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A GUIDED REINVENTION OF RING, INTEGRAL DOMAIN, AND FIELD

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DEPARTMENT OF MATHEMATICS

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For Regina, my wonderful wife.

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Abstract

Abstract algebra enjoys a prestigious position in mathematics and the undergraduate mathematics curriculum. A typical abstract algebra course aims to provide students with a glimpse into the elegance of mathematics by exposing them to structures that form its foundation—it arguably approximates the actual practice of mathematics better than any of the courses by which it is typically preceded. Regrettably, despite the importance and weight carried by the abstract algebra, the educational literature is replete with suggestions that undergraduate students do not appear to be grasping even the most fundamental ideas of the subject. Additionally, many students fail to make the connection between abstract algebra and the algebra they learned at the primary and secondary levels, perpetually blind to any interpretations of the subject beyond surface-level. These discrepancies have two problematic consequences. First, students who were otherwise enthusiastic and interested in mathematics experience a complete reversal and become indifferent and disengaged. Second, future mathematics teachers at the primary and secondary levels do not build upon their elementary understandings of algebra, leaving them unable to communicate traces of any deep and unifying ideas that govern the subject.

To address this problem, it has been suggested that the traditional lecture method be eschewed in favor of a student-centered, discovery-based approach. There have been several responses to this call; most notable and relevant to this project is the work of Larsen (2004, 2009), who developed an instructional theory to support students' reinvention of group and group isomorphism. As no such innovative methods of instruction exist regarding ring field theory, this project details the development of an instructional theory supporting students' reinvention of fundamental structures from ring theory: ring, integral domain, and field.

Rooted in the theory of Realistic Mathematics Education, this dissertation reports on a developmental research project conducted via multiple iterations of the constructivist teaching experiment, wherein the primary goal was to test and revise an instructional theory supporting the guided reinvention of ring, integral domain, and field. The findings include an empirically tested and revised instructional theory, as well as conceptual frameworks detailing the emergence and progressive formalization of the key features in a ring structure.

Chapter 1: Introduction

Most undergraduate curricula include a course in abstract algebra at their apex, and with good reason. The subject matter, containing such concepts as groups, rings, and fields, has extensive consequences and has been declared an "essential component of contemporary mathematics" (NCTM, 2001, p. 1). Despite its overall significance in mathematics, however, students encountering the subject for the first time often struggle and fail to comprehend many of the fundamental ideas (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Hazzan & Leron, 1996; Leron & Dubinsky, 1995). The irony of the situation is that abstract algebra is meant to be the course in which students are given their first glimpse into the elegance of mathematics. No longer should mathematics be an endless string of algorithms and processes; abstract algebra should build upon these rudimentary mathematical methods and supplement them with powerful methods of inquiry and reason. And yet, the literature suggests that this is simply not the case. A more accurate picture, unfortunately, is that many students who are interested in mathematics experience a complete reversal of opinion and become indifferent or disengaged.

Some difficulty is expected, of course, as "abstract algebra is the first course for students in which they must go beyond learning 'imitative behavior patterns' for mimicing [sic] the solution of a large number of variations on a small number of themes (problems)" (Dubinsky et al., 1994, p. 268). Indeed, abstract algebra eschews the algorithmic, step-by-step problem solving techniques developed in previous courses and replaces them with a strong reliance on abstraction and creativity to solve problems. This transition to higher mathematics is doubtless accompanied by a flurry of unfamiliar ideas and concepts. This difficulty prevents many undergraduate students from seeing that topics in abstract algebra provide rationale for why different structures have certain properties that others do not (for example, the integers do not have multiplicative inverses, while the rational numbers do). In other words, they fail to see the connection between abstract algebra and high school algebra (Usiskin, 1988). This is reiterated by Cuoco (2001), who emphasized that "abstract algebra is seen as a completely different subject from school algebra" (p. 169). In the case of primary and secondary mathematics teachers, this disconnect causes a reversion to their own knowledge of high school algebra, which likely emphasized a highly algorithmic approach as opposed to one that encouraged conceptual understanding and critical thinking. The failure to associate abstract algebra with elementary algebra serves to further obfuscate the purpose of the course and thus represents a significant problem in collegiate mathematics education.

Regrettably, traditional classroom practices do not appear to enable students to make the transition to abstract algebra or other courses in advanced mathematics successfully. Leron and Dubinsky (1995) posited that traditional methods of classroom instruction for abstract algebra may not be sufficient to help students overcome these obstacles. In fact, they go so far as to state that the "teaching of abstract algebra is a disaster, and this remains true almost independently of the quality of the lectures" (p. 227). To this end, alternative methods of teaching and learning abstract algebra must be explored.

In response to their own call for innovative instructional methods, Leron and Dubinsky (1995) advocated a discovery-based alternative to the lecture method in which students discover structures themselves. DeVilliers (1998) shared similar sentiments, arguing that students should be actively involved in the process of defining in the classroom. Findell (2001) summarized the discovery method as follows: “Give the students a rich problem situation to explore. They will discover patterns and relationships, develop ideas and concepts, and create objects and processes” (p. 335). In other words, using the discovery method, students should be given the opportunity to use their own reasoning and intuition to identify and “create” mathematical structures. Papick, Reys, Beem, and Reys (1999) advocated a similar approach in abstract algebra by stating that “a rigorous examination of arithmetic properties in various algebraic structures deepens the understanding of traditional arithmetic and accentuates the importance of axiomatic mathematics” (p. 306). Once this has taken place, the students feel responsible for their own learning, as if the instructor is merely summarizing knowledge they have already discovered (Leron & Dubsinsky, 1995).

Indeed, several discovery approaches to instruction have been developed in response to this need. Leron and Dubinsky (1995) developed a technology-based approach with the programming software ISETL which enables students to easily explore specific examples of structures in a group theory course. Subsequent studies were conducted with ISETL to examine student understanding of most typical group theory topics, including subgroups, normal subgroups, cosets and quotient groups, and group homomorphisms (Dubinsky et al, 1994). Larsen (2004) produced an instructional theory grounded in the theory of Realistic Mathematics Education (RME) by which students reinvent the concepts of group and group isomorphism by use of didactically-minded activities. The method of guided reinvention used by Larsen involves students formalizing their intuitive and informal notions of a concept. These efforts have since been expanded to create a complete reinvention-based curriculum for group theory (Larsen, Johnson, Rutherford, & Bartlo, 2009; Larsen, Johnson, & Scholl, 2011). Nevertheless, it is worth noting that all of these innovative approaches occur completely within the arena of group theory, leaving ring theory relatively untouched.

Motivation

My desire to increase the body of knowledge in this field stems from my experience as an undergraduate mathematics major, which mirrored the experience of the typical student who struggles to understand abstract algebra (as depicted in the literature). Like most students, abstract algebra was the first course I took as an undergraduate that exposed me to advanced mathematics. I performed reasonably well in the course despite struggling to comprehend the motivation behind the core concepts of group and ring. This elusive understanding came several years later while I was in graduate school as a result of my work with abstract algebra concepts in subsequent courses. The most disquieting realization was that these ideas were not complicated – groups arose out of notions of permutations and symmetry, while (commutative) rings came forth from a need to generalize the structure of the integers. These are ideas which are well within the grasp of any student in abstract algebra, yet it appears that many students are initially (or perhaps perpetually) blind to them.

Once I was able to grasp the significance of the group and ring axioms for myself, I acquired not only a greater appreciation for the subject in general but also a much more comprehensive understanding of the material. Thus, this dissertation project is partially in response to my undergraduate experience and my desire to have students realize and comprehend the meaning and motivation for the primary concepts in abstract algebra right from the start. The unifying ideas which underlie the notions of group and ring are easily accessible, making this a very attainable goal through the Realistic Mathematics Education heuristic of guided reinvention (Freudenthal, 1973). In short, guided reinvention involves administering tasks designed to elicit powerful informal understandings, which form the foundation for a gradual transition to more formal mathematical activity.

Significance

The significance of this project is rooted equally in the importance of rings in mathematics and the disparity of information regarding how students come to understand rings in the educational literature. First, the concept of ring is a rich subject in its own right and has implications in algebraic geometry, algebraic number theory, field theory, group theory, and real and complex analysis. In short, having a firm foundational understanding of rings (and other similar concepts from abstract algebra) is necessary for continuing study in mathematics.

Second, the importance of rings in mathematics clearly underscores the need for research addressing how undergraduate students develop and understanding of its core properties. As noted previously, many students (presumably some of whom are future mathematics teachers) fail to connect their knowledge of abstract algebra with the high school algebra they are teaching. Thus, a comprehensive understanding of rings is needed not just by undergraduate students intending to major in mathematics but also by prospective primary and secondary mathematics teachers.

Currently, there is a considerable disparity between the significance of this topic in mathematics (and the undergraduate curriculum) and the amount of information known about how students learn it. In fact, only one article (Simpson & Stehlikova, 2006) can be found which directly addresses students learning about rings at all. Since several features and learning mechanisms for groups have direct analogs in ring theory, limited information can be harvested from the group theory literature. Those involving the definition of ring or the ring structure include, for example, binary operation (Brown, DeVries, Dubinsky, & Thomas, 1997; Iannone & Nardi, 2002), student proficiency (or lack thereof) with the group axioms (Dubinsky et al., 1994), confusion of the associative and commutative properties (Findell, 2001; Larsen, 2010), and the use of operation tables (Findell, 2001). Despite any possible application of this knowledge to student learning of rings, however, even introductory ring theory possesses several key, nontrivial features for which there is no analog in ring theory: zero divisors, an additional binary operation, and the distributive property (to name a few). Information regarding these concepts can only be obtained by research which directly examines student learning of rings.

This, compounded with the well-documented fact that students struggle mightily in abstract algebra, highlights a glaring gap in the research literature. This project aims to begin filling this void by contributing findings about how students come to understand rings, integral domains, and fields. Furthermore, the principal goal of this project is to develop an innovative instructional theory in order to support student learning of these topics.

Research Questions

The ultimate purpose of this dissertation project is to develop an instructional theory to support the guided reinvention of the concepts of ring, integral domain, and field in a classroom setting. The overarching question which guides this research project is:

- How might students reinvent the definitions of ring, integral domain, and field?

The following are supporting research questions:

- What models and activities are involved in developing these concepts when the students start with their own reasoning and intuition?
- What models and activities enable students to see the need for, define, and differentiate between additional ring structures like integral domain and field?

Overview of Study

The purpose of this project was to develop a local instructional theory (Gravemeijer, 1999) supporting the guided reinvention of the definitions of ring, integral domain, and field by investigating how these concepts might be reinvented when the students start with their own knowledge and intuitive reasoning. I adopted a constructivist epistemology (Piaget, 1977) and employed the theoretical perspective and instructional design methods of Realistic Mathematics Education (Freudenthal, 1973). The developmental research design (Gravemeijer, 1999) consisted of iterating the constructivist teaching experiment (Cobb, 2000; Steffe & Thompson, 2000), in which each experiment was conducted sequentially with a different pair of students.

The experiments were conducted with pairs of students because it allowed and encouraged the students to work together and communicate with each other. This research design allowed me to continually analyze and revise both the instructional tasks and the emerging local instructional theory.

In Chapter 2, I examine and critique the existing literature that is relevant to this project in an effort to both build upon previous knowledge and contextualize this project within the field. In Chapter 3, I detail and explain the rationale for the theoretical framework and methods I used in executing this project, including my epistemological stance and theoretical perspective. The research design and methods of this project, including participant selection, data collection methods, and tools for data analysis are discussed in Chapter 4. Chapter 5 presents the results from the teaching experiments. This project culminates in Chapter 6, wherein I present the conclusions from this project, including the refined local instructional theory. A sample instructional sequence and guide indicating one possible implementation of the refined instructional theory concludes this dissertation in Chapter 7.

Chapter 2: Literature Review

The purpose of this chapter is to summarize and critique the literature relevant to this project. First, I discuss work with guided reinvention in abstract algebra. Next, I introduce research directly related the teaching and learning of specific topics in group theory which have analogs in ring theory. I then discuss research directly addressing student learning in ring theory. Finally, this review concludes with an overview of research related to other issues involved in the teaching and learning of abstract algebra.

Guided Reinvention Projects in Abstract Algebra

The research literature contains two instructional theories supporting students' reinvention of three different abstract algebra topics: group and group isomorphism (Larsen, 2004), and quotient group (Larsen, Johnson, Rutherford, & Bartlo, 2009). The method of guided reinvention used by Larsen involves students formalizing their intuitive and informal notions of these concepts. These efforts have since been expanded to create a complete reinvention-based curriculum for group theory (Larsen, Johnson, & Scholl, 2011). Many constructs and ideas discussed in this section are also used in this project. These are touched upon briefly here and explained in more detail in subsequent chapters.

Reinvention of group and group isomorphism. Larsen's (2004) dissertation was the first project to apply the theory of Realistic Mathematics Education (RME) and guided reinvention to topics in abstract algebra. Specifically, his project details the development of a local instructional theory which supports the guided reinvention of the concepts of group and group isomorphism. Note that many of the constructs and ideas which are used in Larsen's work are also used in my project. To avoid repetition, I describe them briefly here and then discuss them in more detail in subsequent chapters.

Theoretical perspective. Larsen employed two theoretical perspectives: Realistic Mathematics Education (RME) and the emergent perspective (Cobb, 2000). The theory of Realistic Mathematics Education governed the instructional design while the emergent perspective provided a lens through which the individual and social cognitive processes of the students were interpreted.

Larsen made use of several theoretical RME constructs to support and interpret the reinvention process. Because of their relevance to this dissertation project, I list and briefly explain them here. More detail can be found in subsequent chapters. The relevant constructs include the reinvention principle, emergent models (Gravemeijer, 1999), mathematizing (Treffers, 1987), and the *record-of* to *tool-for* transition (Rasmussen, Marrongelle, & Keynes, 2003). The reinvention principle, more often referred to as guided reinvention, was the overarching idea around which the project was based. The goal of guided reinvention is for students to feel responsible for the mathematics in question. The concepts of group and group isomorphism were viewed as an emergent model, and this perspective was used to design instructional tasks which would elicit informal initial activity with the group structure in the form of an operation table, the idea being that these informal understandings could be harnessed and eventually transformed into formal mathematics. The term *mathematizing* (Treffers, 1987), which refers to the mathematical organization of a content area, was used to describe specific activity, both anticipated and actual, on the part of the students. The *record-of* to *tool-for* construct, a smaller-scale version of the emergent model's transition from informal to formal, helped to explain certain decisions made by the researcher during the teaching experiments as well as certain ideas and notations presented by the students.

Research design and methods. Larsen's goal was to produce an instructional theory supporting the guided reinvention of group and group isomorphism, and the following research questions (and their eventual answers) supported and informed the research design of the project. His research questions were as follows (p. 64):

- How can students reinvent the notions of group and group isomorphism?
- What is involved in developing the concepts of group and group isomorphism when the starting point is the students' own activity and knowledge?
- What kinds of informal knowledge and student strategies can serve as starting points?
- What kinds of mathematical activity can promote the evolution of students' informal knowledge and strategies into more powerful ways of thinking, symbolizing, and acting?

Larsen used a developmental research design (Gravemeijer, 1998) as a means of producing an instructional theory to support the guided reinvention of group and group isomorphism. This theory was tested by way of three iterations of the constructivist teaching experiment (Cobb, 2000). Each teaching experiment consisted of the researcher (serving as the instructor) and two students. The students were selected on the basis of availability and willingness to participate in the study.

Additionally, Larsen wanted to include mathematics and mathematics education majors as well as students who could be expected to both thrive and struggle in an introductory abstract algebra course

Initial local instructional theory. Larsen used the symmetries of an equilateral triangle as an experientially real starting point. Viewing the group structure as an emergent model, he conjectured that the group concept would emerge as a *model-of* the students' activity with the triangle. Furthermore, he anticipated the need for tasks which would promote the development of the group structure in the form of an operation table. Furthermore, Larsen "expected that the evolution of the group concept would lead to the creation of a new mathematical reality, in which generic groups were experientially real objects" (p. 69). He then planned to shift to activities which addressed the issue of whether two groups were the same. Upon reinvention of the definitions of both group and group isomorphism, Larsen assigned tasks in which the students would use their newly reinvented definitions in this new mathematical reality.

Teaching experiment activities. As stated in the initial local instructional theory, activities were initially centered around the symmetries of an equilateral triangle. The first instructional task engaged the students by having them physically manipulate a cutout of an equilateral triangle in order to generate a complete list of possible moves. The students were then prompted to create an operation table for all of the possible moves and asked what rules or properties they would need in order to perform a string of calculations (without directly referencing the operation table).

Subsequent activities involved similar tasks involving different geometric objects, including a square and a rectangle. Additionally, the students encountered both the groups S_3 (isomorphic to the group of symmetries of an equilateral triangle) and $Z_2 \times Z_2$ by means of the game "It's a SNAP" (Huetinck, 1996). The process used by Larsen and the students to turn the results of these activities into the definition of group is detailed below.

Relevant results. Results from Larsen's teaching experiments which have direct applications to this project are detailed in this section. Specifically, these results include how the issue of the associative property was addressed, the process used in the teaching experiments to define group, and how the group's binary operation was included and addressed.

Associativity. Larsen noted that the students did not seem to think that the associative law was necessary to include in their list of rules (the rationale behind the different steps taken to perform the symmetry calculations), even though the need for it appeared almost instantly (from his perspective) in each teaching experiment. This is evident in a dialogue excerpt from the second teaching experiment (p. 117):

> Erika: So then moving the parentheses around is really quite meaningless.
> Mary: Yeah, exactly!

Larsen addressed this issue by pointing out the student's implicit use of associativity, asking if it was necessary, and then encouraging them to include it in their list of rules. Furthermore, once the students in each experiment included the associative property in their list of rules, they discussed how this property governed the "order" of the calculations. In particular, all of the students tended to confuse the order in which the calculations were performed (associativity) with the order in which the moves were performed (commutativity).

Process of defining group. In each experiment, after the students had completed their lists of rules, Larsen asked the students to reduce them to a minimal set. The students responded by performing more informal calculations with their rules in order to eliminate those that were unnecessary and could be deduced from others. After the students had finished with these lists of reduced rules for each of the different activities, attention turned to defining. Larsen described the students' activity as follows:

> In each teaching experiment, I started the process by having the students identify the properties common to all of the situations they had considered. In each teaching experiment, the students quickly stated that in each case there was always an identity, everything had an inverse, and the associative property held. (p. 133)

Additionally, he used a cyclic process in order to support students' revising their attempts at the definition:

1. The students prepared a definition.
2. The moderator (Larsen) read and interpreted this definition, calling attention to particular choices made by the students.
3. The students revised their definitions as necessary and restarted the process.

Including and defining binary operation. Initially, the definitions submitted by all three pairs of students did not include any information about the group operation. Larsen responded to this by asking if anything else was needed, as in this excerpt from the third teaching experiment (p. 135):

SL: So here is my question. If you just have a set, can you have an associative property?
Kim: You have to do something. [same time as below]
SL: … Or is there something else involved?
Nancy: There must be something else involved.
Kim: We have to do something.
SL: Are plusses part of the set? You put plusses up there right?
Kim: That's what I was trying to say. You have to do something to the elements.

After the students included the operation in the definition, attention was turned to explicating what was meant by an operation. The students in the first and second teaching experiments realized that an operation was a function rather quickly, though they initially struggled to identify the domain of the function. Larsen addressed this by asking if the operation could be performed on one element. This prompted discussions which eventually led to the students identifying the set of ordered pairs as the domain.

The students in the third teaching experiment, on the other hand, did not define binary operation in terms of a function at all, as demonstrated by their final definition of binary operation: "A binary operation uses any two elements of a set and performs an operation where the outcome is one element of the set" (p. 138). Even after this was finished and Larsen asked them how they might be able to define it in terms of a function, the students struggled to identify the domain and range.

Conclusions. Larsen noted that his "project was not designed to investigate students' learning of the concepts of group and group isomorphism in general" (p. 199). Rather, the purposes of the conclusions he made regarding student learning of these concepts were to (1) inform his revised local instructional theory, and (2) make connections with the research literature. Specific conclusions made by Larsen which are relevant to ring theory and this project are listed here.

Meaning of the binary operation symbol. The students' often thought of the operation not as a binary operation but as a left to right sequential procedure, agreeing with Kieran's (1979) observations in elementary school students. In particular, "[it] seems more accurate to say that the students were thinking of an operation that links any (finite) number of movements to produce a result" (p. 202). This, in turn, was an obstruction to the motivation for the associative property. Larsen noted that this difficulty may have arisen from the context of symmetry in which the students were working.

Confusion of the associative and commutative properties. Agreeing with the previous findings of Findell (2002), Larsen's students also confused the associative and commutative laws. He noted that his students' confusion of the properties went beyond mere slips of the tongue, suggesting a deeper relationship between these two properties in the minds of students. For example, "[it] appeared to the students that the associative property meant that order did not matter, while the fact that the symmetries did not commute meant that order did matter" (p. 205). The students in the first and third teaching experiments were able to resolve the conflict by realizing that the associative property concerns the order of operations.

Use of operation tables. Larsen described the use of operations tables by the students in his teaching experiments as a *record-of* the notation systems in specific contexts. This, in turn, transitioned into a *tool-for* supporting future reasoning and abstraction, similar to the findings of Findell (2002). Furthermore, Larsen argued that his students' use of the operation tables supported and provided evidence for the *model-of* to *model-for* transition

Refined local instructional theory. In this section, I elaborate the portions of Larsen's refined instructional theory which are relevant to this project. Specifically, this includes all of the instructional theory up through the use of the reinvented definition of group. Thus, the portion of the instructional theory regarding group isomorphism has been omitted.

1. *An experientially real starting point*: the first example of a group should be experientially real to the students; preliminary instructional activities may be needed.
2. *The emergence of the group structure as a model-of*: The reinvention process should start with an activity in an experientially real context and should anticipate the integral aspects of the group structure.
3. *Mathematizing focused on the relational aspects of the model*: The students can mathematize their activity in the original task setting by highlighting the relational components of the students' activity.
4. *"Applying" the model to similar situations*: The beginning of the *model-of* to *model-for* transition can be brought about by having the students consider contexts similar to that of the original task setting.
5. *"Applying" the model in different contexts*: The *model-of* to *model-for* transition can continue by having students consider different yet structurally similar situations.

6. *Comparing the model to familiar non-examples*: Before the process of defining can start, it may be helpful to refine the ideas that comprise the model by having students consider non-examples.
7. *Formulating a definition of group*: The group structure, now having emerged as a *model-for* more formal activity, can be used to define the concept of group.
8. *Using the model for more formal reasoning*: The model should be used to have the students consider examples or non-examples; additionally, the students can consider more formal aspects of the group structure.

Limitations and generalizability. Larsen acknowledged several limitations of his study and cited as the most significant those which related to how such an instructional theory might be implemented in an actual classroom, where the instructional theory would be subject to constraints like shorter class periods, more small groups of students, and required syllabus material. He stated that further work with this local instructional theory should include a developmental research design in a classroom setting. Larsen also drew attention to the fact that all of his participants were female and were accustomed to working on mathematical content in small groups. Additionally, as a result of the large size of his data set, he addressed concerns regarding his lack of fine-grained data analysis.

Larsen stated that his conclusions are not generalizable in the sense that the phenomena he observed would manifest in any abstract algebra course, but rather generalizable in the sense of Clement's (2000) theoretical generalizability. This form of generalizability asserts that a theoretical model developed under a certain set of conditions may be used to explain behavior under a different set of conditions. Larsen added that “the findings of a study of this type do not substitute for an analysis of another similar situation, but can inform such an analysis” (Larsen, 2004, p. 249). In fact, it is this type of generalizability which makes Larsen’s local instructional theory so important: rather than a sequence of instructional tasks being the main result of his project, the local instructional theory is used so that it might have this theoretical generalizability.

Reinvention of quotient group. Larsen, Johnson, Rutherford, and Bartlo (2009) developed an instructional theory supporting the guided reinvention of quotient group. A natural corollary of Larsen’s (2004) dissertation, this study again used the instructional design theory of RME to create a local instructional theory supporting students’ reinvention of a topic from abstract algebra. Following the suggestion from Burn (1996) that the odd and even integers are the simplest example of a quotient group (and therefore somewhat accessible to students), the experientially real starting point involved students’ informal notions of parity. Using one of the groups from the instructional activities in his dissertation (the dihedral group of order 8), the students started the reinvention process by sorting elements of this group based on their intuitive understandings of parity derived from the even and odd integers.

Research design. The research design consisted of (1) a teaching experiment with a pair of undergraduate students that consisted of ten 60-90 minute sessions, and (2) two classroom implementations of the instructional theory which resulted from the teaching experiment. Data were analyzed using the same techniques of multiple iterative analysis used in Larsen (2004).

Initial local instructional theory. For the initial local instructional theory, Larsen et al. (2009) conjectured that the concept of quotient group would emerge as a result of the students' informal activity of searching for parity in the group of symmetries of a square. The students were expected to build upon this partitioning by considering whether it formed a group itself. Then they would be given prompts to generalize to more complex groups of partitions (such as four subsets). The researchers conjectured that, at this point, the students could be presented with examples and meaningful non-examples to determine exactly what conditions were needed for a partition to form a group. It was expected that the definition of normal subgroup would arise in response to this prompt. RME constructs to be used in this process include the reinvention principle, emergent models, and Larsen and Zandieh's (2007) *proofs and refutations*.

Refined local instructional theory. The refined local instructional theory supporting the reinvention of quotient group is given in five succinct steps:

1. Identifying evens and odds in a finite group;
2. Conjecturing and proving that one of the subsets must be a subgroup;
3. Generalizing to a more complex group of subsets;
4. Determining how to partition the rest of the group;
5. Finding a necessary condition for a partition of cosets to form a group.

Though these steps are listed with the corresponding student activity from the teaching experiment, the researchers do not supply rationale for the importance of the given steps. There are no other conclusions set forth in this paper.

Research on Student Learning of Group Theory

Many structural similarities exist between groups and rings (in fact, one might consider rings to be a special kind of group). Indeed, "the surprising fact about rings is that, despite their having *two* operations and being more complex than groups, their fundamental properties follow exactly the same pattern already laid out for groups" (Pinter, 2010, p. 169). To this end, it is reasonable to complement findings in the ring theory literature with group theory findings concerning ideas overlapping the two areas. Thus, in this section I discuss literature concerning binary operations, closure, operations tables, associativity, and commutativity. Before doing so, however, it is necessary to discuss a particular framework which guided the analysis and findings of many of the studies in the group theory literature: the APOS framework.

APOS, an acronym for action, process, object, and schema, is a theoretical perspective developed by Brown, DeVries, Dubinsky, and Thomas (1997) as an extension of Piaget's (1977) concept of reflective abstraction. The framework presents a method for interpreting students' mental progressions through certain topics. Students come to regard a concept as an *action* at first, in which their understanding is primarily procedural with minimal understanding of the mathematics at hand. This gradually evolves into a *process*, by which the procedural understanding is interiorized and can be used in larger and more general situations. When these processes are reflected upon, the student becomes aware of the objects as abstract and encapsulates them as *objects*. Objects can be operated on and transformed without specific attention paid to the underlying processes. Finally, the coordination of several similar concepts in this manner is called a *schema*. This framework was used in many studies to track the students' intellectual progress when learning a particular idea.

Binary operation and closure. Brown, DeVries, Dubinsky, and Thomas (1997) examined how students addressed the role of the binary operation in a group. They found that in order to fully understand the concept of group, students must understand what binary operations and sets are and the role that they play in the group structure. Unfortunately, students often struggle with the notion of an abstract binary operation. Findell (2001), summarizing student activity in his dissertation project, wrote that "the notion of an abstract binary operation presented notational, conceptual, and even linguistic issues" (p. 147).

The literature points to two student errors commonly associated with binary operations. The first is to simply ignore the role of the binary operation in a group and viewing the group as a set instead, the implications of which include ignoring the inner structure of the group that is a direct consequence of the binary operation (Iannone & Nardi, 2002). Secondly, students often view the binary operation as systematic left-to-right procedure (Dubinsky, Dautermann, Leron, & Zazkis, 1994). Note that such a view, apart from being inefficient, makes it difficult for students to see the need for the associative property. Dubinsky et al. (1994) concluded that an important developmental step in understanding the concept of group occurs when students begin to focus on the function aspect of a binary operation. Additionally, they established that this function conception of binary operation is a central connecting link between subgroup and group in the minds of students. Confirming this finding, Hazzan and Leron (1996) asserted that students often ignore the role of the binary operation when determining if a subset is a subgroup by focusing on student responses to the question: is Z_3 a subgroup of Z_6? Findell (2001) called this error *operation confusion*, referring to ignoring or incorrectly using a binary operation when more than one operation is available.

Operation tables. Findell (2001) established that operation tables were often a means used by students to manage the abstraction of the abstract group concept, noting that most of the group axioms could be observed in an operation table. In particular, a student of his, Wendy, used operation tables to verify that Z_6 under multiplication modulo 6 is not a group. Referring to the row of the element 2 in the operation table for multiplication modulo 6, Wendy stated: "…I have tried every element, 0, 1 … 0 through 5, multiplied by 2 to see if I can get the identity, 1, and I can't get it. So therefore, Z_6 is not a group under multiplication" (p. 136). Note that a similar line of reasoning in the context of ring theory could lead a student to see that Z_6 is also not a field.

Operation tables, in addition to displaying the failure of certain group axioms, can also "prove" that certain axioms are satisfied. Findell (2001) called this phenomenon *reasoning from the table*, in which "the group operation table serves metaphorically as the group, supporting students' thinking and reasoning" (p. 334). An excerpt from Findell's conversations with Wendy demonstrates her use of the operation table of addition modulo 6 for Z_6 to show that this is a closed system: "And then it's closed. You can see that there are no elements other than 0 through 5, looking at the chart, because we have all possible combinations on elements in Z_6. So it is closed also" (p. 139).

In a similar fashion, students might identify the identity and inverse axioms in the operation table. Commutativity is also visible, if applicable, through symmetry over the main diagonal (bottom left to upper right, if the table is conventional) of the operation table. When used in this manner, the operation table can be a powerful tool for making the abstract more concrete, but it is not without limitations. For example, one of the chief limitations is that the associative property is not visible in an operation table (Findell proposed that this might contribute to a confusion of associativity and commutativity, as noted below). Findell also found that, when relied upon too heavily, the operation table *became* the group for the students as opposed to being a helpful metaphor, hiding the underlying structure and concepts. He suggested that knowledge of these limitations is crucial for helping students to progress toward higher levels of abstraction.

Associativity and commutativity. Findell (2001) found that, even in an advanced mathematics course like abstract algebra, students often confuse and mentally tangle the associative and commutative properties, both in simple slips of the tongue and also in a manner beyond such surface level interpretations. Interestingly, he found that students often referred to associativity when they meant commutativity, but he found no evidence to support the opposite being true. One possible reason for this, he suggested, is that the most simple examples of noncommutative operations are also nonassociative, leading the two properties to be blended together into a single "order does not matter" property. He proposed that this is due to the commutative property being identifiable in an operation table, whereas the associative property is not. Larsen (2010), confirming Findell's findings, explained that the distinction lies in the fact that the commutative property governs the order in which moves are performed while the associative property governs the order of operations.

Research on Student Learning of Ring Theory

The research literature relating to student learning in ring theory is exceptionally sparse. Fukawa-Connelly (2007), for example, investigated whether an example-driven instructional method is more effective than the traditional lecture method in the contexts of an abstract algebra course (which involved ring theory). Since the overall goal of the paper was to evaluate teaching methods, however, little can be harvested from this paper in terms of student understanding of rings. There are other studies involving student activity with rings, but student understanding of rings, again, is not the primary focus of the paper (see, for example, Brenton & Edwards, 2007).

Despite the lack of ring-theoretic content in the educational literature, it is not difficult to argue that students have the same issues with rings as they do with groups. Considering that "rings are more complicated than groups" (Pinter, 2010, p. 169), it is not unreasonable to assume that students experience similar difficulties with rings as they do with groups. For example, since students have been found to experience difficulty with the abstract definition of a group (Dubinsky et al., 1994), it may be safely assumed that they might encounter similar difficulties with the definition of ring. Compounding this potential issue is the fact that "rings may also have 'optional features' which make them more versatile and interesting" (Pinter, 2010, p. 172). This brings to light a variety of different questions: why are some features "optional" and others not? Why are only certain multiplicative axioms optional while all of the additive axioms remain unchanged across the board?

Need for work exploring student learning of rings. Unfortunately, attempting to find effective methods of addressing these student questions from the research literature leaves many gaps and few answers. While much information can be gained from the group theory literature, more work needs to be done which directly addresses student learning of concepts in ring and field theory. Ring theory introduces notions that group theory, and the corresponding educational literature, are not able to adequately address.

Multiple structures. The primary structure of group theory is a group, of course, but ring theory has several very similar, yet decidedly different, structures upon which its foundation is built: ring, integral domain, and field. Furthermore, though these structures are axiomatically very similar, they are distinguished by several nontrivial properties. Thus, this project not only seeks to support the reinvention of these definitions, but also to investigate a means by which they can be differentiated by students.

Two operations. The definition of a ring requires two binary operations while the definition of a group requires just one. This might be considered the fundamental difference between the structures of ring and group. The presence of two binary operations produces several consequences. For example, the interaction of the two operations through the distributive property is a nontrivial component in the definition of ring which has no analog in group theory.

Units and zero-divisors. Due to their more complex structure, rings have features such as units and zero-divisors, which do not exist in group theory. Though the concept of unit (an element with a multiplicative inverse) is related to the concept of inverse in group theory, zero divisors do not appear (and, in fact, are not possible; if a zero divisor exists, then the structure is not a group) in group theory. Thus, no project which addresses only group theory is able to address these prominent ring theoretic features.

The distinctions listed above are ideas for which information about student learning can only be gained through direct examination. Unfortunately, little is known about these distinctive features that conclusively differentiate rings from groups.

Coming to understand a commutative ring: A case study. Simpson and Stehlikova's (2006) case study of how a student came to understand the commutative ring Z_{99} represents the sole article in the literature that directly addresses student learning in ring theory. The researchers also used the student's explorations of the structure to make a more general commentary on how students apprehend mathematical structure.

Discussion and debate. As a means of explaining the rationale behind the student activity observed during their study, the authors debated the merits of an examples-before-abstraction approach. While they acknowledged that attending to the abstract definitions before the examples may encourage students to work with more abstraction, many students are not able to comprehend abstraction right from the start. On the other hand, the examples-first approach encourages "students to attend to aspects of the particular which will appear as important facets of the general" (p. 349).

The examples-first approach, they noted, agrees with Skemp (1971), who stated that "it must first be ensured that these [examples] are already formed in the mind of the learner" (p. 350; from Skemp, 1971). In contrast, Dreyfus (1991) believed that concrete structures might obstruct the process to abstraction for certain students.

Shifts of attention. Simpson and Stehlikova noted that, in using the examples-first approach, there are five primary shifts of attention as students move gradually toward abstraction:

1. Seeing the elements in the set as objects upon which the operations act (which may involve a process-object shift).
2. Attending to the interrelationships between elements in the set which are consequences of the operations.
3. Seeing the signs used by the teacher in defining the abstract structure as abstractions of the objects and operations, and seeing the names of the relationships amongst signs as the names for the relationships amongst the objects and operations.
4. Seeing other sets and operations as examples of the general structure and as prototypical of the general structure.
5. Using the formal definition to derive consequences and seeing that the properties inherent in the theorems are properties of all examples.

Their study focused on the second shift in attention, for which they coined the term *apprehending structure.* While there are several levels on which shifts of attention can occur, this paper focused on the small scale shifts, such as how the interrelationships between the elements are connected as a result of the operations.

The study. This case study examines the process by which a female student, Molly, 'apprehended' a ring isomorphic to Z_{99} for her undergraduate thesis over a period of three years. The researchers presented this disguised (yet isomorphic) version of Z_{99} to Molly so that she would not immediately recognize it as a familiar one: she was given a set $\underline{A}_2=\{1, 2,\ldots,99\}$, the elements of which were termed *z*-numbers, and two operations $\otimes$ and $\oplus$, called *z*-multiplication and *z*-addition, respectively. The operations were given by $x\otimes y=r(xy)$ and $x\oplus y=r(x+y)$, where r is a reduction mapping on the natural numbers given by $r(n)=n-99\times[n/99]$ (where $[k]$ is the integer part of k). It was expected that Molly would eventually realize that this is an equivalent (isomorphic) realization of Z_{99}, yet she did not.

The researchers used Molly's encounters with inverse operations and zero-divisors to explain how she was able to apprehend this structure on a small scale. Though the researchers occasionally guided Molly in a particular direction, they attempted to influence her as little as possible. Thus, many of the activities in which she engaged were self-initiated. Her primary activities included solving basic linear and quadratic equations and finding Pythagorean triples in Z_{99}.

Inverse operations. Rather than attempt to define a *z*-subtraction, Molly initially used what the researchers called a "strategy of inverse reduction" in order to make use of her knowledge of ordinary arithmetic to solve basic additive equations. For example, to solve $x\oplus 25=6$, she reversed the reduction map to solve x+25=105 with ordinary arithmetic on the integers. Since she had avoided the expected route of development, the researchers prompted her to attempt to define a subtraction for this set. This was eventually done without issue. Notably, however, her use of these *z*-

numbers on a number line to explain subtraction indicates that she had some grasp of the cyclic nature of the structure.

Separately from the development of *z*-subtraction, however, she developed the notion of “opposite numbers” (her term for additive inverses), and concluded that every *z*-number has an opposite. This led to her use of additive inverse notation in basic arithmetic calculations (for example: expressing 4 minus 12 as $4 \oplus (99-12)$. In particular, she wrote:

> In the set of $\underline{A}_2$, classic negative numbers do not exist, that is why we will introduce *the opposite number* x' which will play a function of a negative number. … It holds $x \oplus x' = 99$, where $x' = 99 - x$, or *r(-x)=99-x*, where $x \in \underline{A}_2$. (p. 360)

She eventually used this as a technique to solve equations by defining *z*-subtraction as the addition of the opposite number. The authors used this to note that Molly had, by this point, undergone three shifts of attention between three phases:

1. The objects and operations used as in ordinary arithmetic
2. Explicit focuses on the relationships between the objects
3. Attention to *z*-subtraction and its relationship with the *z*-additive inverse

She proceeded through a similar series of steps with regards to *z*-division and multiplicative inverses.

Zero-divisors. Molly was not asked any specific questions about zero divisors. Instead the concept arose naturally from her work with the objects and operations. She started to notice their presence in the first few interviews with the researchers, noting that some of the objects behaved differently than others when solving multiplicative equations. In particular, she noticed that the elements 3, 6, 9, 11, 12, and 15 create multiple solutions to a multiplicative equation and create problems when trying to cancel by them. At one point, however, after noticing the presence of these zero-divisors, she attempted to use the zero-product property (equivalent to the absence of zero-divisors) to solve a multiplicative equation. Eventually, she began to discuss "divisors of 99" (p. 363) and wrote down part of a solution to the quadratic equation $x \otimes (a \otimes x \oplus b) = 99$ which takes zero divisors into account. For example, this (reproduced) excerpt from her written work shows her acceptance of the presence of zero-divisors: $x \otimes (a \otimes x \oplus b) = 99 \Rightarrow x = 3 \wedge a \otimes x \oplus b = 33$. The researchers noted that this followed a trajectory similar to her discovery of additive inverses: from the implicit to the explicit. In this way,

> a fully worked out example is meant to prepare the student for subsequent shifts of attention, first to the definitional properties of the abstract structure and then to proofs and theorems as simultaneously general and applicable to all examples of the structure. (p. 364)

While the researchers do not go so far as to assert that an examples-first pedagogy is more effective than one relying heavily on abstraction, the above statement asserted their support (and rationale) for such a method.

Conclusion. The researchers suggested that an examples-first instructional method should not be unguided. Rather, it should be a process of joint attention, with the teacher guiding attention to those aspects of the structure which will be abstracted and then identifying the relationships between the abstract definition and the particular example. Despite substantial differences, most notably that the student had encountered the formal definition of a ring previously and thus did not attempt to reinvent a definition, this is an approach which is compatible with the guidelines and heuristics of RME, setting a precedent for a guided discovery and examples-first approach in ring theory.

Research Related to Other Issues Involved in the Teaching and Learning of Algebra

Additional research exists that does not deal with specific topics in abstract algebra but rather with issues involving the subject in general. Specifically, issues from abstract algebra relevant to this project include student understanding of equation solving, student efforts to reduce abstraction in abstract algebra, and common errors made by students when constructing proofs.

Lack of conceptual understanding of equation solving. The focus of most elementary algebra courses is often the task of solving basic linear equations. Kieran (1992) identified six primary techniques students use to find solutions to such equations:

1. just knowing the answer outright (known facts),
2. counting techniques,
3. guess and check,
4. covering up one side of the equation,
5. working backwards, and
6. formal operations (with equivalent equations).

Linsell (2009) posited that the method of formal operations, which involves transforming the given equation into a sequence of equivalent equations, is the most sophisticated because it requires due attention to the algebraic features of the structure on which the equations are being solved. Unsurprisingly, Linsell also concluded that many students do not understand the process of transforming the given equation into a sequence of equivalent equations as anything more than an algorithm to be iterated until a solution is reached (hence ignoring the structure and properties used). This finding is corroborated in Herscovics and Kieran (1999).

Additionally, Sfard and Linchevski (1994) found that students may see the need for operating on the equation in this manner until presented with an equation that has x on both sides, such as $x+3=2x-7$. Note that unless the method of equivalent equations is used, it is likely that the structural aspects of the process are lost in favor of simply finding a solution. Recognizing and calling attention to this disparity, Wagner and Parker (1999) stated that "few students fully appreciate the fact that solving an equation is finding the value(s) of the variable for which the left-hand and right-hand sides are equal" (p. 333). Given that equation solving was one of the intellectual antecedents of modern algebra (Kleiner, 1999), the literature here underscores the inconsistency between the importance of equation solving in mathematics along with its diluted manifestation in the minds of students.

Reducing abstraction. In a study aimed at explaining how undergraduate students cope with abstract algebra concepts, Hazzan (1999) found that student actions can often be explained and interpreted as a means to *reduce abstraction*. She argued that this coping mechanism helps students to deal with the abstraction in an abstract algebra class by making the concepts more mentally accessible. In some cases, reducing abstraction is an effective strategy used on the way towards a complete understanding. Hazzan noted, however, that reducing abstraction can also be misleading to students. The argument for encouraging the reduction of abstraction lies in providing students with a concrete foundation in examples so that they better understand exactly what is being abstracted. In fact, this is an approach supported and used by several popular abstract algebra textbooks (Gallian, 1998; Herstein, 1996; cited in Hazzan, 1999). As a result of reviewing the educational literature, Hazzan set forth three primary types of reducing abstraction that would serve as the foundation for her framework.

