A THEORETICAL STUDY OF SOLITONS AND THEIR IMPLICATIONS

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1973

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
December, 1980

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AND THEIR IMPLICATIONS

Thesis Approved:

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To

Schroeder

PREFACE

In this study I have attempted to demonstrate the important role solitons can play in a physical system. While the concept of soliton is quite recent, it promises to provide a general framework from which to understand nonlinear phenomena in field theories.

I have tried at every stage to present the theory of solitons in the simplest possible terms, keeping the physical ideas to the fore as much as possible. The aim throughout has been to produce a work that can be understood by anyone acquainted with the basics of classical field theory and quantum mechanics. More specialized ideas, like Bäcklund transformations, are developed in the text. At the risk of becoming verbose, I have tried to supply as many calculational details as seemed practical.

I would like to take this opportunity to express my appreciation to Dr. N. V. V. J. Swamy for serving as my major adviser. I am also grateful to Dr. James Lange, Dr. Larry Scott, and Dr. Robert Mulholland for serving on my committee.

I would especially like to acknowledge many helpful discussions on various physical and philosophical matters with my friends and colleagues Wayne Vinson, David Gordon, and Deuard Worthen. All of whom are excellent physicists. I am very grateful to Mrs. Janet Sallee for typing the final copy with her usual excellence and patience. In addition, I should acknowledge considerable financial support in the form of research and teaching assistantships which the people of Oklahoma have made possible through their hard-earned tax dollars.

The completion of this work on my part would never have been possible without the love, support, and encouragement of my parents, Carl and Jimmie Lou Wilson. And finally, I am most indebted to Schroeder for her constant encouragement, love, and understanding which was unwavering throughout the course of this work.

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CHAPTER I

INTRODUCTION

During the last decade the theory of solitons has rapidly emerged as a new unifying concept in pure and applied physics.[1] While the formalized notion of a soliton is quite recent,[2] its origins date back to the early days of hydrodynamics and the study of certain nonlinear waves with remarkable stability properties.[3-4] Since that time, the investigation of a relatively minor curiosity of nonlinear waves has expanded to the point where more than a hundred different soliton systems are known and almost every area of physics has groups involved in their study.[5]

For our purposes, a soliton may be tentatively defined as a stable, finite energy solution to a nonlinear wave equation. A soliton can be roughly thought of as a solitary wave similar in appearance to a traveling wave pulse or wave except that it does not disperse.[6] A more explicitly formulated definition of a soliton will be given later in Chapter II.

It is the primary goal of this work to investigate what effects, if any, the presence of solitons in a physical system can produce. To do this requires a consideration of how the present theory of solitons (which is almost entirely limited to one spatial dimension) extends to the more realistic two and three dimensional cases.

The subject of three dimensional solitons is intimately connected with the seemingly unrelated subjects of magnetic charge[7] and the non-Abelian gauge theories of elementary particles.[8] Goddard and Olive[9] have shown that in order to generalize solitons to a 4-dimensional space-time, one is required to introduce long-range gauge fields. Prior to this Polyakov and 't Hooft had derived extended particle solutions bearing magnetic charge from the field equations of a non-Abelian gauge theory.[7] The 't Hooft monopole, as this solution is now called, can be shown to be a three dimensional soliton.

This unexpected connection between solitons and magnetic charge, coupled with the recent phenomenological successes of the Weinberg-Salam gauge theory,[10] has resulted in the serious consideration of the existence of magnetic charge. In this regard, the effects of such magnetic monopole/solitons on the propagation of electromagnetic waves naturally suggests itself as an area of fundamental importance. This point is investigated classically in the present study as one of the more striking implications of the theory of solitons and leads to a simple test that can in principle detect the presence of magnetically charged particles in the interstellar plasma.

For various reasons there is cause to believe a close relation exists between magnetic charge-type solitons and the Sine-Gordon equation.[10] An investigation of this possibility is developed in the present study. The most promising place to attempt such a relation appears to be in the area of superconductive tunneling and the Josephson junction.[11,12] It is easily shown that the propagation of magnetic flux vortices on a large area Josephson junction is described by the Sine-Gordon equation, the solitons in this case corresponding to the quantized flux vortices. These flux vortices can be thought of as the two dimensional analogues of magnetic monopoles. Viewed in this way the effect of the flux tube/solitons on the propagation of electromagnetic waves in the Josephson junction is considered. Again the presence of solitons is found to produce non-trivial results.

The Sine-Gordon equation itself is one of the most frequently occurring nonlinear equations having soliton solutions and has figured prominently in the emergence of the theory of solitons as well as in the development of nonperturbative solution methods such as the Bäcklund transformation technique.[13] In many respects the Bäcklund transformation equations of the Sine-Gordon equation are suggestive of a Dirac factorization. An investigation of this interpretation concludes the present study.

FOOTNOTES

[1]An excellent early review is that of A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE 61, 1443 (1973).

[2]N. J. Zabrusky and M. D. Kruskal, Phys. Rev. Lett. 15, 240 (1965).

[3]J. Scott-Russell, Proc. Roy. Soc. Edinburg 1, 319 (1844).

[4]D. J. Korteweg and G. deVries, Philos. Mag. 39, 422 (1895).

[5]To obtain an idea of the wide range of soliton physics see the recent Proceedings of the Conference on the Theory of Solitons, ed. by H. Flaska and D. W. McLaughlin, Rocky Mountain J. Math. 8, 1 (1978).

[6]It should be pointed out that several different definitions of soliton occur in the literature. This is source of much confusion and it is matter of some debate as to which definition is best and will ultimately come into general use. The definition used here is the most widely accepted and was originally advocated by T. D. Lee, Phys. Rep. C23, 254 (1974).

[7]For a recent review see P. Goddard and D. I. Olive, Rep. Prog. Phys. 41, 1358 (1978).

[8]A good general review of the non-Abelian gauge theories is that of E. Abers and B. W. Lee, Phys. Rep. C9, 5 (1973).

[9]P. Goddard and D. I. Olive, p. 1379.

[10]Ibid., p. 1375.

[11]L. Solymar, Superconductive Tunneling and Applications (Wiley, New York, 1972).

[12]B. D. Josephson, Adv. Phys. 14, 419 (1965).

[13]See, for instance, Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications, ed. by R. M. Miura (Springer-Verlag, New York, 1976).

CHAPTER II

SOLITONS AND THE SINE-GORDON EQUATION IN 1+1 DIMENSIONS

Since there is some lack of uniformity in the literature as to exactly what is meant by a soliton, it is useful for us to first give a definition of the term as it will be used here. This is perhaps best done within the context of a particular example and for this purpose the Sine-Gordon theory will be used. A discussion in 1+1 dimensions (one space dimension and time) will adequately illustrate all the ideas basic to solitons and simultaneously serve as an introduction to the Sine-Gordon equation. Besides being easier to handle mathematically, the lower dimensional case is of great current interest and has applications in its own right. It is also hoped, of course, that results obtained in 1+1 dimensions will indicate what to expect in higher dimensions and serve as a guide for explorations there.

Definition of Soliton

The term "soliton" was originally introduced by Zabrusky and Kruskal[1] to describe a class of nonlinear waves they observed in their study of nonlinear plasma oscillations. For them, solitons were pulse-like traveling waves that upon scattering from each other emerged with their initial shapes and velocities. Zabrusky and Kruskal were so taken by this remarkable stability that they gave these waves the new name, soliton, to emphasize their particle-like behavior.

This original definition of Zabrusky and Kruskal is still used, particularly in the United States among plasma and solid state physicists. It is this definition that is used by Scott, Chu, and McLaughlin[2] in their oft-cited review article on solitons. In the Soviet literature, on the other hand, the term soliton is rather loosely employed almost synonomously with what Zabrusky and Kruskal, and Scott, et al. would term a solitary wave.[3]

It is now generally held that the first of these usages is too restrictive while the latter is too expansive. An intermediate definition was proposed by T. D. Lee[4] and is the one given here. For us, a soliton is a solution $\phi_S(\vec{x},t)$ to the nonlinear field equation(s) $N[\phi] = 0$ having energy $E[\phi_S]$ such that

(1) The solution ϕ_S has finite, nonzero energy; that is,

$$0 < E[\phi_S] < \infty, \text{ and}$$

(2) The solution ϕ_S is localized at all times to a finite region of space; that is,

$$\phi_S(\vec{x},t) \rightarrow \text{vacuum as } |\vec{x}| \rightarrow \infty$$

for all t.

This definition of soliton is currently the most widely used but is in no sense uniformly accepted. Some typical wave profiles of solitons are shown in Figure 1 which also serves to define the various types of solitons. A soliton may be stationary or translating rigidly with constant velocity. A moving soliton is depicted in Figure 2.

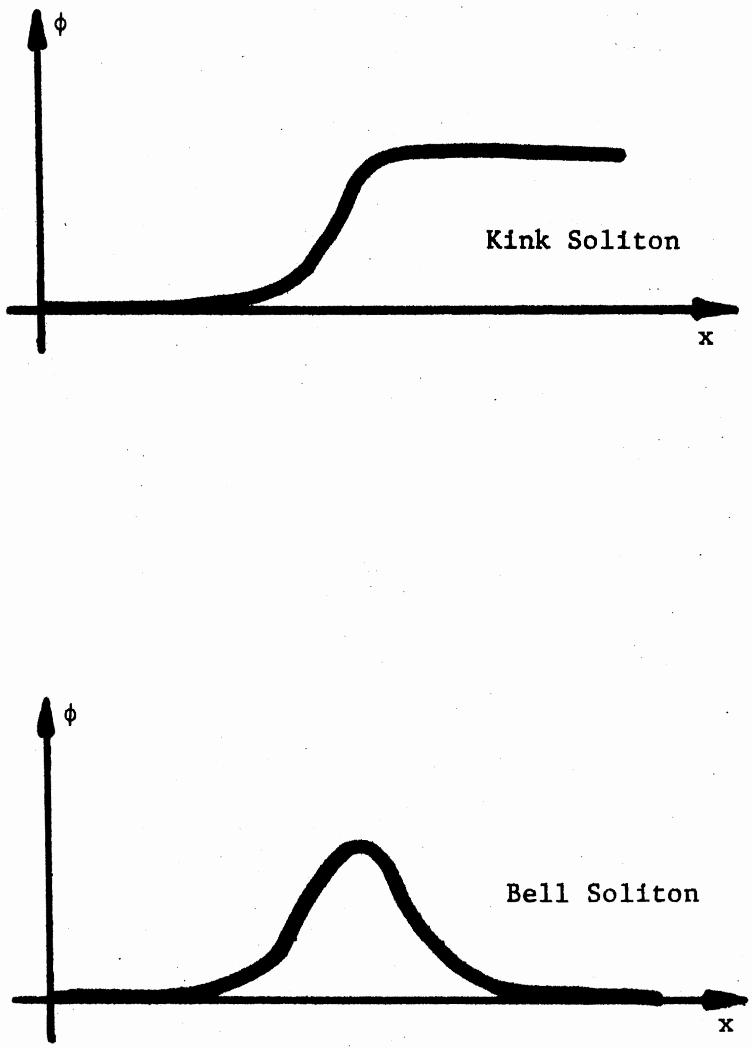

Figure 1. Wave Profiles of Different Soliton Types

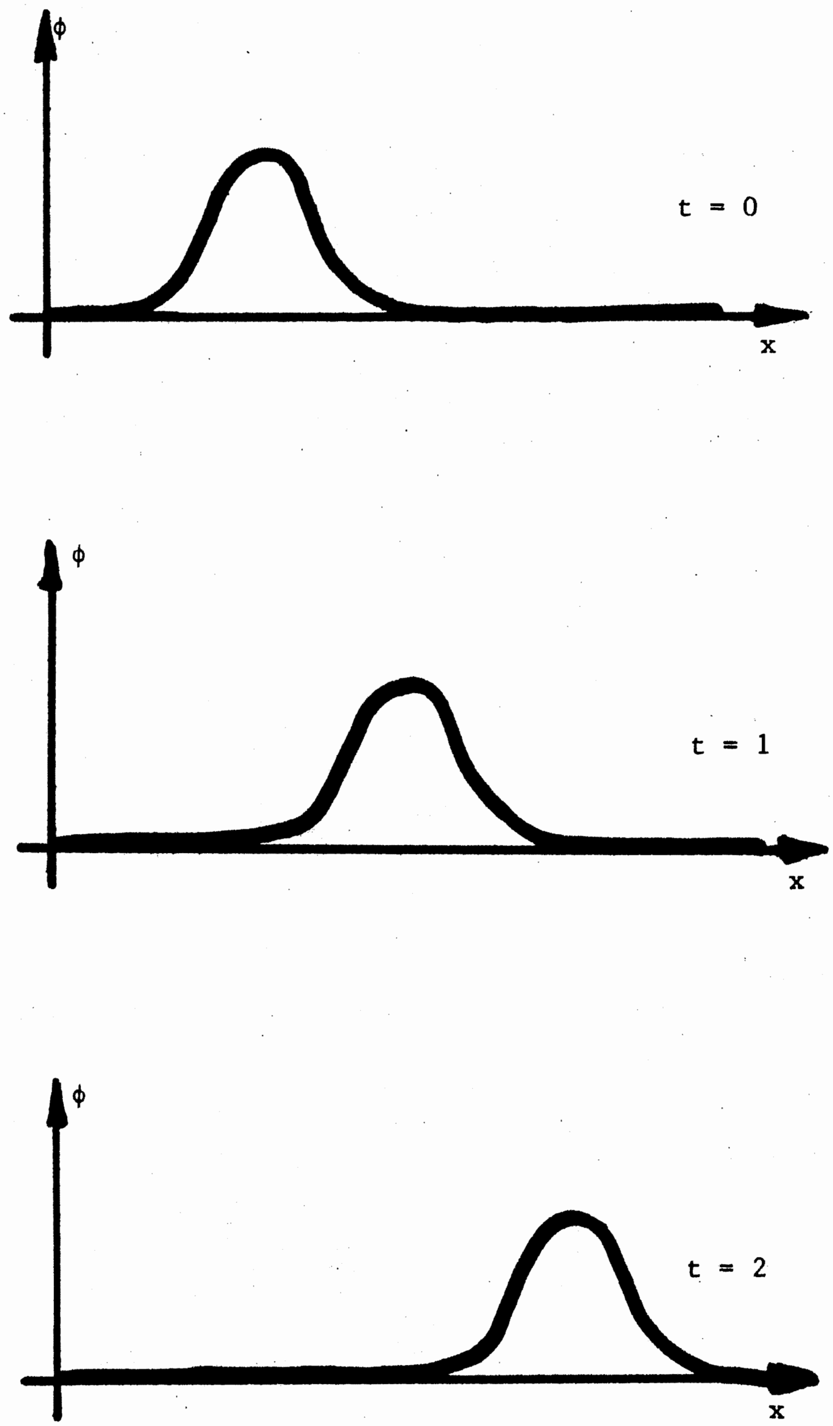

Figure 2. Wave Profiles of a Moving Soliton

Qualitatively, one can think of solitons as being produced by a balance of two effects: (1) Dispersion, which tends to make the solitary wave spread as it propagates, and (2) Nonlinearity, which tends to make the propagating wave pinch and collapse on itself.[2] The combination of these two effects produces a remarkably stable wave with particle-like properties. Some of the particle-like properties of solitons include: the ability of solitons to carry a type of "charge", the existence of antisoliton solutions, the existence of scattering and bound state solutions for pairs of solitons, and the ability to have soliton-antisoliton pair creation (and annihilation).[2,5]

It should be emphasized from the start that the concept of soliton is a purely classical one and has nothing a priori to do with the process of second quantization. The similarity of solitons with such entities as photons, phonons, magnons, exitons, etc., is therefore a similarity in name only. The analogy with these concepts is at times, however, quite suggestive.

The Sine-Gordon Equation

The Sine-Gordon equation,[6]

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{m^3}{g}\sin\left(\frac{g}{m}\phi\right) = 0, \qquad (2\text{-}1)$$

$$\phi = \phi(x,t), \quad m,g \text{ constants},$$

was one of the first systems studied in connection with the theory of solitons. Equation (2-1) arises in many different physical contexts and is known to have a wide range of applications. A large body of literature exists on this system and can be traced from several excellent reviews.[2,7,8] Some of the Sine-Gordon equation's more important properties, particularly those that illustrate the concept of soliton, will be recounted here.

Applications

One of the most striking features of the Sine-Gordon equation, aside from its admitting soliton solutions, is its widespread occurrence in nonlinear physics. Indeed, there seems to be almost no end to the number of physical systems which are known to be reducible to the Sine-Gordon equation. To name but a few, the Sine-Gordon equation has been found to describe (1) Coulomb plasmas,[9] (2) Propagation of crystal defects,[2,7] (3) Bloch wall motion in magnetic crystals,[2,7] (4) Propagation of splay waves along a lipid bio-membrane,[2] (5) Propagation of magnetic flux on a Josephson junction,[2,7,10,11] (6) Elementary particles,[2,7,12,13] and (7) Propagation of ultra-short optical pulses through resonant 2-level laser media.[2,7,14]

Whenever an equation arises with this sort of frequency in such a variety of different physical contexts there is usually some underlying mathematical reason. The Sine-Gordon equation is no exception and arises in pure mathematics in the study of the differential geometry of pseudospherical surfaces (surfaces of constant negative curvature).[15]

It should be noted that the Sine-Gordon equation is closely related to the Klein-Gordon equation as its name would indicate. This can be seen by linearizing Eq. (2-1). For small values of $\frac{g}{m}\phi$, the sine term may be expanded and (2-1) becomes

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{m^3}{g}\left[\frac{g}{m}\phi - \frac{1}{3!}\left(\frac{g}{m}\phi\right)^3 + \cdots\right] = 0$$

Dropping higher order terms yields

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + m^2\phi = \frac{g^2}{6}\phi^3 \qquad (2\text{-}2)$$

which is the Klein-Gordon equation with the nonlinear self-interaction $\frac{g^2}{6}\phi^3$. The linearized equation obtained by dropping the righthand side of (2-2) is thus nothing other than the usual Klein-Gordon equation in 1+1 dimensions.

Soliton Solutions

We can easily obtain nonperturbative, traveling wave solutions of the Sine-Gordon equation by looking for solutions of the form of a traveling wave,

$$\phi = \phi_T(\xi), \qquad \xi = x-vt \qquad (2\text{-}3)$$

where v is a constant, real parameter. It will turn out that v specifies the velocity of the soliton. Using (2-3), the Sine-Gordon equation (2-1) reduces to the form of the simple pendulum equation.

$$\frac{d^2\phi_T}{d\xi^2} = \frac{m^3/g}{1-v^2}\sin\left(\frac{g}{m}\phi\right) . \qquad (2\text{-}4)$$

The solutions to (2-4) are well known and can be written using elliptic functions. For the particular choice of boundary condition.

$$\frac{m}{g}\phi_T \to 0 \ (\text{Mod } 2\pi) \text{ as } |x| \to \infty,$$

the solutions may be expressed in terms of elementary functions. Only two such solutions exist (up to an arbitrary displacement of the origin), namely,

$$\phi_S = \frac{4m}{g} \tan^{-1} [\exp(m \frac{x-vt}{\sqrt{1-v^2}})] \qquad (2\text{-}5)$$

and

$$\phi_{AS} = \frac{4m}{g} \tan^{-1} [\exp(-m \frac{x-vt}{\sqrt{1-v^2}})] \ . \qquad (2\text{-}6)$$

These two solutions correspond respectively to a kink soliton and a kink antisoliton moving with velocity v. They are sketched in Figure 3. It should be noted that both of these soliton solutions are singular in the coupling constant g and therefore cannot be reached by standard perturbation theory.

<u>Symmetries</u>

It is important to observe that the Sine-Gordon equation

$$\frac{\partial^2}{\partial t^2} \phi - \frac{\partial^2 \phi}{\partial x^2} + \frac{m^3}{g} \sin(\frac{g}{m} \phi) = 0$$

is invariant under the Lorentz transformations

$$x \rightarrow x' = \frac{x-\beta t}{\sqrt{1-\beta^2}} \qquad (2\text{-}7a)$$

$$t \rightarrow t' = \frac{t-\beta x}{\sqrt{1-\beta^2}} , \qquad \beta = V/c, \qquad (2\text{-}7b)$$

provided ϕ transforms as a scalar, i.e.,

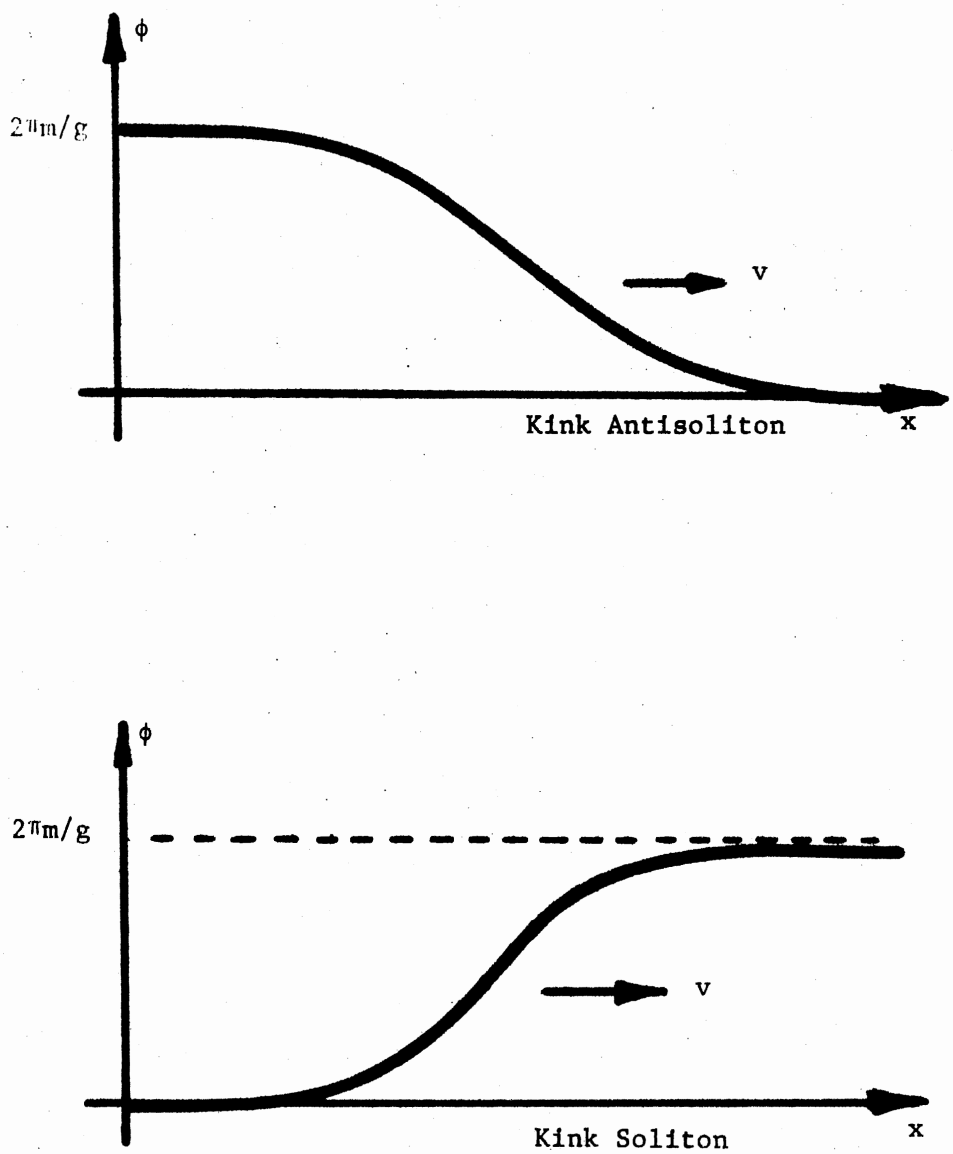

Figure 3. Sketch of the Fundamental Soliton Solutions

$$\phi \rightarrow \phi' = \phi . \qquad (2\text{-}7c)$$

The Sine-Gordon theory is also invariant under the internal symmetries,

$$\phi \rightarrow \phi' = \pm \phi + 2\pi n(\frac{m}{g}) ,$$
$$n = 0, \pm 1, \pm 2, \ldots . \qquad (2\text{-}8)$$

This symmetry is closely related to the fact that the Sine-Gordon equation admits antisoliton as well as soliton solutions. By inspection of the soliton solutions (2-5), (2-6) it is easily seen that the transformation

$$\phi \rightarrow \phi' = - \phi + 2\pi(m/g) \qquad (2\text{-}9)$$

transforms solitons into antisolitons and vice versa. In this respect the transformation (2-9) is analogous to charge conjugation in field theory.

In addition to these rather obvious invariance groups, the Sine-Gordon equation has a hidden SO(2,1) symmetry.[16] The physical interpretation of this symmetry group is not totally understood, but it is thought to be related to the existence of an infinite number of conservation laws for the Sine-Gordon theory.[17]

Variational Formulation

The Sine-Gordon equation can be derived from the Lagrangian density,[18]

$$L[\phi] = L(\phi,\partial_\mu\phi) = \frac{1}{2}(\partial_\mu\phi)(\partial_\mu\phi) - \frac{m^4}{g^2}(1 - \cos\frac{g}{m}\phi)$$

$$= \frac{1}{2}[(\frac{\partial\phi}{\partial t})^2 - (\frac{\partial\phi}{\partial x})^2] - \frac{m^4}{g^2}(1 - \cos\frac{g}{m}\phi) \quad . \qquad (2\text{-}10)$$

Use of (2-10) in the Euler-Lagrange equation,

$$\partial_\mu(\frac{\partial L}{\partial(\partial_\mu\phi)} - \frac{\partial L}{\partial\phi} = 0, \qquad (2\text{-}11)$$

yields Eq. (2-1).

From $L[\phi]$ we may obtain the Hamiltonian density by making the usual Legendre transformation,

$$H[\phi] = \pi\frac{\partial\phi}{\partial t} - L[\phi] \qquad (2\text{-}12a)$$

where

$$\pi \equiv \frac{\partial L}{\partial(\frac{\partial\phi}{\partial t})} = \frac{\partial\phi}{\partial t} \qquad (2\text{-}12b)$$

is the canonical momentum conjugate to ϕ. Doing this we obtain

$$H[\phi] = H(\phi,\partial_\mu\phi) = \frac{1}{2}[(\frac{\partial\phi}{\partial t})^2 + (\frac{\partial\phi}{\partial s})^2] + \frac{m^4}{g^2}(1 - \cos\frac{g}{m}\phi), \qquad (2\text{-}13)$$

which may be interpreted as the energy density of the Sine-Gordon field. The Hamiltonian density is clearly the sum of two distinct terms,

$$T[\phi] \equiv \frac{1}{2}[(\frac{\partial\phi}{\partial t})^2 + (\frac{\partial\phi}{\partial x})^2] \qquad (2\text{-}14)$$

which is often called the kinetic energy-stress density and

$$V[\phi] \equiv \frac{m^4}{g^2}(1 - \cos\frac{g}{m}\phi) \qquad (2\text{-}15)$$

which is the potential energy density.

The total energy is given by the integral

$$E[\phi] \equiv \int_{-\infty}^{\infty} H[\phi]\, dx \qquad (2\text{-}16)$$

and may be considered to be a functional of ϕ. An inspection of (2-13) shows that

$$H[\phi] \geq 0 \qquad (2\text{-}17)$$

which implies

$$E[\phi] \geq 0 \qquad (2\text{-}18)$$

for all ϕ as one would expect for the total energy.

Soliton Mass

Using (2-13) and (2-16) the energy density and total energy can be calculated for any field configuration ϕ. In particular, for the fundamental soliton solution ϕ_S given by (2-5) one finds the energy density

$$H[\phi_S] = \frac{1}{2}[(\frac{\partial\phi_S}{\partial t})^2 + (\frac{\partial\phi_S}{\partial x})^2] + \frac{m^4}{g^2}(1 - \cos\frac{g}{m}\phi_S)$$

$$= \frac{4m^4}{g^2(1-v^2)}\,\mathrm{sech}^2(m\,\frac{x-vt}{\sqrt{1-v^2}}) \qquad (2\text{-}19)$$

which is plotted in Figure 4. As can be seen, the energy is localized to the region of the kink.

The total energy is then given by

$$E[\phi_S] = \int_{-\infty}^{\infty} H[\phi_S]dx$$

$$= \frac{4m^4}{g^2(1-v^2)} \int_{-\infty}^{\infty} \mathrm{sech}^2(m \frac{x-vt}{\sqrt{1-v^2}})\,dx \qquad (2\text{-}20)$$

$$= \frac{8m^3/g^2}{\sqrt{1-v^2}}$$

This last result is often written as an effective soliton mass (since $E=Mc^2$, in $\hbar=c=1$ units, $E=M$).

$$M(v) = \frac{M_o}{\sqrt{1-v^2}} , \qquad M_o = (\frac{8m^2}{g^2})m . \qquad (2\text{-}21)$$

This displays the correct relativistic factor as it should. Furthermore, if one interprets m as the bare soliton mass, m_{BARE}, then the mass renormalization due to the self-interaction with coupling g is given by the second of equations (2-21),

$$M_{PHYSICAL} = (\frac{8m^2_{BARE}}{g^2})m_{BARE} . \qquad (2\text{-}22)$$

It should be noted that the mass renormalization factor is singular in the coupling constant g.

The radius of the soliton is usually defined from the argument of (2-19) as

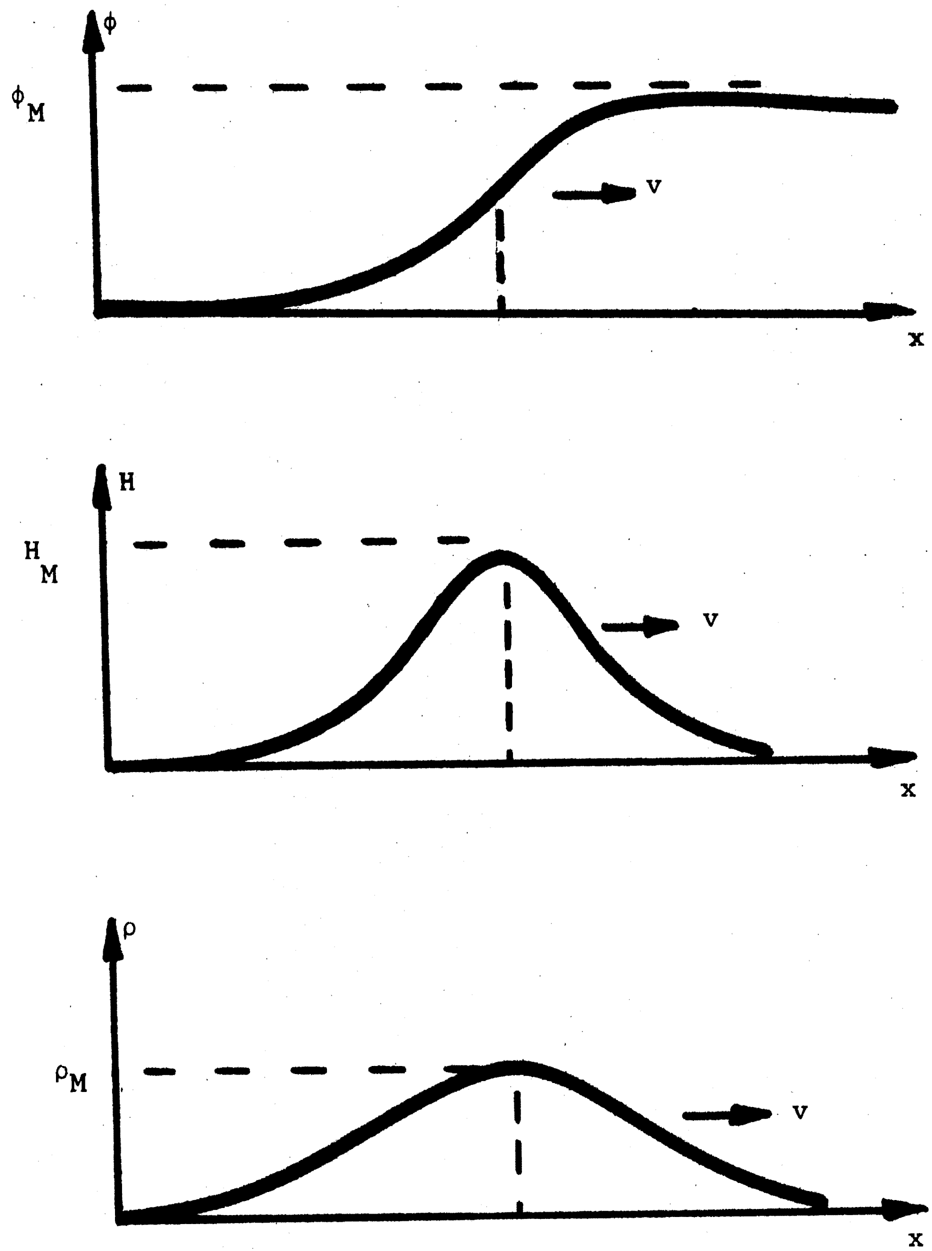

Figure 4. Energy and Charge Density of the Kink Soliton

$$R(v) = R_o \sqrt{1-v^2} \tag{2-23}$$

where

$$R_o = R(o) = \frac{1}{m} \tag{2-24}$$

Outside this radius, the soliton field ϕ_s is essentially negligible. Again, if we interpret v as the velocity of the soliton, then (2-23) displays the correct relativistic factor. This Lorentz contraction is, of course, a consequence of the overall relativistic invariance of the Sine-Gordon theory.

The Sine-Gordon Vacuum

The Sine-Gordon equation possesses an interesting vacuum with non-trivial structure. In linear field theories the vacuum state is unique and one usually scales things so that $\phi_{vacuum} = 0$. In a nonlinear theory, however, this may not be the case and there may, in fact be more than one lowest energy state. This is what happens for the Sine-Gordon equation which has such a multiple vacuum. It turns out that it is just this multiplicity of the vacuum that allows the existence of solitons in the Sine-Gordon theory.

In general, the vacuum state ϕ_o is the solution to the field equation(s) $N[\phi] = 0$ having the lowest energy. Since for physical theories $E[\phi] \geq 0$, this implies $E[\phi_o] = 0$ for the vacuum. It is also customary to require the vacuum to be uniform and isotropic, i.e., $\partial_\mu \phi_o = 0$. From this definition of the vacuum and Eqs. (2-13) and (2-16), the Sine-Gordon vacuum is given by ϕ_{VAC} where

$$E[\phi_{VAC}] = \int_{-\infty}^{\infty} H[\phi_{VAC}]\, dx$$

or

$$\frac{m^4}{g^2} \int_{-\infty}^{\infty} (1 - \cos \frac{g}{m}\, \phi_{VAC})\; dx = 0 \;. \tag{2-25}$$

Eq. (2-25) can hold only if

$$1 - \cos(\frac{g}{m}\, \phi_{VAC}) = 0$$

which implies

$$\phi_{VAC} = 2\pi n\, (\frac{m}{g}),$$

$$n = 0,\ \pm 1,\ \pm 2,\ \ldots \;. \tag{2-26}$$

Thus, the Sine-Gordon equation has a countably infinite number of states corresponding to the vacuum. This degeneracy of vacuum results in a most interesting property which is discussed next.

