THE EFFECT OF DEPENDENT SAMPLING ON THE PERFORMANCE OF NONPARAMETRIC COINCIDENCE DETECTORS

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Submitted to the Faculty of the Graduate School
of the Oklahoma State University of
Agriculture and Applied Science
in partial fulfillment of
the requirements for
the degree of
DOCTOR OF PHILOSOPHY
August, 1965

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PREFACE

The problem of detecting the presence or absence of a signal in noise when the probability distribution of the noise is unknown is one to which nonparametric statistical tests can be applied. Previous investigations have assumed that the input random process to the detector can be sampled in such a manner so that the resulting observations are statistically independent. As the sampling frequency increases, this assumption obviously becomes invalid. The purpose of this study is to determine the effect of correlated input samples on the performance of a particular type of nonparametric detector, the median detector.

The first part of this investigation consists of a statistical analysis of the nonparametric median detector and two parametric detectors for the detection of a constant signal in stationary, additive noise. The computational results based on this analysis are given in the latter portion. Included are comparisons of the asymptotic relative efficiency (ARE) of the three detectors, as well as comparisons of their operating characteristics.

I wish to express my sincere appreciation to my major advisor, Dr. H. T. Fristoe, who was always willing to give his assistance and counsel throughout my graduate studies. His suggestions and criticisms have been most valuable, as well as his encouragement at times when it was certainly needed. I also express thanks to the members of my committee, especially to Dr. Paul E. Long for his sincere interest and concern when asked for help. A special note of thanks is given to Dr. J. L. Folks and to Dr. A. M. Breipohl who, although not on my committee, gave unselfishly of their time to assist me on technical problems. Acknowledgement is given to Dr. D. G. Lainiotis for suggesting the thesis problem.

It is difficult to express in words my appreciation to my wife, Lynn, who stood beside me when things looked the darkest trying to encourage and help me in any way that she could.

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CHAPTER I

INTRODUCTION

The reception of signals in the presence of noise is a central problem of communication theory. Noise to varying degrees always obscures the desired signal or message. Two major subdivisions of this problem are known as detection and extraction. The detection problem involves the design of systems which determine only the presence or absence of a signal in noise. Extraction of signals in noise is the estimation of one or more of the information-bearing features of a signal, for example, the amplitude, frequency, or waveform. This investigation will consider only the detection problem.

The detection problem is encountered quite frequently in practice. Possibly the most common example is the radar detection problem, in which it is desired to determine the presence of a target by detecting the presence of a radar return signal in noise. A second example is a PCM (pulse code modulation) system, in which a message is received by determining whether or not successive pulses are present in noise.

Early approaches to the detection problem were based on the signal-to-noise ratio criterion (1). These methods required relatively little information on the signal and noise statistics -- the correlation function of the noise was usually sufficient. The output of such a detector was certain a posteriori information on the basis of which an observer could come to a decision as to the presence or absence of a signal. Typically, the detector output included the a posteriori probabilities of signal and no-signal. The detection system assisted the observer but left the actual decision to his discretion.

More recently, the detection problem has been studied from the viewpoint of statistical decision theory (2). The resulting detection systems have been classified as parametric detectors and nonparametric detectors. The parametric detector requires a knowledge of the probability distributions of the signal and noise, whereas the non-parametric detector can be designed with a minimum of information about the signal and noise statistics. Thus the nonparametric detector could be applied, for example, if it is known that the noise and the signal plus noise distributions are continuous and differ in location, but nothing else. In such cases the parametric detector could not be obtained.

Consider again the radar (or sonar) detection problem. When the statistical nature of the received process is known for the signal and the no-signal cases, a parametric procedure, such as the Neyman-Pearson test (13) or the sequential probability ratio test (11), may be used. However, if the underlying signal and noise distributions are unknown due to a jamming or countermeasures environment, or lack of physical knowledge of the process, then a nonparametric procedure, such as a detector based on the sign test (18), could be employed.

Comparisons of various parametric and nonparametric detection procedures have been made in the past to show the relative advantages of each procedure. With few exceptions the cardinal assumption has been made that the input samples are statistically independent. That is, the input process is sampled in such a way that the samples are uncorrelated or independent of each other. In a practical detection system the input process is sampled at as high a frequency as possible, since the length of the signal pulse to be detected is short in most cases of interest. Also, a larger quantity of input data will allow the detector to make a more nearly accurate decision. As the sampling frequency increases, the correlation between samples will increase and the performance of the detector will change. Therefore, the present analysis of nonparametric detection procedures should be extended to include the possibility of dependent input samples.

The purpose of this thesis is to study the effect of dependent samples on the performance of a particular nonparametric detector, the coincidence or median detector. (The median detector is based on the sign test and requires only a knowledge of the noise median.) The detection problem will be to detect a constant signal in additive noise with a specified noise correlation function. This requires that the receiver and the transmitter be synchronized and the signal pulse width be known (coherent detection (2)). The median detector was chosen because it is based on one of the few nonparametric statistical tests which can be easily implemented. Also, its efficiency (assuming independent sampling) is comparable -- or at least not grossly inferior -- to that of the more common nonparametric tests.

The median detector will be compared to the conventional Neyman-Pearson detector based on the likelihood ratio test. Both the Neyman-Pearson detector designed for dependent samples as well as the one for independent samples will be considered. The asymptotic relative efficiency and the operating characteristics will be used to provide comparisons of each detection procedure. The question of how much is to be gained by increasing the sampling frequency, as well as the amount of error introduced in previous analyses assuming independence, will be discussed.

CHAPTER II

BACKGROUND OF THE PROBLEM

Theory of Hypothesis Testing and the Detection Problem

The problem of determining whether or not a signal is present in noise can be approached from the viewpoint of statistical decision theory. Recently, several books have been written incorporating statistical decision theory with the detection problem (2, 3, 4, 5). The detection process is a decision system whose output is "yes" - a signal is present - or "no" - only noise occurs. The problem of the detection of a signal in noise is therefore equivalent to one which, in statistical terminology, is called the problem of testing hypotheses (6). Here, the hypothesis that noise alone is present is to be tested, on the basis of some received data, against the hypothesis that a signal is present. The function of the detector is to examine the set of input samples in order to determine whether the null hypothesis H_o (signal is absent) is true, or whether the alternative hypothesis H_1 (signal is present) is true.

To illustrate the detection process, consider the detection of a constant signal A in additive gaussian noise with zero mean. Imagine that the receiver input consists of a single sample Y_t of the received process. On the basis of this sample the receiver chooses between one of two hypotheses H_o and H_1. If the value of Y_t is always zero when hypothesis H_o is true, and A when H_1 is applied, there would be no problem. But on account of noise, the sample Y_t is a random variable that must be described statistically by giving its probability density functions $p_{Y|0}(y)$ and $p_{Y|1}(y)$ under hypotheses H_o and H_1, respectively.

The receiver is based on a particular strategy that divides the range of values of Y_t into two regions R_o and R_1 such that the detector chooses hypothesis H_o when Y_t lies in region R_o and hypothesis H_1 when Y_t lies in R_1. The division of the range of values of Y_t by some threshold value of Y is based on the allowable detection errors. Figure 1 shows the probability density functions assumed in this example and the relationship between the threshold and the detection errors.

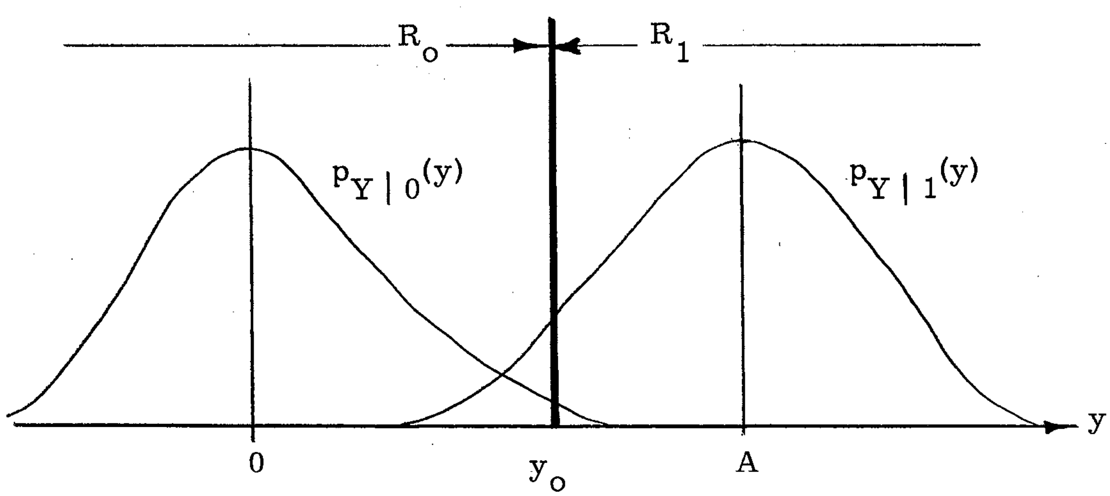

Fig. 1. Probability Density Functions Associated with a Simple Detector.

Basic assumptions and terminology of detection theory and hypothesis testing will now be discussed.

The noise N(t) which is introduced into the transmission medium is assumed to be a sample function of a continuous parameter stochastic process $\{N(t)\}$ and independent of the signal S(t). The output of the transmission medium (the input to the detector) is denoted by Y(t), and is considered to be a sample function of a continuous parameter stochastic process $\{Y(t)\}$. If the signal is not transmitted (H_o), then Y(t) is identical to N(t); if the signal is transmitted (H_1), Y(t) is a mixture of signal and noise (7).

One of the functions of the detector is to sample the input Y(t) at times t_i to obtain the observations $Y_1, \cdots, Y_n$, ($Y_i = Y(t_i)$, $i = 1, \cdots, n$). During the time that these n samples are being obtained it is assumed that the signal is either on or off. Thus, $Y_1, \cdots, Y_n$ is a set of random variables with a certain joint probability distribution function which depends upon whether or not the signal is present. The detector then bases its decision on the observed samples $Y_1, \cdots, Y_n$.

In detection problems it is convenient to introduce a quantity known as the signal-to-noise ratio parameter θ. When the signal S(t) is a constant, the signal-to-noise ratio is a function of the peak signal-to-rms noise ratio. Hence, θ is defined as the magnitude of the constant signal divided by the rms value of the noise. If the signal S(t) is a sample function of a random process, the signal-to-noise ratio is defined as a function of the rms signal-to-rms noise ratio. In either case it is required that θ always be positive and that it go to zero as the peak (or rms) signal-to-rms noise goes to zero.

If the signal is absent (H_o), the peak (or rms) signal-to-rms noise ratio is obviously equal to zero and hence θ is equal to zero. If the signal is present (H_1), then the peak (or rms) signal-to-rms noise ratio and θ are not equal to zero.

The errors committed by the detector can be of the following two exhaustive and mutually exclusive types: (1) the detector says there is a signal present, when in reality the signal is absent; the probability of such an error is denoted by α, and is known as the false alarm probability; (2) the detector says there is no signal present, when in reality there is a signal present; the probability of this type of error is denoted by β, and is known as the false dismissal probability. In statistical terms, α is the level of significance or size of the test (detector), and $1-\beta$ is the power of the test. In communication theory $1-\beta$ is called the detection probability. A schematic representation of the error probabilities is shown in Fig. 2.

A detection procedure bases its decisions concerning the presence of the signal in noise on a statistic $T_n (Y_1, \cdots, Y_n)$, a function which is defined on the observations $Y_1, \cdots, Y_n$. If T_n is a Borel-measurable function of $Y_1, \cdots, Y_n$, then T_n is a random variable. The introduction of T_n serves the useful purpose of mapping the outcome of the measurements on Y_t from the n-dimensional

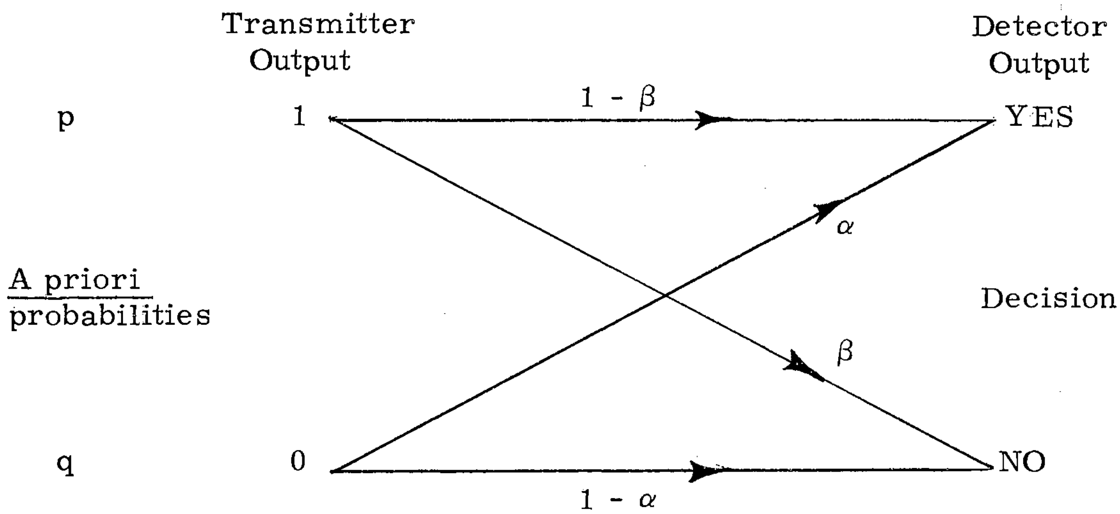

Fig. 2. The Error Probabilities of a Detector.

sample space of $Y_1, \cdots, Y_n$ to the one-dimensional sample space of T_n, i.e., the real line (8).

A detection scheme can be thought of as equivalent to a division of the sample space of the observations into two parts. The null hypothesis H_o is accepted, or rejected, depending on whether the observed sample $Y_1, \cdots, Y_n$ lies in the acceptance or critical region, respectively.

If the function T_n is to serve as a useful mapping of the set of measurements $Y_1, \cdots, Y_n$, it must map the critical and acceptance regions of this n-dimensional space into nonoverlapping regions on the real line. In this event the division of the n-dimensional sample space is reduced to the considerably simpler problem of dividing the one-dimensional sample space of T_n into two regions. If the statistic T_n, as calculated for a particular set of observations, lies in the acceptance region, H_o (signal is absent) is accepted; otherwise H_1 (signal is present) is accepted. The critical region is an interval of the form $(-\infty, T_{n,\alpha})$, and the acceptance region is an interval of the form $(T_{n,\alpha}, \infty)$, where $T_{n,\alpha}$ is a constant chosen to make the false alarm probability of the detector equal to α.

The distinguishing characteristic of a detection system is the procedure used to decide between H_o and H_1. A system is required to handle the receiver input so that the decisions are made with the greatest possible success in a series of observations. The decision strategy will depend not only on the nature of the signals to be detected and on the character of the noise that corrupts them, but also on the particular definition of success. In general, both α and β will depend on the decision procedure.

Decision procedures can be classified as either parametric or nonparametric tests. Parametric statistical tests are those in which the probability distribution underlying the observations is taken to be of a certain form, and the test is concerned only with the value of one or two (or at most a countable number) of the distribution's parameters (18). The term "parameters" refers to a finite number of constants appearing in the specification of the probability distribution of the random variable. Nonparametric statistical tests include those in which the underlying probability distribution function cannot be described by a finite number of parameters. Parametric detectors (statistical tests) will be considered first.

Parametric Detectors

Detectors based on the parametric methods of the theory of hypothesis testing have been termed parametric detectors (2). In order to apply parametric procedures the probability distribution function of the input sampled process must have some simple functional form, such as that of the normal distribution, and must be completely specified by one, two, or at most a countable number of real parameters. The essential feature is the finite or countable number of real parameters that serve as indices or labels for the probability distributions.

Each of the parametric detectors tests the null hypothesis H_o: probability distribution of $Y_1, \cdots, Y_n$ is $P(y_1, \cdots, y_n \mid \theta = 0)$ (signal is absent) against the alternative hypothesis H_1: probability distribution of $Y_1, \cdots, Y_n$ is $P(y_1, \cdots, y_n \mid \theta = \theta_1)$ (signal is present). The actual decision procedure employed by the various parametric detectors is not the same. Three examples of parametric detectors will illustrate this point.

The ideal observer is a parametric detector originated by Siegert (9). This was the first application of the theory of hypothesis testing to the problem of detecting signals in noise. The ideal observer maximizes the probability of a correct decision by minimizing the sum of the error probabilities α and β for a fixed number of observations. The probability of a correct decision or the level of success depends upon the a priori probabilities p and q ($q = 1 - p$) of signal plus noise and noise alone, and is given by (see Fig. 2)

$$P_s = P_s(\theta, n) = p(1 - \beta) + q(1 - \alpha) = 1 - p\beta - q\alpha$$

where n is the sample size and θ is the signal-to-noise ratio.

The relation between the level of success and the input signal-to-noise ratio is expressed in terms of a betting curve (9), which is the essential feature of the more fundamental theory of the detection process. The minimum detectable signal can be defined, using the betting curve, as that signal observed at the output of the receiver for which there is some arbitrarily selected percentage of success. The betting curve then relates the minimum detectable signal to a corresponding input signal-to-noise ratio.

One of the drawbacks associated with the ideal observer is that the decision procedure depends upon the a priori probabilities p and q. If these probabilities are unknown, the ideal observer is not applicable. If the a priori probabilities are known the ideal observer is optimum in the sense described above.

A second parametric detector is the sequential observer discussed by Bussgang and Middleton (10). It is based on a sequential decision procedure originated by Wald (11). In this system the error probabilities α and β are fixed and the observation time or the number of samples is variable. The observations are continued until certain limits, depending on α and β, are exceeded for the first time, at which point the null hypothesis H_o is either accepted or rejected. The sequential observer has the advantage that the decision procedure is independent of the a priori probabilities p and q.

The most common parametric detector is the Neyman-Pearson observer investigated by Middleton (12). The decision procedure was first given by Neyman and Pearson (13). In this case the decisions are made such that for a given number of observations and a given false alarm probability, the false dismissal probability is minimized. In the classical Neyman-Pearson test the observer formulates its decision without any knowledge of the a priori probabilities p and q. Middleton (2, 12) has extended the classical test to include the more general situation encountered in detection problems. This extension emphasizes the role of a priori probabilities and the cost ratio. (The cost ratio is defined as $\frac{C_\alpha - C_{1-\alpha}}{C_\beta - C_{1-\beta}}$ where C_α and C_β are the costs of false alarm and false dismissal errors, respectively, and $C_{1-\alpha}$, $C_{1-\beta}$ are the costs associated with correct decisions.) He has modified the classical procedure so as to minimize the total false dismissal error probability $p\beta$ with the total false alarm error probability $q\alpha$ fixed. The modification reduces to the classical Neyman-Pearson test if it is assumed that the cost ratio equals one and $p = q$.

Various aspects of the Neyman-Pearson detector have been studied by several investigators. Peterson, Birdsall and Fox (14) have considered several practical applications of the detector. Reich and Swerling (15) and Zubakov (16) have applied the Neyman-Pearson observer to the detection of a sine wave in gaussian noise when correlation exists between successive samples. Other references can be found in Middleton and van Meter (17).

The performance of the Neyman-Pearson detector will be compared with that of the nonparametric coincidence detector later in this thesis; therefore, its structure will now be discussed in detail.

It is convenient to introduce a new function called the likelihood ratio, $\Lambda(Y_1, \cdots, Y_n)$, defined as the ratio of the conditional joint probability density functions,

$$\Lambda(Y_1, \cdots, Y_n) = \frac{f(y_1, \cdots, y_n \mid \theta = \theta_1)}{f(y_1, \cdots, y_n \mid \theta = 0)}$$

where $f(y_1, \cdots, y_n \mid \theta = 0)$ is the conditional joint probability density function of the input samples, given that only noise is present; and $f(y_1, \cdots, y_n \mid \theta = \theta_1)$ is the conditional joint probability density function of the input samples, given that signal and noise both are present. It has been assumed here that the <u>a priori</u> probabilities are equal. $\Lambda(Y_1, \cdots, Y_n)$ represents the likelihood that the set of samples was drawn from signal plus noise relative to the likelihood that it was drawn from noise only. Hence, if $\Lambda(Y_1, \cdots, Y_n)$ is sufficiently large, it would be reasonable to conclude that a signal was present.

For a given value of α and a fixed number of samples a threshold Λ_o can be found. The Neyman-Pearson criterion is then to compare the likelihood ratio to this threshold, Λ_o; the null hypothesis H_o is chosen if $\Lambda < \Lambda_o$, and the alternative hypothesis H_1 is chosen if $\Lambda > \Lambda_o$. It can be shown that this procedure will minimize β (12). A schematic representation of the conditional probability density functions for a Neyman-Pearson test of H_o vs H_1 is given in Fig. 3.

Although we have tacitly assumed that $\theta = \theta_1$ has only one particular value for all tests of n observations, the optimum character of the present test is not altered if other choices of θ_1 are made, as long as θ_1 is considered to be small, i.e., the weak signal case. The Neyman-Pearson test has therefore been termed the locally most powerful test of H_o against H_1 no matter what other tests (for a given α) are tried (6). "Locally" refers to the restriction that the test is optimum only for values of θ_1 in the locality of zero.

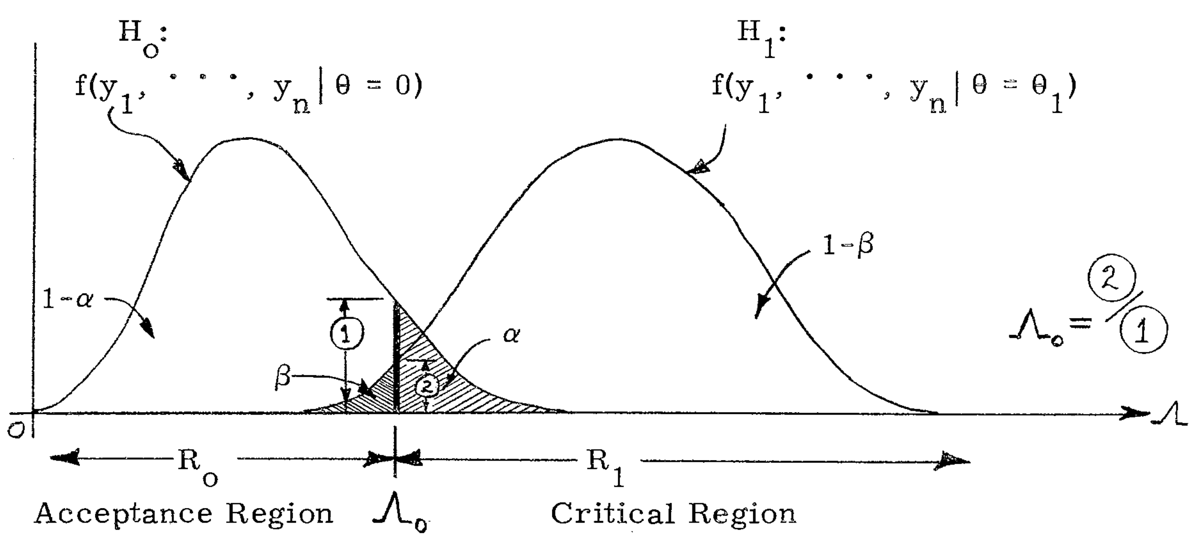

Fig. 3. A Schematic Representation of the Probability Densities in a Neyman-Pearson Test of H_o vs H_1.

The likelihood ratio for the detection of a constant signal in stationary, additive gaussian noise is well known. The input under signal conditions is

$$Y(t) = N(t) + \theta$$

where θ is the input signal-to-noise ratio (in this case the amplitude of the signal to be detected, too), and the noise N(t) is a sample function of a gaussian process $\{N(t)\}$ with a mean of zero and a variance of one. Under no-signal conditions the input is

$$Y(t) = N(t).$$

If the assumption of independent samples $Y_1, \cdots, Y_n$ is made, then likelihood ratio is equivalent to L_n:

$$L_n = \frac{1}{n} \sum_{i=1}^{n} Y_i \qquad [1]$$

The decision rule is if $L_n < L_\alpha$, accept H_o; if $L_n > L_\alpha$, accept H_1, where L_α is the threshold.

If the samples are correlated, the noise correlation matrix $\| R_{ij} \|$ is composed of the elements

$$R_{ij} = R(|i - j|\tau) \qquad (i, j = 1, \cdots, n)$$

where τ is the sampling interval. The decision rule is based on an equivalent likelihood ratio ϕ_n:

$$\phi_n = \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} Y_i \theta \qquad [2]$$

where $\| Q_{ij} \| = \| R_{ij} \|^{-1}$, i.e., $\| Q_{ij} \|$ is the inverse of the noise covariance matrix. If $\phi_n < \phi_\alpha$, accept H_o; if $\phi_n > \phi_\alpha$, accept H_1, where ϕ_α is the threshold.

It should be stressed that the Neyman-Pearson test is optimum only for the particular pair of density functions $f(y_1, \cdots, y_n | \theta = 0)$ and $f(y_1, \cdots, y_n | \theta = \theta_1)$ for which it has been designed. If these density functions are changed, then the optimum test may also be changed.

Nonparametric Detectors

Detectors whose error probabilities α and β remain the same for a large class of possible noise distributions are known as nonparametric (or distribution-free) detectors (7). The parametric detector was designed for a specific noise distribution; whereas the nonparametric detector, based on a nonparametric statistical test, is designed to operate with noise distributions that cannot be completely specified by a finite number of parameters. Nonparametric detectors would be applicable in instances where parametric detectors are inappropriate due to incomplete information concerning the functional form of the noise distribution.

Numerous nonparametric statistical tests have been suggested in the literature (18, 19, 20). Of course, not all of these are applicable; however, many remain to be investigated. Capon (7) was one of the first to apply nonparametric statistical tests to the problem of detecting signals in the presence of noise. Recently, Bell (21) has made a comprehensive study of nonparametric detectors.

In order to briefly summarize the work that has been done, three models of nonparametric detectors will be introduced:

(1) Detectors with one input

(2) Detectors with one input and a reference noise source

(3) Detectors with two inputs.

Statistical tests known as one-sample tests (18) can be applied to the first detector model. One-sample tests should not be confused with the number of observations, $Y_1, \cdots, Y_n$. The term "one-sample" in this case means that one set of data $(Y_1, \cdots, Y_n)$ from one population $\{Y(t)\}$ is available for use in the decision process.

More than twelve one-sample, nonparametric statistics have been studied by Bell (21). Among these are detectors which employ statistics based on differences of the probability distribution function of $Y_1, \cdots, Y_n$ and the pure-noise distribution function. Examples of this type (investigated by Bell) are the Kolmogorov-Smirnov or Vodka detector, the Cramer-von Mises detector, and the sign-quantile detector. The sign-quantile detector is based on one of the oldest and simplest nonparametric tests, the sign test. In its elementary form a sign detector (sometimes called a coincidence or threshold detector) counts the number of times the input observations $Y_1, \cdots, Y_n$ exceed a threshold. If this number is greater than a certain limit, the alternative hypothesis is accepted; otherwise it is rejected. The test statistic has the form

$$S_n = \sum_{i=1}^{n} c\left[Y_i - \xi_p(F_o)\right] \qquad [3]$$

where (*) $c(z) = 1$ if $z > 0$

$= 0$ if $z < 0$

and $\xi_p(F_o) = F_o^{-1}(p)$. F_o is the pure noise probability distribution, and p is some percentile. For a decision threshold S_α, the decision rule is accept H_o if $S_n < S_\alpha$; accept H_1 if $S_n > S_\alpha$. When $p = 0.5$ S_n is sometimes referred to as the median statistic. The more complex sign-quantile detector has more than one threshold level.

A second class of one-sample detectors investigated by Bell is composed of the run-block detectors, which deal with the number of observations $Y_1, \cdots, Y_n$ that fall in certain preassigned intervals and/or the relative spacing of these values relative to the pure-noise distribution function. General forms of detectors of this type are the spacing and the empty-cell detectors.

The third class of one-sample detectors studied by Bell includes the rank-sum detectors, whose decisions are based on sums of percentile ranks of the observations $Y_1, \cdots, Y_n$. One example of this type is the one-sample Mann-Whitney (Wilcoxon) test (22). A second example is the rank sum detector based on the c_1 test (23).

The second detector model has been used by Capon (7, 24) as

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*The possibility of $z = 0$ is ignored since it occurs with probability zero, assuming continuous distribution functions.

well as by Bell (21). The basic assumption is that, in addition to the n independent observations $Y_1, \cdots, Y_n$ of the input process considered above, m independent observations $X_1, \cdots, X_m$ can be obtained under no signal conditions. If the Y's and X's are samples from two independent, stationary random processes with continuous first-order distribution functions F and G, respectively, and if it is desired to detect the presence of a signal component in the Y's with the aid of the X's, then some method of comparison of the X's and Y's is needed which is sensitive to differences between F and G. Two-sample nonparametric statistical tests meet this requirement (18).

Two-sample tests compare data taken from two samples or populations $[(Y_1, \cdots, Y_n), (X_1, \cdots, X_m)]$ and make decisions concerning the distribution functions of the two samples. Bell has applied two-sample versions of the one-sample tests described for the first detector model to this model. Capon has investigated the Mann-Whitney, the Wald-Wolfowitz, the Kolmogorov-Smirov or Vodka, and the rank-sum tests for use as detectors employing a reference noise source.

Probably the most significant of these tests is the two-sample Mann-Whitney test (25). The Mann-Whitney test is based on the statistic

$$V_{mn} = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} c\,(Y_i - X_j)$$

where c(z) has the same meaning as earlier. In essence, the statistic V_{mn} counts the number of times that the magnitude of an observation Y_i exceeds the magnitude of an observation X_j. This test is one of the more efficient nonparametric tests.