Abstraction level as the quality of the relationships between the object of thought and the thinking person. Hazzan cast the concreteness or abstractness of an object as a function of the person's relationship with it as opposed to abstraction being a property of the object itself. In other words, "the closer a person is to an object and the more connections he/she has formed to it, the more concrete (and the less abstract) he/she feels to it" (p. 76). This perspective on abstraction can be used to make sense of students' tendencies to base argument on familiar mathematical entities. In this way, students often completely ignore the meaning of the situation in the problem in favor of the familiar. For example, students tend to rely on their familiar knowledge systems of numbers (such as the real numbers) when constructing arguments involving, for example, permutations or symmetries. Hazzan tied this method of reducing abstraction to a fundamental tenet of constructivism: new knowledge is constructed from existing knowledge.

Abstraction level as reflection of the process-object duality. Though there were (and still are) many existing theories of how a concept transitions from process to object (such as Dubinsky's aforementioned APOS framework), Hazzan characterized a means of reducing abstraction reflective of a process conception of an idea. Namely, it was observed that students often personalize the language of formal logic and mathematics. For example, instead of saying, "there exists a function *f* such that …" students might say "*I* can find a function *f* such that …". Secondly, students were found to resort to canonical procedures to solve problems, even when such procedures were inappropriate or ineffective. Hazzan argues that, since both of these types of actions are intended to make the concept more personal (and thereby more concrete), these are both process conceptions by which students reduce abstraction (recall, however, that not all means of reducing abstraction are effective or even correct).

Abstraction level as the degree of complexity of the concept of thought. This method of reducing abstraction is predicated on the assertion that "the more compound an entity is, the more abstract it is" (p. 82). Students reducing abstraction in this manner often reduce the complexity of a set (such as the set of all groups of prime order) with one of its elements (such as the group Z_5 under addition modulo 5). This, of course, has a great potential for error, as set operations are fundamentally different than operations on specific elements.

Research related to proof in advanced mathematics courses. Since abstract algebra is a proof-based course, I now explicate current relevant knowledge regarding student construction of proof, starting with common student errors in writing proofs and proceeding to strategies.

Common student errors in constructing proofs. While there is a significant body of literature concerning these aspects of proof, Selden and Selden's (1987) *Errors and Misconceptions in College Level Theorem Proving* provides evidence of certain errors specific to abstract algebra courses. Errors which are relevant to this study are discussed here.

Real number laws are universal. Students tend to assume that the properties of the real numbers hold for all number systems. Selden and Selden suggest that this is because "students who take abstract algebra at the junior level have very little idea that mathematics deals with objects other than geometric configurations and real and complex numbers" (p. 8). In response to this deficiency, they proposed that the given examples be simple (such as real numbers, complex numbers, and matrices) yet diverse enough to demonstrate that not everything behaves the way the students might expect.

Ignoring and extending quantifiers. Unsurprisingly, students tend to misuse and ignore the quantifiers "for all" and "there exists." In particular, "often a variable is thought to be universally quantified when it [is not]" (p. 14). Furthermore, students often reverse the order of the quantifiers and see no distinction (Selden & Selden, 1995). For instance, students might not recognize the different between "there exists a 0 in R such that for all x in r, $x+0=0+x=x$" and "for all x in R there exists a 0 in R such that for all x in r, $x+0=0+x=x$." Findell (2001) confirmed this finding and stated that even students who had a comfortable grasp of informal definitions of a concept had trouble stating the corresponding formal mathematical definitions, in no small part due to difficulty with quantifiers. Though the differences are subtle, the logical ramifications of such an error are enough to derail the validity of a proof or statement.

Holes in the proof. This error often entails the student believing a claim to follow immediately from a previous statement when in fact it does not. Selden and Selden (1987) cited an example of this as follows: $k^r \in H$ implies $k \in H$. While there exist conditions under which such a statement is true, it is in general false (for example, $\left(\sqrt{2}\right)^2$ is a rational number, but $\sqrt{2}$ is not).

The four types of mathematical proof. Weber (2002) set forth four basic types of mathematical proofs that either convince, explain, illustrate technique, or justify structure. Those justifying structure are proofs in which "the assumptions that one is making are questionable, but the conclusion is regarded as obvious" (p. 15). He stated that most formal mathematical systems arise intuitively, yet the mathematical community reaches a point for which this intuitive reasoning is no longer acceptable and must be replaced by rigorous proof. Such a proof would, in turn, justify the inclusion of certain axioms in the definition of a mathematical structure. Weber cited Peano's proof that 2+2=4 as an example. Despite the fact that proofs that justify structure, like Peano's proof, are often "lengthy, technical, difficult and decidedly non-intuitive" (p. 15), these proofs may have the potential for pedagogical use. Larsen and Zandieh (2007) suggested that proofs of certain results could be used as a means by which students can justify the inclusion of certain axioms in definitions as part of a reinvention process. This could also take place after students have reinvented (or been presented with) a definition: Larsen (2004) asked his teaching experiment students to prove the uniqueness of the identity and the group cancellation law after the reinvention of the group definition had taken place. Presumably, the goal of these tasks was to provide the students with an opportunity to use their newly reinvented definitions as well as to affirm for the students their inclusion of several of the axioms in the definition (though Larsen made no direct mention of the latter).

Conclusion

Though the literature involving abstract algebra as a whole has seen a flurry of activity in recent years, the literature concerning student learning of rings and fields is exceptionally scarce. In this chapter, I discussed how this can partially, yet not completely, be addressed by the group theory literature. As it stands, the currently available resources in the research literature do not provide adequate insight into how students learn the fundamental notions of ring theory. Indeed, the disparity between the high status of rings in mathematics (and their resulting importance in the undergraduate curriculum) with the lack of educational research to address student difficulty in this area represents a significant problem. And even though research involving discovery-based alternatives to the lecture method exist in group theory, there are no such established methods in ring theory. This dissertation study aims to begin addressing both of these discrepancies by building upon Larsen's (2004, 2009) reinvention efforts and using the idea of equation solving as a means of discovering structure similar to Simpson and Stehlikova (2006).

Chapter 3: Theoretical Framework

Several guiding frameworks were very influential throughout the different stages of this project. Epistemologically, I identify myself as a constructivist (Piaget, 1971). Using Realistic Mathematics Education (RME) as my theoretical perspective and guide for instructional design, I adopted the constructivist teaching experiment methodology (Cobb, 2000; Steffe & Thompson, 2000) as a means of testing my research hypotheses. These notions, as well as my rationale for their inclusion and influence in this project, are discussed in this chapter.

Epistemological Stance: Constructivism

I identify myself as a constructivist in the sense of Piaget (1971) due to my belief that knowledge is constructed by the learner, and that coming to acquire knowledge is "a process of continual construction and reorganization" (p. 2). Indeed, constructivism asserts that all knowledge is constructed by tools resulting from a developmental construction, as opposed to the view that learners passively receive knowledge transferred from an expert. This active construction of knowledge involves both a foundation from which knowledge can be constructed, known as *assimilation*, and a constructive process of formation. This is also accompanied by a method by which the structure of knowledge is revised, known as *accommodation* (Noddings, 1990). Piaget (1977) noted two other constructs that help to describe the constructivist view of learning:

- equilibration: the continual mental balancing between the two acts of assimilation and accommodation, and

- schemes: mental constructions and representations of associated or related thoughts.

Constructivism asserts that learning occurs from these adaptive measures by which we attempt to assimilate and accommodate new information. Also central to the constructivist epistemology is reflective abstraction, the process by which an individual looks back in an attempt to achieve cognitive equilibrium, which is a crucial component of the construction of mathematical knowledge. In agreement with constructivism, I believe that mathematics was created by, and not independent from, human beings. This is in stark contrast to the view of mathematics as a set of objective truths which exist independently of the human mind.

Teaching with a constructivist outlook has critical implications, as it implies a method of teaching which recognizes students as active learners (Noddings, 1990). Confrey (1990) effectively characterized how constructivism guides his teaching:

> As a constructivist, when I teach mathematics I am not teaching students about the mathematical structures which underlie objects in the world; I am teaching them how to develop their cognition, how to see the world through a set of quantitative lenses which I believe provide a way of making sense of the world, how to reflect on those lenses to create more and more powerful lenses and how to appreciate the role these lenses play in the development of their culture. I am trying to teach them to use one tool of their intellect, mathematics. (p. 110)

He went on to state the goals for a constructivist instructor:

> An instructor should promote and encourage the development for each individual within his/her class of a repertoire for powerful mathematical constructions for posing, constructing, exploring, solving and justifying mathematical problems and concepts and should seek to develop in students the capacity to reflect on and evaluate the quality of their constructions. (p. 112)

Confrey asserted that the acceptance of such goals implies three assumptions:

1. Teachers need to build models of students' understanding of mathematics.
2. Instruction is naturally interactive; as such, instructors need to construct a tentative, hypothetical path upon which the students might proceed to construct a mathematical concept. Such a path should be flexible with regards to the ideas of the students.
3. The student must decide on the adequacy of his or her construction.

In this way, my constructivist beliefs informed and are consistent with my choice of theoretical perspective, Realistic Mathematics Education (RME). The primary tenet of RME is that students be given opportunities to reinvent (or construct) the mathematics at hand for themselves. In accordance with Confrey's (1990) second assumption (above), the goal of many RME research projects, in fact, is the constitution of a domain-specific instructional theory which supports students' reinvention of a particular concept. While this is discussed in more detail in the next section, constructivism and the reinvention-minded approach of RME are undeniably compatible.

Second, in my interactions with my student participants, I tried to create a dynamic by which they would be challenged to assimilate and accommodate different ideas into their cognitive schemas. Steffe (1991) explicated a list of recommendations for constructivist researchers in mathematics education, three of which are of crucial relevance to this project:

- learn how to engage students in goal-oriented mathematical activity,

- learn how to encourage reflection and abstraction in a goal-oriented context, and
- learn the content of the mathematical experience of the students.

These recommendations, following from the central tenets of constructivism, guided the process by which I facilitated my interactions with the students. One design heuristic which was particularly helpful in this regard was Larsen and Zandieh's (2007) method of *proofs and refutations*.

Theoretical Perspective: Realistic Mathematics Education

My epistemological stance as a constructivist informed my choice of the theoretical perspective of Realistic Mathematics Education (RME). RME is an approach to mathematics education championed by Freudenthal (1971, 1973) which encouraged the idea that "mathematics can and should be learned on one's own authority and through one's own mental activities" (Gravemeijer, 1999). In agreement with the tenets of constructivism, RME suggests mathematics to be a human activity in which the concepts should become *experientially real* to the student. The primary focus of using an RME perspective is for students to experience formal mathematics in the same way they experience informal mathematics (Gravemeijer, 2000). In fact, the purpose of RME-themed research is to create and analyze instructional sequences which are consistent with this focus (Gravemeijer, Cobb, Bowers, & Whitenack, 2000).

Consistent with his belief that "mathematical activity is . . . an activity of organizing fields of experience" (p. 123), Freudenthal (1973) characterized the reinvention process using a construct he termed mathematizing, the organizing of a mathematical domain. Freudenthal (1971) posited four reasons for which students might engage in mathematizing:

- To generalize: this might be obtained through a structuring or classifying process.
- To achieve certainty: the process of testing and proving conjectures.
- To be exact: examples could include the defining process.
- To be concise: this may involve creating a set of standard procedures and notations.

It is clear from Freudenthal's list, then, that tasks be designed specifically to evoke these types of reasoning from students.

Freudenthal (1973) distinguished between two types of mathematizing: the mathematization of informal activities and intuitions and the mathematization of actual mathematical activity. Treffers (1987) advanced the notion of mathematizing by expounding on Freudenthal's distinction because he believed that it underscores the importance of expanding one's mathematical landscape and continually raising one's mathematical level. For these types of mathematical activity he coined the terms horizontal and vertical mathematization, respectively. In more detail, horizontal mathematization is defined as the process of transforming a starting-point problem or situation into mathematical terms. Vertical mathematization, then, occurs when these starting-points become the subject of further mathematizing. One might view horizontal mathematization as the establishing of an informal mathematical reality to address a problem-specific situation. Vertical mathematization would then be present in any steps taken to advance the informal mathematical reality towards more formal lines of reasoning.

The central tenets of RME and the notion of mathematization provide a general overview of how the reinvention process might take place, but it does not discuss how specific instructional theories or tasks could be designed to support such a process. Prominent in this project are the RME constructs of the principle of guided reinvention, emergent models (Gravemeijer, 1999), and the process of *proofs and refutations* (Larsen & Zandieh, 2007).

The principle of guided reinvention. Gravemeijer (2000) stated the driving force behind the reinvention principle is "to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (p. 159). Freudenthal (1973) suggested that instructional designers ask the question: How might I have been able to invent this myself? Additionally, students' informal problem solving strategies and the history of mathematics can be used as sources of inspiration for instructional design. The historical development in particular can often be a rich source of information. For example, Larsen (2004) designed instructional tasks which centered on symmetries of regular polygons, the historical antecedent of the concept of group, in his development of an instructional theory to support the reinvention of group.

The reinvention principle was crucial to this project because the chief goal revolves around students reinventing the definitions of ring, integral domain, and field. Taking Freudenthal's (1973) advice concerning the potentially rich source of the historical development of the subject, I am using equation solving—noted by Kleiner

(1999) as helping to bring about the axiomatic definition of the field structure—as a means by which students can reinvent the desired definitions. Indeed, equation solving was a historical antecedent of the development of axiomatic algebraic structures, particularly rings and fields. Equation solving, in addition to being a familiar, experientially real mathematical activity to the students, can also serve to motivate and distinguish the basic ring structures. More detail for how equation solving was utilized can be found later in subsequent sections (see, for example, the section in the next chapter dedicated to the initial local instructional theory and the corresponding instructional tasks).

Emergent models. Contrary to the use of models in other approaches, in RME the models are not constructed from the intended mathematics. Rather, they are steeped in contextual problems, and the model emerges from the organizing of these contextual problems. In fact, students oftentimes develop informal procedures of solving context problems which anticipate and serve as a guide for more formal mathematical activity (Gravemeijer, 2000). Thus, the model mediates a shift between informal mathematical activity to a new, more formal mathematical reality. Initially, the model emerges as a *model of* the student's informal mathematical activity, eventually becoming a *model for* more formal mathematics. This is referred to in the literature as the model-of to model-for transition. To further explicate the means by which models-of become models-for, Gravemeijer (1998) delineated the process into a sequence of four phases:

1. The *situational* phase involves (students) working to achieve mathematical goals in an experientially real context.

2. The *referential* phase includes models-of that refer to previous activity in the original task setting.
3. The *general* phase is characterized by models-for that support interpretations independent of the original task setting.
4. The *formal* phase entails student activity that reflects the emergence of a new mathematical reality.

Thus, the model-of to model-for transition occurs between the referential and general phases of the emergence of the model and parallels the shift from reference to the task setting to being completely independent of it. Importantly, once the formal stage has been reached, the student is no longer dependent upon the model. Larsen (2004) provided a characteristic example of how emergent models might be used in the context of reinvention in abstract algebra:

> The starting point for the instructional sequence was the context of the symmetries of an equilateral triangle. The students were to be given a series of tasks in this context (and in similar contexts) with the goal of promoting the emergence of the group concept as a *model of* [emphasis added] the students' activities. Further tasks were to be created to promote the development of productive ways of representing this model including the use of operation tables. It was expected that the evolution of the group concept would lead to the creation of a new mathematical reality, in which generic groups were experientially real objects. (p. 69)

Though it is not addressed in Gravemeijer (1998), it is important to note that instructional tasks may not be completely contained in a single phase (see, for example, Zandieh & Rasmussen, 2010). Furthermore, even though the progression through the four phases is largely sequential, it appears that students may not experience them linearly. For example, some phases may be more heavily "weighted" than others. I viewed these phases as a continuous progression wherein activity within one phase would gradually progress toward the next. Because of the tendency for informal procedures to "anticipate" the emergence of more formal mathematical reasoning (Streefland, 1991), I argue that the progressive formalization within each phase *anticipates* the next. Gravemeijer's four phases can then be expanded (if needed) to accommodate and provide more detail by making use of three *anticipatory*, intermediate phases, occurring between (and yet not necessarily disjointed from) the four original phases. Namely, I introduce and define the following:

- The *situational anticipating referential* phase involves activity still firmly rooted in the original situational setting that lays the groundwork for future referential activity.
- The *referential anticipating general* phase is characterized by models-of that provide an overview of previous work in preparation for abstract or general activity.
- The *general anticipating formal* phase includes models-for which promote more efficient or concise use of the mathematics at hand in preparation for formal use.

I used these seven phases as a lens through which I interpret the results of the teaching experiments and identify the significant milestones of the reinvention process. This, in turn, provided a means by which I began to answer my research questions. Furthermore, it informs the creation of the emerging instructional theory being developed to support the reinvention of ring, integral domain, and field.

Relation to mathematizing. The notion of mathematizing provides a nice analog through which the emergent model transition may be viewed. Horizontal mathematization is analogous to the model emerging as a model-of the student's informal mathematical activity, while the transition to vertical mathematizing is comparable to the model becoming a model-for more formal mathematical activity. In fact, Treffers (1987) acknowledges how mathematization facilitates this transition even before the emergent models construct was formally defined when he wrote that "initial problems become the *model for* [emphasis added] the solution of new problems" (p. 53). In essence, the model (and its transition from model-of to model-for) arises as a result of the student's mathematizing. In fact, a useful criterion when determining the possible effectiveness of a model is its potential for promoting vertical mathematizing (Gravemeijer, 1999).

Chains of signification. Instead of a single model, an emergent model may be viewed as a series of signs in a chain of signification (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer, 1999). In short, a chain of signification initiates when a symbolic representation is used to represent activity with a set of symbols developed beforehand. A series of such symbolic "replacements" forms the chain of signification, which links the original representation with a more formal one. In this way, a chain of signification may be viewed as a series of signs in which each sign lays the foundation for its successor, which uses the original sign but in a more formal environment. This possibly gives the sign a different, more general meaning. As Gravemeijer (2004) noted, "there is not *one* model in a process of emergent modeling but a series of symbolizations or sub-models that together constitute 'the model'" (p. 11). In this way, the term 'model' in RME is flexible; a model could be a diagram, notation, symbol, or activity.

Record-of to tool-for. Not every transition constitutes the appearance of a model-for, however. Gravemeijer (1998) asserted that the transition to the model-for must also reflect the emergence of a new mathematical reality. Otherwise, he noted, every act of defining notation would qualify. A construct derivate of the model-of to model-for transition that addresses similar transitions that occur on a smaller scale is the *record-of* to *tool-for* transition (Rasmussen, Marrongelle, & Keynes, 2003). Using this construct, a form of notation that is used to record or reflect student reasoning is said to be a *record of* their informal activity, while serving as a supporting *tool for* additional mathematical ideas. This construct provides a nice lens through which to detail a transition made by the students without concerns about whether it is accompanied by the emergence of a new mathematical reality. For example, Larsen (2004) uses the *record-of* to *tool-for* construct to describe his students' use of a group operation table, which served as a *record-of* their informal work with the symmetries of a triangle and transformed into a *tool-for* defining the group concept and recognizing group isomorphism.

Applications to this project. For the purposes of this project, I view the activity of equation solving as an emergent model. Equation solving provides a means by which students can interact informally in an experientially real setting with a structure endowed with addition and multiplication (more specifically, a ring). I anticipated that instead of consisting of one model, the emergent model might be comprised of many signs in a chain of signification. The identification of the use of a ring axiom in the solving of an equation might be the first sign in a chain of signification (for example, -*a* might be used to denote the presence of an additive inverse). This notation might pave the way for statements of the axiom itself which become gradually more formal. Thus, what initially arises as a simple piece of notation in the solving of an equation can evolve into a critical component of the targeted definition of ring. The emergent model of equation solving, in this context, would consist of all of the similar chains of signification resulting from the students' coming to terms with each of the different ring and field axioms. Additionally, Gravemeijer's (1998) four phases of the emergence of the model provide a means by which I can interpret the results of the teaching experiments. More information regarding how the instructional theory and tasks were designed using the emergent models construct is available in the statement of the local instructional theory stated later in the next chapter. I use Rasmussen et al.'s (2003) *record-of* to *tool-for* construct to describe significant transitions on a smaller level brought about by a piece of notation or organization of ideas.

Proofs and refutations. Lakatos, in his 1976 book *Proofs and Refutations*, noted that the presentation of mathematics with carefully stated axioms, definitions, and proofs effectively hides the process by which the mathematics was discovered. He outlines several methods of mathematical discovery by which students can be led to discover mathematics for themselves by describing the process in which students and mathematicians formulate conjectures. Lakatos describes the various methods of dealing with counterexamples to proposed conjectures as monster-barring, exception barring, and proof-analysis:

- monster-barring, the outright rejection of a proposed counterexample as invalid by concocting definitions so that the counterexample is not considered to be relevant,
- exception-barring, the listing of exceptions to a conjecture to find an acceptable domain, and
- proof analysis, the most sophisticated method of discovery which focuses on modification of a proposed conjecture so that the given proof will work.

Larsen and Zandieh (2007) suggest that Lakatos' methods of mathematical discovery can be reformulated as a research framework to support reinvention, noting that the ideas presented in *Proofs and Refutations* are consistent with the views of Freudenthal (1971, 1973) that mathematics is a human activity. Larsen and Zandieh reframe Lakatos' three methods of discovery in ways that could be useful to facilitators of reinvention projects as follows:

- monster-barring, the modification of a definition to exclude a counterexample,
- exception-barring, the modification of the proposed conjecture (with no attention given to the proof), and
- proof-analysis: the modification of the conjecture so that the previously proposed proof is valid.

To exemplify each of these methods, they provide an example from an introductory abstract algebra class in which three students, Phil, Steve, and Mike, attempt to generate a conjecture concerning the smallest set of conditions for which a subset of a group is a subgroup. The three students initially assert that a subset of group need only have closure under the group operation to be a subgroup. One of the students, Phil, provides a "proof" of this conjecture. The authors note that the student does not initially acknowledge hidden assumptions made by this proof. The instructor then engages the students in discussion about their conjecture and proof.

Monster-barring. The instructor immediately supplies a counterexample: the positive real numbers under addition. The students' first reaction is to posit that this example does not meet the hypotheses of the conjecture:

Phil:	I forgot to say that it has to have the same group operation.
Teacher:	I didn't change the operation.
Mike:	It's not closed.
Teacher:	Are you sure?

Despite the lack of discussion about an underlying definition, this qualifies as monster-barring (and not the closely-related exception-barring) because it deals solely with the counterexample and not the conjecture itself or the proof.

Exception-barring. In another conversation with the teacher, the students begin modifying their conjecture:

Phil:	Not a subgroup because don't have inverses.
Teacher:	You didn't say I had to have inverses; you said I only had to be closed.
Steve:	He's right.
Phil:	[I'm] trying to think of a way around it.
Steve:	So it's inverses and closure.

Though they are very close to the correct answer, this exchange only qualifies as exception-barring as they are concerned primarily with modifying the conjecture and have yet to address the proposed proof.

Proof analysis. Eventually the students' attention returns to their original conjecture and proof. Phil provides the correct alteration of the original conjecture which makes the proof work (to verify that this was indeed proof analysis, a teaching assistant conferred with him and he explained it works with his proof):

> Phil: If you're talking about an infinite group—you were talking about finite groups before so maybe there's a couple of different cases. If it's finite then you only need closure.

Larsen and Zandieh note that, while proof-analysis is the most sophisticated and comprehensive technique, both monster-barring and exception-barring can still be useful. First, recall that the three students in the above episode successfully improved their conjecture by exception-barring. Additionally, monster-barring might be a useful tool when constructing the needed underlying definitions. For example, should students still be unclear as to the specifications of a definition, the instructor can propose they attempt to conjecture about the nature of certain presented patterns.

Applications to this project. The *proofs and refutations* framework was useful in this research project because it motivated the instructional design and provided me with a means of anticipating and addressing conflicts in the reinvention process. In regards to the instructional design, Larsen and Zandieh (2007) suggested that:

> [The instructor] could begin by identifying important mathematical results that depend on this particular concept (i.e., what proof might be able to generate this concept). Then instruction could be designed to evoke one or more of these results in the form of a primitive conjecture. The students could be asked to propose an argument to support the conjecture, or the teacher could propose one. The students could then be asked whether the conjecture is always true and encouraged to look for counterexamples, or the teacher could propose counterexamples. As the students respond to these counterexamples, they should be encouraged to focus on both the proof and the counterexamples so that through a proof analysis they discover what condition is necessary to make the proof work and as a result reinvent the desired concept. (p. 215)

The mathematical results I used to design the instructional tasks were the additive and multiplicative cancellation laws. I anticipated that the students would devise a primitive conjecture (for example, that the multiplicative cancellation law holds on Z_{12}) and then attempt to prove it. In the case of this example, a proof analysis would reveal that the cancellation law only holds for a small subset of values in Z_{12}. I expected that engaging in proof analysis for proofs of the cancellation laws across a variety of examples of rings would be an effective means of highlighting the ring axioms as well as the features which differentiate rings, integral domains, and fields.

In regards to interaction with the students in the sessions, I used this framework to acknowledge and address issues which develop in order to promote critical thinking on the parts of the students rather than resorting to oversimplified and non-insightful comments such as "that is incorrect." In other words, it allowed me, as the facilitator of the reinvention process, to encourage the students to engage their conflicts mathematically as opposed to simply being told that their work is incorrect. It is not difficult to see how such an approach can be useful in the context of reinventing the definition of a ring. For example, should the students venture a proof which assumes that the zero-product property ($ab = 0$ implies $a = 0$ or $b = 0$) holds for all rings, I can engage them in proofs and refutations by having them examine how the proof might play out in a ring with zero divisors in such rings as Z_{12} (with the familiar addition and multiplication modulo 12). Then, after considering the counterexamples, the students can proceed with proof analysis in an effort to correct their proof.

Methodology: The Constructivist Teaching Experiment

Following Gravemeijer's (1995) suggestion that it is an effective means of producing an instructional theory for Realistic Mathematics Education, I have chosen to adopt the *constructivist teaching experiment* methodology, the primary purpose of which is to examine students' knowledge and how it might be assessed through the teaching of mathematics (Cobb, 2000). Though this methodology was originally devised for teaching mathematics to children, the teaching experiment methodology has recently been applied to dissertation projects at the collegiate level as well (see, for example: Larsen, 2004; Swinyard, 2008). Recall that the purpose of this dissertation project is to develop an instructional theory supporting the guided reinvention of concepts from introductory ring and field theory. In alignment with this goal, Gravemeijer (1995) noted that the development of instructional sequences and their underpinning instructional theories can be effectively addressed by the teaching experiment methodology

Components of a teaching experiment. Steffe and Thompson (2000) gave the following components of a teaching experiment:

- a series of teaching episodes,
- a teacher (who, in the constructivist teaching experiment, is also the researcher),
- one or more students,
- a witness, and
- a method of recording the events of the teaching episode.

The recordings of the episodes may be used for preparation for future sessions as well as for a retrospective analysis. Teaching experiments served as my method of interaction with the participants in this project. In keeping with the guidelines of the constructivist teaching experiment, I served as the teacher and facilitator of each session. Unfortunately, I was not able to have a witness for the teaching episodes.

Role of the researcher. A distinctive feature of the constructivist teaching experiment is that the researcher acts as the teacher. Being immersed in the experiment allows the researchers to promote students' reflection upon their mathematical experiences by administering exploration in a context similar to one already encountered. This is opposed to an experiment with a set agenda, wherein a researcher could act as a passive observer of teaching episodes. Rather, since the interest lies in hypothesizing what might be learned and creating ways to promote this reflection, the researcher must be immersed in this necessarily flexible process (Steffe, 1991). Steffe and Thompson (2000) contended that this allows the researcher to be continuously constructing a model of student thinking about the desired concept. The researcher should be engaged in bringing forth the learning schemes constructed by the students and, importantly, use these schemes to formulate research hypotheses. It is important to note here that, though the evolution of an idea through the course of a teaching experiment as a result of student activity is inexorably unpredictable, the researcher must still formulate, test, and revise a set of research hypotheses.

Testing research hypotheses. Since the word "experiment" in teaching experiment is meant in the scientific sense, Steffe and Thompson highlighted the need for (1) formulating hypotheses before the start of the teaching experiment, and (2) generating and testing of hypotheses while conducting the experiment. In regards to the former, they admonished that "one does not embark on the intensive work of a teaching experiment without having major research hypotheses to test" (p. 277).

These hypotheses help to devise the overall goals of the project and to select the participants. Somewhat ironically, Steffe and Thompson advised that researchers would do best to "forget" these hypotheses during the actual conducting of the experiment, so as not to artificially introduce bias. In fact, they liken this forgetting of the hypotheses to the process of justification in mathematics: the mathematician does not force any conclusions, (s)he focuses on what actually happens in the given situation. Of course, hypotheses may be devised during the teaching experiment as well, and it is these hypotheses which are tested in subsequent sessions.

Application to this project. I use the guidelines of the constructivist teaching experiment in order to engage the students mathematically in an effort to answer my research questions related to the guided reinvention of the definitions of ring, integral domain, and field. In addition to an exploratory pilot study[1], I conducted two teaching experiments with two students each.

Contrary to what it might seem, my goal in the teaching experiments was *not* to have the students reinvent the desired definitions. Rather, my ultimate goal as the researcher was to see *how* they might be able to do so. Thus, in keeping with the above recommendations, I tried to assume a role in the teaching experiment and design instructional tasks which allowed the primary ideas to originate from the students in their own ways as opposed to having the students follow a preordained path I designed. To be sure, as remarked by Simon (1995), "the only thing that is predictable in teaching is that classroom activities will not go as predicted" (p. 133).

[1] Following the advice of Steffe and Thompson (2000), I conducted an exploratory pilot study since I did not have experience conducting teaching experiments prior to my work on this project.

Thus, it is this flexibility I attempted to foster that would enable me to test my research hypotheses in a genuine fashion. In the contexts of developmental research (to be described in the next chapter), the primary method of stating research hypotheses is the formation of an initial instructional theory to support the reinvention process (Gravemeijer, 1998). Testing and revising major research hypotheses in this context, then, essentially amounts to conjecturing and revising an initial instructional theory.

Conclusion

In this chapter, I have explained and described the theoretical framework and lenses through which I approached and interpreted the execution and analysis of this dissertation project. The epistemological views of constructivism provided the overarching framework from which the rest of this project followed. The theoretical perspective and instructional design heuristics of Realistic Mathematics Education provided a lens through which I interpreted the results of the teaching experiments and created the instructional tasks, respectively. The instructional tasks centered on the construct of an emergent model, which was used to model equation solving on different ring structures. I adopted the constructivist teaching experiment methodology (Cobb, 2000; Steffe & Thompson, 2000), through which the research hypotheses, in the form of an instructional theory underpinning the instructional tasks, were subject to revision and analysis.

Chapter 4: Methods

In this chapter I explicate the specific research methods which follow from my choice of theoretical framework. I specify the nature of this project's developmental research design, along with my methods of data collection, participant selection, instructional design, and data analysis. In doing so, I present the initial local instructional theory and discuss the methods by which it was tested and revised.

Research Design

I employed a *developmental research* design (Gravemeijer, 1998), which was compatible with and followed from my theoretical perspective because the primary goal is "the constitution of a domain specific instructional theory for realistic mathematics education" (p. 278). The domain specific instructional theory, also referred to commonly in the literature as a *local instructional theory*, can be likened to Simon's (1995) hypothetical learning trajectory, which he defined as a "prediction as to the path by which learning might proceed" (p. 135). The theory is local in that it describes how the specific mathematical topics should be taught. Rather than being comprised of the instructional tasks themselves, however, this theory consists of the rationale and theoretical notions which underpin and explain the specific tasks. This rationale needs to explain how the choice of activities should agree with the desire to present students with the opportunity to reinvent the mathematics in question. The purpose of creating such a theory, instead of merely producing a set of instructional tasks, is so that the theory is not context-specific. For example, a set of these tasks which are successful for one classroom are not necessarily appropriate for any classroom; such a success would be largely dependent on external factors such as the prior knowledge, mathematical maturity, classroom size, and the amount of time allotted for the class. The theory seeks to abstract the theoretical notions which guide the design of effective instructional tasks so that teachers and researchers in different contexts are able to create an adaptation which works for their own classrooms. The initial local instructional theory I devised for this project is included in the subsequent section devoted to describing the corresponding tasks. Since future revisions of this theory depended on data analysis, they are included with the corresponding results which influenced their revision. The finalized, refined local instructional theory, being the primary conclusion of this dissertation, is discussed in detail in Chapter 7.

A developmental research design consists of a *developmental phase*, which is concerned with developing activities based on the local instructional theory, and a *research phase*, wherein the results of the classroom (teaching experiment) activities are analyzed. Moreover, these phases are necessarily reflexive and iterative: analysis of each phase constantly informs the other. For example, analysis of a classroom teaching activity informs the planning and analysis of future sessions, henceforth influencing the instructional theory. Such alterations to the instructional theory, in turn, then impact future classroom tasks. This process is then iterated as many times as necessary to ensure rigor and completeness. In addition to iterating the process of analysis and revision within each teaching experiment, the literature recommends iterating teaching experiments as well so that the rationale behind instructional tasks can be continuously refined until it reached the status of an instructional theory (Cobb, 2000). In fact, this replication of teaching experiments in general "[contributes] significantly to building models of students' mathematics" (Steffe & Thompson, 2000,

p. 303). Heeding this counsel from the literature, I employed a research design consisting of three iterations of the constructivist teaching experiment: a preliminary, exploratory pilot study followed by two subsequent teaching experiments. In this way, the local instructional theory was subject to revision and analysis not only between each teaching experiment sessions but also between the teaching experiments themselves.

Steffe and Thompson (2000) recommended that any researcher unacquainted with teaching experiments should engage in exploratory teaching beforehand to "become thoroughly acquainted, at an experiential level, with students' ways and means of operating in whatever domain of mathematical concepts and operations are of interest" (p. 275). Otherwise, the teacher-researcher might inadvertently insist that the students learn in a specific way. It is important to note that, were this to happen, very few useful conclusions could be gleaned from the results of such an experiment as it would obscure the students' original reasoning and thinking. Fortunately, this suggestion is compatible with the cyclic nature of developmental research. As a result, I incorporated a preliminary pilot study into my cyclic research design, the purpose of which was twofold. First, it was a means by which I accustomed myself to working with students in a teaching experiment setting. This proved immensely useful, as I gained meaningful experience in attempting to create the cooperative, student-centric dynamic I desired. The pilot study also helped me to gauge exactly what my role in the reinvention process would be and what I could do within the confines of that role without being over- or under-involved.

Second, while I had formed vague hypotheses about the overall learning trajectory in the pilot study, I had not yet constructed the initial local instructional theory because I was unsure of exactly how the students might mathematize and build upon their intuitive notions of equation solving. In this way, the pilot study served as an experimental period for me in which I designed various tasks to see which were best able to encourage the development of the desired concepts. This process was instrumental in helping me develop of the set of instructional tasks I used in the first teaching experiment. Furthermore, it allowed me to form a vague outline of the initial local instructional theory and the subsequent instructional tasks. I omitted the details from the pilot study because it was largely for my own benefit and did not make significant theoretical contributions to this project.

I planned the first few instructional tasks for the pilot study based on a rudimentary *a priori* analysis (Swinyard, 2008) of how I expected the students to proceed. Subsequent tasks resulted from my analysis of how the students interacted with the previous activities. This cyclic process (see the figure above) was iterated until the teaching experiment was completed. This ongoing process helped me to refine the instructional tasks and form the basis for the initial local instructional theory. Once I had formulated this theory based on my experience conducting the pilot study, it was subject to revisions by way of two additional iterations of the constructivist teaching experiment. An overview of my research design is illustrated in the following diagram (the acronym LIT denotes *local instructional theory)*.

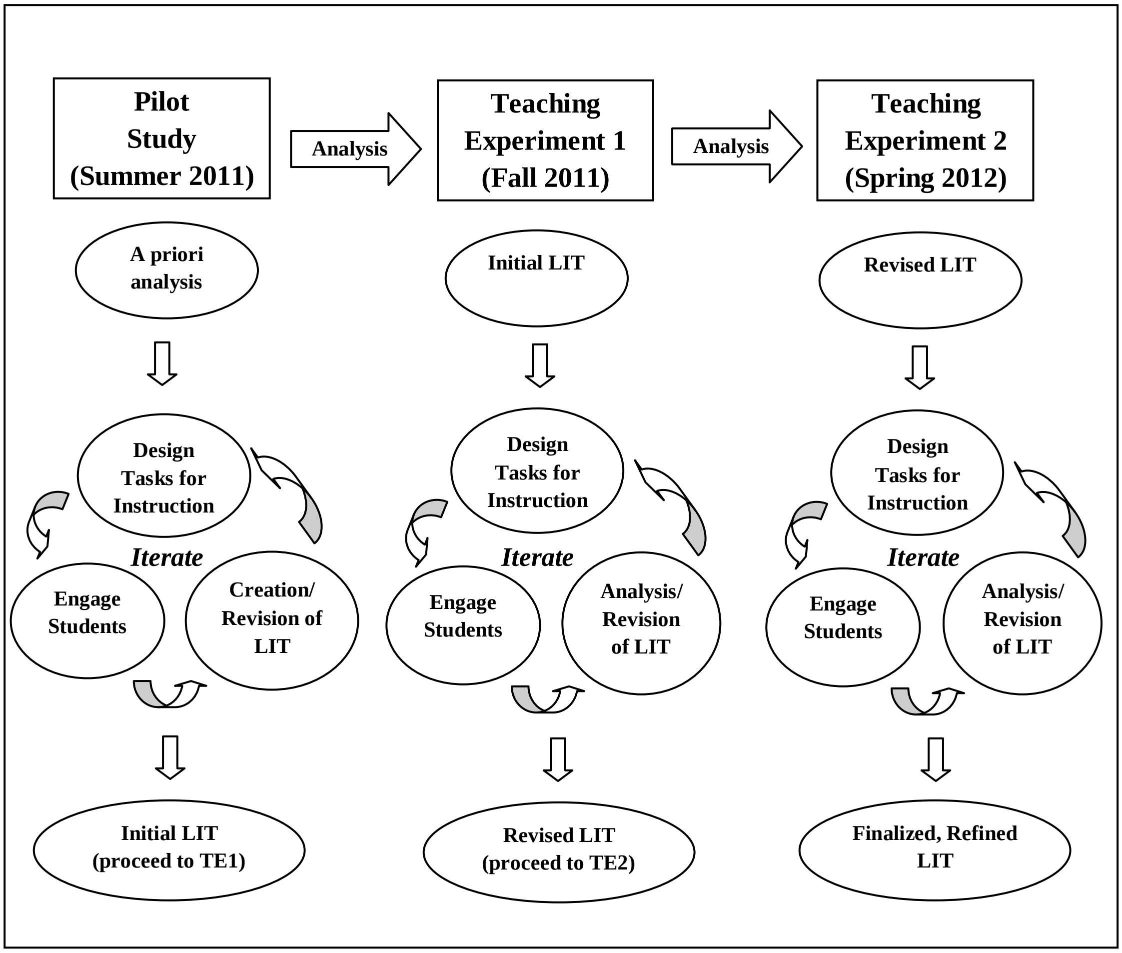

Figure 1. Research design.

I repeated this cyclic process of analysis throughout the first and second teaching experiments, the only difference being that the instructional theory under revision after the pilot study was based upon analysis of student activity instead of my own *a priori* analysis. By analyzing the design process and instructional tasks in this way, I was able to "abstract an instructional theory for 'realistic mathematics education'" (Gravemeijer, 1998, p. 279).

Data Collection

The teaching experiments consisted of a series of a series of sessions. There were nine sessions of up to two hours in the first teaching experiment (TE1), and six sessions of up to two hours in the second teaching experiment (TE2)[2]. Each session was conducted with two students in a classroom on the campus of the university from which the students were recruited (the set of criteria used for student selection is detailed in the next section). The sessions were scheduled in accordance our availability, and they took place during normal school hours. I collected data by videotaping each teaching experiment session and collecting the students' written work. Two pre-stationed digital cameras recorded each experiment from different angles, one to cover all of the students' writing and another to capture my interactions with them.

Participants. Steffe and Thompson (2000) recommended working with students individually or in small groups. I decided to work together with two student participants in each teaching experiment, following the precedent set by similar developmental research projects that investigate guided reinvention in advanced undergraduate mathematics (Larsen, 2004, 2009; Swinyard, 2008). The participant pool from which I recruited and selected the participants was comprised of students who had recently completed a course in discrete mathematics at a large comprehensive research university and had not yet had any formal exposure to abstract algebra (even group theory). I did, however, desire the students to have a functional knowledge of integers, rational numbers, matrices, polynomials, and modular arithmetic. Using these parameters, I selected students who seemed to be enthusiastic about mathematics and participation in the study. Further, I sought to select amiable, mature, and

[2] TE1 consisted of three fewer sessions than TE2 because the students in the second experiment were available to meet for longer periods of time than their predecessors (the mean session time for TE1 was close to 1:15:00, whereas for TE2 it was closer to 1:45:00).

responsible students who demonstrated the ability to proficiently articulate mathematical thoughts without reservation. These qualities and attributes were assessed through personal interaction with me and through the completion of an informational survey after volunteering for participation. The rationale for each of my choices is detailed below.

Working with two students. This decision was largely a function of wanting to create a cooperative dynamic between students. I chose not to work individually with students because I anticipated that, despite any efforts to the contrary, it would be difficult to prevent the teaching experiments from devolving into a situation in which the student would simply tell me what he or she thought I wanted to hear instead of focusing to produce authentic, creative mathematical reasoning. Additionally, this paired-student format diverted some of the attention (and therefore pressure) from each individual student. I hoped that this attempt to encourage students to work together and share ideas with each other would even result in an abundance of situations where my intervention was not needed at all, thus promoting a dynamic wherein the students' ideas governed and directed the session. Moreover, working with two students serves as a better approximation of small-group work in a classroom setting, the means by which other guided reinvention projects in abstract algebra have been implemented (Larsen, Johnson, & Scholl, 2011). On the other hand, I chose to limit the number of participants to two because I felt that any additional participants would only serve to complicate the data analysis process without introducing any critical advantages.