<u>Topological Charge</u>

It can be seen from Figure 3 that the soliton solution ϕ_S approaches two different vacuum states $\phi_{VAC} = 0$ and $\phi'_{VAC} = \frac{2\pi m}{g}$ as $|x| \to \pm\, \infty$. The soliton solution thus interpolates between two vacua. Associated with this fact is the existence of a conserved current

$$J^{\mu} = \frac{1}{2\pi} \frac{g^2}{m}\, \varepsilon^{\mu\nu}\, \partial_{\nu}\phi \tag{2-27}$$

where $\varepsilon^{\mu\nu}$ is the antisymmetric Levi-Civita tensor in two dimensions,

$\epsilon^{00} = \epsilon^{11} = 0$, $\epsilon^{01} = -\epsilon^{10} = 1$. From this definition it follows that J^μ is conserved since

$$\partial_\mu J^\mu = \partial_\mu \left(\frac{1}{2\pi}\frac{g^2}{2m}\epsilon^{\mu\nu}\partial_\nu\phi\right)$$

$$= \frac{1}{2\pi}\frac{g^2}{m}\epsilon^{\mu\nu}\partial_\mu\partial_\nu\phi$$

$$= 0 \,. \qquad (2\text{-}28)$$

From this conservation law arises a conserved topological charge,[19]

$$Q = \int_{-\infty}^{\infty} \rho \, dx \qquad (2\text{-}29)$$

where

$$\rho = J^0 = \frac{g^2}{2\pi m}\frac{\partial\phi}{\partial x} \,. \qquad (2\text{-}30)$$

It is easily shown that the topological charge of any soliton is "quantized". From (2-29) and (2-30)

$$Q = \frac{g^2}{2\pi m}\int_{-\infty}^{\infty}\frac{\partial\phi}{\partial x}\,dx$$

$$= \frac{g^2}{2\pi m}\left[\phi(x = +\infty) - \phi(x = -\infty)\right] .$$

But from our definition of soliton given in the previous section, any soliton solution must approach a vacuum state as $|x| \to \infty$. Thus, from (2-26) this requires

$$Q = \frac{g^2}{2\pi m}\left[2\pi\, n_1\left(\frac{m}{g}\right) - 2\pi\, n_2\left(\frac{m}{g}\right)\right],$$

$$n_1,\, n_2 = 0,\, \pm 1,\, \pm 2,\, \ldots \qquad (2\text{-}31)$$

or

$$Q = \pm\, g\ (\text{Integer}) \qquad (2\text{-}32)$$

and Q is quantized.

Inspection of Figure 3 shows using (2-31) that the charge of the soliton is +g while that of the antisoliton is -g. Using (2-30) and the soliton solution (2-5) gives the charge density

$$\rho = \frac{g}{4\pi\ R(v)}\ \text{sech}\left(m\,\frac{x-vt}{\sqrt{1-v^2}}\right) \qquad (2\text{-}32)$$

which is shown in Figure 4. Again the charge is localized to the area immediately surrounding the kink.

Before closing out this brief discussion of topological charge, several points should be noted:

(1) Topological charges, unlike the more familiar charges of physics, need not be associated with a symmetry of the Lagrangian. Instead, they arise from the non-trivial topology of the manifold of vacuum states of the theory.

(2) It is the conservation of topological charge that assures the stability of the Sine-Gordon solitons.

(3) Since the total topological charge (2-29) is conserved by (2-28), the difference between the number of solitons and the number of antisolitons must be conserved in any physical process. Solitons and

antisolitons must therefore be created and destroyed in pairs.

Multiple Soliton Solutions

Besides the fundamental soliton solutions (2-5) and (2-6), other exact analytical solutions to the Sine-Gordon equation are known. For completeness, some of the more important multiple soliton solutions are given here.

The solution

$$\phi_{SS} = \frac{4m}{g} \tan^{-1}\left[v \frac{\sinh\left(\frac{mvx}{\sqrt{1-v^2}}\right)}{\cosh\left(\frac{mvt}{\sqrt{1-v^2}}\right)} \right] \tag{2-33}$$

can be shown to represent soliton-soliton scattering in the center of mass frame.[8] A similar form,

$$\phi_{SA} = \frac{4m}{g} \tan^{-1}\left[\frac{\sinh\left(\frac{mvt}{\sqrt{1-v^2}}\right)}{v \cosh\left(\frac{mx}{\sqrt{1-v^2}}\right)} \right] \tag{2-34}$$

corresponds to soliton-antisoliton scattering.[8]

From the soliton-antisoliton scattering solution one can calculate the scattering phase shift to be of the form,[5]

$$\delta = \frac{\sqrt{1-v^2}}{v} \ln v \tag{2-35}$$

which is always negative. (The term ln v is always negative because v is measured in units c=1, hence $v = \frac{v}{c} < 1$ always.) The negative phase shift implies the forces between a soliton and antisoliton are and one would suspect a soliton-antisoliton bound state might be possible. Such is indeed the case and an analytic expression for this is known to be[5]

$$\phi_B = \frac{4m}{g} \tan^{-1}\left[\varepsilon \frac{\sin\left(\frac{mt}{\tau}\right)}{\cosh\left(\frac{\varepsilon mt}{\tau}\right)}\right], \qquad (2\text{-}36)$$

$$\varepsilon \equiv \sqrt{\tau^2 - 1}\,, \qquad \tau = \frac{T}{2\pi}$$

This corresponds to a soliton-antisoliton bound state with a relative separation oscillating in time with period

$$T = \frac{2\pi m\sqrt{1 + v^2}}{v}\,.$$

Solutions of the form (2-36) are also occasionally called breathers or bions in the literature.

No soliton-soliton bound states have yet been found. This is consistent with the idea of soliton charge being +g and the existence of a repulsive force between like charges.

In addition to these 2-soliton solutions, exact N-soliton solutions have been found for the Sine-Gordon equation in 1+1 dimensions.[20] As might be expected, these solutions are quite complicated and will not be reproduced here.

Quantization

All of the preceding results are purely classical in content. Even the condition (2-32) on the allowed charges of a soliton is a classical "quantization" condition arising from the topology of the Sine-Gordon vacuum. One can, however, in the usual fashion make the Sine-Gordon theory into a totally consistent (second) quantized field theory. The quantized Sine-Gordon field is not of much relevance to the present study and is mentioned here only in passing. Nevertheless, considerable work has been done on such a quantum theory of solitons.[5,21,22] These investigations into a quantum theory of solitons are motivated by the belief of some that the strongly interacting hadrons may be solitons.[23]

Solitons and Particles

It should be emphasized that while solitons have many properties reminiscent of particles, they are in fact not particles but nonlinear waves. And while it is an interesting conjecture that all elementary particles are solitons of some as yet undiscovered field theory, this is but one (rather minor) application of the theory of solitons. Indeed, the bulk of the theory of solitons is done in the areas of plasma and solid state physics.

It should, perhaps, also be stressed that all of the unusual soliton-related properties of the Sine-Gordon equation apply, albeit with varying physical interpretations, to all of the applications mentioned earlier. For example, the Sine-Gordon equation describes the propagation of magnetic flux on a Josephson line and in this case the solitons are the flux vortices. The condition (2-32) then turns out to be the well known flux quantization condition. The flux vortices can move, scatter from one another, or form bound states as given by Eqs. (2-33)-(2-36).

Derrick's Theorem

A rather obvious generalization of Eq. (2-1) is to increase the number of spatial dimensions to obtain

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + \frac{m^3}{g}\sin\left(\frac{g}{m}\phi\right) = 0, \tag{2-37}$$

which is the 2+1 and 3+1 dimension Sine-Gordon equation for

$$\phi = \phi(x,y,t), \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and

$$\phi = \phi(x,y,z,t), \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

respectively. One might think that since (2-1) has static soliton solutions then its related 2+1 and 3+1 dimensional equations would too. However, this is not the case and, in fact, no stable t-independent solutions exist for Eq. (2-37). The reason for this is essentially embodied in Derrick's theorem.[24,25]

<u>Derrick's Theorem</u>: No stable, t-independent finite energy solutions exist to <u>any</u> scalar wave equation with Lagrangian density of the form

$$L = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x^1}\right)^2 - \left(\frac{\partial\phi}{\partial x^2}\right)^2 - \cdots - \left(\frac{\partial\phi}{\partial x^N}\right)^2 - U(\phi), \tag{2-38}$$

$$U \geq 0$$

for $N \geq 2$, regardless of the specific form of the potential energy $U(\phi)$.

In particular, (2-37) falls into the class of nonlinear theories covered by Derrick's theorem since for it

$$L = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - (\nabla\phi)^2\right] - \frac{m^4}{g^2}\left(1 - \cos\frac{g}{m}\phi\right) .$$

A simplified proof of Derrick's theorem is given in Appendix A. The proof is interesting not only because it is a central result of the theory of solitons, but also because it illustrates how the stability soliton solutions may be investigated.

While Derrick's theorem may at first appear to rule out the existence of solitons in higher dimensions, a more careful inspection shows that this is not the case. There are at least three possible ways around the theorem. Specifically, the theorem does not rule out the existence of (1) time dependent soliton solutions, (2) soliton solutions if other fields, e.g., the electromagnetic field, are present, and (3) multicomponent, i.e., spinor soliton solutions.

The second of these possibilities is illustrated in Chapter III where it is shown that three dimensional solitons can arise provided there is also a magnetic field present so that the soliton represents a magnetic monopole. Some initial investigations into the third possibility are presented in Chapters VII and VIII.

FOOTNOTES

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CHAPTER III

MAGNETIC MONOPOLES AS SOLITONS

One of the basic precepts of present-day electromagnetic theory is that magnetic charges do not exist. Nearly 50 years ago, however, P. A. M. Dirac[1] showed by a brilliant theoretical argument that the existence in the universe of a single magnetic monopole would explain the observed quantized nature of electric charge (be it e or e/3!). Since the discrete quantization of electric charge in multiples of e(1.6 x 10^{-19} coulombs) is a totally unexplained experimental observation, Dirac's argument made the existence of magnetic charge quite appealing on theoretical grounds.

In addition to explaining the observed quantization of electric charge, the existence of magnetic charge would cast the equations of electrodynamics in a form exhibiting complete symmetry between electric and magnetic quantities. In view of the success of symmetry arguments in other areas of physics, it is somewhat surprising that magnetic charges have never been found.

After the pioneering work of Dirac, very little was done on the subject of magnetic charge for many years. Recently, however, there has been a renewed interest in the theory of magnetic monopoles. The reason for this upsurge of interest can be traced to two events. One was the possible experimental observation in cosmic rays of what appeared to be a magnetically charged particle;[2] the second was the discovery that soliton solutions in spontaneously broken non-Abelian gauge theories exhibit magnetic monopole properties.[3-5]

The theory of magnetic monopoles has been the subject of several excellent review articles,[5-7] but for completeness some of the more important aspects of the theory will be recounted here. After this review we will briefly indicate how magnetic monopoles arise as solitons in a non-Abelian gauge theory of elementary particles. The primary purposes of the present chapter, then, are to briefly review the classical theory of magnetic charge, indicate its connection with the theory of solitons, and thereby motivate the investigations discussed in Chapter IV.

Review of Magnetic Charge

The Generalized Maxwell's Equations

The usual equations of electromagnetism exhibiting the absence of magnetic charge are (in CGS-Gaussian units)

$$\nabla\cdot\vec{E} = 4\pi\rho_e, \qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}_e + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$
$$\nabla\cdot\vec{B} = 0, \qquad \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{3-1}$$

with the corresponding Lorentz force density being

$$\vec{F} = \rho_e\vec{E} + \frac{\vec{J}_e}{c}\times\vec{B}\,. \tag{3-2}$$

If we allow for the possible existence of magnetic charge, Maxwell's equations become

$$\nabla\cdot\vec{E} = 4\pi\,\rho_e, \qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}_e + \frac{1}{c}\frac{\partial\vec{E}}{\partial t},$$

$$\nabla\cdot\vec{B} = 4\pi\,\rho_g, \qquad \nabla\times\vec{E} = -\frac{4\pi}{c}\vec{J}_g - \frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \tag{3-3}$$

where ρ_g and $\vec{J}_g$ denote the magnetic charge and current densities respectively. In this case the generalized Lorentz force density is given by

$$\vec{F} = \rho_e\vec{E} + \frac{\vec{J}_e}{c}\times\vec{B} + \rho_g\vec{B} - \frac{\vec{J}_g}{c}\times\vec{E}\,. \tag{3-4}$$

The negative sign of the magnetic current density $\vec{J}_g$ is required in Eq. (3-3) to be consistent with the continuity equation

$$\frac{\partial\rho_g}{\partial t} + \nabla\cdot\vec{J}_g = 0 \tag{3-5}$$

expressing the conservation of magnetic charge. This can be seen by taking the divergence of the equation

$$\nabla\times\vec{E} = -\frac{4\pi}{c}\vec{J}_g - \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

which gives

$$0 = -\frac{4\pi}{c}\nabla\cdot\vec{J}_g - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec{B})$$

since $\nabla\cdot(\nabla\times\vec{E}) = 0$. Using $\nabla\cdot\vec{B} = 4\pi\,\rho_g$, the above equation becomes Eq. (3-5).

The generalized Lorentz force density (3-4) is obtained by performing a Lorentz transformation from the frame of reference where the charge densities are at rest to a frame where they are moving with a velocity $\vec{v}$,

$$\vec{F} = \rho_e \vec{E} + \rho_g \vec{B} \rightarrow \vec{F}' = \rho_e' (\vec{E}' + \frac{\vec{v}}{c} \times \vec{B}') + \rho_g' (\vec{B}' - \frac{\vec{v}}{c} \times \vec{E}')$$

and then identifying $\vec{J}_e' = \rho_e'\vec{v}$, $\vec{J}_g' = \rho_g'\vec{v}$.

Duality Symmetry

It is a straightforward calculation to show that the generalized Maxwell's equations are invariant under the internal U(1) symmetry group given by the transformations

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \rightarrow \begin{pmatrix} \vec{E}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix},$$

$$\begin{pmatrix} \rho_e \\ \rho_g \end{pmatrix} \rightarrow \begin{pmatrix} \rho_e' \\ \rho_g' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \rho_e \\ \rho_g \end{pmatrix}, \tag{3-6}$$

$$\begin{pmatrix} \vec{J}_e \\ \vec{J}_g \end{pmatrix} \rightarrow \begin{pmatrix} \vec{J}_e' \\ \vec{J}_g' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{J}_e \\ \vec{J}_g \end{pmatrix}$$

The transformation of Eqs. (3-6) is usually called a duality transformation and the generalized Maxwell's equations are said to be duality invariant. The duality transformation (3-6) also leaves invariant the generalized Lorentz force law (3-4), the electromagnetic energy density $\frac{\vec{E}^2 + \vec{B}^2}{8\pi}$, the Poynting vector $\vec{S} \equiv \frac{c}{4\pi} \vec{E}\times\vec{B}$, as well as the Maxwell stress tensor

$$\overleftrightarrow{T} \equiv \frac{1}{4\pi} (\vec{E}\vec{E} + \vec{B}\vec{B}) - \overleftrightarrow{1} (\frac{\vec{E}^2 + \vec{B}^2}{8\pi}) .$$

Charge Quantization

From quantum mechanical considerations Dirac[1] was able to show that the existence of magnetic charge would lead to the quantization of electric charge. Using arguments essentially concerning the single-valuedness of the wave function of an electron in the presence of a magnetic monopole, Dirac found the quantization condition

$$eg = \frac{n}{2} \hbar c, \quad n = 0, \pm 1, \pm 2, \ldots , \tag{3-7}$$

where e,g are the electric and magnetic charge and $\hbar$ and c are Planck's constant and the speed of light respectively. A simple derivation of Eq. (3-7) is given in Appendix B. The discrete quantization of electric charge thus follows from the existence of a magnetic monopole.

Schwinger[8] has argued that a proper rotationally invariant derivation leads to the somewhat different (by a factor of two) quantization condition,

$$eg = n \hbar c, \quad n = 0, \pm 1, \pm 2, \ldots . \tag{3-8}$$

The Dirac quantization condition, however, is the most widely accepted.

We may summarize all this by noting that no inconsistencies are known to arise in a carefully constructed quantum mechanics with electric and magnetic charges present provided the quantization condition (3-7), or possibly (3-8), holds.

The Monopole Coupling Constant

Dirac's law of reciprocal electric and magnetic charge quantization can also explain why no monopoles have yet been observed (with the possible exception noted previously). From (3-7) we have for n=1.

$$g = \frac{1}{2}\left(\frac{\hbar c}{e}\right) = \frac{1}{2}\left(\frac{\hbar c}{e^2}\right)e \ . \qquad (3\text{-}9)$$

But $\frac{e^2}{\hbar c} \equiv \alpha \simeq \frac{1}{137} << 1$ is the well known electric fine structure constant which measures the strength of the coupling between two electrons. So using the known value of α we can write (3-9) as

$$g \simeq \frac{137}{2} e$$

which shows the minimum magnetic charge is 137/2 times the electronic charge. Alternatively, we can square Eq. (3-7) to obtain (for n=1)

$$e^2 g^2 = \frac{\hbar^2 c^2}{4}$$

which gives the magnetic fine structure constant

$$\frac{g^2}{\hbar c} = \frac{1}{4}\left(\frac{\hbar c}{e^2}\right) \simeq \frac{137}{4} >> 1 \ . \qquad (3\text{-}10)$$

From (3-10) we see that the coupling between magnetic charges is quite strong, making it difficult to separate out opposite charges from what is ordinarily magnetically neutral matter. This would account for absence of magnetic charges except at very high energies and is in sharp contrast to the electric case where electric charges are easily separated since $\alpha = \frac{e^2}{\hbar c} << 1$.

In this connection, it should be pointed out that with $\frac{g^2}{\hbar c} >> 1$ the usual calculation methods of quantum field theory fail completely in the case of magnetic charges, since these methods rely on a perturbation expansion in the coupling $\frac{g^2}{\hbar c}$. These expansions work for electric charges where $\frac{e^2}{\hbar c} << 1$ and the perturbation expansion is allowed, but for magnetic charges the series expansion would not be valid.

Dually Charged Particles

Dirac considered only the case of particles carrying either electric or magnetic charge but not both. There is nothing in the formalism, however, that prevents the latter case and in general a single particle could possibly carry both magnetic and electric charge. Schwinger[8], in fact, has developed this generalization rather extensively and calls such dually charged particles dyons.

Role of Magnetic Charge in Physics

With the existence of magnetic charge still open to question, their possible role in physics is a matter of some speculation. The existence of magnetic charge would, however, resolve several problems that have plagued particle physics for a number of years. In particular, the existence of magnetic charge would provide an explanation of the violation of CP invariance in particle physics.[8,9] Moreover, the extreme strength of the magnetic coupling and the increasing evidence in favor of some kind of quark theory (which also must have an extremely strong coupling) has led some to suggest that quarks may be nothing more than magnetically charged particles.

Along this line, Schwinger[8,10] has proposed that quarks are spin-½ dyons. The advantage of such an identification is that it supplies a physical realization of the quark model. The problem of quark "confinement" follows quite naturally from the magnetic charge quantization condition. The unknown charge of the quark theory, which the particle physicists have given the name 'color', appears automatically in Schwinger's theory though endowed with a physically significant name--magnetic charge. Once one identifies color with magnetic charge, Schwinger's theory is identical to the standard quark model.

More recently, extended particle solutions bearing magnetic charge have been found for a large class of non-Abelian gauge theories.[3-5] As the gauge theories of elementary particles were based on entirely different physical grounds, it is quite intriguing that they predict the existence of magnetically charged particles. What all this means, of course, remains to be seen and depends on how well the gauge theories bear up to experimental tests. To date at least, some of them have been quite successful. A common feature of all these proposals is that magnetically charged particles are quite massive having a mass

$$M \simeq \left(\frac{g^2}{\hbar c}\right) M_W$$

where M_W is the mass of a typical vector boson. Since $M_W \simeq 60$ GeV we see that for a monopole, $M \simeq 10^5$ GeV. This is much more massive than currently available accelerator energies (~ 20-40 GeV) and explains why such particles have not yet been found in accelerator experiments.

Magnetic Monopole Solitons in Non-Abelian Gauge Theories

To see how extended magnetic monopoles arise as solitons in unified gauge theories, we briefly summarize the work of 't Hooft[3] and Polyakov.[4] Following 't Hooft, we consider the non-Abelian SO(3) gauge theory with the Lagrangian density[11]

$$L = -\frac{1}{8\pi} G^{a}_{\mu\nu} G^{\mu\nu}_{a} + \frac{1}{2} D_{\mu}\phi_{a} D^{\mu}\phi^{a} + \frac{1}{2}\mu^{2}\phi_{a}\phi^{a} - \frac{\lambda}{8}(\phi_{a}\phi^{a})^{2} \qquad \text{(3-11a)}$$

where

$$G^{a}_{\mu\nu} \equiv \partial_{\mu} W^{a}_{\nu} - \partial_{\nu} W^{a}_{\mu} + e\,\varepsilon_{abc}\, W^{b}_{\mu} W^{c}_{\nu} \qquad \text{(3-11b)}$$

and

$$D_{\mu}\phi^{a} = \partial_{\mu}\phi^{a} + e\,\varepsilon_{abc}\, W^{b}_{\mu}\phi^{c} \qquad \text{(3-11c)}$$

is the gauge "covariant" derivative. This Lagrangian describes the interaction of the gauge field $W^{a}_{\mu}(X^{\mu})$ and a scalar Higg's field $\phi^{a}(X^{\mu})$ and when fermion fields are added becomes the Georgi-Glashow model of weak interactions.[12]

The classical Euler-Lagrange field equations that follow from (3-11) are

$$D^{\mu} G^{a}_{\mu\nu} = -4\pi e\,\varepsilon_{abc}\,\phi^{b} D_{\nu}\phi^{c} \qquad \text{(3-12a)}$$

$$D_{\mu} D^{\mu}\phi^{a} = -\frac{\lambda}{4}\left(\phi^{b}\phi_{b} - \frac{4\mu^{2}}{\lambda}\right)\phi^{a} . \qquad \text{(3-12b)}$$

The existence of soliton solutions to this set of coupled nonlinear equations was demonstrated by 't Hooft[3] and Polyakov[4] independently. The solution they found is

$$\phi^a = \frac{-x^a H(r)}{r}, \quad r = (x_1{}^2 + x_2{}^2 + x_3{}^2)^{1/2} \tag{3-13a}$$

$$A_o^a = 0, \quad A_i^a = \frac{\varepsilon_{aij}\, x_j\, [1-K(r)]}{e\, r^2} \tag{3-13b}$$

where H(r), K(r) satisfy the equations

$$r^2 \frac{d^2K}{dr^2} = K(K^2-1) + KH^2 \tag{3-14a}$$

$$r^2 \frac{d^2H}{dr^2} = 2H\,K^2 + \frac{\lambda}{4e^2}\,(H^2 - \frac{4e^2\mu^2}{\lambda}\,r^2)H\,. \tag{3-14b}$$

These last equations can be integrated numerically and show the solution (3-13) to be a soliton located at the origin.[3]

The energy or mass of this soliton can likewise be computed numerically and is found to be $M \simeq (\frac{\hbar c}{e^2})\, M_W$ where $M_W = \frac{2\mu}{\lambda} \simeq 60$ GeV is the mass of the intermediate vector boson in this model.

The electromagnetic field tensor is shown by 't Hooft to be given in this theory by

$$F_{\mu\nu} = \hat{\phi}_a\, G^a_{\mu\nu} - \frac{1}{e}\,\varepsilon_{abc}\, \hat{\phi}^a \mathcal{D}_\mu \phi^b \mathcal{D}_\nu \phi^c,$$
$$\hat{\phi}^a \equiv \frac{\phi^a}{(\phi_b\phi^b)^{1/2}}, \tag{3-15}$$

and insertion of the ansatz (3-13) into (3-15) gives

$$F_{\mu o} = F_{o\nu} = 0, \quad F_{ij} = \frac{\varepsilon_{ijk}\, x^k}{e\, r^3} \tag{3-16}$$

which corresponds to the magnetic field $\vec{B} = \frac{1}{e}\frac{\vec{r}}{r^3}$ of a point charge $g = \frac{1}{e}$. The soliton solution of 't Hooft and Polyakov therefore corresponds to a magnetically charged soliton.

Since the initial work of 't Hooft and Polyakov, a variety of magnetic monopole solitons have also been found in higher gauge groups. In addition, the methods can be easily generalized to allow dyon solutions. Goddard and Olive[5] give an excellent review of these ideas as well as an extensive bibliography of the original sources.

The discovery of soliton-like magnetic monopoles and dyons in non-Abelian gauge theories, which are now considered to be the only viable approach to elementary particle theory, has prompted the serious consideration of the physical consequences of the existence of magnetically charged particles. In this regard, the effects of the presence of magnetic charges on the propagation of electromagnetic radiation naturally suggests itself as an area of fundamental importance and will be discussed in the next chapter.

FOOTNOTES

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[11] The group indices a,b,c range over the values 1,2,3 while the space-time indices $\mu,\nu = 0,1,2,3$, $x^\mu = (x^o, \vec{x}) = (t,x,y,z)$. The summation convention is used for both group and space-time indices. The Minkowski space signature is (1,-1,-1,-1), while the group signature is (1,1,1), so that $x_\mu = (x^o, -\vec{x})$ and $\phi_a = \phi^a$. The coupling constant is denoted by e and ε^{abc} is the usual totally anti-symmetric Levi-Civita symbol with $\varepsilon^{123} = +1$. We have also used the notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$ for the derivatives.

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CHAPTER IV

PLASMAS WITH MAGNETICALLY CHARGED PARTICLES

Due to their large mass and coupling, it would appear that if magnetically charged particles are to exist at all it will only be under extreme physical conditions such as those produced experimentally in high energy accelerators or as occur naturally in astrophysical plasmas. The first of these situations has been rather extensively explored and the considerable literature on the subject may be traced from the references at the end of Chapter III. The second alternative, perhaps because of its cross-disciplinary nature, has not been so thoroughly investigated.

In this chapter we attempt to at least partially remedy this deficiency and look at how some basic plasma physics results are modified if the plasma particles are allowed to carry magnetic as well as electric charge. In particular, the usual plasma wave dispersion relations are found to be significantly modified with new modes appearing if magnetic charges are present. Looking for these effects would be a way of indirectly detecting magnetic charges in plasmas. These modified dispersion relations occur in either the fluid or the kinetic theory description of the plasma which will be discussed in turn.

The Fluid Description

The treatment of plasmas as multi-component, interpenetrating charged fluids is a standard approach used in plasma physics. While the fluid approximation is a rather crude model, it is found adequate to account for most plasma phenomena. The fluid treatment of standard plasmas of electrically charged particles is well known.[1-5] The approach used here closely parallels that of Tanenbaum[5].

Basic Equations

If we allow for the possibility of magnetic charge, the basic fluid description plasma becomes

1) The fluid equations of motion for each charged fluid generalized to include the Lorentz force due to magnetic sources

$$m_\alpha N_\alpha \left[\frac{\partial \vec{V}_\alpha}{\partial t} + (\vec{V}_\alpha \cdot \nabla)\vec{V}_\alpha\right] = e_\alpha N_\alpha \left(\vec{E}_T + \frac{\vec{V}_\alpha}{c} \times \vec{B}_T\right) + g_\alpha N_\alpha \left(\vec{B}_T - \frac{\vec{V}_\alpha}{c} \times \vec{E}_T\right] - \nabla P_\alpha, \tag{4-1}$$

$$\alpha = 1, 2, \ldots, M$$

2) The continuity equations

$$\frac{\partial N_\alpha}{\partial t} + \nabla \cdot (N_\alpha \vec{V}_\alpha) = 0, \quad \alpha = 1, 2, \ldots, M, \tag{4-2}$$

and

3) The generalized Maxwell's equations

$$\nabla \cdot \vec{E}_T = 4\pi \sum_{\alpha=1}^{M} e_\alpha N_\alpha \tag{4-3a}$$

$$\nabla \times \vec{E}_T = -\frac{4\pi}{c} \sum_{\alpha=1}^{M} g_\alpha N_\alpha \vec{V}_\alpha - \frac{1}{c}\frac{\partial \vec{B}_T}{\partial t} \tag{4-3b}$$

$$\nabla \cdot \vec{B}_T = 4\pi \sum_{\alpha=1}^{M} g_\alpha N_\alpha \tag{4-3c}$$

$$\nabla \times \vec{B}_T = \frac{4\pi}{c} \sum_{\alpha=1}^{M} e_\alpha N_\alpha \vec{V}_\alpha + \frac{1}{c} \frac{\partial \vec{E}_T}{\partial t} \qquad (4\text{-}3d)$$

where N_α, $\vec{V}_\alpha$, P_α are the number density, fluid velocity, and pressure of the $\alpha^{\underline{th}}$ specie fluid respectively; e_α, g_α, m_α are the electric charge, magnetic charge, and mass of the particles comprising the $\alpha^{\underline{th}}$ specie fluid; and a total of M different particle species are assumed present. The total electric and magnetic fields $\vec{E}_T$, $\vec{B}_T$ are expressed in CGS-Gaussian units throughout.

One more relation is needed to complete this system of equations and is usually taken to be the thermodynamic equation of state relating P_α to N_α,

$$P_\alpha = A_\alpha (m_\alpha N_\alpha)^{\gamma_\alpha}, \quad \alpha = 1,2,\ldots,M, \qquad (4\text{-}4)$$

where A_α is a constant and γ_α is the ratio of specific heats C_p/C_v for the $\alpha^{\underline{th}}$ specie fluid. As shown in Reference 5, Eq. (4-4) and the ideal gas law $P_\alpha = N_\alpha K_B T_\alpha$ imply

$$\nabla P_\alpha = \gamma_\alpha K_B T_\alpha \nabla N_\alpha, \quad \alpha = 1,2,\ldots,M, \qquad (4\text{-}5)$$

where K_B is Boltzmann's constant and T_α the temperature of the $\alpha^{\underline{th}}$ specie fluid.

The basic equations are thus generalized to include the possibility of magnetic charge on the plasma particles. It should perhaps be pointed out that nothing in our formulation prevents either $e_\alpha = 0$ or $g_\alpha = 0$ for some of the particle species in the plasma. In that event the corresponding number density refers to a purely electric or purely magnetic charge. By the same token, nothing in our formalism prevents a particle carrying both magnetic and electric charge; and in that case the corresponding number density refers to a dually charged particle, or dyon, species in the plasma. If $g_\alpha = 0$ for all α, of course, the equations reduce to the standard plasma equations in which only electric charges are present. The most general case in which some of the plasma particles are pure electric charges, some pure magnetic charges, and some dyons is therefore covered by our basic equations.

Linearized Equations

Since we are interested in the dispersion of small amplitude waves in the plasma, it is appropriate to linearize the basic equations by making the expansion

$$\vec{V}_\alpha = \vec{V}_{o\alpha} + \vec{v}_\alpha \tag{4-6a}$$

$$N_\alpha = N_{o\alpha} + n_\alpha \tag{4-6b}$$

$$\vec{E}_T = \vec{E}_o + \vec{E} \tag{4-6c}$$

$$\vec{B}_T = \vec{B}_o + \vec{B} \tag{4-6d}$$

where we have separated the fields into large average values $\vec{V}_{o\alpha}$, $N_{o\alpha}$, $\vec{E}_o$, $\vec{B}_o$ and small perturbations $\vec{v}_\alpha$, n_α, $\vec{E}$, $\vec{B}$.

In particular, we consider the case of perturbations in a motionless, uniform plasma with no average fields present. That is, we look at the situation where

$$\vec{V}_{o\alpha} = \vec{B}_o = \vec{E}_o = 0 \tag{4-7a}$$

$$\nabla N_{o\alpha} = 0, \quad \frac{\partial N_{o\alpha}}{\partial t} = 0 . \tag{4-7b}$$

In this case Eqs. (4-6) become

$$N_\alpha = N_{o\alpha} + n_\alpha, \; N_{o\alpha} = \text{constant} , \tag{4-8a}$$

$$\vec{V}_\alpha = \vec{v}_\alpha , \tag{4-8b}$$

$$\vec{E}_T = \vec{E} , \tag{4-8c}$$

$$\vec{B}_T = \vec{B} , \tag{4-8d}$$

Substituting (4-8) into our basic plasma equations (4-1) - (4-5) and dropping terms of second order in the perturbed quantities we obtain the linearized equations

$$m_\alpha N_{o\alpha} \frac{\partial \vec{v}_\alpha}{\partial t} = e_\alpha N_{o\alpha} \vec{E} + g_\alpha N_{o\alpha} \vec{B} - \gamma_\alpha K_B T_\alpha \nabla n_\alpha , \quad \alpha = 1,2,\ldots,M , \tag{4-9}$$

$$\frac{\partial n_\alpha}{\partial_t} + N_{o\alpha} \nabla \cdot \vec{v}_\alpha = 0, \quad \alpha = 1,2,\ldots,M , \tag{4-10}$$

$$\nabla \cdot \vec{E} = 4\pi \sum_{\alpha=1}^{M} e_\alpha n_\alpha , \tag{4-11a}$$

$$\nabla x \vec{E} = - \frac{4\pi}{c} \sum_{\alpha=1}^{M} g_\alpha N_{o\alpha} \vec{v}_\alpha - \frac{1}{c} \frac{\partial \vec{B}}{\partial t} , \tag{4-11b}$$

$$\nabla \cdot \vec{B} = 4\pi \sum_{\alpha=1}^{M} g_\alpha n_\alpha , \tag{4-11c}$$

$$\nabla x \vec{B} = \frac{4\pi}{c} \sum_{\alpha=1}^{M} e_\alpha N_{o\alpha} \vec{v}_\alpha + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} , \tag{4-11d}$$

where we also used the overall charge neutrality of the plasma,

$$\sum_{\alpha=1}^{M} e_\alpha N_{o\alpha} = 0, \quad \sum_{\alpha=1}^{M} g_\alpha N_{o\alpha} = 0 . \qquad (4\text{-}12)$$

The Cold Plasma Approximation

Since we are primarily interested in the collective plasma effects, it is appropriate to simplify our equations still further by using the cold plasma approximation. The resulting model of the plasma is often used and provides an overall view of the types of wave motion which can appear without obscuring the essential ideas with mathematical complication. The cold plasma approximation ignores the thermal motion of the plasma particles and focuses on the collective motion. More precisely, if there is a wave in the plasma with frequency ω and wavenumber k, the cold plasma approximation assumes

$$\left(\frac{K_B T_\alpha}{m_\alpha}\right)^{1/2} << \frac{\omega}{k}$$

That is, the plasma particles have thermal speeds much lower than the phase velocity of the wave. This implies that the thermal motion of the plasma particles is on the average so slow that they do not move even a small fraction of a wavelength in one wave period. In this case we may simplify our equations of motion by setting $T_\alpha = 0$, $\alpha = 1,2,\ldots M$, so that Eq. (4-9) becomes

$$\frac{\partial \vec{v}_\alpha}{\partial t} = \frac{e_\alpha}{m_\alpha}\vec{E} + \frac{g_\alpha}{m_\alpha}\vec{B} , \qquad (4\text{-}13)$$

and along with Eqs. (4-10) and (4-11) describes the plasma.

