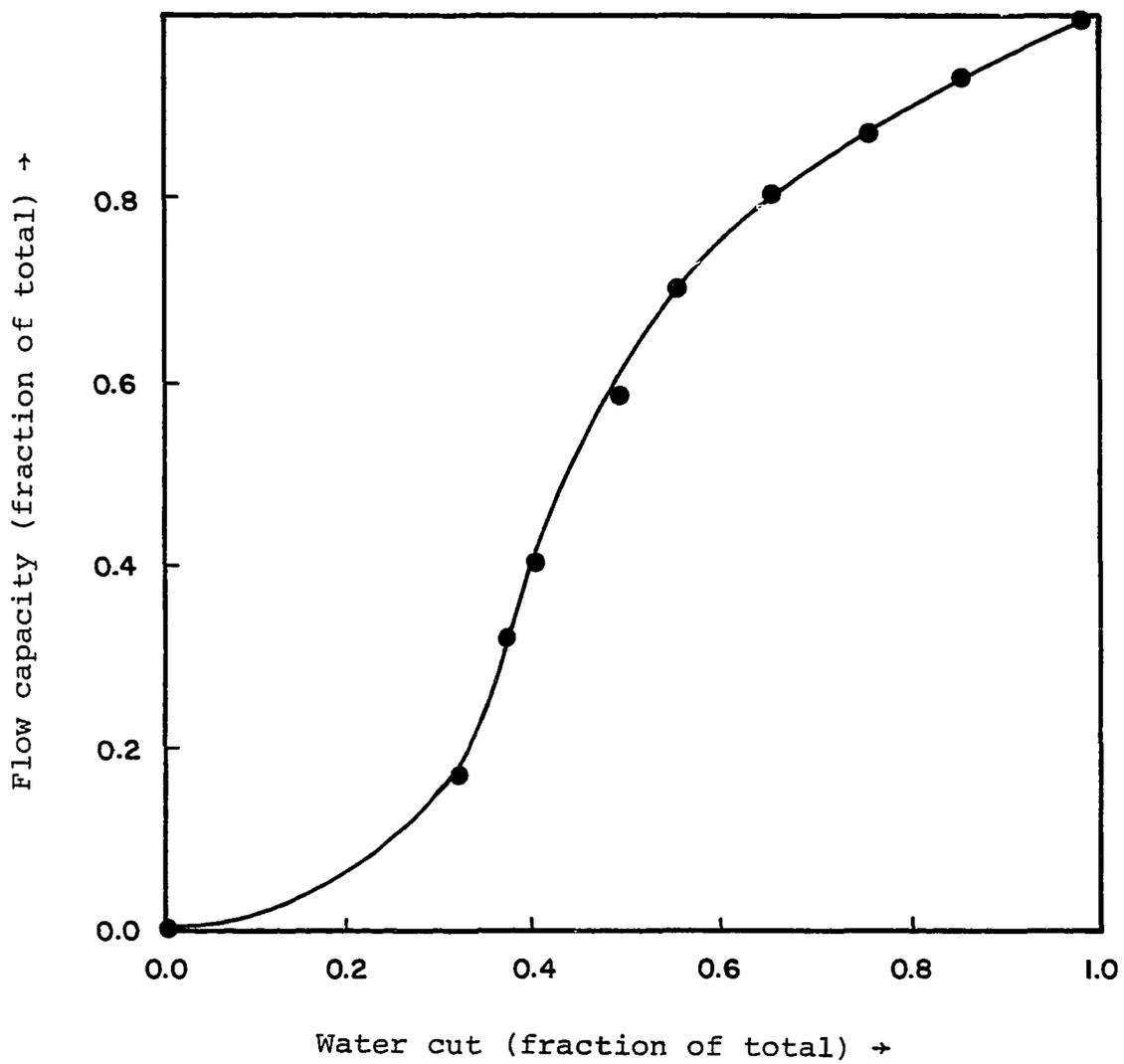
Figure 6.6 Plot of flow capacity versus water cut.

6.2 Determination of Reservoir Productive Volume
(Planimeter Technique

A planimeter (model 80) was used to obtain the bulk

volume (V_b) on successive contour levels of the flow capacity

distribution. The results of the above measurements are

shown in table 6.2 Cumulative values of flow volume (frac-

tion of total) are plotted against cumulative values of flow

capacity. Figure 6.7, the result of this plot, shows that a

small increase in the floodable volume produces a sharp in-

crease in the flow capacity. Similar observation was reported

by Johnson (16) but he used 'ϕh' product instead of actual

reservoir volume.

Table 6.2. Determination of Reservoir Productive Volume

Prod. Cap. Md-ft	Planim- eter units	*Planimeter area (in²)	Area (Aₙ) (acres)	Contour Interval (md-ft)	ΔV** acre (ft-md)
8	1546	16.3	374	6	
14	76	0.8	18.4	6	1178
20	108	1.14	26.1	6	134
26	48	0.51	11.6	6	113
32	15	0.16	3.6	6	46
38	19	0.20	4.6	6	25
44	5	0.052	1.2	6	18
50	8	0.08	1.9	6	9.3
56	20	0.21	4.8	6	20
59	0	0	0	3	15
					1558

* map scale: 1 inch = 1000 ft, one inch square = 22.96 acres.

NOTE: Calibration constant for the planimeter

$$= 95 \text{ units} / (\text{inch})^2 \text{ or } (\text{inch})^2 / 95 \text{ unit}$$

$$= 0.242 \text{ acres/unit}$$

** use the trapezoidal formula

$$\Delta V = \frac{kh}{2} (A_n + A_{n+1}) \qquad 6.7$$
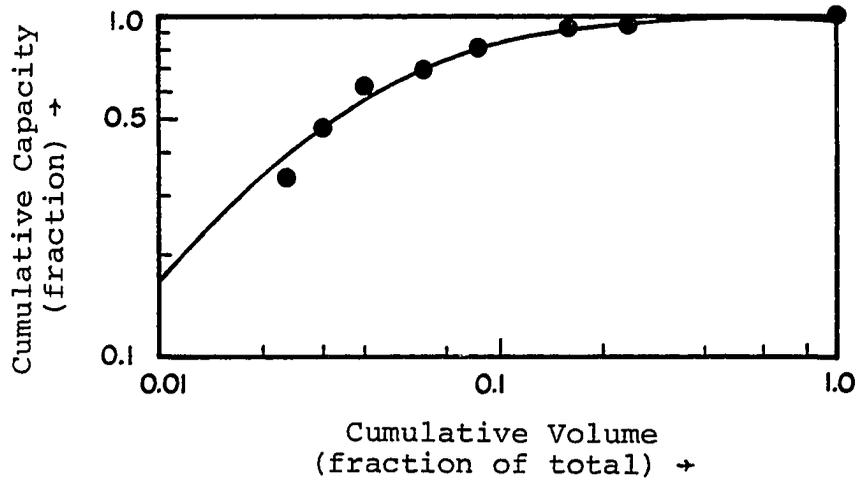
Figure 6.7   Plot of cumulative capacity
versus cumulative volume.