The reference noise source from which $X_1, \cdots, X_m$ are obtained may be difficult to realize in practice, since its characteristics must be identical with those of the signal-bearing channel under no-signal conditions. Of course, the output of the latter channel might itself serve as the reference source during time intervals when it is known that a signal is not being transmitted, provided that these intervals are long enough to permit the establishment of suitable noise records $X_1, \cdots, X_m$, and provided that the channel characteristics do not change when one passes from such a reference interval to an interval in which a signal may be present (22).

It should be stressed that neither Bell (21) nor Capon (7) considered the consequence of correlated or dependent samples in their analysis of various nonparametric detectors. The assumption of independent input observations was made throughout. The validity of this assumption remains to be investigated.

Two-input detection schemes are considered when the signal to be detected is a random process whose presence perturbs both inputs simultaneously; in the absence of a signal, it is assumed that the two inputs are independent random noise processes. Thus, the applicable statistical techniques are those suitable for testing a hypothesis of independence vs an alternative corresponding to the type of dependence introduced by the common signal component.

The most popular two-input nonparametric detector is the polarity coincidence correlator (PCC) detector. It is a simple device to implement and utilizes only the polarity information of the two inputs. Given the input observations $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$, the following test statistic is calculated:

$$S_{pcc} = \sum_{i=1}^{n} \operatorname{sgn}(X_i) \operatorname{sgn}(Y_i)$$

where $\operatorname{sgn}(z) = \pm 1$ according to whether $z > 0$ or $z < 0$. The decision rule is to accept H_o if $S_{pcc} < S_\alpha$ and reject H_o if $S_{pcc} > S_\alpha$, where S_α is the threshold and

H_o: $X(t)$ and $Y(t)$ are independent, statistically identical noise processes (signal absent),

H_1: $X(t)$ and $Y(t)$ contain in addition a common random signal component (signal present).

The polarity coincidence correlator has been discussed by a number of authors. Thomas and Williams (26) considered the performance of the PCC in nonstationary noise. Wolff, Thomas, and Williams (27) compared the PCC with an ordinary correlator and a Neyman-Pearson detector designed for white gaussian noise inputs. Kanefsky (28, 29) described an adaptive PCC for nonstationary processes. The effect of correlated noise samples on the performance of the PCC has been studied by Kanefsky (30) and Ekre (31). Squire (32) analyzed a PCC with biased polarity indicators. The circuit design of a practical

PCC has been given by Rosenheck (33) and a less successful optical model by Collela (34). The results of these investigations indicate that the polarity coincidence correlator can be applied successfully to nonstationary, non-white, and unknown noise environments. However, when the noise distributions are known the Neyman-Pearson detector has superior performance.

The work done by Kanefsky and by Ekre does consider dependent samples; however, it is limited to the PCC detector. Neither uses the methods of comparison considered in this thesis. Ekre considers the detection of a gaussian signal in gaussian noise with identical normalized power spectra. The PCC is compared to the conventional analog correlator, using a signal-to-noise ratio performance criterion. Kanefsky only compares the performance of the PCC under dependent sampling to that under independent sampling. An effective sampling rate is defined as a performance criterion. No mention is made in either paper of the effect of correlated samples on the PCC as compared to the effect on the optimum parametric procedure. Also, the question of the relative performance of the optimum parametric procedure designed assuming independent observations and that designed assuming correlated observations remains unanswered. These points will be studied in this thesis.

To summarize, nonparametric detection schemes offer the following advantages: (1) the false alarm probability can be specified in advance even when detailed information on the input noise process is unavailable; (2) weak-signal performance in gaussian noise may not be too inferior to that of the optimum parametric detector for gaussian noise, while superior performance may be obtained when the noise is actually non-gaussian and/or nonstationary; (3) many times the instrumentation is simpler.

The nonparametric detector to be used in this investigation is the coincidence or median detector. It will be assumed that only one input is available and that the median under no-signal conditions (i.e., the median of the noise distribution) is known. Therefore, the detector is of the first class of detectors discussed previously. As has been pointed out, this detector is based on a special case of the sign statistic.

The median detector was chosen for three reasons. First, it is based on one of the few nonparametric statistical tests which is simple to implement. Second, its statistical analysis is tractable and, hopefully, the results obtained are representative of other tests. Finally, the efficiency of the median detector is comparable or at least not far below that of other nonparametric detectors.

The coincidence detector, due to its simplicity, has been studied and instrumented by several investigators. Not all of the coincidence detectors have been nonparametric, however, because of the manner in which the threshold has been chosen. Schwartz (35) and Capon (8) have studied optimum parametric coincidence procedures for detecting weak signals in additive gaussian noise. The detection of a gaussian signal in correlated gaussian noise by use of a coincidence detector has been investigated by Bunimovich and Morozov (36, 37). The system design and performance of a coincidence detector has been reported by Raether and Bitzer (38).

A coincidence detector with nonparametric properties, hereafter referred to as the median detector, has been studied by Lainiotis (39). The threshold was chosen to be the median of the noise distribution. The input samples were assumed to be statistically independent.

The relative efficiency of the median detector compared to the optimum parametric detector has been calculated assuming independent observations and very small signal-to-noise ratios. For example, the relative efficiency of the median statistic compared to the optimum parametric statistic for detecting a constant signal in additive gaussian noise (the likelihood ratio test) is $2/\pi$ (40). The median detector is twice as efficient as the optimum parametric detector designed under the gaussian assumption but operating under noise with an exponential density function (39). On the other hand, the relative efficiency of the median detector compared against the likelihood ratio test appropriate for detecting a constant signal in a noise with an exponential distribution is zero (41). (The meaning of relative efficiency will be given in Chapter III.)

A block diagram illustrating the median detector for detecting a constant signal in additive noise of median M is given in Fig. 4. The test statistic is

$$S_n = \frac{1}{n} \sum_{i=1}^{n} c\,(Y_i - M) \qquad [4]$$

where

$$c(z) = 1 \quad \text{if } Y_i > M$$
$$= 0 \quad \text{if } Y_i < M \; .$$

The term "constant signal" in practice could imply a pulsed carrier of known phase.

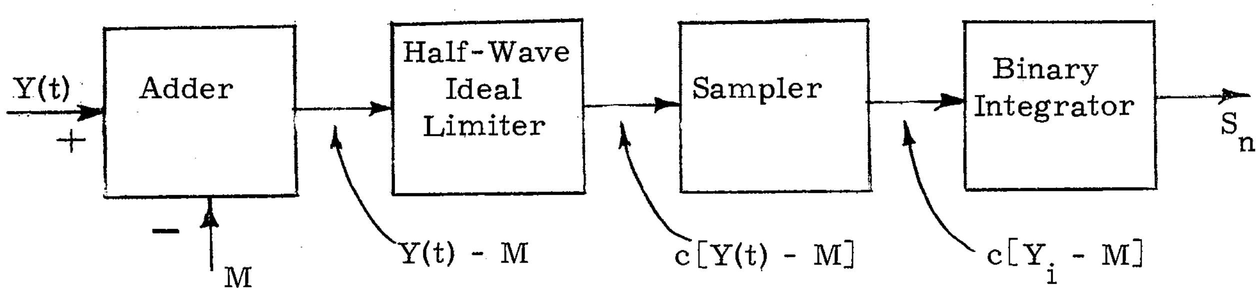

Fig. 4. A Block Diagram of the Median Detector.

In general, individual sample values cannot be treated as statistically independent. This inherent correlation between sample values over the observation period is an essential feature of the detection problem (2). For realizable narrow-band signals, there is no sampling rate for which the independent sample assumption is valid (30). Also, due to the finite bandwidth of the noise, the samples are not statistically independent. If the interval between samples is some-what larger than the reciprocal of the noise bandwidth, the samples are nearly statistically independent (4). It is evident, however, that the more frequent the sampling, the more reliable the signal detection: with an increase in sampling frequency additional measurements are performed, providing additional information concerning the process under observation. Therefore, the effect of correlated samples on the performance of the median detector could be quite important. To determine its importance, certain performance criteria for detectors must be introduced. This will be done in the following chapter.

CHAPTER III

PERFORMANCE CRITERIA FOR DETECTORS

To determine the effect of dependent sampling on a detector, some means of performance comparison must be used. Unfortunately, the comparison methods do not give a sufficiently complete description of the detector, or they impose severe restrictions on the type of operation that can be considered. This chapter will discuss the two methods of performance comparison used in this study: asymptotic relative efficiency and receiver operating characteristics. The relation between these two methods and the input-output signal-to-noise ratios will also be mentioned.

Asymptotic Relative Efficiency

The asymptotic relative efficiency (hereafter designated ARE) of one detector with respect to another is an indication of how many more observations one detector requires than the other to detect a given weak signal with a prescribed accuracy, α, β. It should be stressed that the ARE is a comparison which is valid only for a particular pair of noise and signal plus noise probability distribution functions. If the distribution functions change, the ARE will change. Thus, only comparisons for specific cases can be made; no general performance efficiency can be concluded. The comparison of detectors in the presence of increasingly weak signals is justified by assuming that the strong signal performance is not critical. A more precise definition of ARE will now be given.

Suppose a null hypothesis, $\theta = 0$, is to be tested against the alternative $\theta = \theta_n > 0$, where θ_n approaches zero as n approaches infinity. (θ_n corresponds to the input signal-to-noise ratio defined in Chapter II.) Let the two tests which are to be compared be based on the two statistics T_n and T_n', which are asymptotically normally distributed under both the null and the alternative hypotheses. Suppose that T_n and T_n' require N and N' observations, respectively, to attain the power β at a level of significance α for testing the hypothesis H_o: $\theta = 0$ against the alternative $H_1 : \theta = \theta_n$. The ARE of T_n with respect to T_n' is $\lim_{n \to \infty} \frac{N'}{N}$ (42, 43).

In order for the limit to exist and be independent of α and β, a number of regularity conditions are needed. Let the mean of the test statistic T_n be denoted as $E_\theta [T_n]$ under signal conditions and $E_o [T_n]$ under no-signal conditions; let the variance of T_n be $\sigma_\theta^2 [T_n]$ and $\sigma_0^2 [T_n]$ under signal and no-signal conditions, respectively.

Regularity Conditions:

(A) The distribution of $[T_n - E_\theta [T_n]] / \sigma_\theta [T_n]$ tends to the normal distribution with mean zero and variance one, uniformly in θ (*), with $0 \le \theta \le \theta_1$, for some $\theta_1 > 0$.

(B) $E_\theta'[T_n] = \frac{d}{d\theta} E_\theta [T_n]$ exists for all θ in $(0, \theta_1)$, and is continuous at $\theta = 0$.

(C) $$\lim_{n \to \infty} \frac{1}{n} \left[\frac{E_\theta' [T_n] \big|_{\theta = 0}}{\sigma_0 [T_n]} \right]^2 = E_{T_n} > 0;$$ E_{T_n} is called the efficacy.

(D) There exists a sequence $\{\theta_n\}$, such that

$$\lim_{n \to \infty} \theta_n = 0.$$

(E) $$\lim_{n \to \infty} \frac{\sigma_{\theta_n}^2 [T_n]}{\sigma_0^2 [T_n]} = 1$$

(F) $$\lim_{n \to \infty} \sigma_0^2 [T_n] = 0$$

The regularity conditions are essentially equivalent to requiring that the two tests T_n and T_n' be consistent (**) (7, 42).

---

*Mood (40) states that generally proofs of asymptotic normality do not provide this strong a result, and the validity of the assumption is not completely justified. However, the computations are usually valid.

**The property of consistency is defined as

$$\lim_{n \to \infty} \beta_n = 0,$$

and means that for a fixed alternative θ_1 and α, the decisions concerning the presence or absence of the signal become more and more reliable as more observations are obtained.

If the regularity conditions are satisfied for the detectors T_n and T_n', then it can be shown that (44)

$$\text{ARE } (T_n : T_n') = \frac{E_{T_n}}{E_{T_n'}} \qquad [5]$$

Thus the ARE is given by the ratio of the efficacies of the two detectors (statistics).

The validity of the concept of ARE for comparing statistical tests might be supported by its wide usage in the field of statistics. The concept has been attributed to E. J. G. Pitman, and has been described by several authors (6, 18, 44, 45, 46). Applications of ARE can be found in many papers (47, 48, 49, 50, 51, 52). It is noted that all of these applications have assumed independent observations.

One further comment regarding the efficiency of nonparametric tests is given by Kendall and Stuart (18). They state that a nonparametric test, chosen in ignorance of the form of the underlying distributions, cannot be expected to be as efficient as the test that would have been used had the underlying distributions been known, that is, the optimum parametric test. The only "fair" standard of efficiency for a nonparametric test is that provided by other nonparametric tests. In such cases the most efficient test should naturally be chosen. However, comparisons between parametric and nonparametric tests will be made in this study; hence, these remarks should be kept in mind.

Receiver Operating Characteristics

All statistical performance criteria for detectors are based directly or indirectly on either α and β or cost and risk functions. Cost and risk functions require that the cost of a false alarm and of a miss be known; also, the a priori probabilities must be specified. If a performance criterion based on α and β is used, one usually fixes α and is consequently led to a criterion which (for a fixed α) depends solely on β. Such is the case in this investigation. In so doing it is assumed that values of α and β can be intelligently specified, and that the information concerning the a priori probabilities and the cost functions can somehow be utilized in specifying α and β.

The function $\beta = \beta(\alpha, \theta)$ has been termed the operating characteristic of a receiver, where α, β, and θ are the same as defined previously (5). An alternate form is a plot of the power of the test vs the size of the test ($1 - \beta$ vs α) for θ constant. The operating characteristic depends on the probability density functions of the observations under the two hypotheses, but not on any cost functions or a priori probabilities.

The function to be plotted can be determined from the definitions of α and β. For a threshold T_α,

$$\alpha = \int_{T_\alpha}^{\infty} p_o(y)\, dy \qquad [6]$$

and

$$\beta = \int_{-\infty}^{T_\alpha} p_\theta(y)\,dy \qquad [7]$$

where $p_o(y)$ and $p_\theta(y)$ are the probability density functions of the noise and the signal plus noise observations. For a fixed α, T_α can be found and used in the second equation to find β for various values of θ.

The disadvantage of using the operating characteristic as a performance criterion is that the distribution functions under noise and signal plus noise must be specified, thereby limiting the generality of the comparison.

Signal-to-Noise Ratio Criterion

The definition of signal-to-noise ratio in Chapter II can be used to give input-output signal-to-noise ratio comparisons in terms of loss in db (31, 36). For the detection of a constant signal A in noise with variance σ_N^2, the input signal-to-noise ratio is

$$\text{Input SNR} = \frac{A}{\sigma_N} = \theta .$$

The output signal-to-noise ratio is defined as

$$\text{Output SNR} = \frac{E_\theta[T_n] - E_0[T_n]}{\sigma_\theta[T_n]}$$

where $E_\theta[T_n]$ and $\sigma_\theta^2[T_n]$ are the mean and the variance of the test statistic T_n under signal conditions and $E_0[T_n]$ is the mean under no-signal conditions. If $E_\theta[T_n]$ is expanded in a Maclaurin's series in terms of θ:

$$E_\theta\,[T_n] = E_0\,[T_n] + \theta \frac{d}{d\theta} E_\theta\,[T_n]\Big|_{\theta=0} + \frac{\theta^2}{2}\frac{d^2}{d\theta^2} E_\theta\,[T_n]\Big|_{\theta=0} + \cdots$$

and if $\theta < 1$ such that $\theta^2 \ll 1$, the ratio of the input to output signal-to-noise ratios is

$$\frac{\text{Output SNR}}{\text{Input SNR}} \doteq \frac{\theta \frac{d}{d\theta} E_\theta\,[T_n]\Big|_{\theta=0}}{\theta\,\sigma_\theta\,[T_n]} .$$

Furthermore, if $\sigma_\theta^{\,2}\,[T_n] \doteq \sigma_0^{\,2}\,[T_n]$ for n large and θ small, then

$$\frac{\text{Output SNR}}{\text{Input SNR}} \doteq \frac{E_\theta'\,[T_n]\Big|_{\theta=0}}{\sigma_0\,[T_n]} .$$

Notice that as $n \to \infty$ this becomes the square root of the efficacy of the detector. Therefore, the efficacy of a detector could be interpreted as the ratio of the input and output signal-to-noise ratios squared for n large and θ small. Because of this redundancy, the signal-to-noise criterion will be replaced by the more meaningful efficacy and ARE criteria.

CHAPTER IV

STATISTICAL PROPERTIES OF THE MEDIAN DETECTOR

An introduction to detection theory and a review of the problem have been given in Chapters I and II. It was pointed out that very few studies have been made concerning nonparametric detectors without making the assumption that the observations are independent of each other. No work has been published with respect to the median detector without assuming independent samples. Furthermore, the performance criteria described in Chapter III to be used in the analysis of detectors always requires that the mean and the variance of the test statistic under signal and no-signal conditions be known. Therefore, to investigate the effect of dependent observations on the performance of the median detector, the first step is to derive these quantities. In this chapter the mean and the variance of the median test statistic under signal and no-signal conditions will be found for the cases of independent and of dependent sampling. The optimum parametric test statistic will be similarly considered in Chapter V.

Independent Samples

The mean and variance under signal and no-signal conditions assuming independent samples will first be found for reference purposes. The median test statistic was given in Chapter II as

$$S_n = \frac{1}{n} \sum_{i=1}^{n} c\,(Y_i - M) \qquad [8]$$

where $Y_1, \cdots, Y_n$ are the observations of the input random process $Y(t)$ and M is the median of the noise distribution. The function c(z) is defined as

$$c(z) = 1 \quad \text{if} \quad z > 0$$
$$= 0 \quad \text{if} \quad z < 0.$$

It will be assumed throughout that the random process Y(t) is stationary under no-signal conditions.

The mean of S_n under no-signal conditions and independent observations is (39)

$$E_0[S_n] = \frac{1}{n} \sum_{i=1}^{n} E_0 \quad [c(Y_i - M)]$$

$$= \frac{1}{n} \sum_{i=1}^{n} [1 \cdot P_0 \,(Y_i > M) + 0 \cdot P_0 \,(Y_i < M)]$$

$$= \frac{1}{n} \sum_{i=1}^{n} P_0 \,(Y_i > M) = \frac{1}{n} \sum_{i=1}^{n} \int_M^{\infty} d\,F_{0_i}(y)$$

$$= \frac{1}{n} \sum_{i=1}^{n} [1 - F_{0_i}(M)]$$

where $F_{0_i}(y)$ is the probability distribution function of the random variable Y_i. For a stationary noise median,

$$F_{0_i}(M) = \tfrac{1}{2}$$

and

$$E_0[S_n] = \tfrac{1}{2}. \qquad [9]$$

The variance of S_n under no-signal conditions, assuming independent samples, is (39)

$$\sigma_0{}^2[S_n] = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_0{}^2 \, [c(Y_i - M)] .$$

Let $Z_i = Y_i - M$, and let $E_0[c(Y_i - M)] = m_i$. Then

$$\sigma_0{}^2[S_n] = \frac{1}{n^2} \sum_{i=1}^{n} \left\{ E_0 \, [c(Z_i)^2] - m_i{}^2 \right\} .$$

But from above, $m_i = 1 - F_{0_i}(M) = E_0[c(Z_i)^2]$; furthermore,

$F_{0_i}(M) = \tfrac{1}{2}$. Thus

$$\sigma_0{}^2 \, [S_n] = \frac{1}{4n} . \qquad [10]$$

Note that the mean and the variance are independent of the noise distribution; hence α is constant and the detector is nonparametric.

The mean of the test statistic S_n under signal conditions, assuming independent samples, is (39)

$$E_0 \, [S_n] = \frac{1}{n} \sum_{i=1}^{n} E_\theta \, [c(Y_i - M)]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[1 \cdot P_\theta \, (Y_i > M) + 0 \cdot P_\theta \, (Y_i < M) \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} P_\theta \, (Y_i > M) = \frac{1}{n} \sum_{i=1}^{n} \int_M^\infty dG_{\theta_i}(y)$$

$$= \frac{1}{n} \sum_{i=1}^{n} [1 - G_{\theta_i}(M)] \qquad [11]$$

where $G_{\theta_i}(y)$ is the probability distribution function of the random variable Y_i under signal conditions. It is assumed that under signal conditions the distribution functions of Y_i, $i = 1, \cdots, n$ and Y_j, $j = 1, \cdots, n$, $i \neq j$ differ only through the signal-to-noise ratio parameter, θ (39). Apply the mean value theorem (53) to $G_\theta(M)$.

$$\frac{G_{\theta_i}(M) - G_0(M)}{\theta_i} = \frac{d}{d\theta} G_\theta(M) \Big|_{\theta = \hat{\theta}} \qquad [12]$$

where $0<\hat{\theta}<\theta_i$. Note that $G_0(M) = F_0(M) = \frac{1}{2}$. Substitute these results into Eq. 11;

$$E_\theta [S_n] = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{2} - \theta_i \frac{d}{d\theta} G_\theta(M) \Big|_{\theta = \hat{\theta}} \right].$$

For the weak signal case, $\hat{\theta} \doteq 0$, so

$$E_\theta [S_n] = \frac{1}{2} - \bar{\theta} \frac{d}{d\theta} G_\theta(M) \Big|_{\theta \doteq 0} \qquad [13]$$

where $\bar{\theta} = \frac{1}{n} \sum_{i=1}^{n} \theta_i$.

The variance of S_n under signal conditions, assuming independent sampling, is

$$\sigma_\theta^2 [S_n] = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_\theta^2 [c(Y_i - M)]$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \left[G_{\theta_i}(M) - G_{\theta_i}^{\ 2}(M) \right] .$$

Again apply the mean value theorem of Eq. 12 to $G_\theta(M)$ for the weak signal case.

$$\sigma_\theta^{\ 2}[S_n] = \frac{1}{n^2} \sum_{i=1}^{n} \left[\frac{1}{2} + \theta_i \frac{d}{d\theta} G_\theta(M) \Big|_{\theta \doteq 0} - \left(\frac{1}{2} + \theta_i \frac{d}{d\theta} G_\theta(M) \Big|_{\theta \doteq 0} \right)^2 \right]$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \left[\frac{1}{4} - \theta_i^{\ 2} \frac{d}{d\theta} G_\theta(M) \Big|_{\theta \doteq 0}^{2} \right]$$

$$= \frac{1}{4n} - \frac{1}{n^2} \frac{d}{d\theta} G_\theta(M) \Big|_{\theta \doteq 0}^{2} \sum_{i=1}^{n} \theta_i^{\ 2}$$

$$= \sigma_0^{\ 2}[S_n] - \frac{1}{n} \overline{\theta}^2 \frac{d}{d\theta} G_\theta(M) \Big|_{\theta \doteq 0}^{2}$$

where $\overline{\theta}^2 = \frac{1}{n} \sum_{i=1}^{n} \theta_i^{\ 2}$. Since $\overline{\theta}^2 \ll \overline{\theta}$,

$$\sigma_\theta^{\ 2}[S_n] \doteq \sigma_0^{\ 2}[S_n] . \qquad [14]$$

Equations 9, 10, 13, 14 give the mean and the variance of S_n under signal and no-signal conditions, assuming independent samples.

The mean of S_n can be evaluated for the particular problem of detecting a constant signal in additive, white gaussian noise with zero mean and variance of one. For this problem,

$$G_\theta(y) = F_0(y - \theta)$$

and

$$\left.\frac{d}{d\theta} G_\theta(y)\right|_{\theta \doteq 0} = \left.\frac{d}{d\theta} F_0(y - \theta)\right|_{\theta \doteq 0} = -f(y)$$

where f(y) is the probability density function of the random variable Y_i under no-signal conditions. Since the noise is N (0, 1), then

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2},$$

and

$$\left.\frac{d}{d\theta} G_\theta(M)\right|_{\theta \doteq 0} = \frac{-1}{\sqrt{2\pi}} \qquad [15]$$

since y = 0 at the median. Thus the mean for this example is

$$E_\theta[S_n] = \frac{1}{2} + \frac{\theta}{\sqrt{2\pi}}. \qquad [16]$$

Dependent Samples -- No-Signal Conditions

The mean and the variance of S_n under no-signal conditions, assuming dependent samples, will be derived next. Referring to the procedure in the previous section used to find the mean of S_n under no-signal conditions, it will be observed that the question of independence of $Y_1, \cdots, Y_n$ did not arise. Thus, the mean under no-signal conditions is not affected by the independent assumption, and $E_0[S_n]$ assuming dependent samples is given by Eq. 9.

The variance of S_n under no-signal conditions, assuming dependent samples from a stationary random process, is derived below. Refer to Lee (54) for a discussion of the general procedure used to obtain Eq. 17.

$$\sigma_0^2 [S_n] = \frac{1}{n^2} E_0\left[\sum_{i=1}^{n} c(Z_i) \sum_{j=1}^{n} c(Z_j)\right] - E_0^2 [S_n]$$

$$= \frac{1}{n^2} E_0\left[\sum_{i=1}^{n} c(Z_i)^2\right] + \frac{1}{n^2} E_0\left[\sum_{i \neq j}^{n} \sum^{n} c(Z_i)\, c(Z_j)\right] - \frac{1}{4}$$

$$= \frac{R(0)}{n} - \frac{1}{4} + \frac{2}{n} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) R(k\tau) \qquad [17]$$

where $Z_i = Y_i - M$. R (kt) is defined as

$$R(k\tau) = E_0 [c(Z_0)\, c(Z_k)].$$

This can also be written as

$$R(k\tau) = P_0 (Z_0 > 0 \cap Z_k > 0) \qquad [18]$$

where P_0 is a joint probability function. Thus

$$R(k\tau) = \iint_0^{\infty} f(z_0, z_k)\, dz_0\, dz_k \qquad [19]$$

where $f(z_0, z_k)$ is the joint probability density function of the random variable Z_0, Z_k.

It is clear from Eq. 19 that to proceed further the joint density function must be specified. It is apparent from Eq. 17 and Eq. 18 that the variance of the test statistic S_n is not affected by any instantaneous operation on Y(t) that preserves the "median-crossing level".

Thus, define a class of modified gaussian processes as containing all processes that can be obtained by passing a stationary gaussian process through an instantaneous operation whose characteristic leaves the median value unchanged (30). Furthermore, to make the analysis tractable -- as well as for practical considerations, assume that the input noise process has a mean value of zero. Practically speaking, this assumption is quite valid. (A derivation of $R(k\tau)$ assuming that the mean is nonzero is given in Appendix A.) Since the mean and the median coincide for a gaussian process, the median is zero under these assumptions. The class of modified gaussian processes then becomes the class of all processes that can be obtained by passing a stationary gaussian process through any instantaneous operation whose characteristic passes through the origin.

Kanefsky (30) mentions that there are still definite advantages in using a nonparametric detector for this restricted class of inputs instead of the optimum parametric detector. The main advantages are: 1) simplicity of implementation (the median is zero); 2) invariance with respect to the noise power; and, 3) invariance with respect to certain nonlinear operations on the input signals, especially amplitude limiting operations.

The bivariate gaussian density function for processes with zero mean is (2)

$$f\,(z_0,\ z_k) = \frac{1}{2\pi\sigma_n^{\,2}\sqrt{1-\rho^2}}\exp\left\{-\frac{z_0^{\,2} - 2\rho z_0 z_k + z_k^{\,2}}{2\sigma_n^{\,2}\,(1-\rho^2)}\right\} \qquad [20]$$

where $\rho = \rho(k\tau)$ is the normalized correlation function expressing the correlation between the random variables z_o, z_k and σ_n^2 is the noise power (variance).

Under the assumptions discussed above, the derivation of $R(k\tau)$ can now be completed. The procedure follows that outlined by Rice (55). Substituting $f(z_o, z_k)$ into Eq. 19, evaluate $R(k\tau)$.

$$R\ (k\tau) = \iint_0^\infty \frac{1}{2\pi\sigma_n^2 \sqrt{1-\rho^2}} \exp\left\{-\frac{z_o^2 - 2\rho z_o z_k + z_k^2}{2\sigma_n^2 (1-\rho^2)}\right\} dz_o dz_k \qquad [21]$$

Let $A_1 = \dfrac{1}{2\pi\sigma_n^2 \sqrt{1-\rho^2}}$, $A_2 = \dfrac{1}{2\sigma_n^2 (1-\rho^2)}$.

Then

$$R\ (k\tau) = A_1 \iint_0^\infty \exp\left[A_2 \left(-z_o^2 + 2\rho z_o z_k - z_k^2\right)\right] dz_o dz_k \ .$$

Let $\qquad z_o = y_o + \rho\,(1-\rho^2)^{-\frac{1}{2}}\, y_k$

and $\qquad z_k = (1-\rho^2)^{-\frac{1}{2}}\, y_k$.

It is assumed that $\rho \neq 1$. Since z_k runs from 0 to ∞, so must y_k; y_o runs from $-\rho\,(1-\rho^2)^{-\frac{1}{2}} y_k$ to ∞. This gives

$$R\ (k\tau) = \frac{A_1}{(1-\rho^2)^{\frac{1}{2}}} \int_0^\infty dy_k \int_{\frac{-\rho y_k}{\sqrt{1-\rho^2}}}^\infty \exp\left[-A_2\left(y_o^2 + y_k^2\right)\right] dy_o \ . \qquad [22]$$

In order to carry out the integration of Eq. 22, change to polar coordinates. Hence,

$y_o = r \cos \theta$

$y_k = r \sin \theta$

$dy_k dy_o = r dr d\theta$

$y_k \geq 0$ gives $0 \leq \theta \leq \pi$

$y_o \geq \dfrac{-\rho y_k}{\sqrt{1-\rho^2}}$ gives $\theta \leq \cot^{-1}\left(\dfrac{-\rho}{\sqrt{1-\rho^2}}\right)$

or $\theta \leq \cos^{-1}(-\rho)$

or $\theta \leq \dfrac{\pi}{2} + \sin^{-1}\rho$.