Fourier Transformed Equations

We now take the Fourier transform of Eqs. (4-10), (4-11), and (4-13); that is, we perform the plane wave decompositions

$$\vec{E}(\vec{x},t) = \int_{-\infty}^{\infty} \vec{\mathcal{E}}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\, d^3k\, d\omega \qquad \text{(4-14a)}$$

$$\vec{B}(\vec{x},t) = \int_{-\infty}^{\infty} \vec{\mathcal{B}}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\, d^3k\, d\omega \qquad \text{(4-14b)}$$

$$\vec{v}_\alpha(\vec{x},t) = \int_{-\infty}^{\infty} \vec{\mathcal{V}}_\alpha(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\, d^3k\, d\omega \qquad \text{(4-14c)}$$

$$n_\alpha(\vec{x},t) = \int_{-\infty}^{\infty} \mathcal{N}_\alpha(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\, d^3k\, d\omega \qquad \text{(4-14d)}$$

which have the inverse Fourier transforms

$$\vec{\mathcal{E}}(\vec{k},\omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \vec{E}(\vec{x},t)\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\, d^3x\, dt \qquad \text{(4-15a)}$$

$$\vec{\mathcal{B}}(\vec{k},\omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \vec{B}(\vec{x},t)\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\, d^3x\, dt \qquad \text{(4-15b)}$$

$$\vec{\mathcal{V}}_\alpha(\vec{k},\omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \vec{v}_\alpha(\vec{x},t)\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\, d^3x\, dt \qquad \text{(4-15c)}$$

$$\mathcal{N}_\alpha(\vec{k},\omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} n_\alpha(\vec{x},t)\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\, d^3x\, dt\ . \qquad \text{(4-15d)}$$

Substituting (4-14) into Equations (4-10), (4-11), (4-13) is equivalent to making the substitutions

$$\nabla \rightarrow i\vec{k} \ , \ \frac{\partial}{\partial t} \rightarrow -i\omega$$

$$\vec{E} \rightarrow \vec{\mathcal{E}} \ , \ \vec{B} \rightarrow \vec{\mathcal{B}}$$

$$\vec{v}_\alpha \rightarrow \vec{\mathcal{V}}_\alpha, \ n_\alpha \rightarrow \mathcal{N}_\alpha \ .$$

Doing this we obtain the Fourier transformed equations which are now simple algebraic equations for the transformed variables

$$-i\omega \vec{\mathcal{V}}_\alpha = \frac{e_\alpha}{m_\alpha} \vec{\mathcal{E}} + \frac{g_\alpha}{m_\alpha} \vec{\mathcal{B}} \ , \tag{4-16}$$

$$-i\omega \mathcal{N}_\alpha + N_{o\alpha} \ i\vec{k}\cdot\vec{\mathcal{V}}_\alpha = 0 \ , \tag{4-17}$$

$$i\vec{k}\cdot\vec{\mathcal{E}} = 4\pi \sum_{\alpha=1}^{M} e_\alpha \mathcal{N}_\alpha \ , \tag{4-18a}$$

$$i\vec{k}x\vec{\mathcal{E}} = -\frac{4\pi}{c} \sum_{\alpha=1}^{M} g_\alpha N_{o\alpha} \vec{\mathcal{V}}_\alpha + i \frac{\omega}{c} \vec{\mathcal{B}} \ , \tag{4-18b}$$

$$i\vec{k}\cdot\vec{\mathcal{B}} = 4\pi \sum_{\alpha=1}^{M} g_\alpha \mathcal{N}_\alpha \ , \tag{4-18c}$$

$$i\vec{k}x\vec{\mathcal{B}} = \frac{4\pi}{c} \sum_{\alpha=1}^{M} e_\alpha N_{o\alpha} \vec{\mathcal{V}}_\alpha - i \frac{\omega}{c} \vec{\mathcal{E}} \ . \tag{4-18d}$$

<u>Derivation of the Dispersion Relations</u>

Solving (4-16) for $\vec{\mathcal{V}}_\alpha$, we obtain

$$\vec{\mathcal{V}}_\alpha = \frac{i}{m_\alpha \omega} (e_\alpha \vec{\mathcal{E}} + g_\alpha \vec{\mathcal{B}}) \tag{4-19}$$

which may be used to eliminate $\vec{\mathcal{V}}_\alpha$ from Eqs. (4-18b) and (4-18d) so that they become

$$\vec{k}\times\vec{E} = -\frac{4\pi}{\omega c}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha N_{o\alpha}}{m_\alpha}\right)\vec{E} + \frac{\omega}{c}\left(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_{o\alpha}}{m_\alpha}\right)\vec{B} \tag{4-20a}$$

$$\vec{k}\times\vec{B} = -\frac{\omega}{c}\left(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2 N_{o\alpha}}{m_\alpha}\right)\vec{E} + \frac{4\pi}{\omega c}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_{o\alpha}\right)\vec{B}\ . \tag{4-20b}$$

Solving (4-20a) for $\vec{B}$ and using the result to eliminate $\vec{B}$ from Eq. (4-20b), we find

$$\vec{k}\times(\vec{k}\times\vec{E}) + \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{PH}^2}{\omega^2} + \frac{\omega_o^4}{\omega^2}\right)\vec{E} = 0 \tag{4-21}$$

where we have introduced

$$\omega_{PH}^2 \equiv \sum_{\alpha=1}^{M}\frac{4\pi(e_\alpha^2 + g_\alpha^2)}{m_\alpha}N_{o\alpha} \tag{4-22}$$

$$\omega_o^4 \equiv (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{N_{o\alpha}\ N_{o\beta}}{m_\alpha\ m_\beta}(e_\alpha g_\beta - e_\beta g_\alpha)^2\ . \tag{4-23}$$

The derivation of (4-21) from Eqs. (4-20) is straightforward and the details are given in Appendix C. One can, of course, eliminate $\vec{E}$ from Eqs. (4-20) in favor of $\vec{B}$. If this is done, it is found that $\vec{B}$ satisfies the same equation, (4-21), as one would expect.

Without loss of generality we can focus our attention on a wave propagating along the z-direction so that $\vec{k} = k\hat{z}$. In this case Eq. (4-21) can be written in the matrix form

$$\begin{pmatrix} \omega^4-(\omega_{PH}^2+c^2k^2)\omega^2 + \omega_o^4 & 0 & 0 \\ 0 & \omega^4-(\omega_{PH}^2+c^2k^2)\omega^2 + \omega_o^4 & 0 \\ 0 & 0 & \omega^4-\omega_{PH}^2\omega^2+\omega_o^4 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 . \qquad (4\text{-}24)$$

For a nontrivial solution of Eq. (4-24) one requires the determinant of the coefficient matrix to vanish identically. This leads to the two dispersion relations

$$\omega^4 - (\omega_{PH}^2 + c^2k^2)\omega^2 + \omega_o^4 = 0 \qquad (4\text{-}25)$$

$$\omega^4 - \omega_{PH}^2\,\omega^2 + \omega_o^4 = 0 \qquad (4\text{-}26)$$

which may be seen by recalling $\vec{k} = k\hat{z}$ and inspection of (4-24) to correspond to the transverse and longitudinal modes respectively.

It is of interest to note that the dispersion relations (4-25) and (4-26) are invariant under the duality transformation (3-6) since ω_{PH}^2 and ω_o^4 are simply summations over the duality invariant expressions $e_\alpha^2 + g_\alpha^2$ and $e_\alpha g_\beta - e_\beta g_\alpha$. Also, as may be seen by setting $g_\alpha = 0$, $\alpha = 1,2,\ldots,M$, these dispersion relations reduce to the usual plasma results discussed in References 1-5 where only electric charges are present.

Longitudinal Oscillations

Solving Eq. (4-26) we find for the longitudinal mode

$$\omega^2 = \frac{1}{2}\omega_{PH}^2 \left[1 \pm \sqrt{1 - \frac{4\omega_o^4}{\omega_{PH}^4}}\right] . \tag{4-27}$$

For $\frac{4\omega_o^4}{\omega_{PH}^4} << 1$, we obtain

$$\omega^2 \simeq \begin{cases} \omega_{PH}^2 \\ \\ \omega_{PL}^2 \equiv \dfrac{\omega_o^4}{\omega_{PH}^2} , \end{cases} \tag{4-28}$$

which is plotted in Figure 5.

As discussed in Appendix D, the inequality $\frac{4\omega_o^4}{\omega_{PH}^4} << 1$, is always expected to hold due to the charge quantization condition which requires $g \simeq 137e$. In any event, as shown there, $\omega_o < \frac{1}{2}\omega_{PH}$ so that the higher plasma frequency is ω_{PH} while the lower is ω_{PL}.

It should be noted that in the limit $g_\alpha \to 0$, $\alpha = 1,2,\ldots,M$ (i.e., no magnetic charge), these go over into the corresponding expression for the plasma frequency of a plasma of electric charges only.[1-5] In this case the lower branch $\omega^2 = \omega_{PL}^2$ vanishes completely.

That the two characteristic plasma frequencies should arise when the plasma particles are allowed to carry magnetic as well as electric charge is perhaps to be expected. It is noteworthy, however, that only one of the modes (i.e., ω_{PH}) is expected from duality symmetry arguments.

In summary, then, we have in the cold plasma approximation

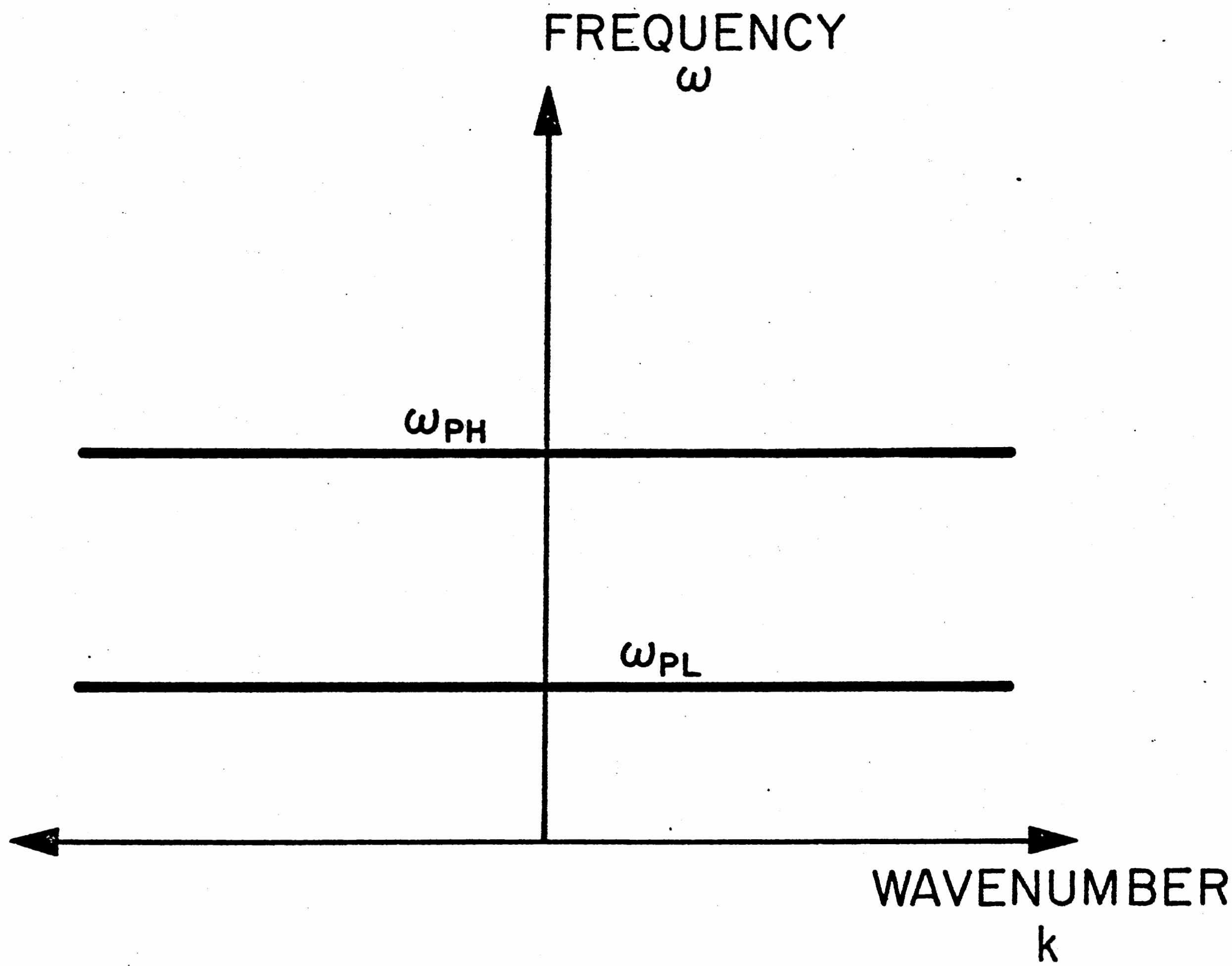


Figure 5. Longitudinal Mode Dispersion Curve

1) The existence of magnetically charged plasma particles introduces an additional longitudinal mode.

2) The eigenfrequency of each mode is independent of the wavenumber, and

3) In both instances the oscillation is a charge density fluctuation and not a true propagating wave since the group velocity $V_g \equiv \frac{d\omega}{dk}$ vanishes.

Transverse Modes

The more interesting dispersion relation is the one for the transverse waves, (4-25). Solving it for ω^2 we obtain

$$\omega^2 = \frac{1}{2}(\omega_{PH}^2 + c^2k^2)\left[1 \pm \sqrt{1 - \frac{4\omega_o^4}{(\omega_{PH}^2 + c^2k^2)^2}}\right] \tag{4-29}$$

For $\frac{4\omega_o^4}{(\omega_{PH}^2 + c^2k^2)^2} << 1$, this last result can be simplified to

$$\omega^2 \simeq \begin{cases} \omega_{PH}^2 + c^2k^2 \\ \\ \dfrac{\omega_o^4}{\omega_{PH}^2 + c^2k^2} \end{cases} \tag{4-30}$$

where we have retained only the leading terms. It is particularly revealing to plot the dispersion curve for (4-30). This is shown in Figure 6. As can be seen, transverse waves can propagate only for frequencies above ω_{PH} <u>and</u> below ω_L. In effect only the frequency band

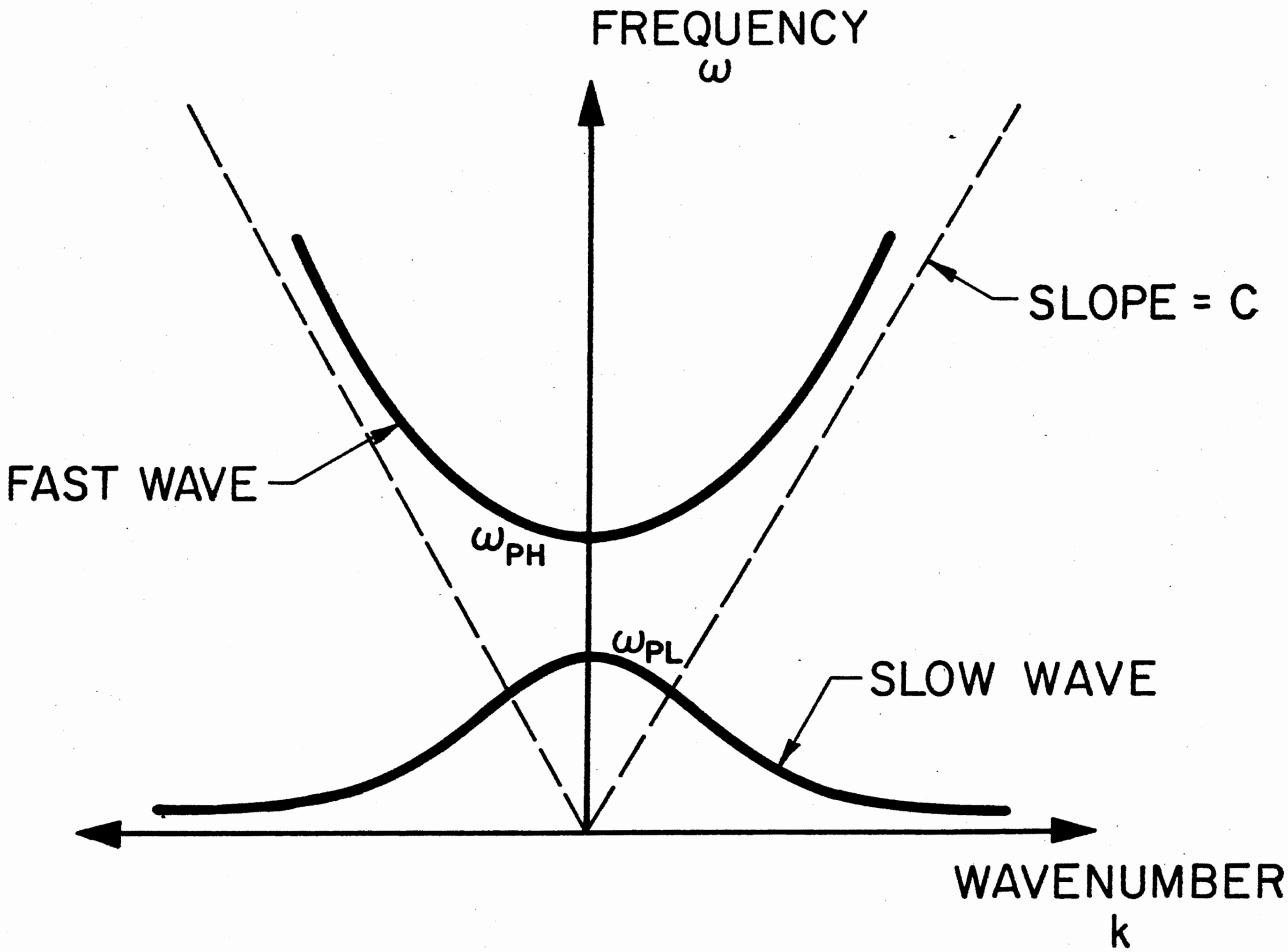


Figure 6. Transverse Mode Dispersion Curve

between ω_{PH} and ω_L cannot propagate.

Again, the appearance of the two branches of the dispersion curve is reasonable when we allow the plasma particles to be dually charged. It is interesting, however, that only one of the two branches (the upper one) may be expected from symmetry arguments. Once again, in the limit $g_\alpha \to 0$, $\alpha = 1,2,\ldots,M$, Eq. (4-30) goes over into the usual plasma result for plasma particles bearing electric charge only.

From (4-30) we see that the wave properties are qualitatively very different for the two branches. In particular, the phase velocity $V_p \equiv \frac{\omega}{k}$ and the group velocity $V_g \equiv \frac{d\omega}{dk}$ will be different depending on whether $\omega > \omega_{PH}$ or $\omega < \omega_{PL}$. From (4-30) we obtain

$$V_p = \begin{cases} c\left(1 - \dfrac{\omega_{PH}^2}{\omega^2}\right)^{-\frac{1}{2}}, & \omega > \omega_{PH} \\ \\ c\,\dfrac{\omega_o^2}{\omega_o^2}\left(1 - \dfrac{\omega^2}{\omega_{PL}^2}\right)^{-\frac{1}{2}}, & \omega < \omega_{PL} \end{cases} \tag{4-31}$$

for the phase velocity and

$$V_g = \begin{cases} c\left(1 - \dfrac{\omega_{PH}^2}{\omega^2}\right)^{\frac{1}{2}}, & \omega > \omega_{PH} \\ \\ c\,\dfrac{\omega^2}{\omega_o^2}\left(1 - \dfrac{\omega^2}{\omega_{PL}^2}\right)^{\frac{1}{2}}, & \omega < \omega_{PL} \end{cases} \tag{4-32}$$

for the group velocity. The phase velocity of the upper branch is superluminal ($V_p > c$) while that of the lower branch is subluminal ($V_p < c$) and we distinguish the two branches as that of a fast wave and a slow wave respectively. The group velocity V_g is, of course, always less than c as it should be.

The Vlasov Treatment

While the fluid plasma approach of the previous section is valid, at least as a first approximation, in most plasma regimes, there are phenomena which it cannot describe (e.g., Landau damping). To see if the introduction of magnetic charges affects these phenomena, it is advisable to look into any modifications of the kinetic theory approach when the plasma particles are allowed to carry magnetic as well as electric charge. In particular, the fluid approach is most applicable when one is studying a cold, relatively dense plasm and the kinetic theory treatment is most useful for plasmas that are relatively high in temperature and diffuse. This latter alternative will now be studied using the familiar Boltzmann description[1-5] of the plasma. The kinetic theory treatment of standard plasmas of electrically charged particles is well known and will not be repeated here. The approach taken here closely parallels that of Tanenbaum.[5]

Basic Equations

Generalizing the usual Boltzmann description of plasmas to include the possibility of magnetic as well as electric charges in the plasma, the basic equations become

1) The generalized Boltzmann equation,

$$\frac{\partial f_\alpha}{\partial t} + (\vec{v}\cdot\nabla) f_\alpha + [\frac{e_\alpha}{m_\alpha}(\vec{E}_T + \frac{\vec{v}}{c} \times \vec{B}_T) + \frac{g_\alpha}{m_\alpha}(\vec{B}_T - \frac{\vec{v}}{c} \times \vec{E}_T)]\cdot\nabla_{\vec{v}} f_\alpha$$

$$= (\frac{\partial f_\alpha}{\partial t})_{collision}, \qquad \alpha = 1,2,\ldots,M, \tag{4-33}$$

and

2) Maxwell's equations

$$\nabla\cdot\vec{E}_T = 4\pi \sum_{\alpha=1}^{M} e_\alpha (\int f_\alpha d^3v) \tag{4-34a}$$

$$\nabla\times\vec{E}_T = -\frac{4\pi}{c} \sum_{\alpha=1}^{M} g_\alpha (\int \vec{v} f_\alpha d^3v) - \frac{1}{c}\frac{\partial \vec{B}_T}{\partial t} \tag{4-34b}$$

$$\nabla\cdot\vec{B}_T = 4\pi \sum_{\alpha=1}^{M} g_\alpha (\int f_\alpha d^3v) \tag{4-34c}$$

$$\nabla\times\vec{B}_T = \frac{4\pi}{c} \sum_{\alpha=1}^{M} e_\alpha (\int \vec{v} f_\alpha d^3v) + \frac{1}{c}\frac{\partial \vec{E}_T}{\partial t} . \tag{4-34d}$$

The electric charge, magnetic charge, and mass of the $\alpha^{\underline{th}}$ specie particle have been denoted by e_α, g_α, m_α respectively. A total of M species are assumed present and $f_\alpha = f_\alpha(\vec{x},\vec{v},t)$ is the usual distribution function of the $\alpha^{\underline{th}}$ specie particle. The total electric and magnetic fields $\vec{E}_T$, $\vec{B}_T$ are taken in CGS-Gaussian units. The integration in the velocity integrals is to be assumed over all possible velocities whenever the limits are omitted.

To take into account collisional effects, at least approximately, we will use a simple Krook-type relaxation model

$$(\frac{\partial f_\alpha}{\partial t})_{collisions} = -\nu_\alpha(f_\alpha - f_{o\alpha}), \quad \alpha = 1,2,\ldots,M, \tag{4-35}$$

where ν_α is the mean collision frequency of the $\alpha^{\underline{th}}$ specie particle and $f_{o\alpha}$ is the equilibrium distribution function.

Except for the generalization to include the possibility of magnetic charge on the plasma particles, these equations are identical to the usual starting point of plasma theory. Again it should be pointed out that nothing in our formulation prevents either $e_\alpha=0$ or $g_\alpha=0$ for some of the particle species of the plasma. In that event, of course, the corresponding distribution function f_α refers to a purely electric or purely magnetic charge. The most general case in which some of the plasma particles are purely electric charges, some pure magnetic charges, and some dyons is therefore covered by our basic equations.

Linearized Equations

Since we are interested in the wave dispersion of small amplitude disturbances in the plasma, it is appropriate to linearize the basic equations by making the decomposition

$$f_\alpha = f_{o\alpha} + f_{1\alpha} \qquad (4\text{-}36a)$$

$$\vec{E}_T = \vec{E}_o + \vec{E} \qquad (4\text{-}36b)$$

$$\vec{B}_T = \vec{B}_o + \vec{B} \qquad (4\text{-}36c)$$

where we have separated the fields into large average values $f_{o\alpha}$, $\vec{E}_o$, $\vec{B}_o$ and small perturbations $f_{1\alpha}$, $\vec{E}$, $\vec{B}$. In particular, we are interested in the case of perturbations in a motionless, uniform plasma with no electromagnetic fields initially present. In this case

$$\vec{E}_o = \vec{B}_o = 0, \qquad f_{o\alpha} = f_{o\alpha}(|\vec{v}|) \text{ only}$$

and Eqs. (4-36) become

$$f_\alpha = f_{o\alpha}(|\vec{v}|) + f_{1\alpha}, \tag{4-37a}$$

$$\vec{E}_T = \vec{E}, \tag{4-37b}$$

$$\vec{B}_T = \vec{B}. \tag{4-37c}$$

Substituting (4-37) into our basic equations (4-33) through (4-35) and dropping terms of second order in the small perturbation quantities, we obtain the linearized equations

$$\frac{\partial f_{1\alpha}}{\partial t} + (\vec{v}\cdot\nabla) f_{1\alpha} + \left[\frac{e_\alpha}{m_\alpha}(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) + \frac{g_\alpha}{m_\alpha}(\vec{B} - \frac{\vec{v}}{c} \times \vec{E})\right]\cdot\nabla_{\vec{v}} f_{o\alpha}$$

$$= -\nu_\alpha f_{1\alpha}, \qquad \alpha = 1,2,\ldots,M, \tag{4-38}$$

$$\nabla\cdot\vec{E} = 4\pi \sum_{\alpha=1}^{M} e_\alpha (\int f_{1\alpha}\, d^3v) \tag{4-39a}$$

$$\nabla\times\vec{E} = -\frac{4\pi}{c}\sum_{\alpha=1}^{M} g_\alpha (\int \vec{v}\, f_{1\alpha}\, d^3v) - \frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{4-39b}$$

$$\nabla\cdot\vec{B} = 4\pi \sum_{\alpha=1}^{M} g_\alpha (\int f_{1\alpha}\, d^3v) \tag{4-39c}$$

$$\nabla\times\vec{B} = \frac{4\pi}{c}\sum_{\alpha=1}^{M} e_\alpha (\int \vec{v}\, f_{1\alpha}\, d^3v) + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{4-39d}$$

where we have also used the overall charge neutrality of the plasma

$$\sum_{\alpha=1}^{M} e_\alpha (\int f_{o\alpha}\, d^3v) = 0, \tag{4-40a}$$

$$\sum_{\alpha=1}^{M} g_\alpha (\int f_{o\alpha}\, d^3v) = 0. \tag{4-40b}$$

These can be further simplified if we observe that for $f_{o\alpha}=f_{o\alpha}(|\vec{v}|)$ only, then

$$\nabla_{\vec{v}} f_{o\alpha} = \frac{\partial f_{o\alpha}}{\partial v} \nabla_{\vec{v}} v, \qquad v = |\vec{v}|$$

$$= \frac{\partial f_{o\alpha}}{\partial v} \nabla_{\vec{v}}\left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right)$$

$$= \frac{\partial f_{o\alpha}}{\partial v} \frac{\vec{v}}{v}$$

or

$$\nabla_{\vec{v}} f_{o\alpha}(|\vec{v}|) = \frac{\vec{v}}{v}\frac{\partial f_{o\alpha}}{\partial v} . \tag{4-41}$$

Using (4-41), Eq. (4-38) can be simplified to

$$\frac{\partial f_{1\alpha}}{\partial t} + (\vec{v}\cdot\nabla) f_{1\alpha} + v_\alpha f_{1\alpha} = - (\frac{e_\alpha}{m_\alpha} \vec{E} + \frac{g_\alpha}{m_\alpha} \vec{B})\cdot\nabla_{\vec{v}} f_{o\alpha} . \tag{4-42}$$

Eq. (4-42) along with (4-39) then comprise the completely linearized equations.

Fourier Transformed Equations

We now take the Fourier transform of Eqs. (4-39) and (4-42); that is, we perform the plane wave decompositions

$$f_{1\alpha}(\vec{x},\vec{v},t) = \int_{-\infty}^{\infty} f_{1\alpha}(\vec{k},\vec{v},\omega)\; e^{i(\vec{k}\cdot\vec{x}-\omega t)}\; d^3k\; d\omega , \tag{4-43a}$$

$$\vec{E}(\vec{x},t) = \int_{-\infty}^{\infty} \vec{\mathcal{E}}(\vec{k},\omega)\; e^{i(\vec{k}\cdot\vec{x}-\omega t)}\; d^3k\; d\omega , \tag{4-43b}$$

$$\vec{B}(\vec{x},t) = \int_{-\infty}^{\infty} \vec{\mathcal{B}}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{x}-\omega t)}\ d^3k\ d\omega\ , \tag{4-43c}$$

which have the inverse Fourier transforms

$$f_{1\alpha}(\vec{k},\vec{v},\omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} f_{1\alpha}(\vec{x},\vec{v},t)\ e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\ d^3x\ dt \tag{4-44a}$$

$$\vec{\mathcal{E}}(\vec{k},\omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \vec{E}(\vec{x},t)\ e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\ d^3x\ dt \tag{4-44b}$$

$$\vec{\mathcal{B}}(\vec{k},\omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \vec{B}(\vec{x},t)\ e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\ d^3x\ dt\ . \tag{4-44c}$$

As before, this is equivalent to making the substitutions

$$\nabla \to i\vec{k} \qquad \frac{\partial}{\partial t} \to -\ i\omega$$

$$\vec{E},\vec{B} \to \vec{\mathcal{E}},\vec{\mathcal{B}} \qquad f_{1\alpha} \to f_{1\alpha}$$

Doing this we obtain the Fourier transformed equations which are now simple algebraic equations for the transformed variables,

$$-i\omega f_{1\alpha} + i(\vec{v}\cdot\vec{k})f_{1\alpha} + v_\alpha f_{1\alpha} = -\left(\frac{e_\alpha}{m_\alpha}\vec{\mathcal{E}} + \frac{g_\alpha}{m_\alpha}\vec{\mathcal{B}}\right)\cdot\nabla_{\vec{v}} f_{o\alpha} \tag{4-45}$$

$$i\vec{k}\cdot\vec{\mathcal{E}} = 4\pi \sum_{\alpha=1}^{M} e_\alpha\left(\int f_{1\alpha}\ d^3v\right) \tag{4-46a}$$

$$i\vec{k} \times \vec{\mathcal{E}} = -\frac{4\pi}{c} \sum_{\alpha=1}^{M} g_\alpha\left(\int \vec{v} f_{1\alpha}\ d^3v\right) + i\ \frac{\omega}{c}\ \vec{\mathcal{B}} \tag{4-46b}$$

$$i\vec{k}\cdot\vec{\mathcal{B}} = 4\pi \sum_{\alpha=1}^{M} g_\alpha\left(\int f_{1\alpha}\ d^3v\right) \tag{4-46c}$$

$$i\vec{k} \times \vec{\mathcal{B}} = \frac{4\pi}{c} \sum_{\alpha=1}^{M} e_\alpha\left(\int \vec{v} f_{1\alpha}\ d^3v\right) - i\ \frac{\omega}{c}\ \vec{\mathcal{E}}\ . \tag{4-46d}$$

Derivation of the Dispersion Relations

Solving (4-45) for $f_{1\alpha}$, we obtain

$$f_{1\alpha} = \frac{-i}{\omega_\alpha - \vec{v}\cdot\vec{k}} \left(\frac{e_\alpha}{m_\alpha}\vec{E} + \frac{g_\alpha}{m_\alpha}\vec{B}\right)\cdot\nabla_{\vec{v}} f_{o\alpha} \tag{4-47}$$

where we have denoted

$$\omega_\alpha \equiv \omega + i\nu_\alpha \, . \tag{4-48}$$

Now using (4-47) to eliminate $f_{1\alpha}$ from Eqs. (4-46b) and (4-46d), they become

$$\vec{k} \times \vec{E} = \frac{4\pi}{c} \sum_{\alpha=1}^{M} g_\alpha \left\{ \int \vec{v} \left[\frac{1}{\omega_\alpha - \vec{v}\cdot\vec{k}} \left(\frac{e_\alpha}{m_\alpha}\vec{E} + \frac{g_\alpha}{m_\alpha}\vec{B} \right)\cdot\nabla_{\vec{v}} f_{o\alpha} \right] d^3v \right\} + \frac{\omega}{c}\vec{B} \tag{4-49a}$$

$$\vec{k} \times \vec{B} = -\frac{4\pi}{c} \sum_{\alpha=1}^{M} e_\alpha \left\{ \int \vec{v} \left[\frac{1}{\omega_\alpha - \vec{v}\cdot\vec{k}} \left(\frac{e_\alpha}{m_\alpha}\vec{E} + \frac{g_\alpha}{m_\alpha}\vec{B} \right)\cdot\nabla_{\vec{v}} f_{o\alpha} \right] d^3v \right\} - \frac{\omega}{c}\vec{E} \tag{4-49b}$$

which can be rearranged into the form

$$\vec{k} \times \vec{E} = -\frac{4\pi}{\omega c} \sum_{\alpha=1}^{M} g_\alpha \left[\left(\frac{e_\alpha}{m_\alpha}\vec{E} + \frac{g_\alpha}{m_\alpha}\vec{B} \right)\cdot\left(\omega \int \frac{(\nabla_{\vec{v}} f_{o\alpha})\vec{v}}{\vec{v}\cdot\vec{k} - \omega_\alpha} d^3v \right) \right] + \omega/c\,\vec{B} \, , \tag{4-50a}$$

$$\vec{k} \times \vec{B} = \frac{4\pi}{\omega c} \sum_{\alpha=1}^{M} e_\alpha \left[\left(\frac{e_\alpha}{m_\alpha}\vec{E} + \frac{g_\alpha}{m_\alpha}\vec{B} \right)\cdot\left(\omega \int \frac{(\nabla_{\vec{v}} f_{o\alpha})\vec{v}}{\vec{v}\cdot\vec{k} - \omega_\alpha} d^3v \right) \right] - \frac{\omega}{c}\vec{E} \, . \tag{4-50b}$$

Defining the dyadic

$$\overleftrightarrow{N}_\alpha(\omega,\vec{k}) \equiv \omega \int \frac{(\nabla_{\vec{v}} f_{o\alpha})\vec{v}}{\vec{v}\cdot\vec{k} - \omega_\alpha} d^3v \tag{4-51a}$$

which in Cartesian components is

$$(\overleftrightarrow{N}_\alpha)_{ij} = \omega \int \frac{\frac{\partial f_{o\alpha}}{\partial v_i} v_j}{\vec{v}\cdot\vec{k} - \omega_\alpha} d^3v \ , \qquad (4\text{-}51b)$$

Eqs. (4-50) can be written in the more elegant form

$$\vec{k}\times\vec{E} = -\frac{4\pi}{\omega c}\sum_{\alpha=1}^{M} g_\alpha \left(\frac{e_\alpha}{m_\alpha}\vec{E} + \frac{g_\alpha}{m_\alpha}\vec{B} \right)\cdot\overleftrightarrow{N}_\alpha + \frac{\omega}{c}\vec{B} \qquad (4\text{-}52a)$$

$$\vec{k}\times\vec{B} = \frac{4\pi}{\omega c}\sum_{\alpha=1}^{M} e_\alpha \left(\frac{e_\alpha}{m_\alpha}\vec{E} + \frac{g_\alpha}{m_\alpha}\vec{B} \right)\cdot\overleftrightarrow{N}_\alpha - \frac{\omega}{c}\vec{E} \ . \qquad (4\text{-}52b)$$

As shown in Appendix E, for the simple case under consideration where $f_{o\alpha} = f_{o\alpha}(|\vec{v}|)$ only, the dyadic tensor $\overleftrightarrow{N}_\alpha$ assumes the especially simple form

$$\overleftrightarrow{N}_\alpha = \begin{pmatrix} N_T^\alpha & 0 & 0 \\ 0 & N_T^\alpha & 0 \\ 0 & 0 & N_L^\alpha \end{pmatrix} \qquad (4\text{-}53)$$

where

$$N_T^\alpha = N_T^\alpha(k,\omega) \equiv -\frac{\omega}{k}\int \frac{f_{o\alpha}}{v_z - \frac{\omega_\alpha}{k}} d^3v \qquad (4\text{-}54)$$

and

$$N_L^\alpha = N_L^\alpha(k,\omega) \equiv \frac{\omega\omega_\alpha}{k^2} \frac{\frac{\partial f_{o\alpha}}{\partial v_z}}{v_z - \omega_\alpha/k} d^3v \qquad (4\text{-}55)$$

In the Eqs. (4-53)-(4-55) we have restricted ourselves to the case where $\vec{k} = k\hat{z}$ (i.e., the wave travels in the z-direction) and used Cartesian coordinates. This can be done without loss of generality since there is no preferred direction in the plasma until the introduction of the wave disturbance.

Eliminating $\vec{B}$ from Eqs. (4-52) is straightforward calculation the details of which are given in Appendix F. Once $\vec{B}$ is eliminated, we find Eqs. (4-52) imply the components of $\vec{E}$ must satisfy the equations

$$\left[\omega^4-\omega^2\left(c^2k^2 + \sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)}{m_\alpha} N_T^\alpha\right) + (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha m_\beta} N_T^\alpha N_T^\beta\right] E_x=0 \tag{4-56a}$$

$$\left[\omega^4-\omega^2\left(c^2k^2 + \sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)N_T^\alpha}{m_\alpha}\right) + (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha m_\beta} N_T^\alpha N_T^\beta\right] E_y=0 \tag{4-56b}$$

$$\left[\omega^4-\omega^2 \sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)N_L^\alpha}{m_\alpha} + (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha m_\beta} N_L^\alpha N_L^\beta\right] E_z=0 \tag{4-56c}$$

which can be put in the matrix equation form

$$\overleftrightarrow{\Omega} \cdot \vec{E} = 0 \tag{4-57}$$

where

$$\overleftrightarrow{\Omega} = \begin{pmatrix} \Omega_T & 0 & 0 \\ 0 & \Omega_T & 0 \\ 0 & 0 & \Omega_L \end{pmatrix} \tag{4-58a}$$

and we have denoted

$$\Omega_T \equiv \omega^4-\omega^2\left[c^2k^2 + \sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)N_T^\alpha}{m_\alpha}\right] + (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha\, m_\beta} N_T^\alpha N_T^\beta \tag{4-58b}$$

$$\Omega_L \equiv \omega^4-\omega^2 \sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)N_L^\alpha}{m_\alpha} + (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha\, m_\beta} N_L^\alpha N_L^\beta \, . \tag{4-58c}$$

For a nontrivial solution of (4-57) with $\overleftrightarrow{\Omega}$ given by (4-58), one requires

$$\det(\overleftrightarrow{\Omega}) = 0 \, .$$

This leads to the two dispersion relations

$$\omega^4-\omega^2\left[c^2k^2 + \sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)}{m_\alpha} N_T^\alpha\right] + (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha\, m_\beta} N_T^\alpha N_T^\beta = 0 \tag{4-59}$$

and

$$\omega^4 \omega^2 \left[\sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)}{m_\alpha} N_L^\alpha\right] + (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha\, m_\beta} N_L^\alpha N_L^\beta = 0 \tag{4-60}$$

which may be seen by recalling $\vec{k} = k\hat{z}$ and inspection of (4-57), (4-58) to correspond to the transverse and longitudinal modes respectively.