Table 6.3. Calculation of Reservoir Volume
(Cum. fraction of total)

Flow Cap. (md)	Cum. Capacity (fraction)	Area (acres)	Vol (Acre ft)	Vol (fraction of total)	Cum vol (fract)
59	0.17	0	15	0.010	0.01
56	0.33	4.8	20	0.013	0.023
50	0.47	1.9	9.3	0.006	0.029
44	0.59	1.2	18.0	0.012	0.041
38	0.7	4.6	25	0.016	0.057
32	0.79	3.6	46	0.03	0.086
26	0.87	11.6	113	0.0725	0.158
20	0.91	26.1	134	0.086	0.244
14	0.97	18.4	1178	.756	1.000
8	0.99	374	0	0	1.000

6.3 Water Injection Plan: Waterflood injection scheme for
a reservoir such as the Sitio Grand field could be planned
as shown in table 6.4. The schedule is such that more in-
jection water is made available to areas of low flow capa-
city. This procedure will reduce problems of channelling
due to differences in permeability or flow capacity. The
underlying assumptions are (1) water injection is a func-
tion of flow capacity, floodable volume, and water cut.

(2) Water injection efficiency is 60%.

The water injection equation is given by :

$$I_{in_j}^i = \frac{1}{0.6} \left[\frac{\rho_i V_i W_i}{\frac{1}{n}\sum_{i=1}^{n} \rho_i V_i} \right] \qquad 6.8$$

A plot of water injection versus watercut is shown
in figure 6.8.
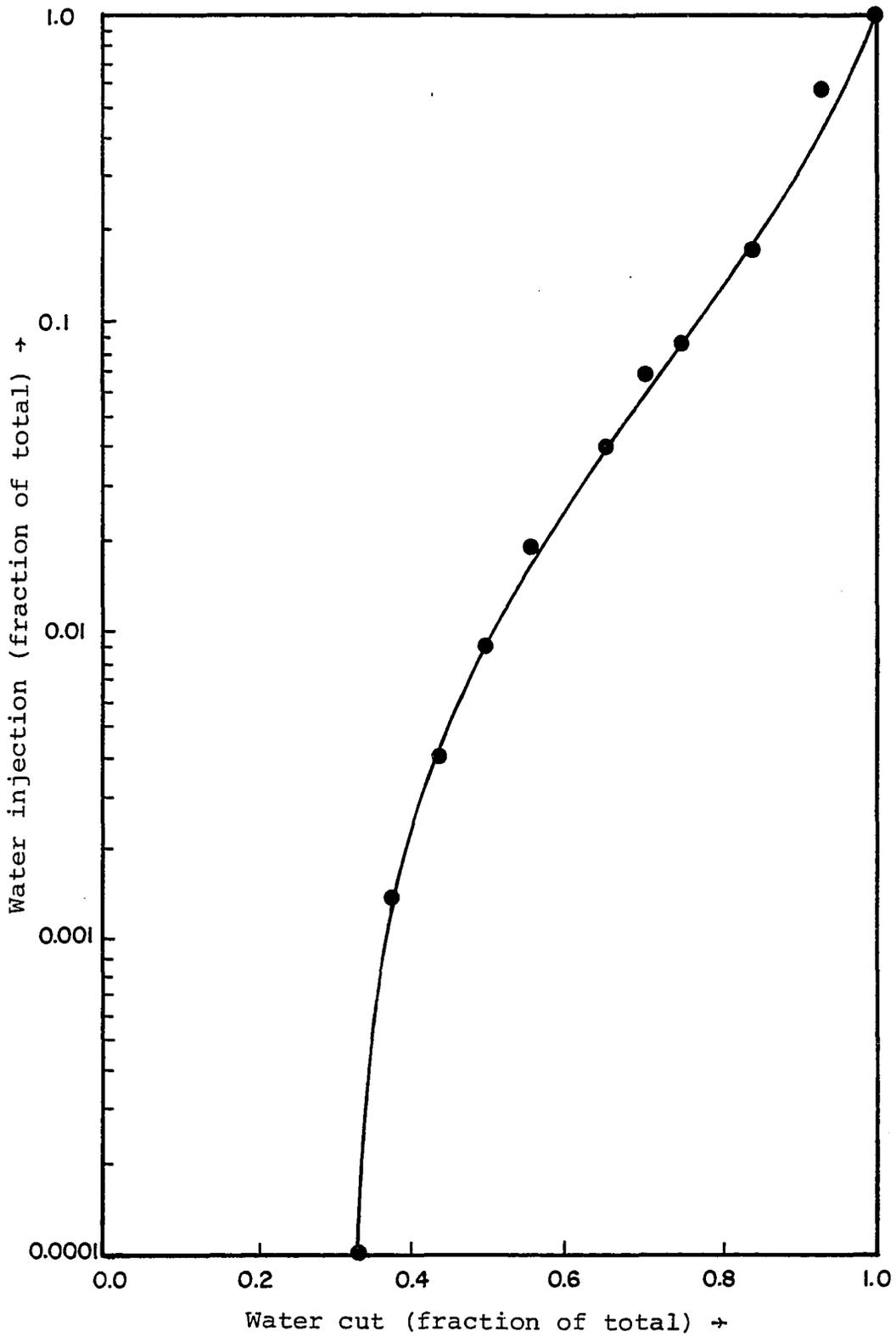
Figure 6.8 Plot of water injection versus water cut.

Table 6.4. Water Injection Plan

ρ (md)	water cut	floodable volume (fraction)	water injection	Cum Water inj (fraction)
0.17	0.33	0.01	0.001	0.0001
0.33	0.37	0.023	0.01	0.0014
0.47	0.43	0.029	0.02	0.004
0.59	0.49	0.041	0.041	0.009
0.7	0.55	0.057	0.075	0.019
0.79	0.65	0.086	0.151	0.039
0.87	0.75	0.158	0.353	0.085
0.92	0.84	0.244	0.65	0.169
0.96	0.93	0.999	3.09	0.57
0.99	0.98	1.000	3.323	1.00

6.4 Limitations of this model

1. Since reservoir performance may be influenced by some
   other unevaluated parameters, this model's accuracy is
   limited.

2. Many simplifying assumptions will tend to reduce the
   quality of the model.

3. Unlike the selective plugging technique, the interpolation
   part of this study does not redistribute injection water
   profile. It only indicates its existence. However, the
   capability of this model to indicate the flow capacity
   profile, and hence the injection water profile, will be
   profitably employed in selective plugging procedure by
   taking advantage of those locations of the reservoir
   that show relatively high flow capacity distributions.
   Thus we can economically inject our costly chemicals only
   in those zones that show abnormally high flow capacity
   values. In that way we can avoid wasting our 'plugging'
   chemicals on the zones where they are not needed.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

1. This model offers an easy and quick means of estimating
    reservoir performance prior to the start of waterflood-
    ing operations by providing flow capacity distribution
    of the reservoir under study.

2. It offers an easy and quick means of estimating actual
    reservoir floodable volume.

3. The model is useful in the estimation and planning of
    waterflood injection schedule.

4. It can be used in the description of reservoir parameter
    distribution.

5. The model can be used to represent data as a continuous
    surface.

6. The model can be used to interpolate data with a con-
    termination level of less than 20%.

RECOMMENDATIONS

1. More reservoir parameters should be incorporated into the program so that better reservoir description could be obtained.

2. Comparison of the performance results of this model should be made with the actual reservoir performance in order to evaluate the reliability of this model.