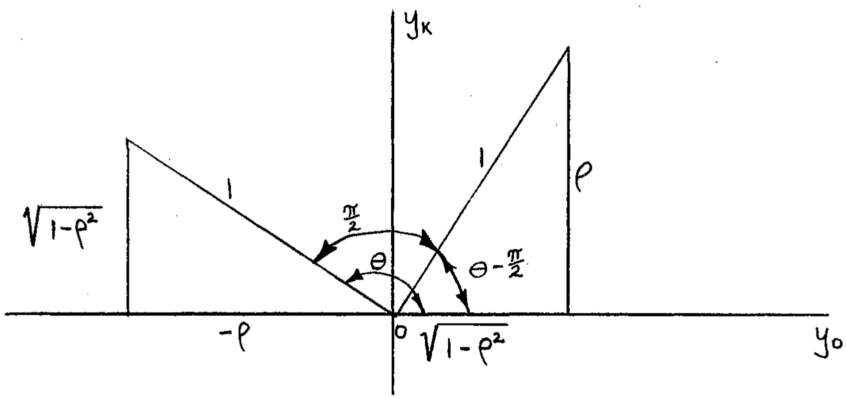

Fig. 5. Transformation of Coordinates for Evaluation of R (kτ).

Substituting these into Eq. 22 and integrating, yields

$$R(k\tau) = \frac{A_1}{(1-\rho^2)^{\frac{1}{2}}} \int_0^{\frac{\pi}{2}+\sin^{-1}\rho} d\theta \int_0^{\infty} r\, e^{-A_2 r^2}\, dr$$

$$= \frac{A_1}{2A_2 (1-\rho^2)^{\frac{1}{2}}} \left[\frac{\pi}{2} + \sin^{-1}\rho\right].$$

Substitute the values of A_1 and A_2 into the above equation and simplify. The result is

$$R\ (k\tau) = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho\,(k\tau)\ . \qquad [23]$$

Also, since $\rho(0) = 1$, then R(0) is

$$R(0) = \tfrac{1}{2}.$$

The value of $R(k\tau)$ and R(0) found in Eq. 23 can be substituted into the expression for the variance, Eq. 17,

$$\sigma_o^{\,2}\,[S_n] = \frac{1}{2n} - \frac{1}{4} + \frac{2}{n} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \left[\frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho\ (k\tau)\right] . \qquad [24]$$

Since (54)

$$\sum_{k=1}^{n} 2\ (n - k) = n\ (n - 1),$$

then Eq. 24 can be simplified to the following form:

$$\sigma_o^{\,2}\,[S_n] = \frac{1}{4n} + \frac{1}{\pi n} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \sin^{-1} \rho\ (k\tau) . \qquad [25]$$

Therefore, the mean and the variance of S_n under no-signal conditions and assuming dependent samples are given by Eq. 9 and Eq. 25.

Dependent Samples -- Signal Conditions

The mean and the variance of the test statistic S_n under signal conditions, assuming dependent samples, will be found next. Again the mean of S_n under signal conditions for independent sampling did not depend upon the independent sample assumption. Hence, the mean of S_n under signal conditions is the same for both dependent and independent sampling, and is given by Eq. 13.

The variance of S_n under signal conditions and under the assumption of dependent samples will now be derived.

$$\sigma_\theta^2 [S_n] = \frac{1}{n^2}\left\{E_\theta\left[\sum_{i=1}^{n} c(Z_i) \sum_{j=1}^{n} c(Z_j)\right] - E_\theta^2\left[\sum_{i=1}^{n} c(Z_i)\right]\right\}$$

$$= \frac{1}{n^2}\left\{\underbrace{E_\theta\left[\sum_{i=1}^{n} c(Z_i)^2\right]}_{\text{FIRST TERM}} - \underbrace{E_\theta^2\left[\sum_{i=1}^{n} c(Z_i)\right]}_{\text{SECOND TERM}} + \underbrace{E_\theta\left[\sum_{i \neq j}^{n}\sum^{n} c(Z_i)\, c(Z_j)\right]}_{\text{THIRD TERM}}\right\} \quad [26]$$

Consider the first term.

$$E_\theta\left[\sum_{i=1}^{n} c(Z_i)^2\right] = \sum_{i=1}^{n} E_\theta\,[c(Z_i)^2]$$

$$= \sum_{i=1}^{n}\left[1^2 \cdot P_\theta\,(Z_i > 0) + 0^2 \cdot P_\theta\,(Z_i < 0)\right]$$

$$= \sum_{i=1}^{n} P_\theta\,(Y_i > M)$$

$$= \sum_{i=1}^{n} [1 - G_{\theta_i}\,(M)]$$

Apply the mean value theorem to $G_\theta(M)$ as in Eq. 12, assuming weak signals, $\theta \doteq 0$. The first term becomes

$$\frac{1}{n^2} E_\theta\left[\sum_{i=1}^{n} c(Z_i)^2\right] = \frac{1}{2n} - \frac{1}{n}\,\overline{\theta}\,\frac{d}{d\theta} G_\theta\,(M)\Big|_{\theta \doteq 0} \quad [27]$$

where $\bar{\theta} = \frac{1}{n} \Sigma \theta_i$.

Consider the second term of Eq. 26.

$$E_\theta^2 \left[\sum_{i=1}^{n} c(Z_i) \right] = \left\{ \sum_{i=1}^{n} E_\theta \left[c(Z_i) \right] \right\}^2$$

$$= \left\{ \sum_{i=1}^{n} \left[1 \cdot P_\theta (Z_i > 0) + 0 \cdot P_\theta (Z_i < 0) \right] \right\}^2$$

$$= \left\{ E_\theta \left[\sum_{i=1}^{n} c(Z_i)^2 \right] \right\}^2$$

Therefore simply square the result in Eq. 27,

$$\frac{1}{n^2} E_\theta^2 \left[\sum_{i=1}^{n} c(Z_i) \right] = \frac{1}{n^2} \left[\frac{n}{2} - n\bar{\theta} \frac{d}{d\theta} G_\theta (M) \Big|_{\theta \doteq 0} \right]^2$$

$$= \frac{1}{4} - \bar{\theta} \frac{d}{d\theta} G_\theta (M) \Big|_{\theta \doteq 0} + \bar{\theta}^{\,2} \frac{d}{d\theta} G_\theta (M) \Big|_{\theta \doteq 0}^{2} \quad . \qquad [28]$$

Consider the third term of Eq. 26.

$$E_\theta \left[\sum_{i \neq j}^{n} \sum^{n} c(Z_i)\, c(Z_j) \right] = \sum_{i \neq j}^{n} \sum^{n} E_\theta \left[c(Z_i)\, c(Z_j) \right]$$

$$= \sum_{i \neq j}^{n} \sum^{n} \left[1 \cdot P_\theta (Z_i > 0 \cap Z_j > 0) + 0 \cdot \text{(remainder of terms)} \right]$$

$$E_\theta \left[\sum_{i \neq j}^{n} \sum^{n} c(Z_i)\, c(Z_j) \right] = \sum_{i \neq j}^{n} \sum^{n} P_\theta (Y_i > M \cap Y_j > M)$$

$$= \sum_{i \neq j}^{n} \sum^{n} [\, 1 + G_{\theta_i \theta_j}(M,M) - G_{\theta_i}(M) - G_{\theta_j}(M)\,] \qquad [29]$$

where $G_{\theta_i \theta_j}(M,M)$ is the joint distribution function of two dependent input random variables, Y_i and Y_j, under signal conditions, and $G_{\theta_i}(M)$ and $G_{\theta_j}(M)$ are marginal distributions. The various distributions differ only through the signal-to-noise ratio parameter, θ. The fundamental expression used to obtain Eq. 29 is (56)

$$P\,(a_1 < X < b_1 \cap a_2 < Y < b_2) = F_{X,Y}(b_1,b_2) + F_{X,Y}(a_1,a_2) - F_{X,Y}(a_1,b_2) - F_{X,Y}(b_1,a_2)\ .$$

Apply the mean value theorem for two functionally independent variables to $G_{\theta_x \theta_y}(M, M)$, (53). The formula is

$$f(a + \Delta x,\ b + \Delta y) - f(a,b) = \Delta x f_1\,(a + \hat{\theta}_1,\ b) + \Delta y f_2\,(a + \Delta x,\ b + \hat{\theta}_2)$$

where $\Delta x^2 + \Delta y^2 < \delta^2$

and $0 < \hat{\theta}_1 < \Delta x,\ 0 < \hat{\theta}_2 < \Delta y$.

In different notation this becomes

$$G_{\theta_i \theta_j}(M, M) - G_{0,0}(M,M) = \theta_i \frac{d}{d\theta_x} G_{\theta_x \theta_y}(M,M) \Bigg|_{\substack{\theta_x = \hat{\theta}_x \\ \theta_y = 0}} + \theta_j \frac{d}{d\theta_y} G_{\theta_x \theta_y}(M,M) \Bigg|_{\substack{\theta_x = \theta_i \\ \theta_y = \hat{\theta}_y}} \qquad [30]$$

where $0<\hat{\theta}_x<\theta_i$ and $0<\hat{\theta}_y<\theta_j$. Assume the small signal case, $\hat{\theta}_x \doteq \hat{\theta}_y \doteq 0$. Also, apply the mean value theorem for one variable from Eq. 12 to $G_{\theta_i}(M)$ and $G_{\theta_j}(M)$. The result is

$$E_\theta\left[\sum_{i}^{n}\sum_{\neq j}^{n} c(Z_i)\ c(Z_j)\right] = \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\left[1 + G_{o,o}(M,M) + \theta_i \frac{d}{d\theta_x} G_{\theta_x\theta_y}(M,M)\Big|_{\substack{\theta_x \doteq 0\\ \theta_y = 0}} + \right.$$

$$+ \theta_j \frac{d}{d\theta_y} G_{\theta_x\theta_y}(M,M)\Big|_{\substack{\theta_x = \theta_i\\ \theta_y \doteq 0}} - G_o(M) - \theta_i \frac{d}{d\theta_x} G_{\theta_x}(M)\Big|_{\theta_x \doteq 0} -$$

$$- G_o(M) - \theta_j \frac{d}{d\theta_y} G_{\theta_y}(M)\Big|_{\theta_y \doteq 0} . \qquad [31]$$

Since

$$P_o(Y_i > M \cap Y_j > M) = 1 + G_{o,o}(M,M) - G_o(M) - G_o(M),$$

then Eq. 31 becomes

$$\frac{1}{n^2} E_\theta\left[\sum_{i}^{n}\sum_{\neq j}^{n} c(Z_i)\ c(Z_j)\right] = \frac{1}{n^2}\sum_{i}^{n}\sum_{\neq j}^{n} P_o(Y_i > M \cap Y_j > M) +$$

$$+ \frac{1}{n^2}\sum_{i}^{n}\sum_{\neq j}^{n}\theta_i \frac{d}{d\theta_x} G_{\theta_x\theta_y}(M,M)\Big|_{\substack{\theta_x \doteq 0\\ \theta_y = 0}} + \frac{1}{n^2}\sum_{i}^{n}\sum_{\neq j}^{n}\theta_j \frac{d}{d\theta_y} G_{\theta_x\theta_y}(M,M)\Big|_{\substack{\theta_x = \theta_i\\ \theta_y \doteq 0}} -$$

$$- 2\left(1 - \frac{1}{n}\right)\bar{\theta}\frac{d}{d\theta} G_\theta(M)\Big|_{\theta \doteq 0} . \qquad [32]$$

In Eq. 32 the fact has been used that the marginal density functions are the same, and that

$$\bar{\theta} = \frac{1}{n}\sum_{i=1}^{n}\theta_i = \frac{1}{n}\sum_{j=1}^{n}\theta_j .$$

Recall that

$$\frac{1}{n^2} \sum_{i}^{n} \sum_{\neq j}^{n} P_o (Y_i > M \cap Y_j > M) = \frac{1}{n^2} \sum_{i}^{n} \sum_{\neq j}^{n} E_o [c(Z_i) \; c(Z_j)]$$

$$= \frac{2}{n} \sum_{k=1}^{n} (1 - \frac{k}{n}) \; R(k\tau)$$

where R(kτ) is the correlation function of the output of the 0, 1 limiter in the detector under no-signal conditions. Substituting this into Eq. 32 yields

$$\frac{1}{n^2} E_\theta \left[\sum_{i}^{n} \sum_{\neq j}^{n} c(Z_i) \; c(Z_j) \right] = \frac{2}{n} \sum_{k=1}^{n} (1 - \frac{k}{n}) \; R(k\tau) -$$

$$-2 \, (1 - \frac{1}{n}) \; \bar{\theta} \frac{d}{d\theta} G_\theta (M) \Big|_{\theta \doteq 0} + \frac{1}{n^2} \sum_{i}^{n} \sum_{\neq j}^{n} \theta_i \frac{d}{d\theta_x} G_{\theta_x \theta_y} (M,M) \Big|_{\substack{\theta_x \doteq 0 \\ \theta_y = 0}} +$$

$$+ \frac{1}{n^2} \sum_{i}^{n} \sum_{\neq j}^{n} \theta_j \frac{d}{d\theta_y} G_{\theta_x \theta_y} (M,M) \Big|_{\substack{\theta_x = \theta_i \\ \theta_y \doteq 0}} . \qquad [33]$$

This is the third term of Eq. 26.

Substitute the results of Eq. 27, Eq. 28 and Eq. 33 into Eq. 26 to find $\sigma_\theta^{\,2} [S_n]$. Simplifying, this yields

$$\sigma_\theta^{\,2} [S_n] = \frac{1}{2n} - \frac{1}{4} + \frac{2}{n} \sum_{k=1}^{n} (1 - \frac{k}{n}) \; R(k\tau) - (1 - \frac{1}{n}) \; \bar{\theta} \frac{d}{d\theta} G_\theta (M) \Big|_{\theta \doteq 0} -$$

$$- \bar{\theta}^2 \frac{d}{d\theta} G_{\theta}(M) \Big|^{2}_{\theta \doteq 0} + \frac{1}{n^2} \sum_{i \neq}^{n} \sum_{j}^{n} \theta_i \frac{d}{d\theta_x} G_{\theta_x \theta_y}(M,M) \Bigg|_{\substack{\theta_x \doteq 0 \\ \theta_y = 0}} +$$

$$+ \frac{1}{n^2} \sum_{i \neq}^{n} \sum_{j}^{n} \theta_j \frac{d}{d\theta_y} G_{\theta_x \theta_y}(M,M) \Bigg|_{\substack{\theta_x = \theta_i \\ \theta_y \doteq 0}} . \qquad [34]$$

For further simplification, assume that θ_i and θ_j can be replaced by their average value, $\bar{\theta}$. The quantity $\sigma_0^2 [S_n]$ from Eq. 25 can be substituted in Eq. 34, thus giving

$$\sigma_\theta^2 [S_n] = \sigma_0^2 [S_n] - \left[(1 - \frac{1}{n}) \bar{\theta} + \bar{\theta}^2\right] \frac{d}{d\theta} G_\theta (M) \Big|_{\theta \doteq 0} +$$

$$+ \frac{\bar{\theta}}{n^2} \sum_{i \neq}^{n} \sum_{j}^{n} \left[\frac{d}{d\theta_x} G_{\theta_x \theta_y}(M,M) \Bigg|_{\substack{\theta_x \doteq 0 \\ \theta_y = 0}} + \frac{d}{d\theta_y} G_{\theta_x \theta_y}(M,M) \Bigg|_{\substack{\theta_x = \theta_i \\ \theta_y \doteq 0}} \right] . \qquad [35]$$

Thus, the mean and the variance of S_n under signal conditions and assuming dependent samples are given by Eq. 13 and Eq. 35.

The mean and the variance of S_n just derived can be evaluated for the specific problem of detecting a constant signal in additive gaussian noise with zero mean, variance of one, and a correlation function $\rho(k\tau)$. Actually, the mean of S_n for this case has been evaluated in Eq. 16. Now consider the variance of S_n. The quantity $\frac{d}{d\theta} G_\theta(M) \Big|_{\theta \doteq 0}$ has been calculated in Eq. 15. The derivative of $G_{\theta_x \theta_y}(x, y)$ is found below.

$$\frac{d}{d\theta_x} G_{\theta_x \theta_y} (x,y) = \frac{d}{d\theta_x} F_{o,o} (x - \theta_x, y - \theta_y) = -f (x - \theta_x, y - \theta_y) \qquad [36]$$

where $F_{o,o}(x,y)$ is the joint noise distribution and $f(x,y)$ is the joint density function. For the case of gaussian noise $f(x,y)$ is given by

$$f (x - \theta_x, y - \theta_y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left\{- \frac{(x - \theta_x)^2 - 2\rho_{ij}(x-\theta_x)(y-\theta_y)+(y-\theta_y)^2}{2 (1 - \rho_{ij}^2)}\right\} \qquad [37]$$

where ρ_{ij} is the correlation between x and y. Thus Eq. 36 and Eq. 37 yield

$$\left.\frac{d}{d\theta_x} G_{\theta_x \theta_y} (M,M)\right|_{\substack{\theta_x \doteq 0 \\ \theta_y = 0}} = \frac{-1}{2\pi\sqrt{1 - \rho_{ij}^2}} ,$$

since x = y = 0 at the median. Also,

$$\left.\frac{d}{d\theta_y} G_{\theta_x \theta_y} (M,M)\right|_{\substack{\theta_x = \theta_i \\ \theta_y \doteq 0}} = \frac{-1}{2\pi\sqrt{1 - \rho_{ij}^2}} \exp\left\{\frac{-\theta_i^2}{2 (1 - \rho_{ij}^2)}\right\} .$$

Substituting these results into Eq. 35 gives

$$\sigma_\theta^2 [s_n] = \sigma_0^2 [s_n] + \frac{1}{\sqrt{2\pi}} [(1 - \frac{1}{n}) \bar{\theta} + \bar{\theta}^2] -$$

$$- \frac{\bar{\theta}}{2\pi n^2} \sum_{i}^{n} \sum_{\neq j}^{n} \frac{1}{\sqrt{1 - \rho_{ij}^2}} \left\{ 1 + \exp\left[\frac{-\bar{\theta}^2}{2 (1 - \rho_{ij}^2)}\right]\right\},$$

which can be written as

$$\sigma_\theta^2 [s_n] = \sigma_0^2 [s_n] + \frac{1}{\sqrt{2\pi}} [(1 - \frac{1}{n}) \bar{\theta} + \bar{\theta}^2] -$$

$$- \frac{\bar{\theta}}{\pi n} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \frac{1 + \exp\left[\frac{-\bar{\theta}^2}{2\left[1 - \rho^2 (k\tau)\right]}\right]}{\sqrt{1 - \rho^2 (k\tau)}} \quad . \qquad [38]$$

Equations 16 and 38 give the mean and the variance of S_n under signal conditions, assuming a constant signal in additive gaussian noise with correlation $\rho(k\tau)$.

The question might arise, either in the analysis or in the implementation of the median detector, as to which is simpler: a detector incorporating an 0 - 1 nonlinear device, such as has been assumed in this chapter, or one using a ± 1 nonlinear device. For the second example the test statistic would be given by

$$S_n' = \frac{1}{n} \sum_{i=1}^{n} \text{sgn}\,(Y_i - M) \qquad [39]$$

where

$$\text{sgn}\,(z_i) = 1 \qquad z_i > 0$$
$$= -1 \qquad z_i < 0.$$

It is shown in Appendix B that the two detectors have the same power under dependent samples, since this has not previously been shown. (Two detectors have the same power if their efficacies are equal.) It can also be seen from Appendix B that the analysis is not simplfied; hence, the median detector with the 0 - 1 nonlinear operation will continue to be used in this report.

The important results of this chapter can be summarized by the following equations: Eq. 9 gives the mean of S_n under no-signal conditions for either dependent or independent samples; Eq. 10 gives the variance of S_n under no-signal conditions and independent samples; Eq. 13 gives the mean of S_n under signal conditions for both dependent and independent samples; Eq. 14 gives the variance of S_n under signal conditions and independent samples; Eq. 16 gives the mean of S_n for a constant signal in gaussian noise for either independent or dependent samples; Eq. 25 gives the no-signal variance of S_n assuming dependent samples; Eq. 35 gives the same variance but under signal conditions; and Eq. 38 specializes this variance to the case of a constant signal in gaussian noise, correlated samples. The means and the variances of the median detector for dependent samples are new results, whereas the results for independent samples are well known (39).

CHAPTER V

STATISTICAL PROPERTIES OF THE NEYMAN-PEARSON DETECTOR

The optimum parametric detector for detecting the presence of a constant signal in additive gaussian noise is the Neyman-Pearson test based on the likelihood ratio. For a given false alarm probability and a fixed number of observations, the Neyman-Pearson detector minimizes the false dismissal probability. The form of the test statistic has been given in Chapter II. Since the Neyman-Pearson detector is parametric, its structure will change depending upon whether the input observations are correlated or independent. If they are correlated, the structure will depend upon the input correlation function. In this chapter the means and the variances of the test statistic will be derived under signal and no-signal conditions for both independent and dependent structures.

Independent Sample Structure

The structure of the Neyman-Pearson detector for detecting constant signals in white, additive gaussian noise was given in Chapter II as

$$L_n = \frac{1}{n} \sum_{i=1}^{n} Y_i \qquad [40]$$

where Y_i, $i = 1, \cdots, n$ are samples of the input random process Y(t). The input under no-signal conditions is

$$Y(t) = N(t) \qquad [41]$$

where N(t) is a gaussian random process with zero mean. Under signal conditions the input is

$$Y(t) = N(t) + \theta \qquad [42]$$

where θ represents the signal to be detected. Although L_n was derived assuming independent observations, the mean and the variance of L_n will be found assuming correlated observations. This would correspond to the case where the independent sample assumption had been made but which was not valid physically.

Consider the no-signal condition first. The mean of L_n is

$$E_0 [L_n] = \frac{1}{n} \sum_{i=1}^{n} E_0 [Y_i] = 0, \qquad [43]$$

assuming the input gaussian random process has mean zero. The variance of L_n is

$$\sigma_0^2 [L_n] = \sigma_0^2 \left[\frac{1}{n} \sum_{i=1}^{n} Y_i \right] = E_0 \left[\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Y_i \, Y_j \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} E_0 [Y_i^2] + \frac{1}{n^2} \sum_{i}^{n} \sum_{\neq j}^{n} E_0 [Y_i \, Y_j]$$

$$= \frac{\sigma_n^2}{n} + \frac{2}{n} \sum_{k=1}^{n} (1 - \frac{k}{n}) \rho (k\tau) \qquad [44]$$

where the input noise variance, σ_n^2, will be assumed equal to one, and the input correlation function as $\rho(k\tau)$.

The mean and the variance of L_n under signal conditions will now be established. The mean of L_n is

$$E_\theta [L_n] = \frac{1}{n} \sum_{i=1}^{n} E_\theta [Y_i] = \theta. \qquad [45]$$

The variance of L_n is

$$\sigma_\theta^2 [L_n] = \sigma_\theta^2 \left[\frac{1}{n} \sum_{i=1}^{n} Y_i \right] = E_\theta \left[\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Y_i Y_j \right] - E_\theta^2 [L_n].$$

But $Y_i = N_i + \theta$, so

$$\sigma_\theta^2 [L_n] = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} E_\theta [(\theta + N_i)(\theta + N_j)] - \theta^2$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} [\theta^2 + E [N_i N_j] + \theta (\bar{N}_i + \bar{N}_j)] - \theta^2 \qquad [46]$$

where $\bar{N}_i = \bar{N}_j = 0$, since the mean value of the noise process was assumed to be zero. Furthermore, $E[N_i N_j]$ is the correlation function of the input noise process. Equation 46 becomes

$$\sigma_\theta^2 [L_n] = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij}$$

which can now be written as

$$\sigma_{\theta}^{2}\,[L_n] = \frac{1}{n} + \frac{2}{n} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \rho\,(k\tau)\ . \qquad [47]$$

Referring to Eq. 44 it is seen that the variance of L_n under signal and no-signal conditions is the same.

Dependent Sample Structure

The optimum parametric detector of a constant signal in additive gaussian noise assuming correlated input observations is based on the test statistic

$$\emptyset_n = \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}\, Y_i\, \theta \qquad [48]$$

where Y_i and θ have the same meaning as in the first part of this chapter. The Q_{ij} are the elements of the inverse of the noise covariance matrix (correlation matrix),

$$\|Q_{ij}\| = \|R_{ij}\|^{-1} \qquad [49]$$

where $R_{ij} = R\,(|i - j|\tau)$. The following derivation of the means and the variances of $\emptyset_n$ are based on procedures stated by Zubakov (5, 16).

Consider first the case of noise only and an input noise correlation function $\rho(k\tau)$. The mean of $\emptyset_n$ under these conditions is

$$E_0\,[\emptyset_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}\, E_0\,[Y_i\theta] = 0\ , \qquad [50]$$

since the gaussian input was assumed to have a zero mean value.

The variance of ϕ_n under no-signal conditions is

$$\sigma_0^2 [\phi_n] = \sum_i \sum_j \sum_k^n \sum_\ell Q_{ij} Q_{k\ell} E_0 [Y_i Y_j] \theta^2 . \qquad [51]$$

Assuming the signal and the noise are uncorrelated, Eq. 51 becomes

$$\sigma_0^2 [\phi_n] = \theta^2 \sum_i \sum_j^n \sum_k \sum_\ell Q_{ij} Q_{k\ell} R_{jk} . \qquad [52]$$

The elements of the inverse matrix $||Q_{ij}||$ satisfy the relation

$$\sum_{k=1}^{n} R_{jk} Q_{k\ell} = \delta_{j\ell} \qquad [53]$$

where $\delta_{j\ell}$ is the Kronecker delta,

$$\delta_{j\ell} = 1 \qquad \text{for } j = \ell$$

$$= 0 \qquad \text{for } j \neq \ell .$$

Using Eq. 53, Eq. 52 reduces to

$$\sigma_0^2 [\phi_n] = \theta^2 \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} . \qquad [54]$$

Finally, find the mean and the variance of ϕ_n under signal conditions. The mean of ϕ_n under signal conditions is

$$E_\theta [\phi_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} E_\theta [Y_i \theta]$$

$$= \theta^2 \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} \ . \qquad [55]$$

The variance of $\emptyset_n$ under signal conditions is

$$\sigma_\theta^{\ 2} [\emptyset_n] = \sum_i \sum_j \sum_k^n \sum_\ell Q_{ij} \ Q_{k\ell} E_\theta \ [Y_i \ Y_j] \ \theta^2 - E_\theta^{\ 2} \ [\emptyset_n]$$

$$= \sum_i \sum_j \sum_k^n \sum_\ell \ Q_{ij} \ Q_{k\ell} \ \overline{(\theta + N_i) \ (\theta + N_k)} \ \ \theta^2 - \left[\theta^2 \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}\right]^2$$

$$= \theta^4 \sum_i \sum_j^n \sum_k \sum_\ell Q_{ij} \ Q_{k\ell} + \theta^2 \ \sum_i \sum_j^n \sum_k \sum_\ell Q_{ij} \ Q_{k\ell} \ R_{ik} -$$

$$- \ \theta^4 \sum_i \sum_j \sum_k^n \sum_\ell Q_{ij} \ Q_{k\ell} \ .$$

Applying Eq. 53 gives

$$\sigma_\theta^{\ 2} [\emptyset_n] = \theta^2 \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} \quad , \qquad [56]$$

which is the same as the variance of $\emptyset_n$ under no-signal conditions given in Eq. 54.

The important results of this chapter can be summarized by the following equations: Eq. 43 gives the mean of L_n under no-signal conditions; Eq. 45 gives the mean of L_n under signal conditions; Eq. 44 states the variance of L_n under no-signal conditions; Eq. 47 gives the variance of L_n under signal conditions; Eq. 50 states the mean of

ϕ_n under no-signal conditions; Eq. 55 gives the mean of ϕ_n under signal conditions; Eq. 51 gives the variance of ϕ_n under no-signal conditions; and Eq. 56 gives the variance of ϕ_n under signal conditions. Note that L_n refers to the test statistic designed for independent samples, and ϕ_n refers to the test statistic designed assuming correlated samples. The test statistic designed assuming independent samples has not previously been studied assuming correlated inputs.

CHAPTER VI

PERFORMANCE OF DETECTORS UNDER DEPENDENT SAMPLING

The necessary means and variances have been found in Chapters IV and V to apply the ARE criterion and to calculate the error probabilities. Before applying the ARE criterion, however, it must be shown that the regularity conditions given in Chapter III are satisfied. This has not been investigated previously. It will be demonstrated in this chapter that the regularity conditions are satisfied by the detectors under consideration. Following this, equations for α, β, and the efficacy of each detector will be stated. The input correlation functions to be used as examples in the computed results will also be discussed. This chapter will conclude with a description of the computations of α, β, and the efficacy for each detector. The effect of correlated samples on the operation of the detectors will be illustrated by these results.

Verification of the Regularity Conditions

It will be shown that the median test statistic satisfies the regularity conditions for the detection of a constant signal in correlated gaussian noise. It will then be pointed out that the Neyman-Pearson detector (both dependent and independent structures) for the same detection problem satisfies the regularity conditions in a similar manner.