The dispersion relations (4-59), (4-60) are formally similar to the dispersion relations obtained in the fluid approach (Eqs. (4-25), (4-26). The primary difference is that Eqs. (4-59) and (4-60) have number densities N_T^α, N_L^α that instead of being constants are complicated functions of ω and k given by the integrals (4-54) and (4-55). To evaluate these integrals and thereby complete the derivation of the dispersion relations, one must specify the equilibrium distribution functions $f_{o\alpha}(|\vec{v}|)$. The usual, and by far the most physically relevant, choice for the equilibrium is a Maxwellian distribution

$$f_{o\alpha}(v=|\vec{v}|) = \frac{N_{o\alpha}}{\pi^{3/2}U_\alpha^3} \mathrm{Exp}\left(-\frac{v^2}{U_\alpha^2}\right) \tag{4-61}$$

where $N_{o\alpha}$ is the mean number density and

$$U_\alpha \equiv \left(\frac{2K_B T_\alpha}{m_\alpha}\right)^{1/2} \tag{4-62}$$

is the thermal velocity of the αth specie particles.

As shown in Reference 5, the integrals (4-54) and (4-55) with $f_{o\alpha}$ given by (4-61) can be reduced to the dispersion integrals

$$N_T^\alpha(\omega,k) = N_{o\alpha}\frac{\omega}{\omega_\alpha}\left[2C_\alpha \int_o^{C_\alpha} e^{x^2-C_\alpha^2} dx - i\pi^{\frac{1}{2}} C_\alpha e^{-C_\alpha^2}\right] \tag{4-63}$$

$$N_L^\alpha(\omega,k) = -2N_{o\alpha}\frac{\omega}{\omega_\alpha} C_\alpha^2\left[1 - 2C_\alpha \int_o^{C_\alpha} e^{x^2-C_\alpha^2} dx + i\pi^{\frac{1}{2}} C_\alpha e^{-C_\alpha^2}\right] \tag{4-64}$$

where we have denoted

$$C_\alpha \equiv \frac{\omega_\alpha/k}{U_\alpha}. \tag{4-65}$$

The Vlasov Plasma

The most interesting case covered by the dispersion relations (4-59), (4-60) with $N_T^\alpha(\omega,k)$ and $N_L^\alpha(\omega,k)$ given by (4-64), (4-65) is when the plasma is hot and diffuse enough so that collisions may be neglected. This corresponds to setting $\nu_\alpha = 0$ to obtain the Vlasov description of the plasma. Doing this we find

$$\omega_\alpha \equiv \omega + i\nu_\alpha \to \omega$$

and

$$C_\alpha \equiv \frac{\omega_\alpha/k}{U_\alpha} \to \frac{\omega_\alpha/k}{U_\alpha}$$

For $C_\alpha = \frac{\omega/k}{U_\alpha} >> 1$, as shown by Tanenbaum[5], we may make the asymptotic expansions

$$N_T^\alpha \simeq N_{o\alpha}\left[1 + \frac{1}{2C_\alpha^2} + \cdots - i\pi^{\frac{1}{2}} C_\alpha e^{-C_\alpha^2}\right] \tag{4-66}$$

$$N_L^\alpha \simeq N_{o\alpha}\left[1 + \frac{3}{2C_\alpha^2} + \cdots - 2i\pi^{\frac{1}{2}} C_\alpha^3 e^{-C_\alpha^2}\right] \tag{4-67}$$

so that to lowest order

$$N_T^\alpha \simeq N_L^\alpha \simeq N_{o\alpha} \tag{4-68}$$

Putting this into the dispersion relations (4-59), (4-60) yields the same results as obtained in the fluid approach, Eqs. (4-25) and (4-26). To lowest order, then, the Vlasov theory gives the same dispersion as the fluid description as it should.

If the next higher order terms in (4-66) and (4-67) are considered we obtain the Bohm-Gross dispersion (from the terms $\sim \frac{1}{C_\alpha^2}$) and Landau damping (from the imaginary part). The introduction of magnetic charges does not introduce any new effects here because N_T^{α}, N_L^{α} are the same as in the standard theory discussed Reference 5. While the analysis could be carried further to obtain explicit expressions for these dispersion relations, they are quite complicated and will not be reproduced here since our interests are simply to ascertain the new effects, if any, produced by the introduction of magnetic charges. As we have shown the Bohm-Gross dispersion and Landau damping in this case is not substantially different from the standard plasma.

Summary

To summarize the results of this chapter, we have found that the presence of magnetic charge on at least some of the particle constituents of a plasma leads to non-trivial consequences. In particular, using the fluid plasma description the number of plasma wave modes was found to be double that of the standard plasma of electric charges only. The presence of magnetic charge allows the existence of an additional longitudinal and transverse mode in the plasma waves.

The kinetic theory approach leads to essentially the same results as the fluid theory description. While the Vlasov treatment gives rise to the usual Bohm-Gross dispersion and Landau damping of the plasma waves, these higher order effects turn out to be substantially the same as in the standard plasma case. The introduction of magnetic charges does not introduce anything new in regard to these particular features. This should have perhaps been expected since the Bohm-Gross dispersion and Landau damping are temperature related effects and arise from having a distribution of plasma particle velocities.

The appearance of new plasma modes when magnetic charges are present is in itself very interesting as it allows one to look for magnetic charges by looking for these new modes. In this regard the application of these results to an astrophysical setting would appear to be the most promising. An example of how these new modes would give rise to observable effects in astrophysics will form the topic of the next chapter.

FOOTNOTES

[1] N. A. Krall and A. W. Trivelpiece, Principles of Plasma Physics (McGraw-Hill, New York, 1973).

[2] T. H. Stix, The Theory of Plasma Waves (McGraw-Hill, New York, 1962).

[3] I. B. Berstein, and S. K. Trehan, Nucl. Fusion 1, 3 (1960).

[4] I. B. Berstein, S. K. Trehan, and M. P. H. Weeink, Nucl. Fusion 4, 61 (1964).

[5] B. S. Tanenbaum, Plasma Physics (McGraw-Hill, New York, 1967), especially Chapter 4.

CHAPTER V

THE EFFECT OF INTERSTELLAR MAGNETIC CHARGES ON PULSAR RADIATION

In the previous chapter we examined how some standard plasma physics results are modified if at least some of the plasma particles are magnetically charged. In particular, it was found that the dispersion of electromagnetic radiation was significantly modified in such a plasma. As an application, we point out in the present chapter how this modified dispersion relation may be used to look for the presence of magnetically charged particles in the interstellar media.

The proposed technique is an extension of a well known astrophysical method for determining the interstellar electron density using pulsar radiation. For comparison the standard theory will be reviewed first and then, using the results of Chapter IV, we will show how the standard theory is modified if magnetic charges are also present.

Since their discovery in 1968, pulsars have been used extensively as a probe of the interstellar medium.[1,2] Because of the presence of electrically charged particles in interstellar space, the signal velocity of electromagnetic waves is slightly less than the velocity of light in free space. This is caused by the dispersive nature of the medium which makes the group velocity of a wave pulse frequency dependent.

For a plasma of electrically charged particles, the transverse electromagnetic waves obey the dispersion relation[1,2]

$$\omega^2 = \omega_p^2 + c^2k^2 \tag{5-1}$$

where the plasma frequency ω_p is given by

$$\omega_p^2 = \sum_{\alpha=1}^{M} \frac{4\pi e_\alpha^2}{m_\alpha} N_{o\alpha} . \tag{5-2}$$

If the wave frequency ω is less than ω_p, then from (5-1) k becomes imaginary and the wave will not propagate. The plasma frequency ω_p is then the cut-off frequency. When ω is greater than ω_p, the wave can propagate and from (5-1) we may calculate the group velocity

$$V_g \equiv \frac{d\omega}{dk} = \frac{d}{dk}\left(\pm\sqrt{\omega_p^2 + c^2k^2}\right)$$

$$= \frac{\pm c^2k}{\sqrt{\omega_p^2 + c^2k^2}} = \pm\frac{c^2k}{\omega}$$

which upon using (5-1) again, may be written in terms of the frequency as

$$V_g = \pm c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} . \tag{5-3}$$

For a continuous monochromatic signal the group velocity is of course not observable, but for modulated signals containing a range of frequencies we can evaluate the degree of dispersion by considering the difference in pulse arrival times of two different frequencies. Applying this to pulsar emission we see that when the emitted pulsar pulse has a range of frequency components, the arrival times of the different frequencies at the earth will vary.

The arrival time of a pulse emitted at t = O that has traveled the distance D from the pulsar to the earth is given by

$$t = \frac{D}{|V_g|} = \frac{D}{c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}} \tag{5-4}$$

where we have used Eq. (5-3). In this pulsar emission at t = O there will be a range of frequencies. The arrival time at the earth of the frequency ω_1 is given by (5-4) as

$$t_1 = \frac{D}{c}\left(1 - \frac{\omega_p^2}{\omega_1^2}\right)^{-1/2}$$

while that of another frequency ω_2 is

$$t_2 = \frac{D}{c}\left(1 - \frac{\omega_p^2}{\omega_2^2}\right)^{-1/2}$$

For the sake of argument we will assume $\omega_2 > \omega_1$. The difference in the arrival time is thus given by

$$\Delta t \equiv t_2 - t_1 = \frac{D}{c}\left[\left(1 - \frac{\omega_p^2}{\omega_2^2}\right)^{-1/2} - \left(1 - \frac{\omega_p^2}{\omega_1^2}\right)^{-1/2}\right]. \tag{5-5}$$

Now if both ω_2 and ω_1 are well above the plasma frequency ω_p, then

$$\frac{\omega_p}{\omega_1} << 1, \quad \frac{\omega_p}{\omega_2} << 1$$

and we may approximate (5-5) by

$$\Delta t \simeq \frac{D}{c}\left[\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega_2^2}\right) - \left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega_1^2}\right)\right]$$

or

$$\Delta t \simeq \frac{D}{2c}\,\omega_p^2\left(\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}\right) = \frac{D\omega_p^2}{2c}\,\frac{\omega_1^2 - \omega_2^2}{\omega_1^2\omega_2^2}\,. \tag{5-6}$$

If $\omega_2\omega_1$ are almost the same frequency we can write

$$\omega_1 = \omega - \frac{\Delta\omega}{2}\,, \qquad \omega_2 = \omega + \frac{\Delta\omega}{2} \tag{5-7}$$

where $\Delta\omega$ is small compared to ω_1, ω_2, ω. Putting (5-7) into (5-6) we find

$$\Delta t \simeq \frac{D\omega_p^2}{2c}\left[\frac{(\omega - \frac{\Delta\omega}{2})^2 - (\omega + \frac{\Delta\omega}{2})^2}{(\omega - \frac{\Delta\omega}{2})^2\,(\omega + \frac{\Delta\omega}{2})^2}\right]$$

which to first order in $\Delta\omega$ is

$$\Delta t \simeq -\frac{D}{c}\left(\frac{\omega_p^2}{\omega^2}\right)\left(\frac{\Delta\omega}{\omega}\right), \qquad \omega \gg \omega_p\,. \tag{5-8}$$

From (5-8) we see that the lower frequencies arrive at a later time than the higher frequencies. In fact, the relation (5-8) has been used extensively to obtain several astrophysical quantities. The delay time Δt of the two frequencies can be measured and if the distance D to the pulsar can be measured using some other technique then formula (5-8) allows one to calculate

$$\omega_p^2 = \frac{4\pi e^2}{m_e} N_{oe}$$

(This assumes either free electrons or the major constituent of the interstellar plasma or that we are looking at frequencies ω where only the electrons have time to respond.) In this way the number density of interstellar electrons can be obtained. On the other hand, if the plasma frequency can be measured by other means then (5-8) can be used to obtain the distance to the pulsar. Both of these techniques are used extensively in astrophysics and are discussed at length in References 1 and 2.

Pulsars as a Probe for Interstellar Magnetic Charges

If magnetic charges exist in the interstellar plasma, then these pulsar dispersion results are significantly modified.[3] Now instead of (5-1), the transverse electromagnetic waves must satisfy the dispersion relation given by Eq. (4-30) of the preceding chapter,

$$\omega^2 \simeq \begin{cases} \omega_{PH}^2 + c^2k^2 & \omega \geq \omega_{PH} \\ \dfrac{\omega_o^4}{\omega_{PH}^2 + c^2k^2} & \omega \leq \omega_{PL} \end{cases} \tag{5-9}$$

which leads to the group velocity (4-32)

$$V_g = \begin{cases} c\left(1 - \dfrac{\omega_{PH}^2}{\omega^2}\right) & \omega \geq \omega_{PH} \\ \\ c\,\dfrac{\omega^2}{\omega_o^2}\left(1 - \dfrac{\omega^2}{\omega_{PL}^2}\right)^{1/2} & \omega \leq \omega_{PL}\ . \end{cases} \tag{5-10}$$

Using the same argument as in the standard plasma case we find that the arrival time of the frequency components $\omega_1 = \omega - \frac{\Delta\omega}{2}$, $\omega_2 = \omega - \frac{\Delta\omega}{2}$ is now given by

$$\Delta t \simeq \begin{cases} -\dfrac{D}{c}\,\dfrac{\omega_{PH}^2}{\omega^2}\,\dfrac{\Delta\omega}{\omega} & \omega >> \omega_{PH} \\ \\ -\dfrac{2D}{c}\,\dfrac{\omega_o^2}{\omega^2}\,\dfrac{\Delta\omega}{\omega} & \omega << \omega_{PH}\ . \end{cases} \tag{5-11}$$

The important point here is that now propagation can occur for $0 < \omega < \omega_{PL}$ as well as for $\omega > \omega_{PH}$. This lower range of frequencies is not allowed if only electrically charged particles exist in the interstellar plasma. Looking for this low frequency behavior would be tantamount to looking for magnetic charges in the interstellar media.

Unfortunately, these observations at low frequencies cannot be carried out below the ionosphere since the ionospheric plasma cuts off frequencies below a few megahertz. Satellite observations, however, would not be subject to this limiation. In view of the inherent simplicity of such observations and the fundamental importance of the issue, it would perhaps be worthwhile to incorporate some low frequency pulsar measurements in the next generation of satellite observations devoted to astrophysical investigations.

FOOTNOTES

[1]R. N. Manchester and J. H. Taylor, Pulsars (Freeman, San Francisco, 1977), Chapter 7 and references therein.

[2]M. Harwitt, Astrophysical Concepts (Wiley, New York, 1973), p. 207f.

[3]W. Wilson and N. V. V. J. Swamy, Phys. Lett. 72A, 188 (1979).

CHAPTER VI

SOLITONS AND THE JOSEPHSON JUNCTION

For various reasons there is cause to believe that a close connection exists between magnetic charges, solitons, and the Sine-Gordon equation.[1] In the present chapter we will develop this rather strange relationship by showing how all three of these ideas arise in the study of superconductive tunneling and the Josephson junction.

It has, of course, been known for some time that the propagation of magnetic flux vortices on a Josephson junction is described by the Sine-Gordon equation with the solitons in this case corresponding to the quantized flux vortices.[2,3] We point out here that these vortices can be viewed as the two-dimensional analogues of magnetic charge. The correctness of the viewpoint is then borne out by a study of electromagnetic radiation propagating along the junction interface. Indeed, the similarity between the results obtained in this case and those of Chapter IV will turn out to be quite striking.

Aside from the insights it provides into these more fundamental ideas, the Josephson tunnel junction is important from the standpoint of applications and device physics.[4] It therefore makes an ideal place to apply on a practical level some of the concepts developed in the theory of solitons.

Flux Vortices in Superconductors

It is well known that a superconductor placed in an external magnetic field exhibits the Meissner effect; that is, the superconductor expels the magnetic field.[5] However, in Type II superconductors when the magnetic field exceeds some critical value B_c, vortices of quantized magnetic flux puncture the superconductor as illustrated in Figure 7. As first discussed by London[6] these magnetic flux vortices must be quantized with flux

$$\Phi = 2\pi n\left(\frac{\hbar c}{q^*}\right), \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6-1}$$

where q* is the effective charge of the superconducting charge carrier. It is now known that q* = 2e and corresponds to the charge of the Cooper pair of superconducting electrons. For this value of q* one obtains from (6-1) the fundamental element of flux

$$\Phi_o = \frac{hc}{2e} \simeq 2 \times 10^{-7} \frac{\text{Gauss}}{\text{cm}^2} . \tag{6-2}$$

The simplest way to describe these effects physically is to regard superconductivity as a macroscopic quantum phenomenon. This viewpoint was first suggested by London and entails assigning a macroscopic wave function ψ to the superconductor so that $|\psi|^2$ is equal to the density of superconducting Cooper pairs.

The physical situation depicted in Figure 7 is not particularly well suited for experimental investigation and a much cleaner experimental arrangement is the one considered by Josephson.[7] This arrangement is now known as a Josephson tunneling junction and consists of

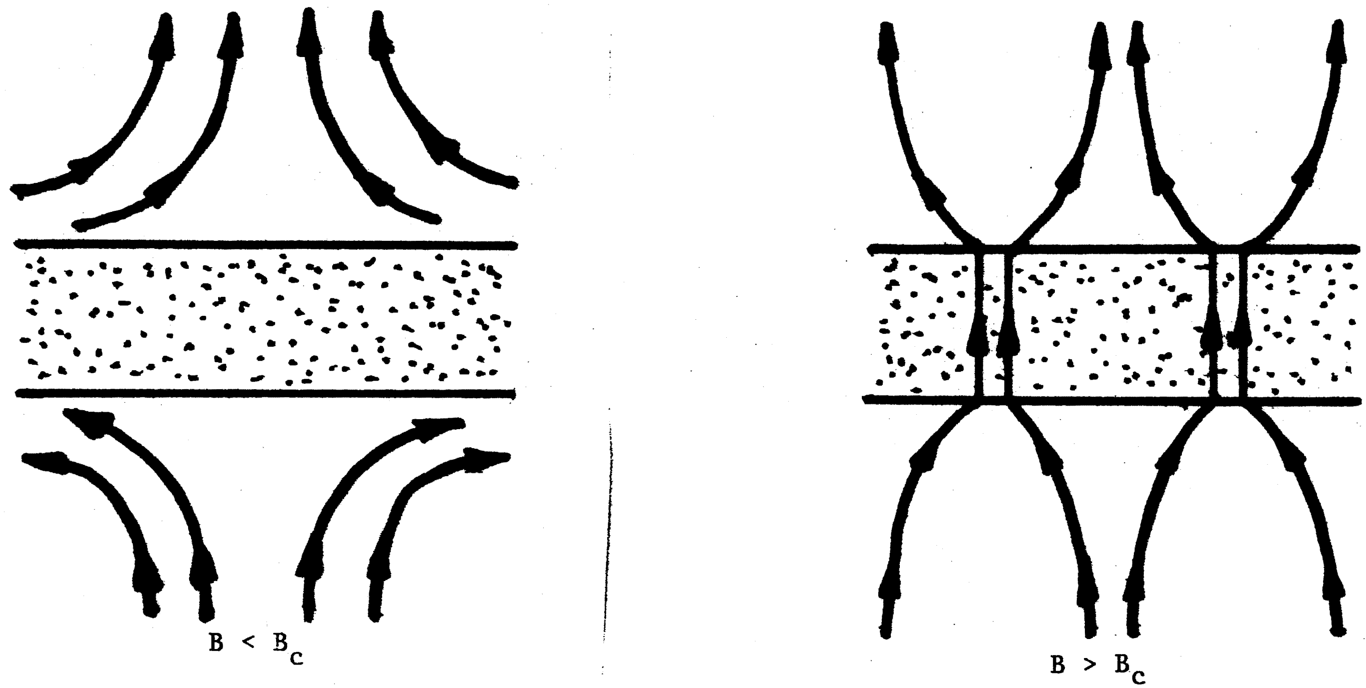


Figure 7. The Meissner Effect and Vortex Formation in Type II Superconductors. For $B > B_c$ the Magnetic Field Punches Holes of Magnetic Flux With a Tubelike Structure Through the Superconductor. These are Vortices of Quantized Magnetic Flux

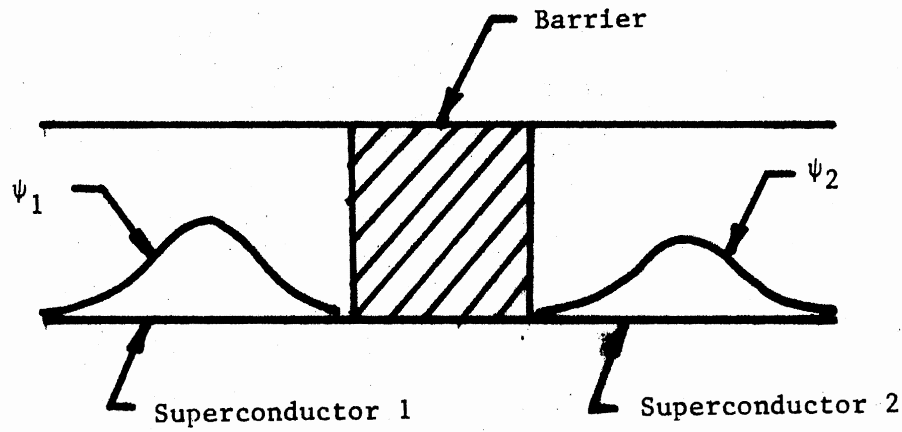
two superconducting metals separated by a nonsuperconducting barrier. As illustrated in Figure 8, these two superconductors may be described by the macroscopic wave functions ψ_1 and ψ_2. If the barrier is thick there is effectively no overlap of the wave functions; but if the barrier is thin (~ 50 Å) there can be an appreciable tunneling amplitude coupling the two superconductors. This coupling makes possible the passage of electrical current between the two superconductors by what is now called Josephson or superconductive tunneling. A typical experimental arrangement is depicted in Figure 9.

Qualitatively, one may think of the overlapping of the wave functions in the thin barrier as changing the nonsuperconducting barrier into a weak, Type II superconductor. Such a thin barrier forms what is called a weak link between the two superconductors. The advantage of this configuration is that it restricts the system to an essentially planar geometry. Again, if $B > B_c$ flux vortices may be formed in the barrier, but now the field lines must exist from the flux tubes along the interfaces of the junction since the superconductors on each side prevent the field lines by the Meissner effect from entering their interior. Instead of appearing as in Figure 7, then, the flux tubes in the Josephson junction are flattened as illustrated in Figure 10.

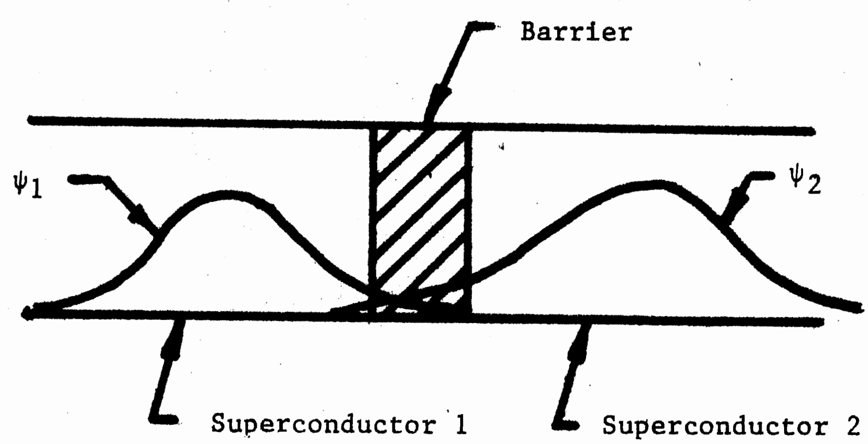
If we restrict ourselves to the plane of the interface between the barrier and the superconductor, the flux tubes appear to be sources and sinks of magnetic field lines and are quantized as we have previously discussed by

$$\Phi = 2\pi n\left(\frac{\hbar c}{q^*}\right), \qquad n = 0, \pm 1, \pm 2, \ldots \quad . \tag{6-3}$$

In this plane the flux would be viewed as being produced by the



(a) Thick Barrier -- No Current Flow



(b) Thin Barrier - Supercurrent Flow

Figure 8. Wavefunctions for the Josephson Junction for Thick and Thin Barriers. For the Thick Barrier There is No Overlap of the Superconductor Wavefunctions ψ_1 and ψ_2 and Hence No Current Flow. As the Barrier Width is Decreased the Wavefunctions are Able to Tunnel Through the Barrier and Allow a Superconductor Flow

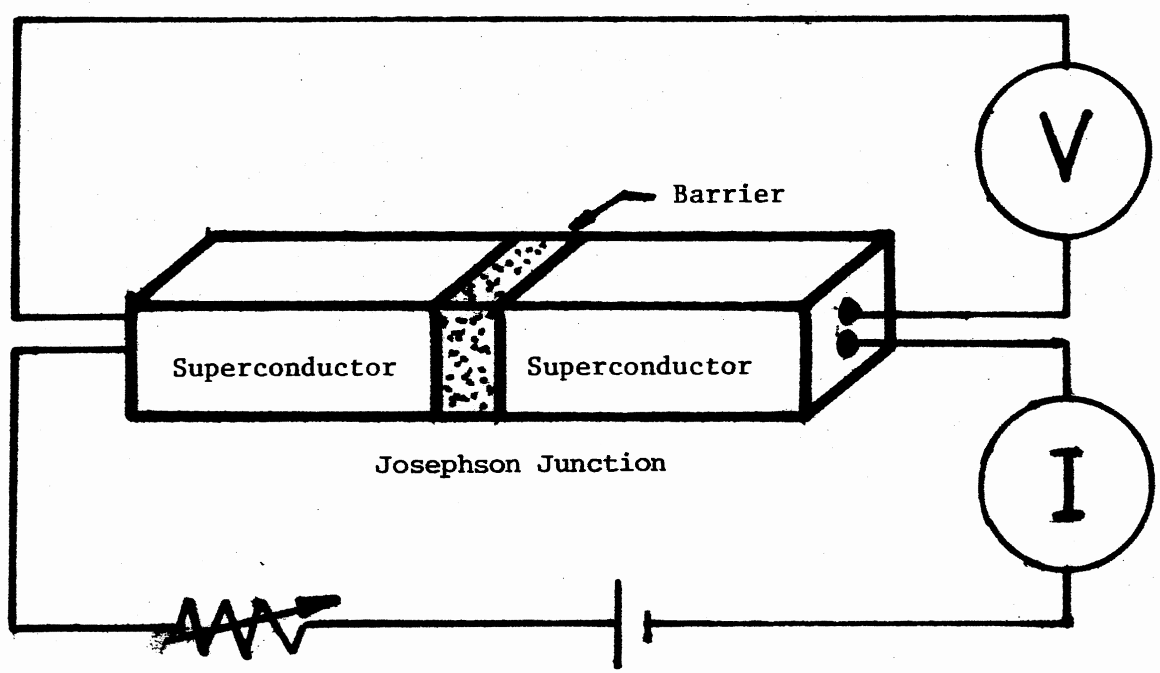
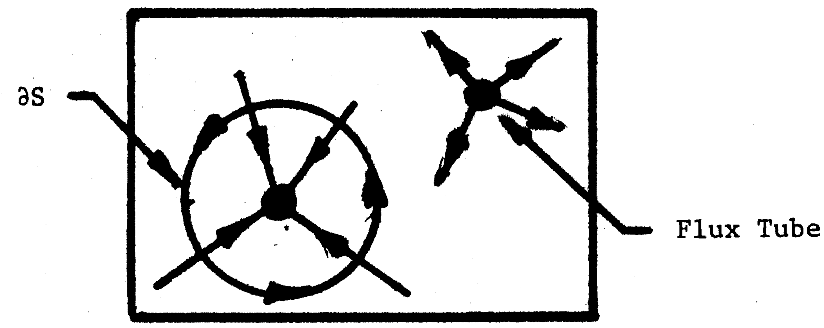
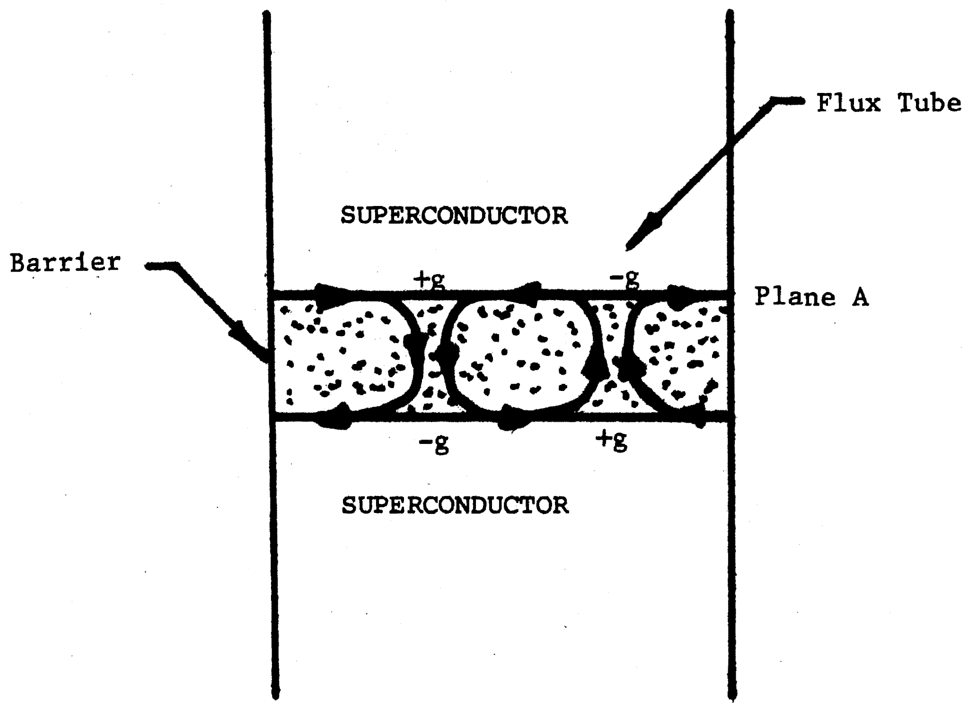


Figure 9. The Basic Experimental Configuration for a Josephson Junction.



Top View of Josephson Junction
at Plane A



Side View of Josephson Junction

Figure 10. Magnetic Flux Vortices in a Josephson Junction. Viewed in Plane A the Flux Tubes Appear to be Source and Sinks of $\vec{B}$ Field Lines

magnetic field of a magnetic charge with vector potential

$$\vec{A} = g \frac{y\hat{x}-x\hat{y}}{x^2+y^2}, \qquad \vec{B} = \nabla x\vec{A} = g \frac{\hat{r}}{r^2}. \tag{6-4}$$

Taking a circular path ∂S of radius a shown in Figure 10, we can calculate the magnetic flux

$$\Phi = \int \vec{A}\cdot d\vec{\ell} = g \int_{\partial S} \frac{ydx-xdy}{x^2 + y^2}$$

Using $x = a \cos\theta$, $y = a \sin\theta$ this becomes

$$\Phi = - g \int_0^{2\pi} d\theta = - g2\pi .$$

So that by putting this into Eq. (6-3) we find

$$eg = n\hbar c, \qquad n = 0, \pm 1, \pm 2, \ldots . \tag{6-5}$$

The Eq. (6-5) may be viewed as the planar analogy of the Dirac quantization condition of Chapter III. And viewed in this way the flux quantization condition is nothing more than a two-dimensional version of magnetic charge.

It should be mentioned that this analogy between flux vortices is superconductors and magnetic monopoles has been noted before.[8,9] A more quantitative treatment is given in the next section.

Derivation of the Basic Equations for the Josephson Junction

In this section we will look at the propagation of magnetic flux on a large area, two-dimensional Josephson tunneling junction. It will be shown that the magnetic flux propagates according to the Sine-Gordon equation, the solitons corresponding to the quantized flux vortices. The model we will use is well known and consists of treating the problem via a macroscopic wave function.[3,4,7,10] While a more exact theory using a BCS many-body approach exists,[5] it is quite complicated however and does not lead to any essentially different results for the Josephson junction.

The Josephson Equations

We consider the Josephson junction in the geometry shown in Figure 11. The two superconductors are separated by a very thin nonsuperconducting barrier of thickness d. We assume the barrier is centered on the xy-plane and has a cross-sectional area A. The voltage drop across the barrier is V(x,y).

Now to a very good approximation the supercurrent charge carriers (Cooper pairs) on each side of the barrier can be described by two macroscopic wave functions. If the superconductors are separated by a thick barrier, then their two wave functions are independent. In this event the circuit is essentially broken at the barrier and no supercurrent can flow.

For a thin barrier, on the other hand, we expect some of the wave function ψ_1 from the first superconductor to extend into the barrier and vice versa some of the wave function ψ_2 of the second superconductor to also extend into the barrier. In this case ψ_1, ψ_2 will overlap in the barrier, weakly coupling the two superconductors, and a supercurrent will flow. The circuit is essentially connected in this case by the quantum tunneling of the charge carriers through the barrier.

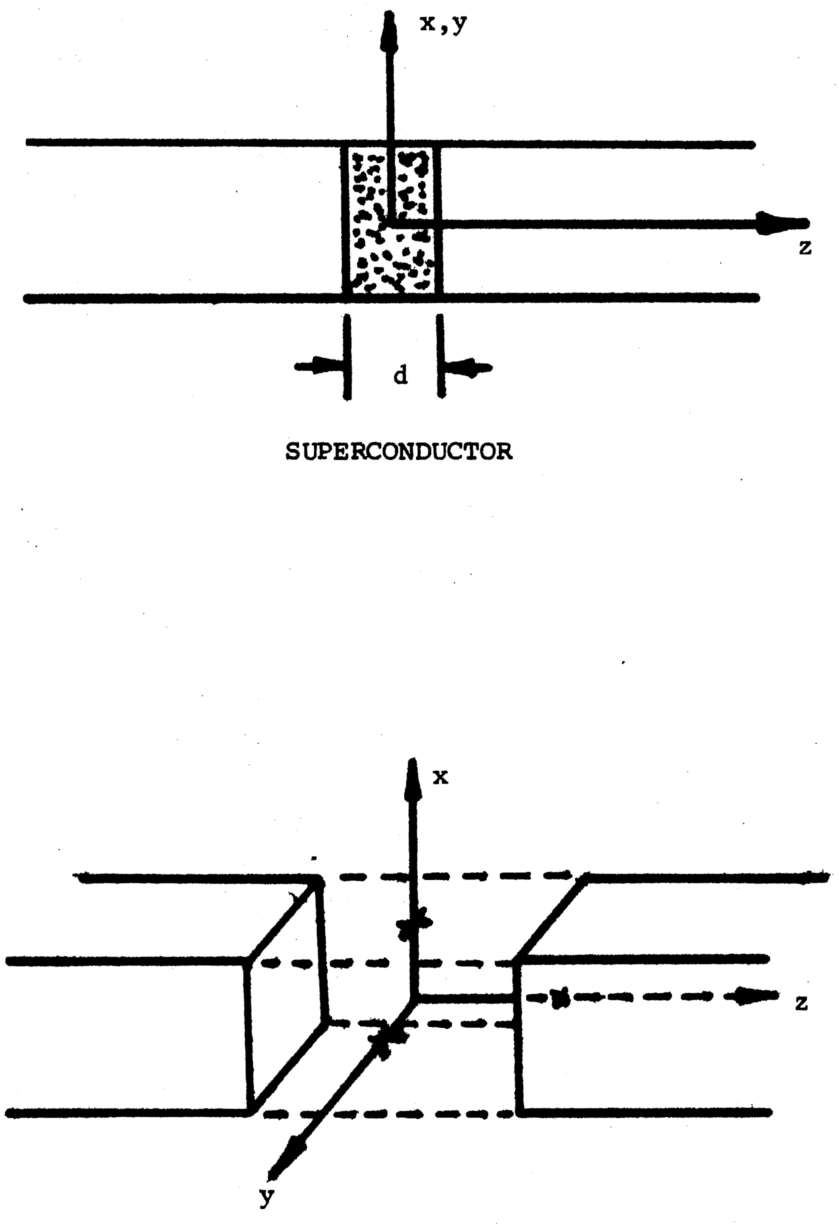


Figure 11. Geometry of the Josephson Junction Used in the Derivation of its Basic Equations

Josephson[2,10] has shown that this process of superconductive tunneling is described by the equations

$$J_z = J_o \sin\theta \tag{6-6}$$

$$\frac{\partial\phi}{\partial t} = \frac{q^*}{\hbar} V \tag{6-7}$$

where

$$\phi \equiv \theta_2 - \theta_1 - \frac{q^*}{\hbar c} \int_1^2 \vec{A}\cdot d\vec{\ell} \tag{6-8}$$

and J_z is the supercurrent density, θ_2, θ_1 are the phases of the wave functions ψ_1, ψ_2 on each side of the barrier, J_o is a phenomenological constant depending on the material properties of the superconductors and the barrier as well as the barrier thickness, and q^* is the effective charge of the supercurrent charge carrier ($q^* = 2e$ for Cooper pairs). The line integral in (6-8) is to be taken over any path across the barrier from the first superconductor's side to the second superconductor. The magnetic vector potential $\vec{A}$ is that due to the magnetic field, $\vec{B} = \nabla x \vec{A}$, which may possibly exist in the barrier. No magnetic field can exist in the superconductors due to the Meissner effect.