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APPENDICES

APPENDIX A

FLOW CHART AND NOMENCLATURE FOR THE

SPLINT PREDICTION ALGORITHM

FLOW- CHART

MAIN

START

Dimension

Data

Initialize
Variables

Call Subroutine
Splint

Store Output
in Disc

Plotting Routines
G Contour, G3D

Write Out
Title

END
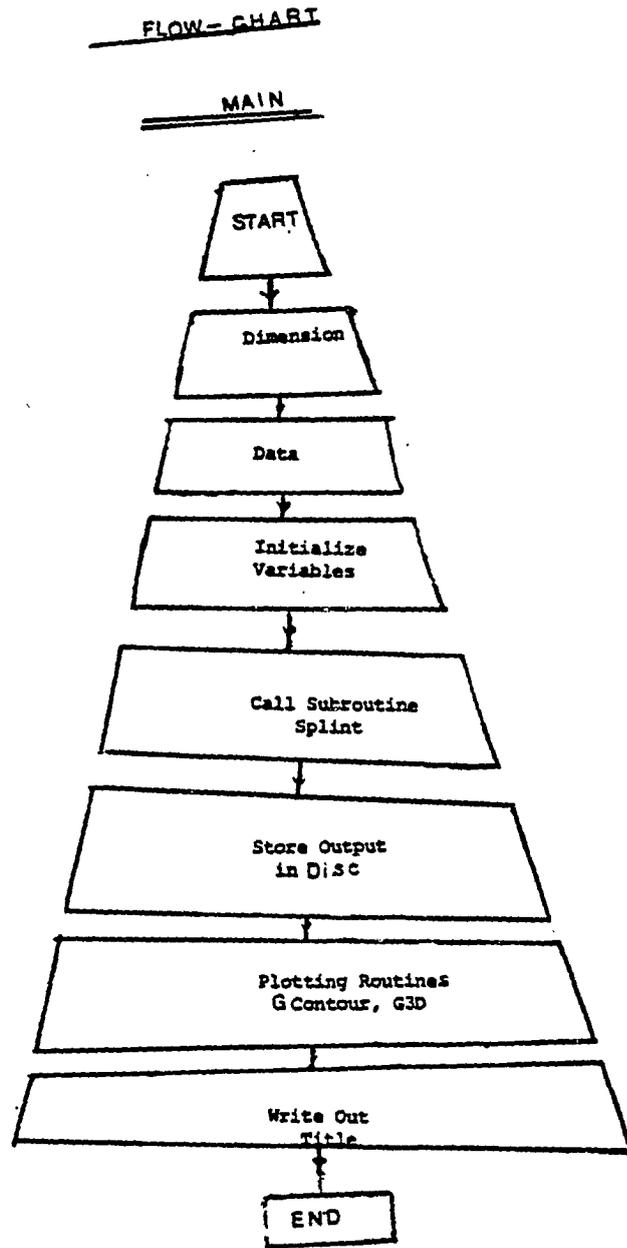
Figure A-1. Flow-chart for this model—Main.

SUB PROGRAM
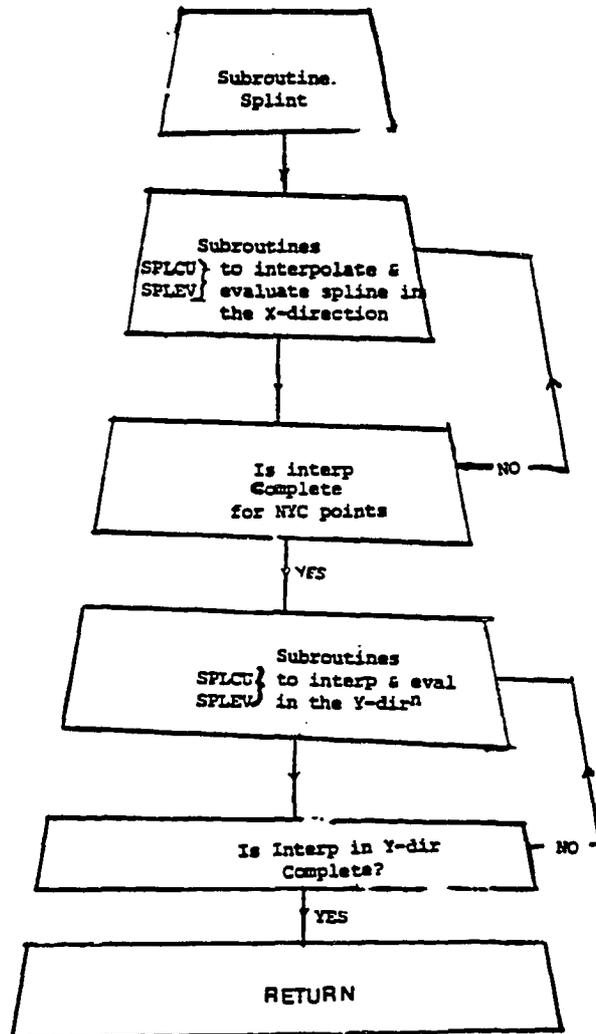
Figure A-2. Flow-chart for Subroutine Splint.

NOMENCLATURE FOR TERMS USED IN THE PROGRAM

NX   Total number of elements in the input vector x

NY   Number of elements in the input vector y

X   Input vector of stretch nx

       (x must be in ascending order)

Y   Input vector of length NY

       (y must be in ascending order)

FS   NX by NY matrix comprising of values at points

       specified by vectors X, and y.

IFD   Number of elements of FW in each row

NXL   Number of points along SXL where interpolation

       are needed

NYL   Number of points along SYL where interpolation

       are needed

SXL   Vector of length NXL--Input co-ord. points.

       (SXL must be order in ascending order)

SYL   Vector of length NYL--input

       (SYL must be ordered in ascending order)

SFL   Matrix NXL by NYL containing the interpolated

       values

SWK   A work vector of max. length the bigger of

       (NX-1)*3 and ((NY-1)*3 + NY))

IFR   Is an error parameter.

APPENDIX B

SAMPLE PROGRAM

Table A-1: Input Data for Sample Program

X	Y 0.00	5.00	10.00	15.00	20.00	25.00	30.00	35.00	40.00
0.	58.20	61.50	47.90	62.30	34.60	45.50	38.20	41.20	41.70
5.	37.20	40.00	0.00	41.30	14.10	24.50	17.20	20.20	20.80
10.	22.40	22.50	14.60	22.50	4.70	7.20	1.80	2.10	2.10
15.	21.80	20.50	12.80	17.60	5.80	7.60	0.80	0.60	0.60
20.	16.80	14.40	8.10	6.90	6.20	0.60	0.10	0.00	0.00
25.	12.00	8.00	5.30	2.90	0.00	0.00	0.00	0.00	0.00
30.	7.40	4.80	1.40	0.10	0.00	0.00	0.00	0.00	0.00
35.	3.20	0.70	0.00	0.00	0.00	0.00	0.00	0.00	0.00
40.	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
45.	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
50.	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

```
SXL    0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0 4.5 5.0 5.5 6.0 6.5 7.0
SXL    7.5 8.0 8.5 9.0 9.510.010.511.011.512.012.513.013.514.014.5
SXL   15.015.516.016.517.017.518.018.519.019.520.020.521.021.522.0
SXL   22.523.023.524.024.525.025.526.026.527.027.528.028.529.029.5
SXL   30.030.531.031.532.032.533.033.534.034.535.035.536.036.537.0
SXL   37.538.038.539.039.540.040.541.041.542.042.543.043.544.044.5
SXL   45.045.546.046.547.047.548.048.549.049.550.0
SYL    0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0 4.5 5.0 5.5 6.0 6.5 7.0
SYL    7.5 8.0 8.5 9.0 9.510.010.511.011.512.012.513.013.514.014.5
SYL   15.015.516.016.517.017.518.018.519.019.520.020.521.021.522.0
SYL   22.523.023.524.024.525.025.526.026.527.027.528.028.529.029.5
SYL   30.030.531.031.532.032.533.033.534.034.535.035.536.036.537.0
SYL   37.538.038.539.039.540.0
```

