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RESIDUAL SPACES OVER COMMUTATIVE RINGS

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RESIDUAL SPACES OVER COMMUTATIVE RINGS

INTRODUCTION

The automorphism theory of the classical groups was begun in 1928 by O. Schreier and B. L. van der Waerden [7] when they described the automorphisms of PSL(V), where V is a vector space over a field.

In 1951, J. Dieudonné [2] published a description of the automorphisms of GL(V) and PGL(V), where V is a vector space of dimension $n \geq 3$ over a division ring k.

E. Artin [1] has studied the structure of the group GL(V), where V is a vector space over a field k. He also considered the symplectic and orthogonal groups.

B. McDonald [3] described a theory similar to that of Artin's except that in this case V is a free module over a local ring R. Recall that all projective R-modules are free R-modules when R is local.

O. T. O'Meara [6] has also studied the structure of GL(V), where V is a vector space over a field k, but he did not use matrix arguments. His approach was to define the fixed and residual spaces of an element of GL(V). Through these spaces he was able to characterize transvections and then redevelop the automorphism theory of GL(V). This is the approach taken in this paper although the automorphism theory is still incomplete.

CHAPTER 1

RESIDUAL SPACES

Introduction

In Chapter 1 we look at the structures of some of the linear groups. The basic approach to this is through what we will call residues. These will be defined in Section 1.4. The elements of GL(V) that we will be working with most often in Chapter 1 and throughout the paper are transvections, which are introduced in Section 1.5. Later in the chapter we consider the problem of generation of subsets of GL(V) by transvections although this is rather difficult in the case where R is a ring and not a field.

Let R be a commutative ring with identity. We will assume that all projective R-modules are free R-modules. All free R-modules are called spaces. Throughout the paper, V will denote a free R-module, U and W will be submodules of V, and n will denote the dimension of V. A submodule U of V is called a subspace if U is a direct summand of V. In this case U is also a free module, so a subspace of V is itself a space. An element x of R is called unimodular if Rx is a direct summand of V. Let R^* represent the multiplicative group of units of R. Let GL(V) represent the group of R-automorphisms of V. A one-dimensional subspace of V is called a line.

1.1 Subspaces

Theorem 1.1.1: (i) If Rb and Rc are lines with $Rb \subset Rc$, then $Rb = Rc$.
(ii) If U_1 and U_2 are subspaces of V with $\dim(U_1) = \dim(U_2)$ and $U_1 \subset U_2$, then $U_1 = U_2$.

Proof: (i) We have $Rb \oplus W_1 = V = Rc \oplus W_2$. So $b = rc$ and $c = sb + w$ for some w in W_1. Then $b = rsb + rw$. So $rs = 1$ and $rw = 0$. Thus $c = src = sb$ is in Rb. Therefore $Rb = Rc$.

(ii) Let $U_1 = Rb_1 \oplus \ldots \oplus Rb_m$, $U_2 = Rc_1 \oplus \ldots \oplus Rc_m$ and $U_1 \oplus W_1 = V = U_2 \oplus W_2$. Then

$$b_i = \sum_{j=1}^{m} r_{ij} c_j \text{ for all } i$$

and

$$c_j = \sum_{k=1}^{m} s_{jk} b_k + w_j \text{ for some } w_j \text{ in } W_1 \text{ for all } j.$$

So

$$\begin{aligned} b_i &= \sum_{j=1}^{m} r_{ij} \left(\sum_{k=1}^{m} s_{jk} b_k + w_j \right) \\ &= \sum_{j=1}^{m} \sum_{k=1}^{m} r_{ij} s_{jk} b_k + \sum_{j=1}^{m} r_{ij} w_j \\ &= \sum_{k=1}^{m} \left(\sum_{j=1}^{m} r_{ij} s_{jk} \right) b_k + \sum_{j=1}^{m} r_{ij} w_j \text{ for all } i. \end{aligned}$$

Thus $\sum_{j=1}^{m} r_{ij} s_{jk} = \delta_{ij}$ for all i and k. So $[r_{ij}][s_{ij}] = I_m$. Thus $[r_{ij}]$ is invertible, so c_j is in U_1 for all j. Therefore $U_1 = U_2$.

Theorem 1.1.2: Let U and W be subspaces of V with $U \subset W$ and $\dim(U) = i$,

$\dim(W) = i + j$ with $j \geq 2$. Then there exist subspaces U_k with $\dim(U_k) = i + k$, $k = 1, \ldots, j - 1$ such that

$$U \subset U_1 \subset \ldots \subset U_{j-1} \subset W.$$

Proof: We have $V = U \oplus M = W \oplus N$. We claim $W = U \oplus (W \cap M)$.

Let x be in $U \cap (W \cap M)$. Then x is in $U \cap M = 0$. So $x = 0$. Let x be in W. Then $x = u + m$, for u in U, m in M. Now $m = x - u$ is in W since $U \subset W$, so m is in $W \cap M$. Thus $W = U + (W \cap M)$. Hence $W = U \oplus (W \cap M)$. Therefore $W \cap M$ is a subspace.

Let $\{b_1, \ldots, b_i\}$ be a basis of U and $\{b_{i+1}, \ldots, b_{i+j}\}$ a basis of $W \cap M$. Let $U_k = Rb_1 \oplus \ldots \oplus Rb_{i+k}$ for $k = 1, \ldots, j - 1$. Then $\dim(U_k) = i + k$ and

$$U \subset U_1 \subset \ldots \subset U_{j-1} \subset W.$$

Theorem 1.1.3: Let U and W be subspaces of V such that $U + W = V$. Then $U \cap W$ is a subspace of V.

Proof: We have the isomorphism

$$(U + W)/U \simeq W/(U \cap W).$$

So $V/U \simeq W/(U \cap W)$. But V/U is free so $W/(U \cap W)$ is free. So we have the split exact sequence

$$0 \to U \cap W \to W \rightleftarrows W/(U \cap W) \to 0.$$

So $W \simeq (U \cap W) \oplus W/(U \cap W)$. Thus $U \cap W$ is a subspace of V.

Theorem 1.1.4: Let U and W be proper subspaces of V. If $W \not\subset U$ then there exists a hyperplane H such that $U \subset H$ and $W \not\subset H$.

Proof: Let $\{b_1, \ldots, b_k\}$ be a basis of U and extend it to a basis $\{b_1, \ldots, b_n\}$ of V. Then $V = U \oplus Rb_{k+1} \oplus \ldots \oplus Rb_n$.

Let $H_i = U \oplus Rb_{k+1} \oplus \ldots \oplus Rb_{k+i-1} \oplus Rb_{k+i+1} \oplus \ldots \oplus Rb_n$, $1 \leq i \leq n - k$. Let x be in V. Then $x = r_1 b_1 + \ldots + r_n b_n$. If x is in H_i, then $r_i = 0$. So if x is in $\bigcap_{i=1}^{n-k} H_i$, then $r_{k+1} = \ldots = r_n = 0$, in which case x is in U. So $\bigcap_{i=1}^{n-k} H_i \subset U$. Clearly $U \subset \bigcap_{i=1}^{n-k} H_i$, so $U = \bigcap_{i=1}^{n-k} H_i$. Therefore, if $\{H_\alpha | \alpha \text{ is in } A\}$ is the set of all hyperplanes that contain U, then $U = \bigcap_{\alpha \in A} H_\alpha$.

Now suppose that $W \not\subset U$, but W is contained in every hyperplane that contains U. Then $W \subset \bigcap_{\alpha \in A} H_\alpha = U$, which is a contradiction. Thus there exists a hyperplane H such that $U \subset H$ and $W \not\subset H$.

1.2 Geometric, Linear, and Projective Transformations

A geometric transformation g of V_1 onto V_2 is a bijection $g: V_1 \to V_2$ which has the property that U is a subspace of V_1 if and only if g(U) is a subspace of V_2.

Theorem 1.2.1: Let g be a geometric transformation of V_1 onto V_2. Let U and W be subspaces of V_1. Then

(i) $g(U \cap W) = g(U) \cap g(W)$;

(ii) (a) If $U + W$ is a subspace, then $g(U + W) \supset g(U) + g(W)$;

(b) If $g(U) + g(W)$ is also a subspace, then $g(U + W) = g(U) + g(W)$;

(iii) $g(0) = 0$, $g(V_1) = V_2$;

(iv) $\dim(U) = \dim(g(U))$.

Proof: (i) It is immediate since g is bijective.

(ii) (a) Note that $g(U) \subset g(U + W)$ and $g(W) \subset g(U + W)$. Now $g(U + W)$ is a subspace so $g(U) + g(W) \subset g(U + W)$.

(b) Now $g(U) + g(W) = g(T)$ for some subspace T of V_1. So $g(U) \subset g(T)$, $g(W) \subset g(T)$. Thus $U \subset T$ and $W \subset T$. So $U + W \subset T$. Therefore $g(U + W) \subset g(T) = g(U) + g(W)$.

(iii) Since 0 is a subspace, $g(0)$ is a subspace. But $g(0)$ contains only one element, so $g(0) = 0$. Also $g(V_1) = g(V_2)$ since g is surjective.

(iv) Let U be a subspace of dimension m. Then $U \oplus W = V_1$. Let $U = Rb_1 \oplus \ldots \oplus Rb_m$ and $W = Rb_{m+1} \oplus \ldots \oplus Rb_n$. We have a chain

$$0 \subset U_1 \subset U_2 \subset \ldots \subset U_{n-1} \subset V_1$$

where $U_i = Rb_1 \oplus \ldots \oplus Rb_i$. So $U_m = U$. Then

$$0 \subset g(U_1) \subset g(U_2) \subset \ldots \subset g(U_{n-1}) \subset g(V_1) \quad = V_2 \,.$$

We have $\dim(V_1) = \dim(V_2) = n$. Suppose $g(U_i)$ and $g(U_{i+1})$ have the same dimension. Then $g(U_i) = g(U_{i+1})$. This is a contradiction. So $\dim(g(U_i)) = i = \dim(U_i)$. In particular, $\dim(g(U)) = \dim(U)$.

Let $P(V)$ be the set of all subspaces of V. Note that $P(V)$ is not necessarily a lattice since $U + W$ and $U \cap W$ may not be free. Let

$$P^i(V) = \{U \text{ in } P(V) \mid \dim(U) = i\}.$$

Call elements of $P^1(V)$ lines, $P^2(V)$ planes, and $P^{n-1}(V)$ hyperplanes.

A projectivity π of V_1 onto V_2 is a bijection $\pi : P(V_1) \to P(V_2)$ with the property that $\pi(U) \subset \pi(W)$ if and only if $U \subset W$.

Theorem 1.2.2: If $\pi : P(V_1) \to P(V_2)$ is a projectivity, then $\dim(V_1) = \dim(V_2)$.