The first regularity condition stated in Chapter III requires that the test statistic be asymptotically gaussian under signal and no-signal conditions. To show that this condition is satisfied, a central limit theorem for sums of dependent random variables must be invoked. Rosenblatt's central limit theorem for dependent processes will be used (57, 58). One requirement of the theorem is that the strong mixing condition hold. The strong mixing condition, introduced by Rosenblatt (57), will be discussed first.

An example of the strong mixing condition is as follows (58): Let A, B be any events determined by conditions on the random variables x_k, $k \leq m$, and x_k, $k \geq n$, respectively, with $n > m$. The process $\{x_k\}$ is said to satisfy the strong mixing condition if

$$\left| P\left\{A \cap B\right\} - P\left\{A\right\} P\left\{B\right\}\right| \leq d(n - m) \qquad [57]$$

for all such events A, B and some function d where d is a function on the positive integers n that decreases to zero as $n \to \infty$. The condition basically requires that two events become independent of each other as the "distance" between them approaches infinity.

Rozanov (59) has stated the strong mixing condition more rigorously: Two σ-algebras $\mathfrak{M}'$ and $\mathfrak{M}''$ of the events A' and A'', respectively, are independent, if for any $A' \in \mathfrak{M}'$ and $A'' \in \mathfrak{M}''$

$$P(A' \cap A'') = P(A')\, P(A'').$$

Define a measure of dependence of the two σ-algebras of events as

$$\alpha(\mathfrak{M}', \mathfrak{M}'') = \sup_{A' \in \mathfrak{M}',\ A'' \in \mathfrak{M}''} [P(A' \cap A'') - P(A')P(A'')] . \qquad [58]$$

Then for a stationary random process $\xi(t)$, $\alpha(\mathfrak{M}_{-\infty}^{t}, \mathfrak{M}_{t+\tau}^{\infty})$ depends only on τ and is designated by $\alpha(\tau)$, where $\mathfrak{M}_{s}^{t}$ denotes the σ-algebra of events which is determined by the quantities $\xi(u)$, $s \le u \le t$. If $\alpha(\tau) \to 0$ for $\tau \to \infty$, then the process $\xi(t)$ is said to possess the property of strong mixing.

Rosenblatt's central limit theorem may be stated as follows: Let $\{x_k\}$ be a stationary process with mean zero, $E[x_k] = 0$, that satisfies the strong mixing condition. Further, let the process satisfy the two moment conditions,

$$E\left[\sum_{i=1}^{n} x_i\right]^2 = h(n) \to \infty \text{ as } n \to \infty , \qquad [59]$$

and

$$E\left[\sum_{i=1}^{n} x_i\right]^4 = \sum_{i}\sum_{j}^{n}\sum_{k}\sum_{\ell} E[x_i x_j x_k x_\ell] = O[h(n)]^2 . \qquad [60]$$

The sums Σx_i are then asymptotically and nontrivially normally distributed when suitably normed as $n \to \infty$. The verification that these three conditions are satisfied for the problem under consideration is shown in Appendix C, subject to the conditions that the input correlation function $\rho(\tau)$ goes to zero as $\tau \to \infty$, and that the input be stationary and gaussian.

Regularity condition B requires that the derivative of the mean of the test statistic exist for all θ in a given interval and be continuous at $\theta = 0$. Referring to $E_\theta [S_n]$ found in Chapter IV, this condition is clearly satisfied.

The third condition requires that the efficacy be positive and finite. The efficacy is given by

$$E_{S_n} = \lim_{n \to \infty} \frac{1}{n} \left[\frac{E'_\theta [S_n]}{\sigma_o [S_n]} \right]^2_{\theta \doteq 0} \quad . \qquad [61]$$

Substitute Eq. 16 and Eq. 25 into Eq. 61 to obtain

$$E_{S_n} = \lim_{n \to \infty} \frac{1}{n} \left[\frac{\frac{1}{2\pi}}{\frac{1}{4n} + \frac{1}{\pi n} \sum_{k=1}^{n} (1 - \frac{k}{n}) \sin^{-1} \rho (k\tau)} \right]$$

$$E_{S_n} = \frac{1}{\frac{\pi}{2} + 2 \sum_{k=1}^{\infty} \sin^{-1} \rho(k\tau)} \quad . \qquad [62]$$

Thus E_{S_n} will be finite and non-zero as long as

$$\sum_{k=1}^{n} \sin^{-1} \rho(k\tau)$$

is a convergent series. Recall that the central limit theorem required that $\rho(\tau) \to 0$ as $\tau \to \infty$; thus, the series will converge under normal circumstances.

For condition D it will be assumed that there exists a sequence θ_n such that

$$\lim_{n \to \infty} \theta_n = 0 \; .$$

Condition E requires that

$$\lim_{n \to \infty} \frac{\sigma_\theta^2 [S_n]}{\sigma_0^2 [S_n]} = 1 \; . \qquad [63]$$

The variances of S_n for the detection of a constant signal in additive gaussian noise have been found in Chapter IV, Eqs. 25 and 38. Substituting Eq. 25 and Eq. 38 into Eq. 63 yields

$$\lim_{n \to \infty} \frac{\sigma_\theta^2 [S_n]}{\sigma_0^2 [S_n]} = 1 +$$

$$+ \lim_{n \to \infty} \frac{\frac{1}{\sqrt{2\pi}} \left[(1 - \frac{1}{n}) \bar{\theta} + \bar{\theta}^2 \right] - \frac{\bar{\theta}}{\pi n} \sum_{k=1}^{n} (1 - \frac{k}{n}) \frac{1 + \exp\left[\frac{-\bar{\theta}^2}{2(1-\rho^2(k\tau))} \right]}{\sqrt{1 - \rho^2 (k\tau)}}}{\frac{1}{4n} + \frac{1}{\pi n} \sum_{k=1}^{n} (1 - \frac{k}{n}) \sin^{-1} \rho(k\tau)} \; . \qquad [64]$$

Retaining only the significant terms in the limiting process gives

$$\lim_{n \to \infty} \frac{\sigma_\theta^2 [S_n]}{\sigma_0^2 [S_n]} = 1 + \lim_{n \to \infty} \frac{\bar{\theta} \left[\frac{1}{\sqrt{2\pi}} - \frac{2}{\pi} \right]}{\frac{1}{4n} + \frac{1}{\pi n} \sum_{k=1}^{n} \sin^{-1} \rho(k\tau)} \; . \qquad [65]$$

But $\bar{\theta} = \frac{1}{n} \Sigma \theta_i$; also, let $C = \frac{1}{\sqrt{2\pi}} - \frac{2}{\pi}$. Then

$$\lim_{n \to \infty} \frac{\sigma_\theta^2 [S_n]}{\sigma_0^2 [S_n]} = 1 + \lim_{n \to \infty} \frac{\frac{C}{n} \sum_{i=1}^{n} \theta_i}{\frac{1}{4n} + \frac{1}{\pi n} \sum_{k=1}^{n} \sin^{-1} \rho(k\tau)}$$

$$= 1 + \frac{C \sum_{1}^{n} \theta_i}{\frac{1}{4} + \frac{1}{\pi} \sum_{1}^{\infty} \sin^{-1} \rho\,(k\tau)} \qquad [66]$$

The second term of Eq. 66 will be approximately zero if θ_i goes to zero much faster than $\rho(k\tau)$, assuming the small signal case and applying condition D. This will be assumed to be possible, thus approximately satisfying condition E.

Condition F requires that

$$\lim_{n \to \infty} \sigma_0^{\,2}\,[S_n] = 0\ . \qquad [67]$$

From Eq. 25 it is possible to write

$$\lim_{n \to \infty} \sigma_0^{\,2}\,[S_n] = \lim_{n \to \infty} \left\{ \frac{1}{4n} + \frac{1}{n\pi} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \sin^{-1} \rho\,(k\tau) \right\}$$

$$= \frac{1}{\pi} \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sin^{-1} \rho\,(k\tau)\ . \qquad [68]$$

In Appendix D it is shown that if $\lim_{n \to \infty} a_n = 0$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0\ .$$

Using this result, Eq. 68 becomes

$$\lim_{n \to \infty} \sigma_o^{\,2}\,[S_n] = 0,$$

and condition F is satisfied.

This completes the list of regularity conditions for the median detector. It follows from Appendix C that condition A also holds true for the dependent and the independent structure Neyman-Pearson detectors. Conditions B and D follow in exactly the same manner as for the median detector. From the results of Chapter V, it is obvious that condition E is satisfied. For the same constraints on the correlation function that have been imposed above, conditions C and F are satisfied.

To summarize, the ARE criterion may be applied to the three detectors under consideration subject only to the following constraints: 1) the input process must be a stationary gaussian process with zero mean; 2) the correlation function $\rho(\tau)$ must go to zero as $\tau \to \infty$; 3) $\sum_{1}^{\infty} \sin^{-1} \rho(k\tau)$ and $\sum_{1}^{\infty} \rho(k\tau)$ must be convergent series; and 4) θ must be small and go to zero much faster than the input correlation function, $\rho(k\tau)$.

Efficacy and Error Probability Equations

The general equation for the efficacy has been given in Chapter III as

$$E_{T_n} = \lim_{n \to \infty} \frac{1}{n} \left[\frac{E'_\theta [T_n]}{\sigma_0 [T_n]} \right]^2_{\theta \doteq 0} . \qquad [69]$$

The equation for the false alarm probability, α, for the detection of a constant signal in additive gaussian noise with a test statistic T_n is (25)

$$\alpha = \frac{1}{\sqrt{2\pi}\,\sigma_o\,[T_n]} \int_{T_\alpha}^{\infty} \exp\left\{ - \frac{\left(x - E_o\,[T_n]\right)^2}{2\sigma_o^{\,2}\,[T_n]} \right\} dx$$

or, putting the integral in standard form,

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{\frac{T_\alpha - E_o\,[T_n]}{\sigma_o\,[T_n]}}^{\infty} e^{-\frac{1}{2}y^2}\, dy \,. \qquad [70]$$

The corresponding equation for the false dismissal probability, β, is

$$\beta = \frac{1}{\sqrt{2\pi}\,\sigma_\theta\,[T_n]} \int_{-\infty}^{T_\alpha} \exp\left\{ - \frac{(x - E_\theta\,[T_n])^2}{2\sigma_\theta^{\,2}\,[T_n]} \right\} dx$$

or, in standard form,

$$\beta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{T_\alpha - E_\theta\,[T_n]}{\sigma_\theta\,[T_n]}} e^{-\frac{1}{2}y^2}\, dy \qquad [71]$$

or

$$\beta = \frac{1}{2} + \frac{1}{2}\left[\int_{-\infty}^{\frac{T_\alpha - E_\theta\,[T_n]}{\sqrt{2}\,\sigma_\theta\,[T_n]}} e^{-y^2}\, dy \right] . \qquad [72]$$

For the median detector and the problem of detecting a constant signal in additive gaussian noise, these equations become

$$E_{S_n} = \frac{1}{\frac{\pi}{2} + 2 \sum_{k=1}^{\infty} \sin^{-1} \rho\,(k\tau)} \qquad [73]$$

from Eq. 62. To find α (or S_α) and β, substitute Eqs. 9, 16, 25, and 38 for $E_0[S_n]$, $E_\theta[S_n]$, $\sigma_0^2[S_n]$, and $\sigma_\theta^2[S_n]$, respectively, into Eq. 70 and Eq. 72. (These substitutions will not be performed here.)

For the Neyman-Pearson detector (independent sample structure) and the same detection problem, the efficacy is

$$E_{L_n} = \lim_{n \to \infty} \frac{1}{n} \left[\frac{1}{\frac{1}{n} + \frac{2}{n} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \rho(k\tau)} \right]$$

or

$$E_{L_n} = \frac{1}{1 + 2 \sum_{1}^{\infty} \rho(k\tau)} . \qquad [74]$$

To find α (or L_α) and β, substitute Eqs. 43, 44, 45, and 47 for $E_0[L_n]$, $E_\theta[L_n]$, $\sigma_0^2[L_n]$, and $\sigma_\theta^2[L_n]$, respectively, into Eq. 70 and Eq. 72.

For the Neyman-Pearson detector (dependent sample structure) and the same detection problem, the efficacy is given by

$$E_{\phi_n} = \lim_{n \to \infty} \frac{1}{n} \frac{\left(E_\theta'[\phi_n] \Big|_{\theta \doteq 0} \right)^2}{\sigma_0^2[\phi_n]} . \qquad [75]$$

The mean value theorem states that

$$E_\theta'[\phi_n] \Big|_{\theta = \hat{\theta}} = \frac{E_\theta[\phi_n] - E_0[\phi_n]}{\theta} \qquad [76]$$

where $0<\hat{\theta}<\theta$. Using regularity condition D that $E_\theta'[\phi_n]$ is continuous at $\theta = 0$ and assuming the small signal case, Eq. 76 becomes

$$E_\theta'[\phi_n]\Big|_{\theta \doteq 0} = \frac{E_\theta[\phi_n] - E_0[\phi_n]}{\theta} . \qquad [77]$$

Substitute Eq. 77 into Eq. 75, along with Eq. 54, to obtain

$$E_{\phi_n} = \lim_{n \to \infty} \frac{1}{n} \frac{\left[\dfrac{E_\theta[\phi_n] - E_0[\phi_n]}{\theta}\right]^2}{\theta^2 \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}}$$

or, using Eq. 50 and Eq. 55,

$$E_{\phi_n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} . \qquad [78]$$

As before, α (or ϕ_α) and β may be obtained by substituting Eqs. 50, 55, 51, and 56 for $E_0[\phi_n]$, $E_\theta[\phi_n]$, $\sigma_0^2[\phi_n]$, and $\sigma_\theta^2[\phi_n]$, respectively, into Eq. 70 and Eq. 72.

Examples of Typical Correlation Functions

Four examples of typical correlation functions encountered in practical situations have been used in the computations to compare the detector performances. These will be discussed in the following paragraphs.

The most common and the simplest correlation function is the decaying exponential. Correlation of this kind is obtained by passing white noise through a simple RC low pass filter with a cutoff frequency ω_c. The resulting normalized autocorrelation function is (2, 31)

$$\rho(\tau) = e^{-\omega_c |\tau|} \quad . \qquad [79]$$

For a sampled random process with sampling frequency ω_s, $\rho(k\tau_s)$ is

$$\rho(k\tau_s) = e^{-\frac{\omega_c}{\omega_s} 2\pi k} \qquad [80]$$

where ω_s/ω_c is the relative sampling rate.

The damped exponential-cosine autocorrelation functions are often used to describe many random noise phenomena of interest. Yaglom (60) states that this type can be used to describe certain types of fading of radio signals. An example of this based on experimental data is given by James, Nichols, and Phillips (61). If white noise is passed through a single-tuned filter with a power spectrum

$$G(\omega) = \frac{\Delta\omega}{\pi} \left[\frac{1}{\Delta\omega^2 + (\omega_o - \omega)^2} + \frac{1}{\Delta\omega^2 + (\omega_o + \omega)^2} \right] ,$$

the normalized autocorrelation function of the filtered noise is (31)

$$\rho(\tau) = e^{-\Delta\omega |\tau|} \cos \omega_o \tau \quad . \qquad [81]$$

Here ω_o is the center frequency and $\Delta\omega$ is the half-bandwidth of the filter, measured in radians per second. For a discrete random process, $\rho(k\tau_s)$ is

$$\rho(k\tau_s) = e^{-\frac{\Delta\omega}{\omega_s} 2\pi k} \cos 2\pi k \frac{\omega_o}{\omega_s} \qquad [82]$$

where ω_s/ω_o will be designated as the relative sampling frequency.

A third example is the sin x/x correlation function. This type of correlation can be obtained by passing white noise through a rectangular bandpass filter with a power spectrum

$$G(\omega) = \frac{1}{2\Delta\omega} \begin{cases} \omega_o - \Delta\omega \leq \omega \leq \omega_o + \Delta\omega \\ \end{cases}$$
$$= 0 \quad \text{elsewhere.}$$

The resulting autocorrelation function is (2)

$$\rho(\tau) = \frac{\sin \Delta\omega\tau}{\Delta\omega\tau} \cos \omega_o \tau. \qquad [83]$$

For a sampled random process with sampling frequency ω_s, Eq. 83 becomes

$$\rho(k\tau_s) = \frac{\sin 2\pi k \frac{\Delta\omega}{\omega_s}}{2\pi k \frac{\Delta\omega}{\omega_s}} \cos 2\pi k \frac{\omega_o}{\omega_s} \qquad [84]$$

where ω_s/ω_o is again the relative sampling frequency.

A fourth type of correlation function has the gaussian form

$$\rho(\tau) = e^{-\left(\frac{\Delta\omega\tau}{2}\right)^2} \cos \omega_o \tau \qquad [85]$$

or

$$\rho(k\tau_s) = e^{-\left(\pi k \frac{\Delta\omega}{\omega_s}\right)^2} \cos 2\pi k\, \omega_o/\omega_s. \qquad [86]$$

These four correlation functions will be used as examples of correlation between input observations when obtaining numerical results in the remainder of this chapter.

Numerical Results

This section will consist of computational results obtained by using the equations derived in the previous chapters. The results of this section show graphically the effect of correlated samples on the performance of the median detector. The performance of the median detector under dependent samples is compared with that under independent samples and with the likelihood detector operating under correlated samples.

The curves to be presented can be divided into three categories: 1) examples of typical correlation functions to be used in the numerical results; 2) curves showing the effect of various types and degrees of correlation of the input observations on the ARE's of the three detectors under consideration; and 3) the receiver operating characteristics showing the effect of correlated samples on the error probabilities of the median detector as compared with that of the likelihood detector (both independent and dependent structures.)

Examples of the correlation functions used in the computations are shown in Fig. 6 through Fig. 11. These are given to point out the extent of correlation that exists between input observations for the range of parameters considered. In Fig. 6 the decaying exponential correlation function (Eq. 80) is shown for values of the parameter ω_s/ω_o (relative sampling frequency) from five to fifty. For $\omega_s/\omega_o = 5$ the correlation only extends over about three samples, whereas for $\omega_s/\omega_o = 50$, $\rho\,(k\tau_s)$ is greater than 0.01 up to k = 38.

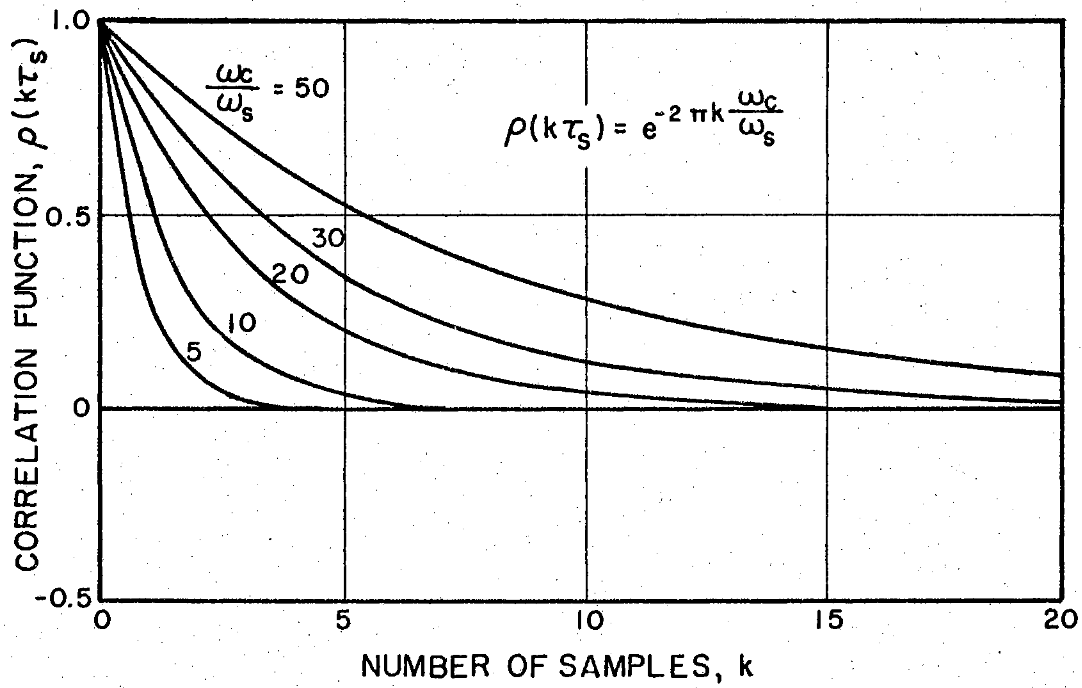

Fig. 6. Decaying Exponential Correlation Function.

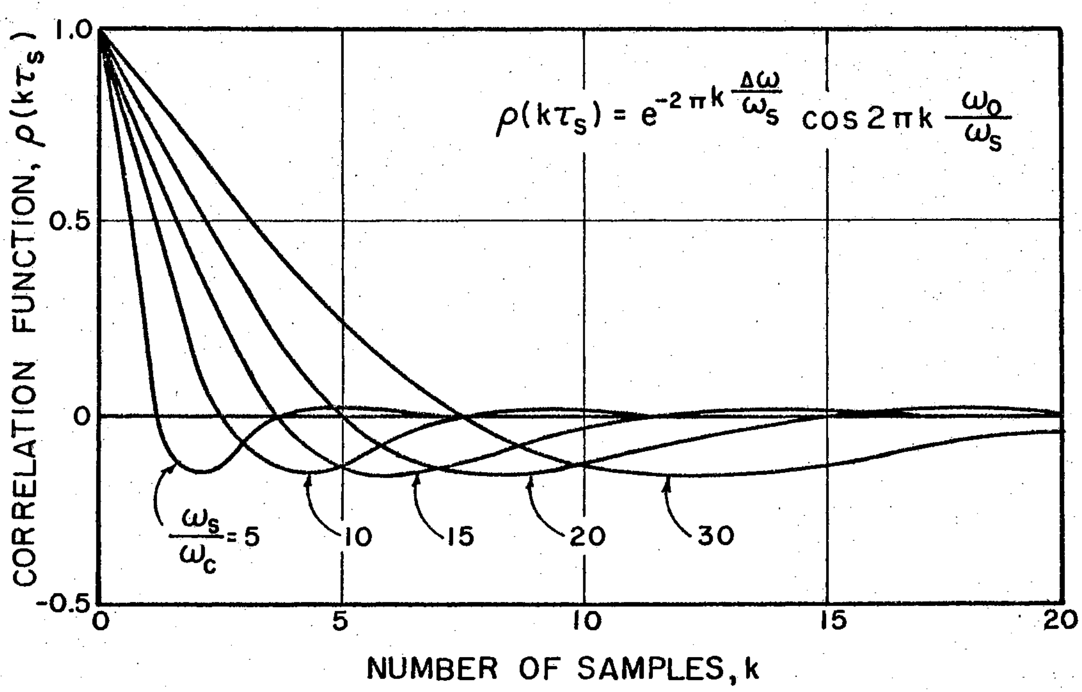

Fig. 7. Damped Cosine Correlation Function, ($\omega_o/\Delta\omega = 1.5$).

Figure 7 shows the damped cosine correlation function (Eq. 82). This correlation function, as well as the next two correlation functions to be considered, is described by two parameters, ω_s/ω_o and $\omega_o/\Delta\omega$. The range of values for these two parameters has been chosen based on practical considerations. The parameter $\omega_o/\Delta\omega$ will be varied from one to five and ω_s/ω_o will be varied from one to a hundred.

The sin x/x correlation function (Eq. 84) is given in Fig. 8 and Fig. 9. This function decays to zero the slowest of the four correlation functions discussed here. Furthermore, it is considerably more difficult to apply over a range of parameter values. This will be discussed in detail later. Figures 10 and 11 display the gaussian correlation function (Eq. 86) for two different values of the parameter $\omega_o/\Delta\omega$.

It is seen that in each case the extent of the correlation increases as either ω_s/ω_o or $\omega_o/\Delta\omega$ is increased. Also, the curves given in Fig. 6 through Fig. 11 can be extrapolated to show that for $\omega_s/\omega_o = \omega_o/\Delta\omega = 1$ the samples are, practically speaking, independent of each other. The values of $\omega_s/\omega_o = \omega_o/\Delta\omega = 1$ will therefore be considered to represent the case of independent observations in the comparisons to be made later.

The ARE's for the median and the likelihood (independent and dependent structure) detectors are given in Fig. 12, as the correlation between observations is varied. The decaying exponential correlation function is used. The ARE_{S_n/L_n} compares the median test statistic

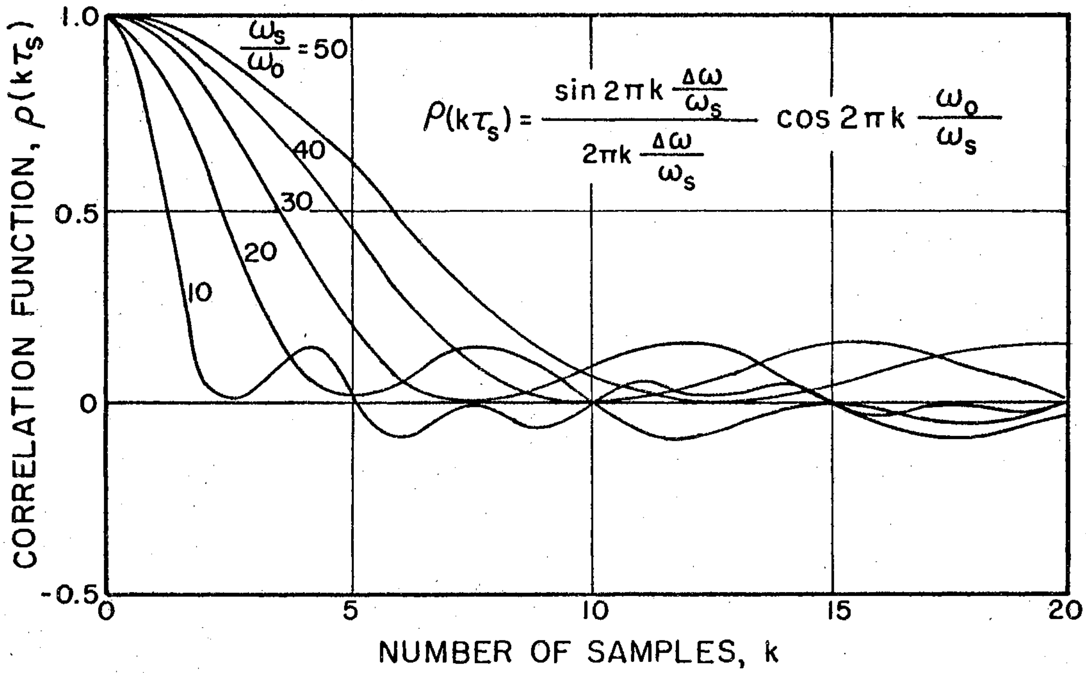

Fig. 8. sin x/x Correlation Function, ($\omega_o/\Delta\omega = 0.5$).

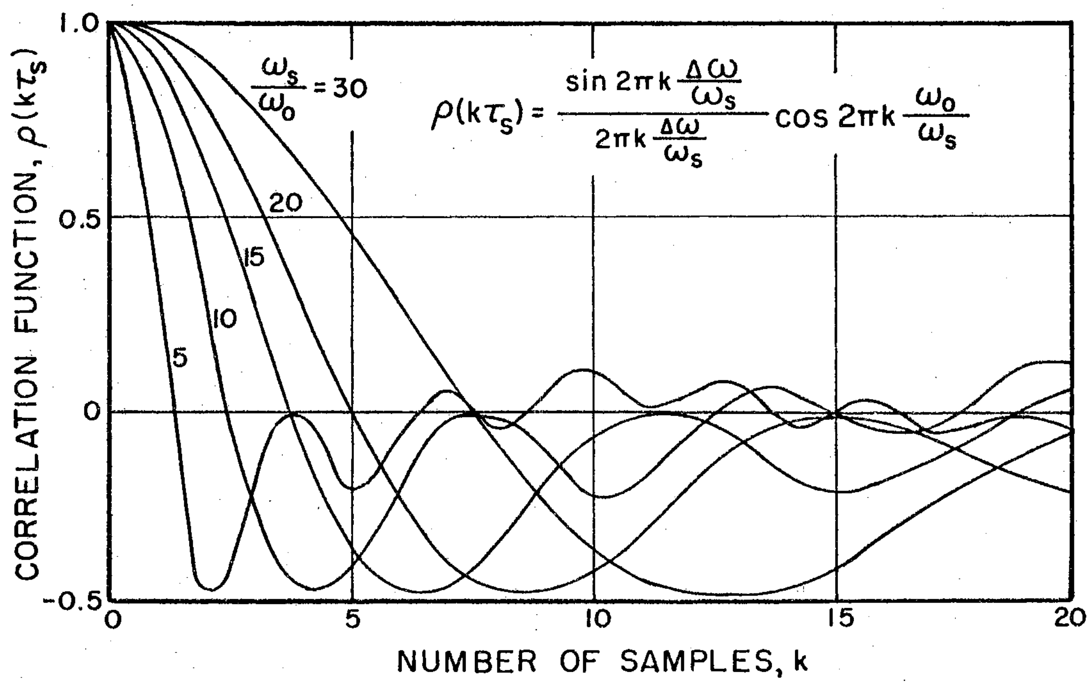

Fig. 9. sin x/x Correlation Function, ($\omega_o/\Delta\omega = 1.5$).