The Josephson relations (6-6)-(6-8) will be taken as the basic phenomenological equations describing the Josephson junction. They were first proposed by B. D. Josephson in 1962.[2,10] For completeness, a heuristic derivation of these relations using a macroscopic wave function approach due to Feynman[11] is given in Appendix G.

The Spatial Variation of ϕ

The formula (6-8) can be used to find the spatial variation of ϕ in the plane of barrier. We first observe that ϕ is invariant under the gauge transformations

$$\theta \rightarrow \theta' = \theta + \frac{q^*}{\hbar c} \chi \tag{6-9a}$$

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \Delta \chi \tag{6-9b}$$

where χ is an arbitrary function. A particular choice of χ corresponds to a particular choice of gauge. Any gauge can be used to calculate the phase difference across the barrier, ϕ.

Now to find the spatial variation of ϕ in the plane of the barrier we let $\vec{x}_1'$, $\vec{x}_2'$ and $\vec{x}_1$, $\vec{x}_2$ be two pairs of points such that $\vec{x}_1'$, $\vec{x}_2'$ are adjacent to each other but on opposite sides of the barrier as shown in Figure 12 and similarly $\vec{x}_1$, $\vec{x}_2$ are adjacent points but on opposing sides. Using (6-8), $\phi(\vec{x}')$ and $\phi(\vec{x})$ are given by

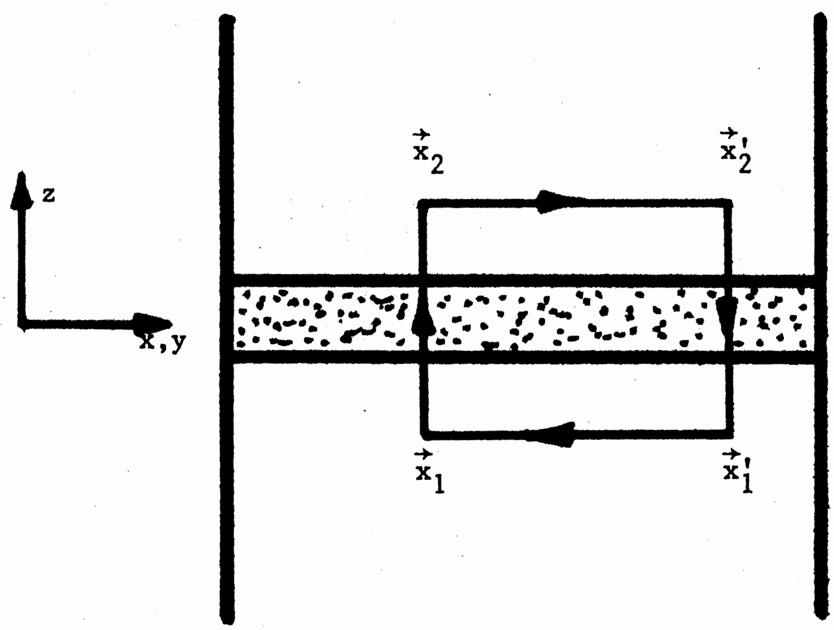
$$\phi(\vec{x}') = \theta(\vec{x}_2') - \theta(\vec{x}_1') - \frac{q^*}{\hbar c} \int_{\vec{x}_1'}^{\vec{x}_2'} \vec{A} \cdot d\vec{\ell} \tag{6-10a}$$

$$\phi(\vec{x}) = \theta(\vec{x}_2') - \theta(\vec{x}_1) - \frac{q^*}{\hbar c} \int_{\vec{x}_1}^{\vec{x}_2} \vec{A} \cdot d\vec{\ell} \tag{6-10b}$$

where the integral are taken over the straight lines joining the end points as shown in Figure 12.

The change in ϕ from $\vec{x}'$ to $\vec{x}$ is thus given by

$$\Delta\phi = \phi(\vec{x}') - \phi(\vec{x}) \tag{6-11}$$



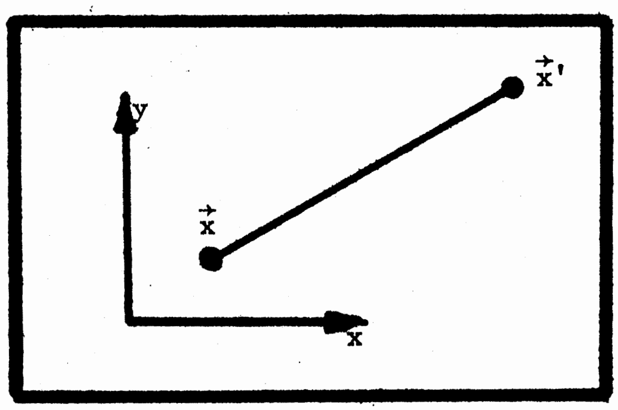


Figure 12. Geometry Involved in the Discussion of the Spatial Variation of $\Delta\phi$

and since by (6-9) any gauge can be used to calculate ϕ, we make the most convenient choice of the gauge in this calculation where $\theta = 0$ everywhere. This corresponds to the London gauge of superconductivity. In this case $\Delta\phi$ becomes

$$\Delta\phi = \frac{q^*}{\hbar c}\left[-\int_{\vec{x}_1'}^{\vec{x}_2'} \vec{A}\cdot d\vec{\ell} + \int_{\vec{x}_1}^{\vec{x}_2} \vec{A}\cdot d\vec{\ell}\right] \tag{6-12}$$

where we have used (6-10) with the choice of gauge $\theta \equiv 0$.

This last expression can be related to the magnetic flux through the surface, Υ, of Figure 12 formed by the closed curve $\partial\Upsilon$ as follows. By definition, the magnetic flux through this surface is Φ_Υ where

$$\Phi_\Upsilon = \iint_\Upsilon \vec{B}\cdot d\vec{S}$$

$$= \iint_\Upsilon (\nabla x\vec{A})\cdot d\vec{S}$$

or

$$\Phi_\Upsilon = \oint_{\partial\Upsilon} \vec{A}\cdot d\vec{\ell} \tag{6-13}$$

where we have used $\vec{B} = \nabla x\vec{A}$ and Stokes theorem. From (6-13) the magnetic flux through is then given by

$$\Phi_\Upsilon = \int_{\vec{x}_1'}^{\vec{x}_2'} \vec{A}\cdot d\vec{\ell} + \int_{\vec{x}_1}^{\vec{x}_2} \vec{A}\cdot d\vec{\ell} + \int_{\vec{x}_2}^{\vec{x}_2'} \vec{A}\cdot d\vec{\ell} + \int_{\vec{x}_2'}^{\vec{x}_1'} \vec{A}\cdot d\vec{\ell}\ . \tag{6-14}$$

Now the first and third terms of (6-14) must vanish since they are over paths that lie entirely inside the superconducting regions, where by the Meissner there can be no $\vec{B}$ field, and thus $\vec{A}$ can be chosen to vanish there too. In this event (6-14) becomes

$$\Phi_\gamma = \int_{\vec{x}_1}^{\vec{x}_2} \vec{A}\cdot d\vec{\ell} - \int_{\vec{x}'_1}^{\vec{x}'_2} \vec{A}\cdot d\vec{\ell} \tag{6-15}$$

where we have reversed the direction of the second integral. Comparing (6-15) and (6-12) we get the desired result for change in ϕ for the two points $\vec{x}', \vec{x}$ in the barrier

$$\Delta\phi \equiv \phi(\vec{x}') - \phi(\vec{x}) = \frac{q^*}{\hbar c} \Phi_\gamma \tag{6-16}$$

It should be emphasized that this last result is simply a direct implication of the Josephson relation (6-8). In practice, it is more convenient to use the differential form of (6-16) which is

$$\frac{\partial\phi}{\partial x} = \frac{q^*(d + 2\lambda)}{\hbar c} B_y \tag{6-17a}$$

$$\frac{\partial\phi}{\partial y} = -\frac{q^*(d + 2\lambda)}{\hbar c} B_x \tag{6-17b}$$

where λ is the penetration depth of the curve $\partial\gamma$ on each side of the barrier. A derivation of (6-17) from their integral form (6-16) is given by Solymar.[4]

Maxwell's Equations

To give a complete description of the Josephson junction one more relation is needed and this is taken to be the z-component of the Maxwell equation for the $\vec{B}$ field in the barrier

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

or

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \frac{4\pi}{c} J_z + \frac{1}{c} \frac{\partial E_z}{\partial t} . \tag{6-18}$$

Now E_z can be related to the voltage drop across the barrier, V, by treating the junction as a parallel plate capacitor, so that

$$E_z = 4\pi (\frac{C}{A}) V$$

where $(\frac{C}{A})$ is the effective capacitance per unit area. Using this (6-18) becomes

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \frac{4\pi}{c} J_z + \frac{4\pi (C/A)}{c} \frac{\partial V}{\partial t} . \tag{6-19}$$

Derivation of the Sine-Gordon Equation for the Josephson Junction

To summarize the results of the previous section, the basic equations describing the physics of a Josephson junction are Eqs. (6-6)-(6-8),

$$J_z = J_o \sin \phi \tag{6-20}$$

$$\frac{\partial \phi}{\partial t} = \frac{q^*}{\hbar c} V \tag{6-21}$$

$$\phi = \theta_2 - \theta_1 - \frac{q^*}{\hbar c} \int_1^2 \vec{A} \cdot d\vec{\ell} , \tag{6-22}$$

Eq. (6-16) or its differential equivalent,

$$\frac{\partial \phi}{\partial x} = \frac{q^* (d + 2\lambda)}{\hbar c} B_y \tag{6-23}$$

$$\frac{\partial \phi}{\partial y} = -\frac{q^*(d + 2\lambda)}{\hbar c} B_x \, , \tag{6-24}$$

and the Maxwell equation

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \frac{4\pi}{c} J_z + \frac{4\pi(C/A)}{c} \frac{\partial V}{\partial t} \, . \tag{6-25}$$

Using (6-20), (6-23), (6-24), and (6-21) to eliminate J_z, B_y, B_x, and V respectively from Eq. (6-25), it becomes the Sine-Gordon equation

$$\frac{\partial}{\partial x}\left[\frac{\hbar c}{q^*(d+2\lambda)} \frac{\partial \phi}{\partial x}\right] - \frac{\partial}{\partial y}\left[- \frac{\hbar c}{q^*(d+2\lambda)} \frac{\partial \phi}{\partial y}\right] = \frac{4\pi}{c} J_o \sin\phi + \frac{4\pi(C/A)}{c} \frac{\partial}{\partial t}\left[\frac{\hbar}{q^*} \frac{\partial \phi}{\partial t}\right]$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{1}{v_o^2} \frac{\partial^2 \phi}{\partial t^2} = \lambda_J^{-2} \sin\phi \tag{6-26}$$

where we have defined

$$v_o \equiv c\left[4\pi(C/A)(d+2\lambda)\right]^{-1/2} \tag{6-27}$$

and

$$\lambda_J = \left[4\pi \, q^* \, J_o \, (d+2\lambda)/\hbar c^2\right]^{-1/2} \, . \tag{6-28}$$

This equation was apparently first derived by Josephson[2] in a manner similar to the derivation presented here. Some typical values for the phenomenological parameters are given in Table I.

The solution of Eq. (6-26) for $\phi(x,y,t)$ subject to appropriate boundary conditions completely determines all the physical quantities of the Josephson junction, since once ϕ is known Eqs. (6-20), (6-21),

TABLE I

SOME TYPICAL VALUES OF THE PHENOMENOLOGICAL PARAMETERS DESCRIBING A JOSEPHSON JUNCTION

Parameter	Typical Value
J_o	$10^3 - 10^6$ A/m^2
d	30 - 100 Å
λ	500 - 1000 Å
λ_J	$5 \times 10^{-5} - 2 \times 10^{-3}$ m
v_o	1-3 x 10^7 m/s
ω_J	$10^{10} - 10^{11}$ Hz

(6-23), and (6-24) can be used to find the corresponding J_z, V, B_x, B_y.

Some Particular Solutions to the Josephson Junction

Meissner Effect

If we consider the static case $\frac{\partial\phi}{\partial t} = 0$, and Eq. (6-26) becomes

$$\nabla^2 \phi = \lambda_J^{-2} \sin\phi \qquad (6\text{-}29)$$

Linearizing this equation by assuming ϕ is small so that $\sin\phi \simeq \phi$, we obtain

$$\nabla^2 \phi = \lambda_J^{-1} \phi \, . \qquad (6\text{-}30)$$

This equation is essentially London's equation and it follows from (6-30) that for small applied fields, where the approximation $\sin\phi \simeq \phi$ is valid, that the currents and magnetic fields are confined to a region near the edges of the barrier and fall off as $\exp(-\gamma/\lambda_J)$ where γ is the distance from the barrier edge. This is essentially the Meissner effect.

Vortex Solutions

As we have seen in Chapter II, Eq. (6-26) admits the particular soliton solution

$$\phi = 4 \tan^{-1}\left[\exp\left(\frac{x-vt}{\lambda_J(1 - v^2/v_o^2)^{1/2}}\right)\right] \qquad (6\text{-}31)$$

where $v < v_o$ is an arbitrary soliton velocity. The total change in ϕ for this solution as x goes from $-\infty$ to $+\infty$ is 2π, so according to Equation (6-16), the magnetic flux associated with the solution (6-31) is exactly on flux quantum, $\frac{\hbar c}{q^*} = \frac{\hbar c}{2e}$. The soliton solution then clearly represents a situation where a single quantized flux vortex propagates across the junction with the velocity in the x-direction.

Electromagnetic Wave Propagation in a Josephson Junction

Another particularly interesting time-varying solution occurs in the study of electromagnetic wave propagation in the Josephson barrier. There are two distinctly different cases to consider: (a) The case when no magnetic field exists in the barrier, and (b) The case when a magnetic field exists in the barrier in the form of the flux vortex/solitons. These will be considered in turn.

Flux Tubes Not Present

Let us consider an electromagnetic wave propagating along the junction barrier in the absence of any initial magnetic field in the barrier so there are no flux tube/solitons present. In this case the electromagnetic wave represents a small disturbance ϕ_1 and we may put

$$\phi(x,y,t) = \phi_o + \phi_1 = \phi_1(x,y,t) \qquad (6\text{-}32)$$

where ϕ_o = constant can be chosen to be zero. Substituting (6-32) into Eq. (6-26) and linearizing, we obtain

$$\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} - \frac{1}{v_o^2}\frac{\partial^2\phi_1}{\partial t^2} = \lambda_J^{-2}\phi_1 . \tag{6-33}$$

For a wave solution of the form

$$\phi_1 = A\,e^{i(kx-\omega t)} \tag{6-34}$$

we obtain from (6-33) the dispersion relation

$$\omega^2 = \omega_J^2 + v_o^2 k^2 \tag{6-35}$$

where

$$\omega_J \equiv v_o^2\,\lambda_J^{-2} \tag{6-36}$$

The dispersion relation (6-35) is sketched in Figure 13.

As can be seen from Figure 13 the electromagnetic wave cannot propagate for $\omega < \omega_J$ which is the cut-off frequency. This is analogous to the behavior of a plasma and in fact the disturbance is usually called the Josephson plasma oscillation. The low frequency ω_J (typically of a few Gigahertz) arises from the relatively low density of charge carriers in the barrier. This plasma resonance of the Josephson junction has been observed by Dahm, et al.[12] This linearized theory of electromagnetic waves propagating in the Josephson barrier is well known and has been extensively developed by Josephson[13] and Kulik.[14]

Flux Tubes Present

Lebwohl and Stephen[15] have investigated the propagation of electromagnetic radiation in the Josephson barrier in the case where flux

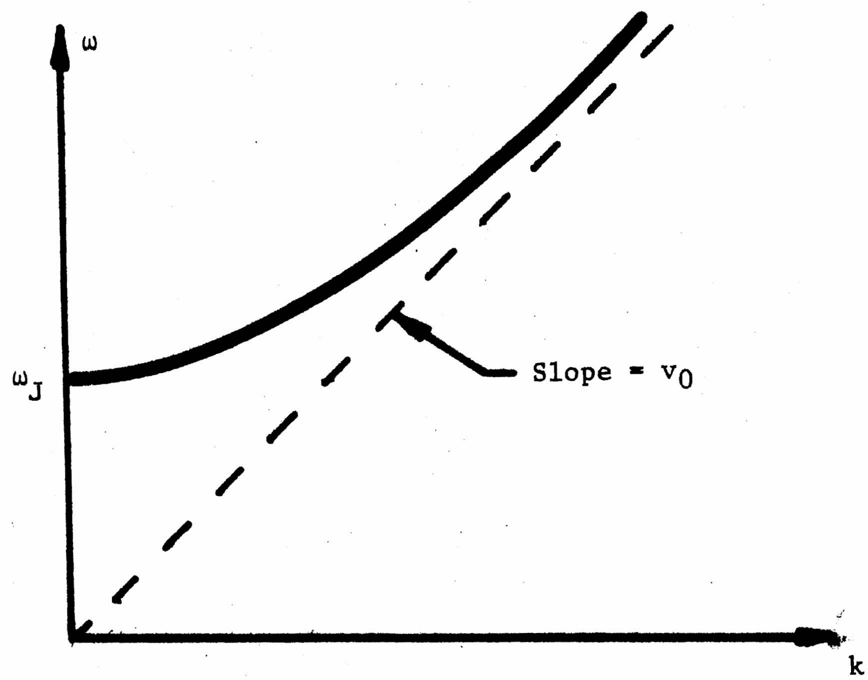


Figure 13. Dispersion Curve for Electromagnetic Wave Propagating in a Flux Free Josephson Junction

tubes/solitons are present. Their results are quite surprising and show the propagation of the electromagnetic wave to be profoundly effected by presence of solitons, although they did not interpret their results in the context of the theory of solitons. The discussion here is essentially the same as the previous no soliton case. More details can be found in Lebohl and Stephen's original paper as well as the book by Solymar.[4]

We consider the propagation of an electromagnetic wave along the barrier as before, except now we assume there is a magnetic field present in the form of flux vortex/solitons. In this case we again represent the electromagnetic wave by a small disturbance ϕ_1 and set

$$\phi(x,y,t) = \phi_o + \phi_1 \qquad (6\text{-}37)$$

where now since flux tubes are present we cannot set $\phi_o = 0$, but instead ϕ_o must be the solution of (6-26) representing a soliton state.

To fix ideas we look at a wave of frequency ω propagating in the x-direction, i.e. we look for solutions of the form

$$\phi = \phi_S(x,t) + U(x)\, e^{i\omega t} \qquad (6\text{-}38)$$

where ϕ_S is the known soliton solution of (6-26). Substituting (6-38) into (6-26) and linearizing with respect to U(x) we obtain

$$\frac{\partial^2 U}{\partial x^2} + \frac{\omega^2}{v_o^2} U = \frac{\cos \phi_S}{\lambda_J^2} U \,. \qquad (6\text{-}39)$$

This equation is a form of Lamé's equation. An exact solution of Lamé's equation exists in terms of eta and theta functions.[15,16] The resulting dispersion relation is of the form shown in Figure 14.

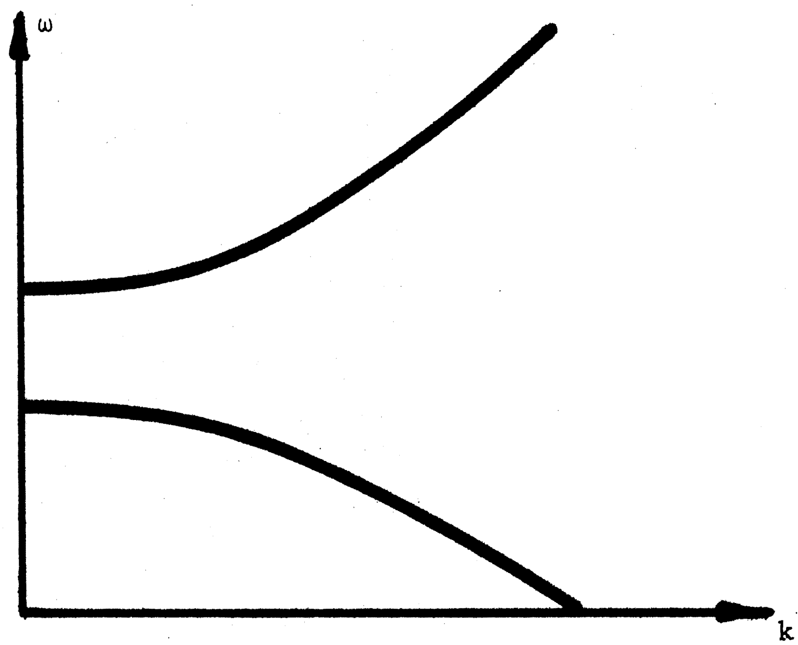


Figure 14. Dispersion Curve for Electromagnetic Waves Propagating in a Josephson Junction in Which Flux Tubes are Present (After Lebwohl and Stephen[15])

The similarity of this dispersion curve and the one obtained in Chapter IV for the propagation of electromagnetic waves in a plasma with magnetic charges present (Figure 6) is quite striking. The reason for this similarity is the flux tube-magnetic charge analogy pointed out earlier in this chapter. From the Meissner effect, Eq. (6-30), we know the electromagnetic wave can propagate only along the barrier edges. But it is just along the barrier edges that the flux tubes appear to be two-dimensional sources and sinks of the $\vec{B}$ field and thus are the two-dimensional analogues of magnetic charge. The propagating electromagnetic wave therefore "sees" the flux tubes as two-dimensional "plasma" of magnetic charges as it propagates in the barrier edge plane and the dispersion curve of Figure 14 is to be expected from the analogy with our results of Chapter IV.

All of this just strengthens the already considerable evidence pointing toward a close connection between magnetic charge, solitons, and the Sine-Gordon equation. In addition, it points out the dramatic effect the presence of solitons can have on the behavior of a physical system. In the case of the Josephson junction the presence of solitons allows a new branch to appear on the dispersion curve. In view of the effects solitons can produce if they are present, it is necessary to study the generation of soliton solutions to the Sine-Gordon and other nonlinear equations. Some initial researches into obtaining such solutions form the subject of the next chapter.

FOOTNOTES

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CHAPTER VII

BÄCKLUND TRANSFORMATIONS AND THE SINE-GORDON EQUATION IN 1+1 DIMENSIONS

The major problem confronting systems which satisfy the Sine-Gordon equation, as with any nonlinear theory, is simply finding solutions. The source of this difficulty can be traced to the fact that no linear superposition principle exists for the Sine-Gordon equation; that is, if ϕ_1 and ϕ_2 are solutions of Eq. (2-1), then in general $\phi_1 + \phi_2$ will not be a solution. Moreover, in the case of the Sine-Gordon theory, as previously mentioned, the most interesting solutions, namely the soliton solutions, cannot be obtained using perturbation theory.

A general method of obtaining soliton solutions to the Sine-Gordon equation uses the fact that it is an equation which admits a Bäcklund transformation.[1] The notion of a Bäcklund transformation is a relatively old mathematical idea which first arose in the study of differential geometry over a hundred years ago.[1-4]

Bäcklund transformations are roughly a generalization of continuous Lie group transformations and may be used to systematically generate solutions to differential equations.[5] In essence a Bäcklund transformation maps a solution surface of one differential equation to a solution surface of another differential equation. Because it is one of the few systematic techniques available for solving nonlinear partial differential equations, the Bäcklund transformation method has in recent years attracted considerable attention.[1,5-8] Bäcklund transformations have been used in fluid dynamics,[9] nonlinear plasma waves described by the Korteweg-deVries equation,[10] the Liouville equations[11] and the Burgers equations which arises in the study of noise and turbulent wave propagation.[12]

Bäcklund transformations were apparently first applied to the solution of the Sine-Gordon equation by Köchendörfer,[13] et al., in their study of crystal defect propagation. Lamb[14] has also used them extensively in his work on the 1+1 dimensional Sine-Gordon equation as it occurs in the propagation of ultra-short optical pulses.

Definition of Bäcklund Transformation

There does not seem to be an authoritative, generally accepted definition of Bäcklund transformation. Indeed, the usual method of defining a Bäcklund transformation is by examples. This will be done subsequently, but first a more formal approach will be taken.

A system of one or more relations of the form

$$B(\phi,\partial_\mu\phi,\partial_\mu\partial_\nu\phi,\ldots;\psi,\partial_\mu\psi,\partial_\mu\partial_\nu\psi,\ldots) = 0 \tag{7-1}$$

is called a Bäcklund transformation if they ensure that the function $\phi = \phi(x^\mu)$ is a solution of the partial differential equation(s)

$$N_1[\phi] = 0$$

whenever the function $\psi = \psi(x^\mu)$ is a solution of the partial differential equation(s)

$$N_2[\psi] = 0$$

and vice versa.

Very often $N_1 = N_2 \equiv N$, so the Bäcklund transformation connects two solutions of the same partial differential equation. As one might expect, not every differential system admits a Bäcklund transformation. Fortunately, the Sine-Gordon equation is one which does. The central problem, if one is to use this technique to generate solutions, is to determine if the system in question admits a Bäcklund transformation and then, if one exists, to find it. Once the Bäcklund transformation (7-1) has been found, however, it may be used to generate a new solution, ϕ, from an old, "known" solution ψ. This technique will be illustrated using the Sine-Gordon equation in 1+1 dimensions.

Equivalent Forms of the Sine-Gordon Equation

Before the concept of Bäcklund transformation is illustrated using the Sine-Gordon equation

$$\frac{\partial^2 \phi'}{\partial t'^2} - \frac{\partial^2 \phi'}{\partial x'^2} + \frac{m^3}{g} \sin(\frac{g}{m} \phi') = 0 \qquad (7\text{-}2)$$

it is convenient to transform (7-2) into several equivalent systems since the Sine-Gordon equation is studied in the literature in various forms.

Making the change to the dimensionless variables

$$t = mt' \qquad (7\text{-}3a)$$

$$x = mx' \qquad (7\text{-}3b)$$

$$\phi = \frac{g}{m} \phi' \qquad (7\text{-}3c)$$

Eq. (7-2) becomes

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \sin\phi = 0 \;. \qquad (7\text{-}4)$$

While making the change to the so-called light cone or characteristic coordinates

$$\tau = \frac{1}{2}(x+t) \qquad (7\text{-}5a)$$

$$\xi = \frac{1}{2}(x-t) \qquad (7\text{-}5b)$$

Eq. (7-4) in turn becomes

$$\frac{\partial^2\phi}{\partial\tau\partial\xi} = \sin\phi \;. \qquad (7\text{-}6)$$

This last form of the 1+1 dimensional Sine-Gordon equation is particularly convenient for a discussion of its Bäcklund transformations.

Bäcklund Transformations

For the Sine-Gordon equation in the form (7-6) the Bäcklund transformation equations are

$$\frac{\partial}{\partial\tau}\left(\frac{\phi_1-\phi_o}{2}\right) = a\sin\left(\frac{\phi_1+\phi_o}{2}\right) \qquad (7\text{-}7a)$$

$$\frac{\partial}{\partial\xi}\left(\frac{\phi_1+\phi_o}{2}\right) = \frac{1}{a}\sin\left(\frac{\phi_1-\phi_o}{2}\right) \qquad (7\text{-}7b)$$

where $\phi_1 = \phi_1(x^\mu)$, $\phi_o = \phi_o(x^\mu)$ and "a" is a constant known as the Bäcklund parameter. The Eqs. (7-7) may be derived either by appealing to their geometric interpretation[3] or by purely algebraic methods.[15]

It is easily shown that if (7-7) holds then ϕ_1, ϕ_o both must satisfy the Sine-Gordon equation (7-6). To see this, take $\frac{\partial}{\partial\xi}$ of (7-7a) and $\frac{\partial}{\partial\tau}$ of (3-7b) to obtain the system

$$\frac{\partial^2}{\partial\xi\partial\tau}\left(\frac{\phi_1-\phi_o}{2}\right) = a\cos\left(\frac{\phi_1+\phi_o}{2}\right)\cdot\frac{\partial}{\partial\xi}\left(\frac{\phi_1+\phi_o}{2}\right) \tag{7-8a}$$

$$\frac{\partial^2}{\partial\xi\partial\tau}\left(\frac{\phi_1+\phi_o}{2}\right) = \frac{1}{a}\cos\left(\frac{\phi_1-\phi_o}{2}\right)\cdot\frac{\partial}{\partial\tau}\left(\frac{\phi_1-\phi_o}{2}\right) \tag{7-8b}$$

which upon using (7-7) again to eliminate the first order derivatives on the right hand side yields

$$\frac{\partial^2}{\partial\xi\partial\tau}\left(\frac{\phi_1-\phi_o}{2}\right) = \cos\left(\frac{\phi_1+\phi_o}{2}\right)\sin\left(\frac{\phi_1-\phi_o}{2}\right) \tag{7-9a}$$

$$\frac{\partial^2}{\partial\xi\partial\tau}\left(\frac{\phi_1+\phi_o}{2}\right) = \cos\left(\frac{\phi_1-\phi_o}{2}\right)\sin\left(\frac{\phi_1+\phi_o}{2}\right) . \tag{7-9b}$$

Adding (7-9a) and (7-9b) one finds

$$\frac{\partial^2}{\partial\tau\partial\xi}\phi_1 = \sin\phi_1$$

while subtracting gives

$$\frac{\partial^2}{\partial\tau\partial\xi}\phi_o = \sin\phi_o .$$

Thus, (7-7) implies both ϕ_o and ϕ_1 satisfy Eq. (7-6) and therefore constitute a Bäcklund transformation for the Sine-Gordon equation in 1+1 dimensions.

It should be pointed out that the form (7-7) for the Sine-Gordon Bäcklund transformation is not unique. A large number of equivalent representations are known and indeed an infinite number are possible. For instance, the form

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi_1}{\partial t} = a \cos\phi_1 \sin\phi_o \tag{7-10a}$$

$$\frac{\partial \phi_o}{\partial t} + \frac{\partial \phi_1}{\partial x} = \frac{1}{a} \sin\phi_1 \cos\phi_o \tag{7-10b}$$

can also be shown to constitute a Bäcklund transformation for the Sine-Gordon equation in the form (7-6).[3]

Generating Solutions With the Bäcklund Transformation

Since both ϕ_o and ϕ_1 satisfy (7-6), the Bäcklund transformations (7-7) can be used to find solutions to the Sine-Gordon equation. From a given, known solution ϕ_o one may obtain a new solution ϕ_1 which contains not only the constant "a" of the Bäcklund transformation but also an integration constant. For instance, the "vacuum" solution $\phi_o = 0$ is by inspection a trivial known solution of (7-6). Putting this into the Bäcklund transformation equations (7-7) they become

$$\frac{\partial}{\partial \tau} (\phi_1/2) = a \sin (\phi_1/2) \tag{7-11a}$$

$$\frac{\partial}{\partial \xi} (\phi_1/2) = \frac{1}{a} \sin (\phi_1/2) \tag{7-11b}$$

which may be easily integrated to give

$$\phi_1(\xi,\tau) = 4 \text{ Arctan}[\exp(\frac{\xi}{a} + a\tau + \delta)] \qquad (7\text{-}12)$$

where δ is a constant of integration. The solution (7-12) is identical to the soliton solution of Chapter III if we identify

$$v = \frac{1-a^2}{1+a^2}$$

or

$$a = \pm\sqrt{\frac{1-v}{1+v}}$$

The Bäcklund parameter "a" is thus a constant which is related to the velocity of the soliton.

Bäcklund transformations can be repeatedly used in this manner to create soliton or antisoliton solutions from the vacuum state. At each stage one "generates" a new solution ϕ_1 by placing a known, old solution ϕ_o into Eqs. (7-7). The resulting equations are of first order and may be integrated by a single quadrature. Repeating this process, multiple soliton solutions can be generated from the vacuum states

$$\phi_{vac} = 2\pi n(\frac{m}{g}), \qquad n = 0,\pm1,\pm2,\ldots$$

through a series of such Bäcklund transformations.[16,17]

The Bäcklund transformation equations (7-7) may be looked upon as mapping a solution ϕ_o of the Sine-Gordon equation into another solution ϕ_1. This is usually indicated symbolically as

$$\phi_1 = \underset{\sim}{B}_a\,\phi_o \qquad (7\text{-}13)$$

where $\underset{\sim}{B}_a$ is known as the Bäcklund operator. For extensive calculations of this sort a graphical representation called a Lamb or Bianchi diagram is often used.[14] The Bianchi diagram corresponding to the Bäcklund transformation (7-13) or (7-7) is shown in Figure 15.

It can be shown that two Bäcklund transformation commute,[18]

$$\underset{\sim}{B}_{a_1} \underset{\sim}{B}_{a_2} = \underset{\sim}{B}_{a_2} \underset{\sim}{B}_{a_1} \tag{7-14}$$

and that this property leads to the nonlinear superposition principle

$$\phi = \phi_o + 4 \operatorname{Arctan}\left[\frac{a_2-a_1}{a_2-a_1} \tan\left(\frac{\phi_2-\phi_1}{2}\right)\right] \tag{7-15}$$

The Bianchi diagram for this process relating the four solutions ϕ, ϕ_o, ϕ_1, ϕ_2 is given in Figure 16. The utility of formula (7-15) is that it allows the algebraic construction of solutions without performing quadratures. Multiple soliton solutions have been constructed in this way by Barnard.[17]

Interpretations of the Bäcklund Transformation

The Bäcklund transformation technique is more than just a clever trick for finding solutions to the Sine-Gordon equation; it is fundamental to the entire theory of solitons. This fact has become clearer in recent years by the discovery of several novel interpretations of the Bäcklund transformation which have clarified their dynamical significance.