Proof: Suppose $\dim(\pi(V_1)) > \dim(\pi(V_2))$. Let

$$U_1 \subset U_2 \subset \ldots \subset U_{n-1} \subset V_1$$

be a chain of subspaces with $\dim(U_i) = i$. Then each $\pi(U_i)$ is a subspace and

$$\pi(U_1) \subset \pi(U_2) \subset \ldots \subset \pi(U_{n-1}) \subset V_2.$$

So there exists i such that $\dim(\pi(U_i)) = \dim(\pi(U_{i+1}))$. But then $\pi(U_i) = \pi(U_{i+1})$, so $U_i = U_{i+1}$, which is a contradiction. Thus $\dim(V_1) \leq \dim(V_2)$.

Now π^{-1} is also a projectivity, so a similar argument using π^{-1} shows that $\dim(V_1) \geq \dim(V_2)$. Thus $\dim(V_1) = \dim(V_2)$.

Theorem 1.2.3: Let $\pi: P(V_1) \to P(V_2)$ be a projectivity.

(i) Let U,W be in $P(V_1)$. Then

(a) If $U \cap W$ is a subspace, then $\pi(U \cap W) \subset \pi(U) \cap \pi(W)$,

(b) If $\pi(U) \cap \pi(W)$ is also a subspace, then $\pi(U \cap W) = \pi(U) \cap \pi(W)$.

(ii) Let U,W be in $P(V_1)$. Then

(a) If $U + W$ is in $P(V_1)$, then $\pi(U + W) \supset \pi(U) + \pi(W)$,

(b) If $\pi(U) + \pi(W)$ is also in $P(V_2)$, then $\pi(U + W) = \pi(U) + \pi(W)$.

(iii) $\pi(0) = 0$, $\pi(V_1) = V_2$,

(iv) $\dim(\pi(U)) = \dim(U)$.

Proof: (i) (a) We have $U \cap W \subset U$ and $U \cap W \subset W$. So $\pi(U \cap W) \subset \pi(U)$ and $\pi(U \cap W) \subset \pi(W)$. Therefore $\pi(U \cap W) \subset \pi(U) \cap \pi(W)$.

(b) If $\pi(U) \cap \pi(W)$ is a subspace, then $\pi(U) \cap \pi(W) = \pi(T)$ for some T in $P(V_1)$. Then $\pi(T) \subset \pi(U)$ and $\pi(T) \subset \pi(W)$. So $T \subset U$ and $T \subset W$. Thus $T \subset U \cap W$, so $\pi(U \cap W) \supset \pi(T) = \pi(U) \cap \pi(W)$. Thus $\pi(U \cap W) = \pi(U) \cap \pi(W)$.

(ii) (a) We have $U \subset U + W$ and $W \subset U + W$ so $\pi(U) \subset \pi(U + W)$ and $\pi(W) \subset \pi(U + W)$. Thus $\pi(U) + \pi(W) \subset \pi(U + W)$.

(b) If $\pi(U) + \pi(W)$ is in $P(V_2)$ then $\pi(U) + \pi(W) = \pi(T)$ for some T in $P(V_1)$. Now $\pi(U) \subset \pi(T)$ and $\pi(W) \subset \pi(T)$, so $U \subset T$ and $W \subset T$. Thus $U + W \subset T$. Thus $\pi(U + W) \subset \pi(T) = \pi(U) + \pi(W)$. Therefore $\pi(U + W) = \pi(U) + \pi(W)$.

(iii) There exists W in $P(V_1)$ with $\pi(W) = 0$. Now $0 \subset W$, so $\pi(0) \subset \pi(W) = 0$. Thus $\pi(0) = 0$. There exists W in $P(V_1)$ with $\pi(W) = V_2$. Now $V_1 \supset W$ so $\pi(V_1) \supset \pi(W) = V_2$. Thus $\pi(V_1) = V_2$.

(iv) Let U be a subspace of dimension m. Then there exist $U_1, \ldots, U_n$ in $P(V_1)$ such that

$$0 \subset U_1 \subset U_2 \subset \ldots \subset U_n \subset V_1$$

with $\dim(U_i) = i$ and $U_m = U$. So

$$0 \subset \pi(U_1) \subset \pi(U_2) \subset \ldots \subset \pi(U_n) = V_2.$$

Suppose $\pi(U_i)$ and $\pi(U_{i+1})$ have the same dimension. Then $\pi(U_i) = \pi(U_{i+1})$, so $U_i = U_{i+1}$, which is a contradiction. So each $\pi(U_i)$ has a distinct dimension. Thus $\dim(\pi(U_i)) = i = \dim(U_i)$. In particular, $\dim(\pi(U)) = \dim(U)$.

So Theorem 1.2.2 tells us that projectivities carry lines to lines, planes to planes, etc. But if L_1, L_2 are in $P^1(V_1)$ with $L_1 \oplus L_2$ in $P^2(V_1)$, we do not necessarily have that $\pi(L_1 \oplus L_2) = \pi(L_1) \oplus \pi(L_2)$.

A transformation $f: V_1 \to V_2$ which maps subspaces to subspaces will induce a transformation $\bar{f}: P(V_1) \to P(V_2)$ defined by

$$\bar{f}(U) = f(U) = \{f(x) | x \text{ is in } U\}.$$

Theorem 1.2.4: Let $g:V_1 \to V_2$ be a geometric transformation. Then the map $\bar{g}$ is a projectivity.

Proof: Suppose $\bar{g}(U) = \bar{g}(W)$. Then $g(U) = g(W)$, so $U = W$.

Now suppose U is in $P(V_2)$. Then there exists W in $P(V_1)$ such that $g(W) = U$. So $\bar{g}(W) = U$.

Suppose $U \subset W$. Then

$$\bar{g}(U) = g(U) \subset g(W) = \bar{g}(W).$$

Suppose $\bar{g}(U) \subset \bar{g}(W)$. Then $g(U) \subset g(W)$, so $U \subset W$. Therefore $\bar{g}$ is a projectivity.

If a projectivity $\pi:P(V_1) \to P(V_2)$ has the form $\pi = \bar{g}$ for some geometric transformation, then we say π is a projective geometric transformation.

Clearly $\overline{g_1g_2} = \bar{g}_1\bar{g}_2$ and $\overline{g^{-1}} = \bar{g}^{-1}$. So composites and inverses of projective geometric transformations are also projective geometric transformations.

Let GG(V) denote the group of geometric transformations of V onto V. Call GG(V) the general geometric group of V. Every element of GL(V) and SL(V) is a geometric transformation so GL(V) and SL(V) are subgroups of GG(V).

If π is a projectivity of V onto V, we say that π is simply a projectivity of V.

The map $\bar{\ }$ is a group homomorphism, $\bar{\ }:GG(V) \to$ group of projectivities of V. Let us write P instead of $\bar{\ }$ and PGG(V) instead of $\overline{GG(V)}$ or $\mathrm{Im}(\bar{\ })$.

Since GL(V) and SL(V) are subgroups of GG(V), we have that PGL(V) and PSL(V) are subgroups of PGG(V).

Theorem 1.2.5: Let $\pi:P(V_1) \to P(V_2)$ be a projectivity. Then π is a bijection from $P^1(V_1)$ to $P^1(V_2)$.

Proof: Let Rb be a line in V_1. Then $\pi(Rb)$ is a subspace of V_2 so $\pi(Rb) = Rb_1 \oplus \ldots \oplus Rb_t$. Now $\pi:P(V_1) \to P(V_2)$ is a bijection so there exist non-zero $U_1, \ldots, U_t$ in $P(V_1)$ with $Rb_i = \pi(U_i)$. Thus $\pi(Rb) = \pi(U_1) \oplus \ldots \oplus \pi(U_t)$. Now $\pi(U_i) \subset \pi(Rb)$ so $U_i \subset Rb$ for all i. Therefore $U_i = Rb$, for if $U_i = 0$ then $\pi(U_i) = 0$, which would be a contradiction. Therefore $U_i = U_j$ for all i and j so $Rb_i = Rb_j$ for all i and j. Thus $\pi(Rb) = Rb_1$.

Suppose $\pi(Rb_1) = \pi(Rb_2)$. Then $Rb_1 \subset Rb_2$ and $Rb_2 \subset Rb_1$, so $Rb_1 = Rb_2$. Let Rb be in $P^1(V_2)$. Then $Rb = \pi(U)$ for some U in $P(V_1)$. So

$$Rb = \pi(U) = \pi(Rb_1 \oplus \ldots \oplus Rb_t) \supset \pi(Rb_1) + \ldots + \pi(Rb_t) \supset \pi(Rb_1) = Rb'$$

for some b' in V_1. So we have equality in each case. So $\pi(U) = \pi(Rb_1)$ implies that $U = Rb_1$. Thus $Rb = \pi(Rb_1)$. Therefore π is a bijection from $P^1(V_1)$ to $P^1(V_2)$.

1.3 Radiations

Let σ be in $GL(V)$. If there exists an α in R^* such that $\sigma(x) = \alpha x$ for all x in V, then σ is called a radiation and is denoted by r_α.

Let σ be in $GL(V)$ and α be in R^*. Then

$$\sigma^{-1}r_\alpha\sigma(x) = \sigma^{-1}(\alpha\sigma(x)) = \sigma^{-1}\sigma(\alpha x) = \alpha x = r_\alpha(x)$$

for all x in V. Therefore the group of radiations, $RL(V)$, is normal in $GL(V)$.

Theorem 1.3.1: Let σ be in $GL(V)$. Then σ is in $RL(V)$ if and only if $\sigma(L) = L$ for all L in $P(V)$. In particular;

(i) $\ker(P|GL(V)) = RL(V)$,

(ii) $\ker(P|SL(V)) = SL(V) \cap RL(V)$,

(iii) $PGL(V) \simeq GL(V)/RL(V)$,

(iv) $PSL(V) \simeq SL(V)/(SL(V) \cap RL(V))$,

where $P: GL(V) \to PGL(V)$ by $P(\sigma) = \bar{\sigma}$.

Proof: Suppose $\sigma(L) = L$ for all L in P(V). Let $\{b_1, \ldots, b_n\}$ be a basis of V. Then $\sigma(b_1) = \alpha b_1$ for some α in R^*. Let b_i be any other basis element. Then $\sigma(b_i) = \beta b_i$ for some β in R^*. Then

$$\alpha b_1 + \beta b_i = \sigma(b_1) + \sigma(b_i) = \sigma(b_1 + b_i) = \gamma(b_1 + b_i) = \gamma b_1 + \gamma b_i$$

for some γ in R^* since $b_1 + b_i$ is unimodular. So $\alpha = \beta = \gamma$ since b_1 and b_i are basis vectors. Thus σ is a radiation.

Conversely, it is clear that $\sigma(L) = L$ for all L in P(V) if σ is a radiation.

(i) Let σ be in $\ker(P|GL(V))$. Then $\bar{\sigma}(L) = L$ for all L in P(V). So $\sigma(L) = L$ for all L in P(V). Thus σ is in RL(V).

Conversely, if σ is in RL(V) then $\sigma(L) = L$ for all L in P(V), so $\bar{\sigma}(L) = \sigma(L) = L$. Thus $\bar{\sigma} = I$, so σ is in $\ker(P|GL(V))$.