Discrete mathematics students with no prior knowledge of abstract algebra. At this university, the discrete mathematics course doubles as an introduction to more advanced mathematics courses and, aside from the course content, focuses on proof construction. The students were only required to have taken discrete mathematics because I wanted the only prerequisite for a course which employs the instructional theory resulting from this dissertation to be reasonable mathematical maturity. In other words, the instructional theory from this project could be implemented with any student who has completed a course in discrete mathematics (or the equivalent), instead of only being available to students who have taken a group theory course (presumably a smaller group). Students who had taken abstract algebra or who had some other formal exposure to the subject were not available for participation in this study in order to ensure the validity and authenticity of the reinvention process.

Functional knowledge of integers, rational numbers, matrices, polynomials, and modular arithmetic. These are topics which are all discussed at length both in the discrete mathematics course and prior mathematics courses at this university. I required that the student participants have a working knowledge of these structures (and their usual operations) specifically because I planned to design instructional tasks with them as a result of my analysis of the results of the pilot study (further rationale for their role in this project is discussed in the section on the instructional theory and tasks). I wanted the students to be proficient with these ideas because I was anticipating that they would use their informal knowledge to make the concepts in the instructional tasks experientially real.

Amiable, mature, and responsible students able to proficiently articulate mathematical thoughts without reservation. I sought out friendly students who seemed to be enthusiastic about participation with the idea that these students would naturally foster an interactive, cooperative dynamic in the teaching experiment sessions. I required that the students be mature and responsible to ensure attendance at the arranged meeting times and to reduce the risk for attrition and drop-out as much as possible (the students were notified in each session that their participation was strictly voluntary). I also hoped to recruit students who were able and willing to voice their mathematical thoughts because I wanted the students to be vocal, participatory, and to share more ideas so that I would have more opportunities to harvest information about their respective cognitive processes with respect to different tasks and concepts, both in the teaching experiment and in subsequent data analysis. The following table includes information on the four selected participants.

Table 1

Teaching Experiment Participants

	Participants (pseudonyms)	Age	Major(s)	Discrete Math Grade
Teaching Experiment 1 (Fall 2011)	Jack	21	Mathematics	B
	Carey	19	Mathematics & Physics	B
Teaching Experiment 2 (Spring 2012)	Laura	18	Mathematics	A
	Haden	19	Mathematics	A

These students were selected for participation based on the criteria described in the previous sections.

Instructional Design

In alignment with my epistemic beliefs as a constructivist and the theoretical perspective of Realistic Mathematics Education, my goal in designing the instructional theory and the corresponding instructional tasks was to provide the students with an environment supporting the assimilation and accommodation of new ideas into their gradually evolving image of a two-operation algebraic structure. The overarching objective for the instructional tasks was to use such cognitive activity to allow the students to reinvent the ring concept.

Target definitions. I intended for the students to reinvent the definitions of ring, integral domain, and field. However, there are many "optional" features for the basic ring structures (such as commutativity of multiplication and presence of a multiplicative identity). Additionally, many textbooks adopt different conventions when defining these basic structures. For example, Dummit and Foote (2003) required integral domains to have a multiplicative identity, yet Herstein (1996) did not. Thus, to eliminate any ambiguity or confusion, I state the exact target definitions in question in this section.

One additional point of clarification is in order. While the reinvention of these definitions was certainly the focal point for the students, it was not rigidly so. For example, if it became clear that the students were headed towards reinventing different, yet very similar, definitions, even if not the exact ones listed above, I would allow and encourage this to occur because it would more accurately reflect the experiences of the students. In this sense, my more general target definitions were actually (1) a basic, general ring structure, (2) a ring structure with no zero divisors, and (3) a ring with no zero divisors that is closed under multiplicative inverses. Thus, because of the undeniable structural similarities, I would have considered the reinvention of, say, a ring, integral domain, and a division ring as a success. Nonetheless, I felt it would be helpful for the purposes of instructional design to delineate specific target definitions.

I selected ring with identity instead of the more general ring because the typical examples of rings with identity are more familiar to students than an example of a general ring which fulfills no additional "optional" properties (such as the multiplicative identity or multiplicative commutativity). Furthermore, I opted for ring with identity over the more specific commutative ring with identity or commutative ring because I planned to incorporate a noncommutative ring (namely, 2x2 matrices over the integers or rational numbers) as an example structure that the students would investigate. Thus, ring with identity seemed to be the most reasonable choice for the most general structure the students would reinvent. For the purposes of this project, I adopted the following definitions of ring with identity, integral domain, and field, respectively:

> A ***ring with identity*** is a set R with two binary operations $+,\cdot : R \times R \to R$ (called addition and multiplication, respectively) such that the following conditions are satisfied:
>
> - There is a $0 \in R$ such that $a + 0 = a = 0 + a$ for any $a \in R$.
> - For every $a \in R$, there is a $-a \in R$ such that $a + (-a) = 0$.
> - $a + (b + c) = (a + b) + c$ for every $a, b, c \in R$.
> - $a + b = b + a$ for every $a, b \in R$.
> - $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in R$.

- $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$ for every $a,b,c \in R$.
- There exists a $1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$ for every $a \in R$.

An ***integral domain*** is a ring with identity *R* such that the following conditions are satisfied:

- If $a \cdot b = 0$ where $a,b \in R$, then either *a*=0 or *b*=0.
- $a \cdot b = b \cdot a$ for any $a,b \in R$.

A ***field*** is an integral domain *R* such that for any nonzero $a \in R$, there exists a $b \in R$ such that *ab*=1.

I also did not require that the axioms be stated exactly as above, as alternative, equivalent statements of these axioms exist. For example, the additive inverse property (the second from the top above) could be stated in the form of an additive cancellation law: $a + c = b + c \Rightarrow a = b$ for any $a,b,c \in R$.

The initial local instructional theory. Gravemeijer (1998) gave four ingredients of a local instructional theory (which may also be used as a guide when constructing the initial local instructional theory):

1. informal knowledge of the students on which instruction can be built,
2. contextual problems which have the potential to evoke powerful informal understandings,
3. tasks meant to foster reflection and abstraction, and
4. the foreshadowing of notions which go beyond the current topic at hand.

The instructional tasks and the (initial) instructional theory which underpins them are detailed in full below. Due to its potential for explaining the ring structure (Kleiner, 1999; Simpson & Stehlikova, 2006), solving linear equations became the focal point of the instructional theory. Viewing equation solving as an emergent model, I anticipated that the ring axioms would emerge as a list of properties needed to solve linear equations, and that these properties would gradually transition in the minds of the students into properties which characterize and differentiate algebraic structures. In line with Zazkis' (1999) recommendation that "working with non-conventional structures helps students in constructing richer and more abstract schemas, in which new knowledge will be assimilated" (p. 651), I planned for the students to solve various linear equations on a set of diverse, yet accessible, rings (the specific structures I chose, along with the rationale for doing so, is detailed in the following section).

I anticipated that solving the additive and multiplicative "cancellation" equations $x+a=a+b$ and $ax=ab$ (a nonzero), respectively, would support the emergence of the ring structure (as a list of the properties needed to solve these equations) and also enable the students to differentiate between general rings, integral domains, and fields. For example, $x+a=a+b$ can be solved on an algebraic structure if and only if its additive structure forms an abelian group. I used the equation $x+a=a+b$ instead of the traditional $x+a=b+a$ to eliminate any ambiguity regarding the necessity of the additive commutativity axiom, which can be derived from the other ring axioms in a ring with identity (see, for example, Dummit & Foote, 2003). The different methods of solving $ax=ab$ make use of all of the multiplicative ring axioms aside from commutativity (including multiplicative inverses). Additionally, $ax=ab$ serves to distinguish rings from integral domains, and integral domains from fields: it has a unique solution ($x=b$) if and only if the structure is an integral domain. In fact, the multiplicative cancellation law holds if and only if the structure is an integral domain.

In fields, this may be shown using multiplicative inverses or the zero-product property. On the other hand, in integral domains that are not fields it may only be proved by the zero-product property.

While the students were required to have a working knowledge of the basic examples of rings in order to be selected for participation, I anticipated that the students might benefit from some additional exploration for several of them. Similarly, I anticipated that there might be structures which needed no additional investigation due to their use in previous courses (for example, I expected that the students would be very familiar with the integers, but perhaps not so with the modular rings). Thus, on structures less familiar to the students (something I gauged based on their entrance surveys and their initial reactions to instructional tasks), I was prepared to give them tasks as necessary centering on solving specific linear equations (for example, $2x+3=11$ instead of $ax+b=c$).

The instructional tasks. The structures upon which the specific linear equations and the cancellation equations would be solved were selected to incorporate examples of rings (that are not integral domains), integral domains (that are not fields), and fields so that each set of examples would be distinct in a meaningful way from the others. The structures I chose for the instructional tasks are the integers modulo 12, integers modulo 5, integers, polynomials in one indeterminate over the integers, and 2x2 matrices over the integers (throughout this paper, assume that these structures are accompanied by their usual operations):

Table 2

Example Ring Structures for TE1

Structure	Z_{12}	Z_5	Z	Z[x]	$M_2(Z)$
Rationale	finite, includes zero divisors	example of a finite field	prototypical ring structure; integral domain that is not a field	prototypical ring structure; integral domain that is not a field	prototypical noncommutative ring, includes zero divisors

Notice that I only included one example of a field, and one that is likely to be unfamiliar to students, at that. I additionally neglected to include the more familiar examples of fields, such as the real or rational numbers, opting instead for an example of a finite field with five elements. Furthermore, I planned for the students to generate their own examples after solving equations on the structures I provided, anticipating that they would introduce the more conventional examples of fields themselves.

The specific equations I gave the students in the instructional tasks were selected to include a list of examples of when the additive and multiplicative cancellation laws hold and do not hold so that I would be able to engage the students with Larsen and Zandieh's (2007) method of *proofs and refutations*. For example, should the students assert that the multiplicative cancellation law holds in a structure when it actually does not, I can turn their attention to an (experientially real) equation they just solved in an effort to have them identify the conflict. Additionally, for each structure, I included linear equations of the type $ax+b=cx+d$ to encourage the students to follow a step-by-step procedure to solve the equations (Sfard & Linchevski, 1994).

<u>Instructional Tasks for Teaching Experiment 1</u> *Each prompt is presented as it was given to the students but has been resized and formatted to conserve space; prompts or activities given verbally are in parentheses.*	
Session 1	1. Think of the way that you add hours of time on a standard, 12 hour clock. Write down a few examples. What does this "clock addition" have in common with normal addition? What is different? 2. Create an operation table for this 12-hour clock addition. Do you notice any additional similarities or differences? 3. Though it may not have a nice real-world analog like adding hours on the clock, do you think, using a similar idea, that we can multiply these elements as well? Show how you might do this by writing out a few examples. What does this "clock multiplication" have in common with normal multiplication? What is different? 4. Create an operation table for this 12-hour clock multiplication. Do you notice any additional differences or similarities? 5. Using the operations you defined in the operation tables, show how you can solve for x in the following equations on Z_{12}: $x+3=9$ $x+8=3$ $x+6=2$ $x+5=3$
Session 2	1. Using the operations you defined in the operation tables, show how you can solve for x in the following equations on Z_{12}: $5x=10$ $4x=8$ $3x=9$ $6x=8$ $8x=4$ $2x=3$ $7x=2$ 2. Again using the operations you defined in the operation tables, write a general solution to multiplicative equations of the form $ax=b$ on Z_{12}.
Session 3	1. We would like to be able to show other people how to algebraically solve the equations $x+a= a+b$ and $ax=ab$ (a nonzero) on Z_{12}. Please write a step-by-step guide which shows how to do this. 2. Are there any rules or properties you used to write the step-by-step guides which do not hold for every element of Z_{12}? If so, give an example for each one. Then compile a list of the rules or properties which hold for any element of the set.

Session 4	1. Now think of the way that you might add the hours on a clock with only 5 hours as opposed to the usual 12. Create an operation table for this 5-hour clock addition. How does this compare to the 12-hour operation table for addition? 2. Similar to how we defined 12-hour clock multiplication, how might we define 5-hour clock multiplication? Create an operation table for this 5-hour clock multiplication. 3. Using the operations you defined in the operation tables, show how you can solve for x in the following equations on Z_5: x+1=5 x+2=1 x+3=4 x+4=3 x+5=2 4. Using the operations you defined in the operation tables, show how you can solve for x in the following equations on Z_5: 1x=4 2x=5 3x=2 4x=3 5x=1 5. We would like to be able to show other people how to algebraically solve the equations $x+a= a+b$ and $ax=ab$ (a nonzero) on Z_5. Please write a step-by-step guide which shows how to do this. 6. Are there any rules or properties you used to write the step-by-step guide that do not hold for every element of the set? If so, give an example for each one. Then compile a list of the rules or properties which hold for any element of the set. 7. Suppose now that we are working with the integers, i.e. a, x, and b are elements of Z. We would like to be able to show other people how to algebraically solve the equations $x+a=a+b$ and $ax=ab$, a nonzero, on the integers. Please write a step-by-step guide which shows how to do this.
Session 5	1. Recall that a polynomial with integer coefficients is an expression of the form $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ where the a_i terms are integers. Suppose that A, B, and X are polynomials of this type (in one variable, x). In a similar fashion to what we have been doing, we would like to be able to show other people how to algebraically solve the equations $X+A=A+B$ and $AX=AB$, A nonzero, for the polynomial X. Please write a step-by-step guide which shows how to do this. 2. Consider the set of 2x2 matrices with integer entries, and suppose that X, A, and B are matrices in this set. Once again, we would

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like to be able to show other people how to algebraically solve the equations $X+A=A+B$ and $AX=AB$, A nonzero, for X. Please write a step-by-step guide which shows how to do this.

3. To summarize the results of your equation solving, we want to display our findings in the following table. Along the top row are the different methods you have identified for solving the equations $x+a=a+b$ and $ax=ab$, a nonzero, and along the left column are the different structures on which we have been solving these equations. Complete the table based upon your knowledge of how and if these equations can be solved on these structures. (Note: the table was drawn by hand in the actual session because it depended on results of the previous activities of the session. It has been reproduced by computer here.)
<table>
<tr><th></th><th>$x+a=a+b$ by additive inverses</th><th>$ax=ab$, a nonzero by mult inverses</th><th>$ax=ab$, a nonzero by distributivity and the zero product property</th></tr>
<tr><td>Z_{12}</td><td></td><td></td><td></td></tr>
<tr><td>Z_5</td><td></td><td></td><td></td></tr>
<tr><td>Z</td><td></td><td></td><td></td></tr>
<tr><td>Polynomials over the integers, $Z[x]$</td><td></td><td></td><td></td></tr>
<tr><td>2x2 matrices over the integers</td><td></td><td></td><td></td></tr>
</table>

4. Do you notice any patterns in the table? Are there any sets on which equations solving techniques are very similar? How might we be able to sort these structures accordingly?
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<td>Session 6</td>
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1. Compile a list of all of the rules/properties used have used to solve equations so far.

2. Which properties are common to the elements of your sorted groups of structures? Can you create a list of criteria for inclusion in each of the groups? In other words, can you create a list of rules/properties that must be true for a structure to be included in that group?
(After this, the students and I worked to formalize their set of criteria – definitions – for each of the groups of structures.)
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Session 7	1. (I name the groups of structures for the students, and they work on formalizing their definitions and writing the definitions in terms of one another.) 2. (The presence of a binary operation is discussed, defined, and included in the above definitions.)
Session 8	1. (Prior to this activity, I explain the definition of a general ring and a commutative ring in terms of their reinvented definition of ring with identity.) On which of the structures we have identified (ring, ring with identity, commutative ring, commutative ring with identity, integral domain, field) is the quadratic equation $x^2 + (a+b)x + ab = 0$, a and b nonzero, solvable by algebraic means? 2. Determine if the given structures are rings. Let $+, \cdot, -$, and $\div$ denote the usual addition, multiplication, subtraction, and division, respectively. Assume the only the following: The usual addition + is both commutative and associative The usual multiplication $\cdot$ is both commutative and associative The usual multiplication $\cdot$ distributes over the usual addition + Matrix addition is both commutative and associative Matrix multiplication is associative and distributive. If a structure is not a ring, state all of the properties that are not satisfied. (**Z**, +, $\div$) $\left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in Z\right\}$ with the usual matrix addition and multiplication (**Z**, $\oplus$, $\cdot$) , where $a \oplus b$=max{a,b} 3. Is it possible for a ring to have more than one additive identity? Multiplicative identity? Prove your assertions. 4. Is it possible for an element to have more than one additive inverse? Multiplicative inverse? Prove your assertions. 5. A *unit* is an element in a ring for which a multiplicative inverse exists. More formally, $a \in R$ is a unit if and only if there exists $a^{-1} \in R$ such that $a \cdot a^{-1} = 1 = a^{-1}a$. Find the set of units for the following rings (for Z_n, addition is $+_n$ and multiplication is $\cdot_n$):

	$(Z_2, +_2, \cdot_2)$ $(Z_3, +_3, \cdot_3)$ $(Z_4, +_4, \cdot_4)$ $(Z_5, +_5, \cdot_5)$ $(Z_6, +_6, \cdot_6)$ $(Z_7, +_7, \cdot_7)$ $(\boldsymbol{Z}_8, +_8, \cdot_8)$ 6. Find the set of units for the following rings: 2x2 matrices with integer entries Polynomials with integer coefficients, ***Z[x]*** ***R*** ***Q*** ***2Z***
Session 9	1. In this exercise, we explore (Cartesian) products (i.e. direct sums) of two rings: a. Recall that Z_3 with addition and multiplication modulo 3 is a field. Consider the product $Z_3 \times Z_3 = \{(a,b) : a,b \in Z_3\}$, with addition and multiplication defined component-wise: $(a,b) +_3 (c,d) = (a +_3 c, b +_3 d)$ $(a,b) \cdot_3 (c,d) = (a \cdot_3 c, b \cdot_3 d)$ Is $Z_3 \times Z_3$ a ring (or, more specifically, a field)? If so, prove your claim and classify the ring (be as specific as possible). Otherwise, show why this structure is not a ring. b. Is $Z \times Z$ a ring component-wise (with operations defined similarly as above). Or, more specifically, is $Z \times Z$ an integral domain? c. Is $R \times R$ with the usual component-wise operations a ring (or more specifically, a field)? What conjectures can you make about the nature of the direct sum of two fields? Integral domains?

Figure 2. Instructional tasks for TE1.

<u>Instructional Tasks for Teaching Experiment 2</u> *Each prompt is presented as it was given to the students but has been resized and formatted to conserve space; prompts or activities given verbally are in parentheses.*	
Session 1	1. Think of the way that you add hours of time. For example, 5 hours added to 9:00 gives 2:00. How is this "clock addition" different from normal addition? 2. Create an operation table for this 12-hour clock addition. Do you notice any additional similarities or differences? 3. Though it may not have a nice real-world analog like adding hours on the clock, using a similar idea, how might you to multiply hours on a clock? Show how you might do this by writing out a few examples. How is this "clock multiplication" different from normal multiplication? 4. Create an operation table for this 12-hour clock multiplication. Do you notice any additional similarities or differences? 5. Using the operations you defined in the operation tables, show how you can solve for x in the following equations: $x+3=9$ $x+12=4$ $x+8=3$ $2x+4=x+7$ $6x+9=7x+11$ $5x=10$ $3x=12$ $11x=4$ $9x=6$ $9x+5=2x+7$ $10x+11=6x+7$
Session 2	1. (I named the structure they were working with in the last session as Z_{12}). In solving the above equations on Z_{12}, could you have "cancelled" any identical additive terms on both sides of the equation? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture. In other words, prove that the additive cancellation law $x+a=a+b \Rightarrow x=b$ holds always, sometimes, or never for this structure. 2. What about cancelling multiplicative terms on Z_{12}? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture. In other words, prove that the multiplicative cancellation law $ax=ab$ (a nonzero) $\Rightarrow x=b$ holds always, sometimes, or never for this structure. 3. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws? Which of these are true for every element of the set? 4. Now, imagine that you have a 5 hour clock instead of a 12 hour

	clock. For example, 4 hours added to 3:00 would be 2:00. Create operations tables for addition and multiplication for these new 5-hour clock operations. Do you notice any similarities to the 12-hour clock operation tables? Any differences? 5. Using the operations you defined in the above operation tables, show how you can solve for x in the following equations: $x+2=4$ $x+5=3$ $x+4=1$ $2x+4=x+3$ $4x+1=3x+4$ $2x=4$ $3x=4$ $4x=2$ $2x=5$ $3x+1=x+4$ $4x+5=x+2$ (At this point, I named this structure as Z_5.) 6. In solving the above equations on Z_5, could you have "cancelled" any identical additive terms on both sides of the equation? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture. In other words, prove that the additive cancellation law $x+a=a+b \Rightarrow x=b$ holds always, sometimes, or never for this structure. 7. What about cancelling multiplicative terms on Z_5? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture. In other words, prove that the multiplicative cancellation law $ax=ab$ (a nonzero) $\Rightarrow x=b$ holds always, sometimes, or never for this structure. 8. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws? Which of these are true for every element of the set? 9. Consider the set of integers $\boldsymbol{Z}$ with the usual addition and multiplication. Do you think the additive cancellation law $x+a=a+b \Rightarrow x=b$ holds for any elements, just for some, or not at all? Prove your conjecture. 10. What about cancelling multiplicative terms on $\boldsymbol{Z}$? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture. In other words, prove that the multiplicative cancellation law $ax=ab$ (a nonzero) $\Rightarrow x=b$ holds always, sometimes, or never for this structure. 11. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws? Which of these are true for every element of Z? 12. (At this point, I engaged the students in discussion about

	similarities and differences between the three structures they had encountered thus far: Z_{12}**,** Z_5**,** and Z.)
Session 3	1. Consider the set of rational numbers $Q = \left\{ \frac{a}{b} : a, b \in Z, b \neq 0 \right\}$ with the usual addition and multiplication. Do you think the additive cancellation law $x + a = a + b \Rightarrow x = b$ holds for any elements, just for some, or not at all? Prove your conjecture. 2. What about cancelling multiplicative terms on Q? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture. In other words, prove that the multiplicative cancellation law $ax = ab$ (*a* nonzero) $\Rightarrow x = b$ holds always, sometimes, or never for this structure. 3. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws? Which of these are true for every element of Q? 4. Is Q similar to or different from any of the previous structures you have encountered? In what ways? 5. Consider the set of polynomials over the integers $Z[X] = \{a_0 + a_1 X + a_2 X^2 + ... + a_n X^n : a_i \in Z, 0 \leq i \leq n\}$ with the usual polynomial addition and multiplication. Do you think the additive cancellation law $x + a = a + b \Rightarrow x = b$ (where *x*, *a*, and *b* represent polynomials in Z[X] here) holds for any elements, just for some, or not at all? Prove your conjecture. 6. What about cancelling multiplicative terms on Z[X]? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture. In other words, prove that the multiplicative cancellation law $ax = ab$ (*a* nonzero) $\Rightarrow x = b$ (where *x*, *a*, and *b* represent polynomials in Z[X] here) holds always, sometimes, or never for this structure. 7. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws? Which of these are true for every element of Z[X]? 8. Is Z[X] similar to or different from any of the previous structures you have encountered? In what ways? 9. Consider the set of 2x2 matrices over the rational numbers

	$M_2(Q)=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in Q\right\}$ with the usual matrix addition and multiplication. Do you think the additive cancellation law $X + A = A + B \Rightarrow X = B$ (where *X*, *A*, and *B* represent elements of $M_2(Q)$) holds for any elements, just for some, or not at all? Prove your conjecture. 10. What about cancelling multiplicative terms on $M_2(Q)$? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture. In other words, prove that the multiplicative cancellation law $AX = AB$ (*A* nonzero) $\Rightarrow X = B$ (where *X*, *A*, and *B* represent matrices in $M_2(Q)$) holds always, sometimes, or never for this structure. 11. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws? Which of these are true for every element of $M_2(Q)$? 12. Is $M_2(Q)$ similar to or different from any of the previous structures you have encountered? In what ways?
Session 4	1. (The students and I began the session by reviewing and discussing the structures they felt were similar to each other out of the six they had encountered.) What different methods did you use to prove the additive cancellation law on these structures? What different methods did you use to prove the multiplicative cancellation law (when possible) on these structures? 2. To summarize the results of your equation solving, we want to display our findings in the following table. Along the top row are the different methods you have identified for proving the additive and multiplicative cancellation laws, and along the left column are the different structures you have investigated. Complete the table with “always”, “sometimes”, or “never” based upon your knowledge of how and if these laws can be proven on these structures. (Note: the table was drawn by hand in the actual session because it depended on results of the previous activity. It has been reproduced by computer here.)

	$x+a=a+b \Rightarrow x=b$ by additive inverses	$ax=ab, a \neq 0 \Rightarrow x=b$ by multiplicative inverses	$ax=ab, a \neq 0 \Rightarrow x=b$ by the zero-product property
$\boldsymbol{Z}_{12}$			
$\boldsymbol{Z}_5$			
$\boldsymbol{Z}$			
$\boldsymbol{Q}$			
$\boldsymbol{Z[x]}$			
$M_2(Q)$			

3. Do the results of the table agree with your identification of similar structures? Why or why not?

4. Make a list of all of the rules/properties used to prove the cancellation laws and solve equations. Can any of these rules be proven from the others (in other words, is listing any of these unnecessary)? (Note: The only rule they had written down to this point which could be proven from the others was $x \cdot 0 = 0 \cdot x = 0$. I also used this as an opportunity to have them prove other rules of arithmetic which they had assumed were true. Among these were $(-1) \cdot a = -a$, $a(-b) = -ab$, and $(-a)(-b) = ab$. To prove $(-a)(-b) = ab$, they proved that additive inverses are unique. A proof that the additive identity is unique followed.)

5. Now suppose we are dealing with a general structure *R* that is closed under addition and multiplication, and let *a* and *x* be elements of *R*. What rules and properties do you need to prove that $x^2 + 2ax + a^2 = 0 \Rightarrow x = -a$? Can you prove this with your list of rules, or do you need any that had not been previously included? Add any additional rules/properties to your list (if any such rules exist). Based on the rules need to complete this proof, on which of the given structures would you be able to solve it?

Session 5

1. Which of the rules/properties/justifications are common to each collection? Use this to create a list of criteria for inclusion in each collection.
   (After this, I verbally directed the students here to start with the set of structures which would produce the most general definition; in this case ring with identity. This began the defining process. The presence and definition of a binary operation was addressed and included. After they had finalized their definition, I named the definition as that of a ring with identity.)

	2. Which of the six structures you have encountered are rings with identity, according to your definition? Can you use this to construct a definition for the next structure? (I directed them to start with the next most general collection. Similarly to the previous step, the definition was given the official name after it was finalized.) 3. Which of the structures you have encountered are integral domains, according to your definition? Can you use this to construct a definition for the remaining collection? (Again, the name "field" was added only after the definition was formalized.) Can you write a definition of field in terms of a ring with identity? (After this was done, I verbally asked them if they could prove the zero-product property from the other field axioms.) 4. Generate a list of examples of sets you know that are closed under addition and multiplication. Are any of your examples rings? If so, prove this conjecture and classify them. If not, provide a counterexample.
Session 6	1. Determine if the following structures are rings. If so, prove it. If not, provide a counterexample. (You may assume that ordinary addition and multiplication of real numbers are associative and that multiplication distributes over addition.) (a) $Z\left[\sqrt{d}\right]=\left\{a+b\sqrt{d} : a,b \in Z\right\}$, for d squarefree, with the usual addition and multiplication. (b) $Q\left(\sqrt{d}\right)=\left\{a+b\sqrt{d} : a,b \in Q\right\}$ (c) $(Z,+,\div)$ (d) $(Q,+,\div)$ (e) $(Z,-,\cdot)$ 2. A unit is an element in a ring that has a multiplicative inverse. More formally, a unit is an element a in R such that there exists an $a^{-1} \in R$ such that $aa^{-1}=1=a^{-1}a$. Find the units of the following rings: Z_{12} Z Z[X] Q Z_6 2Z Examine the sets of units for these structures. Does the set of units of a ring obey any of the ring axioms (it may help to examine the sets of units which are finite first)? Which axioms does it satisfy? Is this true for all of the sets of units? 3. We have previously discussed the notion of a subring, informally referring to it as a ring that is a subset of another ring. How might the notion of a *subring* be formally defined? By your definition, is $\boldsymbol{Z_5}$ a subring of $\boldsymbol{Z_{12}}$?

	4. Recall that Z_3 (with addition and multiplication modulo 3) is a field. Consider the product $Z_3 \times Z_3 = \{(a,b) : a,b \in Z_3\}$, with addition and multiplication defined component-wise. Is $Z_3 \times Z_3$ also a field? Is it an integral domain? If so, prove your claim. If not, conjecture and prove what type of structure it is. Can you use this to make a statement about the direct sums of any two rings?

Figure 3. Instructional tasks for TE2.

Data Analysis

To avoid confusion between the analysis conducted during the teaching experiments and the analysis conducted afterwards, I use Larsen's (2004) terminology in which the former is referred to as the *preliminary analysis* and the latter as the *comprehensive analysis.* My primary method for the comprehensive analysis involves the techniques of iterative analysis of video data (Lesh & Lehrer, 2000), wherein the data is reviewed several times, each time with greater detail and focus. I broke down my analysis of the data into three steps, each detailed below.

Step 1: Viewing and producing a content log of each session. The first step of my data analysis was to create a broad, time-stamped summary of everything that occurred during the session. The content logs served as the primary means of the preliminary analysis conducted during the teaching experiments. Though I was not transcribing yet at this point, I paraphrased dialogue that I felt was important to assist in this initial phase. I included the session time every five minutes and every time where I also paraphrased a bit of dialog. Included in these summaries were copies of the students' work, along with commentary about my observations. An excerpt from a content log from session 3 of TE1 is displayed below in Figure 4.

25:00

I ask them if they can use the same subtraction technique for any element in the set, and Carey responds yes. I tell them that any feature or property that is true for the entire set should be in this list. **I ask them is there a multiplicative version of the inverse property. Carey says no, and says that 1, 5, 7, 11 are the only ones**. I tell them that this is an example of a technique used, but it's not a property of this set because it doesn't hold for everything in the set.

30:00

They start solving the equations at 31:00. I tell them to pretend like they are writing an instruction manual for how to solve these equations to encourage them to not leave anything out. Carey initially uses one step to solve x+a=a+b by using subtraction:

$$(x+a)-a=(a+b)-a$$
$$x=b$$

Jack says that "first you have to assume that you can flip a and b over", hinting that he's starting to get the hang of this rule thing. **Then he says that assoc. can be used after commutativity. Then they discuss how a-a=12**. Carey then restarts. Here we can see the additive ring axioms at work (the only one they forgot to note was commutativity).

Figure 4. Content log excerpt.

After writing the content log, I would immediately review it, emphasizing points I deemed to be important by bolding it (as seen in the above excerpt). I defined "important" as dialogue or student written work which illuminated the students' reasoning about particular features of the ring structure, whether in the context of the equation solving model or otherwise. It is worth noting that what I deemed to be "important" is largely subjective, and it is entirely possible that another researcher, using a different theoretical lens or focus, might have selected a different subset of the data for such attention. Though this was largely subject to my own interpretation, I believe that my involvement in moderating the teaching experiment and interaction with the students enabled me to successfully identify important sections of the teaching episodes. In fact, one method by which I would identify critical moments in the sessions stemmed directly from my involvement in the sessions. For example, if I asked the students a question during the experiment designed to provide me with information about their reasoning and thinking, I would then retroactively use the answer to highlight their conceptual understanding at that stage.

These content logs were written between each session and helped me to prepare for future sessions. Accordingly, I used them as part of my preliminary analyses, which guided and informed the development of instructional tasks as well as the overarching local instructional theory. I would begin the process of transcription at the conclusion of each teaching experiment. By transcribing the boldfaced dialogue (in other words, deemed as important), which I then inserted into the content log in its corresponding place. The example below (Figure 5), the first portion of the previous content log excerpt, demonstrates the end result of this process.

<u>25:00</u>
I ask them if they can use the same subtraction technique for any element in the set, and Carey responds yes.

JP: We had x + 6 = 2 as the original equation. Instead of using subtraction, you guys figured out a way around it. But what was the purpose of that? What was the purpose of all of these steps right here?
Carey: To find x = 8.
JP: Right, so essentially what did you do in the process?
Carey: we subtracted 6 from 2, which on the number line gives us 8.
JP: Could you do that for any element in the set? If this was just x + a = b, for example, could you do that for all of them?
Carey: Yes.

Figure 5. Content log with dialogue excerpt (part 1).

Figure 6 below includes unbolded dialogue that helps to provide context:

<u>27:00</u>
I mention that any feature or property that is true for the entire set should be in this list. **I ask them is there a multiplicative version of the additive inverse property. Carey says no, and says that 1, 5, 7, 11 are the only ones**.

JP: OK, is there a multiplicative version of this property? Look at this one.
Carey: So, multiply both sides by a, and
JP: Right. You use that to solve this. Is that true for any element in the set?
Carey: No.
JP: So which ones did you say that it's true for?
Carey: 1, 5, 7, 11.

We then talk about how this is an example of a technique used, but it's not a rule of this structure because it doesn't hold for any element of the set.

Figure 6. Content log with dialogue excerpt (part 2).

In this way, the content logs not only aided with my preliminary analysis during the teaching experiments, but, integrated with transcription, they also laid the foundation for subsequent phases of data analysis. The non-bolded, summarized dialogue still served a crucial role in providing a surrounding context. Additionally, if it became clear in the future stages of analysis that I needed more dialog related to a particular topic or one for which I had not previously accounted, I could scan the content logs for potentially relevant areas and transcribe them as necessary.

Step 2: Outlining the reinvention process. I conducted this phase of analysis on one teaching experiment at a time, each analysis independent of the other. Once each respective teaching experiment was finished, I used the expanded content logs to inform a theoretical analysis of the data using the construct of an emergent model. More precisely, I utilize Gravemeijer's (1999) four levels of mathematical activity in the emergent model transition along with the three intermediate phases as discussed earlier in this chapter. I used emergent models to reveal the results of these teaching experiments because the progressive formalization and mathematization of the ring structure on the part of the students would be apparent. Additionally, I was able to outline and identify significant milestones in the reinvention process using the seven phases. This was done so that the emergent model transition (and its corresponding seven phases) would lay the foundation for the emerging local instructional theory.

First, I examined the data chronologically via the content logs, making an initial grouping of the data set into the phases according to their descriptions. I required that, in order to be classified into a particular phase, the portion of the data set in question must:

- build upon the activities of the previous phase (aside from the situational phase, the experientially real, accessible task serving as a starting point for the rest of the process),

- exhibit gradual formalization in regards to student reasoning and techniques as the phase progresses,
- anticipate the next phase, and
- involve similar tasks and activities.

After the data set had been partitioned into phases (though I did not necessarily view these phases as disjoint), I attempted to abstract the common themes among the instructional tasks and student activity present in that phase. These themes would be used to lay the framework for a corresponding tenet of the emerging local instructional theory. Analysis in this stage was predominantly provided a means by which I interpreted and presented the results of the teaching experiments, appearing in Chapter 4 and Chapter 5.

Step 3: Conducting a cross-sectional analysis. The primary purpose of this analytical stage was to examine the milestones of the reinvention process across both teaching experiments to further develop the local instructional theory. In particular, I looked for the following:

- commonalities present in both teaching experiments relating to tasks that evoked particularly powerful intuitive understandings,
- notable similarities and differences in the reinvention process between the two teaching experiments, and
- information regarding specific facets of the ring structure and how these facets evolved with respect to the phases of the emergent model transition as a result of the given instructional tasks.

In this way, analysis in this particular stage attempted to find support or disconfirming evidence for potential conclusions, including the refined local instructional theory, by triangulating between the results and analysis of both teaching experiments. Thus, most of the efforts from this analysis can be found in Chapter 6: Conclusions.

Issues of Ethics

Throughout the duration of this project, there were several ethical issues that I recognized and henceforth attempted to address:

Consent. I received proper consent for this project, both from the Institutional Review Board at the university where this study was conducted and from the students who volunteered for participation in this study. All student volunteers gave permission to be videotaped in the teaching experiments and for their ages and course grades to be used and published in this dissertation and subsequent papers.

Anonymity. All student participants were informed that their identities would be kept strictly confidential and that their names would be replaced with pseudonyms in any research presentations, publications, or other papers. Furthermore, their identities were not made known to anyone, including the instructor from the class from which they were recruited.

Lack of coercion. I made an effort to consistently remind the students that participation in this project was purely voluntary and, although I would certainly like for them to participate in the project until its completion, they were under no obligation to do so and could choose to leave the study at any time with absolutely no consequences. It was also made clear to the students that participation in this study had no bearing on their grades or academic standing in any way.

Time commitment. Though there were no risks associated with participation in this study, it did require a significant time commitment from the student participants. As such, I limited the number of sessions per week to 3 and made a conscientious effort to schedule the sessions so that they would be most convenient and unobtrusive for the students. When scheduling future sessions, I often reminded the students that their coursework should take precedence and that I did not want participation in this project to interfere with their grades or any other obligations they might have had.

Perceived evaluation. In the teaching experiment sessions, I highlighted to the students the fact that this study was concerned purely with their reasoning and intuition with the mathematical tasks at hand and that they were not being evaluated in any way on correctness or any other measure. I informed them that, in most cases, their struggles with a particular topic created rich data and would prove useful in determining the conclusions for this project (and, moreover, that these struggles were inherently part of any kind of mathematical activity). Furthermore, I did not want the students to feel as if they were being judged on the perceived quality or insight in any comments made during the sessions. Thus, I attempted to foster an open, accepting environment where the students would feel safe (and even encouraged) to venture conjectures and thoughts about the tasks.

Issues of Validity

I also made efforts to ensure the validity of both the reinvention process and the data analysis.

Ensuring an authentic reinvention. Essentially, validity of the reinvention process could have been threatened if the students had prior knowledge of the topics I was having them rediscover or if they obtained information about the topics, purposefully or otherwise, during the study. I did not want the data skewed (and rendered useless) through the introduction of a prior-knowledge bias: I wanted the students to *reason* instead of *remember* (Swinyard & Lockwood, 2007). I attempted to assess this beyond simply asking for their record of prior coursework by asking for a variety of mathematical definitions in the informational survey. The students were prompted to write out the formal definition, if they knew it, or honestly fill out as much information as they knew about a particular topic. I asked about many definitions, including group, ring, and field. In an effort to disguise the target definitions of the study (since prior knowledge of these would bring the validity of the reinvention process into question), most of the definitions were irrelevant to this project (for example, connected graph, topology, and measure) and served to distract from the focal point (from my perspective) of this particular portion of the survey. Any student who constructed a definition with any hint of understanding of these formal algebraic definitions was disqualified from participation. For example, a student would be rendered ineligible by writing even a recognizable portion of the definition of a group, ring, or field because that would show they had some prior knowledge of the subject. The informational survey can be found in the appendices.

To discourage the students from researching topics covered in the teaching episodes, I would remind them at the conclusion of each session to not seek out any information about the topics covered until the conclusion of the teaching experiment. Correspondingly, before the start of the next session, I would ask them if they had done so. While this method depends on the honesty of the students (who perhaps could have been purposefully deceptive), I believe that such an infiltration of outside knowledge would have been quite obvious. If a case arose in which I suspected one of the students of looking up information, I had planned to ask them about the reasoning they supplied that flagged my attention. Should the student be able to explain their remark(s) naturally and convincingly, the session would continue. Otherwise, I would make a note of this in my content logs and continued the teaching experiment. I would have investigated the matter in a more comprehensive manner at the conclusion of the study. None of this was necessary in this study, however, as I did not once suspect the student participants of such activities.

Lack of session witness. There is also an issue of validity with regards to my analysis of the data, as I was not able to have a witness observe the teaching experiment sessions. I attempted to combat these disadvantages through triangulation with the research literature and ongoing conversation with other researchers in the field (in such conversations, the anonymity of the participants and other similar information was preserved just as it would be in publication). For example, with regards to handling specific situations which arose in the teaching experiments, I consulted Larsen's (2004) dissertation project (and other similar research projects that are noted in the literature review).

Chapter 5: Results

This chapter details the results from the first teaching experiment (TE1, conducted in Fall 2011 with participants Jack and Carey) and the second teaching experiment (TE2, conducted in Spring 2012 with participants Haden and Laura). Due to the similar progression of the teaching experiments, the results are presented in a parallel, cross-sectional manner. Throughout this chapter, “the students” shall henceforth refer to all four of the students across both teaching experiments. To specify, I will always refer to students from a specific teaching experiment by their names or by, for example, the “students from TE1”. Additionally, I have identified each piece of data, whether it is dialogue or written work, as originating from either TE1 or TE2, so that it is clear to which students the excerpt is referring.

Though there were differences in the instructional tasks between the two teaching experiments, the overall outline and sequence of the tasks remained the same. As such, the results have been organized according to the following grouping of similarly-minded instructional tasks:

1. Solving equations on various ring structures.
2. Summarizing results of solving equations.
3. Sorting the structures based on equation solving.
4. The defining process.
5. Using the reinvented definitions.

Due to the gradual process of formalization I attempted to foster during the sessions, many of the students’ initial solutions or responses to instructional tasks were not necessarily complete (or even correct). I show this incomplete work for two primary reasons. First, revealing these student responses provides context for the gradual nature of the reinvention process. Second, I often use the incomplete or incorrect responses to display how the students are thinking about a particular concept at a particular stage in the process. All such solutions were eventually corrected and completed.

A point of clarification is in order regarding the proving of certain (particularly difficult) conjectures. For example, at some point, the students needed to determine if polynomials over the integers were an integral domain. The proof of this fact is lengthy and decidedly off-point from the primary goals of the reinvention process. Other similar cases include proof that the usual addition and multiplication are associative and distributive. Thus, to conserve time and not distract from the actual reinvention process, in cases like this I often asked the students to construct a "sham argument" (Larsen, 2004), wherein they would, for example, make a conjecture and then "prove" it for a specific case that would lend insight into how the overall proof would work in lieu of a complete proof (in the polynomial example above, I had the students verify the zero-product property on a product of two linear factors). I certainly do not mean to imply that such proofs are unimportant – far from it. Rather, I contend that, in the context of a research-based teaching experiment, such proofs detract from the overall goal of developing a local instructional theory (in an actual classroom environment, such proofs could be assigned outside of class, but in the confines of a controlled teaching experiment, this is not possible). Furthermore, unless these proofs provided particular insight into how the students were thinking about an idea, I have not included them in the results (though, when relevant, I do mention that the students did indeed address them). Other, more tractable, proofs of this nature (such as proof that matrix multiplication is associative) were written out in full by the students but have also been omitted for similar reasons.