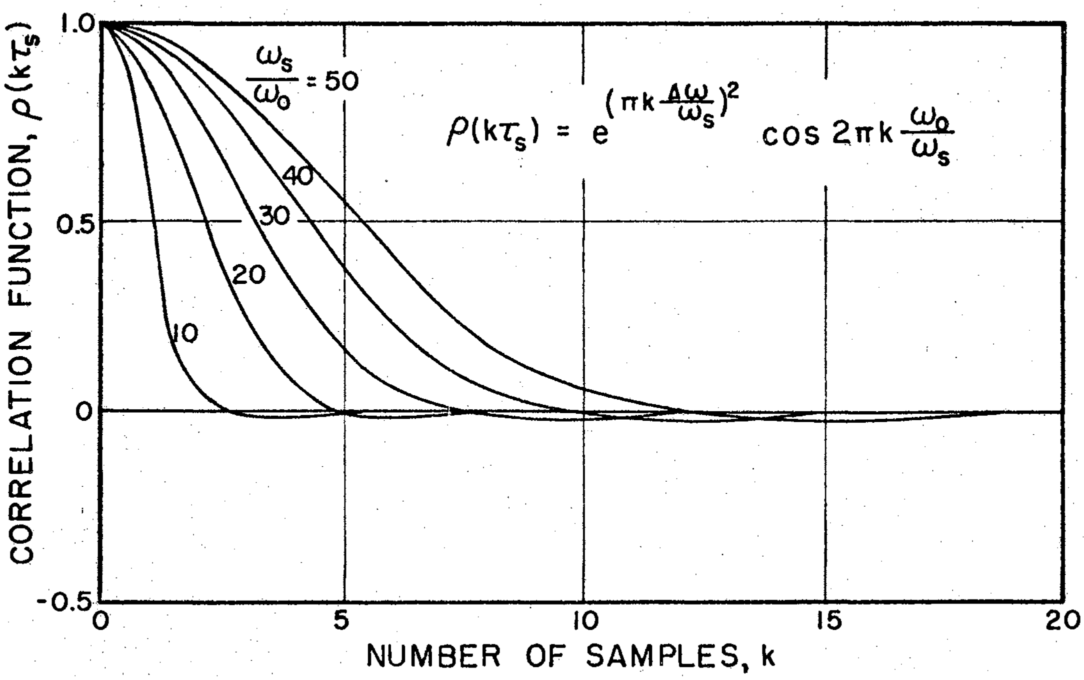

Fig. 10. Gaussian Correlation Function, $(\omega_o/\Delta\omega = 0.5)$.

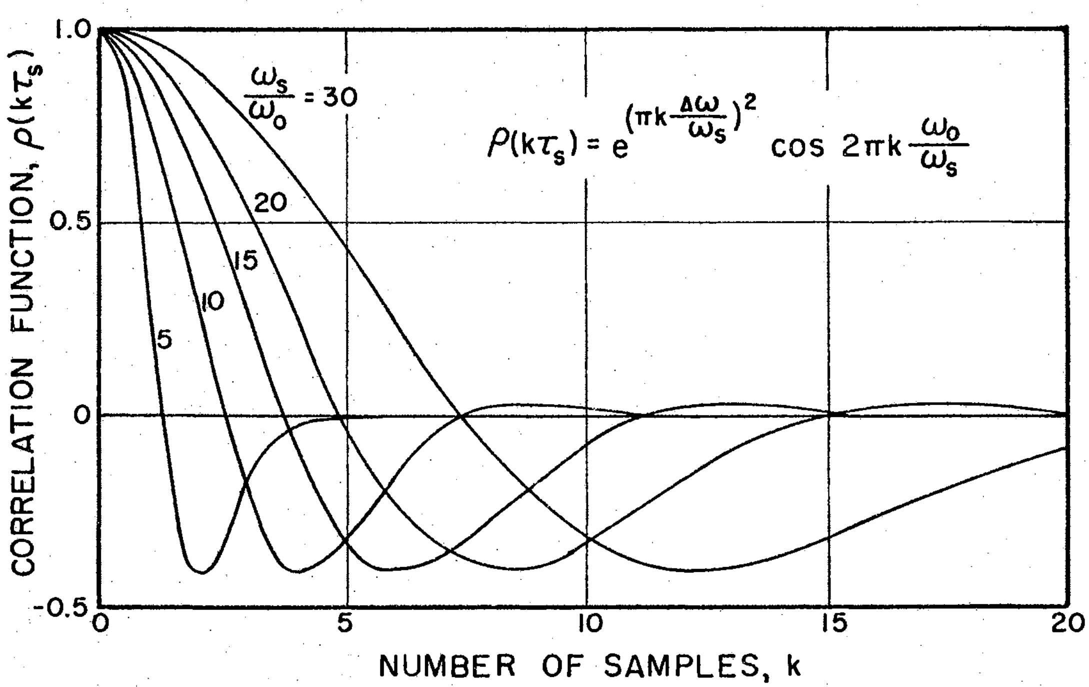

Fig. 11. Gaussian Correlation Function, $(\omega_o/\Delta\omega = 1.5)$.

S_n to the independent structure Neyman-Pearson test statistic L_n. The ARE_{S_n/ϕ_n} compares the median test statistic S_n to the dependent structure Neyman-Pearson test statistic ϕ_n. Finally, the ARE_{L_n/ϕ_n} compares the two Neyman-Pearson statistics. The efficacies of S_n and L_n are also shown. (The efficacy of ϕ_n lies almost on top of that for L_n; hence, it is not shown.)

It will be recalled that the ARE of the median test statistic and the Neyman-Pearson test statistic for the detection of a constant signal in additive gaussian noise, assuming independent observations, was $\pi/2$ or approximately 0.64, which compares quite closely with the value of ARE_{S_n/L_n} for $\omega_s/\omega_o = 1$ in Fig. 12. As the correlation increases to $\omega_s/\omega_o = 100$, the ARE_{S_n/L_n} increases to 0.91. This illustrates the important result that the median detector efficiency improves relative to that of the Neyman-Pearson detector as the correlation between input observations increases. The ARE_{S_n/ϕ_n} increases to 0.78 for $\omega_s/\omega_o = 100$. The ARE_{L_n/ϕ_n} decreases from the expected value of 1.0 at $\omega_s/\omega_o = 1$ to 0.91 at $\omega_s/\omega_o = 100$.

Curves illustrating similar results but based on the damped cosine correlation function are presented in Fig. 13 and Fig. 14. Figure 13 gives the ARE_{S_n/L_n} for several values of $\omega_o/\Delta\omega$. Figure 14 gives the ARE_{S_n/ϕ_n} and the ARE_{L_n/ϕ_n} for two values of $\omega_o/\Delta\omega$. Note the decrease in the ARE_{S_n/L_n} and the ARE_{S_n/ϕ_n} for larger values of

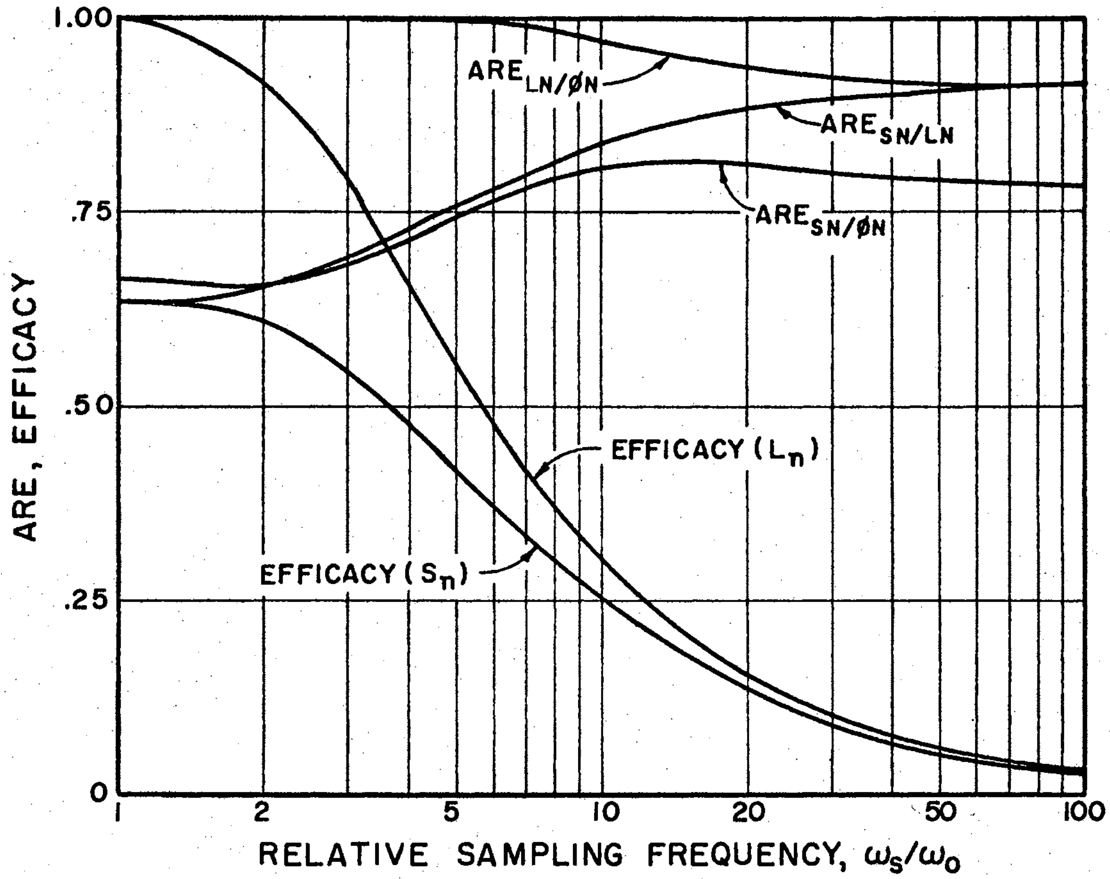

Fig. 12. Efficacy and ARE of the Median and the Likelihood Detectors (Input Correlation Function: Decaying Exponential).

$\omega_o/\Delta\omega$ when ω_s/ω_o is in the range from one to five. Figure 16 shows that the same general shape of the curves holds for the gaussian correlation function. The statistic $\emptyset_n$ is not considered due to the computation time required.

The ARE_{S_n/L_n} based on the sin x/x correlation function is illustrated in Fig. 15. The curve of the ARE_{S_n/L_n} has a discontinuity

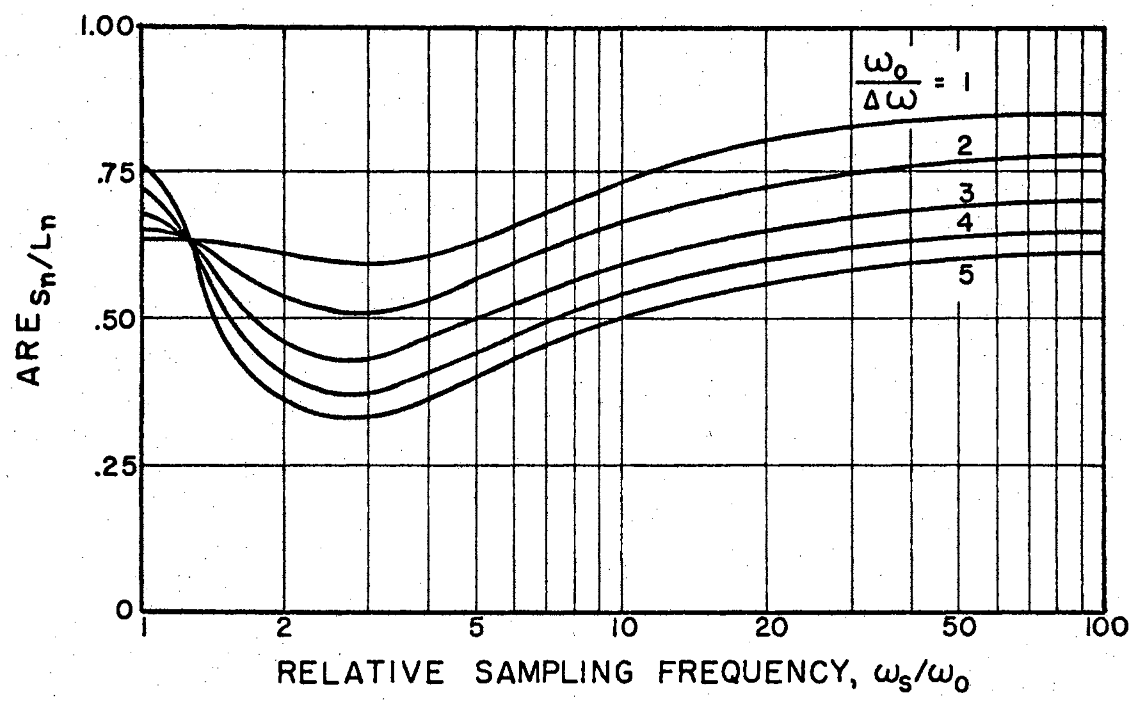

Fig. 13. ARE of the Median and the Likelihood -- Independent Structure Detectors (Input Correlation Function: Damped Cosine).

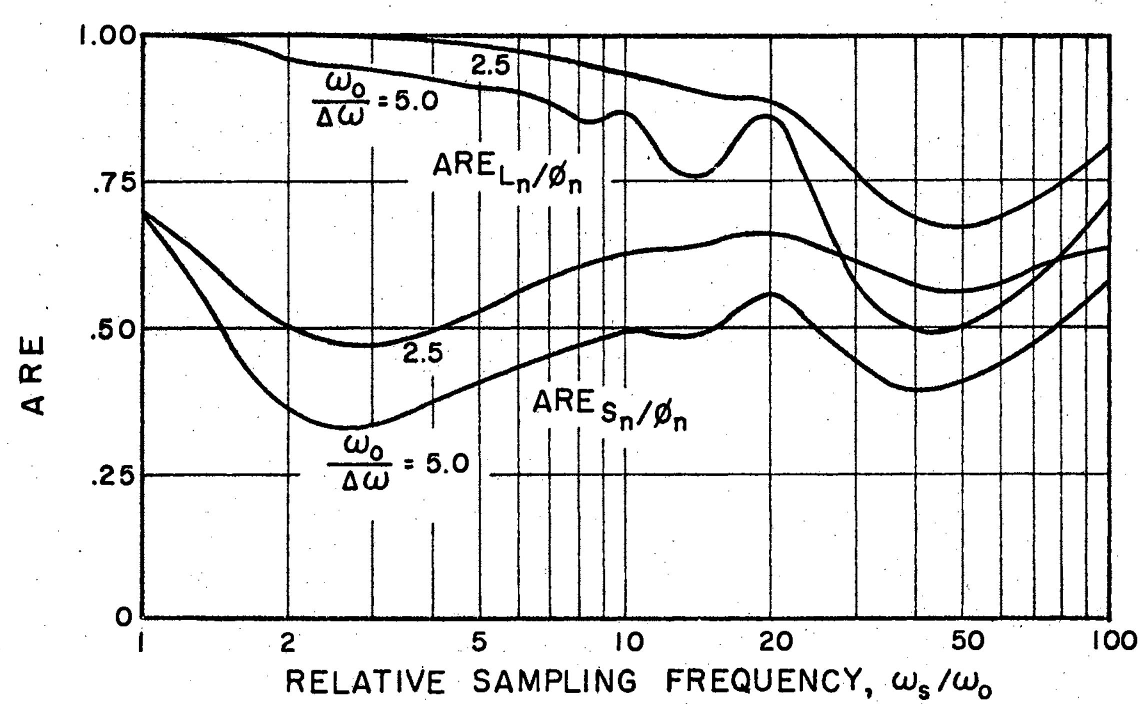

Fig. 14. ARE of the Median and the Likelihood -- Dependent Structure Detectors (Input Correlation Function: Damped Cosine).

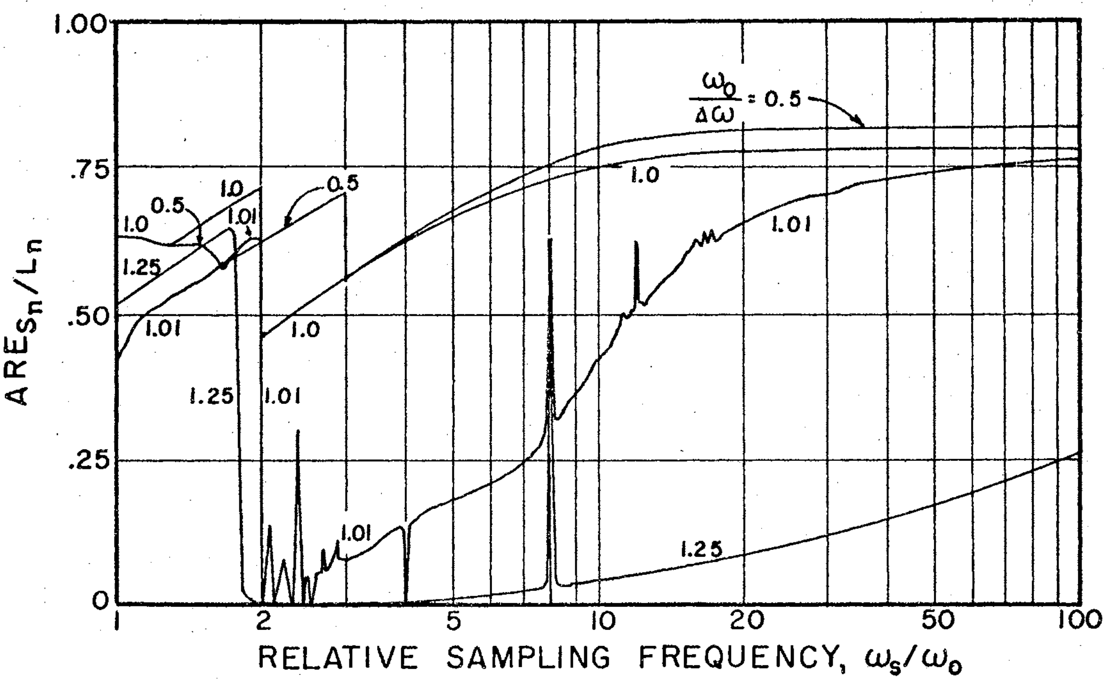

Fig. 15. ARE of the Median and the Likelihood -- Independent Structure Detectors (Input Correlation Function: sin x/x).

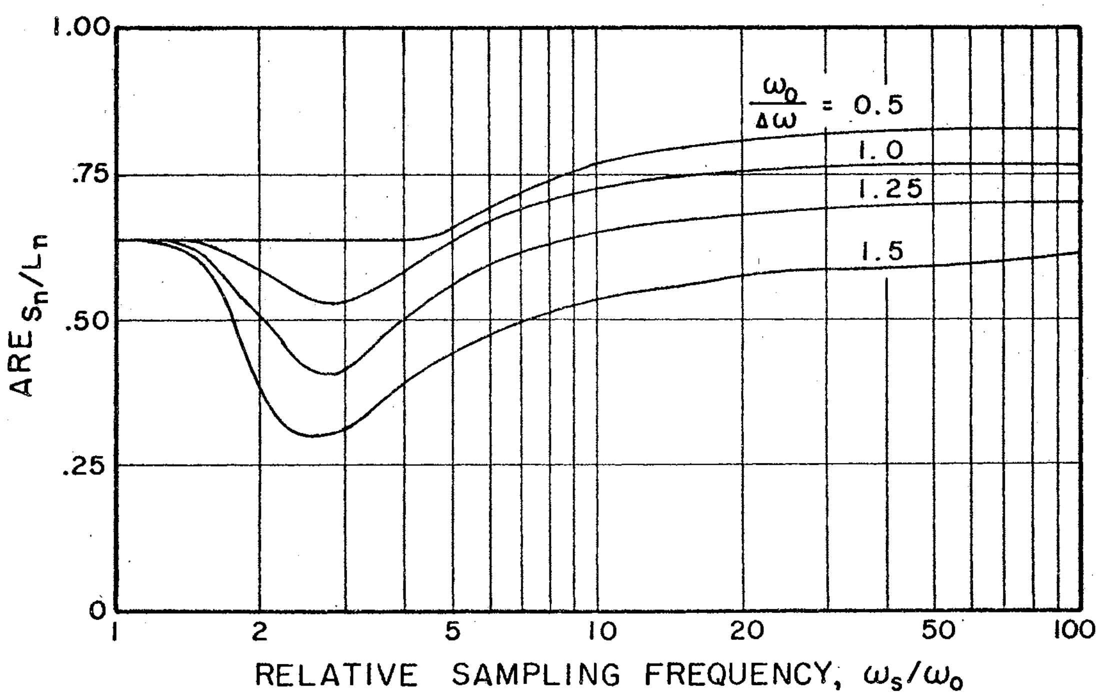

Fig. 16. ARE of the Median and the Likelihood -- Independent Structure (Input Correlation Function: Gaussian).

when the arguments of the sine and cosine of Eq. 84 are $k\pi$. For $\omega_o/\Delta\omega = 0.5$, this occurs at $\omega_s/\omega_o = 3.0$; for $\omega_o/\Delta\omega = 1.0$, it occurs at $\omega_s/\omega_o = 2.0$. Also, when the arguments are varied slightly from these values the resulting curves are very unstable and possess frequent spike discontinuities, as shown for $\omega_o/\Delta\omega = 1.01$. The ARE_{S_n/L_n} changes drastically as $\omega_o/\Delta\omega$ is increased from 1.0 to 1.25. For $\omega_o/\Delta\omega = 1.25$ the performance of the median detector relative to the likelihood detector L_n is grossly inferior.

The results shown in Fig. 12 through Fig. 16 indicate that, in general, the ARE_{S_n/L_n} and the ARE_{S_n/ϕ_n} increase as the correlation between the input observations increases. One exception is with the damped cosine and the gaussian correlation functions for larger values of $\omega_o/\Delta\omega$ when ω_s/ω_o is in the range from one to five. A second and more significant exception occurs with the sin x/x correlation function when $\omega_o/\Delta\omega$ is greater than one. In all cases the ARE_{S_n/L_n} and the ARE_{S_n/ϕ_n} decrease as $\omega_o/\Delta\omega$ increases.

Various receiver operating characteristics are given in Fig. 17 through Fig. 24. Figures 17 and 18 illustrate the effect of increased correlation on the false dismissal probability, β, for varying values of the input signal-to-noise ratio, θ. In both figures $N = 100$ and $\alpha = 0.1$. The curve for $\omega_s/\omega_o = 1$ represents approximately the case of independent sampling. The decaying exponential and the damped cosine correlation functions have been used in Fig. 17 and Fig. 18, respectively.

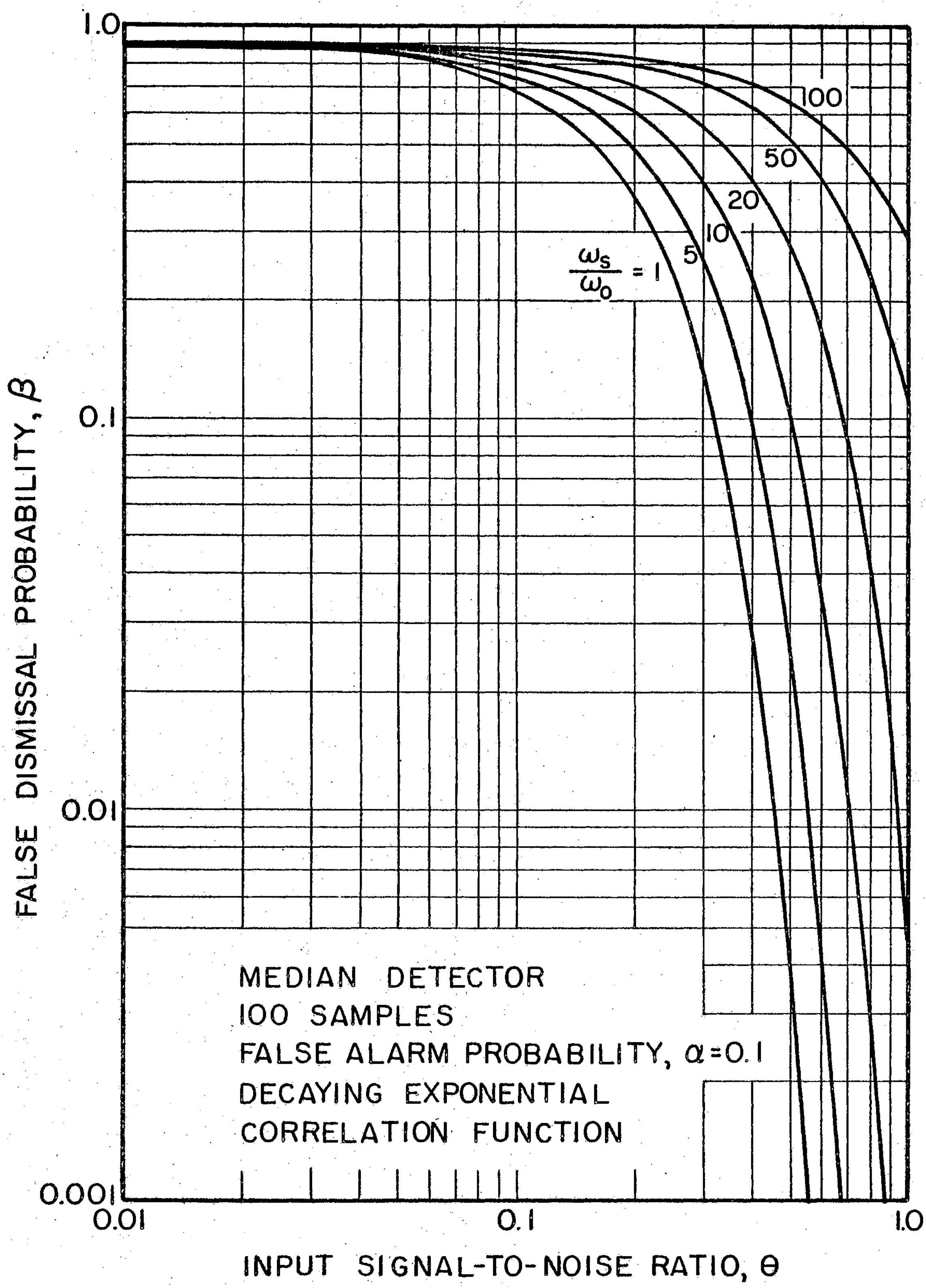

Fig. 17. The Effect of Correlated Samples on the Error Probabilities for the Median Detector (Correlation Function: Decaying Exponential).

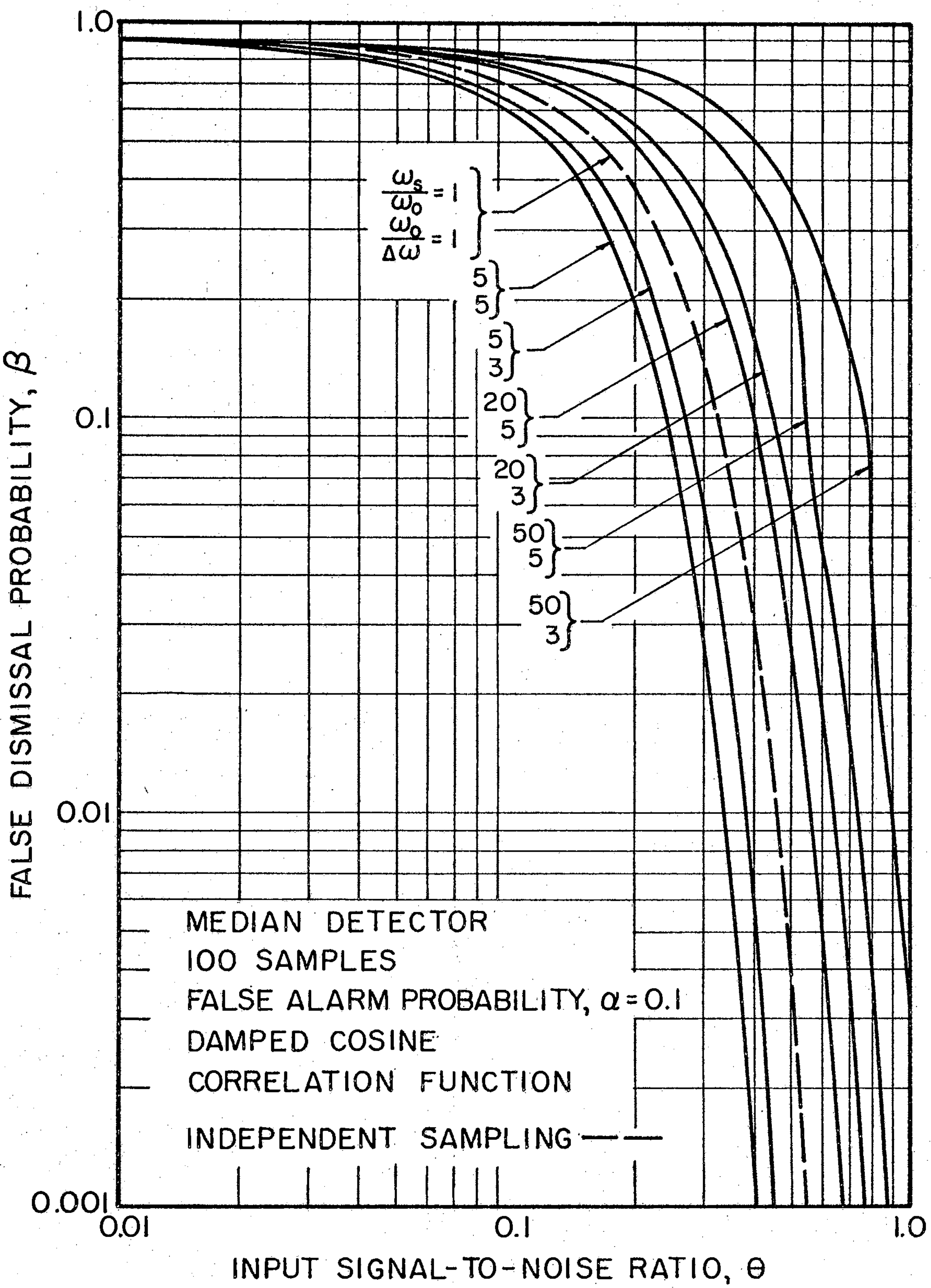

Fig. 18. The Effect of Correlated Samples on the Error Probabilities for the Median Detector (Correlation Function: Damped Cosine).