(ii) Let σ be in $\ker(P|SL(V))$. Then σ is in RL(V) and σ is in SL(V) so σ is in $RL(V) \cap SL(V)$.

If σ is in $RL(V) \cap SL(V)$, then σ is in $\ker(P|GL(V))$, so σ is in $\ker(P|SL(V))$.

(iii) and (iv) It follows that $PGL(V) \simeq GL(V)/RL(V)$ and $PSL(V) \simeq SL(V)/(SL(V) \cap RL(V))$.

Theorem 1.3.2: The group PSL(V) is normal in PGL(V) and

$$PGL(V)/PSL(V) \simeq R^*/(R^*)^n.$$

<u>Proof</u>: Let $\bar{\sigma}$ be in PSL(V) and $\bar{\tau}$ be in PGL(V). Then σ is in SL(V) and τ is in GL(V) so $\tau^{-1}\sigma\tau$ is in SL(V). Thus $\bar{\tau}^{-1}\bar{\sigma}\bar{\tau} = \overline{\tau^{-1}\sigma\tau}$ is in PSL(V). Let $f: PGL(V) \to R^*/(R^*)^n$ be defined by $f: \bar{\sigma} \mapsto \det(\sigma)(R^*)^n$. This map is well defined since $\bar{\sigma}_1 = \bar{\sigma}_2$ implies $\sigma_1 = r\sigma_2$ which implies $\det(\sigma_1) = r^n \det(\sigma_2)$.

Let $\bar{\sigma}$ be in ker(f). Then $\det(\sigma)$ is in $(R^*)^n$. So $\det(\sigma) = r^n$ for some r in R^*. Then $\sigma = \sigma'\sigma_r$ where $\det(\sigma') = 1$. Thus $\bar{\sigma} = \bar{\sigma}'\bar{\sigma}_r = \bar{\sigma}'$ is in PSL(V).

Let $\bar{\sigma}$ be in PSL(V). Then $\sigma = \sigma'\sigma_r$ where σ' is in SL(V). So $f(\sigma) = \det(\sigma)(R^*)^n = \det(\sigma')\det(\sigma_r)(R^*)^n = r^n(R^*)^n = (R^*)^n$. So σ is in ker(f). Thus ker(f) = PSL(V). Hence,

$$PGL(V)/PSL(V) \simeq R^*/(R^*)^n.$$

1.4 Residues

Let σ be in GL(V) and consider the submodules

$$P = \ker(\sigma - I) = \{x \text{ in } V \mid \sigma(x) = x\}$$

and

$$Q = \operatorname{Im}(\sigma - I) = \{x \text{ in } V \mid x = \sigma(y) - y \text{ for some } y \text{ in } V\}.$$

Call P the <u>fixed module of</u> σ and Q the <u>residual module of</u> σ.

We will always use P and Q to denote the fixed and residual modules of σ, respectively. Further, for σ_i, $1 \le i \le n$, P_i and Q_i denote the fixed and residual modules of σ_i. At times it will also be convenient to let P_σ and Q_σ denote the fixed and residual modules of σ.

If P and Q are direct summands of V, then σ is <u>proper</u>. In this case, define the <u>residue of</u> σ, res(σ), to be dim(Q), call P the <u>fixed space of</u> σ and Q the <u>residual space of</u> σ.

We are introducing the fixed and residual spaces of an element of GL(V) in order to avoid complicated arguments involving matrices, although we have used them occasionally. In the study of automorphisms of GL(V), O. T. O'Meara [6] has shown that many of the residual properties of transvections are preserved by automorphisms of GL(V). This suggests that they might provide a matrix-free approach to the study of the automorphisms of GL(V). Indeed, O'Meara has shown that this is the case. As one would expect, difficulties appear when the base field is replaced by a ring. These problems are caused to a great extent by the fact that P and Q are not necessarily subspaces of V, i.e. they need not be direct summands of V. The following example demonstrates this.

Suppose Q is a direct summand of V. We have the split exact sequence

$$0 \to P \xrightarrow{\text{incl.}} V \underset{\tau}{\overset{\sigma-I}{\rightleftarrows}} Q \to 0$$

where $(\sigma - I)\tau = 1$. So $V = P \oplus \tau(Q)$. Thus P is also a direct summand of V. So σ is proper if and only if Q is a direct summand of V.

But if P is a direct summand of V, it may not be true that Q is also a direct summand. For example, let $R = Z$, $V = Z \oplus Z$ and define $\sigma: Z \oplus Z \to Z \oplus Z$ by $\sigma(n,m) = (n + 2m,m)$. Then σ is in $GL_2(Z \oplus Z)$ since $\sigma^{-1}: Z \oplus Z \to Z \oplus Z$ by $\sigma^{-1}(n,m) = (n - 2m,m)$ is its inverse. Then $(\sigma - I)(n,m) = (2m,0)$ so $P = \ker(\sigma - I) = Z \oplus 0$ and $Q = \operatorname{Im}(\sigma - I) = 2Z \oplus 0$. Thus P is a direct summand of V but Q is not.

Theorem 1.4.1: For any σ in GL(V), we have $\sigma(P) = P$ and $\sigma(Q) = Q$. Furthermore, if σ is proper in GL(V), then

(i) $\dim(P) + \dim(Q) = n$,

(ii) $\operatorname{res}(\sigma) = 0$ if and only if $\sigma = I$.

Proof: Let x be in P. Then $x = \sigma(x)$ is in $\sigma(P)$ so $P \subset \sigma(P)$. Let x be in $\sigma(P)$. Then $x = \sigma(y)$ for some y in P. So

$$\sigma(x) = \sigma(\sigma(y)) = \sigma(y) = x .$$

Thus x is in P and $\sigma(P) = P$.

Let x be in Q. Then $x = \sigma(y) - y$ and $y = \sigma(z)$ for some y,z in V since σ is in GL(V). So

$$x = \sigma(y) - y = \sigma(\sigma(z)) - \sigma(z) = \sigma(\sigma(z) - z)$$

which is in $\sigma(Q)$. So $Q \subset \sigma(Q)$.

Let x be in $\sigma(Q)$. Then $x = \sigma(y)$ for some y in Q. Let $y = \sigma(z) - z$. Then

$$x = \sigma(y) = \sigma(\sigma(z) - z) = \sigma(\sigma(z)) - \sigma(z)$$

which is in Q, so $\sigma(Q) = Q$.

(i) If σ is proper, then the exact sequence

$$0 \to P \to V \to Q \to 0$$

is split, so $V \cong P \oplus Q$. Thus

$$\dim(P) + \dim(Q) = \dim(V) = n.$$

(ii) Clearly $\operatorname{res}(\sigma) = 0$ if and only if $P = V$ if and only if $\sigma = I$.

Theorem 1.4.2: Let σ_1 and σ_2 be in GL(V). Let $\sigma = \sigma_1\sigma_2$. Then $P_1 \cap P_2 \subset P$ and $Q \subset Q_1 + Q_2$. If σ_1, σ_2, and σ are proper, then

$$\operatorname{res}(\sigma) \leq \operatorname{res}(\sigma_1) + \operatorname{res}(\sigma_2).$$

Proof: Let x be in $P_1 \cap P_2$. Then

$$\sigma(x) = \sigma_1(\sigma_2(x)) = \sigma_1(x) = x,$$

so x is in P. Thus $P_1 \cap P_2 \subset P$.

Let x be in Q. Then $x = \sigma_1(\sigma_2(y)-y)$ for some y in V. Then $x = (\sigma_1 - I)(\sigma_2(y)) + (\sigma_2 - I)(y)$ is in $Q_1 + Q_2$, so $Q \subset Q_1 + Q_2$.

From this we see that

$$\text{res}(\sigma) \leq \text{res}(\sigma_1) + \text{res}(\sigma_2).$$

whenever σ_1, σ_2, and σ are proper.

Examples:

(1) If σ is the radiation $\sigma(x) = \alpha x$ where $\alpha - 1$ is in R^*, then $Q = V$ since $x = (\sigma - I)((\alpha - 1)^{-1}x)$ is in Q for all x in V.

(2) Let $V = U \oplus W$; $U, W \neq 0$. Set $\sigma_1 = (I_U) \oplus (\alpha I_W)$ for some $\alpha \neq 0$, $\alpha - 1$ in R^*. Let x be in W. Then

$$x = (\alpha - 1)(\alpha - 1)^{-1}(x) = (\sigma_1 - I)((\alpha - 1)^{-1}(x))$$

is in Q_1. So W is in Q_1.

Let x be in Q_1. Then $x = \sigma_1(y) - y$. Let $x = u_1 + w_1$, $y = u_2 + w_2$, where u_1, u_2 are in U and w_1, w_2 are in W. Then

$$\begin{aligned} x &= u_1 + w_1 \\ &= \sigma_1(u_2) + \sigma_1(w_2) - u_2 - w_2 \\ &= \sigma_1(w_2) - w_2 \\ &= \alpha w_2 - w_2 \\ &= (\alpha - 1)w_2 \end{aligned}$$

which is in W. So $Q_1 = W$. Thus Q_1 is a direct summand of V. So P_1 is a direct summand of V. Also,

$$\begin{aligned} \dim(P_1) &= n - \dim(Q_1) \\ &= n - \dim(W) \\ &= \dim(U). \end{aligned}$$

So $P_1 = U$ since $U \subset P_1$. Let $\sigma_2 = (\alpha I_U) \oplus (I_W)$. Then as before, $P_2 = W$ and $Q_2 = U$. Let $\sigma = \sigma_1\sigma_2$. Let $x = u + w$ be in V where u is in U, w is in W. Then

$$\begin{aligned}\sigma(x) &= \sigma(u + w)\\ &= \sigma_1(\sigma_2(u + w))\\ &= \sigma_1(\alpha u + w)\\ &= \alpha u + \alpha w\\ &= \alpha x.\end{aligned}$$

So $Q = V$ and thus $P = 0$. So $Q = Q_1 + Q_2$ and $P = P_1 \cap P_2$.

(3) Let σ_1 be proper in $GL(V)$, $\sigma_1 \neq I$, and $\sigma_2 = \sigma_1^{-1}$. Let $\sigma = \sigma_1\sigma_2 = I$. Then $P = V$ and $P_1 \neq V$, $P_2 \neq V$. So $P \neq P_1 \cap P_2$. Also $Q = 0$ and $Q_1 \neq 0$, $Q_2 \neq 0$. So $Q \neq Q_1 + Q_2$.

<u>Theorem 1.4.3</u>: Let σ_1 and σ_2 be in $GL(V)$ and let $\sigma = \sigma_1\sigma_2$. Then

(i) If $V = P_1 + P_2$, then $Q = Q_1 + Q_2$.

(ii) If $Q_1 \cap Q_2 = 0$, then $P = P_1 \cap P_2$.