Solving Equations on Various Ring Structures

This section details the results of the students' equation solving, including specific[3] linear equations and the general cancellation equations. At first, I gave the students a number of specific additive and multiplicative linear equations on Z_{12} and Z_5 to both gently introduce them to the equation solving tasks and also see if they could make use of their initial intuitions about these two structures from their work with the operations tables. Though these two structures were the only ones for which I thought solving specific equations would be necessary, I had prepared tasks for the other structures with specific equations ready in case the students needed more experience before moving on to the cancellation laws.

Throughout the equation solving activities (including the general cancellation equations), if the students included more than one step between each line of their solution, I asked them questions like

1. How did you get from this step to this step?, and
2. Can you break this down into smaller steps for me?

Specific steps and properties for which the students did not immediately see the need were dealt with individually and are detailed as they arise in the following sections. In fact, one of the fortuitous consequences of having the students solve specific equations

[3] I use "specific" in this sense to differentiate linear equations such as x+3=9 from those without specified values such as $x+a=a+b$.

first was that it made the identification of the rules used to solve the equations more concrete. For example, to motivate associativity, I was able to point to the operation tables as direct, tangible evidence that the operations were defined on (only) two elements at one time (and thus a rule is needed to address how to handle sums or products of more than two elements).

In addition to solving the cancellation equations on Z_{12} and Z_5, I prompted the students to solve them on structures for which they had not yet solved specific equations. For instance, the new structures introduced were Z, Z[x], and matrices over the integers or rational numbers with the usual operations. Despite the fact that they did not solve specific equations on these structures, I anticipated that they would still be able to solve the cancellation equations effectively because of

1. their very recent experience solving these same equations on Z_{12} and Z_5, and
2. their familiarity with the other structures from prior knowledge and courses.

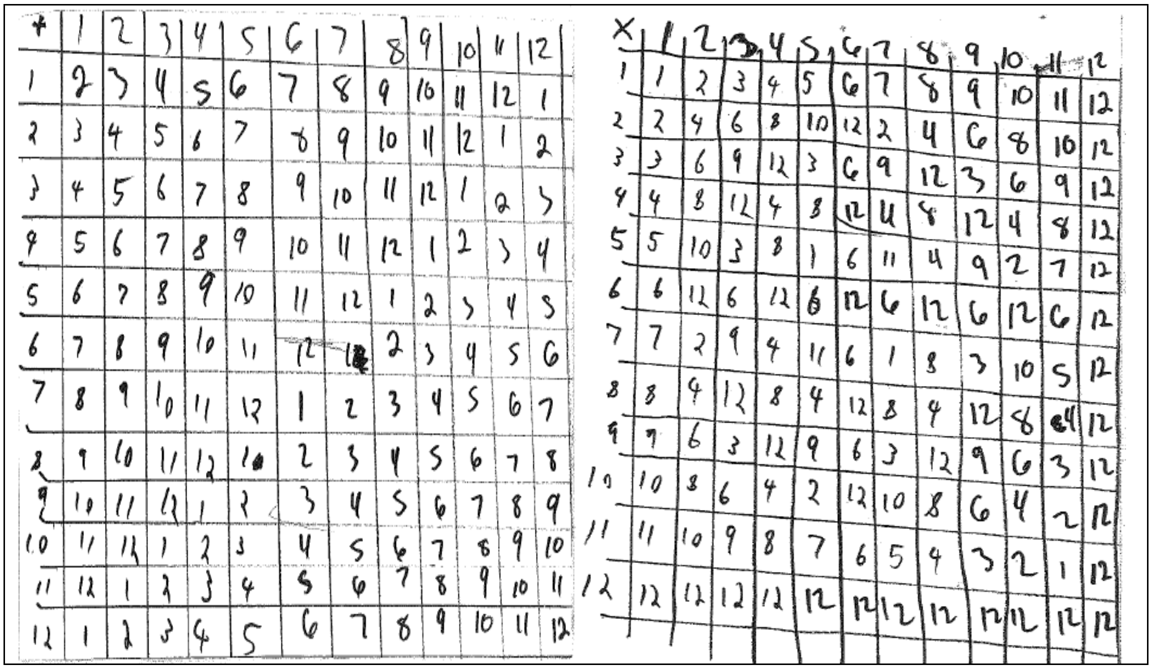
I will demonstrate later that this was a reasonable assumption to make.

When the cancellation equations are reached, the solutions to $x+a=a+b$ are all displayed together, and then all the solutions to $ax=ab$ together (recall that, throughout this paper, a in this equation is assumed to be nonzero, though this notation is often suppressed for brevity). This organization of the data portrays how the students were comparing the different facets of the structures as they solved these equations. The results from the additive equation are displayed first, followed by the multiplicative equation.

Solving specific linear equations on Z_{12} and Z_5. Due to the identical additive structure on all rings, the students' solutions to the additive equations are quite similar.

Several particularly interesting episodes resulting from these tasks are detailed here. Most of the discoveries made by the students took place while working with Z_{12} and were subsequently applied to Z_5. For example, students came to terms with the need for the associative property while solving an additive equation on Z_{12}; this property was then promptly applied to equations on $Z_{5.}$ Unlike the additive structures on Z_{12} and Z_5, which were essentially identical, solving multiplicative equations on these two structures identified several critical, fundamental differences between them.

Operation tables. The students constructed operation tables for addition and multiplication before solving equations on Z_{12} and Z_5, in order that they would explicitly define the relevant operations and encounter a visual representation of the additive and multiplicative structure on these rings. The visual nature of the operations tables enabled the students to recognize certain properties of these structures that would be important once they began solving equations. To this end, the tables were a *record-of* their informal activity with finite rings and then served as a *tool-for* more formal reasoning with solving equations. The students' renditions of the tables for Z_{12} from TE1 are displayed in Figure 7.

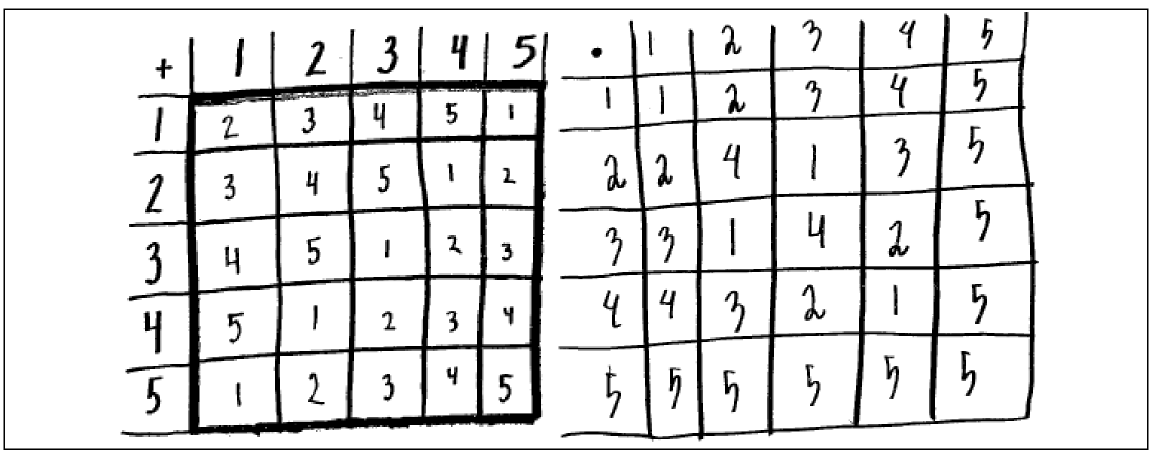


+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	7	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
7	8	9	10	11	12	1	2	3	4	5	6	7
8	9	10	11	12	1	2	3	4	5	6	7	8
9	10	11	12	1	2	3	4	5	6	7	8	9
10	11	12	1	2	3	4	5	6	7	8	9	10
11	12	1	2	3	4	5	6	7	8	9	10	11
12	1	2	3	4	5	6	7	8	9	10	11	12

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	2	4	6	8	10	12
3	3	6	9	12	3	6	9	12	3	6	9	12
4	4	8	12	4	8	12	4	8	12	4	8	12
5	5	10	3	8	1	6	11	4	9	2	7	12
6	6	12	6	12	6	12	6	12	6	12	6	12
7	7	2	9	4	11	6	1	8	3	10	5	12
8	8	4	12	8	4	12	8	4	12	8	4	12
9	9	6	3	12	9	6	3	12	9	6	3	12
10	10	8	6	4	2	12	10	8	6	4	2	12
11	11	10	9	8	7	6	5	4	3	2	1	12
12	12	12	12	12	12	12	12	12	12	12	12	12

Figure 7. Operation tables for Z_{12} (TE1).

Similarly, the operation tables on Z_5 from TE2 are displayed below in Figure 8.



+	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

•	1	2	3	4	5
1	1	2	3	4	5
2	2	4	1	3	5
3	3	1	4	2	5
4	4	3	2	1	5
5	5	5	5	5	5

Figure 8. Operation tables for Z_5 (TE2).

Additive ring structure. The two facets of the additive ring structure that immediately arose as a result of working with the operation tables and specific linear equations were the additive identity and additive inverse.

*Additive identity***.** The students almost immediately noticed the presence of the additive identity in Z_{12}.

> *Teaching Experiment 1*
> Carey: So, 12 is 0.
> Jack: Yeah, adding 12 does nothing.
>
> *Teaching Experiment 2*
> Laura: Either 6 or 6+12, because 12 counts as 0.
> Haden: We don't have 0 in this, that's the only weird thing.

The excerpt from TE2 indicates that Laura was thinking of 12 as an element that acts like 0 (the additive identity), while Haden was associating 0 with the quantity 0 instead of the more abstract notion of additive identity. This conception was amended by the time the students were exploring Z_5, exhibited by a comment Haden made about a difference with Z_{12}: "Well, of course, the 5, i.e. 0, is 5." The students used another property of the additive identity to complete the operation table for multiplication: $x \cdot 0 = 0$.

> *Teaching Experiment 1*
> Carey: Wouldn't all of these just be 12?
> Jack: Yeah, anything times the 0 is 0.
>
> *Teaching Experiment 2*
> Laura: So these are all …
> Haden: Oh yeah, all of these are 12.

The immediate recognition of this property proved to be significant, as both pairs of students made use of it when solving equations (the students initially included this in their list of rules). In TE1, the students were able to find one solution on Z_{12} (x=2) to the equation 8x=4 using this property:

$$
\begin{aligned}
&8(x-2)=12\\
&8x-2\cdot 8\\
&8(x-2)=0\\
&x-2=0
\end{aligned}
$$

Figure 9. Solving $8x=4$ on Z_{12} by the zero-multiplier property (TE1).

The students in TE2 used this property to eliminate one of the *x* terms while solving an equation that had *x* on both sides (by adding 5*x* to both sides):

$$
\begin{aligned}
&6x+9=7x+11\\
&11x+9=12x+11\\
&11x+9=11\\
&\quad\ \ +3 \qquad +3\\
&11x=2\\
&\boxed{x=10}
\end{aligned}
$$

Figure 10. Solving $6x+9=7x+11$ on Z_{12} by the zero-multiplier property (TE2).

Additive inverses. Initially, the students in both teaching experiments solved the equations by simply examining the operation tables. Conveniently, the first equation tackled by both pairs of students was the equation $x+3=9$. Both pairs of students immediately stated $x=6$ to be the solution. At this point, I shifted the focus of the tasks from *finding* the solution(s) to *proving* the solution(s). In this case, the students used this conflict to develop the notion of additive inverse through a form of subtraction.

Carey (TE1) defined subtraction as “adding the negative”. I reminded her that the term “negative” had not yet been explained for this context.

Teaching Experiment 1
JP: Since there is no -3 in this set, you need to explain what you mean.
Jack: We have to define -3.
JP: To be what?
Jack: Uh, plus 9?
JP: So why would you say plus 9?
Jack: [Motioning to the operation table] Every time you go down 3 you go up 9.

Haden and Laura seemed resolved against using subtraction, so I asked them what the purpose of subtraction might be in this case.

Teaching Experiment 2
JP: Is there something that you can do with addition that would have the same effect? You said that you wanted to subtract. What is it that you want to achieve by subtracting?
[The students discussed several ideas in response to this prompt before deciding on the following:]
Laura: We could just add …
Haden: Oh, that’s right, we could use the table.
Laura: …9.

These discussions led to the following (preliminary) solutions to this equation:

Teaching Experiment 1	*Teaching Experiment 2*
$-3 = +9$ $x+3+9 = 9+9$ $x+12 = 6$ $x = 6$	$x+3=9$ $x+3+9 = 9+9$ $x+12 = 12+6$ (note: 12 + any number is that number $12+n \in \mathbb{Z}_{12} = n$) $x=6$

Figure 11. Initial notions of additive inverse on Z_{12} (TE1 and TE2).

To follow-up, I asked them to explain their rationale for choosing to add 9 to both sides.

Teaching Experiment 1

JP: How did you solve that?

Jack: You just add 9.

JP: Why would you add 9?

Jack: Because 3+9=12, and x+12=x.

JP: So what are you doing there?

Carey: We keep finding a number to add to this side to make it 12. And we add the same thing to both sides.

Teaching Experiment 2

Haden: Since we are doing it super step-by-step, I guess we could just put x+3+9=9+9, and then x+12 …

Laura: Yeah, that's …

Haden: I think that the only rule we need is that you can get rid of 12. So we just put everything in terms of something plus 12.

Laura: Yeah, it might be better. 12 plus any number is that number.

These excerpts outline the students' methods for solving these equations and illuminate a preliminary understanding of the importance of additive inverses and the additive identity. Additionally, this is essentially the same method they would use to solve all linear equations: manipulate one side of the equation with the inverse element to obtain the identity (additive or multiplicative, depending on the context). This basic process would be *horizontally mathematized* by the students and applied to a variety of different structures as the teaching experiment progressed.

Following their use of 9 as the additive inverse of 3, I inquired if something similar could be done for each element of Z_{12}. They responded in each case by defining explicitly the "negative" equivalent of each element.

<table>
<tr><td>Teaching Experiment 1[4]</td><td>Negative Number Line
as seen on a Clock

<table>
<tr><td>-12</td><td>-11</td><td>-10</td><td>-9</td><td>-8</td><td>-7</td><td>-6</td><td>-5</td><td>-4</td><td>-3</td><td>-2</td><td>-1</td><td>0</td></tr>
<tr><td>12</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td></tr>
</table></td></tr>
<tr><td>Teaching Experiment 2</td><td>$-1 = 11$ $-2 = 10$ $-3 = 9$ $-4 = 8$
$-5 = 7$ $-6 = 6$ $-7 = 5$ $-8 = 4$
$-9 = 3$ $-10 = 2$ $-11 = 1$ $-12 = 12$</td></tr>
</table>

Figure 12. Defining additive inverses for each element of Z_{12} (TE1 and TE2).

Thus, the solving of $x+3=9$ enabled the students to see the need for subtraction and served as a *model-of* the concept of additive inverses. The students then horizontally mathematized this activity and applied it to additive inverses on Z_5.

<table>
<tr><td>Teaching Experiment 1</td><td>
<table>
<tr><td>-5</td><td>-4</td><td>-3</td><td>-2</td><td>-1</td><td>0</td></tr>
<tr><td>5</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
</table>
$a-a=0=5$</td></tr>
<tr><td>Teaching Experiment 2</td><td>$-1 = 4$ $-2 = 3$ $-3 = 2$ $-4 = 1$ $-5 = 5$</td></tr>
</table>

Figure 13. Defining additive inverses for each element of Z_5 (TE1 and TE2).

[4] Carey's "negative number line as seen on a clock" phrase refers to a previous instructional task designed to increase their familiarity with modular arithmetic by likening addition modulo 12 to clock arithmetic.

Multiplicative ring structure. As expected, the students concentrated much more on the multiplicative structures of these two rings. In these episodes, two themes came to the fore that would also characterize much of the rest of the equation solving (and overall reinvention) process: determining if the zero-product property held and attempting to define division.

Discerning the zero-product property. The presence of zero divisors first came to light through repetition in the multiplicative operations table for Z_{12}. In particular, the students noticed that the rows (or columns) for certain elements repeated (these are the rows and columns of the zero divisors, of course).

> *Teaching Experiment 1*
> Carey: Is there a pattern? Yeah, I guess there is a pattern. Some of the columns repeat themselves.
> Jack: Yeah, that looks good. These are all multiples[5] of 12.
> JP: What about 9?
> Jack: 9 repeats also. It shares 3 with 12.
>
> *Teaching Experiment 2*
> JP: I heard you guys say something about how factors of 12 are different from the other numbers. Which ones specifically are those and what is different about them?
> Laura: If there is regularity, if it repeats itself, then it is a factor of 12? 2, 3, 4, 6.
> Haden: Or it contains factors of 12.

This episode is significant because they have acknowledged the presence of these patterns, and thus, at least informally, noticed the presence of zero divisors. In fact, both pairs of students acknowledged the presence of these elements (though not explicitly) as they attended to solving specific multiplicative equations on Z_{12}:

[5] It is quite clear from the context that Jack meant "factors" or "divisors" of 12 instead of "multiples."

Teaching Experiment 1

Jack: Does there exist an x such that ax=12? That's ultimately what it's going to boil down to.

Carey: So, wait, then, if it does exist, then there's a solution?

Jack: If a, all the ones exist, there are solutions.

Carey: Well, I mean, 2 times 6 is 12.

Teaching Experiment 2

Haden: Get everything on one side.

Laura: We could do that.

Haden: Oh, right, and then factor. That makes sense.

Laura: But we can't have 0 on one side.

JP: Remember, 12 is your 0 element. So, say you factored something into linear factors. What would you do from that point?

Haden: You would solve each factor … right.

Laura: I don't know how that could help us.

Haden: With the 12 thing, I know that we are using the rule backward, like 12*x=12. But the reverse isn't true, since there are other things than 12 that times to equal 12.

The zero-product property was the subject of future discussions as well when the students attempted to prove the multiplicative cancellation law on Z_{12}. In this way, the specific equations brought to light structural features that would prove important when they moved on to the general equations.

Attempting to define division. Before attempting to define division, the students (naturally) attended first to the multiplicative identity. Jack and Carey did not identify the element 1 specifically while filling out the operation table for Z_{12}, but did point it out as a common feature shared by it and Z_5:

Teaching Experiment 1

JP: So, do you guys notice any similarities or differences to the one that we worked with previously? In other words, the mod-12?

Jack: The multiplicative identity carries over.

Curiously, the students in TE2 made no direct references to the multiplicative identity until they devised a method of solving multiplicative equations with inverses, perhaps because 1 being the identity was so familiar that it did not stand out (in contrast to 12 being the additive identity). The use of the multiplicative identity in the context of inverses is discussed in a subsequent section.

A particularly telling episode occurred when the students attempted to make sense of the possibility that linear equations could have multiple solutions (on Z_{12}). I had purposefully included two different equations having x=2 as a solution with the idea that the students would recognize that x=2, while a solution for both, is only unique for one of them. The equations in question for TE1 were $5x=10$ and $4x=8$, and for TE2 they were $5x=10$ and $9x=6$. Instantiating a case of horizontal mathematization (in which the students apply similar methods from solving the additive equations), the students extrapolated their use of subtraction to solve the additive equations in an effort to address division. Both pairs of students used subtraction as a springboard and noticed that division, in the usual sense, does not behave as expected.

Teaching Experiment 1

Jack: To solve this, we need to find an inverse relationship. Well, continuing from my last use of the inverse function with subtraction, we can tell from this not being one-to-one[6] that that won't work.

JP: Could you explain what you mean by that? Why won't inverse functions work in this context?

Jack: Given that these rows are your functions, and these numbers are the elements that you are applying the functions to, this is what they evaluate to. So the inverse function of 6 on 11 would be … [trails off, looking at the operation table]. The problem is when you apply 5 to 8, say … Oh, some of these are, and some of these aren't. So I see. On 6, there are several different numbers. They could give you 11, 9, 7, 5, 3, 1.

[6] Jack is referring to the functions represented by the rows and columns of the operation table for multiplication. For example, 4 is not one-one because it repeats, but 5 is.

Teaching Experiment 2

JP: When you guys came up with the making everything 12 idea, that was in response to dealing with the lack of subtraction, right?

Laura: Correct.

JP: One helpful thing for the multiplicative equations would be … how would you normally solve it?

Laura: Divide.

JP: Well how can you think of division in a similar sense?

Haden: If we multiplied it by 12, then we would get 12 out. 12 times anything is 12, so 12 times 1 is just 12 … so, wait a minute, let's see. 5 times 5 gives 1. That only works for this specifically. Some of them don't ever give 1.

I followed these conversations by asking which elements would have an inverse function. Repetition in the operation table was a common theme:

Teaching Experiment 1

JP: So not all of these have inverse functions. Which ones do?

Jack: 1, 5, 7, and 11.

JP: Right. And how did you figure those out?

Jack: These are the ones that don't repeat. So they are one-to-one.

Teaching Experiment 2

Haden: Some numbers don't have multiplicative inverses. No even ones, which makes sense.

Laura: They never had 1 except the ones that don't repeat.

Haden: You can only rigorously solve it when it is just like, a number times x is another number. When the coefficient of x is 1, 5, 7, 11.

Jack seemed to have made a connection between what he called "inverse functions" and units, and, on the other hand, "inverse relations (that are not also functions)" and zero divisors. Furthermore, this interpretation of addition and multiplication as functions seems to be an initial conception of binary operation. Jack's symbolic representation of this idea, however, makes it clear that he is thinking of the operations on Z_{12} as unary operations (he denotes the domain in question as M), in congruence with his previously stated belief that addition can be viewed as a left-to-right procedure.

$$f\ M \to M$$
$$R\ M \to M$$
$$R^{-1}\ M \to M$$

Figure 14. Initial conception of binary operation and inverse (TE1).

Jack used his idea of functions to make a case that, since each function is not one-one, inverses do not exist for each one (later, in the stages of defining, I have the students recall Jack's notion of function here as a means of motivating the inclusion of the binary operations into the definitions).

Haden and Laura in TE2, while undoubtedly thinking about the concept of inverse functions on some level, did not explicitly state their thoughts in this way. The students' preliminary solutions to 5*x*=10 are displayed below.

	Teaching Experiment 1	*Teaching Experiment 2*
x=2 is unique solution	$5x=10 \Rightarrow$ $5x\cdot 5=10\cdot 5$ $5\cdot 5\cdot x=2$ $1\cdot x=2$ $x=2$	$5\cdot x=10$ $x\cdot(5\cdot 5)=5\cdot 10$ $x=2$

Figure 15. Solving 5*x*=10 on Z_{12} using multiplicative inverses (TE1 and TE2).

Curiously, in both cases, the students' respective solutions to $5x=10$ made use of multiplication on the right (which necessitates the use of commutativity of multiplication) instead of the simpler multiplication on the left. Additionally, the students in TE2 combined the associative and commutative properties, which will be elaborated upon in a subsequent section.

Furthermore, in their attempts to solve the equations for which $x=2$ was not unique, the students struggled to construct a step-by-step solution. Here are their (initial) solutions to $4x=8$ and $9x=6$, respectively.

	Teaching Experiment 1	*Teaching Experiment 2*
$x=2$ is one of several solutions	$4x=8$ $4x-4=8-4$ $4(x-1)=4$	$9x=6$ $\boxed{x=2,6,10}$

Figure 16. Solving equations without unique solutions on Z_{12} (TE1 and TE2).

Jack and Carey attempted a roundabout solution using distributivity, whereas Haden and Laura simply looked at the multiplication table and acknowledged the presence of multiple solutions. After doing so, however, the students came to terms with the absence of division in these cases.

Teaching Experiment 1
Carey: Maybe we can try to find a way to define division?
Jack: It only works for numbers that are not a factor of our base.
JP: Right. So what is it that doesn't work in this other case?
Jack: 4 times any number does not make it 1.

Teaching Experiment 2
Laura: What about $3x$=12?
Haden: 3x is not going be as easy as … there is nothing that makes it be 1.
Laura: Could we define some multiplication that gives 1?
Haden: What do you mean?
Laura: Like nothing times 9 … I'm trying to figure out how to get it to be 1 somehow.

Ultimately, they decided that division was not possible. Because the students noticed that not all multiplicative equations can be solved in the same fashion, inferring that they understand the multiplicative structure varies for different elements. These conversations make it clear that the students were able to differentiate, on a situational level, between the units of the ring and the zero divisors (notice that the modular rings have the unique property that every nonzero element is a unit or a zero divisor). The students in TE1 even wrote a solution to the general case $ax=b$ on Z_{12}:

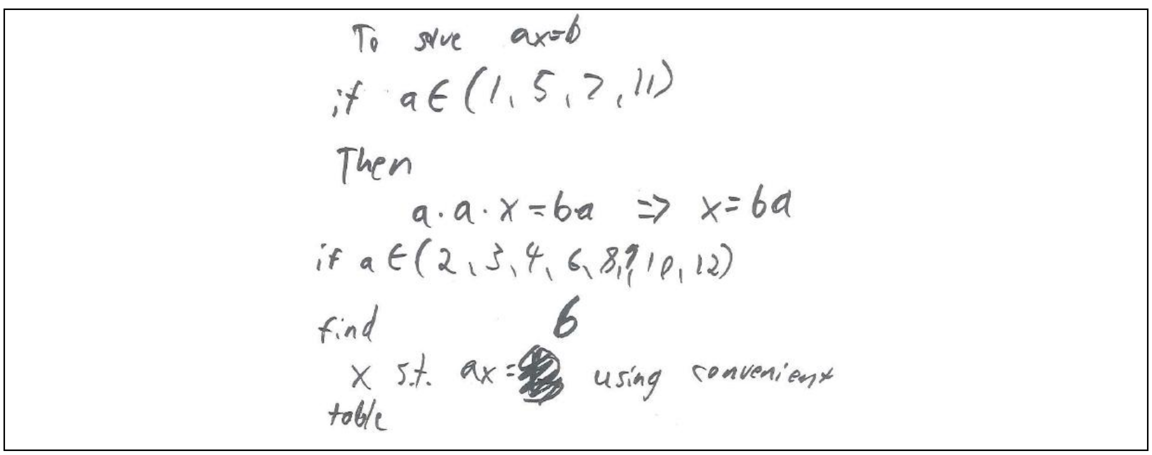


Figure 17. Guide to solving $ax=b$ on Z_{12} (TE1).

Overall, the students' solving of the multiplicative equations were a *model-of* the peculiarities of the multiplicative structure of Z_{12}. Specifically, it identified and modeled the presence and behavior of units and zero divisors.

Both pairs of students attempted to define division on Z_5 as well with much greater success. After constructing the operation table for Z_5, both pairs of students remarked about the primality of 5, suggesting that they are beginning to understand the connection between the structural features and the modulus. Shortly thereafter, they noticed that every nonzero element of Z_5 has a multiplicative inverse.

Teaching Experiment 1

JP: Basically we are doing exactly the same thing, except this time we are pretending we have a five hour clock instead.

Carey: It is a prime number.

[Equation solving activity is introduced]

JP: Remember with mod-12, you used certain arguments that didn't work for everything. Is that the case here?

Carey: The inverse isn't always itself.

Jack: Yeah, in the other one, it was interesting because the inverse was always itself. In this one, there is a multiplicative inverse for every number, except for zero, I guess.

Teaching Experiment 2

JP: Do you notice any differences [from Z_{12}]?

Laura: There aren't any repeats.

Haden: Except for 5, none of the others repeat.

[Equation solving activity is introduced]

JP: It's amazing how much easier it is the second time around.

Haden: Yeah, also with 5, because it's a prime number.

JP: Right, so what are some differences that you noticed?

Haden: There's no situation where you have [a multiplicative equation] that you can't multiply by something.

To this end, the students used the multiplication operations tables as a *tool-for* attending to structural aspects of these two rings that appeared when solving multiplicative equations on these structures.

After realizing that they would be able to use inverses to solve any multiplicative equation (in which the coefficient of *x* was nonzero), the students made quick work of the multiplicative equations (since the solutions were essentially identical). Using similar methods as they used for Z_{12}, the students devised an algorithmic technique by which they would multiply both sides of the equation by a number which would make the *x* coefficient 1. The students displayed this procedure in their respective solutions to the equation 2*x*=5 (the only difference between the two solutions is that the students in TE1 again multiplied on the right).

Teaching Experiment 1	*Teaching Experiment 2*
$2 \cdot x = 5$ $x \cdot 2 = 5$ $(x2)3 = x(2 \cdot 3)$ $x(2)(3) = 5 \cdot 3$ $x \cdot (1) = 5$ $x = 5$	$2x = 5$ $3 \cdot (2 \cdot x) = 3 \cdot 5$ $(3 \cdot 2) \cdot x = 5$ $1 \cdot x = 5$ $\boxed{x = 5}$

Figure 18. Solving 2*x*=5 on Z_5 using multiplicative inverses (TE1 and TE2).

This echoed the same method for solving all equations thus far (which they would continue to use when appropriate): using an inverse to isolate *x*.

Overall ring structure. Through their observations of the operation tables and their experience solving linear equations, several aspects of the ring structure common to both addition and multiplication were brought to light.

Commutativity. Both pairs of students noticed the commutativity of addition and multiplication (modulo 12 and 5) from the operation tables, albeit at different stages. In TE1, when the students were filling out the tables for Z_{12}, Carey mentioned that, after a certain point, "you can just flip it." Later, we discussed the implications of this implied symmetry:

Teaching Experiment 1
Jack: Well, I can observe that the axiom is commutative.
JP: I understand what you are trying to say. What about the table tells you that it is commutative?
Jack: The fact that you can flip it.

Though the students in TE2 did not mention commutativity of these operations while constructing the tables, they referred to it in a later session. I asked them about a step they were making while solving an additive equation on Z_{12}.

Teaching Experiment 2
JP: So I see you've rewritten a+b as b+a. What lets you do that?
Haden: Commutativity.
JP: How do you know that addition is commutative?
Haden: The operation table is symmetric.

Associativity. After they had solved several equations in the manner above, I turned their attention to the following solutions:

Teaching Experiment 1	*Teaching Experiment 2*
$x+8=3$ $x+8+4=3+4$ $x+12=7$ $x=7$	$x+1=4$ $x+1+11=4+11$ $x+12=12+3$ $\boxed{x=3}$

Figure 19. Solutions used to motivate associativity on Z_{12} (TE1 and TE2).

It is not altogether surprising that the students initially did not see the need for associativity – it is a property which was likely underemphasized in their previous algebra courses. Larsen's (2004) students also did not initially recognize the need for the associative property in their interactions with the symmetries of regular polygons. I reminded them that their operations of addition and multiplication, as set forth in the operation tables, were only defined on two elements at one time. In TE1, Jack asserted that it was the commutative property which enabled them to address sums of more than two products.

Teaching Experiment 1

JP: You guys are saying something along the lines of 3+4=4+3. Is that what you are using to make that step? Can you write down what you mean?

Jack: We know that x+8=8+x, and we find that in series of additions you can just swap them. So if you have to … [writes x+8+4=4+8+x].

Notice that Jack was thinking of the notion of binary operation in terms of a left-to-right procedure as well as confusing the associative and commutative properties, misconceptions that are documented in the literature (Brown, DeVries, Dubinsky, & Thomas, 1997; Larsen, 2010). The confusion of the associative and commutative properties amongst both teaching experiments is discussed more in the next section. This misconception manifested itself later as well when the students had moved on to solving general additive equations:

Teaching Experiment 1

Jack: So I can write it as that, too, x+a+(-a). And then we have the identity here.

JP: What do you think, Carey?

Carey: We need parentheses.

Jack: I don't think that I technically need the parentheses, because order of operations is implied by the order that they are in, isn't it?

I again challenged this by repeating my query about how addition could be defined on three elements when their operations tables only define it on two elements at one time. While Jack was somewhat adamant that he could devise a workaround with commutativity and a left-to-right procedure, Carey presented a solution to my proposed conflict by returning to their solution to x+8=3.

Teaching Experiment 1

JP: According to how we defined the operations in the operation tables, how do you come to terms with the left hand side of that equation right there [x+8=3]?

Carey: You can just add the two numbers together.

JP: Which two numbers?

Carey: 8 and 4.

JP: How do you know that that will end up being the same?

Carey: Do you have them in parentheses?

JP: What do you mean by parentheses? Can you show me?

In TE2, however, there was no such difficulty. Haden immediately proposed a solution to the proposed conflict similar to Carey's from TE1.

Teaching Experiment 2

JP: On our operations tables, we have only defined addition on two elements. So x+1+11 doesn't make sense. So we need to address how you are adding three things.

Haden: Do we need parentheses?

JP: What do you mean, can you show me where you would need parentheses?

What followed the respective conversations above was a response to my request that they show what they meant by the use of parentheses:

Teaching Experiment 1	*Teaching Experiment 2*
$(x+8)+4=$ $x+(8+4)$	Associativity: $x+1=4$ $(x+1)+11=4+11$ $x+(1+11)=4+11$

Figure 20. First use of associativity (TE1 and TE2).

The students in the first teaching experiment agreed at this point that, regardless of the legitimacy of the left-to-right procedure, this effectively resolved the conflict. In each case, after writing out the use of the property as above, both pairs of students were familiar with it and its name, but its initial omission from their solutions suggests that they may have been unclear about its use or necessity. Nonetheless, the students incorporated it into subsequent solutions. For example, in addition to the use of inverses, notice how the students incorporated the use of associativity (without specifically being prompted to do so) in their respective solutions to equations on Z_5:

Teaching Experiment 1	*Teaching Experiment 2*
$x+1=5$ $(x+1)-1=5-1$ $x+(1-1)=4$ $x+5=4$ $x=4$	$x+4=1$ $(x+4)+1=1+1$ $x+(4+1)=2$ $x+5=2$ $\boxed{x=2}$

Figure 21. Using associativity on Z_5 (TE1 and TE2).

Entangling of the associative and commutative properties. Jack's difficulty coming to terms with the need for associativity by use of commutativity (as discussed above) suggests the possibility for a confusion of these two properties. Not only did Jack and Carey endure this confusion, but Haden and Laura, who did not exhibit any signs of entangling these two properties at first, committed the error as well. Indeed, the students in both teaching experiments exhibited this confusion even after recognizing and using both properties correctly. Curiously, both misuses occurred on multiplicative equations (in the TE1 excerpt, the mistake occurs in line 3; in the TE2 excerpt, line 2).

Teaching Experiment 1 Solving $3x=2$ on Z_5	*Teaching Experiment 2* Solving $5x=10$ on Z_{12}
$3x=2$ $(3x)\cdot 2 = 2\cdot 2$ $x\cdot(3\cdot 2)=4$ $x(1)=4$ $x=4$	$5\cdot x=10$ $x\cdot(5\cdot 5)=5\cdot 10$

Figure 22. Entangling associativity and commutativity (TE1 and TE2).

The discrepancies above appear to convey some confusion about the nature of the associative and commutative properties, as their solution combined the two in that one step (when one of the two was not even needed). Interestingly, the other student noticed and corrected the error in each case.

Teaching Experiment 1

JP: Jack, you said that you guys might have skipped a step earlier. What was that step?

Jack: From here to here [motions between lines 2 and 3]. You move the 2.

JP: Can you write it in there somewhere?

Jack: She switched these two, I guess.

Teaching Experiment 2

Haden: So there is also associativity of multiplication. You need parentheses in there [writes out the second line].

Laura: We are also moving it around? Is that a different property?

While the errors in both cases were easily corrected, this error recalls a tendency to confuse and combine the two properties that has been demonstrated to manifest itself in a variety of different contexts (Kieran, 1979; Larsen, 2010).

Proving the additive cancellation law. I anticipated that, by solving $x+a=a+b$ on each of the structures, the students would identify (1) the additive ring axioms, and (2) the similar (identical) additive structure common to all of them. The results from these tasks suggest that this is, in fact, what happened.

Additive cancellation on Z_{12}. The equation $x+a=a+b$ on Z_{12} was the first general equation the students had encountered, and their inexperience was at first evident by their unorganized solutions:

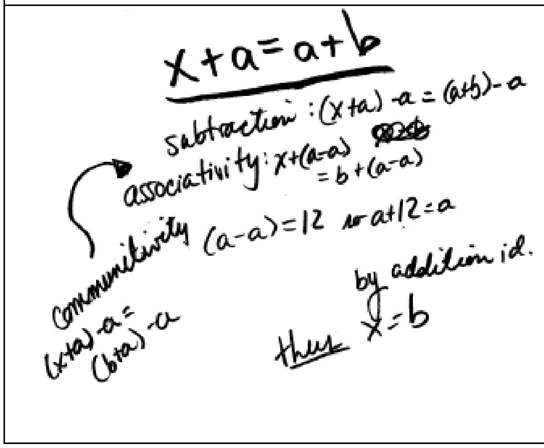
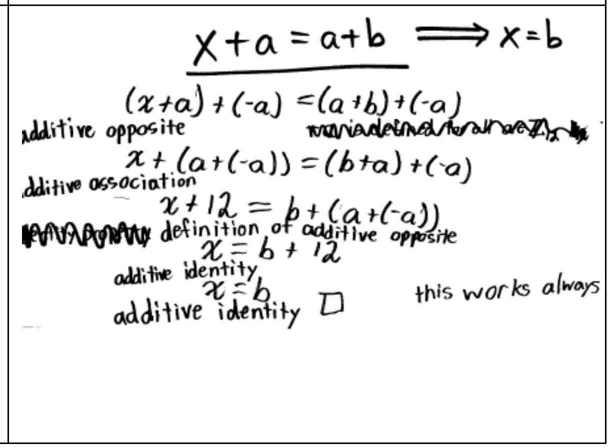
<table>
<tr><th>Teaching Experiment 1</th><th>Teaching Experiment 2</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Figure 23. Proving the additive cancellation law on Z_{12} (TE1 and TE2).

In both cases, the students wrote out their respective solutions with relative ease (and little dialogue), perhaps because of the similarity to the specific equations that they had solved previously. Additionally, I asked them to write out what rules and justifications they were using in each step to gauge their recognition of what constituted a rule. I wanted them to make some note of their justifications, but the rules did not have to be formally stated yet at this point.

Additive cancellation on Z_5. The proofs of additive cancellation followed:

Teaching Experiment 1	*Teaching Experiment 2*
Given $x + a = a + b$ Comm $x + a = b + a$ Subt $(x+a) - a = (b+a) - a$ Asso $x + (a - a) = b + (a - a)$ additive inverse $x + 5 = b + 5$ additive identity $x = b$	$x+a=a+b \Rightarrow x=b$ the proof for this is similar to the one for $\mathbb{Z}_{12}$ except that where 12 appears in that proof, 5 appears in this one. The rules used in the proof are also the same. this works always

Figure 24. Proving the additive cancellation law on Z_5 (TE1 and TE2).

The students proceeded to handle these solutions quite easily as well. In fact, the students in both teaching experiments noted that the solutions to $x+a=a+b$ were virtually identical on Z_{12} and Z_5:

Teaching Experiment 1

Jack: I feel like it's the same.
JP: Does everything hold here that held before? What do you think, Carey?
Carey: It sure looks like it.
Jack: Except for 12 and 5.

Teaching Experiment 2

JP: How would it go for Z_5?
Haden: It doesn't matter.
JP: What do you mean?
Haden: If we were gonna write it out again I would just want to do Z_n or something. Because where 12 appears in that proof, 5 would work as well.

Additive cancellation on Z. The integers were the first structure on which the students had not yet solved specific linear equations. Perhaps due to their familiarity with the integers and to having previously written out almost identical solutions on other structures, the students quickly proved the cancellation law here.

Teaching Experiment 1	*Teaching Experiment 2*
$x+a=a+b$ Given $x+a=b+a$ comm $(x+a)-a=(b+a)-a$ sub $x+(a-a)=b+(a-a)$ Asso $x+0=b+0$ AI $x=b$ AId	$X+a=a+b \Rightarrow x=b$ the proof procedes exactly like the proof for $\mathbb{Z}_{12}$ and makes use of the same rules. this works always

Figure 25. Proving the additive cancellation law on Z (TE1 and TE2).

By this point, the students in TE1 both began referring to the additive equation as the "easy one", indicating that they believed solving this equation to be much more straightforward – a reflection of the much more straightforward nature of the additive ring structure. There was a glimpse of this perceived simplicity in TE2 as well, revealed through this bit of dialogue which took place before they had written out their above solution:

> *Teaching Experiment 2*
> JP: In the integers, does the additive cancellation law hold?
> Haden: Yeah, definitely.
> JP: That was quick. Why so quick?
> Haden: Besides the fact that we have used it for a number of years, negative numbers are defined.

For Haden, the additive cancellation law was closely linked to the existence of additive inverses.

Additive cancellation on Q. The students in TE2 were asked to prove the cancellation laws on *Q* because, after analysis of the events of TE1, I wanted them to have another field with which to pair Z_5 during the sorting activity. Haden and Laura immediately noticed that the proof of the additive cancellation law would be identical to that of the integers.

Teaching Experiment 1	Teaching Experiment 2
The students in TE1 were not given an equation solving task on Q.	$x+a=a+b \Rightarrow x=b$ proof identical to proof for Z

Figure 26. Proving the additive cancellation law on Q (TE2).

Additive cancellation on Z[x]. Starting with Z[x], both pairs of students were fully aware that the solution to the equation $x+a=a+b$ would be largely the same as the previous solution. Specifically, in this case, the students were able to make a connection between the integers and polynomials over the integers. These remarks ensued after they were prompted to solve the cancellation equation on Z[x]:

Teaching Experiment 1
Jack: Adding [polynomials] is basically adding integers.
Carey: So you do the same thing that you did before.

Teaching Experiment 2
Haden: Is this proof going to differ in any way from the proof for Z, because in both cases, it is just add the product of -1 times the number, you add the number where when you add it to equals zero … and then I can't think of anything in the proof for this property for Z that relied on the fact that we were talking about integers.
JP: What do you think, Laura? Is it the same?
Laura: I think so. It don't see anything obvious.

Thus, the students not only recognized the similarities in the additive structures on the rings they had encountered, but also conjectured that polynomials behave similarly to integers. In this way, the previously written solutions were a *tool-for* this similar task.