Canonical Transformation Interpretation

Kodama and Wadati[19] have shown that the Bäcklund transformations

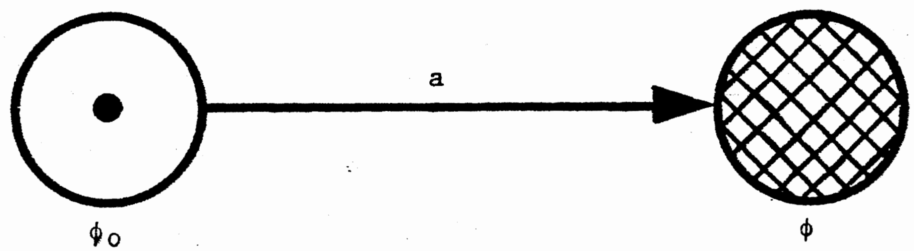

Figure 15. Bianchi Diagram for the Bäcklund Transformation Equations (7-7) and (7-13) Characterized by the Real Parameters a

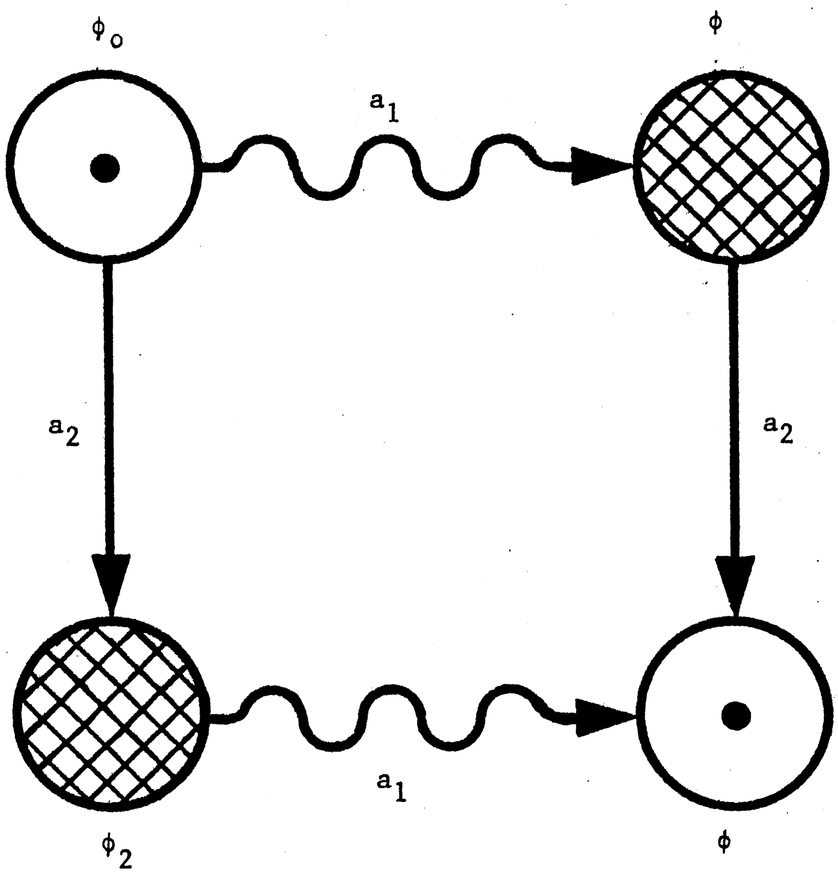

Figure 16. Commuting Bianchi Diagram for Equation (7-14)

of the Sine-Gordon equation are canonical transformations and as such form a group. The Bäcklund transformation thus keeps the Hamiltonian (2-13) form invariant. This symmetry has been used to generate an infinite number of conservation laws for the Sine-Gordon theory.[20]

Creation-Annihilation Operator Interpretation

As discussed previously, the Bäcklund transformation (7-7) represented by

$$\phi_1 = \underset{\sim}{B}_a \phi_o$$

can be used to go from the vacuum state to the one-soliton state, (7-12). To create an N-soliton solution, one has to perform N consecutive Bäcklund transformations on the vacuum

$$\phi_N = \underset{\sim}{B}_{a_N} \underset{\sim}{B}_{a_{N-1}} \cdots \underset{\sim}{B}_{a_2} \underset{\sim}{B}_{a_1} \phi_o \; .$$

This is all very similar to the creation operators of quantum field theory and it has been shown, in fact, that the Bäcklund transformation can be consistently interpreted as a classical creation-annihilation operator for solitons.[19,21,22]

Geometric Interpretation

When referred to suitable coordinates u,v on the surface, the line element of a surface of constant negative curvature may be written[3]

$$ds^2 = R^2 \, (du^2 + 2 \cos\theta \; dudv + dv^2)$$

where $K = -\frac{1}{R^2}$ is the constant total curvature of the surface

(R = radius of curvature of the surface) and $\theta = \theta(u,v)$ is the angle between the asymptotic lines. The Gauss-Codazzi equations of differential geometry can then be shown to require that θ satisfy[3]

$$\frac{\partial^2\theta}{\partial u \partial v} = \sin\theta$$

which is one of the forms of the Sine-Gordon equations.

To each solution of this equation there is a corresponding surface of constant negative curvature. The Bäcklund transformation can therefore be interpreted geometrically as a transformation from one constant negative curvature surface to another. This geometric interpretation can be used to derive the Bäcklund transformation equations.[3,15]

Dirac Factorization Interpretation

Recently Wilson and Swamy[23] have shown that the Bäcklund transformation equations may be looked upon as the Dirac factorization of the Sine-Gordon equation. This interpretation may also be used in deriving the Bäcklund transformation equations and is discussed in detail in the next chapter.

Bäcklund Transformation in Higher Dimensions

If one is to use the Bäcklund transformation equations to generate solutions to the Sine-Gordon equation, it is first necessary to find the Bäcklund transformation equations. While this is easily done in 1+1 dimensions by either geometric[3,5] or analytic methods[5,15,23], in higher dimensions the problem is not so simple. Indeed, for a time it was felt that the Sine-Gordon equation in 3+1 dimensions did not even admit a Bäcklund transformation and that the 1+1 dimensional case was an exception.[24] This work was subsequently invalidated by Leibbrandt's[25] discovery of the Sine-Gordon Bäcklund transformation equations in 2+1 and 3+1 dimensions.

Leibbrandt's technique for finding the higher dimensional Bäcklund transformation equations was apparently based on a skillful guess motivated by analogy with the 1+1 dimensional case. A systematic method for finding the Bäcklund equations based on the Dirac factorization interpretation of the Bäcklund transformation has recently been developed by Wilson and Swamy.[23] Their method leads not only to Leibbrandt's equations but also to an entire class of simpler Bäcklund transformation equations for the Sine-Gordon equation in 3+1 dimensions. This work is discussed in the next chapter.

FOOTNOTES

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CHAPTER VIII

BÄCKLUND TRANSFORMATIONS AND THE SINE-GORDON EQUATION IN 3+1 DIMENSIONS

The utility of the Bäcklund transformation is that it replaces the problem of solving a second-order nonlinear equation with that of solving a set of first-order equations. This is accomplished, however, at the expense of increasing the number of unknown field variables. It was noticed by the author that in this respect the Bäcklund transformation is analogous to Dirac's classic factorization of the Klein-Gordon equation.[1-3] In subsequent investigations it was found that if one takes the point of view that the Bäcklund transformation is a type of Dirac factorization then its derivation follows in a systematic manner. This interpretation is developed in the present chapter and the technique is then used to derive the Bäcklund transformation equations for the 3+1 dimensional Sine-Gordon equation. The Bäcklund transformations found in this manner are much simpler in form than any previously reported in the literature.

Dirac Factorization and the Sine-Gordon Equation in 1+1 Dimensions

It can easily be shown that the Bäcklund transformation equations are equivalent in the 1+1 dimensional case to the Dirac factorization of the Sine-Gordon equation. This gives yet another interpretation to the Bäcklund transformation. For the purposes of demonstrating this equivalence it is convenient to use the Sine-Gordon equation in the dimensionless form.

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} + \sin\psi = 0 \ . \tag{8-1}$$

To attempt a Dirac factorization of Eq. (8-1) we look for a first-order spinor equation which upon iteration requires all its spinor components satisfy (8-1).

For simplicity, we attempt a two-component factorization; that is, we look for a spinor equation of the form

$$(\underset{\sim}{1} \frac{\partial}{\partial t} + \underset{\sim}{\sigma}' \frac{\partial}{\partial x}) \psi = F(\psi) \tag{8-2}$$

where $\underset{\sim}{1}$ is the 2x2 identity matrix, $\underset{\sim}{\sigma}'$ is the Pauli spin matrix,

$$\underset{\sim}{\sigma}' \equiv \underset{\sim}{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

ψ is the two component spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and

$$F(\psi) = \begin{pmatrix} F_1(\psi_1,\psi_2) \\ F_2(\psi_1,\psi_2) \end{pmatrix}$$

is a spinor-valued function of ψ_1, ψ_2 to be determined so that when (8-2) is operated on from the left by the operator

$$\underset{\sim}{1} \frac{\partial}{\partial t} - \underset{\sim}{\sigma}' \frac{\partial}{\partial x}$$

the Sine-Gordon equation (8-1) results for both the spinor components ψ_1, ψ_2. In index notation (8-2) is

$$(\delta_{BC} \frac{\partial}{\partial t} + \sigma'_{BC} \frac{\partial}{\partial x}) \ \psi_C \ = \ F_B(\psi_1, \psi_2), \ B = 1,2 \ , \tag{8-3}$$

where δ_{BC} is the two-dimensional Kronecker delta, the summation convention is used and upper case Latin indices denote spinor components ranging over A,B,C = 1,2. We look for conditions $F_B(\psi_1, \psi_2)$, B = 1,2, must satisfy in order that when (8-2) is operated on by

$$\underset{\sim}{\Lambda}_{AB} \ \equiv \ \delta_{AB} \frac{\partial}{\partial t} - \sigma'_{AB} \frac{\partial}{\partial x} \tag{8-4}$$

it becomes

$$\frac{\partial^2 \psi_A}{\partial t^2} - \frac{\partial^2 \psi_A}{\partial x^2} + \sin \psi_A \ = \ 0, \quad A = 1,2, \tag{8-5}$$

Operating on Eq. (8-3) from the left with $\underset{\sim}{\Lambda}_{AB}$, it becomes

$$(\delta_{AB} \frac{\partial}{\partial t} - \sigma'_{AB} \frac{\partial}{\partial t})(\delta_{BC} \frac{\partial}{\partial t} + \sigma'_{BC} \frac{\partial}{\partial x}) \ \psi_C \ = \ (\delta_{AB} \frac{\partial}{\partial t} - \sigma'_{AB} \frac{\partial}{\partial x}) F_B(\psi)$$

which reduces to

$$\delta_{AC}(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}) \ \psi_C \ = \ \delta_{AB} \frac{\partial F_B}{\partial \psi_C} \frac{\partial \psi_C}{\partial t} - \sigma'_{AB} \frac{\partial F_B}{\partial \psi_C} \frac{\partial \psi_C}{\partial x} \tag{8-6}$$

where we have used the property of the Kronecker delta,

$$\delta_{AB}\,\delta_{AC} = \delta_{AC}$$

and the property

$$\sigma'_{AB}\,\sigma'_{BC} = \delta_{AC}$$

of the Pauli matrices.

Eq. (8-6) can be written in the matrix form

$$\underset{\sim}{1}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\psi = \underset{\sim}{1}\,\underset{\sim}{J}\,\frac{\partial\psi}{\partial t} - \underset{\sim}{\sigma}'\underset{\sim}{J}\,\frac{\partial\psi}{\partial x} \tag{8-7}$$

where we have defined the functional Jacobian matrix by

$$(\underset{\sim}{J})_{BC} \equiv \frac{\partial F_B}{\partial\psi_C}\,. \tag{8-8}$$

Now provided $\underset{\sim}{J}$ anti-commutes with $\underset{\sim}{\sigma}'$ the first order derivatives in (8-7) can be eliminated, since in that case

$$\underset{\sim}{\sigma}'\underset{\sim}{J} = -\,\underset{\sim}{J}\,\underset{\sim}{\sigma}' \tag{8-9}$$

and (8-7) can be written as

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\psi = \underset{\sim}{J}\left(\underset{\sim}{1}\,\frac{\partial}{\partial t} - \underset{\sim}{\sigma}'\,\frac{\partial}{\partial x}\right)\psi \tag{8-10}$$

which upon using (8-2) can be written in the form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\psi = \underset{\sim}{J}\,F(\psi)\,. \tag{8-11}$$

Using (8-8) this last equation can be expressed in the component form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) \psi_A - \frac{\partial F_A}{\partial \psi_B} F_B = 0, \quad A = 1,2 , \tag{8-14}$$

For this to be the Sine-Gordon equation (8-5), we must have F_A such that

$$\frac{\partial F_A}{\partial \psi_B} F_B = - \sin \psi_A, \quad A = 1,2 . \tag{8-13}$$

Thus, provided we can find an F_A such that (8-9) and (8-13) hold, that is, provided we can solve

$$\sigma'_{AB} \frac{\partial F_B}{\partial \psi_C} = - \frac{\partial F_A}{\partial \psi_B} \sigma'_{BC} \tag{8-14}$$

and

$$\frac{\partial F_A}{\partial \psi_B} F_B = - \sin \psi_A \tag{8-15}$$

for the F_A, then Eq. (8-2) with this F_A becomes the Sine-Gordon equation (8-5) when operated on by Λ_{AB}.

It is shown in Appendix H that a solution to the system of Equations (8-14), (8-15) is given by

$$F_1 = - a \sin\left(\frac{\psi_1+\psi_2}{2}\right) + \frac{1}{a} \sin\left(\frac{\psi_1-\psi_2}{2}\right) \tag{8-16a}$$

$$F_2 = a \sin\left(\frac{\psi_1+\psi_2}{2}\right) + \frac{1}{a} \sin\left(\frac{\psi_1-\psi_2}{2}\right) . \tag{8-16b}$$

Putting this into (8-2) we have the Dirac factorization of the 1+1 dimensional Sine-Gordon equation,

$$(\underset{\sim}{1} \frac{\partial}{\partial t} + \underset{\sim}{\sigma}' \frac{\partial}{\partial x})\psi = \begin{bmatrix} -a \sin(\frac{\psi_1+\psi_2}{2}) + \frac{1}{a}\sin(\frac{\psi_1-\psi_2}{2}) \\ \\ a \sin(\frac{\psi_1+\psi_2}{2}) + \frac{1}{a}\sin(\frac{\psi_1-\psi_2}{2}) \end{bmatrix} \qquad (8\text{-}17)$$

which written out in full is

$$\frac{\partial}{\partial t}\psi_1 + \frac{\partial}{\partial x}\psi_2 = -a \sin(\frac{\psi_1+\psi_2}{2}) + \frac{1}{a}\sin(\frac{\psi_1-\psi_2}{2}) \qquad (8\text{-}18a)$$

$$\frac{\partial}{\partial t}\psi_2 + \frac{\partial\psi_1}{\partial x} = a \sin(\frac{\psi_1+\psi_2}{2}) + \frac{1}{a}\sin(\frac{\psi_1-\psi_2}{2}) . \qquad (8\text{-}18b)$$

It is a straightforward calculation to verify that when (8-17) is operated on from the left by $\underset{\sim}{1} \frac{\partial}{\partial t} - \underset{\sim}{\sigma}' \frac{\partial}{\partial x}$, it yields the Sine-Gordon equation for each of its components. By construction, then, Eqs. (8-18) imply both ψ_1, ψ_2 satisfy the Sine-Gordon equation. But by the definition given in Chapter VII this is just what we mean by a Bäcklund transformation. It has thus been shown that at least in 1+1 dimensions the concept of Bäcklund transformation and the Dirac factorization are equivalent for the Sine-Gordon equation.

In fact, Konopelchenko[4] has also recently arrived at the Sine-Gordon Bäcklund transformation equations in essentially the same form (modulo a notation change) as (8-18) by completely different methods. His motivation for using this form for the Bäcklund transformation is that it is Lorentz invariant. It is particularly satisfying that the

Lorentz invariant Bäcklund transformation equations automatically occur if one uses the Dirac factoring technique to arrive at the equations.

The importance of the preceding discussion, in addition to presenting a new physical interpretation of the Bäcklund transformation, is to suggest a systematic technique for finding the Bäcklund transformation equations in 3+1 dimensions. Namely, one attempts to perform a Dirac-type factoring of the Sine-Gordon equation in 3+1 dimensions, the resulting equations should then be the desired Bäcklund transformation. This method of obtaining Bäcklund transformations leads to simpler transformation equations than previously reported in the literature and will be discussed next.

Bäcklund Transformations for the Sine-Gordon Equation in 3+1 Dimensions

Using the method due to Rund[5] of deriving Bäcklund transformations from the equations of motion and the factoring

$$\underset{\sim}{1}\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) = \left(\underset{\sim}{1}\frac{\partial}{\partial t} - \vec{\sigma}\cdot\nabla\right)\left(\underset{\sim}{1}\frac{\partial}{\partial t} + \vec{\sigma}\cdot\nabla\right)$$

we arrive at the system

$$\left(\underset{\sim}{1}\frac{\partial}{\partial t} + \vec{\sigma}\cdot\nabla\right)\left(\frac{\psi+\psi'}{2}\right) = \underset{\sim}{A}\sin\left(\frac{\psi-\psi'}{2}\right) \tag{8-19a}$$

$$\left(\underset{\sim}{1}\frac{\partial}{\partial t} - \vec{\sigma}\cdot\nabla\right)\left(\frac{\psi-\psi'}{2}\right) = -\underset{\sim}{A}^{-1}\sin\left(\frac{\psi+\psi'}{2}\right) \tag{8-19b}$$

where $\vec{\sigma}\cdot\nabla = \sigma^i \frac{\partial}{\partial x^i}$, σ^i being the usual Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta_{ij}$$

$\underset{\sim}{1}$ is the 2x2 identity, and $\underset{\sim}{A}$ is any 2x2 invertible matrix of real entries which are to be interpreted as the Bäcklund transformation parameters. It may easily be shown that Eqs. (8-19) require that both the real scalar functions $\psi(x^\mu)$, $\psi'(x^\mu)$ satisfy

$$\frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi + \sin\phi = 0 \qquad (8\text{-}20)$$

and thus constitute a Bäcklund-type transformation for the 3+1 dimensional Sine-Gordon equation.

In fact, the transformation equations (8-19) are equivalent to the equations recently proposed by Leibbrandt[6] with the important exception that in (8-19), as opposed to Leibbrandt's equations, the σ-matrices are chosen in their usual quantum mechanical representations. Moreover, the Eqs. (8-19) follow systematically by performing a Dirac factorization of the d'Alembertian and then following a well-established method for deriving Bäcklund transformation equations.[5] The actual derivation of (8-19) is rather lengthy and will not be reproduced here since a similar derivation of a much simpler set of Bäcklund transformation equations will be given a little later. Besides, once one has the transformation Eqs. (8-19), it is of little consequence as to how they were found, since it may easily be shown that they are in fact a Bäcklund transformation.

To see this, we operate on (8-19a) from the left with the operator

$$\underset{\sim}{\Lambda}_- \equiv \underset{\sim}{1}\frac{\partial}{\partial t} - \vec{\sigma}\cdot\nabla$$

and on (8-19b) with

$$\underset{\sim}{\Lambda}_+ \equiv \underset{\sim}{1}\frac{\partial}{\partial t} + \vec{\sigma}\cdot\nabla$$

to obtain

$$\underset{\sim}{1}\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(\frac{\psi+\psi'}{2}\right) = \cos\left(\frac{\psi-\psi'}{2}\right)\left[\left(\underset{\sim}{1}\frac{\partial}{\partial t} - \vec{\sigma}\cdot\nabla\right)\left(\frac{\psi-\psi'}{2}\right)\right]\underset{\sim}{A}$$

and

$$\underset{\sim}{1}\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(\frac{\psi-\psi'}{2}\right) = -\cos\left(\frac{\psi+\psi'}{2}\right)\left[\left(\underset{\sim}{1}\frac{\partial}{\partial t} + \vec{\sigma}\cdot\nabla\right)\left(\frac{\psi+\psi'}{2}\right)\right]\underset{\sim}{A}^{-1} .$$

Using (8-19) again to eliminate the terms in brackets, these last two equations become

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(\frac{\psi+\psi'}{2}\right) = \cos\left(\frac{\psi-\psi'}{2}\right)\sin\left(\frac{\psi+\psi'}{2}\right) \qquad \text{(8-21a)}$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(\frac{\psi-\psi'}{2}\right) = -\cos\left(\frac{\psi+\psi'}{2}\right)\sin\left(\frac{\psi-\psi'}{2}\right) . \qquad \text{(8-21b)}$$

Now adding Eqs. (8-21) gives

$$\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \sin\psi = 0$$

while subtracting yields

$$\frac{\partial^2}{\partial t^2}\psi' - \nabla^2\psi' + \sin\psi' = 0 .$$

Thus, Eqs. (8-19) imply both ψ,ψ' satisfy the Sine-Gordon equation (8-20) and as such constitute a Backlund-type transformation of the Sine-Gordon system.

Inasmuchas the d'Alembertian admits other factorizations, there are apparently many other forms of Bäcklund-type transformations possible. Indeed, using the factoring[7]

$$\underset{\sim}{1}\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) = (\gamma^\mu\partial_\mu)(\gamma^\mu\partial_\mu)$$

and the same technique as before we can arrive at the Bäcklund-type transformations

$$\gamma^\mu\partial_\mu\left(\frac{\psi+\psi'}{2}\right) = \underset{\sim}{A}\sin\left(\frac{\psi-\psi'}{2}\right) \qquad (8\text{-}22a)$$

$$\gamma^\mu\partial_\mu\left(\frac{\psi-\psi'}{2}\right) = -\underset{\sim}{A}^{-1}\sin\left(\frac{\psi+\psi'}{2}\right) . \qquad (8\text{-}22b)$$

Here γ^μ are the usual Dirac matrices satisfying

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

and $\underset{\sim}{A}$ is a constant 4x4 real matrix. Again, the Eqs. (8-22) requires that the two real, scalar fields ψ,ψ', satisfy the Sine-Gordon equation (8-20) and thus constitute a Bäcklund transformation. It should be noted that the system of equations (8-22) is apparently over-determined, but due to the properties of the Dirac matrices reduces to two Sine-Gordon equations.

The technique of showing that Eqs. (8-22) imply ψ, ψ' satisfy the Sine-Gordon equation is identical to the previous Bäcklund transformation, (8-19). Operating on Eqs. (8-22) with the operator $\gamma^\mu \partial_\mu$, we obtain

$$(\gamma^\mu \partial_\mu)(\gamma^\nu \partial_\nu)\left(\frac{\psi+\psi'}{2}\right) = \cos\left(\frac{\psi-\psi'}{2}\right) \left[\gamma^\mu \partial_\mu \left(\frac{\psi-\psi'}{2}\right)\right] \underset{\sim}{A}$$

$$(\gamma^\mu \partial_\mu)(\gamma^\nu \partial_\nu)\left(\frac{\psi-\psi'}{2}\right) = -\cos\left(\frac{\psi+\psi'}{2}\right) \left[\gamma^\mu \partial_\mu \left(\frac{\psi+\psi'}{2}\right)\right] \underset{\sim}{A}^{-1}$$

which upon using (8-22) to eliminate the terms in brackets become

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(\frac{\psi+\psi'}{2}\right) = -\cos\left(\frac{\psi-\psi'}{2}\right) \sin\left(\frac{\psi+\psi'}{2}\right) \tag{8-23a}$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(\frac{\psi-\psi'}{2}\right) = -\cos\left(\frac{\psi+\psi'}{2}\right) \sin\left(\frac{\psi-\psi'}{2}\right) \tag{8-23b}$$

where we have used the commutation relations $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ as well as the fact that $\underset{\sim}{A}\underset{\sim}{A}^{-1} = \underset{\sim}{A}^{-1}\underset{\sim}{A} = \underset{\sim}{1}$. Again, by adding (subtracting) Eqs. (8-23), one finds $\psi(\psi^1)$ must satisfy the Sine-Gordon equation (8-20).

Yet a third Bäcklund-type transformation can be found using the fact that

$$\frac{\partial^2}{\partial t^2} - \nabla^2 = \partial^\mu \partial_\mu .$$

This yields the particularly simple Bäcklund transformation equations

$$\partial_\mu \left(\frac{\psi+\psi'}{2}\right) = a_\mu \sin\left(\frac{\psi-\psi'}{2}\right) \tag{8-24a}$$

$$\partial_\mu\left(\frac{\psi-\psi'}{2}\right) = a_\mu \sin\left(\frac{\psi+\psi'}{2}\right) \tag{8-24b}$$

where a_μ is a real, constant 4-vector such that

$$a_\mu a^\mu = -1 . \tag{8-25}$$

Provided the Bäcklund parameters a_μ (only three of which are independent) satisfy the constraint (8-25), Eqs. (8-24) imply both $\psi(x)$, $\psi'(x)$ satisfy the 3+1 dimensional Sine-Gordon equation.

To illustrate the procedure used to derive all the preceding Bäcklund transformations we will outline the derivation of the Bäcklund transformation equations (8-24). The method of derivation essentially follows that of Rund[5].

We introduce the (nonlinear) operator $N[\cdot]$ defined by

$$N[\phi] \equiv \partial^\mu\partial_\mu \phi + \sin\phi \tag{8-26}$$

so that the Sine-Gordon equation can be expressed as

$$N[\phi] = 0 .$$

We then consider the difference Δ defined by

$$\Delta \equiv N[\psi'] - N[\psi] \tag{8-27}$$

which upon using (8-26) may be written as

$$\Delta = \partial^\mu\partial_\mu(\psi'-\psi) + \sin\psi' - \sin\psi . \tag{8-28}$$

Now making the regular change of variable

$$u = \frac{1}{2}(\psi'-\psi) \tag{8-29a}$$

$$v = \frac{1}{2}(\psi'+\psi) \tag{8-29b}$$

which possess the inverse transformation

$$\psi' = u+v \tag{8-30a}$$

$$\psi = u-v \tag{8-30b}$$

Eq. (8-28) becomes

$$\Delta = 2(\partial_\mu \partial^\mu v + \cos u \sin v) \ . \tag{8-31}$$

Now by definition, a pair of relations of the form

$$\partial_\mu v = F_\mu(u) \tag{8-32a}$$

$$\partial_\mu u = G_\mu(v) \tag{8-32b}$$

are a Bäcklund transformation if the 4-vector functions $F_\mu(u)$, $G_\mu(v)$ are such as to ensure that Eqs. (8-32) imply

$$\Delta \equiv 0 \ .$$

Since then, if ψ satisfy the Sine-Gordon equation, ψ' must also and vice versa.

Taking the 4-divergence of Eq. (8-32a) gives

$$\partial^\mu \partial_\mu v = \frac{dF_\mu(u)}{du} \partial^\mu u$$

and using (8-32b) to eliminate $\partial_\mu u$ yields

$$\partial^\mu \partial_\mu v = \frac{dF_\mu(u)}{du} G^\mu(v) . \quad (8\text{-}33)$$

Putting (8-33) into (8-31), it becomes

$$\Delta = 2[\frac{dF_\mu(u)}{du} G^\mu(v) + \cos u \sin v] . \quad (8\text{-}34)$$

This will vanish making Eqs. (8-32) a Bäcklund transformation provided $F_\mu(u)$ and $G_\mu(v)$ are chosen such that

$$\frac{dF_\mu(u)}{du} G^\mu(v) + \cos u \sin v = 0$$

or

$$\left[\frac{\frac{dF_\mu(u)}{du}}{\cos u}\right] \cdot \left[\frac{G^\mu(v)}{\sin v}\right] = -1 . \quad (8\text{-}35)$$

Since u,v are independent variables, Eq. (8-35) can hold provided

$$\frac{dF_\mu(u)}{du} / \cos u = a_\mu \quad (8\text{-}36a)$$

$$G^\mu(v)/\sin v = b^\mu \quad (8\text{-}36b)$$

where a_μ, b^μ are constant 4-vectors such that

$$a_\mu b^\mu = -1 . \quad (8\text{-}37)$$

Eqs. (8-36) can be solved to give

$$F_\mu(u) = a_\mu \sin u \tag{8-38a}$$

$$G_\mu(v) = b^\mu \sin v \tag{8-38b}$$

and putting these into Eqs. (8-32) we find

$$\partial_\mu u = b_\mu \sin v \tag{8-39a}$$

$$\partial_\mu v = a_\mu \sin u \ . \tag{8-39b}$$

These may be written in terms of ψ,ψ' using Eqs. (8-29) to arrive at

$$\partial_\mu\left(\frac{\psi'+\psi}{2}\right) + b_\mu \sin\left(\frac{\psi'-\psi}{2}\right) \tag{8-40a}$$

$$\partial_\mu\left(\frac{\psi'-\psi}{2}\right) + a_\mu \sin\left(\frac{\psi'+\psi}{2}\right) \ . \tag{8-40b}$$

where $a^\mu b_\mu = -1$. For $\psi = 0$, which is a solution of the Sine-Gordon equation, Eqs. (8-40) must reduce to the same equation. This requires $b_\mu = a_\mu$, and we arrive at Eqs. (8-24), (8-25) which was the desired result.

Again, it is a simple matter to check that Eqs. (8-24) constitute a Bäcklund transformation for the Sine-Gordon equation. Operating on Eqs. (8-24) with ∂^μ they become

$$\partial^\mu\partial_\mu\left(\frac{\psi+\psi'}{2}\right) = a_\mu \cos\left(\frac{\psi-\psi'}{2}\right) \cdot \partial^\mu\left(\frac{\psi-\psi'}{2}\right) \tag{8-41a}$$

$$\partial^\mu\partial_\mu\left(\frac{\psi-\psi'}{2}\right) = a_\mu \cos\left(\frac{\psi+\psi'}{2}\right) \cdot \partial^\mu\left(\frac{\psi+\psi'}{2}\right) \tag{8-41b}$$

and using Eqs. (8-24) again to eliminate the first-order derivatives one arrives at

$$\partial^\mu \partial_\mu \left(\frac{\psi+\psi'}{2}\right) = a_\mu a^\mu \cos\left(\frac{\psi-\psi'}{2}\right) \sin\left(\frac{\psi+\psi'}{2}\right)$$

$$\partial^\mu \partial_\mu \left(\frac{\psi-\psi'}{2}\right) = a_\mu a^\mu \cos\left(\frac{\psi+\psi'}{2}\right) \sin\left(\frac{\psi-\psi'}{2}\right) .$$

Using (8-25) these in turn become

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \left(\frac{\psi+\psi'}{2}\right) + \cos\left(\frac{\psi-\psi'}{2}\right) \sin\left(\frac{\psi+\psi'}{2}\right) = 0 \tag{8-42a}$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \left(\frac{\psi-\psi'}{2}\right) + \cos\left(\frac{\psi+\psi'}{2}\right) \sin\left(\frac{\psi-\psi'}{2}\right) = 0 . \tag{8-42b}$$

Adding Eqs. (8-42) one finds

$$\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + \sin\psi = 0$$

while subtracting gives

$$\frac{\partial^2 \psi'}{\partial t^2} - \nabla^2 \psi' + \sin\psi' = 0 .$$

Thus, Eqs. (8-24) with (8-25) imply both ψ, ψ' satisfy the Sine-Gordon equation and hence form a Bäcklund transformation.

Generation of Solutions

Any of the Bäcklund transformations of the previous section may be used to find a "new" solution ψ of the 3+1 dimensional Sine-Gordon equation by starting from an "old", known solution, ψ', which is one of the important applications of the transformation. To illustrate this procedure, we will use the Bäcklund transformation given by Eqs. (8-24). To find a new solution ψ we set $\psi' = 0$ (which is the trivial "vacuum" solution of the Sine-Gordon equation), so that Eqs. (8-24) reduce to the single equation

$$\partial_\mu \psi/2 = a_\mu \sin \psi/2 . \qquad (8\text{-}43)$$

Eq. (8-43) can be immediately integrated to obtain the solution

$$\psi = 4 \tan^{-1} [A \exp(a_\mu x^\mu + \delta)] \qquad (8\text{-}44)$$

where A, δ are integration constants and a_μ are the Bäcklund parameters subject to the constraint $a_\mu a^\mu = -1$. We note that the solution (8-44) has well established soliton-like properties.[6]

If we parameterize a_μ as

$$a_\mu = (a_o, a_1, a_2, a_3)$$

$$= (\sin\theta \sin\phi \sinh \tau,\ \cos\theta,\ \sin\theta \cos\phi,\ \sin\theta \sin\phi \cosh \tau)$$

where the Bäcklund parameters can take on the values

$$0 \leq \theta \leq \pi ,$$

$$0 \leq \phi \leq 2\pi ,$$

$$-\infty < \tau < \infty ,$$

then $a_\mu a^\mu = -1$ and (8-44) is identical to a solution previously reported in the literature.[6] Again the Bäcklund parameters a_μ clearly are related to the propagation velocity of the soliton-like solution we have "generated" from the vacuum.

While we have used the Bäcklund transformation of Eqs. (8-24) to arrive at the solution (8-44), we could have as well used Eqs. (8-19) or (8-22) and in the same fashion arrived at (8-44). Indeed, in Reference 3 we derive the solution (8-44) using the Bäcklund transformation of Eqs. (8-22). The Bäcklund transformation of Eqs. (8-24), however, leads much more directly to the solution due to its inherently simpler structure. In principle this procedure could be repeated indefinitely, always producing a new solution from a known solution by performing a single quadrature. In practice however this is not necessary since a nonlinear superposition principle can be obtained which alleviates even the need to do any integrations and allows the algebraic construction of solutions.

The Nonlinear Superposition Principle

A Bäcklund transformation closely related to Eqs. (8-24) is

$$\partial_\mu \left(\frac{\alpha - i\beta}{2}\right) = a_\mu \sin\left(\frac{\alpha + i\beta}{2}\right) \tag{8-45}$$

where again a_μ is a real 4-vector such that $a_\mu a^\mu = -1$. Eq. (8-45) requires that the real functions $\alpha(x)$, $\beta(x)$ satisfy

$$\frac{\partial^2}{\partial t^2}\alpha - \nabla^2\alpha + \sin\alpha = 0 \tag{8-46a}$$

$$\frac{\partial^2}{\partial t^2}(i\beta) - \nabla^2(i\beta) + \sin(i\beta) = 0 \tag{8-46b}$$

and thus implies a transformation from the "old" solution, α, to the "new" solution $i\beta$. This can be represented symbolically by

$$i\beta(x,a_\mu) = \underset{\sim}{B}(a_\mu),\ \alpha(x) \tag{8-47}$$

where $B(a_\mu)$ is the Bäcklund operator which depends on the Bäcklund parameters a_μ $(a_\mu a^\mu = -1)$. The corresponding Bianchi diagram is shown in Figure 17.