<u>Proof</u>: (i) We have

$$\begin{aligned}Q_1 &= (\sigma_1 - I)(V)\\ &= (\sigma_1 - I)(P_1 + P_2)\\ &= (\sigma_1 - I)(P_2)\\ &= (\sigma_1\sigma_2 - I)(P_2)\\ &\subset (\sigma - I)(V)\\ &= Q.\end{aligned}$$

Now $\sigma^{-1} = \sigma_2^{-1}\sigma_1^{-1}$, so

$$\begin{aligned} Q_2 &= (\sigma_2^{-1} - I)(V) \\ &= (\sigma_2^{-1} - I)(P_1 + P_2) \\ &= (\sigma_2^{-1} - I)(P_1) \\ &= (\sigma_2^{-1}\sigma_1^{-1} - I)(P_1) \\ &\subset (\sigma^{-1} - I)(V) \\ &= Q. \end{aligned}$$

So $Q_1 + Q_2 \subset Q$. Thus $Q = Q_1 + Q_2$ by Theorem 1.4.2.

(ii) Let x be in P. Then

$$\sigma_2(x) - x = -(\sigma_1(\sigma_2(x)) - \sigma_2(x))$$

which is in $Q_1 \cap Q_2 = 0$. So x is in P_2. Thus $P \subset P_2$. So the above gives $-(\sigma_1(x) - x) = 0$. Thus x is in P_1. Thus $P \subset P_1$. So $P \subset P_1 \cap P_2$. Hence $P = P_1 \cap P_2$ by Theorem 1.4.2.

<u>Theorem 1.4.4</u>: Let σ and τ be elements of GL(V). Then the fixed and residual modules of $\tau\sigma\tau^{-1}$ are $\tau(P)$ and $\tau(Q)$, respectively. Also, if $\sigma\tau = \tau\sigma$ then $\tau(P) = P$ and $\tau(Q) = Q$. In particular, if σ is proper then $\tau\sigma\tau^{-1}$ is proper and $\mathrm{res}(\tau\sigma\tau^{-1}) = \mathrm{res}(\sigma)$.

<u>Proof</u>: Let x be in P. Then $x = \tau(y)$ for some y in P. So

$$\tau\sigma\tau^{-1}(x) = \tau\sigma\tau^{-1}\tau(y) = \tau\sigma(y) = \tau(y) = x.$$

So x is in $P_{\tau\sigma\tau^{-1}}$.

Let x be in $P_{\tau\sigma\tau^{-1}}$. Then $\tau\sigma\tau^{-1}(x) = x$ so $\sigma\tau^{-1}(x) = \tau^{-1}(x)$. Thus $\tau^{-1}(x)$ is in P so x is in $\tau(P)$. Hence $P_{\tau\sigma\tau^{-1}} = \tau(P)$.

Let x be in $\tau(Q)$. Then

$$x = \tau(\sigma(y) - y) = \tau\sigma(y) - \tau(y),$$

for some y in V. Let $y = \tau^{-1}(z)$, for some z in V. Then

$$x = \tau\sigma(y) = \tau(y) = \tau\sigma\tau^{-1}(z) - \tau\tau^{-1}(z) = \tau\sigma\tau^{-1}(z) - z.$$

So x is in $Q_{\tau\sigma\tau^{-1}}$.

Let x be in $Q_{\tau\sigma\tau^{-1}}$. Then $x = \tau\sigma\tau^{-1}(y) - y$, for some y in V. Now $y = \tau(z)$ for some z in V, so

$$x = \tau\sigma(z) - \tau(z) = \tau(\sigma - I)(z),$$

which is in $\tau(Q)$. So $Q_{\tau\sigma\tau^{-1}} = \tau(Q)$.

Suppose $\tau\sigma = \sigma\tau$. Then $\tau\sigma\tau^{-1} = \sigma$ so $\tau(P) = P_{\tau\sigma\tau^{-1}} = P$ and $\tau(Q) = Q_{\tau\sigma\tau^{-1}} = Q$.

Suppose σ is proper. Then $V = P \oplus U = Q \oplus W$ for subspaces U and W. So $V = \tau(P) \oplus \tau(U) = \tau(Q) \oplus \tau(W)$. Thus the fixed and residual modules of $\tau\sigma\tau^{-1}$ are free, so $\tau\sigma\tau^{-1}$ is proper. Then clearly $res(\tau\sigma\tau^{-1}) = res(\sigma)$.

<u>Theorem 1.4.5</u>: Let σ_1 and σ_2 be in $GL(V)$. Then if $Q_1 \subset P_2$ and $Q_2 \subset P_1$, then $\sigma_1\sigma_2 = \sigma_2\sigma_1$.

<u>Proof</u>: Let x be in V. Then

$$\begin{aligned}\sigma_1\sigma_2(x) &= \sigma_1(\sigma_2(x) - x) + \sigma_1(x)\\ &= \sigma_2(x) - x + \sigma_1(x)\\ &= \sigma_1(x) - x + \sigma_2(x)\\ &= \sigma_2(\sigma_1(x) - x) + \sigma_2(x)\\ &= \sigma_2\sigma_1(x)\end{aligned}$$

Therefore $\sigma_1\sigma_2 = \sigma_2\sigma_1$.

<u>Theorem 1.4.6</u>: Let σ_1 and σ_2 be in $GL(V)$ with $\sigma_1\sigma_2 = \sigma_2\sigma_1$. Then $Q_1 \subset P_2$ and $Q_2 \subset P_1$ if either $V = P_1 + P_2$ or $Q_1 \cap Q_2 = 0$.

<u>Proof</u>: Suppose $Q_1 \cap Q_2 = 0$. From $\sigma_1\sigma_2 = \sigma_2\sigma_1$, we have $\sigma_1(Q_2) = Q_2$ and

$\sigma_1(P_2) = P_2$. So $(\sigma_1 - I)(Q_2) \subset Q_1 \cap Q_2 = 0$. So $Q_2 \subset P_1$. Similarly, $Q_1 \subset P_2$.

Suppose $V = P_1 + P_2$. Then

$$Q_1 = (\sigma_1 - I)(V) = (\sigma_1 - I)(P_1 + P_2) = (\sigma_1 - I)(P_2) \subset P_2.$$

Similarly, $Q_2 \subset P_1$.

Theorem 1.4.7: Let σ be in $GL(V)$. Then $\sigma^2 = I$ if and only if $\sigma|Q = -I_Q$.

Proof: $\sigma^2 = I$ if and only if $\sigma^2(x) = x$ for all x in V

if and only if $\sigma(\sigma(x) - x) = -(\sigma(x) - x)$ for all x in V

if and only if $\sigma(y) = -y$ for all y in Q

if and only if $\sigma|Q = -I_Q$.

Theorem 1.4.8: Let $\sigma \neq I$ be proper in $GL(V)$ and $V = Q \oplus W$. Then

$$\begin{array}{ccccccccc} 0 & \to & Q & \xrightarrow{I_Q} & V & \xrightarrow{\pi_W} & W & \to & 0 \\ & & \downarrow{\scriptstyle \sigma|Q} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle I_W} & & \\ 0 & \to & Q & \xrightarrow{I_Q} & V & \xrightarrow{\pi_W} & W & \to & 0 \end{array}$$

commutes, where π_W is the projection of V onto W.

Proof: (a) Let x be in Q. Then $\sigma(I_Q(x)) = \sigma(x)$ and $I_Q((\sigma|Q)(x)) = I_Q(\sigma(x)) = \sigma(x)$. Thus the first square commutes.

(b) Let $x = u + w$ be in V with u in Q and w in W. Then $I_W(\pi_W(x)) = I_W(w) = w$ and $\pi_W(\sigma(x)) = \pi_W(\sigma(u) + \sigma(w)) = w$ since $\sigma(u)$ is in Q and $\sigma(w) - w$ is in Q. Thus the second square commutes.

Theorem 1.4.9: Let $\sigma \neq I$ be proper in $GL(V)$. Then

(i) $\det(\sigma) = \det(\sigma|Q)$,

(ii) $tr(\sigma) = tr(\sigma|Q) + t$,

(iii) $\chi(\sigma) = \chi(\sigma|Q)(X - 1)^t$,

where $\chi(\sigma)$ is the characteristic polynomial of σ, $tr(\sigma)$ is the trace of σ, and $t = \dim(W)$.

Proof: Follows from Theorem 1.4.8.

Theorem 1.4.10: If $V = V_1 \oplus V_2$ and $\sigma = \sigma_1 \oplus \sigma_2$ where σ_1 is in $GL_n(V_1)$ and σ_2 is in $GL_m(V_2)$, then σ is in $GL_{n+m}(V)$ and $P = P_1 \oplus P_2$, $Q = Q_1 \oplus Q_2$. If σ_1 and σ_2 are proper then σ is proper.

Proof: Clearly $P_1 \oplus P_2 \subset P$ and $Q_1 \oplus Q_2 \subset Q$. Let $x_1 + x_2$ be in P. Then

$$\sigma(x_1 + x_2) = \sigma_1(x_1) + \sigma_2(x_2) = x_1 + x_2.$$

So $\sigma_1(x_1) = x_1$ and $\sigma_2(x_2) = x_2$. Then x_1 is in P_1 and x_2 is in P_2.

Let $x_1 + x_2$ be in Q. Then

$$\begin{aligned} x_1 + x_2 &= \sigma(y) - y \\ &= \sigma(y_1 + y_2) - (y_1 + y_2) \\ &= (\sigma(y_1) - y_1) + (\sigma(y_2) - y_2) \end{aligned}$$

which is in $Q_1 \oplus Q_2$. Therefore $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$.

1.5 Transvections

Let σ be proper in GL(V). Then σ is a proper transvection if $\sigma = I$ or $\operatorname{res}(\sigma) = 1$ and $\det(\sigma) = 1$; σ is a proper dilation if $\operatorname{res}(\sigma) = 1$ and $\det(\sigma) \neq 1$.

From Theorem 1.4.4 we have $\operatorname{res}(\tau\sigma\tau^{-1}) = \operatorname{res}(\sigma)$ and we also have $\det(\tau\sigma\tau^{-1}) = \det(\sigma)$, so $\tau\sigma\tau^{-1}$ is a proper transvection (dilation) if and only if σ is a proper transvection (dilation).

Theorem 1.5.1: Let σ be proper in GL(V) with $\operatorname{res}(\sigma) = 1$ and $n \geq 2$. Then (i) $Q \subset P$ if and only if σ is a proper transvection,

(ii) $P \cap Q = 0$ if and only if σ is a proper dilation and $\det(\sigma) - 1$ is not a zero divisor,

(iii) $V = P \oplus Q$ if and only if σ is a proper dilation and $\det(\sigma) - 1$ is a unit.

Proof: (i) Suppose $Q \subset P$. Then $\sigma(x) = x$ for all x in Q. So $\det(\sigma) = \det(\sigma|Q) = 1$. Thus σ is a proper transvection. Now suppose σ is a proper transvection, $\sigma \neq I$. Then $\det(\sigma|Q) = \det(\sigma) = 1$. But Q is a line so $Q = Rb$ for some unimodular b in V. Then $\sigma(b) = b$ since $\det(\sigma|Q) = 1$. So $\sigma(x) = x$ for all x in Q. Thus $Q \subset P$.