Here are the solutions on Z[x] (where the elements *x, a, b, X, A, B* each represent arbitrary polynomials in Z[x]):

Teaching Experiment 1	Teaching Experiment 2
$X + A = A + B$ $(X + A) + (-A) = (A + B) + (-A)$ $X + 0 = (A + B) + -A$ $x = (B + A) + -A$ $x = B + (A + -A)$ $x = B + 0$ $x = B$	$x + a = a + b$ $x + a = b + a$ commutative property of addition $(x + a) + (-a) = (b + a) + (-a)$ additive inverse $x + (a + (-a)) = b + (a + (-a))$ associativity $x + 0 = b + 0$ additive inverse $x = b$ identity property of addition

Figure 27. Proving the additive cancellation law on Z[x] (TE1 and TE2).

After they had written the above solutions, I asked them exactly what was meant by the additive identity and the additive inverse in this context. Haden and Laura immediately identified 0 as the additive identity of Z[x], but in TE1 it was not quite so clear, as Jack and Carey had trouble differentiating a polynomial that is 0 for some value of *x* from the zero polynomial itself (which is 0 for all values of *x*), perhaps because of the way I had initially worded the question:

Teaching Experiment 1

JP: What does it mean for a polynomial to be zero?

Jack: It could mean that the sum of the terms is zero. It would mean that there is a solution that is zero.

JP: Let me clarify: what would it mean for an unevaluated polynomial to be 0?

Carey: That all of that [referring to generic polynomial] equals 0. All of the terms would cancel out.

Carey's comment demonstrates that there was still a bit of confusion over the nature of the 0 polynomial – after all, a polynomial written in canonical form can have no "cancelling out". Eventually, the students agreed that the zero polynomial was the polynomial for which every coefficient is 0.

Despite differing initial stances on the zero polynomial, both pairs of students defined the additive inverse of a polynomial in exactly the same way:

Teaching Experiment 1	*Teaching Experiment 2*
$A = a_0 + \ldots + a_n x^n$ $-A = -1 \cdot (a_0 + \ldots + a_n x^n)$	define $-a$ as $-1 \cdot (a_0 + a_1 X \ldots + a_n X^n)$ so that $a + (-a) = 0$

Figure 28. Defining additive inverse on Z[x] (TE1 and TE2).

Interestingly, both pairs of students used multiplication by -1 to define the notion of additive inverse (they were prompted later to prove that (-1)(*a*)=-*a*). This may suggest that additive inverses are tied strongly to the concept of -1.

Short discussions followed about whether polynomial addition is associative and commutative, and the students concluded that both of these properties still held because polynomial addition is "pretty much the same" as regular (integer) addition. The direct comparison of the polynomials to the integers suggests that, in addition to modeling the additive ring axioms, the equation solving model is beginning to model the matching additive structure on all rings.

Additive cancellation on $M_2(Z)$, $M_2(Q)$. While the students in TE1 worked with $M_2(Z)$, I expanded the possible entries to the rational numbers for TE2 (because the TE2 students had previously solved equations on Q and I hoped to avoid any confusion over the determinant[7]).

Teaching Experiment 1 -- $M_2(Z)$	*Teaching Experiment 2* -- $M_2(Q)$
The students successfully argued that this solution was identical to all of the previous ones, and thus I did not require them to write it out.	$\underline{X+A=A+B \Rightarrow X=B}$ $X+A=A+B$ $X+A=B+A$ $(X+A)+(-A)=(B+A)+(-A)$ $X+(A+(-A))=B+(A+(-A))$ $X+\begin{bmatrix}0&0\\0&0\end{bmatrix}=B+\begin{bmatrix}0&0\\0&0\end{bmatrix}$ $X=B$ same as for others

Figure 29. Proving additive cancellation on $M_2(Z)$ (TE1) and $M_2(Q)$ (TE2).

After realizing that matrix addition is commutative, the students in both teaching experiments concluded that the solution to the additive cancellation equation on $M_2(Z)$ (or, respectively, $M_2(Q)$) was the same as the others.

[7] Recall that a matrix in $M_2(Z)$ is invertible if and only if its determinant is equal to 1 or -1. In $M_2(Q)$, the only condition is that the determinant be nonzero.

Teaching Experiment 1

Jack: It is commutative, A + B = B + A.

Carey: You can do the same thing that you did for the addition, because you just add the components.

JP: And they are [matrices] over the integers. So the components are commutative. So you guys are saying that it's exactly the same thing as the others?

Carey: Yeah.

Jack: Yes.

Teaching Experiment 2

Haden: It's exactly the same as all the other ones. The definition of –a was a little different [writes out definition of –a]. After that, it went the same as before. The same properties involved.

Laura: Exactly the same, but with capital letters. [laughter]

This dialogue suggests that, in addition to successfully motivating the need for all of the additive ring axioms, equation solving was an effective *model-of* the identical additive structure in all rings (or, at the very least, all of the examples of rings from these instructional tasks). As such, I did not require the students to write out the same solution for these matrices (though the students in TE2 still did). I did ask them, however, to identify was was meant by –*A* and 0 (there were also short discussions such as the one above to verify that the other additive properties, associativity and commutativity, held; for brevity, these calculation-based proofs have not been included).

Teaching Experiment 1	*Teaching Experiment 2*
$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad -A=\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ Additionally, Carey defined the zero matrix to be “the matrix with all zero entries”.	define $-A$ such that if $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $-A=\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ so $A+(-A)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Figure 30. Defining additive inverse on $M_2(Z)$ (TE1) and $M_2(Q)$ (TE2).

Summary. The results from this section suggest that the students had clear conceptions of the need for the additive ring axioms and of the matching additive structure for each ring. In this way, the additive ring structure has emerged as a *model-of* the students’ informal equation solving activities. In a comparable vein, each of the students’ solutions to these equations, both specific and general, served as a *record-of* their previous activity and a *tool-for* future tasks.

Proving the multiplicative cancellation law. Whereas the students noticed that the additive structures were nearly identical, they soon discovered that the multiplicative structure is a different story. What resulted from the following tasks were intuitive understandings of (1) how the multiplicative structures on these rings differed, and (2) which structures behaved similarly. As the students attempted to solve $ax=ab$ on each of these structures, two principal ideas came to the foreground for the students: attempting to define division (multiplicative inverse), and determining if the zero-product property held.

Multiplicative cancellation on Z_{12}. Being the first structure on which they would solve $ax=ab$, Z_{12} provided a template by which the students would apprehend other structures. Continuing the methods used to solve the specific equations, the students identified division (multiplicative inverses) and the zero-product property as critical techniques that could be used to solve $ax=ab$.

Attempting to define division. The students' first recourse was to use multiplicative inverses to solve this equation. Even though the students had identified earlier that multiplicative inverses held for only a subset of the elements, it still took some deliberation to interpret the significance and meaning of that fact in the context of proving a general statement (like this cancellation law). Eventually, both pairs of students, recalling their previous experience with Z_{12}, realized that division would not be possible for each element:

Teaching Experiment 1

Jack: 5 over 5 equals 1. Here's the thing. It's not necessarily true that 2 over 2 equals 1.

Carey: Well … yeah. Because it doesn't have an inverse, there's no way that's true.

Jack: I don't know, I don't like it.

Teaching Experiment 2

Haden: The problem is that we couldn't in all cases. With 3, there is nothing that we could multiply by that would be the same as multiplying by 1/3 …

JP: Right.

Haden: … that equals 1, I would think.

JP: But you did do that here for 5. The question is: does this hold all the time?

Laura: It doesn't … because of 9.

After these discussions, the students recalled that 1, 5, 7, and 11 were the only elements for which there is a multiplicative inverse. Comments about how this structural property affected the truth of the cancellation law followed:

> *Teaching Experiment 1*
> Carey: x is only equal to b when a is 1, 5, 7, and 11.
> Jack: It's only necessarily[8] equal to b.
> Carey: Yeah.
> Jack: It could be equal to b and other things, too.
>
> *Teaching Experiment 2*
> JP: Could you solve this equation [motions to $ax=ab$] simply by using the cancellation law?
> Haden: If this [motions to multiplicative inverse axiom] is true, it forces this [motions to cancellation law] to be true. Whereas, if you had other solutions[9], it could be that x doesn't equal b. OK, that makes sense.
> Laura: So that's [motions to the cancellation law] false.
> Haden: Not only is it not provable, but it's actually false.

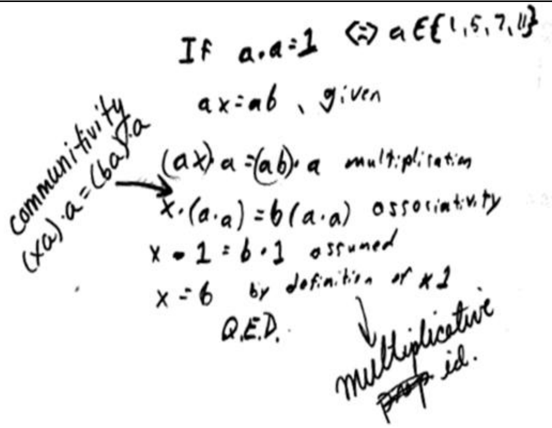
At this point, I engaged the students in a form of *proofs and refutations* (Larsen & Zandieh, 2007), wherein I asked them if the cancellation law held for certain values of a instead of just all nonzero values. In other words, I asked them if the hypotheses of the proposed cancellation law could be modified so that the result would be true and would follow the same procedure with multiplicative inverses that they had used to solve the specific multiplicative equations. In each case, the students identified that a could be 1, 5, 7, or 11:

---

[8] Jack's insertion of "necessarily" was meant to differentiate how those values of a necessitate that $x=b$ is unique.

[9] Haden is referring to the cases which have multiple solutions, such as 3x=6 (in Z_{12}), which has 2, 6, and 10 as solutions. In this case, he is arguing (correctly) that 3x=6 need not imply that x is only 2.

Teaching Experiment 1



Teaching Experiment 2

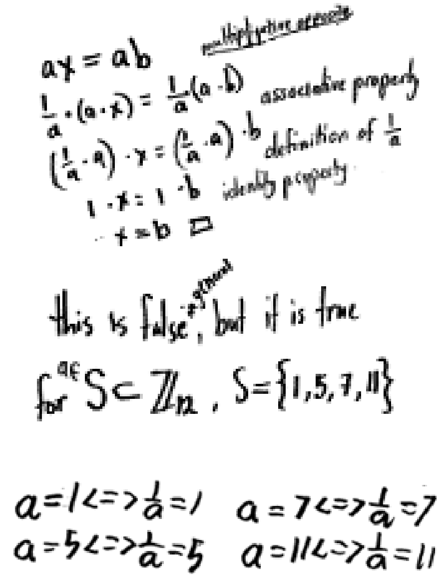


Figure 31. Proving multiplicative cancellation on Z_{12}.

The students in TE1 avoided the use of inverse notation due to the fact that each unit in Z_{12} is its own multiplicative inverse. Notice that, perhaps because using inverse notation was unnecessary, the rule they use for justification of line 3 is "multiplication" instead of "multiplicative inverse," a term they had used several times to this point. Also, just as they did with the specific linear equations, Jack and Carey multiplied on the right.

The students in TE2 took an alternative route and justified their use of the notation 1/*a* by explicitly defining it for the units of Z_{12} (shown below their proof in Figure 31).

Discerning the zero-product property. Whereas Haden and Laura in TE2 did not revisit the zero-product property while proving the cancellation law on Z_{12}, the students in TE1, in an attempt to work around the absence of division, attempted to solve the equation by moving everything over to one side (and re-examination became necessary):

$$ax=ab$$
$$ax-ab=12$$
$$a(x-b)=12$$

Figure 32. Discerning the zero-product property on Z_{12} (TE1).

This presented another road block for the students: Jack and Carey knew that $x=b$ was a solution from this point (after all, a times 12 is 12) but were unsure about what came next. Again, this is another issue previously addressed that the students needed to reconceptualize for the general case. This led to a discussion about the zero-product property:

Teaching Experiment 1

Carey: Then we could say a=12 or 0, and we could say x-b=12 or 0.
Jack: How did we conclude a=12?
Carey: Because … say 12 is 0. Then if a=0, then … it's like, you know, 12 times anything will equal 12.
Jack: It acts a lot like 0. Multiply anything by 0, you get 0.
Carey: 12 is 0.
Jack: From that we can conclude … can 12 only be reached by multiplying. I think, like, if you time numbers that aren't 0, you can't get 0. Right?
Carey: What?
Jack: Well, find two nonzero numbers that will be zero when you take their product.
Carey: Mmm.
Jack: But that's not true for x, though. Or for 12, rather. You can multiply two numbers and get 12.
Carey: Yeah.
Jack: So the rules that apply to 0 don't apply to 12.

Jack's comment concluding this excerpt makes it clear that he distinguished between 0, presumably as the zero for the integers or real numbers, and 12, the "zero" for this structure. Moreover, this episode brought the zero-product property into question in the general case, the same way that multiplicative inverses were earlier (and thus, equation solving is a *model-of* the absence of the zero-product property on Z_{12}). Haden and Laura would not revisit the zero-product property on Z_{12} until they were identifying which properties held for each of the given structures. I anticipated that this would highlight the importance of the property in cases when it actually did hold (such as on the next two structures, Z_5 and Z).

Multiplicative cancellation on Z_5. When presented with proving the cancellation law in Z_5, each pair of students was able to reference their prior experience with the specific linear equations on Z_5 (as well as their experience solving $ax=ab$ on Z_{12}). Without much discussion, they wrote up their solutions.

Teaching Experiment 1	*Teaching Experiment 2*
$ax = ab$ Given $a \neq 5$ MI $(ax)a^{-1} = (ab)a^{-1}$ Comm $(xa)a^{-1} = (ba)a^{-1}$ Asso $x(aa^{-1}) = b(aa^{-1})$ MI $x \cdot 1 = b \cdot 1$ MId $x = b$	$a \cdot x = a \cdot b$ $\frac{1}{a} \cdot (a \cdot x) = \frac{1}{a} \cdot (a \cdot b)$ $(\frac{1}{a} \cdot a) \cdot x = (\frac{1}{a} \cdot a) \cdot b$ $1 \cdot x = 1 \cdot b$ $x = b$ $\square$ this works for all $a \neq 5$

Figure 33. Proving multiplicative cancellation on Z_5 (TE1 and TE2).

A point of interest here is that the students recognized, almost immediately, that these arguments held for any nonzero element of Z_5. The students from TE1 again used right multiplication. Additionally, as the students in TE2 did not explicitly define $1/a$ in their proof at first (as they had done for Z_{12}), I prompted them to define each inverse similarly for Z_5:

$a = 1 \Leftrightarrow \frac{1}{a} = 1$	$a = 3 \Leftrightarrow \frac{1}{a} = 2$
$a = 2 \Leftrightarrow \frac{1}{a} = 3$	$a = 4 \Leftrightarrow \frac{1}{a} = 4$

Figure 34. Defining multiplicative inverses on Z_5 (TE2).

Multiplicative cancellation on Z. While the students were undoubtedly familiar with the integers, they were not familiar with the nuances of solving multiplicative equations on them. Most of their experience with linear equations certainly involved integers, but they were viewed as elements in the larger structure of, for example, the real numbers, wherein division is allowed. This may help to explain why division was one of their first recourses when prompted to solve $ax=ab$.

Attempting to define division. The students made initial attempts to prove the cancellation law for the integers (which they knew from experience to be true) using the methods that had previously worked for them with Z_{12} and Z_5. In each case, the students conclude that 1 and -1 are the only integers for which division is possible.

Teaching Experiment 1
Carey: Did we define division?
JP: What would happen if you did that?
Carey: Like x over a equals x times 1 over a.
Jack: The problem is what is this? 1 over a. It's not going to exist over the integers necessarily. That's not necessarily going to be in the integers unless it's 1.
JP: What else could it[10] be?
Jack: Negative 1, I guess.

Teaching Experiment 2
Haden: It's true if we get to divide, but based on what we did before, we can't do that, because there is no integer times another integer that equals 1. Except 1.
JP: Are there any others?
Haden: No … [mumbles] -1. 1 and -1.

At this point, the students initially concluded that the cancellation law in TE1 could not be proved because division and multiplicative inverses do not hold. In other words, they arrived at a conclusion contradictory to their previous knowledge. In TE2, on the other hand, the students conjectured that it was true but could not be proved.

Teaching Experiment 1
Carey: I guess that I can't cross out.
JP: Why not?
Carey: Can't multiply by the reciprocal.

[10] "It" refers to the element *a*.

Teaching Experiment 2

Laura: This is true, right? We just can't prove it.

Haden: It depends on what kind of system you are setting up.

JP: So we are using the integers with the normal addition and multiplication, similar to what we have been doing [with the other structures].

Haden: Then you can't do it. It's true, but you can't prove it.

To encourage them to attempt the solution in a different manner, I asked the students how they might solve a quadratic equation. Both pairs of students responded by moving all terms over to one side of the equation, prompting discussions about the zero-product property.

Discerning the zero-product property. Faced with the prospect that division could only be defined on two elements out of an infinite set, the possibility of setting the equation equal to zero and using distributivity proved to be a much more attractive option. Shown in Figure 35 are the students' initial attempts at a solution:

Teaching Experiment 1	*Teaching Experiment 2*
$ax=ab$ $ax-ab=0$ $a(x-b)=0$ $\cancel{a}x-b=0$ $x=b$	$ax=ab \Rightarrow ax-ab=ab-ab$ $\Downarrow$ $ax-ab=0$ $\Updownarrow$ $a(x-b)=0$ $a\neq 0 \Updownarrow$ $x-b=0$ $\Updownarrow$ $x=b$

Figure 35. Initial proofs of multiplicative cancellation on Z (TE1 and TE2).

I asked the students to explain their implicit use of the zero-product property in addition to asking them if the same steps would have worked for Z_{12}.

Teaching Experiment 1

JP: How did you get from line 3 to line 4?
Carey: Since a is not 0, we know that x-b=0.
Jack: It's using the property of multiplication that says that … [writes ab=0 implies a=0 or b=0].
JP: There you go. Is it true in all of the sets that we have been dealing with?
Jack: Yes. All numbers times 12 are 12.
JP: What about 3 and 4?
Jack: Oh, that's right. There were other ways to construct 12.
Carey: Right.
Jack: So there's a property of the integers that doesn't hold in our modular system. You prove this easily on the rational numbers by dividing by a on both sides.

Teaching Experiment 2

JP: Is 0 the only value that x-b can take here?
Laura: Yes …
JP: Has that always been the case?
Haden: Um, I guess not. Remember 2 times 6 was 12 [in Z_{12}].
Laura: Oh, yeah.

I saved Jack's comment about being able to easily prove the zero-product property on the rational numbers and brought it up again when they were defining the field concept. After this discussion, I encouraged the students to include this in their list of rules. Because this property does not hold for every structure, they agreed it was noteworthy and should henceforth be included in their list of justifications. Despite the fact that they had implicitly discussed that it does not hold on Z_{12}, this was the first time that the students used (and acknowledged) the zero-product property (while it certainly holds in Z_5, their use of multiplicative inverses precluded its use and thus went unnoticed). Here are the students' revised solutions that make note of their newly credited property:

Teaching Experiment 1	*Teaching Experiment 2*
$ax=ab \leftarrow ax-ab=ab-ab$ $ax-ab=0$ Additive inverse $a(x-b)=0$ Distributive prop. $x-b=0$ given that $a\neq 0$ $x=b$ $(x-b)+b=0+b$ $x(-b+b)=b$ $x=b$ using: $ab=0 \Leftrightarrow a=0$ or $b=0$ if $a\neq 0$ then $b=0$	$ax=ab$ $ax-ab=ab-ab$ additive inverse $ax-ab=0$ additive inverse $a(x-b)=0$ distributive property $x-b=0$ if $a\cdot b=0$, then $a=0$ or $b=0$ $(x-b)+b=0+b$ additive opposite $x+(-b+b)=b$ associative property of addition, identity $x+0=b$ definition of the additive inverse $x=b$ identity

Figure 36. Final proofs of multiplicative cancellation on Z (TE1 and TE2).

These solutions, in addition to identifying the necessity of the distributive and zero-product property, also helped the students to mentally differentiate the integers (and, eventually, polynomials) from the modular rings with which they had worked previously.

Multiplicative cancellation on Q. The students in TE2 quickly attended to this proof, recognizing that each (nonzero) element has a multiplicative inverse[11]. They also made some noteworthy comments about Q's similarity to Z_5 and how the zero-product property holds for Q:

[11] Recall that the students in TE1 were not given an equation solving task on Q.

Teaching Experiment 2

Haden: Now this one will be different, because we can actually define multiplicative inverses for everything. So I guess we should start off with that definition. This will actually be similar to the proof on Z_5, since we can define multiplicative inverses. We could also use the zero-product property, but that wouldn't be interesting.

Laura: It was a direct copy[12].

Haden: Yeah, it was. This part, too. So in this sense it was more similar to the one for Z_5 than the one for the integers. We could have done it the same way that we did it for the integers.

As they noted in the above conversation, the proof was identical to the proof for Z_5.

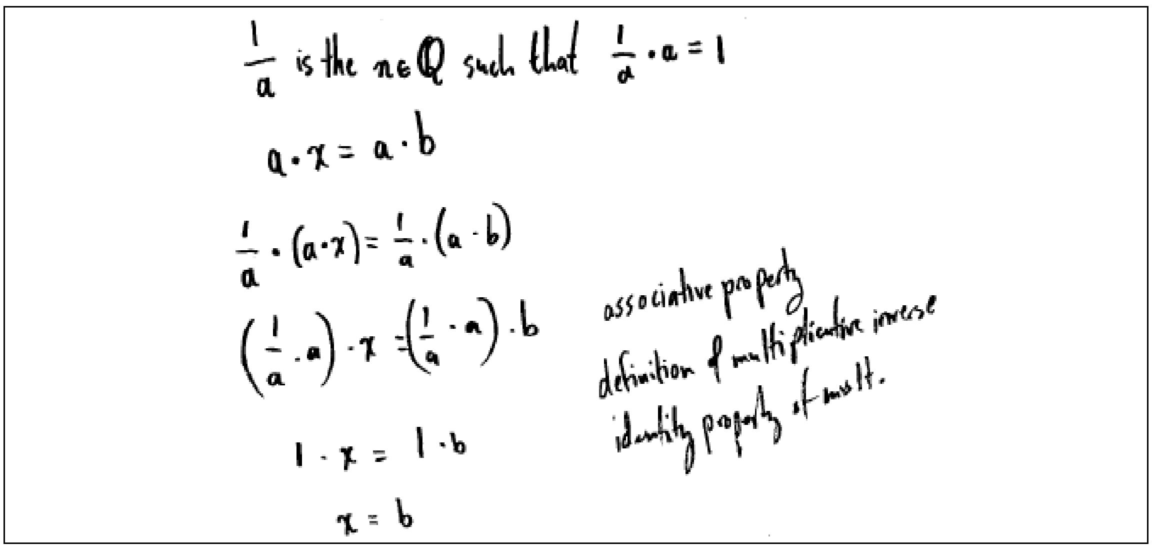


Figure 37. Proving multiplicative cancellation on Q (TE2).

Multiplicative cancellation on Z[x]. Rather than deal with division and the zero-product property on Z[x] separately (as they had done for all previous structures), these notions were intertwined in the case of polynomials over the integers (to demonstrate this correlation, these two topics are discussed simultaneously). Interestingly, division, or an attempt to investigate inverses, did not arise first as they had previously. In TE1, Carey detected that the multiplicative structure of Z[x] is

[12] Laura appears to have been referring to the proof of the multiplicative cancellation law for Z_5.

integer-like (just as she did for the additive polynomial structure) and said, "So it's just like the same as last time." Similarly, in TE2, Haden noted that "we could just do the zero-product property." I asked about their proposed use of the zero-product property.

Teaching Experiment 1

JP: The question, of course, is does the zero-product property hold?
Jack: You're basically writing one over a polynomial, which would hold. Because you are basically multiplying by 1 over a, and then you're cancelling out by a over 1.[13]
JP: What is 1 over a?
[students think for 15-20 seconds]
Jack: So we don't have division defined.

Teaching Experiment 2

Haden: We could just do the zero-product property. We don't even have to bother with inverses.
JP: Could you bother with inverses?
Haden: Yeah. Just thinking about it algebraically. Oh wait, because it's Z, not Q. Actually we couldn't bother with inverses.
JP: Suppose that it was Q[x] we are dealing with. With would the multiplicative inverse of x^2 be?
Haden: x to the negative 2.
JP: Is that a polynomial?
Haden: [re-reading definition of a polynomial] Oh, in that case, no.
JP: Which polynomials would have multiplicative inverses, if any?
Laura: The ones that only have this term.
Haden: Yeah, just the constant term.

The students rightly asserted that the zero-product property held for Z[x], but Jack's reasoning for why this is true, polynomial division, was initially flawed. Haden made a similar mistake despite the fact that the students were provided (and had prior experience with) the definition of a polynomial. This episode may suggest an inherent confusion over what constitutes a polynomial.

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[13] In this comment, Jack is proposing a proof [albeit incorrect] for the zero-product property on Z[x].

The students in each experiment both know from their experience with polynomials that the zero-product property held, although they were not asked to do this directly due to the lengthy and technical nature of the proof, they were not asked to do this directly (in a classroom setting, this would have made a nice homework problem; as a substitute, I asked them to verify that it held for quadratic polynomials). After coming to terms with the absence of division and the presence of the zero-product property, the students wrote out their solutions (*A, X, B, a, x, b* represent polynomials):

Teaching Experiment 1	*Teaching Experiment 2*
$AX = AB$ $AX + (-AB) = AB + (-AB)$ $AX - AB = 0$ $A(X - B) = 0$ $X - B = 0$ $X = B$	$a \cdot x = a \cdot b \Rightarrow x = b$ $ax - ab = ab - ab$ definition of $-n$ $(n \in \mathbb{Z}[x])$ $ax - ab = 0$ distributive property $a(x - b) = 0$ zero product property $x - b = 0$ $(x - b) + b = 0 + b$ associative property $x + (-b + b) = b$ definition of additive inverse $x + 0 = b$ identity property of addition $x = b$

Figure 38. Proving multiplicative cancellation on Z[x] (TE1 and TE2).

Multiplicative cancellation on $M_2(Z)$, $M_2(Q)$. Working with the multiplicative structure over these matrices illuminated some interesting properties for the students and forced them to reassess certain assumptions that they might have made regarding inverses and the zero-product property.

Attempting to define division. Before they attempted to ascertain the validity of inverses or division, I asked them about a multiplicative identity. They affirmed its existence and defined it as follows:

Teaching Experiment 1	*Teaching Experiment 2*
$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} A = A$	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Figure 39. Addressing the multiplicative identity for matrices (TE1 and TE2).

In each teaching experiment, I had planned to avoid the specific conditions for invertible matrices unless the students resorted to solving specific matrix equations. I felt that simply knowing that some matrices are invertible and some are not was sufficient for these exercises (should specific matrix equations have been solved, this would almost certainly need to be fleshed out in more detail). The students in TE1 only briefly mentioned the determinant while it came up several times in TE2. Next, the students discussed how inverses might come into play in this situation. In fact, both pairs of students concluded that, when inverses are assumed to exist, the proof proceeds similarly to ones they had devised before (for the students in TE1, this was true with one minor exception, discussed below).

Teaching Experiment 1

Jack: Is A invertible? That's really the question.

JP: Not necessarily. Why did you ask whether it is invertible?

Jack: Given that there is an inverse …

JP: So how would that help you solve the equation? Maybe you can tell me, what would it look like if you assumed that A was invertible.

Jack: You could say that … it looks exactly like that one [motions to solution for Z_5] with capital letters.

Teaching Experiment 2

Laura: That's just a special case of the determinant. The determinant is zero, so that's why you don't have numbers that work. I think it would be true if we had cases where the determinant was not equal to zero.
[The students work on the details to fix the proof.]

Laura: It's the same.[14]

For the students in TE1, this did not completely resolve the problem, as they used right multiplication in their proof for Z_5. I brought this conflict to light by asking students to guide me through the proof for Z_5 as if the elements were matrices. Using their prior knowledge that matrix multiplication is not commutative, they instantly identified the problem and devised a solution using left multiplication (I also had the students construct arguments demonstrating that matrix multiplication was associative and distributive). The students' proofs of the cancellation law (for *A* invertible) are included in Figure 40.

Teaching Experiment 1 – $M_2(Z)$	*Teaching Experiment 2* – $M_2(Q)$
Given $\exists A^{-1}$ $AX = AB$ $A^{-1}(AX) = A^{-1}(AB)$ $(A^{-1}A)X = (A^{-1}A)B$ $(ID)X = (ID)B$ $X = B$	$\det A \neq 0$, $\exists A^{-1}$ s.t. $A^{-1} \cdot A = 1$ $A \cdot X = A \cdot B$ $A^{-1} \cdot (A \cdot X) = A^{-1} \cdot (A \cdot B)$ $(A^{-1} \cdot A) \cdot X = (A^{-1} \cdot A) \cdot B$ $1 \cdot X = 1 \cdot B$ $X = B$

Figure 40. Proving multiplicative cancellation on $M_2(Z)$, $M_2(Q)$ for *A* invertible (TE1 and TE2).

[14] Laura said this after examining the other proofs of the cancellation laws by inverses. While it was not clear exactly the one to which she was referring, it can be safely assumed that she was acknowledging that this proof proceeds similarly to the others that presume the existence of an inverse.

With my prompting, the students proceeded to attempt to find a solution that did not require using inverses, which led them to attempting a solution with the zero-product property.

Discerning the zero-product property. Before attempting an alternative proof, Jack (TE1) stated that "we can't work with matrices the same way we can work with the integers." As if they were trying to demonstrate that this structure is indeed different from the integers, the students wrote out a "proof" of the cancellation law using the zero-product property.

Teaching Experiment 1	*Teaching Experiment 2*
$Ax = AB$ $Ax - AB = 0$ $A(x - B) = 0$ $x - B = 0$ $x = B$	$A \cdot X = A \cdot B$ $A \cdot X - A \cdot B = A \cdot B - A \cdot B$ $A(X - B) = 0$ $X - B = 0$ $(X - B) + B = 0 + B$ $X + (-B + B) = B$ $X + 0 = B$ $X = B$

Figure 41. Attempting to prove multiplicative cancellation on $M_2(Z)$, $M_2(Q)$ by using the zero-product property.

After writing out these proposed solutions, the students each had second thoughts.

> *Teaching Experiment 1*
> JP: So if you had the product of two matrices equal to 0, does one of them necessarily equal zero?
> Carey: No.
> JP: Could you give me an example?

Teaching Experiment 2

Haden: Was this a mistake? Does the zero-product property hold for matrices?

JP: What would make you say that?

Haden: It seems like there is a way to set it up so that even though you are multiplying factors and adding their products the sum is zero even if the individual factors are not opposites.

JP: Can you come up with an example where this wouldn't necessarily hold?

The students immediately went to work and produced the desired counterexamples.

Teaching Experiment 1	*Teaching Experiment 2*
$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0$	$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Figure 42. Instances of zero-divisors in $M_2(Z)$, $M_2(Q)$ (TE1 and TE2).

After making these realizations, I asked the students if they could revise their solutions. Jack and Carey simply inserted an arrow with the word "sometimes", whereas Haden and Laura actually proved that the zero-product property holds if *A* is invertible.

Teaching Experiment 1	*Teaching Experiment 2*
$AX = AB$ $Ax - AB = 0$ $A(x-B) = 0$ sometimes $x - B = 0$ $x = B$	$A^{-1} \cdot (A \cdot (x - B)) = A^{-1} \cdot 0$ $(A^{-1} \cdot A) \cdot (x - B) = 0$ $1 \cdot (x - B) = 0$ $x - B = 0$ $(x - B) + B = 0 + B$ $x + (-B + B) = B$ $x + 0 = B$ $x = B$

Figure 43. Addressing the zero-product property for matrices (TE1 and TE2).

Since the students in TE1 had not specified to what "sometimes" referred, I asked them to elaborate.

> *Teaching Experiment 1*
> Carey: Yeah, you could put that one here and this one there [pointing out matrix components]. And X doesn't equal B, but they both equal zero.
> JP: So what are the conditions?
> Jack: I was just thinking in terms of classifying something as invertible and whether you can say that this is invertible.

Thus, through the use of *proof analysis*, the students modified their hypotheses so that their previously used proofs were valid. Interestingly, even in their attempts to work around assuming the existence of inverses, they still arrived at the conclusion that, for the cancellation law to hold, *A* must be invertible. Otherwise, they concluded that, if *A* was not invertible, not only does the zero product property not hold but the equation is not even guaranteed to have a single solution.

Summary. There were two predominant features attended to by the students in apprehending the multiplicative structures of these rings: (1) the existence of division and multiplicative inverses, and (2) the truth of the zero-product property (though this was not always done separately, as these notions are linked together). In doing so for each structure, they used each of the multiplicative ring axioms, including distributivity. Thus, the equation solving model went beyond modeling the multiplicative ring axioms, but also emerged as a *model-of* the ways in which the structures can vary.

Summarizing Results of Solving Equations

After the equation solving activities were completed, I gave the students a task which prompted them to organize their solving of the equations $x+a=a+b$ and $ax=ab$. Specifically, they were asked to identify the different methods used to solve the equations, and whether the given equation could be solved in that manner always, sometimes, or never on each of the structures. Once they had discussed the different methods for solving the equations on each of the examples, I had them organize their results in a chart by writing "A" for "always works", "S" for "sometimes works", and "N" for "never works". In TE1, this task represented the first time I had prompted them directly to compare their equation solving results across the different structures. In TE2, I took a different approach by prompting the students to assess how the different methods of equation solving compared to those on the other structures as they progressed through the instructional tasks.

Though this activity was designed to summarize their previous work, the students were also required to extrapolate it. Some notable examples of such extrapolation include determining if multiplicative inverses can be used to solve *ax=ab* on Z and Z[x] and whether the zero-product property holds on Z_5. All other aspects of the activity were either previously determined or instantly assessed by the students.

Multiplicative inverses on Z and Z[x]. Though both pairs of students had discussed the elements of Z and Z[x] with multiplicative inverses previously, the students had not yet considered directly whether *ax=ab* could be solved on the integers or polynomials by this method. Haden and Laura (in TE2) wrote the following argument, recalling that the only elements with inverses Z and Z[x] were 1 and -1:

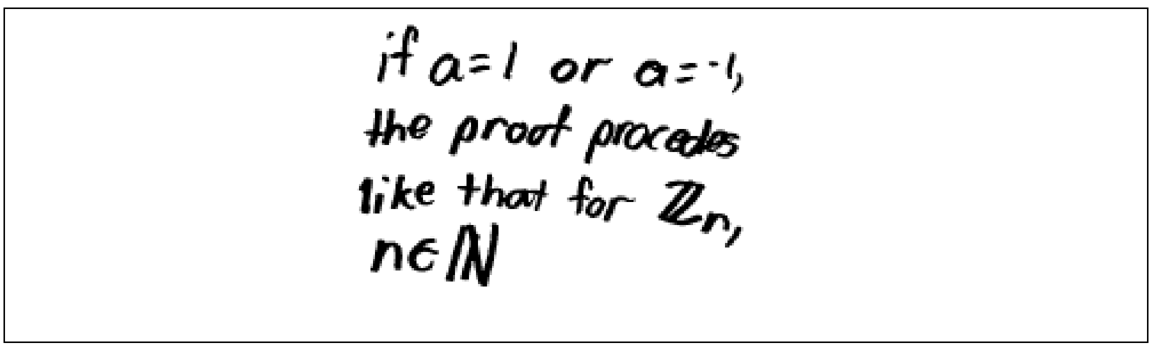


Figure 44. Units on Z and Z[x] in relation to multiplicative cancellation (TE2).

During this task, the students in TE1 initially concluded that no elements in the integers had inverses:

Teaching Experiment 1
Jack: There are no multiplicative inverses for the integers.
Carey: Right, so it never works.
Jack: You can't have one over a number.

After some discussion, they expressed similar thoughts for polynomials:

Teaching Experiment 1
Jack: Polynomials over integers. Inverses held, didn't it?
Carey: We didn't do it that way.
JP: What would happen if you tried to construct a multiplicative inverse for a polynomial? $1/x^2$. Is that a polynomial, based on how we defined it?
Carey: Basically, we are starting with n = 0, which we are.
Jack: So we can't do things with polynomials.

I followed this up by asking if a specific element had a multiplicative inverse, leading to an insightful discovery by Jack regarding the relationship between integers and polynomials.

Teaching Experiment 1
JP: Do you have a multiplicative inverse?
Jack: Oh, so I guess that -1 is the only number that does have a multiplicative inverse.
JP: And, trivially, 1 itself.
Jack: So this is "sometimes." Anything that holds for this [integers] at least sometimes, holds for polynomials at least sometimes.

Jack's comment in TE1 and the fact that Haden and Laura grouped Z and Z[x] together for their argument above add more evidence to suggest that the integers and polynomials were intimately connected in the minds of the students.

Zero-product property on Z_5. Because each nonzero element of Z_5 has a multiplicative inverse, the students did not examine whether $ax=ab$ could be solved using the zero-product property until this task. In both cases, the operation table was of critical importance.

Teaching Experiment 1
Carey: Would that [points to Z_5] be all As?
JP: You have the multiplication table for Z_5 right there. What can you conclude?
Jack: That a=5 or x-b=5.
JP: So what can you say about Z_5 using the zero-product property?
Carey: That it always works.

Teaching Experiment 2

JP: Would this also be true for Z_5? Could you use the method that you also used for Z?

Haden: Then if you define the 5 product property: if two factors multiplied together equal 5, then one of them must equal 5.

JP: How would you know that that is true?

Laura: From the table.

Haden: You can just sort of look at it.

Again, the operation table, originally a *record-of* multiplication modulo 5, was a *tool-for* determining if the zero-product property held on Z_5.

Summarizing results in chart form. The completed charts are shown below (across the top row: $x+a=a+b$; $ax=ab, a \neq 0$ using multiplicative inverses; $ax=ab, a \neq 0$ using distributivity and the zero-product property):

	$x+a=a+b$	$ax=ab, a\neq0$ using mult. inverses	$ax=ab, a\neq0$ using distributivity and zero-product property
$\mathbb{Z}_{12}$	A	S	S
$\mathbb{Z}_5$	A	A	A
$\mathbb{Z}$	A	S	A
polynomials over integers	A	S	A
2×2 matrices w/ integer entries	A	S	S

All proofs of $x+a=a+b \Rightarrow x=b$ identical.

	$x+a=a+b \Rightarrow x=b$ by additive inv	$ax=ab$, a nonzero $\Rightarrow x=b$ by mult. inv	$ax=ab$, a nonzero $\Rightarrow x=b$ by the ZPP
$\mathbb{Z}_{12}$	A	S	S
$\mathbb{Z}_5$	A	A	A
$\mathbb{Z}$	A	S	A
$\mathbb{Q}$	A	A	A
$\mathbb{Z}[x]$	A	S	A
$M_2(\mathbb{Q})$	A	S	S

Figure 45. Sorting charts from TE1 (top) and TE2 (bottom).

A number of interesting patterns emerged in the chart, both from my perspective and the students'. First, they recognized that there is essentially only one way to solve the additive equation. Jack noticed this (in TE1) during the activity by referencing their previous work solving $x+a=a+b$, remarking, "I think that this method works in all of the cases." Haden noticed this as well (in TE2) and noted it above the chart. Second, notice that the sets with "identical ratings" do indeed have substantial features in common. The always-sometimes-sometimes rating appears for Z_{12} and the matrix structures ($M_2(Z)$ and $M_2(Q)$) , which are the structures containing zero-divisors. The always-sometimes-always rating appears for Z and Z[x], which are the integral domains that are not fields. Lastly, Z_5 (and, in TE2, Q) has ratings of always-always-always (modeling the fact that, on fields, equations are *always* solvable by a variety of techniques).

At this point in TE1, I encouraged the students to generate their own examples of structures upon which the given equations could be solved (in other words, sets endowed with addition and multiplication). Then I prompted them to fill out a similar chart for their new examples. This activity in TE2 was postponed until after the reinvention process because I wanted to see how they might classify the examples as one of their first exercises using the definitions.

	$x+a=a+b$	$ax=ab$, $a\neq 0$ by mult. inverses	$ax=ab, a\neq 0$ distributivity and ZPP
$\mathbb{R}$	A	A	A
$\mathbb{C}$	A	A	A
$\mathbb{Q}$	A	A	A
$\{0\}$	A	N	N
$\mathbb{Z}_n$ $n\in\mathbb{N}$, n prime	A	A	A
$\mathbb{Z}_n$ $n\in\mathbb{N}$, n composite	A	S	S

Figure 46. Sorting chart for student-generated examples (TE1).

As I had previously anticipated, the students' own examples were dominated by fields. In fact, four of the six student-generated examples were fields (specifically, the real numbers, complex numbers, rational numbers, and integers modulo a prime). The students automatically wrote all A's for the real, complex, and rational numbers without verifying the individual properties, perhaps because of their familiarity with these structures. Notice also that, in keeping with their comments earlier about the modulus being prime, they differentiated between Z_n for n prime and composite (this occurred before the chart activity as a result of generalizing their reasoning about Z_5 and Z_{12}). As expected, the fields and their always-always-always ratings agree with the ratings for Z_5 on the previous chart. The only example I had not anticipated was {0}, the trivial ring. Because this example is markedly different from the other examples and does not lend any insight into the ring structure, I intervened and removed it from further consideration (it was, however, brought back into consideration after the definitions had been reinvented).

Sorting the Structures Based on Methods of Solving the Cancellation Equations

Now that the students had organized the results of their equation solving, I encouraged them to sort the structures based on common characteristics that they had identified (using a combination of their intuition, prior experience solving equations, and the chart). Thus, I anticipated that equation solving would emerge at this point as a *model-for* defining ring, integral domain, and field. This activity proved quite useful, because the students performed the bulk of the mathematical activity for this task by filling out the charts. The students sorted the structure quite differently because in TE2 I had them use the chart to affirm their previous intuitions about the differences between the structures, while in TE1 the students could directly reason from the chart. As a result, the sorting activity for each teaching experiment is discussed separately.