The error probability, β, increases as the degree of correlation is increased for fixed values of θ, with one exception. In Fig. 18 β is decreased slightly for $\omega_s/\omega_o = 5$ and $\omega_o/\Delta\omega = 3, 5$.

Figures 19 and 20 compare the error probabilities of the three detectors for the case of the decaying exponential correlation function. The median detector and the likelihood detector (independent structure) are seen to have nearly the same error probabilities for $\omega_s/\omega_o = 25$ in Fig. 19; however, the distance between the two curves increases when $\omega_s/\omega_o = 1$. This corresponds to the fact that the ARE_{S_n/L_n} increased as ω_s/ω_o was increased in Fig. 12, ($ARE_{S_n/L_n} = 0.64$ for $\omega_s/\omega_o = 1.0$ compared with $ARE_{S_n/L_n} = 0.89$ for $\omega_s/\omega_o = 25$). Thus, the error probabilities for the two detectors would be expected to approach each other as the correlation increases. Figure 20 compares all three detectors. The curves for the likelihood detector (independent structure) and the likelihood detector (dependent structure) lie almost on top of each other for $\omega_s/\omega_o = 1$ and $\omega_s/\omega_o = 10$; hence, only the one curve has been drawn. Again, the relative performance of the three detectors agrees with the results obtained using the ARE criteria.

The error probabilities of the median detector are considered in Fig. 21 for independent ($\omega_s/\omega_o = 1$) and dependent ($\omega_s/\omega_o = 25$) samples, where the number of samples varies from 25 to 500. For a fixed α ($\alpha = 0.1$) and a decaying exponential correlation function, a wide variation of β is obtained for a particular value of θ. Similarly, in

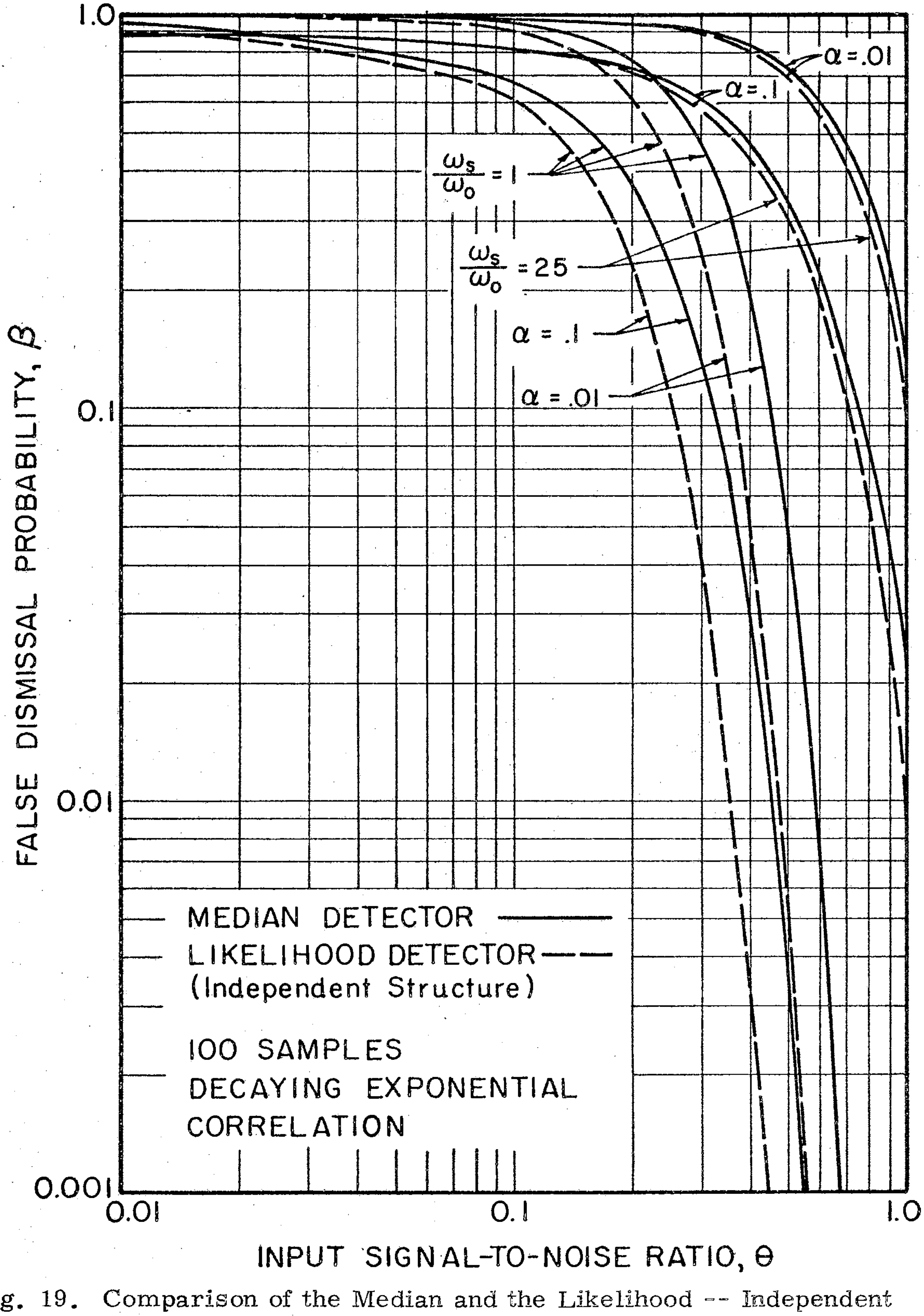

Fig. 19. Comparison of the Median and the Likelihood -- Independent Structure Detectors' Error Probabilities Assuming Correlated Samples.

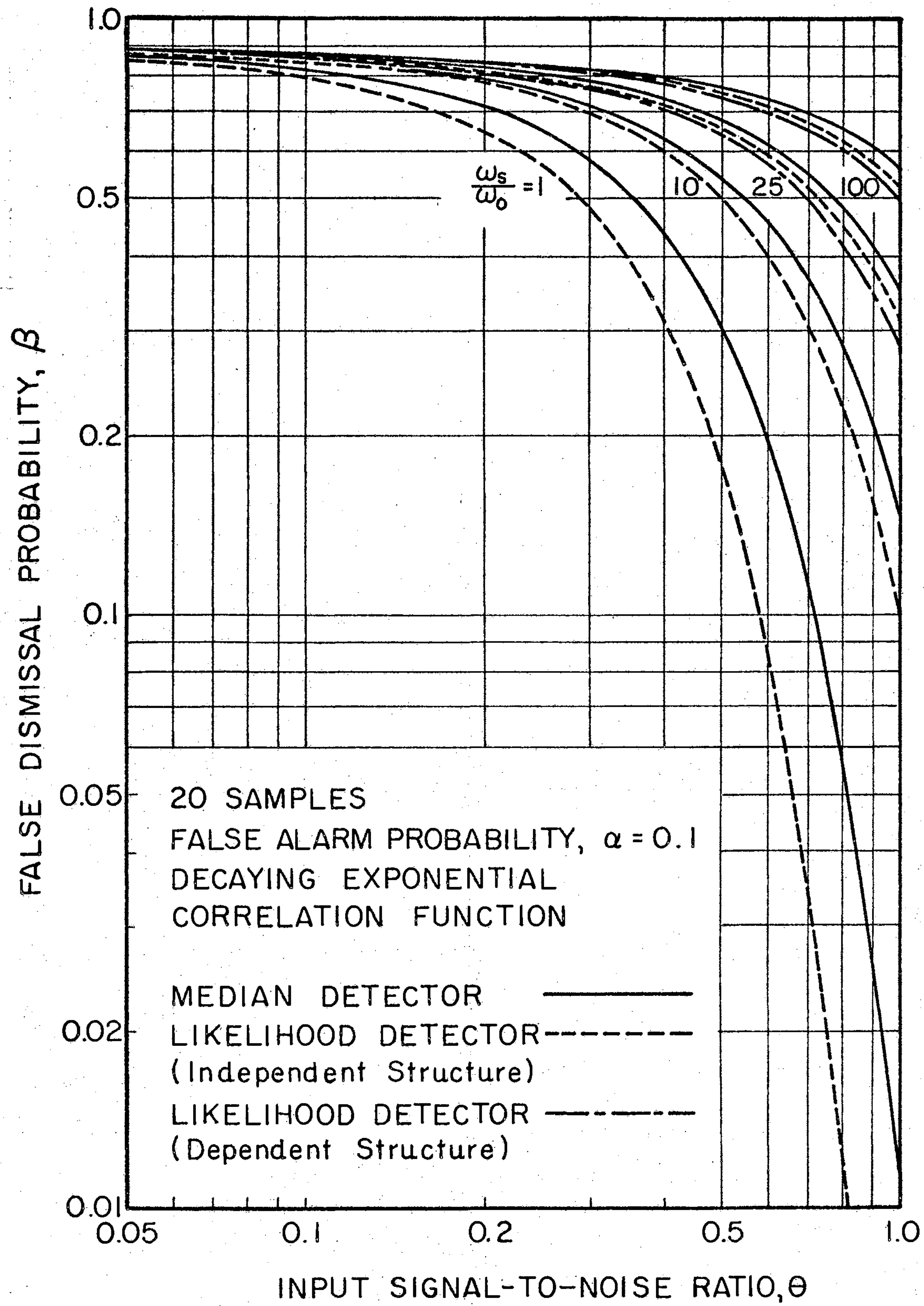

Fig. 20. Comparison of Error Probabilities for Three Detectors with Correlated Input Samples.

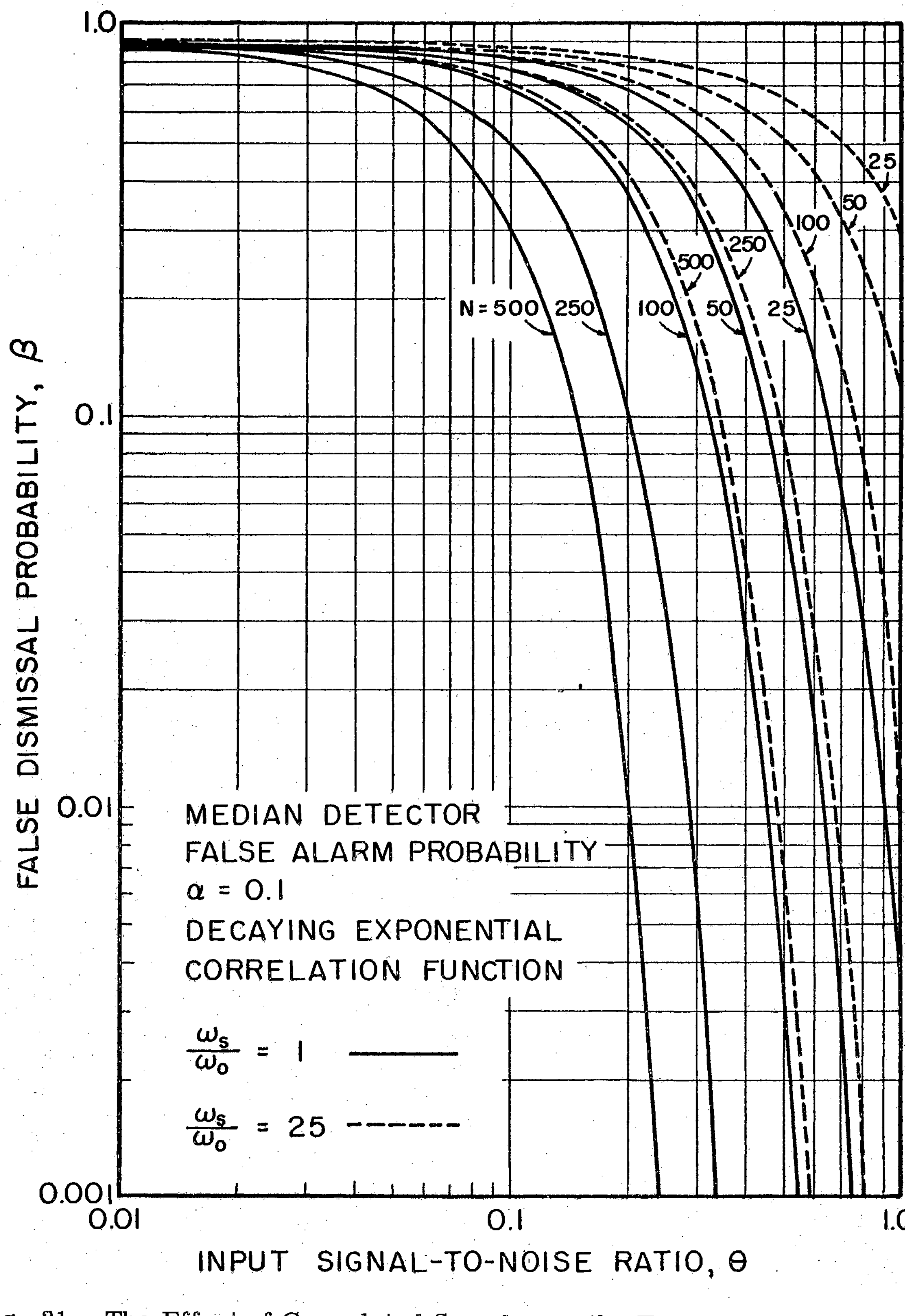

Fig. 21. The Effect of Correlated Samples on the Error Probabilities of the Median Detector for Various Numbers of Samples.

Fig. 22 the number of samples is held constant (N = 100) and α is varied from 0.001 to 0.5. The effect of correlated samples is again noticed.

The family of curves in Fig. 23 illustrates the effect of correlation on the number of samples required to maintain a particular set of error probabilities. The strongly correlated sample cases require five to ten times as many samples to maintain the same error probability as for the case of independent samples ($\omega_s/\omega_o = 1$). For example, $\omega_s/\omega_o = 1$ and N = 100 has the same set of error probabilities as the case where $\omega_s/\omega_o = 22.5$ and N = 500.

The answer to the question of how much is to be gained by sampling faster -- if any -- can be answered in the manner shown in Fig. 24. As the sampling frequency is increased, both N and ω_s/ω_o increase. Whether it would pay to sample faster would depend upon the noise correlation function. For weak correlation it would be profitable to sample faster; however, if strong correlation exists between the input observations, little might be gainedby increasing the sampling rate.

To consider a specific example, assume a decaying exponential correlation function and assume that doubling N results in doubling the relative sampling frequency, ω_s/ω_o. (The length of the signal pulse or the observation time will be assumed constant.) The gain in performance as the sampling frequency is doubled, increased five times, and increased ten times (x2, x5, x10) is shown in Fig. 24 for two different cases. In the first case the original sampling frequency results in

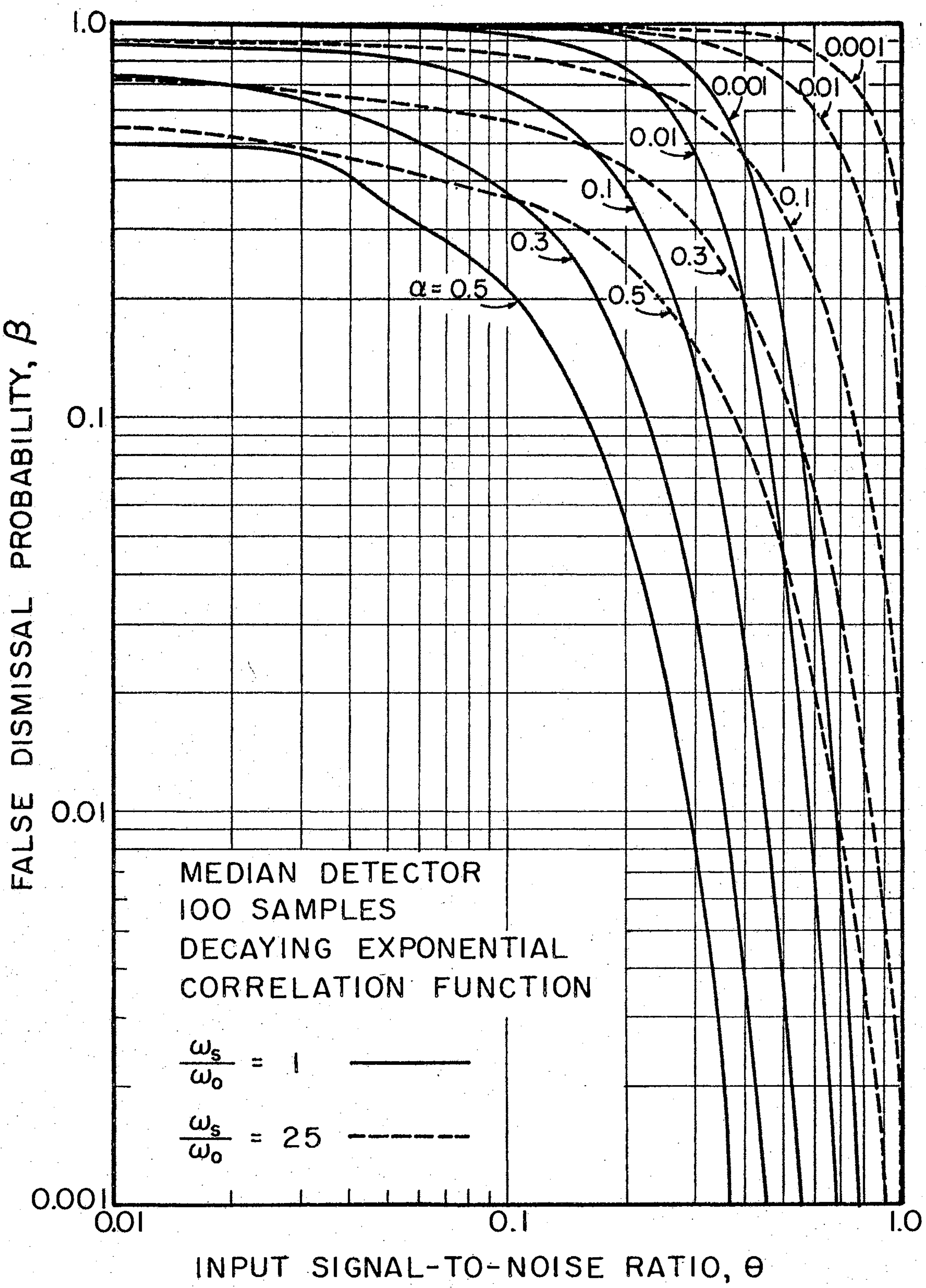

Fig. 22. The Effect of Correlated Samples on the Error Probabilities of the Median Detector for Various Values of α.

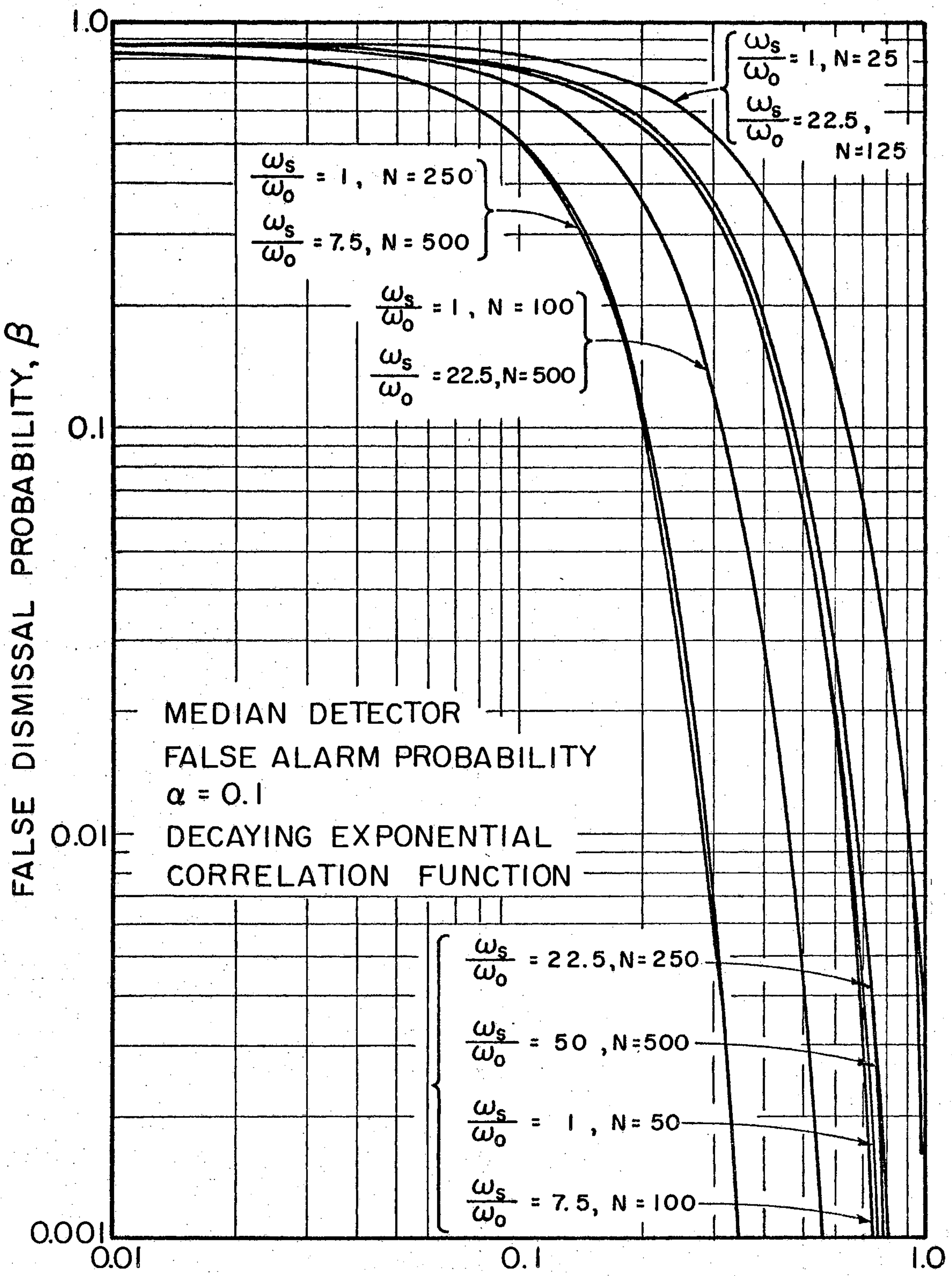

Fig. 23. The Relationship between the Degree of Input Correlation and the Number of Samples Required to Maintain a Fixed Set of Error Probabilities.

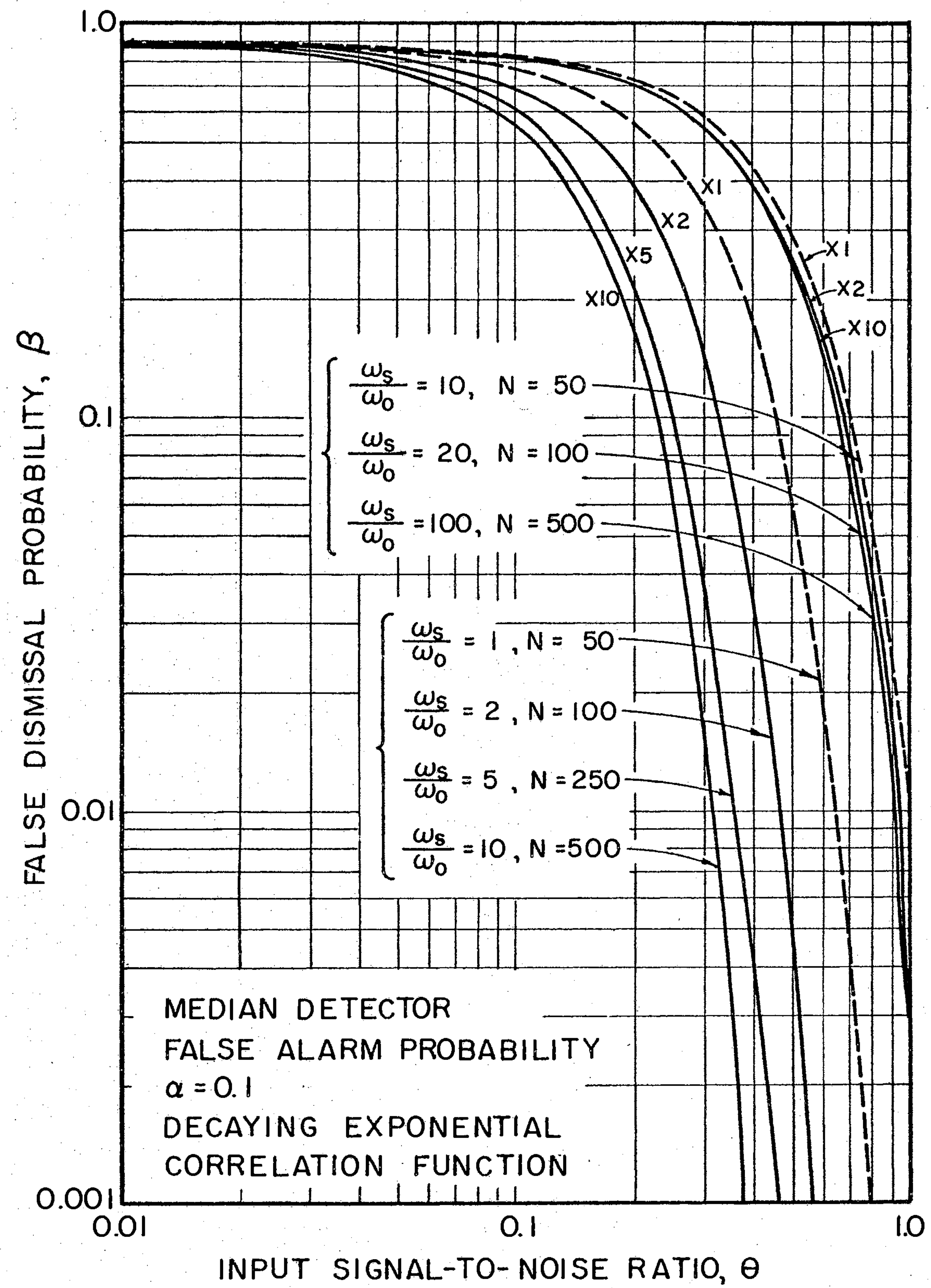

Fig. 24. An Example Showing the Decrease in Error Probabilities Gained by Increasing the Sampling Frequency.

very little correlation between samples (x1). Increasing the sampling frequency (x2, x5, x10) in this situation produces considerable improvement in the error rate. However, in the second case shown moderate correlation between observations is present initially (x1). Increasing the sampling frequency (x2, x10) results in very little improvement in the error rates.

CHAPTER VII

CONCLUSIONS

A statistical analysis of the nonparametric median detector has been presented in the preceding chapters, extending previous results to include the case of dependent as well as independent input observations. The median detector, a form of the coincidence detector based on the sign test, was chosen because of its nonparametric properties and its easily implemented structure. The purpose of this study was to determine the effect that correlated samples have on the performance of the median detector. This has been accomplished by comparing the ARE and the operating characteristics of the median detector under dependent samples with the median detector under independent samples, with the Neyman-Pearson detector designed assuming independent samples but operating under dependent samples, and with the Neyman-Pearson detector designed assuming dependent samples. The problem of detecting a constant signal in additive gaussian noise was chosen to make these comparisons.

It may be concluded that the relative efficiency of the median detector with respect to the Neyman-Pearson detector increases as the amount of correlation between input samples increases. For the case of a decaying exponential correlation between the input samples the ARE of the median detector and the Neyman-Pearson detector (designed assuming independent samples) increases from 0.64 for independent samples to greater than 0.91 for strongly correlated samples. Similar but less significant results were found for the damped cosine and the gaussian correlation functions. For the sin x/x input correlation function the ARE of the median detector with respect to the Neyman-Pearson detector drops to a very low value over a range of values of the correlation function parameters; however, it is questionable whether or not this range of parameter values represents a form of correlation that would be encountered in practice.

The operating characteristics of the three detectors illustrated the effect that correlation between the input samples has on the error probabilities. The results supported the conclusions drawn using the ARE criteria. The operating characteristics were also used to investigate the question of how much is gained by increasing the sampling frequency (and hence the degree of correlation). A simple example illustrated the amount by which the probability of error can be reduced by increasing the sampling rate.

Several areas for further investigation can be stated as a result of research done for this thesis. One area concerns extending the analysis to non-gaussian noise distributions. The modified gaussian noise distribution is considerably more general than the gaussian distribution; however, distributions such as the exponential and the Rayleigh should also be applied to the analysis. As it was pointed out in Chapter II, the ARE of the median detector with respect to the Neyman-Pearson detector designed for gaussian noise distributions is greater than one when the noise becomes exponentially distributed (assuming independent sampling). Carrying out the analysis for dependent sampling would possibly further emphasize the advantages of the median detector over the Neyman-Pearson detector when the underlying noise distribution is unknown.

An extension of the analysis of this thesis to detection problems other than that of detecting a constant signal in additive noise could be another area of investigation. The first step would be to consider the detection of a signal of unknown phase (incoherent detection).