(ii) Suppose $P \cap Q = 0$. Then $Q \not\subset P$ so σ is not a proper transvection. So $\det(\sigma) \neq 1$. Thus σ is a proper dilation. Let $\det(\sigma) = r$. Then $\det(\sigma|Q) = r$. So $\sigma(x) = rx$ for all x in Q since Q is a line. Suppose $(r - 1)s = 0$ for some s in R. Then $rs = s$ so $rsb = sb$ where $Q = Rb$. Then $\sigma(sb) = rsb = sb$ so sb is in P. Thus $sb = 0$ and $s = 0$. Therefore $r - 1$ is not a zero divisor.

Let σ be a proper dilation such that $\det(\sigma) - 1$ is not a zero divisor. Let $r = \det(\sigma)$. Let x be in $P \cap Q$. Then $\sigma(x) = x$ and $\sigma(x) = rx$. So $x = rx$. Let $\{b_1, \ldots, b_n\}$ be a basis of V and $x = \Sigma a_i b_i$. Then $\Sigma a_i b_i = \Sigma r a_i b_i$ so $(r - 1)a_i = 0$ for all i, thus $a_i = 0$ for all i. So $x = 0$. Thus $P \cap Q = 0$.

(iii) Suppose $V = Q \oplus P$. Then $P \cap Q = 0$ so σ is a proper dilation. Let $\{b_1, \ldots, b_n\}$ be a basis of V where $Q = Rb_1$, and $P = Rb_2 \oplus \ldots \oplus Rb_n$. Since $V = P \oplus Q$,

$$\operatorname{mat}(\sigma) = \begin{bmatrix} r & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \text{ so } \operatorname{mat}(\sigma - I) = \begin{bmatrix} r-1 & & 0 \\ & 0 & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix}.$$

So

$$(\sigma - I)V = (r - 1)Rb_1 = Rb_1.$$

Thus $r - 1$ is a unit.

Now suppose σ is a proper dilation. By Theorem 1.4.8 there exists a basis $\{b_1, \ldots, b_n\}$ of V such that

$$\mathrm{mat}(\sigma) = \begin{bmatrix} \mathrm{mat}(\sigma|Q) & * \\ 0 & I_{n-1} \end{bmatrix}$$

where $Q = Rb_1$, $\sigma(b_1) = rb_1$ and $r \neq 1$. Then

$$\mathrm{mat}(\sigma) = \begin{bmatrix} r & a_2 \cdots a_n \\ 0 & 1 \quad 0 \\ \vdots & \ddots \\ 0 & 0 \quad 1 \end{bmatrix} \quad \text{for some } a_2, \ldots, a_n \text{ in } R.$$

So $\sigma(b_i) = b_i + a_i b_1$, $2 \leq i \leq n$.

Suppose $\det(\sigma) - 1$ is in R^*. Then $r - 1$ is in R^* since $\det(\sigma) = r$. Now σ is invertible so $\{rb_1, b_2 + a_2 b_1, \ldots, b_n + a_n b_1\}$ is a basis of V. Define σ_1 by

$$c_1 = \sigma_1(b_1) = b_1$$

and

$$c_i = \sigma_1(b_i) = (r-1)b_i - a_i b_1 = rb_i - (b_i + a_i b_1)$$

for $i = 2, \ldots, n$. The matrix of this transformation is

$$\begin{bmatrix} 1 & -a_2 & \cdots & -a_n \\ 0 & (r-1) & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & (r-1) \end{bmatrix}$$

whose determinant is $(r-1)^{n-1}$ which is in R^*. So $\{c_1, \ldots, c_n\}$ is a basis of V. Now $\sigma(c_1) = rc_1$ and

$$\begin{aligned} \sigma(c_i) &= (r-1)\sigma(b_i) - a_i\sigma(b_1) \\ &= (r-1)(b_i + a_i b_1) - a_i(rb_1) \\ &= rb_i + ra_i b_1 - b_i - a_i b_1 - ra_i b_1 \\ &= (r-1)b_i - a_i b_1 \\ &= c_i. \end{aligned}$$

Therefore

$$\operatorname{mat}(\sigma) = \begin{bmatrix} r & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

with respect to $\{c_1, \ldots, c_n\}$ and $Q = Rc_1$, $P = Rc_2 \oplus \ldots \oplus Rc_n$ and $V = P \oplus Q$.

When $n = 1$, I is the only proper transvection. When $R = Z_2$, there are no proper dilations.

Let a be in V and $\rho: V \to R$ be a surjective R-module morphism. Then

$$0 \to \ker(\sigma) \to V \overset{\rho}{\to} R \to 0$$

is split exact, so $\ker(\rho)$ is a hyperplane.

Let $\tau_{a,\rho}: V \to V$ be given by $\tau_{a,\rho}(x) = x + \rho(x)a$. A map of the form $\tau_{a,\rho}$ is defined to be a <u>transvection</u> when $\det(\tau_{a,\rho}) = 1$ and is defined to be a <u>dilation</u> when $\det(\tau_{a,\rho}) \neq 1$, but is a unit. We will show later (Theorem 1.5.5) that when $\tau_{a,\rho}$ is proper the definitions above conform with those given earlier for proper transvections and proper dilations.

Let x be in V. Then $(\tau_{a,\rho} - I)(x) = \rho(x)a$ which is in Ra. So $(\tau_{a,\rho} - I)(V) = Ra$ since ρ is surjective. If $\det(\tau_{a,\rho})$ is in R^*, then $\tau_{a,\rho}$ is in GL(V). Note that $\tau_{a,\rho} = I$ if and only if $a = 0$.

<u>Theorem 1.5.2</u>: Suppose a,b are in V, $\rho: V \to R$, and $\phi: V \to R$, where ρ and ϕ are surjective. Then

(i) $\tau_{a,\rho} = \tau_{b,\rho}$ if and only if $a = b$.

(ii) $\sigma\tau_{a,\rho}\sigma^{-1} = \tau_{\sigma a,\rho\sigma^{-1}}$ for all σ in GL(V).

If a and b are unimodular, then

(iii) $\tau_{a,\rho} = \tau_{a,\phi}$ if and only if $\rho = \phi$,

(iv) $\tau_{a,\rho} = \tau_{b,\phi}$ if and only if there exists u in R^* such that $a = ub$ and $\rho = u^{-1}\phi$.

Proof: (i) Suppose $\tau_{a,\rho} = \tau_{b,\rho}$. Then $\rho(x)a = \rho(x)b$ for all x in V. There exists y in V such that $\rho(y) = 1$. So

$$a = \rho(y)a = \rho(y)b = b.$$

(ii)
$$\begin{aligned}\sigma\tau_{a,\rho}\sigma^{-1}(x) &= \sigma(\sigma^{-1}(x) + \rho(\sigma^{-1}(x))(a)) \\ &= \sigma\sigma^{-1}(x) + \rho\sigma^{-1}(x)\sigma(a) \\ &= x + (\rho\sigma^{-1})(x)\sigma(a) \\ &= \tau_{\sigma a,\rho\sigma^{-1}}\end{aligned}$$

(iii) Suppose a is unimodular and $\tau_{a,\rho} = \tau_{a,\phi}$. Then $\rho(x)a = \phi(x)a$ for all x in V. So $\rho(x) = \phi(x)$ for all x in V. Thus $\rho = \phi$.

(iv) Suppose a and b are unimodular and $\tau_{a,\rho} = \tau_{b,\phi}$. Then $\rho(x)a = \phi(x)b$ for all x in V. There exists y in V such that $\rho(y) = 1$, so $a = \phi(y)b$. There exists z in V such that $\phi(z) = 1$, so $\rho(z)a = b$. Thus $a = \phi(y)\rho(z)a$. But a is unimodular, so $\phi(y)\rho(z) = 1$. Let $\phi(y) = u$. Then $a = ub$, and $u\rho(x)b = \phi(x)b$. But b is unimodular, so $u\rho(x) = \phi(x)$ for all x in V. So $u\rho = \phi$.

Theorem 1.5.3: For the map $\tau_{a,\rho}$, we have that

$$\det(\tau_{a,\rho}) = 1 + \rho(a).$$

Proof: Let $\{b_1, \ldots, b_{n-1}\}$ be a basis of $\ker(\rho)$. So $\rho(b_1) = \ldots = \rho(b_{n-1}) = 0$.

Case I: Suppose a is in $H = \ker(\rho)$. Let b_n be unimodular such that $V = H \oplus Rb_n$. Then

$$\rho(\tau_{a,\rho}(b_n) - b_n) = \rho(\rho(b_n)a) = \rho(b_n)\rho(a) = 0.$$

So $\tau_{a,\rho}(b_n) - b_n$ is in H. Let

$$\tau_{a,\rho}(b_n) - b_n = a_1b_1 + \ldots + a_{n-1}b_{n-1}.$$

Then $\tau_{a,\rho}(b_n) = a_1b_1 + \ldots + a_{n-1}b_{n-1} + b_n$. Also, $\tau_{a,\rho}(b_i) = b_i$ for $1 \leq i \leq n - 1$. So

$$\mathrm{mat}(\tau_{a,\rho}) = \begin{bmatrix} 1 & 0 & a_1 \\ & \ddots & \vdots \\ 0 & 1 & a_{n-1} \\ 0 \ldots & 0 & 1 \end{bmatrix}$$

Thus $\det(\tau_{a,\rho}) = 1 = 1 + \rho(a)$.

Case II: Suppose a is not in H. Again, let b_n be such that $V = H \oplus Rb_n$. Now,

$$\begin{aligned} \rho(\tau_{a,\rho}(b_n) - b_n - \rho(a)b_n) &= \rho(\rho(b_n)a - \rho(a)b_n) \\ &= \rho(b_n)\rho(a) - \rho(a)\rho(b_n) \\ &= 0. \end{aligned}$$

So $\tau_{a,\rho}(b_n) - b_n - \rho(a)b_n$ is in H. Let

$$\tau_{a,\rho}(b_n) - b_n - \rho(a)b_n = c_1b_1 + \ldots + c_{n-1}b_{n-1}.$$

So

$$\tau_{a,\rho}(b_n) = c_1b_1 + \ldots + c_{n-1}b_{n-1} + (1 + \rho(a))b_n.$$

So

$$\mathrm{mat}(\tau_{a,\rho}) = \begin{bmatrix} 1 & 0 & c_1 \\ & \ddots & \vdots \\ 0 & 1 & c_{n-1} \\ 0 \cdots & 0 & 1+\rho(a) \end{bmatrix}$$

Thus $\det(\tau_{a,\rho}) = 1 + \rho(a)$.

Thus $\tau_{a,\rho}$ is a transvection if and only if a is in $\ker(\rho)$ and $\tau_{a,\rho}$ is a dilation if and only if a is not in $\ker(\rho)$ and $1 + \rho(a)$ is in R^*.

If $\tau_{a,\rho}$ is proper, with $1 + \rho(a)$ a unit, then by Theorem 1.5.1, we see that $Q \subset P$ if and only if $\rho(a) = 0$; $P \cap Q = 0$ if and only if $\rho(a)$ is not a zero divisor; and $V = P \oplus Q$ if and only if $\rho(a)$ is a unit.