Teaching experiment 1. Jack started this activity off by commenting, "If we are not categorizing them by the first column, which is trivial, we are categorizing them by the second column and the third column," suggesting that the equation solving chart is now a *record-of* the identical additive structure for all rings (ratings in the $x+a=a+b$ column are all "always") as well as the differing multiplicative structure (differing ratings for the *ax=ab* columns). This realization enabled the students to sort based on the ratings for *ax=ab*:

Group 1: $\mathbb{Z}_5$ $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}_{n}$, n prime

Group 2: $\mathbb{Z}$, polynomials

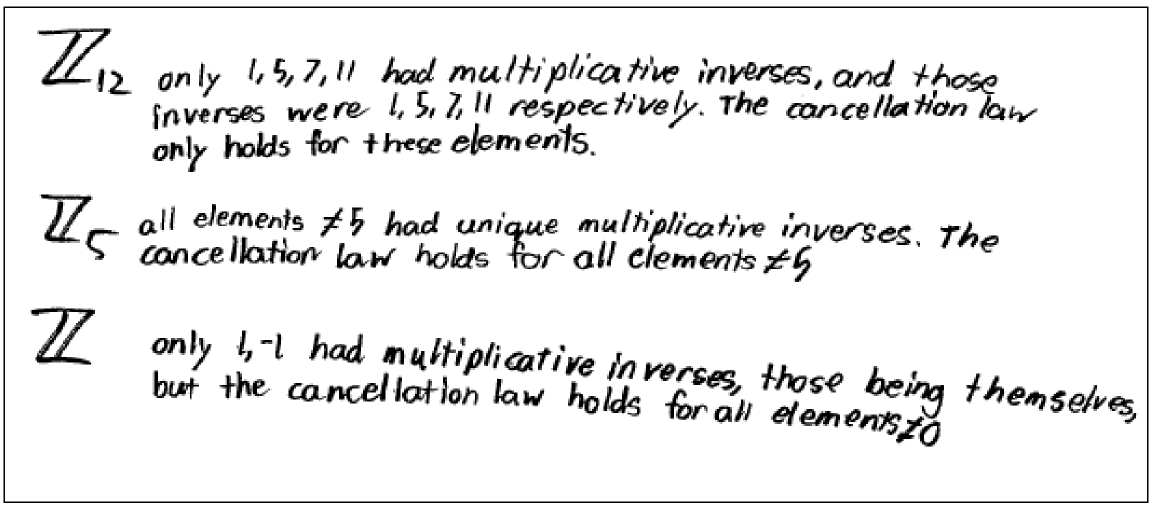
Group 3: $\mathbb{Z}_{12}$, matrices $\mathbb{Z}_{n}$, n: composite

Figure 47. Results from sorting activity (TE1).

Jack and Carey did not have much discussion during the actual sorting of the structures, and, while common threads amongst their sorted groups are apparent, their criterion for sorting the structures during the activity is not entirely clear. Nonetheless, I believe that their previous and subsequent activity indicate that Jack and Carey sorted the structures on the basis of sound (albeit not entirely formal) mathematical reasoning as opposed to merely identifying structures with similar ratings. Specifically, I believe the criteria used by the students was based on the following: (a) Group 1 contains structures whose (nonzero) elements always have multiplicative inverses, (b) Group 2 contains structures whose elements have the zero-product property but not always multiplicative inverses, and (c) Group 3 contains structures whose elements have the zero-product property and multiplicative inverses only sometimes.

I revised this activity for TE2 because I wanted more insight into the students' reasoning about how the structures were being sorted. Additionally, I hoped the students would gradually recognize which structures were similar and which were different. It is worth noting that they may have been doing this regardless (nonverbally); nonetheless, I chose to place more emphasis here so that I would have a chance to engage the students in a conversation about their choices.

Teaching experiment 2. This process was different for Haden and Laura because I prompted them to discuss similarities and differences of the given structures after working with each one. As I previously mentioned, the chart served more as confirming evidence for their previous assertions. After working with Z_{12}, Z_5, and Z, the students summarized their previous work by stating the following differences between these structures:



$\mathbb{Z}_{12}$ only 1, 5, 7, 11 had multiplicative inverses, and those inverses were 1, 5, 7, 11 respectively. The cancellation law only holds for these elements.

$\mathbb{Z}_5$ all elements $\neq 5$ had unique multiplicative inverses. The cancellation law holds for all elements $\neq 5$

$\mathbb{Z}$ only 1, -1 had multiplicative inverses, those being themselves, but the cancellation law holds for all elements $\neq 0$

Figure 48. Identifying structural differences (TE2).

Subsequently, I asked them if any of the following three structures were similar to the preceding ones. The rational numbers followed the integers:

Teaching Experiment 2

JP: Q is similar, out of the structures we've dealt with so far, what would you say Q is most similar to?

Haden: I'd say Z_5.

Laura: Yeah, because it has the multiplicative inverse.

Haden: Whereas Z_{12} had it part of the time and didn't have it part of the time and the integers never[15] had it.

Thus, the primary structural feature linking Z_5 and Q for Haden and Laura were the existence of multiplicative inverses. The rationals were followed by Z[x]:

Teaching Experiment 2

JP: Have dealt with four basic structures at this point. Which one of them, if any, do you think Z[x] is closest to?

Haden: I would say Z.

JP: Why Z?

Laura: It doesn't have multiplicative inverses.

Haden: So it actually, like the proof here is exactly the same as the proof for Z, in that the only multiplicative inverses are 1 and -1, same exact values as in Z, and you just have to prove it by the zero-product property.

Z and Z[x] appear to be connected by (1) the absence of multiplicative inverses for all elements except for 1 and -1, and (2) the fact that the zero-product property must be used to solve the multiplicative cancellation law. For the last structure, 2x2 matrices over the rationals, Haden noticed a similarity to Z_{12} that was unprompted:

Teaching Experiment 2

JP: So, you've just shown that requiring determinant of A to be nonzero fixes your original proof.

Haden: It makes sense. So determinant A not being zero is the analog of the coefficient a not being or containing a factor of 12 in Z_{12}.

JP: That's a good observation!

When they reached the point at which they were prompted to sort the structures, then, I asked them to summarize these results:

[15] Recall that the integers do have 2 units, 1 and -1, yet Haden seemed to be overlooking or ignoring these.

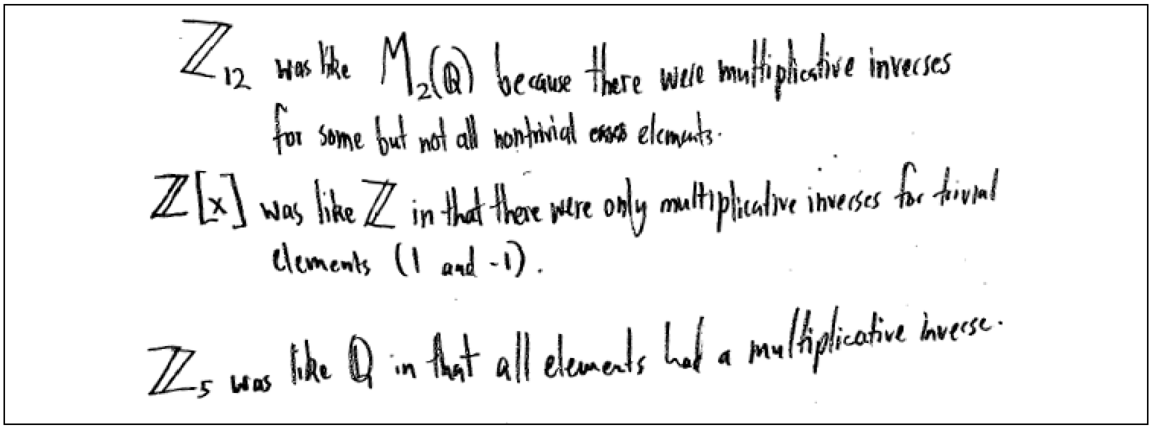


Figure 49. Results from sorting activity (TE2).

In this way, the students had actually already performed the reasoning for how they sorted the structures when they reached this point. I asked them if their equation solving summary chart agreed with their previous assertions (above) about similar structures:

> *Teaching Experiment 2*
> JP: Alright, so now my question is, does your chart summary agree with your conjectures about which sets are similar?
> Laura: Yeah …
> Haden: Yes it does. Z and Z[x], which are ASAs, and Z_5 and Q, which are AAAs. Z_{12} and the matrices, which are always for additive cancellation, and then sometimes you can prove multiplicative cancellation with inverses and sometimes you can do it with the zero-product property, but in both cases there are times when multiplicative cancellation does not hold.

Recall that, in TE2, the student-generated examples activity was not administered until after the reinvention process was completed. Now that the students had sorted like structures, I turned their attention to abstracting common features and beginning the process of defining.

The Process of Defining

The process of defining was shaped around the idea of writing a list of criteria for inclusion into each of the three groups by means of the rules and properties the students had used to solve equations. This, of course, required them to identify the common characteristics of each collection.

A missing axiom. Before starting this process, one issue needed to be discussed. The students in TE1, through their insistence on multiplying on the right in their solutions to the multiplicative equations, made use of the commutativity of multiplication axiom with regularity. However, the students in TE2 multiplied each time on the left, and thus had not made use of this property. As the moderator, this presented a bit of a conflict[16]: commutativity of multiplication, after all, is a necessary axiom for the definitions of integral domain and field. Thus, to simplify matters, I devised an activity around factoring and solving a basic quadratic equation ($x^2 + 2ax + a^2 = 0$) that would necessitate the use of commutativity. While Haden and Laura were writing out a step-by-step solution to this equation, I asked what rules this generic structure would need to have so that all of their steps would be allowed:

[16] I briefly entertained simply ignoring it for TE2, and having them reinvent the definition of a division ring (which does not require commutative multiplication) instead (the definition of ring with identity would obviously remain unchanged). However, I could not reconcile the absence of the property from the integral domain definition, as no mainstream sources could be found which define integral domain (or any analogous structures) without commutativity of multiplication.

Teaching Experiment 2

Haden: We're going to want distributivity, the zero-product property, and additive inverses. We have to factor it, say that one of them has to equal zero – we're going to have the same one in this case. We're going to have to use the additive inverse in this case to get one of them to equal a.

Laura: We also might need commutativity of multiplication.

Haden: We'll definitely need commutativity of addition.

Laura: In order to do that distributive operation, in order for that to equal what's above, this will have to be equal.

Haden: Isn't that just distributivity?

Laura: Well this is what you get when you distribute, and then you would have to switch the terms around.

Here is their solution, complete with the rules used in each step[17]:

$$x^2 + 2ax + a^2 = 0$$

$$x \cdot x + a \cdot x + a \cdot x + a^2 = 0$$

commutativity of multiplication

$$x \cdot x + x \cdot a + a \cdot x + a \cdot a = 0$$

distributivity

$$x \cdot (x+a) + a \cdot (x+a) = 0$$

$$(x+a) \cdot (x+a) = 0$$

z.p.p.

$$x+a=0$$

additive inverse

$$(x+a)+(-a) = 0 + (-a)$$

associativity of addition, additive identity

$$x + (a+(-a)) = (-a)$$

$$x + 0 = (-a)$$

$$x = -a$$

Figure 50. Solving a quadratic equation (TE2).

After the students had completed this solution, I asked them on which of the structures they had dealt with so far could this equation be solved (based on the rules they had used).

[17] This solution was not solved entirely step-by-step. For example, to show that 2ax=ax+ax, the students would have technically needed to show that 2ax=(1+1)ax=ax+ax. Because this was irrelevant to the primary goal of the task, however, I did not pursue it.

Teaching Experiment 2

Laura: Well, there's commutativity of multiplication.

Haden: Which rules out matrices.

Laura: Yeah, I think all the rest of them are OK, because we didn't use multiplicative inverse.

Haden: That's true. Multiplicative inverse would have gummed it up for Z_{12}. Also Z and Z[x].

JP: What about the zero-product property?

Haden: The zero-product property does not hold for Z_{12}.

This discussion pointedly displays how the structures are sorted in the students' minds as a result of the recently completed sorting activity. Additionally, they were now ready to proceed.

Constructing a master list of rules. To start, I had them construct a "master list" of all of the properties that they had used.

Teaching Experiment 1	*Teaching Experiment 2*
Additive Identity Multiplicative Identity Associativity of addition Communitivity of addition Distributivity Zero-product property Associativity of multiplication Communitivity of multiplication Multiplicative inverse Additive inverse	commutitivity of addition additive inverse additive identity associativity of addition distributivity multiplicative identity multiplicative inverse associativity of multiplication zero-product property commutativity of multiplication

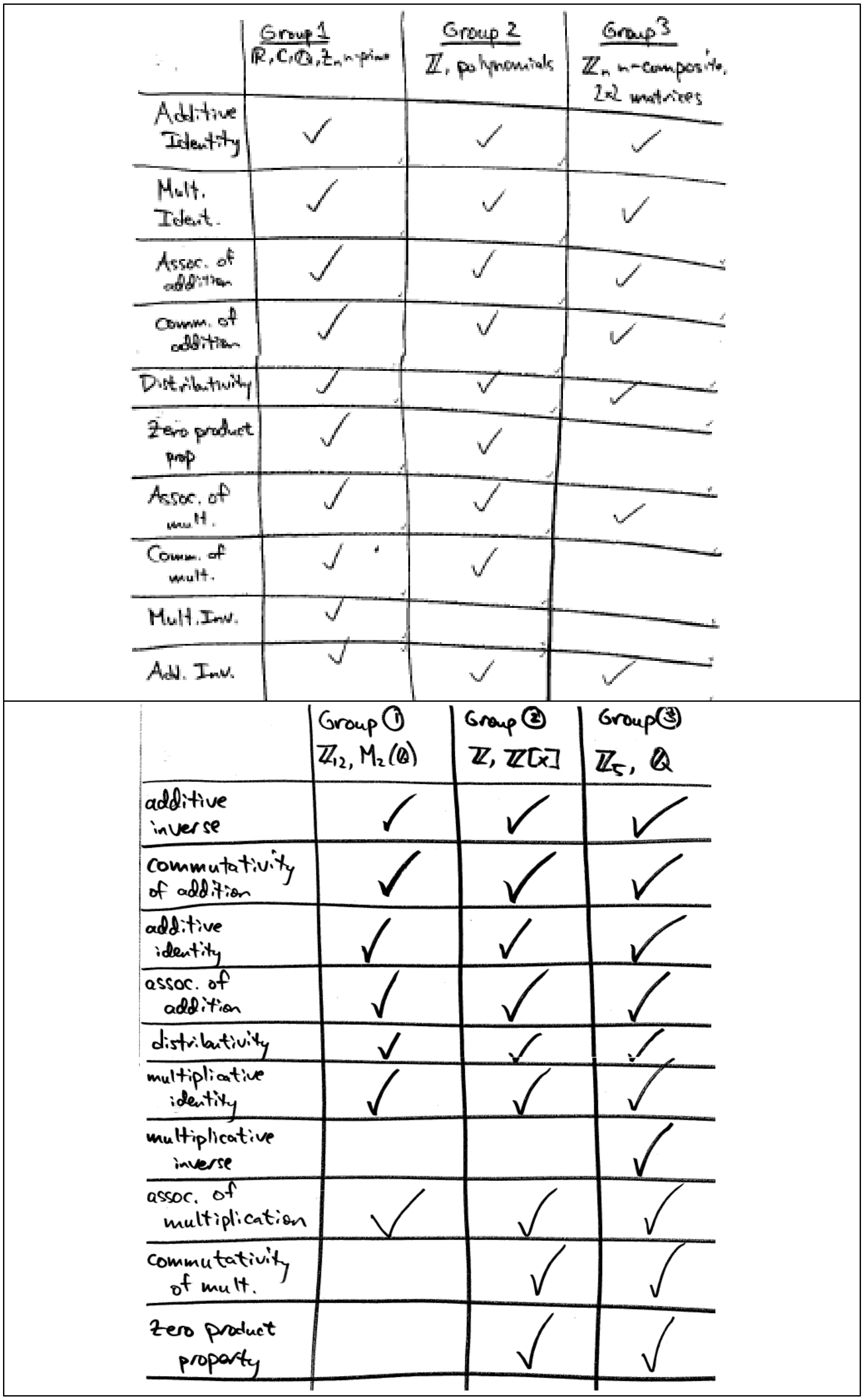
Figure 51. Compiling a list of rules (TE1 and TE2).

The students then began examining their previous work with equation solving to determine which of the rules held for each of the groups. To help the students write their list of criteria for each group of structures, I provided a table by which they could summarize their findings (see Figure 52). Along the left are all of the rules the students had agreed upon, and across the top are the different groups of structures from the sorting activity (this, like the previous activity, was exactly the same across both teaching

experiments, up to a reordering of the rules and the labeling of the groups).

This served as a springboard for the process of defining. I followed Larsen's (2004) guidelines for supporting a cyclic process of presenting and revising a definition:

4. The students prepared a definition.
5. I read and interpreted the definition, calling attention to particular choices made by the students.
6. The students revised their definitions as necessary and restarted the process.



	Group 1 $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}_n$ n-prime	Group 2 $\mathbb{Z}$, polynomials	Group 3 $\mathbb{Z}_n$ n-composite, 2x2 matrices
Additive Identity	✓	✓	✓
Mult. Ident.	✓	✓	✓
Assoc. of addition	✓	✓	✓
Comm. of addition	✓	✓	✓
Distributivity	✓	✓	✓
Zero product prop	✓	✓	
Assoc. of mult.	✓	✓	✓
Comm. of mult.	✓	✓	
Mult. Inv.	✓		
Add. Inv.	✓	✓	✓

	Group 1 $\mathbb{Z}_{12}, M_2(\mathbb{Q})$	Group 2 $\mathbb{Z}, \mathbb{Z}[x]$	Group 3 $\mathbb{Z}_5, \mathbb{Q}$
additive inverse	✓	✓	✓
commutativity of addition	✓	✓	✓
additive identity	✓	✓	✓
assoc. of addition	✓	✓	✓
distributivity	✓	✓	✓
multiplicative identity	✓	✓	✓
multiplicative inverse			✓
assoc. of multiplication	✓	✓	✓
commutativity of mult.		✓	✓
zero product property		✓	✓

Figure 52. Abstracting common properties for TE1 (top) and TE2 (bottom).

Ring with identity. I suggested that they start with the group containing Z_{12} and either $M_2(Z)$ or $M_2(Q)$ because I reasoned that starting with the most general set of structures would provide them with the option of defining subsequent structures in terms of this one. The students' first attempts at a definition for this structure (ring with identity) are shown below. For TE1, these structures were contained in what they named group 3; for TE2, group 1.

R

A set that you can add and multiply them is part of group 3 iff the following properties are satisfied

Additive identity: $\exists 0 \in S$ s.t. $\forall A \in S$ $A+0=A$

Multiplicative identity $\exists 1 \in S$ s.t. $\forall A \in S$ $1 \cdot A = A = A \cdot 1$

Additive inverse $\forall A \in S$ $\exists (-A) \in S$ s.t. $A + -A = 0$

Association of addition $\forall A, B, C \in S$ $(A+B)+C = A+(B+C)$

Communitivity of addition $\forall A, B \in S$ $A+B = B+A$

Distributivity $\forall A, B, C \in S$ $A(B+C) = AB+AC$ and $(B+C)A = BA+CA$

Associativity of multiplication $\forall A, B, C \in S$ $(AB)C = A(BC)$

A structure is in Group 1 if and only if the following conditions hold a, b, c are elements of the structure $\in R$

additive inverse $\forall a\ \exists (-a)$; $a + (-a) = 0$

associativity of addition $\forall a,b,c$ $(a+b)+c = a+(b+c)$

commutativity of addition $\forall a,b$ $a+b = b+a$

additive identity $\forall a$, $a+0=a$

distibutivity $\forall a,b,c$ $ab+ac = a\cdot(b+c)$

multipliactive identity $\forall a$, $a \cdot 1 = a$

assoc. of mult. $\forall a,b,c$ $(a \cdot b)\cdot c = a \cdot (b \cdot c)$

Figure 52. Initial definitions for ring with identity for TE1 (top) and TE2 (bottom).

This, of course, is a preliminary definition of a ring with identity. At this point, there were still a number of issues to be tended to, the most notable of which were (1) misuse of quantifiers, (2) consequences of the absence of commutativity of multiplication, and (3) absence of the two binary operations. In regards to revising the statements of these axioms, Larsen and Zandieh's (2007) method of *proofs and refutations* were especially helpful.

Issues with quantifiers. As can be seen in the above definitions, the primary errors the students made in the definition of ring with identity were issues of ordering and absence of quantifiers. Confirming the findings of Selden and Selden (1995) and Findell (2001), for example, the students in TE1 initially saw no issue with a statement like "for every a in S there exists a 0 in S such that $a+0=a$"[18]. A similar error was committed in TE2, wherein Haden and Laura neglected to use an existential quantifier in their statement of the additive identity property. I brought these respective issues to their attention.

Teaching Experiment 1

JP: So do you want the 0 that works for a to also work for any other element?
Carey: Sure!
JP: As you have it stated right now, that's not necessarily going to be true.
Jack: I feel like that's the same thing, though. I don't see the problem.
JP: The way it's stated right now includes the possibility that each element could have its own additive identity. And, while that's true in a sense, that's not what you want to say. What do you want to say?
Carey: That 0 works for all of them.
JP: Exactly.
Carey: So we can put zero first?

[18] Such a statement, of course, implies that the "0" element for which $a+0=a$ may not work for another element. In other words, with this phrasing, it may not be concluded that $b+0=b$, for some b not equal to a.

Teaching Experiment 2

JP: So, do you want zero to be an element of your set?

Laura: Yeah … isn't it?

Haden: Yeah, that's what the additive identity property says.

JP: Well, right now it says that a+0=a, but you haven't required 0 to be contained in the set.

Haden: I think it's implied.

JP: What about the natural numbers? For any n, n+0=n, but 0 isn't a natural number.

Haden: Oh yeah, that makes sense.

Laura: Maybe we could just say that there is a zero in the set at the beginning.

This occurred and was dealt with similarly for the statements of the multiplicative identity, additive inverse, and multiplicative inverse axioms (if needed).

Consequences of the absence of commutativity of multiplication.

Immediately after the revision of statements in which quantifiers were used incorrectly, the statements of the multiplicative identity and distributivity axioms assumed the use of commutativity of multiplication, such as these initial statements of the axioms (reproduced from videotape) from TE1 (the statements in TE2 were similar):

Multiplicative identity: $\exists\, 1 \in S \ \mathrm{s \cdot t} \ \ \forall A \in S \quad A \cdot 1 = A$

Distributivity: $\forall A, B, C \in S \ \ A(B+C)=AB+AC$

I challenged these statements by asking about multiplication on the right:

Teaching Experiment 1
JP: So, does it follow that 1 times A is A?
Jack: Uhh … yeah.
JP: How do you know?
Jack: Because … that's just what it is. [Laughs]
JP: Not according to your rules.
Carey: We can just flip it.
JP: Can you? Can you use that rule here?
Carey: Oh, well, yeah … So … well, it's still true on matrices, too.
JP: But now we are talking about a general structure, R. If you were to solve ax=ab on some R …
Jack: Does this mean we have to include the other side in our rules?

The issue of distributivity was taken care of when I asked the students if there were similar issues with any of the other properties. The students in TE2 handled both issues as a part of the same conversation:

Teaching Experiment 2
JP: What does this say about what 1 times a is equal to?
Haden: Oh, we couldn't say, because we don't have commutativity.
JP: So that has to be taken into account somewhere.
Haden: So we need to have two multiplicative identity rules?
JP: Not necessarily. You can combine them. Are there any other ones that might be affected by the lack of commutativity?
Laura: Distributivity, maybe.
JP: How would it affect distributivity?
Laura: If you had (b+c)a.

Including and defining binary operation. In TE1, I introduced the concept of binary operation by framing it around a comment that Jack had made in a previous session about the rows of the operations tables being functions.

Teaching Experiment 1

JP: I want to focus on something that Jack said earlier about the operation tables from the beginning. You said that the set can't satisfy these properties on its own.

Jack: Right … function over the set.

JP: OK, so you can think of addition and multiplication as functions. What is the domain of these functions?

Jack: It would be the … set.

JP: Let's call the set R.

Jack: So we have a function that maps R to R, right?

Jack proceeded to write down his preliminary definition of these functions on the set *R*.

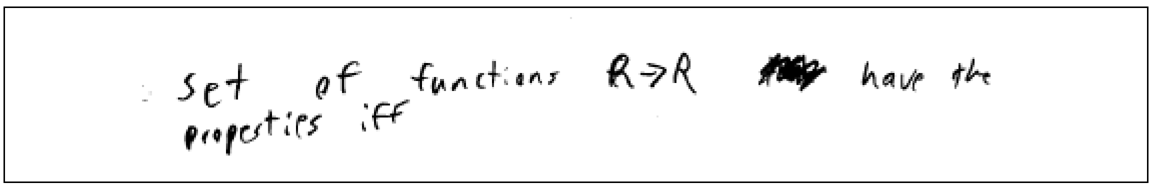

set of functions $R \to R$ have the
properties iff

Figure 54. An initial conception of binary operation (TE1).

Jack's assertion that the domain is *R* is consistent with his earlier views that the rows of the operation tables are functions, or, more specifically, unary operations. In fact, it appears that he believed the structure to be a set of functions when he stated after the above conversation that "we basically have two sets of functions, right? We have the addition and the multiplication?".

In TE2, since there had been no previous mention of functions over the given sets, I used Larsen's (2004) technique of asking the students if a set on its own can have any of these properties:

Teaching Experiment 2

JP: Do these structures, can these sets have these properties on their own? Can just the elements of these sets have these properties, or is there something else that needs to be included?

Laura: Like, are you saying that there needs to be an operation called multiplication?

JP: Yeah, otherwise we haven't defined addition and multiplication. What would you say that addition and multiplication are?

Laura: Functions.

JP: Yeah, so how do we determine functions? What are some of the things that we consider?

Laura: Domain and range.

JP: Right. What is the domain of addition? If we call the structure R.

Laura: R.

Thus, the students were both thinking of the domain of the operations as simply R from separate routes. I challenged this similarly to how I challenged the manner in which they ignored associativity by asking how many elements are added at one time.

Teaching Experiment 1

JP: When you add elements together, how many elements are you adding together at one time?

Carey: Two.

JP: And where do both of these elements come from?

Carey: R.

JP: So you are starting with two elements. But only single elements can come from a domain of R. R to R just means you are taking one value of R to another.

Jack: Yeah.

JP: Do you see the problem?

Jack: Couldn't you say function mapping a to b if a and be were elements of R?

JP: But, as you've said, you are actually taking two elements instead of just one.

Even though Jack and Carey have identified that two elements are taken from the domain each time the addition function is applied, they still seem to believe that the domain is still *R* instead of the set of ordered pairs. I ask them about other sets in which two "elements" are considered:

Teaching Experiment 1

JP: Can you think of other sets that you always take two elements from?
Carey: Umm … the coordinate plane?
JP: And what is the coordinate plan?
Carey: R^2. (*Author's note: 2 is an exponent here rather than a footnote)*
JP: Exactly!

A similar conversation occurred in TE2, but Haden and Laura quickly realized that the domain would be the set of ordered pairs over R instead of just R itself. It is intriguing, however, to note that Laura also used the real coordinate plane as a metaphor in this situation.

Teaching Experiment 2

JP: When you are adding something, how many things are you adding together?
Haden: Two.
Laura: Then wouldn't that just be a 2 right there?
JP: What do you mean?
Laura: I'm confused with notation. When you have the real numbers, you have something like R^2. Can we do that here?

I told the students that functions of this type were called *binary operations*, and I then asked them to formally define the term based on our previous discussion:

Teaching Experiment 1[19]	*Teaching Experiment 2*
R^2 $R \times R \to R$ (a,b) s.t. $a, b \in R$	binary operation : function that maps 2 elements of R to 1 element of R. i.e. $f : R^2 \to R$

Figure 55. Definitions of binary operation (TE1 and TE2).

At this point, the students continued to incorporate this into their revised definition (along with the revised statements of the axioms for additive identity, additive inverse, and multiplicative identity). It was also at this point that I named the structure for them so that they could write what turned out to be their final definition of a ring with identity:

[19] Notice that the TE1 definition does not include the term "function", something I did not notice at the time (perhaps because they had been using the term with regularity). Nonetheless, it is clear from their previous commentary that they do regard it as a function.

Teaching Experiment 1	A set R with $+: R \times R \to R$ and $\cdot : R \times R \to R$ is a ring (with identity) iff the following properties are met: Additive identity $\exists\, 0 \in R$ s.t. $\forall A \in R$ $A + 0 = A$ Multiplicative identity $\exists\, 1 \in R$ s.t. $\forall A \in R$ $1 \cdot A = A$ Association of addition $\forall A, B, C \in R$ $(A+B)+C = A+(B+C)$ Commutativity of addition $\forall A, B \in R$ $A + B = B + A$ Distributivity $\forall A, B, C \in R$ $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$ Associativity of multiplication $\forall A, B, C \in R$ $(AB)C = A(BC)$ Additive inverse $\forall A \in R \;\exists (-A) \in R$ s.t. $A + (-A) = 0$
Teaching Experiment 2	A ring with identity is a set R… with binary operations $+: R^2 \to R$ and $\cdot : R^2 \to R$ such that 1. ADDITIVE INVERSE $\forall a\; \exists (-a);\; a + (-a) = 0$ 2. ASSOCIATIVITY OF ADDITION $\forall a,b,c \quad (a+b)+c = a+(b+c)$ 3. COMMUTATIVITY OF ADDITION $\forall a,b \quad a+b = b+a$ 4. ADDITIVE IDENTITY $\exists 0 \in R; \forall a \quad a + 0 = a$ 5. DISTRIBUTIVITY $\forall a,b,c \quad a \cdot b + a \cdot c = a \cdot (b+c)$; $b \cdot a + c \cdot a = (b+c) \cdot a$ 6. MULTIPLICATIVE IDENTITY $\exists 1 \in R; \forall a \quad a \cdot 1 = a = 1 \cdot a$ 7. ASSOCIATIVITY OF MULTIPLICATION $\forall a,b,c \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Figure 56. Revised definitions of ring with identity (TE1 and TE2).

Integral domain. After this definition was completed, they focused their attention on group 2 (which, in both teaching experiments, were the integral domains that were not also fields). Before starting the process of defining these structures, I gave the students the name *integral domain.* Because they had already written out multiple drafts of the previous definition (a rather lengthy process), I used this as an opportunity to engage them in a conversation about how they could shorten the process:

Teaching Experiment 1

JP: So, as you guys have correctly noted, writing all of these out is a huge [inconvenience], so if we wanted to write out, say the next one, knowing that we have this definition down now, what's a way that we could shorten the next one.

Jack: We just say, if it's A ring[20], and has the following properties.

JP: Okay. So, how would you do that? …

Jack: Uh, oh if you wrote the main rings then the difference between a ring with identity is that [an integral domain] has a few more properties.

Teaching Experiment 2

JP: The structures in group 2 are called integral domains. It's obvious to see where that comes from. How do you think you could define an integral domain?

Laura: Could we define it in terms of rings?

Haden: If we are allowed to do this, we could say, it's a ring that also has … and then list the additional properties.

JP: Yeah, you can start with a ring with identity and go from there.

The students also contended that, due to the presence of commutativity of multiplication, they need not worry about writing the "double-sided" versions of the

[20] The students across both teaching experiments often shortened the term "ring with identity" to "ring" in conversation.

axioms. Since they had already attended to all issues with quantifiers and binary operations for the previous definition, this one quickly followed in one take. The finalized version of this definition follows:

Teaching Experiment 1	A set R with $+: R \times R \to R$ and $\cdot: R \times R \to R$ is an integral domain iff it is a ring (with iden and has the following properties: Zero product property: $\forall A, B \in R$ $AB = 0 \Rightarrow a = 0 \vee b = 0$ Commutativity of multiplication $\forall A, B \in R$ $AB = BA$
Teaching Experiment 2	An integral domain is a ring with identity R such that 8. COMMUTATIVITY OF MULTIPLICATION: $\forall a \forall b$ $a \cdot b = b \cdot a$ 9. ZERO PRODUCT PROPERTY: $a \cdot b = 0 \Rightarrow a = 0 \vee b = 0$

Figure 57. Definitions of integral domain (TE1 and TE2).

Field. Once they had moved on to fields, both pairs of students quickly recognized that they could define the structures in the remaining group by using a method similar to what they had done for integral domains:

Teaching Experiment 1	A set R with $+: R \times R \to R$ and $\cdot : R \times R \to R$ is a field iff it is an integral domain & satisfies the following ~~propose~~ property: Multiplicative inverse: $\forall A \in R, \exists A^{-1} \in R$ s.t. s.t. $A \neq 0$, $A(A^{-1}) = 1$
Teaching Experiment 2	A FIELD is an INTEGRAL DOMAIN such that 10. MULTIPLICATIVE INVERSE $\forall a^{\neq 0} \exists a^{-1} ; a \cdot a^{-1} = 1 = a^{-1} \cdot a$

Figure 58. Definitions of field in terms of integral domain (TE1 and TE2).

There was one only one conflict with the students' definition for field: stating that *every* element of the set had a multiplicative inverse (the fact that the definitions currently show the "nonzero" qualifier are the result of the students revisions; these were not initially included).

Including zero in the multiplicative inverse axiom. As may be apparent by examining the above student writing closely, the original statements of the multiplicative inverse axiom across both teaching experiments failed to exclude 0 from having a multiplicative inverse. To achieve this, I simply prompted them to review their equation solving activities. Though the students were initially confused by being asked to do this, they eventually recognized their error and excluded zero from the axiom. In retrospect, a more direct way to compel the students to recognize and address the conflict might have been for them to explicitly name the multiplicative inverses for every element of a finite field, such as Z_5.

Eliminating unnecessary rules. To my surprise, throughout the entire reinvention process, the students across both teaching experiments had only introduced 2 rules which were unnecessary: (1) $x \cdot 0 = 0$, and (2) the inclusion of the zero-product property in the definition of a field (in terms of a ring with identity). I addressed the former by directly asking the students if they could prove it from their other axioms. Shown below is the proof presented by the students in TE2 (the proof from TE1 was very similar):

$$
\begin{aligned}
&= x \cdot 0 + 0 \\
&= x \cdot 0 + (x + (-x)) \\
&= (x \cdot 0 + x) + (-x) \\
&= x(0 + 1) + (-x) \\
&= x(1) + (-x) \\
&= x + (-x) \\
&= 0
\end{aligned}
$$

Figure 59. Eliminating an unnecessary axiom (TE2).

For the zero-product property in a field, I asked the students if they could state the process of defining a field in terms of a ring with identity. Since they had already defined it in terms of an integral domain, this proved to be quite simple.

Teaching Experiment 1[21]	A set R with $+: R \times R \to R$ & $\cdot: R \times R \to R$ is a field iff R is a ring (with identity) & satisfies the following properties: ~~Zero product property~~ ~~$\forall A, B \in R$~~ ~~[illegible]~~ Commutativity of multiplication $\forall A, B \in R$ $AB = BA$ Multiplicative inverse $\forall A \in R$ s.t. $A \neq 0$, $\exists A^{-1} \in R$ s.t. $A(A^{-1}) = 1$
Teaching Experiment 2	A FIELD is a ring with identity R such that 8. Commutativity of multiplication: $\forall a \forall b \; a \cdot b = b \cdot a$ 9. Zero product property: $a \cdot b = 0 \Rightarrow a = 0 \vee b = 0$ 10. Multiplicative inverse: $\forall a^{\neq 0} \exists a^{-1}; \; a \cdot a^{-1} = 1 = a^{-1} \cdot a$

Figure 60. Definitions of field in terms of ring with identity (TE1 and TE2).

In TE1, I reminded the students of a previous comment that Jack had made:

> *Teaching Experiment 1*
>
> Jack: Well, if you are using the real numbers, then zero-product property is easy [writes ab=0]. Divide by b. Since you are using real numbers, the division only holds if b is nonzero. So you can conclude that if b is not zero, then a is zero.

In TE2, I simply asked the students if they could prove it in a field. Their proofs of this fact are shown in Figure 61 (again, the students in TE1 use right multiplication):

[21] The property that has been crossed out is the zero-product property. This crossing-out was done after the students had proved that it was unnecessary.

Teaching Experiment 1	*Teaching Experiment 2*
$(AB)(A^{-1}) = 0\,(A^{-1})$ $(BA)(A^{-1}) = 0$ $B(A(A^{-1})) = 0$ $B1 = 0$ $B = 0$	proof: $a \cdot b = 0$ if $a = 0$, the result is trivial so let $a \neq 0$ $a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$ $(a^{-1} \cdot a) \cdot b = 0$ $1 \cdot b = 0$ $b = 0 \quad \square$

Figure 61. Proving the zero-product property in a field (TE1 and TE2).

These proofs, in addition to the other ring-arithmetic proofs that the students constructed (which have been omitted) are examples of *proofs that justify structure* (Weber, 2002) because the use of the axioms to demonstrate that the proposition is true justifies their inclusion in the original definitions.

Using the Reinvented Definitions

Upon the reinvention of the definitions of ring with identity, integral domain, and field, I turned the students' attention to tasks that required the use of their reinvented definitions. These tasks were selected to either test certain research hypotheses that arose during the course of the teaching experiment or evoke certain types of reasoning with the definitions.

Classifying different ring structures. Large parts of the instructional theory and instructional tasks up to this point focused on being able to differentiate between various ring structures; therefore, it was natural to present the students with an opportunity to classify concrete examples according to these definitions.

***$Z_3 \times Z_3$*.** In both teaching experiments, I gave the students a task which asked them to classify $Z_3 \times Z_3$ (with the usual component-wise operations modulo 3). I anticipated that they would first conjecture that it is a field (since Z3 is a field), but upon further investigation, they would find that this was not the case. This conjecture proved to be partially true. While both groups of students did not explicitly state that they believed the new structure to be a field, the axioms on which they concentrated were those characteristic of a field: multiplicative inverses and the zero-product property. Indeed, after verifying that all of the axioms for a ring with identity (plus commutativity of multiplication) held, they once again turned their attention to multiplicative inverses and the zero-product property. Now independent of the original task setting, the equation solving model emerges as a *model-for* classifying structure.

Attempting to define division. The students in both teaching experiments, after quickly identifying (1,1) to be the multiplicative identity, decided to try to find multiplicative inverses for all nonzero elements. They started this process by writing down all of the elements that clearly had inverses:

Teaching Experiment 1	*Teaching Experiment 2*
$\langle a,b\rangle \cdot \langle c,d\rangle = \langle 1,1\rangle$ $ac=1$ $bd=1$ everything $\langle a,b\rangle$ where a & $b \neq 0$	multiplicative inverses (1,1) (2,1) (1,2) (2,2)

Figure 62. Finding units in $Z_3 \times Z_3$ (TE1 and TE2).

Jack and Carey (TE1) immediately realized that no element with a 0 in one of the components could have an inverse because of the conditions, *ac*=1 and *bd*=1. Haden and Laura (TE2) recognized this after they had written down four elements that they knew had inverses. I asked Haden and Laura if they could prove that this was a complete list.

assume

$(a,b)\cdot(0,c)=(1,1)$, $c\neq 0$

then $a\cdot 0=1$

$0=1$

which is a contradiction. so the assumption is false.

So not all nonzero elements have mult. inverses.

Figure 63. Proving that $Z_3 \times Z_3$ is not a field (TE2).

Thus, the students had successfully proven that $Z_3 \times Z_3$ is not a field. The question was then posed: could it be an integral domain?

Discerning the zero-product property. The students, now having worked with two examples of rings for which the zero-product property does not hold, knew instantly to look for counterexamples to the property, or zero divisors.

Teaching Experiment 1	*Teaching Experiment 2*
$\langle 0,a\rangle \cdot \langle b,0\rangle = \langle 0,0\rangle$ $a,b \neq 0$ So zero-product prop. doesn't hold.	$(0,1)\cdot(1,0) = (0,0)$

Figure 64. Zero-Divisors in $Z_3 \times Z_3$ (TE1 and TE2).

Thus, having checked or assumed the truth of the other axioms, they concluded that $Z_3 \times Z_3$ is a ring with identity that also has a commutative multiplication (in both teaching experiments, I used this opportunity to introduce the notion of a *commutative ring*).

Z×Z. As a follow-up in TE1, I asked the students to classify the infinite ring $Z \times Z$. The students then discussed the zero product property:

Teaching Experiment 1
Jack: Z cross Z would be…
Carey: Don't we, like, have a similar problem?
Jack: It would still be a ring.
Carey: Yeah.
Jack: It would have the exact same problem with zero-product property 'cause there's going to be…you can just take pairs of zeros out of it.

They concluded that, since the zero-product property did not hold, that $Z \times Z$ could not be an integral domain or a field.

***R*×*S*.** In TE2, I followed-up with a more general question: for any two commutative rings with identity R and S, could their direct sum ever be an integral domain or a field?

Teaching Experiment 2

Haden: The properties in question are the zero product property and having a multiplicative inverse.

Laura: I think no.

Haden: Because none of our arguments were specific to Z_3. The fact that we can find some multiplicative inverse might not always hold.

JP: If you are saying that they don't hold, just find a counterexample.

Haden: Right.

JP: Or if they do hold, then try to prove them.

Haden: For multiplicative inverses, you can use exactly the same proof, because nothing in here says anything about Z_3, so that's the same proof. If we switch one out.

Laura: That is of course assuming that the commutativity proof depended on the original being commutative.

Haden: But that could only decrease its qualifications.

Laura: I mean as long as its commutative the Cartesian product is commutative.

Haden: But it's definitely never an integral domain or a field. And it's the same proof if you switch one variable. So it's the same proof. On the zero product property, you could switch 1 out for a variable, in case you were doing the even integers and didn't have 1.

Haden and Laura here mathematize their previous mathematical activity with a finite product of rings to answer a question about general products of rings. Overall, these excerpts demonstrate that the students had a functional, working knowledge of the definitions they reinvented in addition to displaying their activity in a new mathematical reality.

***Classifying student-generated examples*.** The students in TE2 had an additional task at this point: generate and classify their own examples (recall that the students in TE1 had done this previous to their reinvention of the definitions). Haden and Laura came up with the following examples (*F* below refers to a generic field):

$\mathbb{Q}[x], \mathbb{R}, \mathbb{N}, \mathbb{Z}_{n\,\text{composite}}, \mathbb{Z}_{n\,\text{prime}}, M_n(F)$

Figure 65. Student-generated examples of two-operation structures (TE2).

Recall that Q[x] had been previously proposed to be a structure similar to Q (before the definitions had been reinvented), but the students arrived at the conclusion that it was indeed similar to Z[x] more than anything else. Notice also that, just as the students in TE1 had done, Haden and Laura distinguished between the integers modulo a prime and modulo a composite as a result of their experiences with Z_5 and Z_{12}. Several interesting conversations ensued regarding how these structures would be classified.