APPENDIX C

OUT PUT

SFL(SXL,SYL)

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58.2 61.8 64.5 66.3 67.3 67.6 67.2 66.4 65.1 63.4 61.5 59.4 57.3 55.1 53.1
51.2 49.6 48.4 47.7 47.5 47.9 49.0 50.7 52.7 54.9 57.2 59.3 61.0 62.2 62.7
62.3 60.9 58.7 55.8 52.4 48.9 45.3 41.8 38.8 36.3 34.6 33.8 33.9 34.7 36.0
37.6 39.4 41.3 43.0 44.5 45.5 46.0 45.9 45.5 44.7 43.7 42.5 41.3 40.1 39.0
38.2 37.7 37.4 37.4 37.6 37.9 38.4 39.0 39.7 40.5 41.2 41.9 42.6 43.1 43.6
43.9 43.9 43.8 43.4 42.7 41.7 56.3 61.6 65.5 68.1 69.4 69.7 69.0 67.6 65.5
62.8 59.8 56.5 53.1 49.7 46.4 43.4 40.8 39.8 37.4 36.9 37.3 38.8 41.1 43.9
47.2 50.6 53.8 56.7 59.0 60.4 60.8 59.9 57.9 55.6 51.6 47.8 43.9 40.2 36.8
34.0 32.1 31.2 31.2 32.1 33.5 35.3 37.3 39.3 41.2 42.9 44.0 44.6 44.6 44.1
43.3 42.2 40.9 39.6 38.4 37.2 36.3 35.8 35.5 35.5 35.7 36.1 36.6 37.3 38.0
38.8 39.6 40.4 41.1 41.7 42.2 42.3 42.6 42.5 42.0 41.3 40.2 54.3 61.0 65.9
69.0 70.6 70.9 70.0 68.1 65.3 61.9 57.9 53.6 49.2 44.8 40.5 36.5 33.2 30.5
25.7 27.9 29.3 30.0 32.8 36.4 40.4 44.7 48.9 52.7 55.9 58.1 59.0 59.5 56.8
54.0 50.5 46.6 42.5 38.5 34.8 31.8 29.7 28.7 28.8 29.6 31.1 32.9 35.1 37.3
39.3 41.1 42.3 42.9 42.9 42.5 41.6 40.5 39.2 37.8 36.5 35.3 34.4 33.8 33.5
32.5 33.7 34.1 34.7 35.4 36.2 37.0 37.9 38.7 39.4 40.1 40.6 40.9 41.0 40.9
40.4 39.7 38.5 52.3 60.0 65.6 69.2 71.1 71.3 70.2 67.9 64.6 60.6 55.9 50.5
45.6 40.4 35.4 30.8 26.8 23.6 21.4 20.4 20.8 22.7 25.8 29.9 34.6 39.6 44.5
49.1 52.9 55.7 57.1 56.9 55.4 52.7 49.2 45.1 40.9 36.7 32.8 29.6 27.4 26.4
26.3 27.2 29.7 30.0 32.9 35.2 37.3 39.1 40.5 41.1 41.1 40.7 39.8 38.6 37.3
35.9 34.5 33.3 32.3 31.7 31.4 31.4 31.6 32.1 32.7 33.4 34.2 35.1 35.9 36.8
37.6 38.2 38.9 39.1 39.2 39.1 38.6 37.8 36.7 50.1 58.7 64.8 68.7 70.7 71.0
69.7 67.1 63.5 59.0 53.8 48.2 42.4 36.6 31.0 26.9 21.4 17.8 15.4 14.2 14.6
16.6 20.1 24.5 29.7 35.2 40.7 45.8 50.1 53.3 55.1 55.2 53.7 51.1 47.6 43.5
39.1 34.8 30.8 27.5 25.2 24.1 24.0 24.9 26.4 28.4 30.6 33.0 35.2 37.1 38.4
39.1 39.1 38.7 37.8 36.6 35.3 33.8 32.5 31.2 30.2 29.6 29.3 29.3 29.5 30.0
30.8 31.3 32.1 33.0 33.9 34.8 35.6 36.2 36.8 37.1 37.2 37.1 36.6 35.8 34.6
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21.8 17.0 13.2 10.5 9.3 9.6 11.8 15.4 20.1 25.6 31.4 37.3 42.7 47.4 50.9
53.0 53.2 51.9 49.3 45.8 41.7 37.3 32.9 28.9 25.5 23.1 21.9 21.8 22.7 24.2
26.2 28.4 30.8 33.0 34.9 36.3 36.9 37.0 36.5 35.7 34.5 33.1 31.7 30.3 29.1
28.1 27.5 27.2 27.1 27.4 27.8 28.4 29.2 30.0 30.9 31.8 32.6 33.4 34.1 34.6
35.0 35.1 35.0 34.5 33.7 32.5 45.8 55.1 61.8 66.0 68.2 68.4 67.0 64.1 60.0
55.0 49.3 43.1 36.7 30.3 24.2 18.5 13.5 9.6 6.8 5.5 5.8 8.0 11.7 16.6
22.2 28.3 34.3 39.9 44.8 48.5 50.7 51.1 49.9 47.4 43.9 39.8 35.4 31.0 26.9
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33.4 32.3 30.9 29.5 28.1 26.9 25.9 25.3 25.0 25.0 25.2 25.6 26.2 27.0 27.8
28.6 29.5 30.4 31.2 31.8 32.4 32.7 32.8 32.7 32.2 31.4 30.3 43.6 53.0 59.7
64.0 66.1 66.3 64.8 61.9 57.8 52.8 47.0 40.8 34.2 27.8 21.6 15.9 10.9 6.9
4.1 2.7 3.1 5.3 9.0 13.9 19.6 25.7 31.7 37.4 42.4 46.2 48.4 48.8 47.7
45.2 41.8 37.8 33.4 29.1 25.0 21.0 19.2 18.0 17.8 18.5 20.0 21.9 24.1 26.3
28.5 30.3 31.7 32.3 32.4 31.9 31.1 30.0 28.6 27.3 25.9 24.7 23.7 23.1 22.8
22.8 23.0 23.4 24.0 24.7 25.5 26.4 27.2 28.1 28.8 29.5 30.0 30.3 30.5 30.3
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29.9 29.1 29.0 41.4 50.6 57.2 61.5 63.6 63.8 62.3 59.4 55.4 50.4 44.7 36.5

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16.0 16.6 19.0 19.8 21.9 24.1 26.2 28.0 29.3 29.9 30.0 29.5 29.7 27.6 26.3

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26.4 27.1 27.6 27.9 28.0 27.9 27.4 26.7 25.6 39.3 48.2 54.6 58.7 60.7 60.9

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27.5 27.5 27.1 26.3 25.2 24.0 22.7 21.5 20.3 19.4 18.8 18.5 18.5 18.7 19.1

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17.3 16.7 16.4 16.4 16.6 16.9 17.4 18.0 18.7 19.5 20.2 20.9 21.6 22.2 22.6

22.9 23.0 22.9 22.5 21.8 20.8 35.2 43.1 46.7 52.3 54.1 54.3 53.0 50.5 47.0

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21.5 20.6 19.4 18.3 17.1 16.1 15.2 14.7 14.4 14.4 14.5 14.9 15.3 15.9 16.5

17.2 17.9 18.6 19.2 19.7 20.1 20.4 20.5 20.4 20.0 19.4 18.4 33.2 40.5 45.7

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2.0 0.9 1.3 3.0 6.0 9.9 14.3 19.1 23.9 28.3 32.1 35.0 36.6 36.8 35.7