Let P and Q be the fixed and residual modules of $\tau_{a,\rho}$, respectively. Then $P = \ker(\rho)$ and $Q = Ra$. If $\tau_{a,\rho}$ and $\tau_{b,\rho}$ are transvections, then

$$\begin{aligned}\tau_{a,\rho}\,\tau_{b,\rho}(x) &= \tau_{a,\rho}(x + \rho(x)b)\\ &= x + \rho(x)b + \rho(x + \rho(x)b)a\\ &= x + \rho(x)b + \rho(x)a + \rho(x)\rho(b)a\\ &= x + \rho(x)(b + a)\\ &= \tau_{a+b,\rho}(x) \quad \text{for all } x \text{ in } V.\end{aligned}$$

So $\tau_{a,\rho}\,\tau_{b,\rho} = \tau_{a+b,\rho}$.

If $\tau_{a,\rho}$ and $\tau_{a,\phi}$ are transvections then

$$\begin{aligned}\tau_{a,\rho}\,\tau_{a,\phi}(x) &= \tau_{a,\rho}(x + \phi(x)a)\\ &= x + \phi(x)a + \rho(x + \phi(x)a)a\\ &= x + \phi(x)a + \rho(x)a + \phi(x)\rho(a)a\\ &= x + (\phi + \rho)(x)a\\ &= \tau_{a,\rho+\phi}(x) \text{ for all } x \text{ in } V.\end{aligned}$$

So $\tau_{a,\rho}\,\tau_{a,\phi} = \tau_{a,\rho+\phi}$.

So if $\tau_{a,\rho}$ is a transvection and n is an integer, then $(\tau_{a,\rho})^n = \tau_{na,\rho} = \tau_{a,n\rho}$.

<u>Theorem 1.5.4</u>: Let $n \geq 2$. Let L be a line and H a hyperplane in V. If $L \subset H$, then there exists a proper transvection with $P = H$ and $Q = L$.

<u>Proof</u>: Suppose $L \subset H$. Let $L = Ra$. Let $V = H \oplus Rb$. Define $\rho : V \to R$ by $\rho(H) = 0$, $\rho(b) = 1$. Then $\tau_{a,\rho}$ is a proper transvection with $P = H$ and $Q = L$.

Theorem 1.5.5: Let σ be proper in GL(V) with res(σ) = 1. Then there exist $\rho: V \to R$, where ρ is onto, and unimodular a in V such that $\sigma = \tau_{a,\rho}$.

Proof: Let H = P. Then $V = H \oplus Rb$. Define $\rho: V \to R$ by $\rho(H) = 0$ and $\rho(b) = 1$. Let $a = \sigma(b) - b$. Let $x = h + rb$ be in V, h in H, r in R. Then

$$\begin{aligned}\tau_{a,\rho}(x) &= \tau_{a,\rho}(h + rb)\\ &= h + rb + \rho(h + rb)a\\ &= h + rb + r\rho(b)a\\ &= h + rb + ra\\ &= h + rb + r(\sigma(b) - b)\\ &= h + rb + \sigma(rb) - rb\\ &= h + \sigma(rb)\\ &= \sigma(h + rb)\\ &= \sigma(x)\end{aligned}$$

Thus $\tau_{a,\rho} = \sigma$.

Now $Q \subset Ra$ so $Q = Ra$ since Q is a free summand. Therefore a is unimodular.

By this theorem we see that

$$\{\tau_{a,\rho} \mid a \text{ unimodular is in } \ker(\rho) \text{ and } \rho: V \to R \text{ is surjective}\}$$

is exactly the set of proper transvections.

Theorem 1.5.6: (i) If dim(V) = 2, then there exists a proper transvection τ such that $\alpha\tau$ is also a proper transvection if and only if $\alpha^2 = 1$ and $(\alpha - 1)^2 = 0$.

(ii) If dim(V) > 2, and if there exists a proper transvection τ such that $\alpha\tau$ is also a proper transvection, then $\alpha^n = 1$ and $(\alpha - 1)^2 = 0$.

(iii) If n is not a zero divisor and τ is a proper transvection, then $\alpha\tau$ is a proper transvection if and only if $\alpha = 1$.

(iv) Suppose R has no nontrivial nilpotents and τ is a proper transvection. Then $\alpha\tau$ is a proper transvection if and only if $\alpha = 1$.

Proof: (ii) Let $\dim(V) \geq 2$. Let τ be a proper transvection and suppose $\alpha\tau$ is also a proper transvection. Clearly α is a unit and $\alpha^n = 1$.

Let $\tau = \tau_{a,\rho}$ and $\alpha\tau = \tau_{b,\phi}$. Then $\alpha\tau_{a,\rho}(x) = \tau_{b,\phi}(x)$ which implies

$$\alpha x + \alpha\rho(x)a = x + \phi(x)b$$

for all x in V. So

$$\alpha a + \alpha\rho(a)a = a + \phi(a)b \text{ and } (\alpha - 1)a = \phi(a)b.$$

Then

$$(\alpha - 1)\phi(a) = \phi((\alpha - 1)a) = \phi(\phi(a)b) = \phi(a)\phi(b) = 0.$$

So

$$(\alpha - 1)^2 a = (\alpha - 1)\phi(a)b = 0.$$

Thus $(\alpha - 1)^2 = 0$.

(i) Suppose $\dim(V) = 2$, $\alpha^2 = 1$ and $(\alpha - 1)^2 = 0$. Let $\{b_1, b_2\}$ be a basis for V. Let $a = b_1$, $b = \alpha b_1 + (\alpha - 1)b_2$. Define $\rho : V \to R$ by $\rho(b_1) = 0$ and $\rho(b_2) = 1$ and define $\phi : V \to R$ by $\phi(b_1) = \alpha - 1$ and $\phi(b_2) = 1$. Then a and b are unimodular and $\rho(a) = 0$ and

$$\phi(b) = \alpha(\alpha - 1) + (\alpha - 1) = (\alpha + 1)(\alpha - 1) = \alpha^2 - 1 = 0.$$

So $\tau_{a,\rho}$ and $\tau_{b,\rho}$ are proper transvections. Then

$$\tau_{a,\rho}(b_1) = b_1 + \rho(b_1)b_1 = b_1$$
$$\tau_{a,\rho}(b_2) = b_2 + \rho(b_2)b_1 = b_2 + b_1$$

and

$$\tau_{b,\phi}(b_1) = b_1 + \phi(b_1)(\alpha b_1 + (\alpha - 1)b_2)$$
$$= b_1 + \alpha(\alpha - 1)b_1 + (\alpha - 1)^2 b_2$$
$$= (\alpha^2 - \alpha + 1)b_1$$
$$= ((\alpha - 1)^2 + \alpha)b_1$$
$$= \alpha b_1.$$

$$\tau_{b,\phi}(b_2) = b_2 + \phi(b_2)(\alpha b_1 + (\alpha - 1)b_2)$$
$$= b_2 + \alpha b_1 + (\alpha - 1)b_2$$
$$= \alpha(b_2 + b_1).$$

Thus $\alpha\tau_{a,\rho} = \tau_{b,\phi}$.

(iii) Suppose n is not a zero divisor and τ and $\alpha\tau$ are proper transvections. Then $tr(\tau) = n$, so $tr(\alpha\tau) = \alpha n$, where $tr(\tau)$ is the trace of τ. But if $\alpha\tau$ is also a proper transvection, then $tr(\alpha\tau) = n$. So $\alpha n = n$. Hence $\alpha = 1$ since n is not a zero divisor.

(iv) From proof of part (ii) we have that $(\alpha - 1)^2 = 0$. Since R has no nontrivial nilpotents then $\alpha - 1 = 0$ so $\alpha = 1$.

<u>Theorem 1.5.7</u>: Let σ_1 and σ_2 be proper in GL(V) with $res(\sigma_1) = res(\sigma_2) = 1$ and $\sigma_1\sigma_2$ proper, but $\sigma_1\sigma_2 \neq I$. Then $res(\sigma_1\sigma_2) = 1$ if $P_1 = P_2$ or $Q_1 = Q_2$.

<u>Proof</u>: Let $\sigma = \sigma_1\sigma_2$. Suppose $Q_1 = Q_2$. Then $Q \subset Q_1 + Q_2 = Q_1$. So $\dim(Q) \leq \dim(Q_1) = 1$. But $\dim(Q) \neq 0$, so $\dim(Q) = 1$. Hence $res(\sigma) = 1$. Suppose $P_1 = P_2$. Then $P \supset P_1 \cap P_2 = P_1$. So $\dim(P) \geq \dim(P_1) = n - 1$. But $\dim(P) \neq n$, so $\dim(P) = n - 1$. Thus $\dim(Q) = 1$ and $res(\sigma) = 1$.

<u>Theorem 1.5.8</u>: Let σ_1 and σ_2 be proper in GL(V) with $res(\sigma_1) = res(\sigma_2) = 1$ and $res(\sigma_1\sigma_2) = 1$, but $\sigma_1\sigma_2 \neq I$.

(i) If $Q_1 \cap Q_2 = 0$, then $P_1 = P_2 = P$.

(ii) If $P_1 + P_2 = V$, then $Q_1 = Q_2 = Q$.

Proof: (i) If $Q_1 \cap Q_2 = 0$, then $P = P_1 \cap P_2 \subset P_1$ by Theorem 1.4.3. But $\dim(P) = \dim(P_1)$, so $P = P_1 \cap P_2 = P_1$. Thus $P_1 \subset P_2$. Similarly $P_2 \subset P_1$. Thus $P_1 = P_2 = P$.

(ii) If $P_1 + P_2 = V$, then $Q = Q_1 + Q_2 \supset Q_1$. But $\dim(Q) = \dim(Q_1)$, so $Q = Q_1 + Q_2 = Q_1$ thus $Q_2 \subset Q_1$. Similarly $Q_1 \subset Q_2$. Thus $Q_1 = Q_2 = Q$.

Theorem 1.5.9: Let σ_1 and σ_2 be nontrivial proper transvections in $GL(V)$.

(i) If $Q_1 \subset P_2$, $Q_2 \subset P_1$, then $\sigma_1\sigma_2 = \sigma_2\sigma_1$.

(ii) If $\sigma_1\sigma_2 = \sigma_2\sigma_1$ and either $V = P_1 + P_2$ or $Q_1 \cap Q_2 = 0$, then $Q_1 \subset P_2$, $Q_2 \subset P_1$.

(iii) If R has no nontrivial nilpotents and $\sigma_1\sigma_2 = \sigma_2\sigma_1$, then $Q_1 \subset P_2$ and $Q_2 \subset P_1$.

Proof: (i) This follows from Theorem 1.4.5.

(ii) This follows from Theorem 1.4.6.