> *Teaching Experiment 2*
> Laura: R is definitely a field.
> JP: What about N?
> Haden: No additive inverse. I'm sure that there are some others that it knocks out.
> JP: You mentioned earlier that it doesn't have the additive identity, either.
> Haden: Right, because there's no zero.
> [students examine structures]
> Haden: Z_n composite would be rings.
> Laura: Yeah, not an integral domain.
> JP: Why not an integral domain?
> Laura: Because it doesn't have … what property was it? Commutativity or zero-product?
> Haden: It's multiplicative inverse that's not defined on all of them. Zero-product property is what we proved from that.

Interestingly, despite the fact that a similar conversation had taken place earlier, the students again asserted that Q[x] is a field.

Teaching Experiment 2
JP: What would Q[x] be?
Haden: I think it would be a field.
Laura: I think so … maybe not, actually.
JP: How did we define a polynomial?
Laura: There's no multiplicative inverse.
Haden: Don't you with Q though, because if Q is the rational numbers …
JP: So, x^2 is a polynomial in Q[x].
Haden: … there is no x^{-2}.
JP: Right, you've got it.

This raises interesting questions for future research about what students intuitively believe about the properties inherited by polynomial rings from their "base structures".

Apprehending the set of units. In both teaching experiments, I gave the students an exercise prompting them to find the units of a given ring. A sample of the results from each teaching experiment is shown below:

Teaching Experiment 1	*Teaching Experiment 2*
$(\mathbb{Z}_2, +_2, \cdot_2)$ 1 $(\mathbb{Z}_6, +_6, \cdot_6)$ 1, 5 $(\mathbb{Z}_3, +_3, \cdot_3)$ 1, 2 $(\mathbb{Z}_7, +_7, \cdot_7)$ 1, 2, 3, 4, 5, 6 $(\mathbb{Z}_4, +_4, \cdot_4)$ 1, 3 $(\mathbb{Z}_8, +_8, \cdot_8)$ 1, 3, 5, 7 $(\mathbb{Z}_5, +_5, \cdot_5)$ 1, 2, 3, 4	$\mathbb{Z}_6$ 1, 5 $\mathbb{Z}_{12}$ 1, 5, 7, 11 $\mathbb{Z}$ 1, -1 $\mathbb{Q}$ all $n \in \mathbb{Q}, n \neq 0$ $\mathbb{Z}[x]$ 1, -1 $2\mathbb{Z}$ none

Figure 66. Finding and apprehending the set of units (TE1 and TE2).

At the completion of this exercise, I prompted the students to examine a possible ring structure on these sets of units. In both cases, the students found that, in addition to multiplication being the only function which was a binary operation on the set of units, four of the ring axioms held:

Teaching Experiment 1	*Teaching Experiment 2*
Associativity of multiplication Commutativity of multiplication Multiplicative inverse Multiplicative identity	mult. inverses mult. identity mult. commutativity mult. associativity

Figure 67. Preliminary definition of an abelian group (TE1 and TE2).

These axioms, of course, are the primary axioms for an abelian group. This approach to developing a group structure was not explored any further in this study, but may be an effective means of introducing the group structure in a situation where rings have been learned first. On the other hand, if groups were learned first, an exercise like this could give students an opportunity to identify the group structure.

Definition of subring. The notion of a subring had come up several times during the course of the second teaching experiment, primarily when Haden and Laura were considering the nature of Z[x] and Q[x]. Thus, after they had reinvented the definitions, I asked them if they had any inclinations about how the term *subring* could be defined. They started with the following definition, no doubt using the definition of subset as a guide:

a ring whose elements are fully contained in another ring.

Figure 68. Preliminary definition of subring (TE2).

Using a modification of Hazzan and Leron's (1996) question "Is Z_3 a subgroup of Z_6?", I asked the students "Is Z_5 a subring of Z_{12}?".

> *Teaching Experiment 2*
> JP: The question is, is Z_5 a subring of Z_{12}?
> Haden: Well, Z_5 is a ring because it's a higher thing so it has to at least be a ring. By that definition, yes.
> JP: By your definition it is. The question is: should it be?
> Haden: Depends on what you want from your definition.

The students, particularly Haden, did not initially see any conflict, perhaps because of an incomplete idea of what the definition should encompass (as I had not given them any concrete examples of subrings; instead, I anticipated that they would be able to infer it from the context). Perhaps this could have been addressed by asking about other examples (as opposed to counterexamples like the one above). I challenged this by asking them to evaluate the sum 3+3 both in Z_{12} and Z_5. Once they had recognized that the sum was inconsistent, I encouraged them to revise their initial definition to address this discrepancy.

> *Teaching Experiment 2*
> Laura: The difference is just that there is no 6 in Z_5.
> Haden: It seems like you are doing the same operation except with Z_5 you loop back at some point.
> JP: So how can you revise this definition to make sure that something like this doesn't happen?

I was anticipating that they would append a qualifier like “with respect to the operations of the larger ring” to their previous definition. While their method was valid and effectively addressed the conflict, it was also unconventional.

A subring R_s of a given ring R is a ring that obeys

1. $R_s \subset R$
2. $\forall a \forall b \forall c \; a+b=c$ on $R_s \iff a+b=c$ on R
3. $\forall a \forall b \forall c \; a \cdot b=c$ on $R_s \iff a \cdot b=c$ on R

Figure 69. Revised definition of subring (TE2).

Summary

This chapter has detailed the results of both teaching experiments wherein the students reinvented and used the definitions of ring (with identity), integral domain, and field. The results in this chapter were organized in a cross-sectional manner as follows (written in terms of the corresponding student tasks):

1. Solving equations on various ring structures;
2. Summarizing results of solving equations;
3. Sorting the structures based on equation solving;
4. The defining process;
5. Using the reinvented definitions.

The next stage of analysis uses Gravemeijer’s (1999) phases of the emergent model transition to lay the groundwork for a local instructional theory supporting the guided reinvention of these definitions.

Chapter 6: Conclusions

In this chapter, I present conclusions that are supported by the empirical results detailed in Chapter 5. In particular, I describe the specific phases of the emergence of the equation solving model using my expanded version of Gravemeijer's (1999) phases of the emergent model transition[22]. I then use this to delineate the evolution of the concepts of additive inverse and identity, zero-divisor, and unit throughout the reinvention process. The phases of the emergent model transition also lay the foundation for the refined local instructional theory, the principal finding from this project. Finally, I discuss issues related to limitations and generalizability for this project and conclude with recommendations and possible avenues and for future research.

Phases of the Emergent Model Transition

In this section, in order to identify the milestones of the reinvention process and lay the groundwork for the refined local instructional theory, I use Gravemeijer's (1999) phases of the *model-of* to *model-for* transition along with the three intermediate phases detailed in Chapter 3.

Preliminary: Constructing operation tables for finite rings. Though "preliminary" is not named as a phase of the emergent model transition and could have easily been classified as situational, I include it here to illuminate the importance of the students' work with the operation tables and their informal work solving equations. The students were able to anticipate certain facets of the ring structure from these tables, including commutativity of the operations, the additive identity, the

[22] The phases of the emergent model transition are detailed in Chapter 3.

multiplicative identity, units, and zero divisors. In this way, the operation tables for Z_{12} and Z_5 served as a *record-of* the additive and multiplicative structures of these respective rings and then as a *tool-for* the equation solving that followed.

Situational: Solving specific linear equations on various rings. In addition to being purposefully designed as the original task setting, I classified the solving of specific linear equations on these structures as situational because it involved the students working towards a mathematical goal in an experientially real context (Gravemeijer, 1999). In this way, solving these equations not only provided a means by which the students identified important aspects of the ring structure, but also helped to establish equation solving as an experientially real starting point.

Situational anticipating referential: Proving the cancellation laws for various rings. While this activity could be classified as simply situational (as the cancellation equations could have been the focus of the original task setting on their own), it also anticipated the referential phase because the students often referenced their work solving the specific linear equations to help them prove (or disprove) the cancellation laws. In other words, the specific equations were used as a paradigm the students could reference to solve $x+a=a+b$ and $ax=ab$. Consequently, I classified this activity in the intermediate stage of situational anticipating referential. Some important features the students referenced included associativity and the possibility of multiplicative inverses and zero-divisors. It is important to note that these features might have been easier to ignore or overlook had the students not solved specific equations first. Solving these general equations was also designed to promote the summarization of their previous activity, thus anticipating their need to reference these results at a later stage.

Referential: Summarizing the proofs of the cancellation laws. After making primitive conjectures and proving statements regarding the cancellation laws on each of the examples, the students were prompted to summarize their work (so that the similarities and differences between the structures would be more apparent). I classified this task as referential because it was distinct from the original task setting yet referenced the students' prior experience from it. Additionally, the students' equation solving was still a *model-of* their informal activity.

Referential anticipating general: Sorting the structures. The students then mathematized their prior activity and sorted the various ring structures based on characteristics that they had deemed important. I categorized this as part-referential because it involved referencing their previous activity (summarizing their results in a chart and, to a lesser extent, the actual equation solving tasks). In this way, the task was not yet "independent of the original task setting," a characteristic of the general phase. Sorting based on common features, however, anticipated the mathematical activity of abstraction, which certainly qualifies as general activity.

General: The process of defining. To initiate the process of writing a definition, I prompted the students to devise a list of the rules and properties common to the structures in the most general category from the sorting activity. I deemed the process of defining as general because, finally, the equation solving model was completely independent of the original task setting and had emerged as a *model-for* writing a formal mathematical definition.

General anticipating formal: Writing nested definitions. The students' attention then turned to writing definitions for the other two sets of structures from the sorting activity. I categorized the activity of nesting these definitions as general anticipating formal because it still involved the defining of mathematical structure (which I previously argued was general) while it also served as a tool for classifying other ring structures. Additionally, constructing them in this manner emphasized the interrelationships between the three definitions. Thus, nesting the definitions prepared them for use in a more formal setting.

Formal: Using the definitions for more formal activity. The students were then given tasks to promote the use of the reinvented definitions. These tasks included determining if a given structure is a ring, classifying different examples of rings, apprehending the structure of the set of units, and constructing a definition of subring. These tasks qualify as formal because they reflect the emergence of a new mathematical reality.

On the Emergence of Structure: Additive Inverses, Units, and Zero-Divisors

The additive inverses, units, and zero-divisors are all of vital importance when determining if and when the cancellation law holds. These three components of the ring structure were present at some level throughout the entire reinvention process. As such, in this section I detail how the students in this study related to and identified these features during each phase of the emergent model transition. In turn, the breakdown of these concepts may be used as a framework for gauging students' thinking in an instructional sequence similar to those set forth in this project (including the sequence proposed in the next chapter).

Additive inverses and identity. The notion of additive inverses was present in the students' thinking from the very beginning, starting with the additive operation table for Z_{12}. The notion of identity, inextricably tied to that of inverse, was also inherently present in their thought process. While these ideas did not provide a means by which the students could differentiate between and sort the examples of rings, they were nonetheless present throughout all stages of the reinvention process. The following table demonstrates how these concepts were developed as the equation solving model transitioned from *model-of* to *model-for*.

Table 3

Conceptual Framework for the Emergence of Additive Inverse

Emergent Model Phase	Role of Additive Inverse
Preliminary: operation tables for finite rings	The zero element in each row and column
Situational: solving specific equations	The method used to isolate the x term in additive equations
Situational anticipating referential: proving the cancellation laws	The property underpinning the additive cancellation law (linked to the element -1)
Referential: summarizing the methods used to prove the cancellation laws	The unique method used to prove the additive cancellation law
Referential anticipating general: sorting structures	A basis on which no structures may be sorted since it holds for all examples
General: process of defining	An axiom appearing in all definitions that characterizes the identical additive structure
General anticipating formal: writing nested definitions	
Formal: using reinvented definitions	A means of eliminating non-examples of rings (such as the natural numbers); can be used to prove propositions (such as uniqueness of identity)

Units and zero-divisors. Across both teaching experiments, the students focused their attention on two crucial components of the multiplicative structure of each ring they encountered in an effort to define division and determine if the zero-product property held: units and zero-divisors, respectively. The students' tendency to attempt to define division seemed to be closely linked to their notions of additive inverses and subtraction. That is, the idea of additive inverses was horizontally mathematized by the students as they attempted to transfer the idea of inverses to the multiplicative ring structure.

The students were not directly prompted to attend to these structural features. Rather, in a manner similar to the student in Simpson and Stehlikova's (2006) case study, their interest arose as a result of the equation solving tasks they were given. In other words, these concepts emerged naturally as the students attempted to solve equations on the different ring structures.

As the students progressed through the various stages of the reinvention process, their conceptions of zero-divisors and units evolved accordingly. The following table summarizes the progression of these ideas as the students observed the operation tables, solved equations, defined the various ring structures, and used the concepts in a formal mathematical setting.

Table 4

Conceptual Framework for the Emergence of Unit and Zero-divisor

Emergent Model Phase	Role of Zero-divisors	Role of Units
Preliminary: operation tables for finite rings	The elements whose rows and columns have repeating patterns	The elements whose rows and columns do not repeat
Situational: solving specific equations	The elements a for which $ax=b$ has multiple solutions; division to isolate x is not possible	The elements a for which $ax=b$ has a unique solution; division to isolate x is possible
Situational anticipating referential: proving the cancellation laws	The elements that disrupt multiplicative cancellation	The elements that uphold multiplicative cancellation
Referential: summarizing the methods used to prove the cancellation laws	Emerge as a means to differentiate structures for which the zero-product property holds always, sometimes, or never	Emerge as a means to differentiate structures on which division is possible always, sometimes, or never
Referential anticipating general: sorting structures		
General: process of defining	Zero-product property is defining characteristic of integral domains	Division (multiplicative inverse) is defining characteristic of fields
General anticipating formal: writing nested definitions		
Formal: using reinvented definitions	Used to determine if new structure is an integral domain	Used to determine if new structure is a field

A conceptual framework. The identifications set forth in the above tables provide a functional conceptual framework for the development of the concepts of additive inverse (and identity), multiplicative inverse, and zero divisor. Originating as repetition in the operation tables and transitioning into a means by which one can distinguish and characterize structure, these ideas at the crux of proving the cancellation laws provide more insight and detail about the emergent model transition. More specifically, this framework further delineates how equation solving (and, to a lesser degree, the operation tables) served (or can serve) as a *model-of* the behavior of the additive and multiplicative structures of rings that eventually transformed into a *model-for* more formal mathematical activity.

The Refined Local Instructional Theory

The stages of the emergent model transition explicated previously in this chapter laid the framework for the refined local instructional theory presented in this section. The instructional theory, rooted in empirical evidence brought about through the developmental research cycle in this project, represents *one* way that reinvention of these concepts could be accomplished. No claims are made that this represents the *best* way to proceed in reinventing ring, integral domain, and field.

Additionally, recall that a local instructional theory is an abstraction of the rationale and methods used in the specific instructional tasks (rather than consisting of the instructional tasks themselves). Thus, the local instructional theory is meant as a guide that instructors can use to design instruction that is specific to and appropriate for their own students. For example, the experientially real starting point for reinventing need not be equation solving, so long as some other accessible starting point anticipates the important facets of the ring structure (as well as means for differentiating rings, integral domains, and fields). The framework for the local instructional theory is given in the following table.

Table 5

Framework for the Refined Local Instructional Theory

Task	Rationale
1. Establishing an experientially real starting point	Before beginning the reinvention process, an accessible starting point must be established for the students. The starting point should have the potential to evoke powerful informal understandings and anticipate critical aspects of the ring structure.
2. Apprehending various examples of ring structures	The model emerges as a *model-of* the students' informal activity of attending to the various intricacies of the ring structure by examining specific elements in the established experientially real context. The examples of ring structures should include at least one example of each type of structure and be as varied as possible.
3. Applying the model in a more general context	Tasks should be given which encourage the students to expound upon and summarize (in a more general context) the relationships previously identified. These tasks should anticipate the differentiation of the different structures.
4. Considering meaningful examples and non-examples of rings	The model may be refined (and omissions or misconceptions overcome) by considering carefully chosen additional examples or non-examples of rings.
5. Identifying similarities and differences among the various examples	Mathematizing their experience from the original task setting, the students should be prompted to identify similarities and differences among the different structures.
6. Sorting ring structures based on identified features of interest	The model, now a *model-of* the distinctions between ring, integral domain, and field, provides a means for the students to sort the structures.
7. Formulating definition of most general ring structure	The model, now completely independent of the original task setting, becomes a *model-for* defining the most general ring structure possible.
8. Formulating "nested" definitions for subsequent structures	The definitions of integral domain and field can be formulated in terms of previous definitions (when possible), enabling the students to explicitly and formally differentiate between the various structures.
9. Using the model for more formal mathematical activity	Tasks should be given which reflect the emergence of a new mathematical reality; the model has now emerged as a *model-for* formal mathematical activity

Each step of the refined local instructional theory is further explicated in the following subsections.

Step 1: Establishing an experientially real starting point. Following the guidelines set forth by Larsen (2004), an informal, meaningful, experientially real starting point must be established before proceeding. For example, if equation solving is the desired starting point (as it was in the instructional sequences in this project), then the instructor must first ensure that this activity is familiar to and meaningful for the students. Additionally, all of the examples of structures on which the students will solve equations must be experientially real. Possible deficiencies can be accounted for by designing preliminary activities whose purpose is to familiarize the students with the activity or structure. The students in this project, for instance, were very familiar with basic equation solving techniques but were only vaguely accustomed to modular arithmetic. Consequently, I administered the operation table tasks for Z_{12} and Z_5 so that their experience in completing these activities would eventually result in the students effectively solving equations on these structures.

Step 2: Apprehending various ring structures. Simpson and Stehlikova (2006) coined the term *apprehending structure* as the act of attending to interrelationships between the elements in the set which are consequences of the operations. These tasks and the set of examples upon which the tasks are performed should allow the students to "attend to aspects of the particular which will appear as important facets of the general" (Simpson & Stehlikova, 2006, p. 349). To this end, the instructional tasks (and model in question) should also necessitate the use of all desired ring axioms.

In this study, solving specific linear equations allowed the students to discover and address particular patterns that arose as a consequence of the specific elements. These equations anticipated the “important facets of the general” because they effectively highlighted the need for most of the ring axioms and other features like units and zero-divisors.

The choice of examples to be examined by the students is of critical importance in the instructional design process. Specifically, the set of examples needs to include at least one distinct instance of a general ring structure, a ring with no zero divisors, and a ring in which every nonzero element is a unit (though two of each example is recommended so that the students can pair similar structures during the sorting activity). It is also important that each of these instances be distinct from the “next” category so that the students are able to distinguish amongst each type of structure. For instance, even though the real, rational, and complex numbers are indeed examples of a ring with identity, integral domain, and field, respectively, these examples would not enable the students to clearly distinguish between, say, rings with identity and fields. It is worth noting that the structures reinvented in this particular study (ring with identity, integral domain, and field) are not the only structures that satisfy these requirements. The following table details alternative possibilities for structures that could result from using this instructional theory (depending on the choice of examples in the instructional design and how students differentiate the structures in the sorting activity):

Table 6

Possible Definitions Resulting from the Reinvention Process

Structure	Possibilities for Reinvention
General ring	Ring, ring with identity, commutative ring, commutative ring with identity
Ring with no zero divisors	Integral domain, domain
Ring in which every nonzero element is a unit	Field, division ring

Recall Zazkis' (1999) recommendation that "working with non-conventional structures helps students in constructing richer and more abstract schemas, in which new knowledge will be assimilated" (p. 651). To this end, the instructor should choose an acceptable number of non-conventional structures as appropriate for his or her students. For instance, in this study, I could have had the students solve equations on the quaternions, an example of a division ring[23]. Evidence from this study suggests that varying on the following properties (and combinations thereof) provides an acceptably wide collection of examples: finite, noncommutative, and varying amounts of zero-divisors and units.

Finite. Finite rings provide a functionally accessible starting point because important facets of their additive and multiplicative structures are plainly visible through patterns in the operation tables. In this way, the operation tables are a *record-of* these facets of the ring structure and serve as a *tool-for* future instructional tasks (in the case of this project, equation solving). Transitioning to finite rings, then, encouraged the students to expand and adapt reasoning that may have been specific to

[23] I decided not to include this example because the students had no prior experience with this structure (and, by my estimation, could not have easily acclimated themselves with it due to time constraints).

finite rings, such as reasoning directly from the operation tables. For example, in this project, the students, only vaguely familiar with modular arithmetic, were prompted to construct operation tables for Z_{12} and Z_5 using a clock-arithmetic analogy. From these tasks, the students gained preliminary notions of zero-divisor, as well as additive and multiplicative inverse, identity, and commutativity.

Noncommutative multiplication. While a lack of commutativity does not cause too much of a disturbance in the overall ring structure (as the absence of, say, associativity might), introducing noncommutative examples does encourage students to investigate the ramifications of the absence of other properties. In other words, it promotes the exploration of which properties might be considered unnecessary in certain contexts.

The inclusion of 2x2 matrices (with the usual operations) as a noncommutative example was particularly useful in this study because it encouraged the students to reconsider their assertion that left- and right-multiplication were the same (in TE1) and consider whether a generic quadratic equation could be factored (in TE2). Additionally, the absence of commutativity caused both pairs of students to re-examine their statements for several of the multiplicative axioms.

Zero-divisors. Zero-divisors are features of rings that are likely to be unfamiliar to most students. To this end, the results from this study suggest that the example structures include each of the following: a structure with no zero divisors, finitely many zero divisors, and infinitely many zero-divisors. Inclusion of a structure with an infinite number of zero-divisors (like a matrix ring or Cartesian product of infinite rings) allows the students to notice that zero-divisors are not common to the modular rings. Additionally, including two examples of rings with zero-divisors allows the students to identify the two structures as similar during the sorting activity.

Units. While the concept of multiplicative inverses may be very familiar to students, they may be less familiar with the possibility that nonzero elements may not have them. In this way, using the same reasoning as for zero-divisors, the evidence from this study suggests that the example structures include each of the following: a structure with only trivial (or no) units[24], finitely many units, and infinitely many units. Providing an array of unit structures like this allows the students to notice the direct connection between the existence of units and whether the multiplicative cancellation law holds on a given structure. Additionally, the students may also be able to ascertain the structure on the set of units (which may prove valuable whether they have previously taken a course in group theory or otherwise).

Step 3: Applying the model in a more general context. Because the students will attend to patterns arising as interrelationships between specific elements, the emergence of the model continues by having the students investigate how these specific relationships affect the overall ring structure. The task that promotes general activity should be the same for each example so that the students are able to easily compare their results. It may be helpful if the task involves the proof of a primitive conjecture that is related to the initial task setting. If this approach is taken, the process of *proofs and refutations* (Larsen & Zandieh, 2007) might prove to be especially helpful.

[24] In this study, I only included (nontrivial) rings with identity (because most conventional textbook definitions of integral domain require an identity), and thus each of the examples had at least two units, 1 and -1.

The tasks promoting more general activity should also anticipate the differentiation of the various ring structures in preparation for the sorting activity. In other words, the task or proofs of the respective conjectures should hinge on key characteristics of the example structures (such as the presence or absence of zero-divisors). For example, if the instructional design focuses on equation solving, the students might notice that certain elements cause multiplicative equations to have multiple solutions whereas others guarantee unique solutions. Switching to a more general lens could be achieved by asking the students to make conjectures about and prove the cancellation laws on their example structures. The cancellation laws also anticipate the sorting activity because units and zero-divisors play key roles in these proofs.

Step 4: Expanding the model by considering meaningful examples and non-examples. Although this step was not prominent in this study, the teacher may clarify misconceptions and refine the model before the process of defining begins by considering carefully chosen non-examples of rings. For example, if the students do not recognize commutativity of multiplication as an important axiom, they could be presented with a task that requires them to solve a simple quadratic equation on the various examples (similar to the students in TE2 in this study). This task, in addition to being very similar to the original task setting, could enable the students to see the utility of the commutativity of multiplication since it is necessary to factor a quadratic equation.

Non-examples may also play an important role. For instance, consideration of the natural numbers as an example with no additive or multiplicative inverses could help to emphasize the importance of these axioms in the ring structure. Consideration of the integers with subtraction or division could be a helpful non-example to highlight the utility of closure or associativity.

Step 5: Identifying the similarities and differences among the various ring examples. In preparation for the sorting activity, the students should be prompted to identify both similar and distinguishing features among the various examples. The equation solving model at this point is a *model-of* the similarities and differences in structure (made apparent by referencing the previous task settings). If the students struggle to identify significant features, the instructor can ask them what features caused the structures to behave differently in the (original or general) task setting.

Instead of waiting entirely until this step to compare the example structures, after each new structure is introduced, it may prove useful to ask the students how they perceived it to be similar to or different from the previous examples. Then this step would consist of referencing (and possibly adding to) any realizations made during their previous activity. For example, after the students in TE2 completed a proof of the cancellation laws for a particular structure, I asked them to compare and contrast the hypotheses of their proof with those previously examined.

Step 6: Sorting the structures based on characteristics of interest. The students, using the distinguishing characteristics they had previously identified, intuitively sort the example structures. Thus, the students are mathematizing their previous activity from the task setting in an effort to engage in more formal mathematical activity. It might be useful to have the students display all of their results visually, such as in chart form, so that the common similarities and differences among the structures are more readily apparent.

This activity is arguably the most important and influential task in the entire reinvention process. As such, it is crucial that the students provide mathematically sound and concrete justifications for their choices. Furthermore, these justifications must be based upon critical ideas that originated from the original task setting, for two reasons. First, it is consistent with the idea of the progressive formalization of intuitive ideas (a hallmark of guided reinvention). Second, it prevents the students from sorting the structures for irrelevant reasons (such as pairing Z_{12} with Z_5 because they are both finite, for instance).

Step 7: Formulating a definition for the most general ring structure. The equation solving model has now emerged as a *model-for* defining a mathematical structure. The instructor should prompt them to start with the most general structure first so that subsequent definitions may be formulated using this definition (this is discussed in the next step). The most general ring structure, in keeping with the recommendations for example selection, is likely to be one of the following: ring, ring with identity, commutative ring, or commutative ring with identity.

Similar to Larsen (2004), the results of this study suggest that this process may consist of several steps (not necessarily in chronological order): (a) abstracting the axioms[25] that are common to each of the sorted sets of structures, (b) creating formal statements of the appropriate axioms as part of a preliminary definition, and (c)

[25] The axioms for the definitions are the rules and properties used and identified by the students from the original task setting.

defining the term *binary operation* and including the two binary operations in the definition. After these three steps are completed, an iterative process of writing a concise, formal mathematical definition can occur:

1. The students formulate and present a draft of a definition.
2. The instructor reads over the definition, pointing out consequences of their choices and asking how the current definition accommodates examples or non-examples so that the students might identify any potential conflicts.
3. The students identify conflicts and revise their definition accordingly.

As with other components of the reinvention process, the method of *proofs and refutations* can be an especially effective tool for testing and revising a definition.

Step 8: Formulating nested definitions for subsequent structures. The definitions of the next two structures, presumably integral domain and field[26], can be formulated in terms of the previous definition by using the model as a *model-for* differentiating structure. There are two notable advantages in this approach. First, it encourages the students to identify exactly which axioms differentiate the three structures and reinforces the fact that the definitions are, to a certain degree, overlapping. Second, it is much more efficient than having the students simply re-copy all of the axioms from their previous definition(s). For example, instead of listing out the lengthy list of field axioms, the students should be able to define a field as an integral domain R for which every nonzero element a has a multiplicative

[26] Again, depending on the choice of examples, it is possible for slightly different structures to be reinvented, such as a division ring instead of a field.

inverse (followed, of course, by the formal statement of the axiom). The definition of field could also be stated equivalently in terms of, say, a ring with identity.

Step 9: Using the model for more formal mathematical activity. Tasks should now be administered that reflect the emergence of a new mathematical reality. In this context, the new mathematical reality consists of activities that require use of the reinvented definitions. Tasks in which students apprehend ring (or non-ring) structures, develop a definition of subring, and prove basic ring-theoretic propositions are appropriate for this stage. They could also be administered in an effort to develop other subsequent topics such as ring isomorphism and ideal.

Limitations

This study was designed to develop a local instructional theory supporting the guided reinvention of the definitions of ring, integral domain, and field. As might be expected from a study with a small number of participants, there are several limitations that need to be acknowledged and discussed. In this section, I address these limitations in light of the primary goal of this project. The limitations are consistent with those of qualitative research and do not threaten the studies validity or integrity.

The chief limitation of this project is that all of the data was collected in the very controlled environment of a teaching experiment, which contrasts significantly with a traditional classroom. I worked with only two students at one time, who had my undivided attention. It remains to be seen how the instructional theory developed in such a controlled setting could be applicable in a classroom environment. For example, the students could have many differing ideas with regards to one instructional task. The instructor, not being able to work closely with each student, may need to make adjustments to the instructional tasks to ensure that certain concepts are developing appropriately for the entire class.

Additionally, the students in this study worked with me for 12-18 hours each. This is a significant amount of time, and the fact that this instructional theory has not been tested in a classroom setting would seem to exacerbate the issue. While this is a well-founded concern, there are several factors that alleviate it. First, I did not require my participants to do any work outside of the teaching experiment sessions. In a classroom setting, students would be expected to work outside of class, lessening the amount of material needing to be discussed in class. Second, recall that my first goal as session facilitator was to see how the students *might* be able to reinvent these concepts—having them actually reinvent the concepts was secondary. Consequently, I allowed the students more freedom to explore their own ideas, even if I recognized that a particular idea was incorrect or perhaps not the most efficient method of approaching the material. A classroom setting, in which the primary goal is to have the students reinvent the desired concepts, would require that the instructor take a more active role. And by participating more in the guiding process, the teacher will lessen the burden with time constraints.

Even though the reinvention process may not take as long as it did for the students in this project, the fact that this represents a significant time commitment is undeniable (especially when considering that these three definitions can be covered much more quickly in a lecture-style class). As a result, fewer ideas and concepts can be covered. Some students and instructors may find this approach too time-consuming and inefficient for their particular needs. For many students, on the other hand, spending more time developing a firm understanding of the building blocks of the subject will be more beneficial and will enable them to learn additional topics with less difficulty.

Another limitation stems from my choice of participants. All of the students reported upon in this study were above-average discrete mathematics students. Additionally, they were also outgoing students who were willing to articulate their mathematical thoughts without reservation[27]. Indeed, the very definition of the word *average* ensures that not all students in a classroom will fit this mold. While the choice of these students was appropriate for the goals of this study, it does inhibit the potential applicability of these findings. Ideally, the instructional theory provides a framework that can be adapted to fit the needs of most classrooms and groups of students.

The size of the data set represents another limitation of the study. In total, the data from TE1 and TE2 consisted of up to 32 hours of video containing conversations between as many as three people. As a result, I did not transcribe each session in full, nor did I devote the same amount of analysis to each portion of the video data[28]. While I believe that my data analysis techniques did enable me to create a comprehensive timeline for the reinvention process (to create the local instructional theory), it also prevented me from exploring other possibly rich topics. For example, it might have been useful to document how the collaboration between the students

[27] A complete discussion of participant selection is available in Chapter 4.

[28] The data analysis techniques and methods can be found in Chapter 4.

contributed to their overall understanding and how the ideas of one student affected the thoughts and learning trajectory of the other.

Issues of Generalizability

Recall that neither guided reinvention nor the instructional theory presented are necessarily the *best* way for students to learn these concepts. I have not quantitatively evaluated or assessed any aspect of the students' learning nor have I compared the effectiveness of this approach versus the traditional lecture (attempting to answer such questions, however, would be an productive avenue for future research). Rather, I maintain that this instructional theory is *one* possible method grounded in empirical data that can be used for instructors who wish to engage their students in the method of guided reinvention.

Many of these limitations, of course, also directly influence the generalizability of the findings of this study. I invoke the same stance on generalizability as Larsen (2004), who cited Cobb's (2002) discussion of how findings of this type might be applicable to other situations:

> The theoretical analysis developed when coming to understand one case is deemed to be relevant when interpreting other cases. Thus, what is generalized is a way of interpreting and acting that preserves the specific characteristics of individual cases. For example, I and my colleagues conjecture that much of what we learned when analyzing first-graders' modeling can inform analyses of other students' mathematical learning in a wide range of classroom situations…It is this quest for generalizability that distinguishes analyses whose primary goal is to assess a particular instructional innovation from those whose goal is the development of theory that can feed forward to guide future research and the development of activities. (pp. 327-328; cited in Larsen, 2004)

Thus, as the goal of this study was to develop a theory to guide the development of activities (and not to assess an instructional innovation), I maintain that this study is generalizable in the sense that it fosters the development of instructional tasks that are appropriate for different contexts. Furthermore, the findings in this study are able to influence and inform future research in the area of ring and field theory.

Contributions to the Field

This dissertation project provides several significant contributions to the field of mathematics education. First, as the literature in ring and field theory is rather barren, this project helps to explain how students come to terms with the subject's introductory definitions. Second, this project addresses the "serious educational problem" (Dubinsky, Dautermann, Leron, & Zazkis, 1994, p. 268) that has been reported regarding the struggles of undergraduate students with abstract algebra by proposing an innovative method of instruction that actively engages the students in learning new, advanced mathematics.

As previously discussed, Simpson and Stehlikova's (2006) case study of how a student came to understand a commutative ring structure informed this study by setting a precedent that solving basic linear (and quadratic) equations could serve as a powerful mechanism for introducing students to the intricacies of the ring structure. This project, in turn, confirms this suggestion and builds upon it by explicitly detailing how equation solving might be used in this manner. Furthermore, I have used the emergent models construct to create a detailed conceptual framework for how the ideas of zero-divisor, unit, and additive inverse could arise as a result of using equation solving as a means of exploring the ring structure.

The findings from this work also represent the first empirically-based alternative to the traditional lecture in ring theory by building upon Larsen's (2004, 2009, 2010) group theory work with guided reinvention. Though the two projects have produced similar findings, the local instructional theories have marked differences. First, Larsen's (2004) instructional theory for reinventing the definition of group had only one target definition. On the other hand, this project provided a means of reinventing and differentiating between three similar, yet conceptually distinct, definitions. It also provides new information for algebraic concepts for which there is no group theory analog: units and zero-divisors (for example). In doing so, this project joins a growing body of research that aims to establish guided reinvention instructional theories in advanced mathematics.

Implications for Future Research

In addition to making notable contributions to the field, this study brings to light many avenues for future research. First and foremost, future projects could be undertaken to expand this instructional theory to include subsequent ring theory topics such as ideals, quotient rings, and ring isomorphisms. Additionally, future research is needed to address how the instructional theory from this study can be implemented in a classroom setting.

Recall that a limitation of this study is that no substantive claims can be made regarding its effectiveness – in comparison to the lecture method or otherwise. Thus, quantitative research could be conducted to formally assess student understanding when taught using the methods of guided reinvention. Though this information could certainly be useful on its own, it could also then be compared with similar data collected from courses taught in a more conventional manner.

As the amount of literature available to use to construct an initial instructional theory was extremely limited, this project also underscores the need for additional research concerning student understanding in ring theory. To this end, more research needs to be conducted that explores student understanding on topics in ring theory. While this information is certainly not necessary to develop innovative methods of instruction, research of this nature would provide more resources for instructional designers to use when planning a reinvention project.

Lastly, an effort needs to be made to implement projects of this type through working with instructors. The findings from this study and others involving different instructional methods will fall far short of their ultimate goals if efforts are not made to use these methods in the classroom. Thus, future work needs to be done that supports the integration of innovative instructional methods into the classroom through collaboration with mathematicians and mathematics instructors.

Chapter 7: Sample Instructional Sequence and Instructor Guide

In this chapter, I present a sample instructional sequence that follows from and is underpinned by the refined local instructional theory explicated in the previous chapter. Additionally, as a guide to instructors, I include thoughts on possible student responses and conflicts that may arise in response to the presented instructional sequence. It is important to note that this instructional sequence represents just *one* possible learning trajectory for a classroom of students with a background similar to the participants in this study. Specific implementations of the instructional theory (and of this sequence in particular) will depend largely on the contexts of the classroom for which it is intended.

Activity 1: Solving Specific Equations on Various Ring Structures

This activity may or may not be needed, as its primary goal is to help establish equation solving as an experientially real starting point. Furthermore, it may only be needed for a subset of the structures being used in the instructional design (for example, equation solving on the rational numbers is likely to already be experientially real for most students from school algebra). I recommend that specific equations be solved on at least two of these structures, if only to familiarize the students with the rigorous step-by-step procedure and identification of rules that will be utilized when they move on to proving the cancellation laws. As with the instructional materials used in this study, I present sample activities only for the finite rings.

Activity 1-1: Z_{12}. I start with Z_{12} so that clock arithmetic may be used as an accessible starting point for modular arithmetic if needed[29].

Activity 1-1: Solving equations on Z_{12}

1. Think about how you add hours on a clock. For example, 5 hours added to 9:00 is 2:00. If we remove time notation, we could express this statement as 9+5=2. Using the same idea, create a "clock addition" operation table that includes all of the hours on a clock (i.e. 1, 2, 3, … , 12). The operation is called *addition modulo 12*. What are some things you notice about this operation in the table?

2. In the same way, how might we define "clock multiplication"? Create a "clock multiplication" operation table that includes all of the hours on a clock. This operation is called *multiplication modulo 12.* What are some things you notice about this operation in the table?

3. The set {1, 2, 3, …, 12} with the above operations is called the integers modulo 12, typically denoted as Z_{12}. Using the operations you defined in the operation tables (addition and multiplication modulo 12), show how you can solve for x in the following equations on Z_{12}:
   $x+3=9$ $x+12=4$ $x+8=3$
   $5x=10$ $3x=12$ $11x=4$ $9x=6$
   If needed/desired: $2x+4=x+7$ $6x+9=7x+11$ $9x+5=2x+7$ $10x+11=6x+7$
   If needed/desired: $x^2+10x+1=12$, $x^2+2x+1=12$

4. What properties did you make use of while solving these equations? In other words, what properties of Z_{12} did you use to justify your line-by-line solutions?

Figure 70. Sample activity 1-1: solving equations on Z_{12}.

The instructor should take note that this activity may take the longest amount of time of any of the activities in this proposed learning trajectory, due to the fact that the tasks require the students to (1) apprehend a new (and possibly unfamiliar) ring structure, (2) recognize trends in the operation tables and how they relate to equation solving, (3) familiarize themselves with rigorous equation solving, and (4) identify

[29] Throughout these instructional tasks, I am viewing Z_n as the set of integers $\{1, \ldots, n\}$ (so that there is a clear analogy with clock arithmetic) instead of as a set of equivalence classes of integers.

rules and properties used to solve the equations. Most of these activities will be repeated when investigating other structures, and their experience here in the first task should not only inform but also improve their future activity.

Encouraging step-by-step solutions. The students might forego a step-by-step approach the first time though in favor of simply writing down the solution(s). This is fine as long as they notice that some of the multiplicative equations have multiple solutions (whereas all others have unique solutions). The instructor can also encourage the students to produce more rigorous solutions to the equations by asking them to "prove" that the solutions they have written are indeed solutions. Additional useful questions could include "how did you move from this step to this step?" and "can you break this into smaller steps for me?". In an additional effort to promote rigorous solutions, the equations of the form $ax+b=cx+d$ are included in the activity above because data from this study along with the literature suggest that these equations encourage a step-by-step approach (instead of simply writing the solution), but they are not necessary.

What constitutes a rule? As the instructor for the teaching experiments in this project, I found it helpful to introduce the notion of "rules" at this early stage because it familiarizes the students with the idea before moving on to the more general equations. The primary purpose of asking the students to identify the rules and properties used to solve the equations, of course, is that it gives rise to the ring axioms. However, there may be some initial confusion regarding what is meant by a "rule". What I found to be a helpful approach, instead of asking the students to identify the rules, was asking the students to write down their justification for each of their steps.

For example, if lines 1 and 2 of a solution were x+2=5 and x+2+10=5+10, respectively, I would ask the students for their justification for the second line. At this stage, it is not critical (or even recommended) that the students write out formal statements of the axioms. Rather, simply having them acknowledge the rules that they are making use of in some form, such as "subtraction" or "additive inverse", is sufficient. Essentially, the primary goal is to have the students recognize the properties that are being used to solve the equations. Eventually, the question regarding these rules will emerge as "what rules are needed to solve equations?" (*model-of*) and transition to "how can these rules define and characterize mathematical structure?" (*model-for*).

The role of operation tables. The operation tables for finite rings, in particular, have been demonstrated by the data in this study to be a powerful resource by which the students can visually identify several of the ring axioms. Specifically, it is possible for the students to easily notice the commutativity of both operations, the presence of the identities, zero-divisors (as elements that cause repetitive rows and columns), and units (as elements with rows in which every element is used exactly once). It is fine if the students do not immediately notice these concepts at this point (the ideas will arise as the students solve the equations), but as they do notice characteristics of the ring structure, the instructor can ask them if the property is visually present in the operation table. The students might also solve the given equations the first time through the operation tables – an approach that reinforces the operation tables' role as a *record-of* their initial activity and a *tool-for* more formal activity.

Additive identity and inverse. The students may immediately identify 12 with 0, or they may think of them as distinct elements (both cases arose in this study). There are a number of benefits to both ways of thinking. Immediately associating 12 with 0 is likely to be accompanied with realizing that 12 is the additive identity. On the other hand, thinking of the two as separate may encourage the students to notice the failure of the zero-product property on Z_{12} (since nonzero elements multiplying to give 0 may be difficult to comprehend based on their experience, whereas two elements multiplying together to give 12 should not be unexpected). Regardless of their specific thinking about the two elements, the most important idea that should arise is that 12 functions as the additive identity.

This preliminary understanding of the additive identity is crucial to the students' conceptual understanding of the additive inverse. Initially, additive inverse is present in the addition table (12 is present one time in every row and column). Whether this pattern is noticed or not, given their previous experience with solving equations, the students will probably want to subtract. In this case, the instructor should remind them that using subtraction is perfectly acceptable, but they must define what this new operation means first. In the case of Z_{12}, the students may write out explicitly what subtraction (or, equivalently, additive inverse) means for each element.

Associativity. The operation tables provide a context that the instructor can use to motivate associativity (and, eventually, binary operation). Even after the students have identified most of the rules used to solve these equations, the students still might ignore associativity (and have triple sums or products without parentheses indicating order). The instructor can challenge the students by asking how an expression such as

x+1+11 is defined given that the operations are only defined on two elements at one time. The thought that all expressions may be evaluated left-to-right may also be challenged with a similar example because x+1+11 is not able to be simplified in this manner.