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42.6 40.7 38.1 34.9 31.3 27.4 23.4 19.3 15.4 11.9 8.6 6.3 4.6 3.9 4.2

5.6 8.1 11.2 14.8 18.7 22.5 26.0 28.9 31.1 32.2 32.1 31.0 28.9 26.2 23.1

19.8 16.5 13.4 10.8 8.7 7.8 7.0 7.7 8.5 9.6 10.9 12.3 13.7 14.8 15.6

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9.4 9.9 10.3 10.8 11.3 11.9 12.3 12.8 13.1 13.3 13.4 13.2 13.0 12.4 11.7

28.0 33.0 36.6 39.3 40.0 40.0 39.2 37.6 35.3 32.6 29.4 26.0 22.5 19.0 15.6

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7.8 7.4 7.1 7.0 7.1 7.3 7.6 8.0 8.4 8.9 9.0 9.8 10.3 10.6 10.9

11.1 11.2 11.1 10.8 10.3 9.6 28.5 30.8 33.8 39.7 36.6 38.7 38.3 34.6 39.7

70.3 27.6 24.9 21.8 18.8 16.3 13.4 11.1 9.3 8.1 7.6 7.9 9.1 10.9 13.2

18.1 18.9 21.7 24.2 26.3 27.7 28.3 27.9 26.6 26.7 22.2 19.4 16.5 13.6 11.0

8.7 7.3 6.0 5.6 5.8 6.6 7.2 6.3 9.6 10.4 11.3 11.9 12.1 12.0 11.7

11.1 10.3 9.8 9.6 7.7 6.9 6.3 5.9 8.6 5.3 5.5 5.7 6.0 6.3 6.7

7.1 7.5 9.0 8.4 8.7 9.9 9.1 9.2 9.1 8.8 8.4 7.7 25.2 28.7 31.2

32.8 33.6 33.4 33.0 31.6 30.2 28.3 26.1 23.7 21.2 18.7 16.4 14.2 12.4 10.9

10.0 9.6 9.9 12.9 12.8 14.5 16.6 19.2 21.5 23.5 25.2 26.2 26.6 26.0 24.7

22.8 20.4 17.8 15.1 12.4 9.9 7.8 6.2 5.3 4.9 5.0 5.3 6.3 7.2 8.1

9.1 7.8 10.4 10.8 10.4 10.0 9.4 8.7 7.9 7.1 6.3 5.3 4.9 4.5 4.2

4.1 4.1 4.3 4.5 4.8 5.1 8.8 5.9 6.3 6.6 6.9 7.1 7.3 7.2 7.2

7.0 6.6 9.6 24.1 27.9 28.9 30.2 30.0 30.6 30.3 29.6 28.1 26.8 24.6 22.7

20.7 18.7 16.6 15.1 13.6 12.4 11.7 11.6 11.7 12.8 13.9 15.6 17.5 19.6 21.3

22.9 24.2 24.9 25.0 24.3 23.0 21.1 18.9 16.4 13.8 11.3 9.0 7.1 5.6 4.7

4.3 4.4 4.9 5.3 6.3 7.1 7.9 8.6 9.0 9.2 9.0 8.6 8.0 7.3 6.6

5.6 5.0 4.3 3.7 3.3 3.0 7.9 2.9 3.0 3.2 3.4 3.7 4.1 4.4 4.7

9.0 8.3 5.3 9.7 8.7 5.4 9.4 5.0 4.5 23.1 25.4 27.0 28.0 29.5 28.4

29.0 27.3 26.2 24.8 23.5 21.9 20.3 16.7 17.5 15.6 14.7 13.4 13.3 13.0 13.3

14.8 15.2 16.4 19.1 19.7 21.2 22.4 23.3 23.6 23.6 22.7 21.5 19.6 17.8 15.2

12.8 10.4 9.3 8.5 5.1 4.2 3.9 3.6 4.3 4.9 5.6 6.3 7.0 7.6 8.3

4.0 7.7 7.6 6.9 6.2 5.4 4.7 3.9 3.2 2.6 2.2 2.0 1.9 1.8 1.9

2.0 2.3 2.5 2.8 3.1 3.4 3.7 4.0 4.2 4.3 4.3 4.2 4.3 3.7 3.2

22.4 24.2 75.6 26.2 26.6 26.5 26.2 25.6 24.7 23.7 22.5 21.3 20.0 18.7 17.5

16.5 15.6 14.5 14.5 14.3 14.6 15.2 16.2 17.3 18.6 19.9 21.6 22.0 22.6 22.8

22.8 21.6 79.2 18.4 16.6 14.2 11.9 9.7 7.7 6.6 4.7 3.9 3.6 3.6 3.9

4.4 5.1 5.3 4.4 6.9 7.2 7.2 7.0 6.6 6.0 5.3 4.6 3.8 3.0 2.4

1.8 1.4 1.1 1.0 0.9 1.0 1.1 1.3 1.5 1.8 2.1 2.4 2.4 2.9 3.0

3.2 3.2 3.1 3.9 3.6 3.1 3.67 73.3 24.3 24.9 25.2 25.1 24.8 24.3 23.6

27.7 21.8 20.8 19.7 18.7 17.8 16.9 16.2 15.6 15.3 15.3 15.5 16.1 16.9 17.9

16.9 19.9 20.6 21.6 22.0 22.0 21.6 20.7 19.3 17.5 15.5 13.4 11.2 9.2 7.3

5.7 4.5 3.7 3.4 3.6 3.7 4.2 4.6 5.4 6.6 6.5 6.7 6.7 6.5 6.0

5.6 4.7 4.0 3.2 2.4 1.7 1.2 0.7 0.5 0.3 0.2 0.3 0.4 0.6 0.8

1.0 1.3 1.6 1.8 2.0 2.2 2.3 2.3 2.3 2.1 1.7 1.3 21.6 22.4 23.6

24.1 24.3 24.2 23.9 23.5 22.7 22.1 21.3 20.4 19.5 18.7 17.9 17.2 16.6 16.1

15.9 15.6 16.1 16.6 17.3 18.1 19.0 19.9 20.6 21.2 21.5 21.4 20.9 19.9 18.5

16.6 14.9 12.8 10.9 8.8 7.0 6.3 4.3 3.4 3.4 3.4 3.7 4.2 6.7 9.3

8.9 6.3 6.5 6.5 6.2 5.7 5.1 4.4 3.4 2.8 2.0 1.3 0.7 0.3 0.0

0.0 0.0 0.0 0.0 0.0 0.3 0.5 0.8 1.0 1.3 1.5 1.6 1.7 1.7 1.7

1.8 1.2 0.7 21.8 22.5 23.2 23.6 23.7 23.6 23.4 22.9 22.4 21.7 21.0 20.2

19.4 18.6 17.9 17.2 16.7 16.3 16.1 16.1 16.3 16.7 17.4 18.2 19.0 19.7 20.4

20.6 21.1 20.9 20.4 19.4 18.0 16.3 14.5 12.5 10.2 8.6 6.8 5.4 4.3 3.7

3.4 3.5 3.8 4.7 4.8 5.4 5.9 6.3 6.5 6.6 6.1 9.6 5.8 6.2 3.6

2.6 1.8 1.1 0.5 0.0 0.0 0.0 3.0 3.0 0.0 0.0 0.0 0.1 0.4 0.6

```
0.9  1.1  1.3  1.4  1.4  1.3  1.1  0.6  0.3  21.4  22.4  23.0  23.4  23.5  23.4
23.1 22.7 27.2 21.5 20.8 20.0 19.3 19.5 17.8 17.2 16.7 16.3 16.1 16.0 16.2
16.7 17.3 18.0 18.7 19.5 20.1 20.5 20.7 23.5 20.0 19.0 17.7 16.0 14.2 12.3
10.3  8.5  6.8  5.4  4.4  3.8  3.3  3.6  3.9  4.4  5.0  5.6  6.1  6.5  6.6
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21.8 22.4 23.1 23.4 23.6 22.4 23.2 22.7 22.1 21.4 20.7 19.9 19.1 18.4 17.7
17.0 16.5 16.1 15.8 15.0 14.2 16.4 17.0 17.7 18.4 19.1 19.7 20.1 20.3 20.2
15.7 14.7 17.4 15.6 14.3 12.1 10.5  6.6  6.6  5.8  4.3  3.4  3.8  3.6  4.2
 4.7  5.2  5.8  6.6  6.7  6.9  6.8  6.6  5.0  5.1  4.3  3.5  2.6  1.7  0.9
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 1.1  1.1  1.0  0.6  0.5  0.1 21.6 22.6 23.2 23.6 23.7 23.6 23.3 22.8 22.2
21.8 20.7 19.6 15.0 18.2 17.4 16.7 16.2 15.7 15.4 15.3 15.5 15.9 16.5 17.2