The Bäcklund transformation of Eq. (8-45) is particularly simple in form and may be used to find the generating formula (nonlinear superposition principle) which enables us to derive new solutions without performing additional quadratures. Referring to the Bianchi diagram of Figure 18 and abbreviating $\beta(x,a^i_\mu) = \beta_i$, $i = 1,2$, we have the corresponding four transformations

$$\partial_\mu\left(\frac{\alpha_o - i\beta_1}{2}\right) = a^1_\mu \sin\left(\frac{\alpha_o + i\beta_1}{2}\right) \tag{8-48a}$$

$$\partial_\mu\left(\frac{\alpha_o - i\beta_2}{2}\right) = a^2_\mu \sin\left(\frac{\alpha_o + i\beta_2}{2}\right) \tag{8-48b}$$

$$\partial_\mu\left(\frac{i\beta_1 - \alpha}{2}\right) = a^2_\mu \sin\left(\frac{\alpha + i\beta_1}{2}\right) \tag{8-48c}$$

$$\partial_\mu\left(\frac{i\beta_2 - \alpha}{2}\right) = a^1_\mu \sin\left(\frac{\alpha + i\beta_2}{2}\right) . \tag{8-48d}$$

It is shown in Appendix I that from Eqs. (8-48) one can obtain the nonlinear superposition principle

$$\tan\left(\frac{\alpha - \alpha_o}{4}\right) = \pm \left[(1 - a^1_\mu a^{2\mu})(1 + a^1_\mu a^{2\mu})^{-1}\right]^{1/2} \tanh\left(\frac{\beta_1 - \beta_2}{4}\right) \tag{8-49}$$

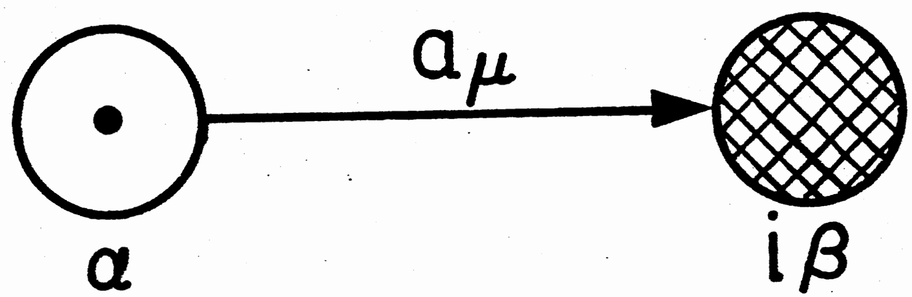

Figure 17. Bianchi Diagram for the Bäcklund Transformation Equation (8-47) Characterized by the Real Parameter a_μ With $a^\mu a_\mu = -1$

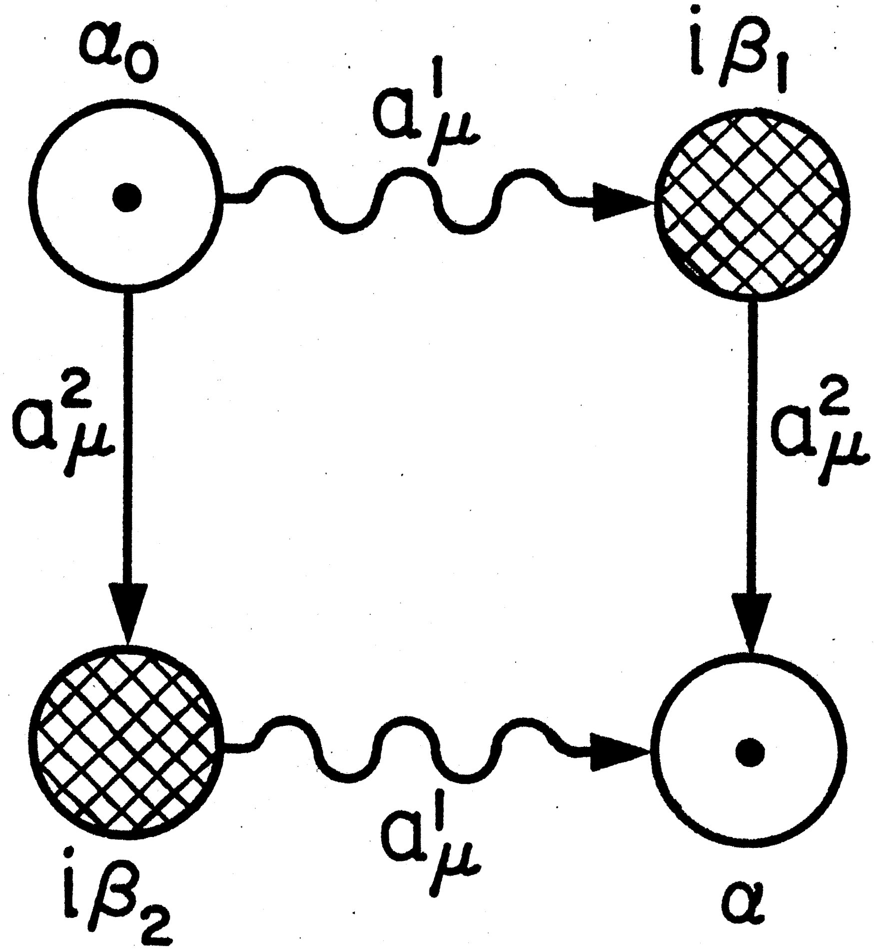

Figure 18. Bianchi Diagram Used in Deriving Formula (8-49). The Bäcklund Parameters a^i_μ ($i=1,2$) are Subject to the Constraint $a^i_\mu a^{i\mu} = -1$ (No Sum on i)

from which a new solution, α, may be obtained from any three known solutions α_o, $i\beta_1$, $i\beta_2$.

By choosing the parameterization (which identically satisfies $a^i_\mu a^{i\mu}$, no sum on i),

$$a^i_\mu = (\sin\theta_i \sin\phi_i \sinh\tau_i, \cos\theta_i, \sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i \cosh\tau_i) \ , \qquad (8\text{-}50)$$

$$0 \leq \theta_i \leq \pi \ ,$$

$$0 \leq \phi_i \leq 2\pi \ , \qquad i = 1,2$$

$$-\infty < \tau_i < \infty \ ,$$

one finds that the generating formula (8-49) is identical to the one obtained by Leibbrandt.[6] Using (8-49) a new solution α may be obtained by combining old solutions.

Of course, in deriving (8-49) we have implicitly assumed that the Bianchi diagram of Figure 18 holds, that is, that the Bäcklund transformations of that diagram commute. This can be proven using the same method used to prove the corresponding result in the 1+1 dimensional case[9,10] by exploiting the formal similarity of the Bäcklund transformation (8-45) and the Bäcklund transformation of the 1+1 dimensional Sine-Gordon equation.

FOOTNOTES

[1]P. A. M. Dirac, Proc. Roy. Soc. London <u>A117</u>, 610 (1928).

[2]Ibid., <u>The Principles of Quantum Mechanics</u>, 4th ed. (Oxford, England, 1958), Chapter 11.

[3]W. Wilson and N. V. V. J. Swamy, Nuo. Cim. <u>A56</u>, 41 (1980).

[4]B. G. Konopelchenko, J. Phys. A<u>12</u>, 1937 (1979).

[5]H. Rund, in <u>Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications</u>, ed. by R. M. Miura (Springer-Verlag, New York, 1976), 199f.

[6]G. Leibbrandt, Phys. Rev. Lett. <u>41</u>, 435 (1978).

[7]We use the notation $x = x^{\mu} = (x^{o}, x^{i}) = (t, \vec{x})$ with $c = 1$ and metric $g^{\mu\nu}$ with signature (1,-1,-1-1). We write ∂_{μ} for $\frac{\partial}{\partial x^{\mu}}$ and use the summation convention throughout.

[8]Reference 6, Eq. (9).

[9]D. W. McLaughlin and A. C. Scott, J. Math. Phys. <u>14</u>, 1817 (1973).

[10]H. Steudel, Ann. Physik <u>7</u>, 205 (1975).

CHAPTER IX

SUMMARY AND CONCLUSIONS

In this investigation we have shown that the presence of soliton structures in a physical system produces important observable effects. We have demonstrated that the apparently unrelated concepts of magnetic charge and soliton are both described by the Sine-Gordon equation and hence are in fact closely related. We have also presented a new physical interpretation of the Bäcklund transformation and used this interpretation in deriving new Bäcklund transformations for the (3+1)-dimensional Sine-Gordon equation.

After introducing the concept of a soliton, we reviewed how magnetically charged particles arise as solitons in the gauge theories of elementary particles. This led us to consider the propagation of electromagnetic radiation in a plasma where at least some of the plasma particles may be magnetically charged. It was found that the presence of magnetic charge on the plasma particles leads to non-trivial consequences. In particular, the number of plasma wave modes was found to be double that of a standard plasma of electrically charged particles only. In effect, the presence of magnetic charges allows the existence of an additional longitudinal as well as transverse mode in the plasma waves. The appearance of these new modes can be used as the basis for searching for magnetically charged particles and in Chapter V we outline a way to use pulsar emissions as a probe for the presence of magnetic charges in the interstellar plasma. It should be noted, however, that while this method can in principle detect magnetic charges, it is incapable of distinguishing between the Dirac and Schwinger quantization conditions relating the electric and magnetic couplings. Its value, then, is as a qualitative test for the sheer detection of magnetic charges.

In Chapter VI we looked at the Josephson junction and showed how the Sine-Gordon equation arises as its basic equation with the solitons corresponding to quantized magnetic flux vortices. An analogy was drawn between the flux vortex/solitons and two-dimensional magnetic charge. Again, it was found that the presence of solitons dramatically affected the propagation of electromagnetic radiation in the Josephson barrier, introducing an entirely new mode of propagation.

The importance of these two results lies not so much in the particular phenomena they attempt to describe as in their following general implications. First, since solitons behave as though they were particles they should be expected to exhibit "plasma-type" collective oscillations. And second, the interaction of solitons with elementary exitations (e.g., phonons, photons, etc.) can produce a significantly different spectrum than the one obtained for the case when no solitons are present. These two areas are just beginning to be explored but are clearly of the utmost importance.

Finally we turned to the Sine-Gordon equation itself and applied the idea of Bäcklund transformation to it. A hitherto unsuspected physical interpretation of the Bäcklund transformation as the Dirac factorization of the Sine-Gordon equation was given and this interpretation was used to derive several new Bäcklund-type transformations in

3+1 dimensions. These new transformations are much simpler than any previously reported in the literature. The real use of these transformations, besides revealing the underlying structure of the nonlinear equation, is in the generation of new solutions starting from known solutions.

It is not inconceivable that this analogy between the Dirac factorization technique and Bäcklund transformations may be deeper than a mere formal interpretation. It may be that the Bäcklund transformations, inasmuchas they involve the use of Pauli spin matrices at any rate, are introducing something in the nature of internal degrees of freedom heretofore unsuspected in the different phenomena which the Sine-Gordon equation describes. There seems to be a case for studying this point further and the results presented here should be considered only a first step in this direction.

As Schwinger has pointed out, the existence of quarks should imply the existence of magnetic charges or dyons and the next fundamental discovery in physics may lie in this direction. At least Schwinger's observation ought to make the search for magnetic charges a serious quest. We hope the studies and suggestions made in this work will be a contribution to this quest.

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APPENDIX A

A PROOF OF DERRICK'S THEOREM

A simple heuristic proof of Derrick's theorem follows from an elementary scaling argument. A more rigorous demonstration essentially along the same lines has been given in References 24 and 25 of Chapter II.

The Lagrangian density (2-38) leads in the usual way to the Hamiltonian density

$$\mathcal{H}[\phi] = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 + (\nabla\phi)^2\right] + U(\phi), \quad U(\phi) \geq 0, \tag{A-1}$$

where we have denoted the N-dimensional gradient by

$$\nabla\phi \equiv \left(\frac{\partial\phi}{\partial x^1}, \frac{\partial\phi}{\partial x^2}, \ldots, \frac{\partial\phi}{\partial x^N}\right) .$$

The total energy is then given by

$$E[\phi] = \int \mathcal{H}[\phi]\, d^N x \tag{A-2}$$

where $d^N x \equiv dx^1 dx^2 \ldots dx^N$ is the N-space volume element and the integral is to be taken over all space.

Suppose now that $\phi_S = \phi_S(\vec{x}) = \phi_S(x^1, x^2, \ldots, x^N)$ is a known t-independent, finite energy solution of the wave equation resulting from (2-38). We would like to find conditions under which ϕ_S is not a stable solution. To investigate this we form the energy of ϕ_S,

$$E \equiv E[\phi_S] = \int H[\phi_S]\, d^N x$$

$$= \int [\tfrac{1}{2}(\nabla\phi_S)^2 + U(\phi_S)]\, d^N x \qquad \text{(A-3)}$$

since $\phi_S = \phi_S(\vec{x})$ only. The integral (A-3) exists by our finite energy hypothesis.

Separating the total energy E into its kinetic and potential energy parts we have,

$$E = KE + PE \qquad \text{(A-4)}$$

where

$$KE = \int \tfrac{1}{2}(\nabla\phi_S)^2\, d^N x \geq 0 \qquad \text{(A-5)}$$

$$PE = \int U(\phi_S)\, d^N x \geq 0 \qquad \text{(A-6)}$$

Since $U(\phi) \geq 0$ by assumption. Now if ϕ_S is stable, a slight perturbation, say

$$\phi_S \rightarrow \phi_{PER} = \phi_S + \delta\phi$$

should be energetically unfavorable. In particular, the specific perturbation of scale

$$\phi_S \rightarrow \phi_{PER} \equiv \phi(\lambda) = \phi_S(\lambda\vec{x}) \qquad \text{(A-6)}$$

should yield an energy minimum for $\lambda = 1$, since then $\phi_{PER} = \phi_S$. Computing the energy of $\phi(\lambda)$, we find using (A-1) and (A-2)

$$E(\lambda) \equiv E[\phi(\lambda)] = \int [\tfrac{1}{2}(\nabla\phi(\lambda))^2 + U(\phi(\lambda))]\, d^N x$$

$$= \int [\tfrac{1}{2}(\nabla\phi_s(\lambda\vec{x}'))^2 + U(\phi_s(\lambda\vec{x}'))]\, d^N x'$$

Making the change of variables

$$\vec{x} = \lambda\vec{x}'$$

this gives

$$E(\lambda) = \lambda^{-N} \int [\tfrac{\lambda^2}{2}(\nabla\phi_s(\vec{x}))^2 + U(\phi_s(\vec{x}))]\, d^N x$$

$$= \lambda^{2-N} \int \tfrac{1}{2}(\nabla\phi_s)^2\, d^N x + \lambda^{-N} \int U(\phi_s)\, d^N x$$

or

$$E(\lambda) = \lambda^{2-N}\, KE + \lambda^{-N}\, PE \qquad \text{(A-7)}$$

where use has been made of (A-5) and (A-6).

But as we have argued, $\lambda = 1$ is an energy minimum and this requires from elementary calculus that

$$\frac{d}{d\lambda} E(\lambda)\Big|_{\lambda=o} = 0 \qquad \text{(A-8)}$$

and

$$\frac{d^2 E(\lambda)}{d\lambda^2}\Big|_{\lambda=o} > 0\,. \qquad \text{(A-9)}$$

Condition (A-8) with (A-7) implies

$$\frac{d}{d\lambda} E(\lambda)\Big|_{\lambda=o} = (2-N)\ KE - N\cdot PE = O$$

or

$$KE = \frac{N}{2-N} PE\ . \tag{A-10}$$

This result in itself is a virial-type theorem for theories of the form (2-38) since Eq. (A-10) relates the kinetic and potential energies of the solution ϕ_S.

Now condition (A-9) with (A-7) on the other hand requires

$$\frac{d^2}{d\lambda^2} E(\lambda)\Big|_{\lambda=o} = (2-N)(1-N)\ KE + N(1+N)\ PE > O$$

and with the use (A-10) we can write this in the form

$$2(2-N)\ KE > O \tag{A-11}$$

Since by (A-5) KE $\geq$ O, this last result demands

$$2-N > O$$

if the perturbation is stable. Or, in other words, the number of space dimensions must be less than two; i.e., only for N = 1 are t-independent, finite energy, stable solutions possible. This result is known as Derrick's theorem.

APPENDIX B

A "DERIVATION" OF THE DIRAC QUANTIZATION CONDITION

A naive argument in support of the Dirac quantization condition (3-7) can be made from elementary considerations. We consider the motion of a dually charged particle of mass m, electric charge e_1, and magnetic charge g_1 moving in the field of similar particle of electric charge e_2, magnetic charge g_2 assumed fixed at the origin. An equivalent analysis follows if the second particle is allowed to move provided one looks at the relative motion and replaces m by the reduced mass.

The non-relativistic equation of motion is

$$m \frac{d\vec{v}}{dt} = e_1(\vec{E}_2 + \frac{\vec{v}}{c} \times \vec{B}_2) + g_1(\vec{B}_2 - \frac{\vec{v}}{c} \times \vec{E}_2) \qquad \text{(B-1)}$$

where $\vec{r}$ is the position of the first particle, $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{E}_2$, $\vec{B}_2$ are the fields due to the dyon at the origin

$$\vec{E}_2 = \frac{e_2\vec{r}}{r^3}, \quad \vec{B}_2 = \frac{g_2\vec{r}}{r^3} \qquad \text{(B-2)}$$

Putting these into Eq. (B-1), it becomes

$$m \frac{d^2\vec{r}}{dt^2} = (e_1e_2 + g_1g_2) \frac{\vec{r}}{r^3} + (e_1g_2 - e_2g_1) \frac{\vec{v}}{c} \times \frac{\vec{r}}{r^3} . \qquad \text{(B-3)}$$

Taking $\vec{r}\times$ of Eq. (B-3) we find

$$m\,\vec{r}\times\frac{d\vec{v}}{dt} = (e_1g_2 - g_1e_2)\,\vec{r}\times\left(\frac{\vec{v}}{c}\times\frac{\vec{r}}{r^3}\right) \qquad \text{(B-4)}$$

since $\vec{r}\times\vec{r} = 0$.

Now using the vector identities

$$\frac{d}{dt}(\vec{r}\times m\vec{v}) = m\,\vec{r}\times\frac{d\vec{v}}{dt}$$

and

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \vec{r}\times\left(\vec{v}\times\frac{\vec{r}}{r^3}\right),$$

Eq. (B-4) can be written as

$$\frac{d}{dt}(\vec{r}\times m\vec{v}) = \frac{d}{dt}\left[(e_1g_2 - g_1e_2)\frac{\vec{r}}{cr}\right]$$

Upon identifying the orbital angular momentum $\vec{L}\equiv\vec{r}\times m\vec{v}$ we have

$$\frac{d}{dt}\left[\vec{L} - (e_1g_2 - g_1e_2)\frac{\vec{r}}{cr}\right] = 0\,. \qquad \text{(B-5)}$$

This last result suggests we should define the total conserved angular momentum

$$\vec{J} = \vec{L} + \vec{S} \qquad \text{(B-6)}$$

where we have denoted

$$\vec{S} = -(e_1g_2 - g_1e_2)\frac{\vec{r}}{cr}\,. \qquad \text{(B-7)}$$

The physical interpretation of the second term $\vec{S}$ can be seen by calculating the total angular momentum of the electromagnetic field. From classical electromagnetic theory this is given by

$$\vec{S} = \int \vec{r} \times \vec{P}_{FIELD}\, d^3r \tag{B-8}$$

where

$$\vec{P}_{FIELD} = \frac{1}{4\pi c} \vec{E}_{TOTAL} \times \vec{B}_{TOTAL} \tag{B-9}$$

is the electromagnetic field momentum density and the integral is to be taken over all space.

In our case

$$\vec{E}_{TOTAL} = \vec{E} + \frac{e_2\vec{r}}{r^3}\,, \quad \vec{B}_{TOTAL} = \vec{B} + \frac{g_2\vec{r}}{r^3} \tag{B-10}$$

where $\vec{E},\vec{B}$ is the field of the particle at $\vec{r}$, so we have

$$\vec{P}_{FIELD}(\vec{r}\,') = \frac{1}{4\pi c}\left[\vec{E}(\vec{r}\,') + \frac{e_2\vec{r}\,'}{r'^3}\right] \times \left[\vec{B}(\vec{r}\,') + \frac{g_2\vec{r}\,'}{r'^3}\right]$$

$$= \frac{1}{4\pi c}\left[\vec{E}(\vec{r}\,') \times \frac{g_2\vec{r}\,'}{r'^3} + \frac{e_2\vec{r}\,'}{r'^3} \times \vec{B}(\vec{r}\,')\right]. \tag{B-11}$$

To arrive at (B-11) we have used the fact that

$$\vec{E}(\vec{r}\,') \times \vec{B}(\vec{r}\,') = \frac{r_1(\vec{r}\,'-\vec{r})}{|\vec{r}\,'-\vec{r}|^3} \times \frac{g_1(\vec{r}\,'-\vec{r})}{|\vec{r}\,'-\vec{r}|^3} = 0$$

and

$$\frac{e_2 \vec{r}\,'}{r'^3} \times \frac{g_2 \vec{r}\,'}{r'^3} = 0 .$$

Putting (B-11) into (B-8) we obtain

$$\vec{S} = \frac{1}{4\pi c} \int \vec{r}\,' \times \left[\left(g_2 \vec{E}(\vec{r}\,') - e_2 \vec{B}(\vec{r}\,') \right) \times \frac{\vec{r}\,'}{r'^3} \right] d^3r'$$

$$= \frac{1}{4\pi c} \int \left\{ \left[g_2 \vec{E}(\vec{r}\,') - e_2 B(\vec{r}\,') \right] \frac{1}{r'} - \frac{\vec{r}\,'}{r'^3} \left[\vec{r}\,' \cdot \left(g_2 \vec{E}(\vec{r}\,') - e_2 \vec{B}(\vec{r}\,') \right) \right] \right\} d^3r'$$

which in Cartesian components is

$$S_i = \frac{1}{4\pi c} \int \frac{1}{r'} (g_2 E_j - e_2 B_j)(\delta_{ij} - \frac{r'_i r'_j}{r'^2}) \, d^3r'$$

$$\frac{\partial}{\partial r_j} (\frac{r_i}{r}) = \frac{1}{r} (\delta_{ij} - \frac{r_i r_j}{r^3}) \qquad \text{(B-12)}$$

this can be written as

$$S_i = \frac{1}{4\pi c} \int \left[g_2 E_j - e_2 B_j \right] \frac{\partial}{\partial r'_j} \left(\frac{r'_i}{r'} \right) d^3r'$$

$$= - \frac{1}{4\pi c} \int \frac{r'_i}{r'} \frac{\partial}{\partial r'_j} \left[g_2 E_j(\vec{r}\,') - e_2 B_j(\vec{r}\,') \right] d^3r'$$

where we have integrated by parts and use the fact that $\vec{E}(\vec{r})$ and $\vec{B}(\vec{r})$ vanish as $|\vec{r}| \to \infty$. This last result can be rewritten in the form

$$\vec{S} = - \frac{1}{4\pi c} \int \frac{\vec{r}\,'}{r'} \left[g_2 \nabla' \cdot \vec{E} - e_2 \nabla' \cdot \vec{B} \right] d^3r'$$

$$= - \frac{1}{4\pi c} \int \frac{\vec{r}'}{r'} \left[g_2 \; 4\pi e_1 \; \delta(\vec{r}'-\vec{r}) - e_2 \; 4\pi g_1 \; \delta(\vec{r}'-\vec{r}) \right] d^3r'$$

giving

$$\vec{S} = - (e_1 g_2 - g_1 e_2) \frac{\vec{r}}{cr} \qquad \text{(B-13)}$$

where we have used the fact that

$$\nabla' \cdot \vec{E}(\vec{r}') = 4\pi e_1 \; \delta(\vec{r}'-\vec{r})$$

$$\nabla' \cdot \vec{B}(\vec{r}') = 4\pi g_1 \; \delta(\vec{r}'-\vec{r}) \; .$$

Thus, the total angular momentum conserved is the sum of the orbital angular momentum of the particle and the angular momentum of the electromagnetic field,

$$\vec{J} = \vec{r} \times m\vec{v} - (e_1 g_2 - g_1 e_2) \frac{\vec{r}}{cr} \; . \qquad \text{(B-14)}$$

From (B-14), we see that the radial component of the total angular momentum arises only from the field and is

$$J_r \equiv \hat{r} \cdot \vec{J} = - (e_1 g_2 - g_1 e_2)/c \; . \qquad \text{(B-15)}$$

This result is purely classical. Quantum mechanical considerations show that any angular momentum must be quantized in unit multiples of $\hbar/2$. In this case (B-15) would lead us to suspect the quantization condition.

$$\frac{e_1 g_2 - g_1 e_2}{c} = (\text{Integer}) \times \hbar/2$$

or

$$e_1 g_2 - g_1 e_2 = n \frac{\hbar c}{2} , \quad n = 0, \pm 1, \pm 2, \ldots , \tag{B-16}$$

to follow from the quantization of angular momentum.

In fact, the result of Eq. (B-16) reduces to the Dirac quantization condition if we assume the particles are an electron ($e_1 = -e$, $g_1 = 0$) and a pure magnetic monopole ($e_2 = 0$, $g_2 = g$). It should be stressed that the argument given here in support of the quantization condition (B-16) is meant to be a plausibility argument only and is not intended to be rigorous. For a rigorous derivation see the references at the end of Chapter III.

APPENDIX C

DERIVATION OF EQUATION (4-21)

It is straightforward to show that Eq. (4-21) follows from Eqs. (4-20):

$$\vec{k}\times\vec{E} = -\frac{4\pi}{\omega c}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_{o\alpha}\right)\vec{E} + \frac{\omega}{c}\left(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_{o\alpha}}{m_\alpha}\right)\vec{B} \qquad \text{(C-1a)}$$

$$\vec{k}\times\vec{B} = -\frac{\omega}{c}\left(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2 N_{o\alpha}}{m_\alpha}\right)\vec{E} + \frac{4\pi}{\omega c}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_{o\alpha}\right)\vec{B}\ . \qquad \text{(C-1b)}$$

Solving (C-1a) for the $\vec{B}$ term and (C-1b) for the $\vec{E}$ term, we obtain

$$\frac{\omega}{c}\left(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_{o\alpha}}{m_\alpha}\right)\vec{B} = \vec{k}\times\vec{E} + \frac{4\pi}{\omega c}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_{o\alpha}\right)\vec{E} \qquad \text{(C-2)}$$

$$-\frac{\omega}{c}\left(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2 N_{o\alpha}}{m_\alpha}\right)\vec{E} = \vec{k}\times\vec{B} - \frac{4\pi}{\omega c}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_{o\alpha}\right)\vec{B}\ . \qquad \text{(C-3)}$$

Now multiplying Eq. (C-1a) by

$$-\frac{\omega}{c}\left(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2 N_{o\alpha}}{m_\alpha}\right)$$

and Eq. (C-1b) by

$$\frac{\omega}{c}\left(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_{o\alpha}}{m_\alpha}\right)$$

they become

$$\vec{k} \times [-\frac{\omega}{c}(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2 N_{o\alpha}}{m_\alpha})\vec{E}] = -\frac{4\pi}{\omega c}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha N_{o\alpha}}{m_\alpha})[-\frac{\omega}{c} \times$$

$$(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2 N_{o\alpha}}{m_\alpha})\vec{E}] - \frac{\omega^2}{c^2}(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2 N_{o\alpha}}{m_\alpha})$$

$$\times (1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_{o\alpha}}{m_\alpha})\vec{B} \ . \tag{C-4a}$$

$$\vec{k} \times [\frac{\omega}{c}(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_{o\alpha}}{m_\alpha})\vec{B}] = -\frac{\omega^2}{c^2}(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_{o\alpha}}{m_\alpha})$$

$$\times (1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2 N_{o\alpha}}{m_\alpha})\vec{E} + \frac{4\pi}{\omega c}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha N_{o\alpha}}{m_\alpha})$$

$$\times [\frac{\omega}{c}(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha} N_{o\alpha})]\vec{B} \ . \tag{C-4b}$$

So that using (C-3) to eliminate $\vec{E}$ from Eq. (C-4a) and (C-2) to eliminate $\vec{B}$ from Eq. (C-4b) we find

$$\vec{k} \times [\vec{k} \times \vec{B} - \frac{4\pi}{\omega c}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha} N_{o\alpha})\vec{B}] = -\frac{4\pi}{\omega c}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha})[\vec{k} \times \vec{B}$$

$$- \frac{4\pi}{\omega c}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha} N_{o\alpha})\vec{B}] - \frac{\omega^2}{c^2}(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2 N_{o\alpha}}{m_\alpha})(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_{o\alpha}}{m_\alpha})\vec{B}, \tag{C-5a}$$

$$\vec{k} \times [\vec{k} \times \vec{E} + \frac{4\pi}{\omega c}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha} N_{o\alpha})\vec{E}] = -\frac{\omega^2}{c^2}(1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha} N_{o\alpha}) \times$$

$$\times \left(1 - \frac{4\pi}{\omega^2} \sum_{\alpha=1}^{M} \frac{e_\alpha^2}{m_\alpha} N_{o\alpha}\right)\vec{E} + \frac{4\pi}{\omega c} \left(\sum_{\alpha=1}^{M} \frac{e_\alpha g_\alpha}{m_\alpha} N_{o\alpha}\right)$$

$$\times \left[\vec{k} \times \vec{E} + \frac{4\pi}{\omega c} \left(\sum_{\alpha=1}^{M} \frac{e_\alpha g_\alpha}{m_\alpha} N_{o\alpha}\right)\vec{E}\right] , \qquad \text{(C-5b)}$$

which may be simplified to

$$\vec{k} \times (\vec{k} \times \vec{B}) = \left[\left(\frac{4\pi}{\omega c}\right)^2 \left(\sum_{\alpha=1}^{M} \frac{e_\alpha g_\alpha}{m_\alpha} N_{o\alpha}\right)^2 - \frac{\omega^2}{c^2} \left(1 - \frac{4\pi}{\omega^2} \sum_{\alpha=1}^{M} \frac{g_\alpha^2 N_{o\alpha}}{m_\alpha}\right) \times \right.$$

$$\left. \left(1 - \frac{4\pi}{\omega^2} \sum_{\alpha=1}^{M} \frac{e_\alpha^2 N_{o\alpha}}{m_\alpha}\right)\right]\vec{B} \qquad \text{(C-6a)}$$

$$\vec{k} \times (\vec{k} \times \vec{E}) = \left[\left(\frac{4\pi}{\omega c}\right)^2 \left(\sum_{\alpha=1}^{M} \frac{e_\alpha g_\alpha}{m_\alpha} N_{o\alpha}\right)^2 - \frac{\omega^2}{c^2} \left(1 - \frac{4\pi}{\omega^2} \sum_{\alpha=1}^{M} \frac{g_\alpha^2 N_{o\alpha}}{m_\alpha}\right) \times \right.$$

$$\left. \left(1 - \frac{4\pi}{\omega^2} \sum_{\alpha=1}^{M} \frac{e_\alpha^2 N_{o\alpha}}{m_\alpha}\right)\right]\vec{E} \ . \qquad \text{(C-6b)}$$

Thus, both $\vec{E}$ and $\vec{B}$ satisfy the same wave equation.

The term in brackets can be simplified as follows:

$$\left[\left(\frac{4\pi}{\omega c}\right)^2 \left(\sum_{\alpha=1}^{M} \frac{e_\alpha g_\alpha}{m_\alpha} N_{o\alpha}\right)^2 - \frac{\omega^2}{c^2} \left(1 - \frac{4\pi}{\omega^2} \sum_{\alpha=1}^{M} \frac{g_\alpha N_{o\alpha}}{m_\alpha}\right)\left(1 - \frac{4\pi}{\omega^2} \sum_{\alpha=1}^{M} \frac{e_\alpha^2 N_{o\alpha}}{m_\alpha}\right)\right]$$

$$= -\frac{\omega^2}{c^2}\left[1 - \frac{4\pi}{\omega^2} \sum_{\alpha=1}^{M} \frac{(e_\alpha^2+g_\alpha^2) N_{o\alpha}}{m_\alpha} + \left(\frac{4\pi}{\omega^2}\right)^2 \left(\sum_{\alpha=1}^{M} \frac{e_\alpha^2 N_{o\alpha}}{m_\alpha}\right)\right.$$

$$\times \left(\sum_{\beta=1}^{M} \frac{g_\beta^2 N_{o\beta}}{m_\alpha}\right) - \left(\frac{4\pi}{\omega^2}\right)^2 \left(\sum_{\alpha=1}^{M} \frac{e_\alpha g_\alpha N_{o\alpha}}{m_\alpha}\right)\left(\sum_{\beta=1}^{M} \frac{e_\beta g_\beta}{m_\beta} N_{o\beta}\right.$$

$$= -\frac{\omega^2}{c^2}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{(e_\alpha^2+g_\alpha^2)N_{o\alpha}}{m_\alpha} + \left(\frac{4\pi}{\omega^2}\right)^2 \sum_{\alpha,\beta=1}^{M}\frac{(e_\alpha^2 g_\beta^2 - e_\alpha e_\beta g_\alpha g_\beta)N_{o\alpha}N_{o\beta}}{m_\alpha\, m_\beta}\right. . \tag{C-7}$$

The third term of (C-7) may be further reduced by recognizing

$$\sum_{\alpha,\beta=1}^{M}\sum \frac{(e_\alpha^2 g_\beta^2 - e_\alpha e_\beta g_\alpha g_\beta)}{m_\alpha\, m_\beta} N_{o\alpha}\, N_{o\beta} = \sum_{\alpha,\beta=1}^{M}\sum \frac{e_\alpha g_\beta (e_\alpha g_\beta - e_\beta g_\alpha)}{m_\alpha\, m_\beta} N_{o\alpha}\, N_{o\beta}$$

$$= \frac{1}{2}\left[\sum_{\alpha,\beta=1}^{M}\sum \frac{e_\alpha g_\beta (e_\alpha g_\beta - e_\beta g_\alpha)}{m_\alpha\, m_\beta} N_{o\alpha}\, N_{o\beta}\right.$$

$$\left. + \sum_{\alpha,\beta=1}^{M}\sum \frac{e_\beta g_\alpha (e_\beta g_\alpha - e_\alpha g_\beta)}{m_\beta\, m_\alpha} N_{o\beta}\, N_{o\alpha}\right]$$

$$= \frac{1}{2}\sum_{\alpha,\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - e_\beta g_\alpha)^2}{m_\alpha\, m_\beta} N_{o\alpha}\, N_{o\beta}$$

$$= \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - e_\beta g_\alpha)^2}{m_\alpha\, m_\beta} N_{o\alpha}\, N_{o\beta} . \tag{C-8}$$

Combining these results, Eqs. (C-6) assume the desired forms

$$\vec{k}\times(\vec{k}\times\vec{B}) = -\frac{\omega^2}{c^2}\left[1 - \frac{\omega_{PH}^2}{\omega^2} + \frac{\omega_o^4}{\omega^4}\right]\vec{B} \tag{C-9a}$$

$$\vec{k}\times(\vec{k}\times\vec{E}) = -\frac{\omega^2}{c^2}\left[1 - \frac{\omega_{PH}^2}{\omega^2} + \frac{\omega_o^4}{\omega^4}\right]\vec{E} \tag{C-9b}$$

where we have used the definitions of Eqs. (4-22), (4-23) of Chapter IV for ω_{PH}^2 and ω_o^4.

APPENDIX D

ON THE RELATIVE MAGNITUDES OF ω_o AND ω_{PH}

It can easily be shown that $\omega_o^4 << \frac{\omega_{PH}^2}{2}$ regardless of the charge quantization condition. To see this consider the quantity $(e_\alpha g_\beta - g_\alpha e_\beta)^2$. Clearly,

$$(e_\alpha g_\beta - e_\beta g_\alpha)^2 \leq (e_\alpha g_\beta - e_\beta g_\alpha)^2 + (e_\alpha e_\beta + g_\alpha g_\beta)^2 \tag{D-1}$$

since the right hand side differs from the left only by a squared term. If we expand out the term on the right we find Eq. (D-1) becomes

$$(e_\alpha g_\beta - g_\alpha e_\beta)^2 \leq (e_\alpha^2 + g_\alpha^2)(e_\beta^2 + g_\beta^2) \quad . \tag{D-2}$$

Multiplying (D-2) by $\frac{(4\pi)^2 N_{o\alpha} N_{o\beta}}{m_\alpha m_\beta}$ and summing as indicated we arrive at

$$\sum_{\alpha\neq\beta=1}^{M}\sum \frac{(4\pi)^2 (e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha m_\beta} N_{o\alpha} N_{o\beta} \leq \sum_{\alpha\neq\beta=1}^{M}\sum \frac{(4\pi)^2 N_{o\alpha} N_{o\beta}}{m_\alpha m_\beta} (e_\alpha^2+g_\alpha^2)(e_\beta^2+g_\beta^2) \tag{D-3}$$

Comparing the left side with Eq. (4-23) we see that it is nothing other than $2\omega_o^4$, so (D-3) is

$$2\omega_o^4 \le \sum_{\alpha\ne\beta=1}^{M}\sum \frac{(4\pi)^2 N_{o\alpha} N_{o\beta}}{m_\alpha m_\beta} (e_\alpha^2 + g_\alpha^2)(e_\beta^2 + g_\beta^2) \;. \tag{D-4}$$

Now increasing the summation to range over all α,β only adds in more positive terms and (D-4) becomes

$$2\omega_o^4 << \sum_{\alpha,\beta=1}^{M}\sum \frac{(4\pi)^2 N_{o\alpha} N_{o\beta}}{m_\alpha m_\beta} (e_\alpha^2+g_\alpha^2)(e_\beta^2+g_\beta^2)$$

$$= \left[\sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)}{m_\alpha} N_{o\alpha}\right]^2$$

which upon using the definition (4-22) gives

$$\omega_o^4 << \frac{1}{2}\,\omega_{PH}^4 \tag{D-5}$$

as claimed.

Note that the argument leading to (D-5) is independent of the particular numerical values assigned to $N_{o\alpha}$, e_α, g_α, or m_α. If, in addition, we require $g \simeq 137e$ as implied by the quantization condition, then one would expect the ratio $\frac{\omega_o}{\omega_{PH}}$ to be even smaller.