Finally, other nonparametric detectors could be analyzed assuming dependent observations. It is felt that simplicity of the detector's structure should be a paramount factor in selecting other nonparametric tests, since many are extremely difficult to implement. One example could be the sign-quantile detector mentioned in Chapter II, which utilizes more than one threshold level in its operation.

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APPENDIX A

DERIVATION OF THE AUTOCORRELATION FUNCTION OF A GAUSSIAN PROCESS WITH NON-ZERO MEAN PASSED THROUGH A LIMITING DEVICE

The correlation function $R(k\tau)$ was derived in Chapter IV for a gaussian process with <u>zero</u> mean passed through a limiter whose output was either zero or one. The same problem will be considered in this appendix assuming an input gaussian process with a non-zero mean. Since these results have not previously been derived, a complete description of the procedure used will be given here. $R(k\tau)$ was given in Chapter IV, Eq. 19, as

$$R(k\tau) = \iint_0^\infty f\,(z_o, z_k)\, dz_o\, dz_k \qquad [A-1]$$

where, for non-zero input means, $f\,(z_o, z_k)$ is

$$f\,(z_o, z_k) =$$

$$= \frac{1}{2\pi\,\sigma^2\,(1-\rho^2)^{\frac{1}{2}}}\ \exp\left\{-\ \frac{(z_o-\mu)^2 - 2\rho\,(z_o-\mu)(z_k-\mu) + (z_k-\mu)^2}{2\sigma^2\,(1-\rho^2)}\right\} \qquad [A-2]$$

where $\mu = \mu_o = \mu_k$, $\sigma^2 = \sigma_o^{\ 2} = \sigma_k^{\ 2}$, and $\rho = \rho(k\tau)$ is the input correlation function. Instead of substituting Eq. A-2 into Eq. A-1 and integrating, as was done in Chapter IV, the characteristic function method will be used to find $R(k\tau)$ (55, 62).

The characteristic function method developed by Rice gives the output correlation function of a random process passed through some nonlinear device. The method is based on the equation

$$R(k\tau) = \int_C F(ju)\, du \int_C F(jv)\, g(u, v, \tau)\, dv \qquad [A-3]$$

where $g(u, v, \tau)$ is the characteristic function of the input random process and C, F (ju), and F (jv) are chosen to fit the particular nonlinear device used.

The nonlinear device is described by the contour integral

$$I = \frac{1}{2\pi} \int_C F(ju)\, e^{jVu}\, du \qquad [A-4]$$

where V is the input and I is the output of the device. To describe the nonlinear operation

$$I = \begin{cases} 0 \text{ if } V < 0 \\ 1 \text{ if } V > 0 \end{cases} \qquad [A-5]$$

which applies to the problem under consideration, the function F (ju) is chosen as $\frac{1}{ju}$ and the contour C as the real u axis from $-\infty$ to $+\infty$ with a downward indentation at u = 0. To verify that the integral will represent the 0 - 1 limiter, evaluate the contour integral

$$I = \frac{1}{2\pi j} \int_C \frac{1}{u}\, e^{jVu}\, du.$$

Let $\lambda = ju$, then

$$I = \frac{1}{2\pi j} \int_L \frac{e^{V\lambda}}{\lambda} d\lambda$$

where L is an infinite contour from $-j\infty$ to $+j\infty$, passing to the right of all singularities. By application of Jordan's lemma it can be shown that (63)

$$I = \frac{1}{2\pi j} \int_L \frac{e^{V\lambda}}{\lambda} d\lambda = \begin{cases} 1 \text{ if } V > 0 \\ 0 \text{ if } V < 0 . \end{cases}$$

Hence, the contour integral does represent the 0 - 1 limiter.

Evaluate $R(k\tau)$ using the fundamental equation, Eq. A-3. The characteristic function for the gaussian process described in Eq. A-2 is (2)

$$g(u, v, \tau) = \exp\left[j\mu(u + v) - \tfrac{1}{2}\sigma^2(u^2 + 2\rho uv + v^2)\right]. \qquad [A\text{-}6]$$

Substitute $g(u, v, \tau)$ and $F(iu)$ into Eq. A-3 to obtain

$$R(k\tau) = \frac{1}{4\pi^2} \iint_C \left(\frac{1}{ju}\right)\left(\frac{1}{jv}\right) \exp\left[j\mu(u + v) - \tfrac{1}{2}(\sigma^2 v^2 + 2\sigma^2 \rho uv + \sigma^2 u^2)\right] du\, dv$$

$$= \frac{-1}{4\pi^2} \iint_C \frac{1}{uv}\, e^{j\mu(u + v)}\, e^{-\frac{1}{2}\sigma^2 (u^2 + v^2 + 2\rho uv)}\, du\, dv \qquad [A\text{-}7]$$

where C is the infinite contour on the real axis from $-\infty$ to $+\infty$ with a downward indentation at the origin. Expand the cross product term of the second exponential in Eq. A-7 into a power series to give

$$R(k\tau) = \frac{-1}{4\pi^2} \iint_C \frac{1}{uv}\, e^{-\frac{1}{2}\sigma^2 u^2 + j\mu u}\, e^{-\frac{1}{2}\sigma^2 v^2 + j\mu v}\, e^{-\sigma^2 \rho uv}\, du\, dv$$

$$R(k\tau) = \frac{-1}{4\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n (\sigma^2\rho)^n}{n!} \left\{ \iint_C u^{n-1} e^{-\frac{1}{2}\sigma^2 u^2 + j\mu u} du \right\}^2 . \quad [A-8]$$

The integral in Eq. A-8 can be evaluated using Eq. A.1.55(*) of Middleton (page 1079) (2). The result is

$$R(k\tau) = \frac{-1}{4\pi^2} \sum_{n=0}^{\infty} \frac{(-\sigma^2\rho)^n}{n!} \left\{ \frac{\pi j^{-n+1}}{(\frac{1}{2}\sigma^2)^{\frac{1}{2}n}} \left[{}_1F_1\left(\frac{n}{2}; \frac{1}{2}; \frac{-\mu^2}{2\sigma^2}\right) \Big/ \Gamma\left(\frac{1-n+1}{2}\right) + \right.\right.$$

$$\left.\left. + \frac{\sqrt{2}\mu}{\sigma}\, {}_1F_1\left(\frac{n+1}{2}; \frac{3}{2}; \frac{-\mu^2}{2\sigma^2}\right) \Big/ \Gamma\left(\frac{1-n}{2}\right) \right] \right\}^2$$

$$R(k\tau) = \frac{1}{4} \sum_{n=0}^{\infty} \frac{[2\rho(k\tau)]^n}{n!} \left\{ {}_1F_1\left(\frac{n}{2}; \frac{1}{2}; \frac{-\mu^2}{2\sigma^2}\right) \Big/ \Gamma\left(1 - \frac{n}{2}\right) + \right.$$

$$\left. + \frac{\sqrt{2}\mu}{\sigma}\, {}_1F_1\left(\frac{n+1}{2}; \frac{3}{2}; \frac{-\mu^2}{2\sigma^2}\right) \Big/ \Gamma\left(\frac{1-n}{2}\right) \right\}^2 \quad [A-9]$$

where ${}_1F_1(a;b;c)$ is the confluent hypergeometric function and $\Gamma(n)$ is the gamma function.

Note that for $\mu = 0$ Eq. A-9 reduces the result obtained in Eq. 23; that is,

$$R(k\tau)\Big|_{\mu=0} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(2\rho)^n}{n!\, \Gamma^2\left(1 - \frac{n}{2}\right)}$$

---

* Equation A.1.55 of Middleton (2) was stated incorrectly in the first printing of the book; however, by the third printing it had been corrected. Also, the formula is stated incorrectly in the source reference of Middleton (64).

$$= \frac{1}{4} + \frac{1}{2\pi} \left(\rho + \frac{\rho^3}{2 \cdot 3} + \cdots \right)$$

$$= \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho(k\tau).$$

The expression for $R(k\tau)$ with $\mu \neq 0$ obtained in Eq. A-9 could be substituted in Eq. 25 to find the variance of S_n under dependent sampling.

APPENDIX B

THE POWER OF THE MEDIAN DETECTOR FOR TWO DIFFERENT LIMITERS

The question arose in Chapter IV concerning what effect the type of limiter output had on the analysis and instrumentation of the median detector; that is, does it make any difference if the output of the limiter is 0 - 1 or ± 1. First, it is necessary to show that the power of the median detector is the same regardless of which limiter is used. This will be shown in the following paragraphs, assuming dependent sampling. Obviously, if the powers are the same for dependent samples, they will also be the same for independent samples. It will be observed that the analysis of the detector is not noticeably simplified by using the ± 1 limiter instead of the 0 - 1 limiter. Thus, the question of which limiter to use depends upon the implementation problems.

To show that two detectors have the same power is equivalent to showing that their efficacies are equal, i.e.,

$$\lim_{n \to \infty} \frac{1}{n} \left[\frac{E_\theta' [S_n]}{\sigma_0 [S_n]} \right]^2_{\theta = 0} = \lim_{n \to \infty} \frac{1}{n} \left[\frac{E_\theta' [S_n']}{\sigma_0 [S_n']} \right]^2_{\theta = 0} \qquad [B-1]$$

where S_n refers to the median detector using the 0 - 1 limiter and S'_n refers to the detector using the ± 1 limiter. To show that Eq. B-1 is true, the mean of S'_n under signal conditions and dependent samples and the no-signal variance for dependent samples must be found.

The mean value under signal conditions, assuming dependent sampling, of S'_n is found as follows. Let

$$S'_n = \frac{1}{n} \sum_{i=1}^{n} \text{sgn}\,(Y_i - M) \qquad [B-2]$$

where
$$\begin{aligned} \text{sgn}\,(Y_i - M) &= 1 \text{ if } Y_i > M \\ &= -1 \text{ if } Y_i < M \end{aligned}$$

and where M is the median of the noise distribution. The mean of S'_n under signal conditions is derived from Eq. B-2 by writing

$$\begin{aligned} E_\theta\,[S'_n] &= \frac{1}{n} \sum_{i=1}^{n} E_\theta\,[\text{sgn}\,(Y_i - M)\,] \\ &= \frac{1}{n} \sum_{i=1}^{n} [P_\theta\,(Y_i > M) - P_\theta\,(Y_i < M)\,] \\ &= \frac{1}{n} \sum_{i=1}^{n} [\int_M^\infty d\,G_{\theta_i}\,(y) - \int_{-\infty}^M d\,G_{\theta_i}\,(y)\,] \\ &= \frac{1}{n} \sum_{i=1}^{n} [1 - 2\,G_{\theta_i}\,(M)\,] \end{aligned} \qquad [B-3]$$

where $G_{\theta_i}(y)$ is the probability distribution function of the random variable Y_i. Apply the mean value theorem of Eq. 12 to $G_\theta(M)$. Assuming the small signal case, Eq. B-3 becomes

$$E_\theta [S_n'] \doteq \frac{1}{n} \sum_{i=1}^{n} \left[1 - 2 \left(\tfrac{1}{2} + \theta_i \frac{d}{d\theta} G_\theta (M) \Big|_{\theta \doteq 0} \right) \right]$$

$$= -2 \bar{\theta} \frac{d}{d\theta} G_\theta (M) \Big|_{\theta \doteq 0} \qquad [B-4]$$

where $\bar{\theta} = \frac{1}{n} \Sigma \theta_i$. Taking the derivative of Eq. B-4 with respect to θ gives

$$E' [S_n'] = -2 \frac{d}{d\theta} G_\theta (M) \Big|_{\theta \doteq 0} . \qquad [B-5]$$

The variance of S_n' under dependent samples and no-signal conditions can be found by referring to Eq. 17 and re-defining $R(k\tau)$. From Eq. 17,

$$\sigma_0^2 [S_n'] = \frac{R(0)}{n} - E_0^2 [S_n'] + \frac{2}{n} \sum_{i=1}^{n} (1 - \frac{k}{n}) R(k\tau) \qquad [B-6]$$

where $R(k\tau) = E_0 [\text{sgn}(z_o) \text{ sgn}(z_k)]$. [B-7]

Furthermore,

$$E_0 [S_n'] = \frac{1}{n} \sum_{i=1}^{n} E_0 [\text{sgn}(Y_i - M)]$$

$$E_0 [S_n'] \doteq \frac{1}{n} \sum_{i=1}^{n} [1 \cdot P_0 (Y_i > M) - 1 \cdot P(Y_i < M)]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\int_{M}^{\infty} dF_{0_i}(y) - \int_{-\infty}^{M} dF_{0_i}(y) \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} [1 - F_{0_i}(M) - F_{0_i}(M)]$$

or $E_0 [S_n'] = 0$ [B-8]

since $F_{0_i}(y)$ is the probability distribution function of the random variable Y_i under no-signal conditions, and $F_{0_i}(M) = \frac{1}{2}$.

The output correlation function, $R(k\tau)$, of a gaussian random process with zero mean and an input correlation function, $\rho(k\tau)$, passed through a $\pm$ 1 hard-limiting device is well known (62). Thus $R(k\tau)$ of Eq. B-7 can be evaluated as

$$R(k\tau) = \frac{2}{\pi} \sin^{-1} \rho(k\tau) \ . \qquad [B-9]$$

Since $\rho(0) = 1$, then $R(0) = 1$. Substituting the results of Eq. B-8 and Eq. B-9 into Eq. 6 gives

$$\sigma_0^2 [S_n'] = \frac{1}{n} + \frac{4}{\pi n} \sum_{i=1}^{n} (1 - \frac{k}{n}) \sin^{-1} \rho(k\tau) \ . \qquad [B-10]$$

The efficacies of the two detectors will be calculated and compared; hence,

$$E_{S_n} = \lim_{n \to \infty} \frac{1}{n} \left[\frac{E_\theta' [S_n]}{\sigma_0 [S_n]} \right]^2_{\theta \doteq 0} \ . \qquad [B-11]$$

Substitute Eq. 13 and Eq. 25 into Eq. B-11,

$$E_{S_n} = \lim_{n \to \infty} \frac{1}{n} \left\{ \frac{\left. \frac{d}{d\theta} G_\theta (M) \right|^2_{\theta \doteq 0}}{\frac{1}{4n} + \frac{1}{\pi n} \sum_{k=1}^{n} (1 - \frac{k}{n}) \sin^{-1} \rho(k\tau)} \right\} . \qquad [B-12]$$

Similarly,

$$E_{S'_n} = \lim_{n \to \infty} \frac{1}{n} \left[\frac{E'_\theta [S'_n]}{\sigma_0 [S'_n]} \right]^2_{\theta \doteq 0} . \qquad [B-13]$$

Substituting Eq. B-5 and Eq. B-10 into Eq. B-13, gives

$$E_{S_n}' = \lim_{n \to \infty} \frac{1}{n} \left\{ \frac{4 \left. \frac{d}{d\theta} G_\theta (M) \right|^2_{\theta \doteq 0}}{\frac{1}{n} + \frac{4}{\pi n} \sum_{k=1}^{n} (1 - \frac{k}{n}) \sin^{-1} \rho(k\tau)} \right\} . \qquad [B-14]$$

Comparing Eq. B-12 and Eq. B-14 it can be seen that the power of the median detector is the same regardless of which limiter is used.

APPENDIX C

A CENTRAL LIMIT THEOREM FOR DEPENDENT RANDOM PROCESSES

One form of Rosenblatt's central limit theorem for sums of dependent random variables was stated in Chapter VI. It will be shown in this appendix that the hypotheses of the theorem are satisfied by the median test statistic for the problem of detecting a constant signal in additive gaussian noise.

The first requirement of the theorem is that the strong mixing condition be satisfied. Rozanov (59, 65) has shown that the strong mixing condition is satisfied in the case of discrete time if for a stationary gaussian process the spectral density is continuous and never vanishes. Since the spectral density and the correlation function are Fourier transform pairs (54) this implies that the correlation function $\rho(\tau)$ must approach zero as $\tau \to \infty$. However, the median statistic sums a binomial process of ones and zeros, not a gaussian process. The statement made above can be extended to include the binomial process (*).

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*The generous help of Dr. J. L. Folks in extending these results is acknowledged.

Given two sets of A and B of samples from a gaussian process such that

$$A \in \{-\infty, \cdots, k\} = \{\mathcal{M}_{-\infty}^{k}\}$$

and

$$B \in \{k + n, \cdots, \infty\} = \{\mathcal{M}_{k+n}^{\infty}\}.$$

The strong mixing condition states that

$$\sup_{A,B} [P(A \cap B) - P(A)\ P(B)] \leq \alpha(n) \downarrow 0 \text{ as } n \to \infty .$$

As stated above, this condition will be satisfied for the gaussian process if $\rho(\tau) \to 0$ as $\tau \to \infty$. Consider now a binomial process of ones and zeros obtained by passing the gaussian process through a 0 - 1 limiting operation. The resulting samples of the two processes are shown in Fig. 25.

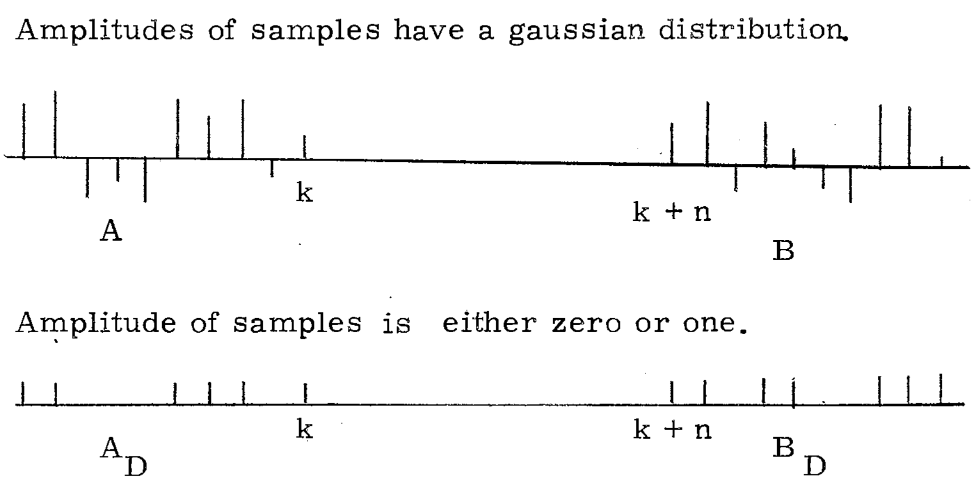

Fig. 25. Binomial and Gaussian Samples.

In terms of probabilities, the probability for the binomial random process expressed in terms of that for the gaussian process is

$$P\,(A_D) = P\,(\text{hyperquadrant of gaussian process}),$$

i.e., the probabilities of A_D and of B_D are hyperquadrants of the gaussian process. So

$$\sup_{A_D, B_D} \left[P\,(A_D \cap B_D) - P\,(A_D)\,P\,(B_D)\right] =$$

$$= \sup_{\substack{\text{subset} \\ \text{of } A,B}} \left[P\,(A_D \cap B_D) - P\,(A_D)\,P\,(B_D)\right]$$

$$\leq \sup_{A,B} \left[P\,(P\,(A \cap B) - P\,(A)\,P\,(B)\,\right].$$

Therefore, the strong mixing condition will also hold for the binomial process described.

The second condition of the central limit theorem hypothesis concerns the second moment, namely,

$$E \left| \sum_{i=1}^{n} x_i \right|^2 = h(n) \to \infty \text{ as } n \to \infty .$$

In the problem under consideration, x_i is a binary random variable which represents the output of the limiter of the detector. Let the random variable y_i represent the input gaussian random process to the detector. Then

$$E\left[\sum_{i=1}^{n} x_i\right]^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} E\,[x_i\, x_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} P\,(y_i > 0 \cap y_j > 0). \qquad [C\text{-}1]$$

This can be evaluated to give (66).

$$E\left[\sum_{i=1}^{n} x_i\right]^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \tfrac{1}{2}\left[1 - \frac{\cos^{-1} \rho_{ij}}{\pi}\right] \qquad [C-2]$$

where ρ_{ij} is the input correlation function. Eq. C-2 can be rewritten as

$$E\left[\sum_{i=1}^{n} x_i\right]^2 = \frac{n^2}{2} - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\cos^{-1} \rho_{ij}}{2\pi}$$

$$= \frac{n^2}{2} - \frac{1}{\pi} \sum_{k=1}^{n} (n - k) \cos^{-1} \rho(k\tau). \qquad [C-3]$$

It is clear from Eq. C-3 that the second moment condition of the hypothesis is satisfied.

Now consider the third condition of the central limit theorem hypothesis. In his original statement of the theorem Rosenblatt (57) required that

$$E\left|\sum_{i=1}^{n} x_i\right|^{2+\delta} = o\,[h\,(n)]^{1+\delta/2} \qquad [C-4]$$

as $n \to \infty$, for some $\delta > 0$. However, in a later statement of the theorem (58) he required that

$$E\left|\sum_{i=1}^{n} x_i\right|^{4} = O\,[h\,(n)]^{2} \qquad [C-5]$$

as $n \to \infty$ (*). (All other conditions of the theorem remain the same.)

The problem in showing that either Eq. C-4 or Eq. C-5 is satisfied is the difficulty involved in evaluating the higher order moments of the summation. Thus, one is tempted to let $\delta = 1$ in Eq. C-4 and calculate the third moment. However, the result shows that the third moment is of comparable order with $[h(n)]^{3/2}$ instead of inferior order. Comparing the two conditions given in Eq. C-4 and Eq. C-5, one sees an obvious difference in the conditions for $\delta = 2$. Hence, one might strongly suspect that it is sufficient for the moment in Eq. C-4 to be of comparable order with $[h(n)]^{1 + \delta/2}$. No further information has been found in the literature. Since the third order moment can be stated in closed form (but not the fourth order moment), the condition stated in Eq. C-4 will be shown to be true if "o" is replaced by "O" for $\delta = 1$. Following this, a more complicated derivation will show that Eq. C-5 can be satisfied.

---

*The O and o notations are defined as follows (67):

$$f(x) = o(g(x)) \text{ as } x \to x_o \text{ whenever}$$

$$\lim_{x \to x_o} \frac{f(x)}{g(x)} = 0$$

i.e., f(x) is of smaller order than g(x). If f and g are two real-valued functions defined on a set S of real numbers and if g is nonnegative, then $f(x) = O(g(x))$ for x in S whenever there exists a positive constant M such that

$$f(x) \leq Mg(x) \qquad \text{for every } x \text{ in } S;$$

i.e., f(x) is of comparable order with g(x).

The third order moment is given by

$$E\left|\sum_{i=1}^{n} x_i\right|^3 = \sum_i \sum_j^n \sum_k E[x_i\, x_j\, x_k]$$

$$= \sum_i \sum_j^n \sum_k P(y_i > 0 \cap y_j > 0 \cap y_k > 0) \qquad [C\text{-}6]$$

where the binary random variable x_i is the output of the limiter, and y_i is a random variable from the input gaussian random process. Using the results of David (66), Eq. C-6 becomes

$$E\left|\sum_{i=1}^{n} x_i\right|^3 = \sum_i \sum_j^n \sum_k \frac{1}{4\pi}\left[2\pi - \cos^{-1}\rho_{ij} - \cos^{-1}\rho_{ik} - \cos^{-1}\rho_{jk}\right]$$

$$= \frac{n^3}{2} - \frac{n}{4\pi}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} \cos^{-1}\rho_{ij} + \sum_{i=1}^{n}\sum_{k=1}^{n} \cos^{-1}\rho_{ik} + \sum_{j=1}^{n}\sum_{k=1}^{n} \cos^{-1}\rho_{jk}\right\}$$

$$= \frac{n^3}{2} - \frac{3n}{2\pi}\left\{\sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \cos^{-1}\rho(k\tau)\right\}. \qquad [C\text{-}7]$$

These results are used to find

$$\lim_{n\to\infty} \frac{E\left|\sum^{n} x_i\right|^3}{\left[E\left|\sum^{n} x_i\right|^2\right]^{3/2}} = \lim_{n\to\infty} \frac{\frac{n^3}{2} - \frac{3n^2}{2\pi}\sum_1^n \left(1 - \frac{k}{n}\right)\cos^{-1}\rho(k\tau)}{\left[\frac{n^2}{2} - \frac{n}{\pi}\sum_1^n \left(1 - \frac{k}{n}\right)\cos^{-1}\rho(k\tau)\right]^{3/2}}$$

$$= \lim_{n \to \infty} \frac{\frac{n^3}{2} - \frac{3n^2}{2\pi} \sum_{1}^{n} \cos^{-1} \rho(k\tau)}{\left[\frac{n^2}{2} - \frac{n}{\pi} \sum_{1}^{n} \cos^{-1} \rho(k\tau)\right]^{3/2}}$$

$$= \sqrt{2},$$

assuming $\rho(k\tau) \to 0$ as $k \to \infty$. This shows that

$$E \left| \sum_{1}^{n} x_i \right|^3 = 0 \left[h(n)\right]^{3/2} \qquad [C\text{-}8]$$

as $n \to \infty$. If the original statement of the theorem can be extended such that the condition in Eq. C-8 replaces that in Eq. C-4, then the third condition of the central limit theorem would be satisfied. However, if this is not possible, the following alternative derivation using the fourth order moment could be used.

The fourth order moment of Σx_i is evaluated as follows. The results are based upon equations derived by Kendall (68) and Moran (69). Thus,

$$E \left| \sum_{i=1}^{n} x_i \right|^4 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} E\left[x_i x_j x_k x_\ell\right]$$

$$= \sum_{i} \sum_{j} \sum_{k} \sum_{\ell}^{n} P\left(y_i > 0 \cap y_j > 0 \cap y_k > 0 \cap y_\ell > 0\right)$$

$$= \sum_{i,j,k,\ell}^{n} \sum_{m,n,p,q,r,s}^{\infty} \left\{ \frac{\rho_{ij}^{\,m} \rho_{ik}^{\,n} \rho_{i\ell}^{\,p} \rho_{jk}^{\,q} \rho_{j\ell}^{\,r} \rho_{k\ell}^{\,s}}{m!\, n!\, p!\, q!\, r!\, s!} \right. .$$

$$\cdot (-1)^{m+n+p+q+r+s} \cdot G_{m+n+p} \cdot G_{m+q+r} \cdot G_{n+q+s} \cdot G_{p+r+s} \qquad [C-9]$$

where $G_0 = \frac{1}{2}$

$G_t = 0$ when t is even

$G_t = \dfrac{(2\lambda)!}{j\sqrt{2\pi}\;2^{\lambda}\;\lambda!}$ when t is odd and equal to $(2\lambda + 1)$ where $\lambda = 0,1,2,\cdots$.

Looking at the first order terms of this expansion, Eq. C-9 yields

$$E\left|\sum_{1}^{n} x_i\right|^4 = \sum_{i,j,k,\ell}^{n}\left\{\frac{1}{16} + \frac{1}{8\pi}(\rho_{ij} + \rho_{ik} + \rho_{i\ell} + \rho_{jk} + \rho_{j\ell} + \rho_{k\ell}) + \right.$$

$$+ \frac{1}{4\pi^2}(\rho_{ij}\,\rho_{k\ell} + \rho_{ik}\,\rho_{j\ell} + \rho_{i\ell}\,\rho_{jk}) - \frac{1}{4\pi^2}(\rho_{ij}\,\rho_{ik}\,\rho_{i\ell} +$$

$$+ \rho_{ij}\,\rho_{jk}\,\rho_{j\ell} + \rho_{i\ell}\,\rho_{j\ell}\,\rho_{k\ell} + \rho_{ik}\,\rho_{jk}\,\rho_{k\ell}) +$$

$$\left. + \rho_{ij}\,\rho_{ik}\,\rho_{i\ell}\,\rho_{jk}\,\rho_{j\ell}\,\rho_{k\ell} + \text{higher order terms in } \rho\right\}. \qquad [C-10]$$

Equation C-10 can be simplified to the following form,

$$E\left|\sum_{1}^{n} x_i\right|^4 = \frac{n^4}{16} + \frac{6n^2}{8\pi}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij} + \frac{3}{4\pi^2}\sum_{i}\sum_{j}\sum_{k}^{n}\sum_{\ell}\rho_{ij}\,\rho_{k\ell} -$$

$$- \frac{1}{\pi^2}\sum_{i}\sum_{j}\sum_{k}^{n}\sum_{\ell}\rho_{ij}\,\rho_{ik}\,\rho_{i\ell} +$$

$$+ \frac{1}{4\pi^2}\sum_{i}\sum_{j}\sum_{k}^{n}\sum_{\ell}\rho_{ij}\,\rho_{ik}\,\rho_{i\ell}\,\rho_{jk}\,\rho_{j\ell}\,\rho_{k\ell} + \cdots \quad . \qquad [C-11]$$

If $\rho(\tau) \to 0$ as $\tau \to \infty$, then $E\left|\Sigma x_i\right|^4$ has n^4 as its leading term which is of comparable order with $\left(E\left[\sum x_i^{\,2}\right]\right)^2$, thereby satisfying the condition given in Eq. C-5. This then shows that the three conditions of the central limit theorem are satisfied.

APPENDIX D

EVALUATION OF THE LIMIT OF A SUM

The following problem arose in satisfying the regularity conditions: Given that $\lim_{n \to \infty} a_n = 0$, show that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} a_i = 0.$$

The solution to this problem could not be found in the literature, hence it will be stated as a theorem and proved in this appendix (*).