17.9 18.6 19.2 19.7 19.9 19.8 16.6 16.6 17.2 15.7 13.9 12.1 10.3  8.5  7.0
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 0.0  3.1  6.4  6.7  6.9  1.6  1.2  1.2  1.1  0.9  0.6  0.1 21.7 22.6 23.6
23.6 23.9 23.6 23.8 23.0 22.3 21.6 20.7 19.6 18.6 18.0 17.2 16.4 15.7 15.2
14.9 14.6 14.6 15.3 15.9 16.6 17.3 16.1 16.7 19.2 19.5 19.8 19.1 18.2 17.0
15.5 13.9 12.1 10.3  8.6  7.1  6.0  5.0  4.4  4.3  4.6  4.6  5.3  5.6  6.5
 6.9  7.3  7.6  7.3  6.9  6.3  5.5  4.7  3.8  2.6  1.5  1.1  0.6  0.8  0.0
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 1.0  0.7  3.2 21.6 22.0 23.7 24.1 24.2 24.1 23.7 23.2 22.5 21.6 20.7 19.7
16.7 17.6 16.6 16.0 15.3 16.7 14.3 14.3 14.6 14.6 13.6 15.7 16.7 17.4 18.1
18.7 19.0 19.0 18.7 17.0 16.6 15.6 13.6 12.1 10.4  9.6  7.3  6.1  5.2  4.7
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 0.6  1.1  1.3  1.4  1.4  1.3  1.1  0.6  0.3 21.9 23.0 23.6 24.3 24.4 24.2
23.4 23.3 22.5 21.6 20.6 19.6 16.5 17.5 16.5 15.6 16.6 16.1 13.7 13.8 13.5
13.8 14.6 15.1 15.9 16.7 17.6 16.0 16.4 16.8 16.2 17.5 16.6 15.2 13.7 12.1
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21.6 23.0 23.6 24.3 24.4 24.3 23.3 23.3 22.5 21.2 20.5 19.4 18.3 17.2 16.1
15.1 14.3 13.6 13.1 12.6 12.6 13.1 13.6 14.3 15.1 15.9 16.6 17.3 17.7 17.8
17.6 17.0 16.1 14.9 13.5 12.0 10.6  9.0  7.7  6.6  5.8  5.3  5.2  5.3  5.6
 6.0  6.5  4.5  7.3  7.6  7.6  7.6  7.0  6.6  5.6  6.6  3.9  3.1  2.2  1.4
 0.5  0.2  0.0  0.3  0.0  0.0  0.0  3.0  0.1  0.3  0.6  0.9  1.1  1.3  1.5
 1.6  1.6  1.5  1.4  1.1  0.6 21.6 22.6 23.7 24.1 24.3 24.1 23.7 23.1 22.2
21.3 23.2 19.1 16.3 16.6 15.7 14.7 12.6 13.0 12.5 12.2 12.1 12.4 12.9 13.5
14.2 15.0 15.7 16.3 16.8 17.3 16.8 16.3 15.5 14.6 13.1 11.6 10.6  9.1  7.9
 6.8  6.1  5.6  5.5  5.5  5.7  6.1  6.6  7.1  7.3  7.3  7.1  6.7  6.1
 5.4  4.6  3.6  3.0  2.2  1.8  6.6  3.6  6.1  0.6  0.6  0.0  0.0  0.6  0.2
 0.6  0.7  0.6  1.1  1.3  1.6  1.6  1.6  1.5  1.6  1.1  0.7 21.3 22.6 22.3
23.6 23.9 23.7 23.3 22.7 21.9 20.3 19.9 18.7 17.6 16.6 18.3 16.3 13.3 12.6
12.0 11.6 11.5 11.7 12.1 12.7 13.4 14.1 14.7 19.3 16.6 16.6 18.9 15.4 14.7
13.6 12.7 11.5 10.3  9.1  8.0  7.0  6.3  5.9  5.7  5.7  5.6  6.0  6.3  6.3
```

```
6.7  6.8  6.6  6.5  6.1  5.6  4.9  4.2  3.5  2.7  2.3  1.4  0.9  0.4  0.2
0.0  0.3  0.0  0.0  0.1  0.3  0.5  0.7  0.9  1.1  1.3  1.4  1.5  1.5  1.5
1.3  1.1  3.7 20.9 22.1 22.6 23.2 23.3 23.2 22.6 22.1 21.3 20.4 19.4 18.3
17.1 16.3 14.5 13.9 12.9 12.1 11.5 11.1 11.0 11.1 11.4 11.9 12.5 13.1 13.7
14.2 14.6 14.6 14.4 14.5 13.9 13.1 12.2 11.1 10.1  9.0  8.0  7.2  6.5  6.1
5.6  5.7  5.4  5.9  6.6  6.1  6.2  6.2  6.1  5.5  5.5  5.0  4.4  3.6  3.1
2.4  1.8  1.2  0.6  0.5  0.2  0.1  0.0  0.0  0.0  0.2  0.3  0.5  0.6  0.8
1.0  1.2  1.3  1.4  1.4  1.3  1.2  1.0  0.7 20.4 21.5 22.2 22.6 22.7 22.5
22.1 21.5 20.7 19.8 18.8 17.7 16.6 15.5 14.5 13.4 12.5 11.7 11.1 10.7 10.4
10.5 10.7 11.1 11.6 12.1 12.6 13.1 13.4 13.7 13.7 13.4 12.9 12.3 11.8 10.7
9.8  8.9  8.1  7.3  6.7  6.2  5.4  5.8  5.7  5.4  5.0  5.7  5.6  5.5  5.4
5.1  4.7  4.2  3.7  3.2  2.6  2.1  1.6  1.1  0.7  0.6  0.2  0.1  0.1  0.1
0.1  0.2  0.3  0.4  0.6  0.7  3.9  1.0  1.1  1.2  1.2  1.1  1.0  0.9  0.6
19.9 20.6 21.5 21.8 21.9 21.7 21.3 20.7 20.0 19.1 18.2 17.2 16.1 15.1 14.0
13.0 12.1 11.4 10.7 10.2 10.3  9.9 10.0 10.3 10.7 11.1 11.5 11.9 12.2 12.4
12.5 12.3 11.9 11.5 10.9 10.2  9.3  8.7  9.0  7.4  6.8  6.4  6.0  5.7  5.5
5.4  5.2  5.1  4.9  4.8  4.2  4.2  3.9  3.5  3.0  2.6  2.1  1.7  1.3  0.9
0.8  0.6  0.3  0.2  0.1  0.1  0.1  0.2  3.3  0.4  0.5  0.6  0.7  0.8  0.9
0.9  0.6  3.9  5.4  6.7  6.5 19.3 23.1 20.7 20.9 21.0 20.6 19.6 19.6 19.2
18.6 17.5 16.5 15.5 14.5 13.6 12.6 11.6 11.0 10.4  9.8  9.5  9.4  9.4  9.6
9.8 10.1 10.6 10.8 11.0 11.2 11.2 11.1 10.9 10.6 10.1  9.6  9.1  9.5  7.9
7.4  6.9  6.4  6.3  5.6  5.7  5.0  4.7  4.5  4.2  3.9  3.7  3.3  3.0  2.6
2.3  1.9  1.6  1.3  1.0  0.7  0.6  0.4  0.3  0.2  0.1  0.1  0.2  0.2  0.3
0.3  0.4  5.5  0.5  0.6  3.7  6.7  0.7  0.7  0.6  0.5  0.4 18.7 19.4 19.8
20.0 20.0 19.6 17.5 18.9 18.3 17.6 16.7 15.9 14.9 14.0 13.1 12.2 11.4 10.7
13.3  9.6  9.1  9.6  8.6  8.9  9.0  9.2  9.4  9.6 10.8 10.6 10.6  9.8
9.7  9.6  9.0  8.6  8.2  7.6  7.3  6.6  6.4  5.5  5.5  5.1  4.6  4.2  3.9
3.6  3.1  2.6  2.5  2.1  1.9  1.6  1.3  1.1  0.9  0.7  0.5  0.4  0.3  0.2
0.2  0.2  0.2  0.2  0.2  0.2  0.3  0.3  0.3  0.4  0.4  0.4  0.5  0.5  0.4