From their definitions (4-22), (4-23) we can write

$$\frac{\omega_o^4}{\omega_{PH}^4} = \frac{\frac{1}{2}\sum_{\alpha,\beta=1}^{M}\sum \frac{(4\pi)^2 (e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha m_\beta} N_{o\alpha} N_{o\beta}}{\sum_{\alpha,\beta=1}^{M}\sum \frac{(4\pi)^2 (e_\alpha^2+g_\alpha^2)(e_\beta^2+g_\beta^2)}{m_\alpha m_\beta} N_{o\alpha} N_{o\beta}} \;. \tag{D-6}$$

If we compare the two sums in (D-6) term by term, we see that for each term,

$$\frac{(4\pi)^2 \dfrac{(e_\alpha g_\beta - e_\beta g_\alpha)^2 N_{o\alpha} N_{o\beta}}{m_\alpha m_\beta}}{(4\pi)^2 \dfrac{(e_\alpha^2+g_\alpha^2)(e_\beta^2+g_\beta^2) N_{o\alpha} N_{o\beta}}{m_\alpha m_\beta}} = \frac{(e_\alpha g_\beta - e_\beta g_\alpha)^2}{(e_\alpha^2+g_\beta^2)(e_\beta^2+g_\beta^2)}$$

$$= \frac{n(\hbar c/2)^2}{(e_\alpha^2+g_\alpha^2)(e_\beta^2+g_\beta^2)}$$

$$\simeq \frac{n^4}{(\frac{1}{137} + 137)(\frac{1}{137} + 137)}$$

$$\simeq n^4/(137)^2, \qquad n = 0, \pm 1, \pm 2, \ldots ,$$

where we have used the charge quantization condition (B-16) of Appendix B and recognized that $\frac{e^2}{\hbar c} \simeq \frac{1}{137}$, $\frac{g^2}{\hbar c} \simeq 137$, as discussed in Chapter III. Since this ratio is maintained term by term in (D-5) one expects roughly

$$\frac{\omega_o^4}{\omega_{PH}^4} \simeq \frac{1}{2}\left(\frac{1}{137}\right)^2$$

or

$$\frac{\omega_o^4}{\omega_{PH}^4} <<< 1$$

provided n is small which it should be.

APPENDIX E

PROPERTIES OF THE DYAD $\overleftrightarrow{N}_\alpha$

It can easily be shown that the dyadic tensor defined by Eq. (4-51)

$$\overleftrightarrow{N}_\alpha = \omega \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\vec{v}(\nabla_{\vec{v}} f_{o\alpha})}{\vec{v}\cdot\vec{k} - \omega_\alpha} d^3v \qquad \text{(E-1)}$$

is diagonal provided

$$f_{o\alpha} = f_{o\alpha}\ (v = |\vec{v}|) \qquad \text{(E-2)}$$

only. For $f_{o\alpha}$ of the form of Eq. (E-2) we have

$$\nabla_{\vec{v}} f_{o\alpha} = \frac{\partial f_{o\alpha}}{\partial v} \nabla_{\vec{v}} |\vec{v}|$$

$$= \frac{\partial f_{o\alpha}}{\partial v} \nabla_{\vec{v}} \left(\sqrt{v_x^2 + v_y^2 + v_z^2} \right)$$

or

$$\nabla_{\vec{v}} f_{o\alpha} = \frac{\partial f_{o\alpha}}{\partial v} \frac{\vec{v}}{v} . \qquad \text{(E-3)}$$

Using (E-3), Eq. (E-1) becomes

$$\overleftrightarrow{N}_\alpha = \omega \int_{-\infty}^{\infty} \frac{1}{v} \frac{\partial f_{o\alpha}}{\partial v} \frac{\vec{v}\vec{v}}{\vec{v}\cdot\vec{k} - \omega_\alpha} d^3v \qquad \text{(E-4)}$$

and without loss of generality we can choose

$$\vec{k} = k\hat{z} \tag{E-5}$$

so this can be written as

$$(\overleftrightarrow{N}_\alpha)_{ij} = \omega \int_{-\infty}^{\infty} \frac{1}{v} \frac{\partial f_{o\alpha}}{\partial v} \frac{v_i v_j}{\vec{v}\cdot\vec{k}-\omega_\alpha} d^3v \tag{E-6}$$

where i,j = 1,2,3. We have used the notation $v_1 = v_x$, $v_2 = v_y$, $v_3 = v_z$.

Now clearly for $i\neq j$ the integrand of (E-6) is odd with respect to either v_i or v_j (possibly both) since $\frac{1}{v}\frac{\partial f_{o\alpha}}{\partial v}$ is even in any v_i and for $i\neq j$ at least one of v_i, v_j is not v_z. In this case (E-6) is an integral of an odd function over an even interval for either v_i or v_j (possibly both) and therefore vanishes. As a result $\overleftrightarrow{N}_\alpha$ is zero for $i\neq j$ and thus is diagonal.

Since $\overleftrightarrow{N}_\alpha$ is diagonal we can write

$$(\overleftrightarrow{N}_\alpha)_{ij} = \omega\, \delta_{ij}\left(\int_{-\infty}^{\infty} \frac{1}{v} \frac{\partial f_{o\alpha}}{\partial v} \frac{v_j^{\,2}}{v_z k-\omega_\alpha} d^3v\right), \quad \text{no sum on } j. \tag{E-7}$$

From the symmetry of the integral (E-7) we see that

$$(\overleftrightarrow{N}_\alpha)_{11} = (\overleftrightarrow{N})_{22} \equiv N_T^\alpha \tag{E-8}$$

and we can write (E-7) as

$$\overleftrightarrow{N}_\alpha = \begin{pmatrix} N_T^\alpha & 0 & 0 \\ 0 & N_T^\alpha & 0 \\ 0 & 0 & N_L^\alpha \end{pmatrix} \tag{E-9}$$

where we have defined

$$N_T^\alpha = N_T^\alpha(\omega,k) \equiv \omega \int_{-\infty}^{\infty} \frac{1}{v} \frac{\partial f_{o\alpha}}{\partial v} \frac{v_x^2}{v_z k - \omega_\alpha} d^3v \tag{E-10}$$

and

$$N_L^\alpha = N_L^\alpha(\omega,k) \equiv \omega \int_{-\infty}^{\infty} \frac{1}{v} \frac{\partial f_{o\alpha}}{\partial v} \frac{v_z^2}{v_z k - \omega_\alpha} d^3v \; . \tag{E-11}$$

These last two expressions can be simplified somewhat. Using (E-3) we can write (E-10), (E-11) as

$$N_T^\alpha = \frac{\omega}{k} \int_{-\infty}^{\infty} \frac{v_x \frac{\partial f_{o\alpha}}{\partial v_x}}{v_z - \omega_\alpha/k} d^3v \tag{E-12}$$

and

$$N_L^\alpha = \frac{\omega}{k} \int_{-\infty}^{\infty} \frac{\frac{\partial f_{o\alpha}}{\partial v_z}}{v_z - \omega_\alpha/k} d^3v \; . \tag{E-13}$$

Integrating (E-12) once by parts we obtain

$$N_T^\alpha = \frac{\omega}{k} \left[\int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \left(\left. \frac{v_x f_{o\alpha}}{v_z - \frac{\omega_\alpha}{k}} \right|_{v_x=-\infty}^{v_x=+\infty} - \int_{-\infty}^{\infty} \frac{f_{o\alpha}}{v_x - \frac{\omega_\alpha}{k}} dv_x \right) \right] ,$$

or provided $v_x f_{o\alpha} \to 0$ as $v_x \to \pm \infty$, which is the usual case (as for example when $f_{o\alpha}$ is a Maxwellian distribution) we have

$$N_T^{\alpha} = -\frac{\omega}{k}\int_{-\infty}^{\infty} \frac{f_{o\alpha}}{v_x - \omega_\alpha/k}\, d^3v \; . \tag{E-14}$$

The integral of (E-13) can be simplified in a similar manner by noting

$$\frac{v_z}{v_z - \omega_\alpha/k} = 1 + \frac{\omega_\alpha/k}{v_z - \omega_\alpha/k} \tag{E-15}$$

so that (E-13) becomes

$$N_L^{\alpha} = \frac{\omega}{k}\left[\int_{-\infty}^{\infty} \frac{\partial f_{o\alpha}}{\partial v_z}\, d^3v + \frac{\omega_\alpha}{k}\int_{-\infty}^{\infty} \frac{\frac{\partial f_{o\alpha}}{\partial v_z}}{v_z - \omega_\alpha/k}\, d^3v\right]$$

$$= \frac{\omega}{k}\left[\int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y\, f_{o\alpha}\Bigg|_{v_z=-\infty}^{v_z=+\infty} + \frac{\omega_\alpha}{k}\int_{-\infty}^{\infty} \frac{\frac{\partial f_{o\alpha}}{\partial v_z}}{v_z - \omega_\alpha/k}\, dv^3\right] .$$

(E-16)

Provided $f_{o\alpha} \to 0$ as $v_z \to \pm \infty$, which again is the usual physical situation (as, for example, when $f_{o\alpha}$ is a Maxwellian distribution) we then have from Eq. (E-16),

$$N_L^{\alpha} = \frac{\omega}{k}\frac{\omega_\alpha}{k}\int_{-\infty}^{\infty} \frac{\frac{\partial f_{o\alpha}}{\partial v_z}}{v_z - \omega_\alpha/k}\, d^3v \; . \tag{E-17}$$

APPENDIX F

DERIVATION OF EQS. (4-56)

Writing out the Cartesian components of Eqs. (4-52) they become

$$-\,kE_y = -\frac{4\pi}{c\omega}\sum_{\alpha=1}^{M} g_\alpha\left(\frac{e_\alpha}{m_\alpha}E_x + \frac{g_\alpha}{m_\alpha}B_x\right)N_T^\alpha + \frac{\omega}{c}B_x \qquad \text{(F-1a)}$$

$$kE_x = -\frac{4\pi}{c\omega}\sum_{\alpha=1}^{M} g_\alpha\left(\frac{e_\alpha}{m_\alpha}E_y + \frac{g_\alpha}{m_\alpha}B_y\right)N_T^\alpha + \frac{\omega}{c}B_y \qquad \text{(F-1b)}$$

$$0E_z = -\frac{4\pi}{c\omega}\sum_{\alpha=1}^{M} g_\alpha\left(\frac{e_\alpha}{m_\alpha}E_z + \frac{g_\alpha}{m_\alpha}B_z\right)N_L^\alpha + \frac{\omega}{c}B_z \qquad \text{(F-1c)}$$

and

$$-\,kB_y = \frac{4\pi}{c\omega}\sum_{\alpha=1}^{M} e_\alpha\left(\frac{e_\alpha}{m_\alpha}E_x + \frac{g_\alpha}{m_\alpha}B_x\right)N_T^\alpha - \frac{\omega}{c}E_x \qquad \text{(F-2a)}$$

$$kB_x = \frac{4\pi}{c\omega}\sum_{\alpha=1}^{M} e_\alpha\left(\frac{e_\alpha}{m_\alpha}E_y + \frac{g_\alpha}{m_\alpha}B_y\right)N_T^\alpha - \frac{\omega}{c}E_y \qquad \text{(F-2b)}$$

$$0B_z = \frac{4\pi}{c\omega}\sum_{\alpha=1}^{M} e_\alpha\left(\frac{e_\alpha}{m_\alpha}E_z + \frac{g_\alpha}{m_\alpha}B_z\right)N_L^\alpha - \frac{\omega}{c}E_z \qquad \text{(F-2c)}$$

where we have used (4-53) and chosen $\vec{k} = k\hat{z}$. These may be rearranged into the equations

$$- kE_y = \frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_T^\alpha}{m_\alpha}\right]B_x - \frac{4\pi}{c\omega}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha\right)E_x \qquad \text{(F-3a)}$$

$$kE_x = \frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_T^\alpha\right]B_y - \frac{4\pi}{c\omega}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha\right)E_y \qquad \text{(F-3b)}$$

$$0E_z = \frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_T^\alpha\right]B_z - \frac{4\pi}{c\omega}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_L^\alpha\right)E_z \qquad \text{(F-3c)}$$

and

$$- kB_y = -\frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2}{m_\alpha}N_T^\alpha\right]E_x + \frac{4\pi}{c\omega}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha\right)B_x \qquad \text{(F-4a)}$$

$$kB_x = -\frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2}{m_\alpha}N_T^\alpha\right]E_y + \frac{4\pi}{c\omega}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha\right)B_y \qquad \text{(F-4b)}$$

$$0B_z = -\frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2}{m_\alpha}N_L^\alpha\right]E_z + \frac{4\pi}{c\omega}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_L^\alpha\right)B_z . \qquad \text{(F-4c)}$$

Now solving Eqs. (G-3) for the term containing B_i, we obtain

$$\frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_T^\alpha\right]B_x = - kE_y + \frac{4\pi}{c\omega}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha\right)E_x \qquad \text{(F-5a)}$$

$$\frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{a=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_T^\alpha\right]B_y = kE_x + \frac{4\pi}{c\omega}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha\right)E_y \qquad \text{(F-5b)}$$

$$\frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_L^\alpha\right]B_z = 0E_z + \frac{4\pi}{c\omega}\left(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_L^\alpha\right)E_z . \qquad \text{(F-5c)}$$

Multiplying (F-4a), (F-4b) by

$$\frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_T^\alpha\right]$$

and (F-4c) by

$$\frac{\omega}{c}\left[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_L^\alpha\right]$$

and then using Eqs. (F-5) to eliminate all terms involving B's we obtain

$$-k[kE_x + \frac{4\pi}{c\omega}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha)E_y] = -\frac{\omega^2}{c^2}[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_T^\alpha}{m_\alpha}]$$

$$\times [1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2}{m_\alpha}N_T^\alpha]E_x + \frac{4\pi}{c\omega}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha)[-kE_y + \frac{4\pi}{c\omega}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha)E_x] \tag{F-6a}$$

$$k[-kE_y + \frac{4\pi}{c\omega}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha)E_x] = -\frac{\omega^2}{c^2}[1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2 N_T^\alpha}{m_\alpha}]$$

$$\times [1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2}{m_\alpha}N_T^\alpha]E_y + \frac{4\pi}{c\omega}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha)[kE_x + \frac{4\pi}{c\omega}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha)E_y] \tag{F-6b}$$

$$0 = -\frac{\omega^2}{c^2}[1 - \frac{4\pi}{\omega^2}(\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_L^\alpha)][1 - \frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2}{m_\alpha}N_L^\alpha]E_z$$

$$+ \frac{4\pi}{c\omega}(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_L^\alpha)(\frac{4\pi}{c\omega}\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_L^\alpha)E_z\ . \tag{F-6c}$$

This last result can be simplified to

$$\{\frac{\omega^2}{c^2}[1-\frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_T^\alpha][1-\frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2}{m_\alpha}N_T^\alpha]$$

$$-(\frac{4\pi}{c\omega})^2(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha)^2-k^2\}E_x=0\ , \qquad \text{(F-7a)}$$

$$\{\frac{\omega^2}{c^2}[1-\frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_T^\alpha][1-\frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2}{m_\alpha}N_T^\alpha]$$

$$-(\frac{4\pi}{c\omega})^2(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_T^\alpha)^2-k^2\}E_y=0\ , \qquad \text{(F-7b)}$$

$$\{\frac{\omega^2}{c^2}[1-\frac{4\pi}{\omega^2}(\sum_{\alpha=1}^{M}\frac{g_\alpha^2}{m_\alpha}N_L^\alpha)][1-\frac{4\pi}{\omega^2}\sum_{\alpha=1}^{M}\frac{e_\alpha^2}{m_\alpha}N_L^\alpha]$$

$$-(\frac{4\pi}{c\omega})^2(\sum_{\alpha=1}^{M}\frac{e_\alpha g_\alpha}{m_\alpha}N_L^\alpha)^2\}E_z=0\ , \qquad \text{(F-7c)}$$

or upon expanding out the sums

$$\{\omega^4-\omega^2\sum_{\alpha=1}^{M}\frac{4\pi(e_\alpha^2+g_\alpha^2)N_T^\alpha}{m_\alpha}+(4\pi)^2[\sum_{\alpha,\beta=1}^{M}\sum\frac{(e_\alpha^2g_\beta^2-e_\alpha g_\alpha e_\beta g_\beta)}{m_\alpha\ m_\beta}N_T^\alpha N_T^\beta]$$

$$-c^2k^2\omega^2\}\ E_x=0 \qquad \text{(F-8a)}$$

$$\{\omega^4-\omega^2\sum_{\alpha=1}^{M}\frac{4\pi(e_\alpha^2+g_\alpha^2)}{m_\alpha}N_T^\alpha+(4\pi)^2[\sum_{\alpha,\beta=1}^{M}\sum\frac{(e_\alpha^2g_\beta^2-e_\alpha g_\alpha e_\beta g_\beta)}{m_\alpha\ m_\beta}N_T^\alpha N_T^\beta]$$

$$-c^2k^2\omega^2\}\ E_y=0 \qquad \text{(F-8b)}$$

$$\{\omega^4 - \omega^2 \sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)}{m_\alpha} N_L^\alpha + (4\pi)^2 [\sum_{\alpha,\beta=1}^{M}\sum \frac{(e_\alpha^2 g_\beta^2 - e_\alpha g_\beta e_\beta g_\alpha)}{m_\alpha\ m_\beta} N_L^\alpha N_L^\beta]\}$$

$$\times E_z = 0 . \qquad \text{(F-8c)}$$

Recognizing that the terms

$$\sum_{\alpha,\beta=1}^{M}\sum \frac{(e_\alpha^2 g_\beta^2 - e_\alpha g_\alpha e_\beta g_\beta)}{m_\alpha\ m_\beta} N_{T,L}^\alpha\ N_{T,L}^\beta = \frac{1}{2} \sum_{\alpha,\beta=1}^{M}\sum \frac{(e_\alpha^2 g_\beta^2 - e_\alpha g_\alpha e_\beta g_\beta)}{m_\alpha\ m_\beta} N_{T,L}^\alpha\ N_{T,L}^\beta$$

$$+ \frac{1}{2} \sum_{\alpha,\beta=1}^{M}\sum \frac{(e_\beta^2 g_\alpha^2 - e_\beta g_\beta e_\alpha g_\alpha)}{m_\beta\ m_\alpha} N_{T,N}^\alpha\ N_{T,L}^\beta$$

$$= \frac{1}{2} \sum_{\alpha,\beta=1}^{M}\sum \frac{N_{T,L}^\alpha}{m_\alpha} \frac{N_{T,L}^\beta}{m_\beta} [e_\alpha g_\beta (e_\alpha g_\beta - g_\alpha e_\beta) - e_\beta g_\alpha (e_\alpha g_\beta - g_\alpha e_\beta)]$$

$$= \frac{1}{2} \sum_{\alpha,\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha\ m_\beta} N_{T,L}^\alpha\ N_{T,L}^\beta$$

$$= \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha\ m_\beta} N_{T,L}^\alpha\ N_{T,L}^\beta , \qquad \text{(F-9)}$$

we can write Eqs. (F-8) as

$$[\omega^4 - \omega^2 (c^2k^2 + \sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)N_T^\alpha}{m_\alpha}) + (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha\ m_\beta} N_T^\alpha N_T^\beta] E_x = 0$$

(F-10a)

$$[\omega^4 - \omega^2 (c^2k^2 + \sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)}{m_\alpha} N_T^\alpha) + (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta - g_\alpha e_\beta)^2}{m_\alpha\ m_\beta} N_T^\alpha N_T^\beta] E_y = 0$$

(F-10b)

$$\left[\omega^4-\omega^2 \sum_{\alpha=1}^{M} \frac{4\pi(e_\alpha^2+g_\alpha^2)}{m_\alpha} N_L^\alpha + (4\pi)^2 \sum_{\alpha<\beta=1}^{M}\sum \frac{(e_\alpha g_\beta-g_\alpha e_\beta)^2}{m_\alpha m_\beta} N_L^\alpha N_L^\beta\right]E_z = 0 \qquad \text{(F-10c)}$$

which is the desired form.

APPENDIX G

A "DERIVATION" OF THE JOSEPHSON EQUATIONS

The macroscopic wave function description of superconductors can be used to give a heuristic argument for the validity of Eqs. (6-6) through (6-8). Considering the Josephson junction in the geometry of Figure 11 of Chapter VI, we can assign the wave functions ψ_1, ψ_2 to the superconductors 1 and 2 respectively. If the barrier is very thick, we expect the two wave functions to be essentially independent and evolve according to their own Schrödinger-like equations,

$$i\hbar \frac{\partial}{\partial t} \psi_1 = E_1 \psi_1 \qquad \text{(G-1a)}$$

$$i\hbar \frac{\partial}{\partial t} \psi_2 = E_2 \psi_2 \; . \qquad \text{(G-1b)}$$

When the two superconductors are separated by a thin barrier, however, we expect tunneling to occur and thus weakly coupld the two superconductors. In other words, for thin barriers we expect the time evolution of ψ_1, ψ_2 to be perturbed by equal amounts of the other superconductor's wave function. In this event, Eqs. (G-1) become for thin barriers

$$i\hbar \frac{\partial}{\partial t} \psi_1 = E_1 \psi_1 + \varepsilon \, \psi_2 \qquad \text{(G-2a)}$$

$$i\hbar \frac{\partial}{\partial t} \psi_2 = E_2 \psi_2 + \varepsilon \, \psi_1 \qquad \text{(G-2b)}$$

where ε is a small (compared to E_1,E_2) coupling energy term.

If the voltage drop across the barrier is $V(x,y)$, then we can place the zero voltage such that

$$E_1 = \frac{q^*V}{2}, \quad E_2 = -\frac{q^*V}{2} .$$

Doing this Eqs. (G-2) assume the form

$$i\hbar \frac{\partial\psi_1}{\partial t} = \frac{q^*V}{2}\psi_1 + \varepsilon\,\psi_2 \tag{G-3a}$$

$$i\hbar \frac{\partial\psi_2}{\partial t} = -\frac{q^*V}{2}\psi_2 + \varepsilon\,\psi_1 \tag{G-3b}$$

Substituting

$$\psi_1 = \sqrt{\rho_1}\; e^{i\theta_1}$$

$$\psi_2 = \sqrt{\rho_2}\; e^{i\theta_2}$$

into Eqs. (G-3) and separating the real and imaginary parts, we find that (G-3) is equivalent to the four real equations

$$\frac{\partial\rho_1}{\partial t} = \frac{2\varepsilon}{\hbar}\sqrt{\rho_1\rho_2}\,\sin(\theta_2-\theta_1) \tag{G-4a}$$

$$\frac{\partial\rho_2}{\partial t} = -\frac{2\varepsilon}{\hbar}\sqrt{\rho_1\rho_2}\,\sin(\theta_2-\theta_1) \tag{G-4b}$$

$$\frac{\partial\theta_1}{\partial t} = -\frac{q^*}{2\hbar}V - \frac{\varepsilon}{\hbar}\sqrt{\frac{\rho_2}{\rho_1}}\,\cos(\theta_2-\theta_1) \tag{G-5a}$$

$$\frac{\partial}{\partial t}\theta_2 = \frac{q^*}{2\hbar}V - \frac{\varepsilon}{\hbar}\sqrt{\frac{\rho_1}{\rho_2}}\cos(\theta_2-\theta_1) \ . \tag{G-5b}$$

Eqs. (G-4) require

$$\frac{\partial\rho_1}{\partial t} = -\frac{\partial\rho_2}{\partial t}$$

that is, the charge lost from superconductor 1 is gained by superconductor 2. But we maintain a steady state flow with the battery connected in the circuit and hence require

$$\rho_1 = \rho_2 = \rho \tag{G-6}$$

so that Eqs. (G-4), (G-5) reduce to

$$\frac{\partial\rho}{\partial t} = \frac{2\varepsilon\rho}{\hbar}\sin(\theta_2-\theta_1) \tag{G-7}$$

$$\frac{\partial\theta_1}{\partial t} = -\frac{q^*}{2\hbar}V - \frac{\varepsilon}{\hbar}\cos(\theta_2-\theta_1) \tag{G-8a}$$

$$\frac{\partial\theta_2}{\partial t} = \frac{q^*}{2t}V - \frac{\varepsilon}{\hbar}\cos(\theta_2-\theta_1) \ . \tag{G-8b}$$

Subtracting Eqs. (G-8) we see that

$$\frac{\partial}{\partial t}(\theta_2-\theta_1) = \frac{q^*}{\hbar}V \ . \tag{G-9}$$

Letting τ be the time it takes a typical charge carrier q^* to move the distance d across the barrier, we see from (G-7) that the total charge per unit volume flowing across the barrier in time τ is

$$q^* \, \Delta\rho = \frac{2\varepsilon\rho q^*\tau}{\hbar} \sin(\theta_2 - \theta_1)$$

so that the current density J_z is

$$J_z = \begin{pmatrix} \text{charge} \\ \text{density} \end{pmatrix} \times \begin{pmatrix} \text{velocity} \end{pmatrix} = (q^* \, \Delta\rho) \times \frac{d}{\tau}$$

or

$$J_z = \frac{2\varepsilon\rho q^* d}{\hbar} \sin(\theta_2 - \theta_1) \; .$$

Defining

$$J_o \equiv \frac{2\varepsilon\rho q^* d}{\hbar} \qquad \text{(G-10)}$$

this becomes

$$J_z = J_o \sin(\theta_2 - \theta_1) \qquad \text{(G-11)}$$

and along with (G-9) describe the Josephson junction provided no magnetic field is present in the barrier.

In the event that a magnetic field described by the vector potential $\vec{A}$ exists in the barrier, Eqs. (G-9), (G-11) must be altered in a gauge-invariant way by the introduction of $\vec{A}$. This is accomplished in quantum mechanics by the well-known change in phase

$$\theta \rightarrow \theta - \frac{q^*}{\hbar c} \int_1^2 \vec{A} \cdot d\vec{\ell} \; .$$

Doing this Eqs. (G-9), (G-11) become the desired results

$$\frac{\partial\phi}{\partial t} = \frac{q^*}{\hbar} V \tag{G-12}$$

$$J_z = J_o \sin\phi \tag{G-13}$$

where

$$\phi \equiv \theta_2 - \theta_1 - \frac{q^*}{\hbar c}\int_1^2 \vec{A}\cdot d\vec{\ell} \ . \tag{G-14}$$

APPENDIX H

DERIVATION OF EQUATIONS (8-16)

Using the fact that

$$\sigma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Eqs. (8-14) are easily shown to be equivalent to the system

$$\frac{\partial F_1}{\partial \psi_1} = -\frac{\partial F_2}{\partial \psi_2} \qquad \text{(H-1a)}$$

$$\frac{\partial F_1}{\partial \psi_2} = -\frac{\partial F_2}{\partial \psi_1}\,, \qquad \text{(H-1b)}$$

while Eqs. (8-15) may be written in full as

$$\frac{\partial F_1}{\partial \psi_1} F_1 + \frac{\partial F_1}{\partial \psi_2} F_2 = -\sin \psi_1 \qquad \text{(H-2a)}$$

$$\frac{\partial F_2}{\partial \psi_1} F_1 + \frac{\partial F_2}{\partial \psi_2} F_2 = -\sin \psi_2\,. \qquad \text{(H-2b)}$$

The Eqs. (H-1), (H-2) may now be looked upon as a system of partial differential equations to be solved for $F_1(\psi_1,\psi_2)$ and $F_2(\psi_1,\psi_2)$.

We shall solve these equations in two stages. First, we will show that Eqs. (H-1) require F_1,F_2 to be of the form

$$F_1 = f\left(\frac{\psi_1+\psi_2}{2}\right) + g\left(\frac{\psi_1-\psi_2}{2}\right) \quad \text{(H-3a)}$$

$$F_2 = -f\left(\frac{\psi_1+\psi_2}{2}\right) + g\left(\frac{\psi_1-\psi_2}{2}\right) \quad \text{(H-3b)}$$

where f and g are arbitrary functions; and, second, we will use Eqs. (H-2) to solve for the specific forms of f,g.

Taking $\frac{\partial}{\partial\psi_1}$ of (H-1a) and $\frac{\partial}{\partial\psi_2}$ of Eq. (H-1b), they become

$$\frac{\partial^2 F_1}{\partial\psi_1^{\,2}} = -\frac{\partial^2 F_2}{\partial\psi_1\partial\psi_2} \quad \text{(H-4a)}$$

$$\frac{\partial^2 F_1}{\partial\psi_2^{\,2}} = -\frac{\partial^2 F_2}{\partial\psi_2\partial\psi_1} \,. \quad \text{(H-4b)}$$

Subtracting these equations gives

$$\frac{\partial^2 F_1}{\partial\psi_1^{\,2}} - \frac{\partial^2 F_1}{\partial\psi_2^{\,2}} = 0 \,. \quad \text{(H-5)}$$

This is the wave equation in 1+1 dimensions and is well known to have the general solution

$$F_1(\psi_1,\psi_2) = f\left(\frac{\psi_1+\psi_2}{2}\right) + g\left(\frac{\psi_1-\psi_2}{2}\right) \quad \text{(H-6)}$$

where f,g are arbitrary functions.

Putting this form back into Eqs. (H-1) gives the system

$$\frac{\partial F_2}{\partial \psi_2} = -\frac{1}{2} f'(\frac{\psi_1+\psi_2}{2}) - \frac{1}{2} g'(\frac{\psi_1-\psi_2}{2}) \qquad \text{(H-7a)}$$

$$\frac{\partial F_2}{\partial \psi_1} = -\frac{1}{2} f'(\frac{\psi_1+\psi_2}{2}) + \frac{1}{2} g'(\frac{\psi_1-\psi_2}{2}) \qquad \text{(H-7b)}$$

where the primes denote differentiation with respect to the argument. Now Eq. (H-7b) may be integrated at once to give

$$F_2(\psi_1,\psi_2) = - f(\frac{\psi_1+\psi_2}{2}) + g(\frac{\psi_1-\psi_2}{2}) + h(\psi_2) \qquad \text{(H-8)}$$

where h is an arbitrary function of ψ_2. Putting this result back into Eq. (H-7a) we find that it implies

$$h'(\psi_2) = 0 ,$$

that is,

$$h = \text{constant} .$$

This constant can be absorbed into the arbitrary functions f,g so we have the desired result, namely that Eqs. (H-1) have a solution of the for (H-3).

To complete the solution, we now put these forms for F_1,F_2 into Eqs. (H-2) so that they become

$$f'(u)\ g(v) + f(u)\ g'(v) = - \sin(u+v) \qquad \text{(H-9a)}$$

$$f(u)\ g'(v) - f'(u)\ g(v) = - \sin(u-v) \qquad \text{(H-9b)}$$

where we have made the regular change of variable

$$u = \frac{\psi_1+\psi_2}{2} \tag{H-10a}$$

$$v = \frac{\psi_1-\psi_2}{2} \,. \tag{H-10b}$$

Adding Eq. (H-9) gives after using some trigonometric identities

$$f(u)\ g'(v) = -\sin u \cos v$$

or separating the variables

$$\frac{f(u)}{-\sin u} = \frac{\cos v}{g'(v)} \tag{H-11}$$

Now Eq. (H-11) can hold for independent variables u,v only provided

$$\frac{f(u)}{-\sin u} = a \tag{H-12a}$$

and

$$\frac{\cos v}{g'(v)} = a \tag{H-12b}$$

hold where "a" is an arbitrary constant. Solving Eqs. (H-12) we obtain

$$f(u) = -a \sin u \tag{H-13a}$$

$$g(v) = \frac{1}{a} \sin v \tag{H-13b}$$

where the integration constant has been set to zero. Putting these back into (H-3) and using (H-10), we arrive at the final solution

$$F_1 = -a \sin\left(\frac{\psi_1+\psi_2}{2}\right) + \frac{1}{a} \sin\left(\frac{\psi_1-\psi_2}{2}\right) \tag{H-14a}$$

$$F_2 = a \sin\left(\frac{\psi_1+\psi_2}{2}\right) + \frac{1}{a} \sin\left(\frac{\psi_1-\psi_2}{2}\right) \qquad \text{(H-14b)}$$

which is the desired result.

APPENDIX I

DERIVATION OF THE NONLINEAR SUPERPOSITION PRINCIPLE

Adding Eqs. (8-48) according to the combination [(8-48a) + (8-48c) - (8-48b) - (8-48d)] gives

$$0 = a_\mu^1 \sin\left(\frac{\alpha+i\beta_1}{2}\right) + a_\mu^2 \sin\left(\frac{\alpha+i\beta_1}{2}\right) - a_\mu^2 \sin\left(\frac{\alpha_o+i\beta_2}{2}\right) - a_\mu^1 \sin\left(\frac{\alpha+i\beta_2}{2}\right) \quad \text{(I-1)}$$

or

$$a_\mu^1\left[\sin\left(\frac{\alpha+i\beta_2}{2}\right) - \sin\left(\frac{\alpha_o+i\beta_1}{2}\right)\right] = a_\mu^2\left[\sin\left(\frac{\alpha+i\beta_1}{2}\right) - \sin\left(\frac{\alpha_o+i\beta_2}{2}\right)\right. . \quad \text{(I-2)}$$

Using the trigonometric identity

$$\sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$$

Eq. (I-2) can be written as

$$a_\mu^1 \, 2 \sin\left(\frac{\alpha+i\beta_2-\alpha_o-i\beta_1}{4}\right) \cos\left(\frac{\alpha+i\beta_2+\alpha_o+i\beta_1}{4}\right)$$

$$= a_\mu^2 \, 2 \sin\left(\frac{\alpha+i\beta_1-\alpha_o-i\beta_2}{4}\right) \cos\left(\frac{\alpha+i\beta_1+\alpha_o+i\beta_2}{4}\right)$$

or

$$a_\mu^1 \sin\left[\frac{(\alpha-\alpha_o)+i(\beta_2-\beta_1)}{4}\right] = a_\mu^2 \sin\left[\frac{(\alpha-\alpha_o)-i(\beta_2-\beta_1)}{4}\right] . \qquad \text{(I-3)}$$

Expanding the sines, Eq. (I-3) becomes

$$a_\mu^1\left[\sin\left(\frac{\alpha-\alpha_o}{4}\right) \cos i\left(\frac{\beta_2-\beta_1}{4}\right) + \cos\left(\frac{\alpha-\alpha_o}{4}\right) \sin i\left(\frac{\beta_2-\beta_1}{4}\right)\right]$$

$$= a_\mu^2\left[\sin\left(\frac{\alpha-\alpha_o}{4}\right) \cos i\left(\frac{\beta_2-\beta_1}{4}\right) - \cos\left(\frac{\alpha-\alpha_o}{4}\right) \sin i\left(\frac{\beta_2-\beta_1}{4}\right) .$$

Dividing this last result by

$$\cos\left(\frac{\alpha-\alpha_o}{4}\right) \cos i\left(\frac{\beta_2-\beta_1}{4}\right)$$

we obtain

$$a_\mu^1\left[\tan\left(\frac{\alpha-\alpha_o}{4}\right) + i \tanh\left(\frac{\beta_2-\beta_1}{4}\right)\right] = a_\mu^2\left[\tan\left(\frac{\alpha-\alpha_o}{4}\right) - i \tanh\left(\frac{\beta_2-\beta_1}{4}\right)\right]$$

which upon rearrangement gives

$$(a_\mu^2-a_\mu^1)\tan\left(\frac{\alpha-\alpha_o}{4}\right) = i(a_\mu^1+a_\mu^2)(\tanh\left(\frac{\beta_2-\beta_1}{4}\right) \qquad \text{(I-4)}$$

Now multiplying Eq. (I-4) by its contravariant complex conjugate and summing over μ, we find

$$(a_\mu^2-a_\mu^1)(a^{2\mu}-a^{1\mu})\tan^2\left(\frac{\alpha-\alpha_o}{4}\right) = (a_\mu^1-a_\mu^2)(a^{1\mu}-a^{2\mu})\tanh^2\left(\frac{\beta_2-\beta_1}{4}\right)$$

or

$$(1+a_\mu^2 a^{1\mu})\tan^2\left(\frac{\alpha-\alpha_o}{4}\right) = (1-a_\mu^2 a^{1\mu})\tanh^2\left(\frac{\beta_2-\beta_1}{4}\right) \qquad \text{(I-5)}$$

where we have used the fact that

$$a_\mu^1 \, a^{1\mu} = -1$$

$$a_\mu^2 \, a^{2\mu} = -1 .$$

Solving (I-5) for $\tan\left(\frac{\alpha-\alpha_o}{4}\right)$ we find

$$\tan\left(\frac{\alpha-\alpha_o}{4}\right) = \pm \left[(1-a_\mu^2 a^{1\mu})(1+a_\mu^2 a^{1\mu})^{-1}\right]^{1/2} \tanh\left(\frac{\beta_2-\beta_1}{2}\right)$$

which is the desired result.

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