(iii) Let $\sigma_1 = \tau_{a,\rho}$, $\sigma_2 = \tau_{b,\phi}$. Then

$$\tau_{a,\rho}\, \tau_{b,\phi}(x) = \tau_{b,\phi}\, \tau_{a,\rho}(x)$$

$$\tau_{a,\rho}(x + \phi(x)b) = \tau_{b,\phi}(x + \rho(x)a).$$

Thus,

$$x + \phi(x)b + \rho(x + \phi(x)b)a = x + \rho(x)a + \phi(x + \rho(x)a)b$$

$$x + \phi(x)b + \rho(x)a + \phi(x)\rho(b)a = x + \rho(x)a + \phi(x)b + \rho(x)\phi(a)b$$

and consequently, $\phi(x)\rho(b)a = \rho(x)\phi(a)b$ for all x in V. Thus $\phi(a)\rho(b) = 0$. So

$$0 = \phi(x)\phi(a)\rho(b)a = \rho(x)(\phi(a))^2 b$$

for all x in V. Choose $x = c$ where $\rho(c) = 1$. Then $(\phi(a))^2 b = 0$ so $(\phi(a))^2 = 0$. Then $\phi(a) = 0$ since R has no nontrivial nilpotents.

Similarly, $\rho(b) = 0$. Thus a is in $\ker(\phi)$ and b is in $\ker(\rho)$. So $Q_1 \subset P_2$ and $Q_2 \subset P_1$.

Theorem 1.5.10: Let $V = H \oplus Rb$ where b is unimodular and H is a hyperplane in V. Let a be in V such that H contains $a - b$ and $a - b$ is unimodular. Then there exists a proper transvection σ such that $P = H$ and $Q = R(a - b)$ and $\sigma(b) = a$.

Proof: Define $\rho: V \to R$ by $\rho(H) = 0$ and $\rho(b) = 1$. So $\rho(a - b) = 0$. Let $\sigma = \tau_{a-b,\rho}$. This is a transvection with $Q = R(a - b)$ and $P = H$. Also

$$\begin{aligned}\sigma(b) &= \tau_{a-b,\rho}(b) \\ &= b + \rho(b)(a - b) \\ &= b + a - b \\ &= a.\end{aligned}$$

1.6 Matrices

We shall use $GL_n(R)$ to denote the multiplicative group of invertible $n \times n$ matrices over R and $SL_n(R)$ for the subgroup of those matrices of determinant 1. The group of scalar matrices (matrices of the form $\alpha I_{n\times n}$, with α in R^*) will be denoted $RL_n(R)$.

If we fix a basis for the n-dimensional vector space V, then we have

$$GL(V) \cong GL_n(R)$$
$$SL(V) \cong SL_n(R)$$
$$RL(V) \cong RL_n(R)$$

by mapping σ in $GL(V)$ to $mat(\sigma)$ in $GL_n(R)$, where $mat(\sigma)$ is the matrix of σ with respect to the chosen basis. Let P be the natural surjection

$$P: \ GL_n(R) \to GL_n(R)/RL_n(R).$$

Then

$$PSL_n(R) = (SL_n(R) \cdot RL_n(R))/RL_n(R) \cong SL_n(R)/(SL_n(R) \cap RL_n(R))$$

So $\ker(P|SL_n(R)) = SL_n(R) \cap RL_n(R)$. By Theorem 1.3.1 we have

$$\begin{aligned} PGL(V) &\cong GL(V)/RL(V) \\ &\cong GL_n(R)/RL_n(R) \\ &\cong PGL_n(R) \end{aligned}$$

and

$$\begin{aligned} PSL(V) &\cong SL(V)/(SL(V) \cap RL(V)) \\ &\cong SL_n(R)/(SL_n(R) \cap RL_n(R)) \\ &\cong PSL_n(R). \end{aligned}$$

Let $n \geq 2$, $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \neq j$, λ in R. Let $t_{ij}(\lambda)$ denote the $n \times n$ matrix with 1's on the diagonal, λ in the (i,j)-position, and 0's elsewhere.

Let $\{b_1, \ldots ,b_n\}$ be a basis of V and define $\rho_i : V \to R$ by $\rho_i(b_j) = \delta_{ij}$. Define an <u>elementary transvection</u> with respect to the basis $\{b_1, \ldots ,b_n\}$ to be a transvection of the form $\tau_{\lambda b_i, \mu\rho_j}$ for λ,μ in R. Note that

$$\tau_{\lambda b_i,\mu\rho_j}(x) = x + (\mu\rho_j(x))(\lambda b_i)$$
$$= x + \lambda\mu\rho_j(x)b_i$$
$$= \tau_{b_i,\lambda\mu\rho_j}$$
$$= x + \rho_j(x)(\lambda\mu b_i)$$
$$= \tau_{\lambda\mu b_i,\rho_j}(x)$$

for all x in V, so every elementary transvection can be written as $\tau_{\lambda b_i,\rho_j}$.

Let $\tau_{a,\rho}$ be a proper transvection. Then a is unimodular, so extend $\{a\}$ to a basis $\{a, x_2, \ldots ,x_n\}$ of V, choosing $x_2, \ldots ,x_{n-1}$ in $\ker(\rho)$ and choosing x_n such that $\rho(x_n) = 1$. Then $\tau_{a,\rho}$ is an elementary transvection with respect to this basis. Therefore every proper transvection is also an elementary transvection with respect to some basis. Now

$$\tau_{\lambda b_i,\rho_j}(b_k) = b_k + \lambda\rho_j(b_k)b_i$$
$$= b_k + \delta_{jk}\lambda b_i$$
$$= \begin{cases} b_k \text{ if } k \neq j \\ b_j + \lambda b_i \text{ if } k = j. \end{cases}$$

So $\mathrm{mat}(\tau_{\lambda b_i,\rho_j}) = t_{ij}(\lambda)$ with respect to $\{b_1, \ldots ,b_n\}$. So $(\tau_{\lambda b_i,\rho_j} - I)(V) = R(\lambda b_i)$. Thus an elementary transvection is a proper transvection if and only if λ is in R^*.

If E_{ij} is a matrix with a 1 in the (i,j) position and 0's elsewhere then $t_{ij}(\lambda) = I + \lambda E_{ij}$. So

$$t_{ij}(\lambda)t_{ij}(\mu) = (I + \lambda E_{ij})(I + \mu E_{ij})$$
$$= I + \mu E_{ij} + \lambda E_{ij}$$
$$= I + (\lambda + \mu)E_{ij}$$
$$= t_{ij}(\lambda + \mu).$$

Also

$$[\tau_{\lambda b_i,\rho_k}, \tau_{\mu b_k,\rho_j}] = [t_{ik}(\lambda), t_{kj}(\mu)]$$
$$= (I + \lambda E_{ik})(I + \mu E_{kj})(I - \lambda E_{ik})(I - \mu E_{kj})$$
$$= (I + \lambda E_{ik} + \mu E_{kj} + \lambda\mu E_{ij})(I - \lambda E_{ik} - \mu E_{kj} + \lambda\mu E_{ij})$$
$$= I - \lambda E_{ik} - \mu E_{kj} + \lambda\mu E_{ij} + \lambda E_{ik}$$
$$- \lambda\mu E_{ij} + \mu E_{kj} + \lambda\mu E_{ij}$$
$$= I + \lambda\mu E_{ij}$$
$$= t_{ij}(\lambda\mu)$$
$$= \tau_{\lambda\mu b_i,\rho_j}.$$

From this point on we will assume that 2 is a unit in R.

<u>Theorem 1.6.1</u>: If σ in GL(V) is an involution then σ is proper and $V = P \oplus Q$.

<u>Proof</u>: Let σ be in GL(V) with $\sigma^2 = I$. Let $P = \{x \text{ in } V | \sigma(x) = x\}$. Let x be in Q. Then $x = \sigma(y) - y$ for some y in V. Then

$$-\sigma(x) = -\sigma(\sigma(y) - y) = -\sigma^2(y) + \sigma(y) = \sigma(y) - y = x.$$

So $\sigma(x) = -x$.

Now let x be any element of V and suppose $\sigma(x) = -x$. Then $x = \sigma(-\frac{1}{2}x) - (-\frac{1}{2}x)$ which is in Q. So $Q = \{x \text{ in } V | \sigma(x) = -x\}$.

Let x be in $P \cap Q$. Then $\sigma(x) = x = -x$ so $x = 0$. Let x be in V.

Then $\sigma(x) + x$ is in P and $\sigma(x) - x$ is in Q and $x = \frac{1}{2}(\sigma(x) + x) - \frac{1}{2}(\sigma(x) - x)$ so x is in P + Q. Thus $V = P \oplus Q$.

Theorem 1.6.2: If X is any set of pairwise commuting involutions in GL(V), then there is a basis of V in which $\text{mat}(\sigma) = \text{diag}(\pm 1, \pm 1, \ldots, \pm 1)$ for all σ in X.

Proof: We use induction on n. If n = 1, $\sigma_1^2 = I$, V = Rb, and $\sigma_1(b) = rb$, then $\sigma^2(b) = r^2 b = b$, so $r^2 = 1$. Since 2 is a unit and all projectives of V are free, then $r = \pm 1$. So $\text{mat}(\sigma_1) = [\pm 1]$.

Suppose n > 1. Let σ_1 be in X, $\sigma_1 \neq \pm I$. By Theorem 1.6.1, $V = P_1 \oplus Q_1$. We also have $P_1 \neq 0$ and $Q_1 \neq 0$. Let σ_2 be in X. Then $\sigma_1\sigma_2 = \sigma_2\sigma_1$ so $\sigma_2(P_1) = P_1$ and $\sigma_2(Q_1) = Q_1$. So $\{(\sigma|P_1) | \sigma \text{ is in } x\}$ and $\{(\sigma|Q_1) | \sigma \text{ is in } x\}$ are each families of pairwise commuting involutions, so by induction, there exists a basis $\{b_1, \ldots, b_s\}$ of P_1 such that $\text{mat}(\sigma|P_1) = \text{diag}(\pm 1, \ldots, \pm 1)$ for all σ in X and there exists a basis $\{b_{s+1}, \ldots, b_n\}$ of Q_1 such that $\text{mat}(\sigma|Q_1) = \text{diag}(\pm 1, \ldots, \pm 1)$ for all σ in X. Thus $\text{mat}(\sigma) = \text{diag}(\pm 1, \ldots, \pm 1)$ with respect to the basis $\{b_1, \ldots, b_n\}$ for all σ in X.

1.7 Generation by Transvections

Let σ be in GL(V) and $\sigma = \tau_1 \ldots \tau_k\sigma_1$ where each τ_i is a transvection and σ_1 is either a transvection or a dilation. Then

$$\begin{aligned}\sigma &= \tau_1 \ldots \tau_{k-1}(\tau_k\sigma_1\tau_k^{-1})\tau_k \\ &= \tau_1 \ldots \tau_{k-2}(\tau_{k-1}\tau_k\sigma_1\tau_k^{-1}\tau_{k-1}^{-1})\tau_{k-1}\tau_k \\ &\;\;\vdots \\ &= (\tau_1 \ldots \tau_k\sigma_1\tau_k^{-1} \ldots \tau_1^{-1})\tau_1 \ldots \tau_k.\end{aligned}$$

In each case $\tau_i \ldots \tau_k\sigma_1\tau_k^{-1} \ldots \tau_i^{-1}$ is still a transvection or dilation if σ_1 is a transvection or dilation, resp. So the position of σ_1 really does not matter; we may assume it always appears as the right-most factor.