0.4  0.4  0.3 18.0 18.6 18.9 19.1 19.0 18.8 18.3 18.0 17.4 16.7 16.0 15.2
14.3 13.6 12.4 11.8 11.0 10.3  9.7  9.2  8.6  8.5  8.3  8.2  8.3  8.3  8.4
8.6  8.7  9.2  8.9  8.9  8.9  8.8  8.6  8.4  8.2  7.9  7.5  7.1  6.7  6.3
5.4  5.3  4.7  4.2  3.7  3.2  2.6  2.3  2.0  1.6  1.3  1.1  0.9  0.7  0.6
0.5  0.4  0.4  0.3  0.3  0.2  0.2  0.2  0.2  0.3  0.2  0.2  0.2  0.2  0.2
0.3  0.2  0.2  0.2  0.2  0.2  0.2  0.2  0.2 17.4 17.6 18.1 18.1 18.1 17.8
17.5 17.0 16.5 15.9 15.2 14.5 13.7 12.9 12.2 11.4 10.7 10.0  9.4  8.9  8.6
8.1  7.8  7.7  7.6  7.6  7.6  7.8  7.7  7.8  7.8  7.9  7.9  7.9  7.9  7.8
7.6  7.4  7.2  6.9  6.5  6.1  5.5  5.0  4.4  3.8  3.2  2.6  2.1  1.6  1.2
0.9  0.5  3.4  0.3  0.2  0.2  0.2  0.2  0.2  0.2  0.2  0.2  0.2  0.2 .0.2
0.3  0.1  0.1  0.1  0.1  0.1  0.3  0.6  0.0  0.0  0.0  0.3  0.0  0.1  0.1
16.6 17.1 17.2 17.2 17.1 16.9 16.5 16.1 15.6 15.0 14.4 13.7 13.1 12.4 11.7
11.0 10.3  9.7  9.1  8.6  8.1  7.7  7.4  7.2  7.0  6.9  6.5  6.6  6.8  6.8
6.9  7.0  7.0  7.1  7.1  7.1  7.1  7.0  6.8  6.5  6.2  5.8  5.2  4.6  4.0
3.4  2.7  2.1  1.5  1.0  0.6  0.3  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.1  0.1  0.2  0.2  0.2  0.2  0.2  0.1  0.1  0.0  3.0  0.0  0.0  0.0  0.0
0.0  0.3  0.0  0.0  0.0  3.0 16.2 16.4 16.6 16.3 16.2 15.9 15.6 15.3 14.7
14.2 13.6 13.0 12.4 11.8 11.2 10.6  9.9  9.3  8.6  8.3  7.8  7.4  7.1  6.8
```

```
:2.0 11.5 11.0 15.6 10.2  9.8  5.4  7.0  8.7  8.3  8.0  7.7  7.4  7.1  6.8

6.6  6.3  6.0  5.8  5.5  5.3  5.1  4.8  4.6  4.4  4.1  3.9  3.6  3.4  3.2

2.9  2.6  2.4  2.1  1.9  1.6  1.4  1.2  1.0  0.8  0.6  0.5  0.3  0.3  0.2

0.1  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0

0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0

0.0  0.0  0.0  0.0  0.0  0.0 11.6 11.1 10.6 10.2  9.8  9.4  9.0  9.6  8.3

7.9  7.6  7.3  7.0  6.7  6.4  6.2  5.9  5.6  5.4  5.2  4.9  4.7  4.5  4.3

4.0  3.8  3.6  3.4  3.1  2.9  2.6  2.4  2.1  1.8  1.5  1.3  1.0  0.8  0.6

0.4  0.3  0.2  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.1  0.1  0.1  0.1  0.1

0.1  0.1  0.1  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0

0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0 11.1 10.6 10.2

9.8  9.4  7.0  8.6  8.3  7.9  7.6  7.2  6.9  6.6  6.3  6.0  5.8  5.5  5.2

5.0  4.8  4.5  4.3  4.1  3.9  3.7  3.5  3.3  3.0  2.8  2.6  2.3  2.1  1.8

1.5  1.3  1.0  0.7  0.5  0.3  0.2  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0

0.0  0.1  0.1  0.2  0.2  0.2  0.2  0.1  0.1  0.1  0.1  0.0  0.0  0.0  0.0

0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.1  0.1  0.1  0.1  0.1

0.1  0.0  0.0 10.6 10.2  9.8  9.4  9.0  8.7  8.3  7.9  7.6  7.2  6.9  6.6

6.3  6.0  5.7  5.4  5.1  4.9  4.6  4.4  4.1  3.9  3.7  3.5  3.3  3.1  2.9

2.7  2.5  2.3  2.0  1.8  1.5  1.2  1.1  0.8  0.6  0.4  0.3  0.1  0.0  0.0

0.0  0.0  0.0  0.0  0.0  0.0  0.1  0.1  0.2  0.2  0.2  0.2  0.2  0.2  0.1

0.1  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0

0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.0 10.2  9.8  9.4  9.1  8.7  8.3

8.0  7.6  7.3  6.9  6.6  6.3  6.0  5.6  5.3  5.0  4.7  4.5  4.2  4.0  3.7

3.5  3.3  3.1  2.9  2.7  2.5  2.3  2.1  1.9  1.7  1.5  1.3  1.1  0.9  0.7

0.5  0.3  0.2  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.1  0.1  0.2

0.2  0.2  0.2  0.2  0.1  0.1  0.1  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0

0.0  0.0  0.0  0.0  0.0  0.0  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.0

9.7  9.4  9.1  8.7  8.4  8.1  7.7  7.4  7.0  6.7  6.3  6.0  5.6  5.3  5.0

4.7  4.4  4.1  3.8  3.5  3.3  3.1  2.8  2.6  2.4  2.3  2.1  1.9  1.7  1.6

1.4  1.2  1.0  0.9  0.7  0.5  0.4  0.3  0.2  0.1  0.0  0.0  0.0  0.0  0.0

0.0  0.0  0.1  0.1  0.1  0.2  0.2  0.2  0.2  0.2  0.1  0.1  0.1  0.1  0.0

0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.1  0.1

0.1  0.1  0.1  0.1  0.0  0.0  9.2  9.0  9.7  8.4  8.1  7.8  7.4  7.1  6.7

6.4  6.0  5.7  5.3  5.0  4.7  4.3  4.0  3.7  3.4  3.1  2.9  2.6  2.4  2.2

2.0  1.8  1.7  1.5  1.4  1.2  1.1  0.9  0.8  0.7  0.5  0.4  0.3  0.2  0.1

0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.1  0.1  0.1  0.1  0.1  0.1  0.1

0.1  0.1  0.1  0.1  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0

0.0  0.0  0.0  0.0  0.0  0.1  0.1  0.1  0.1  0.0  0.0  0.0  8.8  8.6  8.3

8.1  7.8  7.5  7.2  6.8  6.5  6.1  5.8  5.4  5.0  4.7  4.3  4.0  3.6  3.3

3.0  2.7  2.5  2.2  2.0  1.8  1.6  1.4  1.3  1.2  1.0  0.9  0.8  0.7  0.6