<u>Theorem</u>: If $\lim_{n \to \infty} a_n = 0$

then $$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0.$$

<u>Proof</u>: For an $\epsilon > 0$, choose N_o such that if $k > N_o$ then $a_k < \epsilon$. This is possible since $\lim_{k \to \infty} a_k = 0$. Hence,

$$\frac{1}{n} [a_1+a_2+\cdot\cdot\cdot+a_n] = \frac{1}{n} [(a_1+a_2+\cdot\cdot\cdot+a_{N_o}) + a_{N_o+1}+\cdot\cdot\cdot+a_n]$$

$$< \frac{1}{n} [C + (n - N_o) \epsilon]$$

---

*The assistance of Dr. Paul E. Long in solving this problem is gratefully acknowledged.

where $C = a_1 + a_2 + \cdots + a_{N_o}$. Thus,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} a_k < \lim_{n \to \infty} \frac{1}{n} [C + (n - N_o) \epsilon] = \epsilon$$

or $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} a_k < \epsilon$.

Since $\epsilon > 0$ is arbitrary, the result follows,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0.$$

Q.E.D.

APPENDIX E

COMPUTATIONAL PROCEDURE AND COMPUTER PROGRAMS

The numerical results of Chapter VI were obtained using the methods described in this appendix. Before listing the computer programs, approximations for evaluating the error function and the inverse error function will be given.

Evaluation of the error function and its inverse was necessary in order to calculate the operating characteristics. For a given α, the threshold value was found. β was calculated using this threshold level with some signal-to-noise ratio value.

Approximations for evaluating the error function and its inverse have been given by Hastings (70). For the inverse error function (used to find the threshold for a given α), the form of the function is

$$q = \frac{1}{\sqrt{2\pi}} \int_{X(q)}^{\infty} e^{-t^2/2}\, dt$$

over the range $0 < q \leq 0.5$. (The range can be changed to $0.5 < q < 1$ by a simple change in the above equation.) The approximation is

$$X^*(q) = \gamma - \left\{ \frac{a_0 + a_1\gamma + a_2\gamma^2}{1 + b_1\gamma + b_2\gamma^2 + b_3\gamma^3} \right\}$$

where $\gamma = \sqrt{\ell n \frac{1}{q^2}}$

and

$$a_0 = 2.515517 \qquad b_1 = 1.432788$$

$$a_1 = .802853 \qquad b_2 = .189269$$

$$a_2 = .010328 \qquad b_3 = .001308 .$$

The error between the approximation and the actual value of the function is within ± 0.0005.

An approximation for the error function was used to find β, given the threshold value obtained from the approximation above. The form of the function is

$$\Phi(X) = \frac{2}{\sqrt{\pi}} \int_0^X e^{-t^2}\, dt$$

over the range $0 \leq X < \infty$. (The range can be altered for the negative values of X by a simple change of the above equation.) The approximation is

$$\Phi^*(X) = 1 - \frac{1}{(1 + a_1X + a_2X^2 + a_3X^3 + a_4X^4 + a_5X^5 + a_6X^6)^{16}}$$

where

$$a_1 = .0705, 2307, 84 \qquad a_4 = .0001, 5201, 43$$

$$a_2 = .0433, 8201, 23 \qquad a_5 = .0002, 7656, 72$$

$$a_3 = .0092, 7052, 72 \qquad a_6 = .0000, 4306, 38 .$$

The error between the approximation and the actual value of the function is within $\pm 3 \times 10^{-7}$.

A brief description of the computer programs will now be given. The computations were performed on an IBM 1401 digital computer using Fortran IV compiler language. Two programs were written. The purpose of the first program was to compare the median detector and the Neyman-Pearson (independent sample structure) detector. The second program compared all three detectors, but was limited to a maximum of twenty input observations. Inversion of the correlation matrix, whose size depended upon the number of observations, limited the second program to only twenty samples. However, this was sufficient to obtain valid comparisons.

Each of the programs was divided into two phases. The first phase of the first program (Program 101) calculated the efficacies and the ARE of the median and the Neyman-Pearson (independent sample structure) detectors. The input data specified the type of correlation function to be used, its parameters, and the number of observations. The second phase (Program 102) calculated β vs θ for a given α and correlation function. The input data specified the range of α and θ to be computed as well as the correlated function and its parameters, and the number of observations.

The first phase of the second program (Program 201) computed the efficacies and the ARE of all three detectors studied for a number of input samples not greater than twenty. Note that the variable name DN was used for $\emptyset_n$, the Neyman-Pearson test statistic designed assuming dependent samples. Part of the output of this phase was used as input for phase two (Program 202), which calculated β vs θ for a given α and correlation function. In each program the correlation function to be used was specified by a "computed go to" statement in the program. The format statements indicate the form of the input data.

Listings of the programs just described are given on the following pages. The corresponding equations have been given in Chapter VI.

PROGRAM 1 - PHASE 1 (FORTRAN IV)

EFFICACY AND ARE OF SN/LN DETECTORS

```
300     FORMAT(1H1,15X,33HARE OF MEDIAN DETECTOR/LIKELIHOOD,
       19H DETECTOR)
301     FORMAT(15X,33H(DEPENDENT SAMPLING - INDEPENDENT,
       111H STRUCTURE))
302     FORMAT(I2,F6.0,F6.2,F6.2,I4,F6.2,F6.2,I4)
303     FORMAT(1HL,5X,21HCORRELATION FUNCTION ,I2,12X,F6.0,
       18H SAMPLES)
304     FORMAT(1HT,5X,6HWSWO1=,F6.2,1X,5HDEL1=,F6.2,1X,3HM1=,
       1I3,5X,6HWODW1=,F5.2,2X,5HDEL2=,F5.2,2X,3HM2=,I3)
305     FORMAT(10X,5HWO/DW,5X,5HWS/WO,5X,8HEFFICACY,5X,
       18HEFFICACY,8X,3HARE)
306     FORMAT(30X,23H(MEDIAN)   (LIKELIHOOD))
307     FORMAT(10X,F6.2)
308     FORMAT(20X,F6.2,F12.5,F13.5,F13.5)
        WRITE(3,300)
        WRITE(3,301)
1       READ(1,302)JJ,AN,WSWO1,DEL1,M1,WODW1,DEL2,M2
        WRITE(3,303)JJ,AN
        WRITE(3,304)WSWO1,DEL1,M1,WODW1,DEL2,M2
        WRITE(3,305)
        WRITE(3,306)
        WODW=WODW1
        DO 200 I=1,M2
        WSWO=WSWO1
        WRITE(3,307)WODW
        DO 100 J=1,M1
        WSDW=WSWO*WODW
        JINX=1
        SUM1=0.0
        SUM2=0.0
        AK=1.0
9       TPK=6.2831852*AK
        GO TO(10,20,30,40),JJ
10      R=EXP(-TPK/WSWO)
        GO TO 50
20      R=EXP(-TPK/WSDW)*COS(TPK/WSWO)
        GO TO 50
30      R=SIN(TPK/WSDW)*COS(TPK/WSWO)*WSDW/TPK
        GO TO 50
40      R=EXP(-((3.1415926*AK/WSDW)**2))*COS(TPK/WSWO)
50      IF(ABS(R).GT..0001) GO TO 55
        IF(JINX.NE.1) GO TO 62
        JINX=2
        GO TO 60
55      RR=ATAN(R/SQRT(1.-R**2))
        SUM1=SUM1+R*(1.-AK/AN)
        SUM2=SUM2+RR*(1.-AK/AN)
```

```
      JINX=1
60    IF(AK.GT.AN) GO TO 62
      AK=AK+1.0
      GO TO 9
62    EFFSN=1./(1.5707952+2.0*SUM2)
      EFFLN=1./(1.0+2.0*SUM1)
      ARE=EFFSN/EFFLN
      WRITE(3,308)WSWO,EFFSN,EFFLN,ARE
100   WSWO=WSWO+DEL1
200   WODW=WODW+DEL2
      GO TO 1
      END
```

*   **   ****   **   *

PROGRAM 1 - PHASE 2 (FORTRAN IV)

CALCULATION OF ALPHA AND BETA FOR SN AND LN DETECTORS

```
600    FORMAT(1H1,22X,29HCALCULATION OF ALPHA AND BETA)
601    FORMAT(26X,23HFOR SN AND LN DETECTORS)
602    FORMAT(I2,F6.0,F6.1,F6.1,I3)
603    FORMAT(F6.1,F6.1,I3,F6.4,F6.4,I3,F6.4,F6.4,I3)
604    FORMAT(1HL,3X,21HCORRELATION FUNCTION ,I2,12X,F6.0,
      18H SAMPLES)
605    FORMAT(4X,6HWSWO1=,F5.1,6H DEL1=,F5.1,4H M1=,I3,8X,
      16HALPH1=,F6.4,7H  DEL3=,F6.4,5H  M3=,I3)
606    FORMAT(10HT   WODW1=,F5.1,6H DEL2=,F5.1,4H M2=,I3,8X,
      16HSNR1 =,F6.4,7H  DEL4=,F6.4,5H  M4=,I3)
607    FORMAT(4X,27HWO/DW   WS/WO     THRESHOLD,6X,
      113HALPHA     S/N,12X,4HBETA)
608    FORMAT(1HS,21X,10HSN       LN,25X,2HSN,9X,2HLN)
609    FORMAT(3X,F6.1,F8.1,F8.2,F8.2,F9.4)
610    FORMAT(45X,F6.3,3X,E12.4,E15.4)
611    FORMAT(3X,F6.1,F8.1,16H    NSQR    NSQR,F9.4,
      128H   GO TO NEXT VALUE OF ALPHA)
612    FORMAT(3X,F6.1,F8.1,F8.2,8H    NSQR,F9.4)
613    FORMAT(3X,F6.1,F8.1,8H    NSQR,F8.2,F9.4)
614    FORMAT(45X,F6.3,6X,4HXXXX,4X,E12.4)
615    FORMAT(45X,F6.3,6X,4HXXXX,7X,4HNSQR)
616    FORMAT(45X,F6.3,3X,E12.4,6X,4HXXXX)
617    FORMAT(45X,F6.3,3X,E12.4,6X,4HNSQR)
       WRITE(3,600)
       WRITE(3,601)
1      READ(1,602)JJ,AN,WSWO1,DEL1,M1
       READ(1,603)WODW1,DEL2,M2,ALPH1,DEL3,M3,SNR1,DEL4,M4
       WRITE(3,604)JJ,AN
       WRITE(3,605)WSWO1,DEL1,M1,ALPH1,DEL3,M3
       WRITE(3,606)WODW1,DEL2,M2,SNR1,DEL4,M4
       WRITE(3,607)
       WRITE(3,608)
       WODW=WODW1
       DO 500 I=1,M2
       WSWO=WSWO1
       DO 490 J=1,M1
       WSDW=WSWO*WODW
       SUM1=0.0
       SUM2=0.0
       JINX=1
       AK=1.0
9      TPK=6.2831852*AK
       GO TO(10,20,30,40),JJ
10     R=EXP(-TPK/WSWO)
       GO TO 50
20     R=EXP(-TPK/WSDW)*COS(TPK/WSWO)
       GO TO 50
```

```
30     R=SIN(TPK/WSDW)*COS(TPK/WSWO)*WSDW/TPK
       GO TO 50
40     R=EXP(-((3.1415926*AK/WSDW)**2))*COS(TPK/WSWO)
50     IF(ABS(R).GT..0001) GO TO 55
       IF(JINX.NE.1) GO TO 62
       JINX=2
       GO TO 60
55     RR=ATAN(R/SQRT(1.-R**2))
       SUM1=SUM1+R*(1.-AK/AN)
       SUM2=SUM2+RR*(1.-AK/AN)
       JINX=1
60     IF(AK.GT.AN) GO TO 62
       AK=AK+1.0
       GO TO 9
62     ALPH=ALPH1
       DO 480 K=1,M3
       IF(ALPH.NE..5) GO TO 75
       SA1=0.0
       SA2=0.5
       NC=1
       GO TO 100
75     IF(ALPH.GT..5) GO TO 80
       NA=1
       GO TO 82
80     NA=2
       ALPH=1.-ALPH
82     Z=SQRT(ALOG(1./ALPH**2))
       XQ=Z-(2.515517+Z*(.802853+.010328*Z))/(1.+Z*(1.432788+
      1Z*(.189269+Z*.001308)))
       GO TO(87,85),NA
85     ALPH=1.-ALPH
       XQ=-XQ
87     TEST1=1./AN+2.*SUM1/AN
       TEST2=0.25/AN+SUM2/(AN*3.1415926)
       IF(TEST1.GT.0.0) GO TO 91
       IF(TEST2.GE.0.0) GO TO 89
       WRITE(3,611)WODW,WSWO,ALPH
       GO TO 480
89     NC=3
       SA2=XQ*SQRT(TEST2)+0.5
       WRITE(3,612)WODW,WSWO,SA2,ALPH
       GO TO 101
91     IF(TEST2.GE.0.0) GO TO 93
       NC=2
       SA1=XQ*SQRT(TEST1)
       WRITE(3,613)WODW,WSWO,SA1,ALPH
       GO TO 101
93     SA1=XQ*SQRT(TEST1)
       SA2=XQ*SQRT(TEST2)+0.5
       NC=1
100    WRITE(3,609)WODW,WSWO,SA1,SA2,ALPH
101    SNR=SNR1
       DO 470 L=1,M4
       SUM3=0.0
```

```
      AK=1.0
      JINX=1
      JJJ=1
108   GO TO(109,161),JJJ
109   TPK=6.2831852*AK
      GO TO(110,120,130,140),JJ
110   R=EXP(-TPK/WSWO)
      GO TO 150
120   R=EXP(-TPK/WSDW)*COS(TPK/WSWO)
      GO TO 150
130   R=SIN(TPK/WSDW)*COS(TPK/WSWO)*WSDW/TPK
      GO TO 150
140   R=EXP(-((3.1415926*AK/WSDW)**2))*COS(TPK/WSWO)
150   IF(ABS(R).GT..01) GO TO 160
      IF(JINX.NE.1) GO TO 153
      JINX=2
      GO TO 162
153   C5=1.+EXP(-.5*SNR**2)
      JJJ=2
      GO TO 161
160   C5=(1.+EXP(-.5*SNR**2/(1.-R**2)))/SQRT(1.-R**2)
161   SUM3=SUM3+(1.-AK/AN)*C5
      JINX=1
162   IF(AK.GT.AN) GO TO 171
      AK=AK+1.0
      GO TO 108
171   C7=.25/AN-SNR**2/6.2831852+(1.-1./AN)*SNR/2.506629
      TEST3=C7+(SUM2-SNR*SUM3)/(3.1415926*AN)
      GO TO(175,180,185),NC
175   IF(TEST3.LE.0.0) GO TO 177
      NE=5
      GO TO 190
177   X2=0.0
      NE=4
      GO TO 190
180   X2=0.0
      NE=3
      GO TO 190
185   X1=0.0
      IF(TEST3.GT.0.0) GO TO 187
      X2=0.0
      NE=2
      GO TO 200
187   NE=1
      GO TO 191
190   X1=(SA1-SNR)/SQRT(TEST1)
      IF(NE.NE.5) GO TO 200
191   X2=(SA2-.5-SNR/2.506629)/SQRT(TEST2)
200   JJJJ=1
      X=X1/1.414214
275   IF(X.GE.0.0) GO TO 277
      NB=1
      X=-X
      GO TO 278
```

```
277    NB=2
278    C8=1.+X*(.070523078+X*(.042282012+X*.0092705272))
       C9=X**4*(.0001520143+X*(.0002765672+X*.0000430638))
       GO TO(280,281),NB
280    BETA=.5-(1.-1./(C8+C9)**16)/2.0
       GO TO 282
281    BETA=.5+(1.-1./(C8+C9)**16)/2.0
282    GO TO(283,284),JJJJ
283    JJJJ=2
       BETA1=BETA
       X=X2/1.414214
       GO TO 275
284    BETA2=BETA
       GO TO(290,291,292,293,294),NE
290    WRITE(3,614)SNR,BETA2
       GO TO 470
291    WRITE(3,615)SNR
       GO TO 470
292    WRITE(3,616)SNR,BETA1
       GO TO 470
293    WRITE(3,617)SNR,BETA1
       GO TO 470
294    WRITE(3,610)SNR,BETA2,BETA1
470    SNR=SNR+DEL4
480    ALPH=ALPH+DEL3
490    WSWO=WSWO+DEL1
500    WODW=WODW+DEL2
       GO TO 1
       END
```

*   **   ****   **   *

PROGRAM 2 - PHASE 1 (FORTRAN IV)

EFFICACY AND ARE OF SN/LN/DN DETECTORS

```
      DIMENSION Q(20,20),R(20,20)
500   FORMAT(1H1,25X,35HEFFICACY AND ARE OF MEDIAN DETECTOR,
     120H/LIKELIHOOD DETECTOR)
501   FORMAT(24X,36H(DEPENDENT SAMPLING - DEPENDENT AND ,
     123HINDEPENDENT STRUCTURES))
502   FORMAT(I2,F6.0,F6.1,F6.1,I3,F6.1,F6.1,I3)
503   FORMAT(1HL,22X,21HCORRELATION FUNCTION ,I2,11X,F6.0
     18H SAMPLES)
504   FORMAT(23X,6HWSWO1=,F5.1,7H  DEL1=,F5.1,5H  M1=,I3,5X,
     16HWODW1=,F5.1,7H  DEL2=,F5.1,5H  M2=,I3)
505   FORMAT(1HL,6X,14HWO/DW    WS/WO,7X,13HEFFICACY     ,
     121HEFFICACY     EFFICACY,9X,3HARE,10X,3HARE,10X,3HARE)
506   FORMAT(30X,4H(SN),9X,4H(LN),9X,4H(DN),9X,7H(SN/LN),6X,
     17H(SN/DN),6X,7H(LN/DN))
507   FORMAT(6X,F6.1)
508   FORMAT(15X,F6.1,5X,F10.4,3X,F10.4,3X,F10.4,3X,F10.4,
     13X,F10.4,3X,F10.4)
509   FORMAT(I2,F6.0,F6.1,F6.1,F16.8,F16.8,F16.8)
      WRITE(3,500)
      WRITE(3,501)
1     READ(1,502)JJ,AN,WSWO1,DEL1,M1,WODW1,DEL2,M2
      WRITE(3,503)JJ,AN
      WRITE(3,504)WSWO1,DEL1,M1,WODW1,DEL2,M2
      WRITE(3,505)
      WRITE(3,506)
      WODW=WODW1
      N=AN
      DO 490 II=1,M2
      WSWO=WSWO1
      WRITE(3,507)WODW
      DO 480 KK=1,M1
      WSDW=WSWO*WODW
      DO 10 I=1,N
      DO 10 J=1,N
      IF(I.EQ.J) GO TO 5
      Q(I,J)=0.0
      GO TO 10
5     Q(I,J)=1.0
10    CONTINUE
      DO 60 I=1,N
      DO 60 J=1,N
      IF(I.NE.J) GO TO 11
      R(I,J)=1.0
      GO TO 60
11    IF(I.GT.J) GO TO 12
      AK=J-I
      GO TO 14
```

```
12      AK=I-J
14      TPK=6.2831852*AK
        GO TO(21,22,23,24),JJ
21      R(I,J)=EXP(-TPK/WSWO)
        GO TO 60
22      R(I,J)=EXP(-TPK/WSDW)*COS(TPK/WSWO)
        GO TO 60
23      R(I,J)=SIN(TPK/WSDW)*COS(TPK/WSWO)*WSDW/TPK
        GO TO 60
24      R(I,J)=EXP(-((3.1415926*AK/WSDW)**2))*COS(TPK/WSWO)
60      CONTINUE
        M=1
        DO 85 L=1,N
        M=M+1
        IF(M.GT.N) GO TO 75
        DO 70 I=M,N
70      R(L,I)=R(L,I)/R(L,L)
75      DO 80 J=1,N
80      Q(L,J)=Q(L,J)/R(L,L)
        DO 85 J=1,N
        IF(L.EQ.J) GO TO 85
        IF(M.GT.N) GO TO 84
        DO 83 I=M,N
83      R(J,I)=R(J,I)-R(L,I)*R(J,L)
84      DO 85 LL=1,N
        Q(J,LL)=Q(J,LL)-Q(L,LL)*R(J,L)
85      CONTINUE
        SUMQ=0.0
        DO 100 I=1,N
        DO 100 J=1,N
100     SUMQ=SUMQ+Q(I,J)
        CONTINUE
        JINX=1
        SUM1=0.0
        SUM2=0.0
        AK=1.0
109     TPK=6.2831852*AK
        GO TO(110,120,130,140),JJ
110     RR=EXP(-TPK/WSWO)
        GO TO 150
120     RR=EXP(-TPK/WSDW)*COS(TPK/WSWO)
        GO TO 150
130     RR=SIN(TPK/WSDW)*COS(TPK/WSWO)*WSDW/TPK
        GO TO 150
140     RR=EXP(-((3.1415926*AK/WSDW)**2))*COS(TPK/WSWO)
150     IF(ABS(RR).GT..0001) GO TO 155
        IF(JINX.NE.1) GO TO 162
        JINX=2
        GO TO 160
155     RRR=ATAN(RR/SQRT(1.-RR**2))
        SUM1=SUM1+RR*(1.-AK/AN)
        SUM2=SUM2+RRR*(1.-AK/AN)
        JINX=1
160     IF(AK.GT.AN) GO TO 162
```

```
      AK=AK+1.0
      GO TO 109
162   EFFSN=1./(1.5707952+2.0*SUM2)
      EFFLN=1./(1.0+2.0*SUM1)
      EFFDN=SUMQ/AN
      ARESL=EFFSN/EFFLN
      ARESD=EFFSN/EFFDN
      ARELD=EFFLN/EFFDN
      WRITE(2,509)JJ,AN,WSWO,WODW,SUM1,SUM2,SUMQ
      WRITE(3,508)WSWO,EFFSN,EFFLN,EFFDN,ARESL,ARESD,ARELD
480   WSWO=WSWO+DEL1
490   WODW=WODW+DEL2
      GO TO 1
      END
```

*   **   ****   **   *

PROGRAM 2 - PHASE 2 (FORTRAN IV)

CALCULATION OF ALPHA AND BETA FOR SN, LN, AND DN DETECTORS

```
500    FORMAT(1H2,25X,33HCALCULATION OF ALPHA AND BETA FOR,
      116H THREE DETECTORS)
501    FORMAT(12X,32H(MEDIAN, LIKELIHOOD-INDEPENDENT ,
      146HSTRUCTURE, AND LIKELIHOOD-DEPENDENT STRUCTURE))
502    FORMAT(I2,F6.0,F6.1,F6.1,F16.8,F16.8,F16.8)
503    FORMAT(F6.4,F6.4,I3,F6.3,F6.3,I3)
504    FORMAT(1HL,20X,21HCORRELATION FUNCTION ,I2,11X,F6.0,
      18H SAMPLES)
505    FORMAT(1H ,20X,6HWSWO =,F6.1,25X,6HALPH1=,F6.4,
      18H    DEL3=,F6.4,6H   M3=,I3)
506    FORMAT(1HT,20X,6HWODW =,F6.1,25X,6HSNR1 =,F6.3,
      18H    DEL4=,F6.3,6H   M4=,I3)
507    FORMAT(21X,9HTHRESHOLD,13X,5HALPHA,8X,3HS/N,20X,
      17HB E T A)
508    FORMAT(1H/,15X,2HSN,7X,2HLN,7X,2HDN,31X,2HSN,11X,2HLN,
      111X,2HDN)
509    FORMAT(12X,F7.2,F9.2,F9.2,5X,F6.4,5X,F6.3,F15.8,F13.8,
      1F13.8)
       WRITE(3,500)
       WRITE(3,501)
1      READ(1,502)JJ,AN,WSWO,WODW,SUM1,SUM2,SUMQ
       READ(1,503)ALPH1,DEL3,M3,SNR1,DEL4,M4
       WRITE(3,504)JJ,AN
       WRITE(3,505)WSWO,ALPH1,DEL3,M3
       WRITE(3,506)WODW,SNR1,DEL4,M4
       WRITE(3,507)
       WRITE(3,508)
       WSDW=WSWO*WODW
       ALPH=ALPH1
       DO 480 I=1,M3
       SNR=SNR1
       DO 470 J=1,M4
       IF(ALPH.NE..5) GO TO 5
       SA1=0.0
       SA2=0.5
       SA3=0.0
       GO TO 99
5      IF(ALPH.GT..5) GO TO 8
       NA=1
       GO TO 10
8      NA=2
       ALPH=1.-ALPH
10     Z=SQRT(ALOG(1./ALPH**2))
       XQ=Z-(2.515517+Z*(.802853+.010328*Z))/(1.+Z*(1.432788+
      1Z*(.189269+Z*.001308)))
       GO TO(25,20),NA
20     ALPH=1.-ALPH
```

```
      XQ=-XQ
25    TEST1=1./AN+2.*SUM1/AN
      TEST2=0.25/AN+SUM2/(AN*3.1415926)
      TEST3=SNR**2*SUMQ
      NW=0
      NX=0
      NY=0
      NZ=0
      IF(TEST1.GT.0.0) GO TO 30
      TEST1=1.0
      NX=1
30    IF(TEST2.GT.0.0) GO TO 35
      TEST2=1.0
      NY=1
35    IF(TEST3.GT.0.0) GO TO 40
      TEST3=1.0
      NZ=1
40    SA1=XQ*SQRT(TEST1)
      SA2=XQ*SQRT(TEST2)+0.5
      SA3=XQ*SQRT(TEST3)
99    SUM3=0.0
      AK=1.0
      JINX=1
      JJJ=1
108   GO TO(109,161),JJJ
109   TPK=6.2831852*AK
      GO TO(110,120,130,140),JJ
110   R=EXP(-TPK/WSWO)
      GO TO 150
120   R=EXP(-TPK/WSDW)*COS(TPK/WSWO)
      GO TO 150
130   R=SIN(TPK/WSDW)*COS(TPK/WSWO)*WSDW/TPK
      GO TO 150
140   R=EXP(-((3.1415926*AK/WSDW)**2))*COS(TPK/WSWO)
150   IF(ABS(R).GT..01) GO TO 160
      IF(JINX.NE.1) GO TO 153
      JINX=2
      GO TO 162
153   C5=1.+EXP(-.5*SNR**2)
      JJJ=2
      GO TO 161
160   C5=(1.+EXP(-.5*SNR**2/(1.-R**2)))/SQRT(1.-R**2)
161   SUM3=SUM3+(1.-AK/AN)*C5
      JINX=1
162   IF(AK.GT.AN) GO TO 171
      AK=AK+1.0
      GO TO 108
171   C7=.25/AN-SNR**2/6.2831852+(1.-1./AN)*SNR/2.506629
      TEST4=C7+(SUM2-SNR*SUM3)/(3.1415926*AN)
      IF(TEST4.GT.0.0)GO TO 173
      TEST4=1.0
      NW=1
173   X1=(SA1-SNR)/SQRT(TEST1)
      X2=(SA2-.5-SNR/2.506629)/SQRT(TEST2)
```

```
      X3=(SA3-SUMQ*SNR**2)/SQRT(TEST3)
200   JJJJ=1
      X=X1/1.414214
275   IF(X.GE.0.0) GO TO 277
      NB=1
      X=-X
      GO TO 278
277   NB=2
278   C8=1.+X*(.070523078+X*(.042282012+X*.0092705272))
      C9=X**4*(.0001520143+X*(.0002765672+X*.0000430638))
      GO TO(280,281),NB
280   BETA=.5-(1.-1./(C8+C9)**16)/2.0
      GO TO 282
281   BETA=.5+(1.-1./(C8+C9)**16)/2.0
282   GO TO(283,284,285),JJJJ
283   JJJJ=2
      BETA1=BETA
      X=X2/1.414214
      GO TO 275
284   JJJJ=3
      BETA2=BETA
      X=X3/1.414214
      GO TO 275
285   BETA3=BETA
      IF(NX.EQ.0) GO TO 300
      SA1=999.99
      BETA1=99.9999
300   IF(NY.EQ.0) GO TO 301
      SA2=999.99
      BETA2=99.9999
301   IF(NZ.EQ.0) GO TO 302
      SA3=999.99
      BETA3=99.9999
302   IF(NW.EQ.0) GO TO 303
      BETA2=99.9999
303   WRITE(3,509)SA2,SA1,SA3,ALPH,SNR,BETA2,BETA1,BETA3
470   SNR=SNR+DEL4
480   ALPH=ALPH+DEL3
      GO TO 1
      END
```

*   **   ****   **   *

VITA

Gene Lee Armstrong

Candidate for the Degree of

Doctor of Philosophy

Thesis: THE EFFECT OF DEPENDENT SAMPLING ON THE PERFORMANCE OF NONPARAMETRIC COINCIDENCE DETECTORS

Major Field: Electrical Engineering

Biographical:

Personal Data: Born in Oklahoma City, Oklahoma, March 23, 1939, the son of Floyd L. and Delma L. Armstrong.

Education: Attended grade school in Midwest City, Oklahoma; attended high schools in Del City, Oklahoma, Amarillo, Texas, and Bartlesville, Oklahoma; graduated from College High School, Bartlesville, Oklahoma in 1957; received the Bachelor of Science degree in Electrical Engineering at Oklahoma State University in May, 1961; received the Master of Science degree in Electrical Engineering at Oklahoma State University in August, 1962; completed the requirements for the Doctor of Philosophy degree in August, 1965.

Experience: Employed part-time as radio-TV technician by Matthews TV-Appliance, Bartlesville, Oklahoma, from 1956 to 1960; employed as a student engineer by Southwestern Bell Telephone Company during the summer of 1960; employed as an assistant engineer by Labko Scientific, Stillwater, Oklahoma, during the summer of 1961; employed as a graduate research assistant (half-time) by the School of Electrical Engineering of OSU from September, 1961 to June, 1962; employed from September, 1962 to June, 1964 as an instructor (quarter-time) by the School of Electrical Engineering, Oklahoma State University.

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