Recognizing equations with multiple solutions and zero-divisors. Zero-divisors should appear almost immediately in the multiplication table as elements whose rows and columns display a repetitive pattern. The students will likely notice this on their own, but otherwise the instructor can ask them to compare and contrast this operation with one that is more familiar or ask them to identify patterns they see in the table. The students can then be prompted to find the common thread or criterion used to characterize zero-divisors.

Since students are accustomed to linear equations having only one unique solution, the instructor may need to point out the presence of additional solutions. For example, if the students conclude that x=2 is the *only* solution to 9x=6, the instructor can suggest another solution: "what about x=6?". After confirming that 6 is indeed a solution, the students will likely notice the presence of other solutions as well. If needed, they can then be asked if any of the other equations have multiple solutions. It may be helpful to encourage them to draw a connection between the elements with repeating patterns in the operation table and the equations with multiple solutions (or, equivalently, elements with a row or column in which every element is used once and multiplicative equations with unique solutions). The instructor can also ask the students if they notice a pattern that can be used to distinguish equations with multiple solutions from those with unique solutions.

Once the students start solving the given equations, they should notice a direct parallel between the elements with repetition and the equations with multiple solutions. This can be accomplished by having the students solve the equations directly from the operation table before they are prompted to solve them algebraically.

Units and attempting to define division. Just as the students identify zero-divisors and equations with multiple solutions, they should make the same connection with units and equations with unique solutions. The concept of units may arise organically as the students attempt to define division. The instructor may promote this activity by encouraging them to create an analogue to how they defined subtraction or additive inverses when solving the additive equations. The instructor can ask the students if division is possible for each element and have them explain their answers. For example, "why can you divide by 5 but not by 3?", or, equivalently, "why can you multiply to get rid of 5 but not of 3?". The instructor can also capitalize on their prior experience by asking them how such a setup differs from the equation solving to which they are most accustomed (likely on the real numbers). The students' responses can then lay the foundation for their understanding of the distinction between units and zero-divisors.

Some students, however, may not immediately recognize the importance of units and equations with only one solution because they expect the sets on which they have solved linear equations in the past guaranteed a single solution. Thus, it may prove helpful to address units with them after a discussion about zero-divisors by asking about the elements that give one solution.

Commutativity and distributivity. There are three important ring axioms that may not arise as a result of this initial activity: commutativity of addition, commutativity of multiplication, and distributivity[30]. First, commutativity of addition will appear later when the students attempt to prove a modified version of the additive cancellation law. Next, if the students have not yet made use of commutativity of multiplication or distributivity (which is likely), an equation that requires these rules to be solved may need to be introduced. Perhaps the most natural task of this sort is to have the students solve a basic quadratic equation on Z_{12}. To accommodate such a situation, the equations $x^2+10x+1=12$ and $x^2+2x+1=12$ are included in the above activity. Both are easily factorable as perfect squares (though they both factor in two different ways), have only two solutions in Z_{12}, and could serve as simple examples that motivate commutativity of multiplication and distributivity[31].

Unnecessary rules. Among others, the rule $x \cdot 0 = 0$ may appear as the students solve these equations. This is entirely acceptable, as the students do indeed make use of property to solve multiplicative equations. If the students have identified enough rules at this point to prove that certain rules like $x \cdot 0 = 0$ follow from the others, the instructor may encourage them to do so. However, it may be more effective to wait until they have a larger arsenal and are more familiar with the notion of proof in this context. Furthermore, throughout the reinvention process, the students can be prompted to prove propositions such as *(-a)(-b)=ab* as they appear (though it is

[30] If these rules do arise, then the suggested tasks which follow are not necessary.

[31] Other examples could be used, of course, but having the students attempt to solve quadratic equations with infinitely many solutions may detract from the focus of the reinvention process.

likely that, since these properties are so familiar, they will not see the need to prove them).

Activity 1-2. Z_5. The next activity with Z_5 is nearly identical to that for Z_{12}. Since most of the important features of the ring structure were identified in the previous activity, the primary focus of this activity is having the students horizontally mathematize these notions and apply them in this new setting.

Activity 1-2: Solving equations on Z_5

1. Suppose now that we want to add numbers using a 5-hour clock. For example, 4+3=2. Using the same idea, create an operation table for this 5-hour addition that includes the numbers 1, 2, 3, 4, 5. The operation is called *addition modulo 5*. What are some things you notice about this operation in the table?

2. In the same way, how might we define a 5-hour multiplication? Create an operation table for this 5-hour multiplication. This operation is called *multiplication modulo 5*. What are some things you notice about this operation in the table?

3. The set {1, 2, 3, 4, 5} with the above operations is called the integers modulo 5, typically denoted as Z_5. Using the operations you defined in the operation tables (addition and multiplication modulo 12), show how you can solve for x in the following equations on Z_5:
   $x+2=4$ $x+5=3$ $x+4=1$
   $2x=4$ $3x=4$ $4x=2$
   $2x=5$ $3x+1=x+4$ $4x+5=x+2$
   If needed/desired: $2x+4=x+3$ $4x+1=3x+4$ $3x+1=x+4$ $4x+5=x+2$
   If needed/desired: $x^2+4x+3=5$, $x^2+4=12$

Figure 71. Sample activity 1-2: Solving equations on Z_5.

The additive structure of Z_5 works nearly identically to that of Z_{12}. Thus, the students should be able to attend to all important aspects of the additive structure of Z_5 by applying their previous methods in this new context. This includes additive identity, inverse, and commutativity. Other notions directly transfer, as well, including:

distributivity, commutativity of multiplication, and associativity of both operations. These can all be dealt with rather easily by referencing previous activity (in fact, the instructor may find it helpful to review some of the students' previous work before or during this new activity). Moreover, this activity can be used as a means to reinforce ideas among this list that may have not been clear to the students during the previous activity. For example, if the students are still struggling with associativity, giving them the opportunity to use it in this new context may strengthen their understanding.

The results from this project suggest that the most critical (and perhaps difficult) structural features to apply in this new context, once again, are units (attempting to define division) and zero-divisors (discerning the zero-product property).

Zero-divisors. While constructing the operation tables, the instructor can ask the students to compare and contrast these tables with those for Z_{12}. Ideally, the students will note that there are no repeating patterns in the multiplication table. The instructor can use this to ask them what this might mean when they solve equations and how this might relate to equations with multiple solutions. The students should recognize, in turn, that zero-divisors do not exist on this new structure.

Units and attempting to define division. Whether the students notice the absence of zero-divisors first or not, they should notice that every nonzero element can be multiplied by some other element to get 1. If desired, the instructor can directly ask them how this feature compares with Z_{12} after they have made this realization.

Activity 1-3: Compiling a list of rules. After the students have solved equations on Z_{12} and Z_5, they should compile a list of all of the rules and properties that they used to do so. Preferably, they will have been identifying these as they went along, so this activity should merely be summarizing their previous efforts. It can also be used, however, to have the students identify any rules or properties that had not yet been addressed.

Activity 1-3

1. Make a list of all of the rules, properties, and line-by-line justifications that you used to solve the equations on Z_{12} and Z_5.

2. Using this list of rules, compare Z_{12} to Z_5. Which properties do they have in common? Which properties differentiate them?

Figure 72. Sample activity 1-3: compiling a list of rules and properties.

There are two significant components of the ring definition that the students may not have identified yet: binary operation, and the zero-product property. If the students have commented on the importance of the binary operation, the instructor may certainly promote efforts to define the term (though I would think this unlikely). Otherwise, the students will define binary operation after writing an initial draft for the definition of ring (or ring with identity). Regarding the zero-product property, the students may have recognized zero-divisors, but unless they tried to solve the multiplicative equations by setting them equal to the additive identity, the connection to the zero-product property may not have been made. The instructor need not introduce it yet, either – it will arise naturally as the only way that multiplication equations can be solved on Z and Z[x].

Activity 2: Proving the Cancellation Laws on Each Structure

The instructor should inform the class that the focus of the tasks will be proving a mathematical statement rather than solving an equation. Because the multiplicative cancellation laws do not always hold for each structure in question, the instructor can invoke the process of proofs *and refutations* (Larsen & Zandieh, 2007) in order to encourage the students to revise their conjectures and corresponding proofs.

Activities 2-1 and 2-2. Because these two activities involve proving the cancellation laws on two structures with which the students have just previously worked, I discuss both in this section. These activities expect the students to reference and horizontally mathematize their previous activity in a slightly more general and formal mathematical setting.

Conceptually, these proofs share a considerable amount with the equations the students solved in the previous task. Thus, my commentary and suggestions for this task pertain more to how the instructor can use the *proofs and refutations* process instead of aspects of the ring structure.

Activity 2-1

13. Consider Z_{12} with addition and multiplication modulo 12. Do you think the additive cancellation law ($x + a = a + b \Rightarrow x = b$) holds for any elements, just for some, or not at all? Prove your conjecture.

14. What about the multiplicative cancellation law ($ax = ab$ (a nonzero) $\Rightarrow x = b$) on Z_{12}? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture.

15. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws for Z_{12}? Include only those rules that work in every case (i.e. for any possible values of a and b).

Activity 2-2

1. Consider Z_5 with addition and multiplication modulo 5. Do you think the additive cancellation law ($x + a = a + b \Rightarrow x = b$) holds for any elements, just for some, or not at all? Prove your conjecture.

2. What about the multiplicative cancellation law ($ax = ab$ (a nonzero) $\Rightarrow x = b$) on Z_5? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture.

3. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws for Z_5? Include only those rules that work in every case (i.e. for any possible values of a and b).

4. How do the cancellation laws on Z_5 behave in relation to Z_{12}? Which properties account for any similarities or differences?

Figure 73. Sample activities 2-1 and 2-2: cancellation on Z_{12} and Z_5.

Additive cancellation. The students should be able to prove both of the additive cancellation laws with relative ease since they always hold in a ring and can be proven using the same sequence of steps in any ring. This study suggests that the only issues the students could experience are ignoring the acknowledgement of relevant properties. For example, even though they have previously discussed associativity by this point, it may not be clear to them that it needs to be invoked again. For help, the instructor can use the same questions as before (such as "can you break this into smaller steps for me?") and also have the students follow the outline ordained by their previous solutions to specific equations. Furthermore, the students may struggle with notation in this new, general setting. The instructor can help the students with the notation (for example, suggesting $-a$ for the additive inverse of a) but should leave the students to define precisely what it means.

Multiplicative cancellation. Before attempting a proof, it is crucial that the instructor have the students formulate conjectures (even if the conjecture is as primitive and simple as "the cancellation law always holds" – the students must have an initial conjecture to be able to revise it if necessary. This is especially true in the case of Z_{12}, wherein the multiplicative cancellation law only holds part of the time. Because they have solved equations very recently on this set, their initial conjecture may be accurate (in which case the instructor can encourage them to prove it). It might be more reasonable to expect, however, that the students will initially assert that the cancellation law does indeed hold on Z_{12}. In this case, the instructor can then encourage them to attempt to prove their (incorrect) conjecture and engage them in *proofs and refutations*. For example, if the students assume that each element has a multiplicative inverse, the instructor can ask "What if a=4?" and "Are there any other such values that don't have an inverse?" Attention can then be turned to the cases of a for which there is an inverse, and the hypotheses of the proposition from the original conjecture can be modified appropriately (an example of *proof analysis*). Subsequently, if the students tackle Z_{12} first, they will likely prove the cancellation law on Z_5 easily since there are no exceptional cases.

Activity 2-3. The next activity involves Z, as proving the cancellation laws on the integers necessitates the use of the zero-product property (a property that was likely not invoked in the previous activities). Another reason for introducing Z here is to provide an example of an integral domain that is not also a field, thus providing the students with distinct examples of each of the three target definitions right from the start. The students should then be able to identify each subsequent example with one of the first three they encountered.

Activity 2-3

1. Consider the set of integers Z with the usual addition and multiplication. Do you think the additive cancellation law ($x+a=a+b \Rightarrow x=b$) holds for any elements, just for some, or not at all? Prove your conjecture.

2. What about the multiplicative cancellation law ($ax = ab$ (a nonzero) $\Rightarrow x = b$) on Z? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture.

3. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws for Z? Include only those rules that work in every case (i.e. for any possible values of *a* and *b*). Are any other rules true for Z that might not have been used in these proofs?

4. Compare Z to Z_{12} and Z_5. Which properties do they have in common? Which properties differentiate them?

Figure 74. Sample activity 2-3: cancellation on Z.

Additive cancellation. Even though this task sequence did not include solving specific equations on Z, the students should be able to prove the additive cancellation law easily. The presence of negative numbers, for one, simplifies the notion of additive inverse in a more familiar and intuitive fashion. Additionally, the students could notice that all proofs of the additive cancellation law are virtually the same. If not, the instructor can ask them how it compares to the other proofs after they have written it out. This is a critical identification for the students to make, as the fact that the same proof works for each example is a *model-of* the identical additive structure on each ring.

Multiplicative cancellation. While the cancellation law does hold on Z, the students may struggle in an effort to prove that this is indeed the case. Two stumbling blocks that appeared in this study were attempting to define division and failing to set the equation equal to zero. Additionally, Z is the first structure with which they work that is not finite, preventing them from reasoning with an operation table (also related to the infinite of Z is that, unlike the previous examples, there are elements that are neither units nor zero-divisors).

In the case of attempting to define division, the instructor may again use *proof analysis*. After it has been established that division is not possible (in general) on the integers, the instructor can ask the students about the values of *a* for which a proof by division (multiplicative inverses; in essence, a proof in the same vein as for Z_5) would work. In fact, even if the students do not explore division on the integers on their own, prompting them to explore it anyways may serve to further differentiate Z from the previous examples.

A useful method to turn the students attention to a universal proof is to have them access their prior equation solving knowledge, because students are generally very familiar with the "set the equation equal to zero" technique. Prompts such as "what are some other ways that you have used to solve equations" or "how might you solve a quadratic equation" could be helpful to guide the students in the right direction. After the students have completed the proof, directing the students to recognize the zero-product property is crucial (the distributive property will also be used a cancellation law proof for the first time, but the students should notice this immediately). This can be accomplished in two ways. First, the instructor can simply ask for their justification for moving from the step with $a(x-b)=0$ to $x-b=0$ and see if it arises without further discussion. Otherwise, the instructor can turn to the second method: asking the students if such a step (or proof as a whole) could also work for Z_{12}. Discussions can then follow about how, in Z_{12}, for example, 3 times 4 gives the zero element, but there is no such pair in the integers. The students should conclude that this property is important since it does not hold for all examples.

Activities 2-4, 2-5, and 2-6. The ordering in which the remaining structures appear in the task sequence is unimportant. However, it is important for the instructor to prompt the students to compare each new structure with those they investigated previously. These comparisons will form the foundation of reasoning for the sorting activity that will soon follow.

Activity 2-4

1. Consider the set of rational numbers Q with the usual addition and multiplication. Do you think the additive cancellation law ($x + a = a + b \Rightarrow x = b$) holds for any elements, just for some, or not at all? Prove your conjecture.

2. What about the multiplicative cancellation law ($ax = ab$ (a nonzero) $\Rightarrow x = b$) on Q? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture.

3. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws for Q? Include only those rules that work in every case (i.e. for any possible values of *a* and *b*). Are any other rules true for Q that might not have been used in these proofs?

4. Compare the previous structures you have studied. Which properties do they have in common? Which properties differentiate them? To which structure is Q the most similar?

Activity 2-5

1. Consider the set of 2x2 square matrices with rational entires, $M_2(Q) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in Q \right\}$, with matrix addition and multiplication. Do you think the additive cancellation law ($X + A = A + B \Rightarrow X = B$) holds for any elements, just for some, or not at all? Prove your conjecture.

2. What about the multiplicative cancellation law ($AX = AB$ (A nonzero) $\Rightarrow X = B$) on $M_2(Q)$? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture.

3. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws for $M_2(Q)$? Include only those rules that work in every case (i.e. for any possible values of *a* and *b*). Are any other rules true for $M_2(Q)$ that might not have been used in these proofs?

4. Compare the previous structures you have studied. Which properties do they have in common? Which properties differentiate them? To which structure is $M_2(Q)$ the most similar?

Figure 75. Sample activities 2-4 and 2-5: cancellation on Q and $M_2(Q)$.

Activity 2-6

1. Consider the set of polynomials in variable *X* with integer entries, Z[X] $=\{a_0 + a_1X + a_2X^2 + ... + a_nX^n : a_i \in Z, 0 \le i \le n\}$, with matrix addition and multiplication. Do you think the additive cancellation law ($x + a = a + b \Rightarrow x = b$, where *a*, *b*, *x* are elements of Z[X]) holds for any elements, just for some, or not at all? Prove your conjecture.

2. What about the multiplicative cancellation law ($ax = ab$ (*A* nonzero) $\Rightarrow x = b$) on Z[X]? Do you think this could be done for any elements, just for some, or none at all? Prove your conjecture.

3. What rules/justifications/properties do you use in your step-by-step proofs of the cancellation laws for Z[X]? Include only those rules that work in every case (i.e. for any possible values of *a* and *b*). Are any other rules true for Z[X] that might not have been used in these proofs?

4. Compare the previous structures you have studied. Which properties do they have in common? Which properties differentiate them? To which structure is Z[X] the most similar?

Figure 76. Sample activity 2-6: cancellation on Z[X].

Additive cancellation. Since the proofs of the additive cancellation law will be virtually the same, the emphasis for these tasks should be on the fact that they are indeed the same and verifying that this is the case. For verification that the proofs are identical, the students should explain the proof in each context. For example, they should explain what the additive identity and what the additive inverse of a general element of the set is in each context. The instructor can also have the students verify some of the other rules, such as commutativity and associativity of addition. Depending on the level of the students, however, proofs of associativity for certain structures, which are often lengthy and non-intuitive, might not be appropriate. The utility of such proofs can be left up to the instructor's discretion.

Multiplicative cancellation. As with the first three structures, the method of *proofs and refutations* is exceptionally useful here. The instructor can also make use of the fact that the students have worked with a structure similar to each of these three. For example, regarding Z[x], the students may notice that division is not possible. The instructor could then ask the students if any of the previous structures behaved similarly and have them use the proofs from that structure as a guide. In this way, the students might be able to mathematize their previous experience with a structure that behaves similarly to generate conjectures for new structures.

Just as with the additive cancellation laws, the students should explicitly define how each rule works in each new structure. For example, they will need to define what notions like multiplicative inverse and identity means in the contexts of matrices and polynomials (they will also need to demonstrate that other rules do indeed hold, like associativity and distributivity; again, this is left to the discretion of the instructor).

Specifically, regarding multiplicative inverses in matrices, the focus of these discussions need not be too heavily steeped in determinants (unless the instructor deems it necessary to have the students gain experience with matrices by solving specific matrix equations in the original task setting). Rather, the students only need to acknowledge and realize that some matrices have multiplicative inverses and some do not (it is obviously preferred that they do know the specific criterion involving the determinant, but the point is that this is not a necessity). In fact, once this distinction has been made, the instructor may even remind the students about the specific criterion for an inverse.

Another possible point of confusion involves the definition of a polynomial. Evidence from this study suggests that students may have an unclear image of the definition of a polynomial and may mistakenly include expressions with negative exponents as polynomials (it is specifically for this reason, in fact, that I have included an explicit definition for a polynomial in the statement of the activity). Such an inconsistency may initially cause the students to assert that each polynomial does indeed have a multiplicative inverse. If this occurs, the students need only be prompted to re-examine the definition of a polynomial (following questions can include "what polynomials *do* have a multiplicative inverse?).

Comparing structures. Notice that, at the end of each of the activities discussed in this section, the students are tasked with identifying similar structures for each. The cancellation laws and their corresponding proofs bring to light definitive characteristics of three different kinds of structures: (1) structures with zero-divisors ($M_2(Q)$ and Z_{12}), (2) structures in which every nonzero element has a multiplicative inverse (Q and Z_5), and (3) structures with no zero-divisors which contain elements without multiplicative inverses ($Z[X]$ and Z). Slight variations in this pattern are acceptable, but the students should have solid evidence to support their actions. *The students' rationale for identifying similar structures must be sound mathematical reasoning based on their experience with solving equations and the cancellation laws*. This rationale will lay the groundwork for the sorting activity, which is arguably the most critical and foundational task in the entire reinvention process (going about the identification of similar structures in this way makes for a progressive sorting). Thus, flimsy rationale, such as "Z_{12} is most similar to Z_5 because they are both finite," or even correct identifications without proper justification, such as "Z[X] is most similar to Z because it contains the integers," should not be allowed to stand.

Activity 2-7: Adding to the master list of rules. The purpose of this activity is merely to have the students add any new rules from the tasks involving the cancellation laws to their master list of properties. The zero-product property may be the only one that had not been included to this point, but if there are others, the students should identify them and include them in the list.

Activity 2-7
Examine the proofs of the additive and multiplicative cancellation laws you have just written. Are there any rules or properties used in these proofs that are not included in your master list of properties? If so, add them to the list. Do these new rules also hold for the previous structures?

Figure 77. Sample activity 2-7: adding to the list of rules.

Activity 3: Summarizing the Proofs of the Cancellation Laws

The goal of this task is to provide the students with a summary of their prior activity and provide evidence to support the sorting activity. Thus, this activity is ideal if the students were unsure of which structures were similar or if they struggled with the rationale for their choices in the previous activities. If the students successfully sorted the six structures and provided sound reasoning for these choices, the instructor may opt to bypass this activity.

Activity 3

1. List the different methods used to prove the additive cancellation law on each of these structures.

2. List the different methods used to prove the multiplicative cancellation law on each of these structures.

3. Summarize your work with these structures by filling in "always", "sometimes", or "never" as appropriate in the following chart:

	$x+a=a+b \Rightarrow x=b$ by additive inverses	$ax=ab, a \neq 0 \Rightarrow x=b$ by multiplicative inverses	$ax=ab, a \neq 0 \Rightarrow x=b$ by the zero-product property
$\boldsymbol{Z_{12}}$			
$\boldsymbol{Z_5}$			
$\boldsymbol{Z}$			
$\boldsymbol{Q}$			
$\boldsymbol{Z[x]}$			
$M_2(Q)$			

4. What patterns do you notice in this chart? Based on this chart, which structures are similar? Does this agree with how you grouped similar structures in the previous activity? Does it provide you with any additional evidence to explain your choices?

Figure 78. Sample activity 3: summarizing the cancellation laws and sorting.

The first two tasks are simply asking the students to identify the different methods of proof. These will include (1) proving additive cancellation using additive inverses (the essentially unique way to prove this result), (2) proving multiplicative cancellation using multiplicative inverses, and (3) proving multiplicative cancellation using the zero-product property. These methods, in turn, form the top row of the table in the next activity.

Filling in the chart should largely be completed by referencing their previous work. However, there may be some scenarios for which the students had not previously accounted. For example, they may not have accounted for the fact that Z and Z[X] contain two elements with multiplicative inverses (and should thus be "sometimes" instead of "never"). Or multiplicative inverses on Z_5 may have precluded an investigation of the zero-product property on the structure (of course, for both of these examples and any possible others, the instructor could anticipate this activity back in those task settings and ask the students to address these issues). Whatever the case, the students may have to engage in new (albeit very similar) mathematical activity to complete the chart.

The "ratings" in the chart, when filled out correctly, should make identifying the similar structures obvious. If the students do reason directly (and correctly) from the chart, the instructor should affirm their choices but also expect them to discern the underlying reasons for these choices. In other words, the students should not be allowed to conclude this activity with a superficial statement like "Z and Z[X] are similar because their ratings are both ASA". Rather, use a statement like this to launch a discussion about how the zero-product property holds for each structure even though both of them have only two units.

Activity 4: Consideration of Meaningful Non-Examples

Depending on the progress of the students and their recognition of the important ring features, some meaningful non-examples might need to be introduced. The following represents just one possibility of such an activity that could be used to reinforce the importance and necessity of additive inverses (task 1), associativity of multiplication (task 2), and closure of the operations (task 3).

Activity 4

1. Consider the natural numbers N with the usual addition and multiplication. Would any of your proofs for the cancellation laws hold for N? What steps (and properties) hold? Are there any that do not?

2. Consider the rational numbers Q with addition and division. Would an analogous version of the multiplicative cancellation law ($a \div x = a \div b \Rightarrow x = b$) hold? Would modified versions of any of your proofs hold? Do any of the properties used in your proofs hold for division? Are there any that do not?

3. Consider the set of odd integers { … -5, -3, -1, 1, 3, 5 … } with the usual operations. Would any of your proofs of the cancellation laws work for this structure? What steps (and properties) hold? Are there any that do not?

Figure 79. Sample activity 4: consideration of meaningful non-examples.

Similar activities for different aspects of the ring structure can be tailored to the specific needs of the students in the class. The key ingredient in activities of this kind is providing the students with a situation wherein they identify the conflict, which serves to emphasize the importance of certain aspects of the ring structure.

Activity 5: Defining

Now that the students have sorted the six structures according to identified characteristics of interest, the process of abstracting the common rules for each sorted group begins the process of defining.

Activity 5-1. In this study, the students and I found it helpful to have this activity presented visually in the form of a chart[32].

Activity 5-1

1. Which rules (from the master list you have compiled) are common to each of the sets of structures? Display your results in the following chart (if a rule holds for all structures in the given group, mark it with an X):

	Group 1 Z_5, Q	Group 2 Z, $Z[x]$	Group 3 Z_{12}, $M_2(Q)$
Additive identity			
Multiplicative identity			
Associativity of addition			
Commutativity of addition			
Distributivity			
Zero-product property			
Associativity of multiplication			
Commutativity of multiplication			
Multiplicative inverse			
Additive inverse			

Figure 80. Sample activity 5-1: abstracting common properties.

Again, this is an example of an activity in which the students should be mostly summarizing their previous activity. However, there may be some structures on which certain rules have not been tested. The instructor should assess whether formal proofs will be required or not as dictated by the mathematical maturity of the students and the difficulty of the proposed proof. In any case, the students should provide *some* form of sound reasoning for their assertions, whether it involves concrete proofs or a sham argument. If the students struggle with determining if a certain property holds on a

[32] The chart displayed in this activity shows what is most arguably the most natural sorting of the structures. This does not necessarily preclude, however, an acceptable alternative sorting (though, whatever the case, the students should have sound reasons for their choices).

given example, the instructor might find it helpful to suggest specific situations that necessitate the use of the rule so that the students have some context in which to test it.

Activity 5-2. Before the students start writing formal definitions, they should formalize their statements of their properties (if they have not already done so). Taking care of this ahead of time will cause the process of defining to proceed much more smoothly.

Activity 5-2
Write formal, explicit statements for each of your rules. For example, for *additive inverse*, write down exactly what you mean when you say that this rule holds for a particular structure.

Figure 81. Sample activity 5-2: formalizing statements of rules.

Once an initial statement has been written, the instructor can engage the students in a Larsen's (2004) cyclic revision process:

1. The students prepare a statement of a rule.
2. The instructor reads and interprets the rule, calling attention to particular choices made by the students.
3. The students revise their statement as necessary and restart the process.

The following table presents some possible issues and proposed resolutions, each of which centers on having the students identify and acknowledge the conflict:

Table 7

Potential Issues with Proposed Resolutions

Issue	Resolution
Failure to exclude zero from multiplicative inverse axiom	Ask the students to identify the multiplicative inverses for each element of one of the smaller finite groups (like Z_5)
Quantifiers are in reverse order	Propose a situation that could result from such a statement and have the students compare it to their previous work
Failure to include "double-sided" axioms when commutativity of multiplication is not present	Ask them to solve an equation on a structure without a commutative multiplication that requires use of the missing statement of the property (such as distribution on the right)

Activity 5-3. The students should now be ready to write their first definition. The first structure that the students define should be the most general so that subsequent definitions may be formulated in terms of this one. This is based upon Larsen's (2004) activity for defining group.

Activity 5-3

The six structures you have been investigating are all examples of *rings* (more specifically, they are *rings with identity* to emphasize the presence of the multiplicative identity).

1. What features are needed to have this kind of structure? In other words, what does a ring (with identity) consist of?

2. What rules and properties do rings (with identity) have? In other words, which rules and properties are common to all six structures?

3. Use your responses to the previous questions to write a formal definition for ring (with identity): A *ring (with identity)* is …

Figure 82. Sample activity 5-3: defining ring with identity.

Alternatively, the instructor can simply ask the students about the rules in common to group 3 (the most general group from the previous activity), but this may cloud the notion that *all* of the given structures are rings with identity.

The instructor can decide whether a formal definition of binary operation should be formulated before or after the rest of the axioms are written. In this study, I found it helpful to ask the students "What is missing?" and "Can a set just have these properties on its own?" after an initial definition had been written as a means of defining binary operation. Once they acknowledge that it needs to be included in the definition, the students need to realize that it is a function (at which point they can be asked about the domain and range). If they do not initially use function terminology, the instructor can ask if it may be thought of as a function, but it may be more meaningful to the students if they write their own definition in informal language first.

In the event that extra or unnecessary rules are present in the definition, the students can be asked if the rule can be proven from the others. If the rule is not correct or not true, the instructor should ask if the rule applies to all six example structures. In order to revise the definitions presented, the instructor can make use of the same cyclic process of revision that was used for the formal statements of the axioms. Once a complete formal definition has been produced, the students can continue with the definition of the next structure: integral domain.

Activity 5-4. After ring with identity has been defined, the students may proceed with the definition of integral domain[33]. The same cyclic revision process as before can be used if needed.

Activity 5-4

1. A *commutative* ring with identity is a ring with identity R in which $ab=ba$ for every a, b in R. Which of the six example structures are rings with identity or commutative rings with identity that fulfill no additional properties?

2. The remaining structures are all examples of *integral domains.*
   a. Which rules and properties do integral domains have? In other words, which rules and properties are common to the remaining four structures?
   b. Write a formal definition for integral domain. You may find it helpful to write it in terms of one of your previous definitions. An *integral domain* is …

Figure 83. Sample activity 5-4: defining integral domain.

Again, the instructor could simply ask about the properties common to Group 2, but challenging the students in this manner reinforces the idea that all of the remaining structures are integral domains, not just those in Group 2.

Activity 5-5. The last structure to be defined is field.

Activity 5-5

Which of the six structures fulfill properties that go beyond integral domains and rings with identity? These remaining structures are examples of *fields.*

1. Which rules and properties do fields have?

2. Write a formal definition for field. You may find it helpful to write it in terms of one of your previous definitions. A *field* is …

Figure 84: Sample activity 5-5: defining field.

[33] The instructor can insert an additional activity to define *commutative* ring with identity here if desired.

It may be helpful to have the students formulate the field definition in terms of both integral domain and ring with identity – formulating in terms of the latter provides motivation for having the them prove the zero-product property from the other field axioms (specifically, the multiplicative inverse axiom). This rule could then be eliminated from that definition.

Activity 5-6. This last activity in the defining process aims to clarify exactly how the examples fit into their newly reinvented definitions.

Activity 5-6

1. Of the six example structures, which of them may be classified as a …

   a. Ring[34]?
   b. Ring with identity?
   c. Commutative ring with identity?
   d. Integral domain?
   e. Field?

2. Generate a list of more structures endowed with addition and multiplication. Of this list, which of these structures may be classified as a …

   a. Ring?
   b. Ring with identity?
   c. Commutative ring with identity?
   d. Integral domain?
   e. Field?
   f. None of the above?

Figure 85: Sample activity 5-6: classifying examples of ring structures.

[34] The instructor can explain to the students at this point that the more general definition of "ring" is simply a ring with identity with the multiplicative identity condition removed. Otherwise, this first task may be omitted.

Attention may now be turned to using these definitions in a more formal mathematical setting.

Activity 6: Using the Reinvented Definitions

Now that the target definitions have been reinvented, the students' attention should be directed to using and applying these definitions in a formal mathematical setting. How this is actually done may depend drastically on the plans for the rest of the course. For example, if the ultimate goal of the course is to study Galois theory, then these exercises can be catered to fields and field extensions. If the ultimate goal is to provide a solid foundation in ring theory, then more exercises anticipating ideals, quotient rings, and isomorphisms can be administered. For the purposes of this instructional sequence, I am going to provide sample exercises related to the following topics:

1. Determining if a structure is a ring,
2. Apprehending and classifying new ring structures,
3. Elementary calculations and properties involving units and zero-divisors, and
4. Defining the notion of subring.

These exercises are intended as examples of exercises for which the students should be adequately prepared after completing the reinvention process. Exercises out of relevant sections of a textbook may also be appropriate at this stage.

Activity 6-1: Apprehending Two-Operation Structures

1. Determine if the following structures are rings. Prove your assertions. (You may assume that matrix addition and multiplication and the usual addition and multiplication are associative. You may also assume that these multiplications distribute over their respective additions.)[35]

   a. $M_2(2Z)$ with matrix addition and multiplication
   b. $(Z,+,\div)$
   c. $(2Z, +, \cdot)$
   d. $(nZ, +, \cdot)$ where n is a nonzero integer
   e. $(Q,+,\div)$
   f. $(Z,-,\cdot)$

2. Consider the set {0, 1, x, y} with operations defined as follows:

+	0	1	x	y
0	0	1	x	y
1	1	0	y	x
x	x	y	0	1
y	y	x	1	0

*	0	1	x	y
0	0	0	0	0
1	0	1	x	y
x	0	x	y	1
y	0	y	1	x

   Call this set with these operations F_4. Is F_4 a ring? Prove your assertion. If the structure is a ring, classify it (be as specific as possible). You may assume that + and * are associative and that * distributes over +.

3. You may assume that each of the following examples are rings. Classify exactly what kind of ring each one is. Be as specific as possible (i.e. commutative ring with identity). Prove your assertions.

   a. Z_n (where n is composite) with addition and multiplication modulo n
   b. Z_n where n is prime with addition and multiplication modulo n
   c. $Z_3 \times Z_3$ with the usual component-wise operations
   d. $Z \times Z$ with the usual component-wise operations
   e. $M_2(nZ)$ with matrix addition and multiplication
   f. $\left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a,b \in Q\right\}$ with matrix addition and multiplication
   g. $Z\left[\sqrt{d}\right] = \left\{a + b\sqrt{d} : a,b \in Z\right\}$, for d squarefree, with the usual operations

[35] The instructor can decide exactly which axioms can be assumed to be true and which can be proven by the students (for example, it may be inappropriate or unnecessary to have the students prove that an operation is associative).

h. $Q(\sqrt{d}) = \{a + b\sqrt{d} : a, b \in Q\}$, for d squarefree, with the usual operations

4. Prove or disprove the following statements.

   a. If R is a field then $R[x]$ is a field.
   b. If R is a ring then $R[x]$ is an integral domain.

Figure 86. Sample activity 6-1: exercises promoting formal activity.

Next, attention could be turned to units and zero-divisors.

Activity 6-2: Units and Zero-divisors

5. A unit is an element in a ring that has a multiplicative inverse. More formally, a unit is an element a in R such that there exists an $a^{-1} \in R$ such that $aa^{-1} = 1 = a^{-1}a$. Find the units of the following rings:

   Z_{12} Z Z[X] Q Z_6 $M_2(Z_2)$ F_4

   Examine the sets of units for these structures. Does the set of units of a ring obey any of the ring axioms? Which axioms does it satisfy? Is this true for all of the sets of units?

6. A *zero-divisor* is a nonzero element a of a ring for which there exists a nonzero element b such that ab=0. For example, 3 and 4 are zero-divisors of Z_{12} since 3*4=12, which is the 0 element. Find the zero-divisors of the following rings:

   Z_{12} Z Z[X] Q Z_6 $M_2(Z_2)$ F_4

Figure 87. Sample activity 6-2: exercises with units and zero-divisors.

Finally, the students can be prompted to define the notion of subring, but this may only be appropriate if the idea came up previously.

Activity 6-3: Defining Subring

We have previously discussed the notion of a subring, informally referring to it as a ring that is a subset of another ring. How might the notion of a *subring* be formally defined?

By your definition, is Z_5 a subring of Z_{12}?

Figure 88. Sample activity 6-3: defining subring.

The question that follows the subring prompt can serve as a catalyst for a discussion about the operations being included in the definition of subring, as the students might answer "a subring is a ring that is completely contained in another ring". The question "is Z_5 a subring of Z_{12}?" can help to introduce possible conflicts with such a definition. For example, 3+3=1 in Z_6 but 3+3=6 in Z_{12} (and 6 is not an element of Z_5).

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Appendix A: Statement of Subjectivity

This research project studies how students might be able to reinvent the definitions of ring, integral domain, and field. The purpose of this research is twofold: to develop a local instructional theory for the reinvention of the above definitions which could be implemented in an undergraduate mathematics classroom, and to better understand how students come to understand and apprehend ring structures.

My first exposure to a reinvention-based teaching method occurred as a student in high school geometry. The teacher led a discovery-based, Socratic-style method of conducting the classroom. Students were not only encouraged and expected to voice conjectures and prove them (as opposed to being given the results outright), but also to examine the arguments of other students and present a counterexample if appropriate. It was certainly a nontraditional approach, eschewing the usual definitions and theorems-first approach for one centered on encouraging critical (mathematical!) thinking to construct the main ideas of the course. And I loved it. Not coincidentally, this is the course which honed my focus and eventually directed me towards majoring in mathematics as an undergraduate.

While I was fortunate enough in subsequent years to have many instructors and professors who positively contributed to my education, I did not encounter a course taught with the same methods as my high school geometry course. The essence of the approach, however, stayed with me. To this day, this inquisitive approach I inherited has characterized the method by which I learn and teach mathematics.

Having spent a significant portion of my doctoral coursework in pure mathematics courses (particularly in algebra), this was naturally my area of mathematical interest upon my initial foray into the world of undergraduate mathematics education. I thus began my literature review in the arena of teaching and learning abstract algebra and was immediately drawn to the idea of guided reinvention in the work of Sean Larsen, who created an instructional theory supporting the guided reinvention of group and group isomorphism for his dissertation project. As I browsed the pages of his dissertation, I saw how the group concept arose organically in the students' minds as a result of their encounters with the symmetries of an equilateral triangle. This, of course, also agrees with the historical development of the group concept, so the students were being presented not only with the opportunity to learn these basic concepts but also to grapple actively with the same ideas from which it spawned.

In contrast to the experience of the students about whom I was reading, I struggled with the concept of group in my first algebra course. Not because the idea was beyond my comprehension – the group axioms are simple enough on their own – but because I was blind to any underlying context and reasons for the inclusion of these axioms. After all, I thought, why is associativity so important? Why do we require every element to have an inverse? Why is commutativity not included? I gradually constructed the motivation for the group axioms on my own—but it was a result of the exposure to abstract algebra I obtained in subsequent courses.

The most disquieting realization was that the unifying ideas I sought (that proved to be very elusive) were not complicated – a simple generalization of symmetry, an idea well within the grasp of any abstract algebra student, yet it appears that many students are initially (or perhaps perpetually) blind to them.

My affinity for guided reinvention and my experience with it as a student provide me with a number of strengths I attempted to exploit while conducting this study. For example, my inclination towards learning in this manner on my own part provided me with valuable insight into anticipating student reactions while planning for and conducting the teaching experiment sessions.

That having been said, these same advantages had the potential to be limitations as well. Because I understand these topics in a particular way (more specifically, the way in which I understand them dictated the initial design of the instructional activities), this served as a slight obstacle when the students directed the reinvention process in a way I had not anticipated. Being aware of this potential bias, I constantly engaged in a process of revision of the instructional theory and instructional tasks based on my interactions with the students.

Ultimately, I believe that guided reinvention is an effective tool for mathematics instruction, but I believe that I was able to exploit my strengths and minimize the impact of my limitations in the execution of this project.

Appendix B: Recruitment Survey

SECTION 1: DIRECTIONS

Please carefully read and follow the directions for each section. Please answer all questions. You are encouraged to write down anything that comes to your mind while answering any of the questions on this questionnaire.

SECTION 2: BACKGROUND INFORMATION

Please legibly write down your answer to each question.

1) Name:

2) Age: ______

3) Gender (circle one): male female

4) Classification (circle one): Freshman Sophomore Junior Senior

5) Which mathematics courses have you completed? Starting with Calculus I, please list each mathematics course you have taken. Additionally, please list the grade you received for each course (note: the researcher will keep this information strictly private; in any subsequent use of this information, your name will be replaced by a pseudonym).

Mathematics Course	Grade

SECTION 3: EXAMPLES AND DEFINITIONS

If you are able, give examples and definitions (as indicated) of the following concepts as best you can. If you do not know, please write, “I do not know.”

1) Definition of a connected graph:

2) Example of a connected graph:

3) Definition of a topology

4) Example of a topology:

5) Definition of a group:

6) Example of a group:

7) Definition of a ring:

8) Example of a ring:

9) Definition of a measure:

10) Example of a measure:

11) Definition of a field:

12) Example of a field:

13) Definition of a polynomial:

14) Example of a polynomial:

SECTION 4: CALCULATIONS AND SHORT ANSWER

Answer the following questions and perform the indicated calculations as best you can. If you do not know, please write, “I do not know.”

1) Add the matrices $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$:

2) Multiply the matrices $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$:

3) Complete the sentence: A two by two (2x2) matrix has a multiplicative inverse if …

4) Find the multiplicative inverse of the matrix $\begin{bmatrix} 6 & 3 \\ 1 & 0 \end{bmatrix}$:

5) Calculate the following:

a. 3+4 mod 6 =___________

b. -1+8 mod 5=___________

c. $5 \cdot 4$ mod 7=___________

d. $-2 \cdot 8$ mod 9=___________

6) Write down the elements from each of the following sets (the first one has been done as an example) and/or show examples. If you do not know, write "I do not know."

a. The integers, ***Z***= { … -3, -2, -1, 0, 1, 2, 3, …}

b. The rational numbers,
Q=____

c. The complex numbers,
C=_______________________________________

d. The integers modulo 5,
Z/5Z=_____________________________________

e. Polynomials with integer coefficients,
Z[x]=_________________________

SECTION 5: CONCLUSION

The questionnaire is now complete. If you have any questions that you would like to ask about this research project, please ask the questionnaire administrator, John Paul Cook.

I certify that I am the only person to have worked on this questionnaire, and that I did not consult any sources while completing it nor will I intentionally attempt to consult any source after its completion.

______________________________ ________________________

Signature Date

Please legibly print your e-mail address for further contact:

____

E-mail Address

Thank you very much for your willingness to participate in this research project!