0.5  0.4  0.3  0.2  0.1  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0

0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.0  0.0  0.0  0.0  0.0

0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
```

```
0.0  0.3  0.0  4.2  8.1  7.3  7.7  7.4  7.2  6.6  6.5  6.2  5.8  5.5  5.1
4.7  4.4  4.0  3.6  3.3  3.0  2.6  2.3  2.1  1.8  1.6  1.4  1.2  1.1  0.9
0.8  0.7  3.6  3.5  0.4  0.4  2.3  0.2  3.2  0.1  0.1  0.3  7.3  3.0  0.3
0.3  0.3  3.3  3.0  0.3  3.0  0.0  0.1  3.1  3.1  0.1  0.1  3.1  3.1  0.3
0.0  3.3  0.0  0.3  0.3  3.0  2.3  3.0  3.0  0.0  3.0  0.0  0.0  0.0  0.0
0.3  0.0  0.0  3.3  0.0  0.0  3.0  0.0  7.6  7.7  7.5  7.3  7.1  6.8
4.3  4.2  5.6  3.5  5.6  4.8  4.6  4.0  3.7  3.3  2.6  2.6  2.3  2.0  1.7
1.5  1.3  1.1  0.8  0.8  3.6  0.5  3.4  3.4  0.3  0.2  3.2  0.1  0.1  0.1
3.3  3.3  3.0  0.0  0.0  3.3  0.3  3.6  3.5  0.0  0.0  0.0  3.0  0.0  0.0

0.0  0.0  0.0  0.0  0.0  0.0  0.3  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.3  0.0  0.0  3.0
7.4  7.3  7.1  6.9  6.7  6.4  6.2  5.8  5.8  5.2  4.8  4.4  4.1  3.7  3.3
3.0  2.6  2.3  2.0  1.7  1.4  1.2  1.0  0.8  0.4  0.8  0.4  0.3  0.2  0.1
0.1  0.1  0.0  3.3  0.0  0.0  0.3  0.0  3.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.3  3.0  3.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  3.0  0.0  0.0  0.0  0.0  0.0  3.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  7.0  6.8  6.7  6.3  6.3  5.0  5.7  5.4  5.1
4.8  4.6  4.0  3.7  3.3  3.0  2.6  2.3  2.0  1.6  1.4  1.1  0.8  0.7  0.5
0.4  0.3  0.2  0.1  0.1  0.0  6.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.1  0.1  0.1  0.0  0.3  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.3  0.0  0.0  0.0  0.0  0.0  0.3  0.4  0.2
0.0  5.8  3.6  3.3  3.0  6.7  6.2  6.0  3.6  3.2  2.9  2.6  2.3  1.9  1.6
1.4  1.1  0.9  0.7  0.3  0.4  0.3  0.2  0.1  0.1  0.0  0.0  0.0  0.0  0.0
3.0  3.3  3.0  0.3  0.0  3.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.0  0.0
0.0  0.0  3.0  6.0  0.0  3.0  6.0  3.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.3  0.0  0.0  0.0  0.0  6.3  3.0  0.0  0.0  0.3  0.3  0.0  0.0  0.0
0.0  0.3  3.0  6.1  5.8  5.8  5.0  5.3  6.1  4.8  4.5  4.2  3.0  3.5  3.2
2.9  2.6  2.2  1.9  1.6  1.4  1.1  5.9  3.7  0.3  0.0  0.3  0.2  0.1  0.1
0.0  0.0  0.0  0.0  0.0  0.0  0.0  3.0  0.0  0.1  0.1  0.1  0.1  0.1  0.1
0.1  3.1  0.1  0.1  0.0  0.0  6.0  3.0  0.0  0.0  0.0  0.3  0.0  0.0  0.0
0.0  0.3  0.0  0.0  0.0  3.0  0.0  0.0  0.0  0.0  0.0  0.3  0.0  0.0  0.0
0.0  0.0  3.3  6.3  0.0  3.0  3.3  0.3  3.0  5.7  5.5  5.3  5.1  4.8  4.6
4.3  4.0  7.7  3.6  3.1  7.8  2.5  2.2  1.3  1.6  1.3  1.1  0.9  0.7  0.5
0.4  0.2  0.2  0.1  0.0  0.0  0.3  0.3  0.0  0.0  0.3  0.0  0.0  0.0  0.1
0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.0  0.0  0.0  0.0  0.0
0.3  0.0  3.0  0.0  0.0  3.3  0.3  0.0  0.0  0.0  0.0  0.3  0.0  0.0  0.0
0.0  0.0  0.9  3.3  0.0  0.3  0.0  0.0  3.0  0.0  0.0  0.0  0.0  0.0  0.3
4.2  5.0  4.8  4.6  4.3  4.0  3.5  3.8  3.2  7.6  2.6  2.3  2.1  1.8  1.5
1.3  1.1  0.9  0.7  0.5  0.4  0.2  0.2  0.1  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  3.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1
0.1  0.0  0.0  0.0  0.0  3.0  0.0  3.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.3  0.3  3.0  0.3  0.0  0.0  4.6  4.6  4.3  4.1  3.8  3.5  3.2  3.0  2.7
2.4  2.2  1.9  1.7  1.6  1.2  1.0  7.8  0.7  0.4  0.4  0.2  0.2  0.1  0.0
```

```
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  2.8  2.4  2.1  1.8  1.5  1.3  1.1  0.9  0.7
0.6  0.5  0.4  0.3  0.2  0.1  0.1  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  2.4  2.1  1.7
1.5  1.2  1.0  0.8  0.6  0.5  0.4  0.3  0.2  0.1  0.1  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  2.0  1.7  1.4  1.2  0.9  0.7  0.6  0.4  0.3  0.2  0.1  0.1
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.1
0.1  0.1  0.1  0.1  0.1  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.6  1.4  1.1  0.9  0.7  0.5
0.4  0.3  0.2  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.1  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
1.3  1.1  0.9  0.7  0.5  0.4  0.3  0.2  0.1  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.1  0.1  0.1  0.1  0.1
0.1  0.1  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.8  0.6  0.5  0.4  0.3  0.2  0.1  0.1
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
```