Theorem 1.7.1: If σ is proper in GL(V) and is of the form $\sigma = \sigma_1 \ldots \sigma_t$ where each σ_i is proper in GL(V) and $\mathrm{res}(\sigma_i) = 1$ for each i, then $t \geq \mathrm{res}(\sigma)$. If $t = \mathrm{res}(\sigma)$ then $Q = Q_1 \oplus \ldots \oplus Q_t$ and $P = P_1 \cap \ldots \cap P_t$.

Proof: By Theorem 1.4.2 we have $\mathrm{res}(\sigma) \leq t$. Let $Q_i = Rb_i$ for each i. Suppose $\mathrm{res}(\sigma) = t$. Now $Q \subset Q_1 + \ldots + Q_t = Rb_1 + \ldots + Rb_t$ and Q has dimension t, so $Q = Q_1 \oplus \ldots \oplus Q_t$. Thus $Q_1 \cap Q_2 = 0$ so $P_{\sigma_1\sigma_2} = P_1 \cap P_2$.

Suppose $P_{\sigma_1 \ldots \sigma_k} = P_1 \cap \ldots \cap P_k$ for some $k < t$. Then $Q_{\sigma_1 \ldots \sigma_k} \cap Q_{k+1} = (Q_1 \oplus \ldots \oplus Q_k) \cap Q_{k+1} = 0$ so $P_{\sigma_1 \ldots \sigma_{k+1}} = P_{\sigma_1 \ldots \sigma_k} \cap P_{k+1} = P_1 \cap \ldots \cap P_{k+1}$. Therefore $P = P_1 \cap \ldots \cap P_t$.

Let U be a subspace of V. Define

$$G(U) = \{\sigma \text{ in } GL(V) \mid U \subset P\}$$
$$= \{\sigma \text{ in } GL(V) \mid \sigma(x) = x \text{ for all } x \text{ in } U\},$$

where $P = \ker(\sigma - I)$. Now consider the R-module V/U. We have $r(x + U) = rx + U$. Let $x + U$ be denoted by $\tilde{x}$. Then $r\tilde{x} = \widetilde{rx}$. Suppose $\tilde{x} = \tilde{y}$. Then $x - y$ is in U, so $rx - ry$ is in U since U is a subspace. Thus $\widetilde{rx} = \widetilde{ry}$. So $\sim : V \to V/U$, $x \mapsto \tilde{x}$ is an R-morphism.

Now U is a subspace of V so there exists a subspace W such that $U \oplus W = V$. Then $W \simeq V/U$, so V/U is projective and hence free. Also $\dim(V/U) = \dim(V) - \dim(U)$. For each σ in G(U), we can define an R-morphism $\tilde{\sigma}: V/U \to V/U$ by $\tilde{\sigma}(\tilde{x}) = \widetilde{\sigma(x)}$. If $\tilde{x} = \tilde{y}$, then $x - y$ is in U so $\sigma(x) - \sigma(y) = \sigma(x - y) = x - y$ which is in U. Thus $\widetilde{\sigma(x)} = \widetilde{\sigma(y)}$.

Therefore $\tilde{\sigma}$ is well defined. For any $\tilde{x}$, $\tilde{y}$ in V/U,

$$\begin{aligned}\tilde{\sigma}(\tilde{x} + \tilde{y}) &= \tilde{\sigma}(\widetilde{x + y}) \\ &= \widetilde{\sigma(x + y)} \\ &= \widetilde{\sigma(x) + \sigma(y)} \\ &= \widetilde{\sigma(x)} + \widetilde{\sigma(y)} \\ &= \tilde{\sigma}(\tilde{x}) + \tilde{\sigma}(\tilde{y}).\end{aligned}$$

Let $U \oplus W = V$ and let $\{b_1, \dots, b_m\}$ be a basis of W. Then $\{\tilde{b}_1, \dots, \tilde{b}_m\}$ is a basis of V/U. Let $\tilde{x}$ be in V/U, $\tilde{x} = x + U$. Then $x = u + w$, so $x + U = u + w + U = w + U$. Thus we may assume that x is in W. So $x = r_1b_1 + \dots + r_mb_m$. Then $\tilde{x} = r_1b_1 + \dots + r_mb_m + U$ $= r_1\tilde{b}_1 + \dots + r_m\tilde{b}_m$. Suppose $r_1\tilde{b}_1 + \dots + r_m\tilde{b}_m = 0$. Then $r_1b_1 + \dots + r_mb_m$ is in U so $r_1 = r_2 = \dots = r_m = 0$.

<u>Theorem 1.7.2</u>: Let σ be in G(U). Then the diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & U & \xrightarrow{\lambda} & V & \xrightarrow{\pi} & V/U & \rightarrow & 0 \\ & & \downarrow{\scriptstyle I_U} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tilde{\sigma}} & & \\ 0 & \rightarrow & U & \xrightarrow{\lambda} & V & \xrightarrow{\pi} & V/U & \rightarrow & 0 \end{array}$$

commutes, where $\lambda(u) = u$ and $\pi(u + x) = x + U$.

<u>Proof</u>: Let u be in U. Then $\sigma(\lambda(u)) = \sigma(u) = u$ and $\lambda(I_U(u)) = \lambda(u) = u$.

Let $u + x$ be in V. Then $\pi(\sigma(u + x)) = \pi(\sigma(u) + \sigma(x)) = \pi(u + \sigma(x))$. Let $\sigma(x) = u_1 + y$ where u_1 is in U. Then $\pi(u + \sigma(x)) = \pi(u + u_1 + y)$ $= y + U$. Also

$$\tilde{\sigma}(\pi(u + x)) = \tilde{\sigma}(x + U) = \sigma(x) + U = u_1 + y + U = y + U.$$

Thus the diagram commutes.

<u>Theorem 1.7.3</u>: Let σ be in G(U). Then

(i) $\det(\sigma) = \det(\tilde{\sigma})$,

(ii) $\mathrm{tr}(\sigma) = \mathrm{tr}(\tilde{\sigma}) + t$,

(iii) $\chi(\sigma) = \chi(\tilde{\sigma})(X - 1)^t$,

where $\chi(\sigma)$ is the characteristic polynomial of σ, tr is the trace of σ, and $t = \dim(U)$.

Proof: This follows from Theorem 1.7.2.

Thus $\tilde{\sigma}$ is in GL(V/U). Let $\tilde{\sigma}_1, \tilde{\sigma}_2$ be in GL(V/U). Then

$$\tilde{\sigma}_1\tilde{\sigma}_2(\tilde{x}) = \tilde{\sigma}_1(\widetilde{\sigma_2(x)}) = \widetilde{\sigma_1\sigma_2(x)} = \widetilde{\sigma_1\sigma_2}(\tilde{x}).$$

So $\tilde{\sigma}_1\tilde{\sigma}_2 = \widetilde{\sigma_1\sigma_2}$.

Let σ be in GL(V/U). Then $\sigma(\tilde{b}_i) = \sum_{j=1}^{m} a_{ji}\tilde{b}_j$. Define σ' in GL(V) by $\sigma(b_i) = \sum_{j=1}^{m} a_{ji}b_j$ for $i = 1, \ldots, m$ and $\sigma(b_i) = b_i$ for $i > m$. Then $\tilde{\sigma}' = \sigma$. So $\sim: G(U) \to GL(V/U)$ is a surjective determinant preserving homomorphism. We will use $\sim$ to denote the map $U \to V/U$ and the map $G(U) \to GL(V/U)$. Each map is called the tilda map going with reduction modulo U.

Let σ be in GL(V) with $U \subset P$. Now

$$\begin{aligned}\tilde{Q} &= \{x + U \mid x \text{ is in } Q\} \\ &= \{\sigma(y) - y + U \mid y \text{ is in } V\} \\ &= \{\widetilde{\sigma(y) - y} \mid y \text{ is in } V\} \\ &= \{\tilde{\sigma}(\tilde{y}) - \tilde{y} \mid \tilde{y} \text{ is in } V/U\} \\ &= \mathrm{Im}(\tilde{\sigma} - I).\end{aligned}$$

So $\mathrm{Im}(\tilde{\sigma} - I) = \tilde{Q} = (Q + U)/U$. Suppose $\tilde{\sigma}$ is proper in GL(V/U). Then $V/U = \tilde{Q} \oplus \tilde{W}$ for some $\tilde{W}$. Now clearly $\dim(\tilde{Q}) \leq \dim(Q)$, so $\mathrm{res}(\tilde{\sigma}) \leq \mathrm{res}(\sigma)$ where $\mathrm{res}(\tilde{\sigma}) = \dim(\tilde{Q})$.

Suppose $Q \cap U = 0$. Then $(Q + U)/U = (Q \oplus U)/U \cong Q$ so

$\dim((Q + U)/U) = \dim(Q)$. Thus $\mathrm{res}(\tilde{\sigma}) = \mathrm{res}(\sigma)$.

Now suppose $\mathrm{res}(\tilde{\sigma}) = \mathrm{res}(\sigma)$. Then $\dim((Q + U)/U) = \dim(Q)$. Let $\{b_1, \ldots, b_m\}$ be a basis of Q. Then $\{\tilde{b}_1, \ldots, \tilde{b}_m\}$ is a basis of $\tilde{Q}$. Suppose x is in $Q \cap U$. Then $x = r_1 b_1 + \ldots + r_m b_m$ is in U, so $r_1\tilde{b}_1 + \ldots + r_m\tilde{b}_m = 0$. Thus $r_1 = \ldots = r_m = 0$. So $x = 0$. Thus $Q \cap U = 0$. Therefore $\mathrm{res}(\tilde{\sigma}) = \mathrm{res}(\sigma)$ if and only if $Q \cap U = 0$.

Suppose $\tilde{\sigma} = I$. Then

$\tilde{\sigma}(\tilde{y}) = \tilde{y}$ if and only if $\tilde{\sigma}(\tilde{y}) - \tilde{y} = 0$ for all y in V/U.

if and only if $\widetilde{\sigma(y) - y} = 0$ for all y in V

if and only if $\sigma(y) - y$ is in U for all y in V

if and only if $Q \subset U$.

So $\tilde{\sigma} = I$ if and only if $Q \subset U$.

We say that an element σ in GL(V) is a <u>big dilation</u> if there is a splitting $V = U \oplus W$ with $W \neq 0$ such that $\sigma = (I_U) \oplus (\alpha I_W)$ for some $\alpha \neq 0$ with $\alpha - 1$ in R^*. Then by a previous example, $P = U$ and $Q = W$.

Note that dilations are big dilations, but big dilations need not be dilations, nontrivial radiations are big dilations, I is not a big dilation, and a big dilation may be in SL(V). Note also that big dilations are proper.

<u>Theorem 1.7.4</u>: Let $\sigma \neq I$ be an element of GL(V), let $\sim$ be the tilda mappings going with reduction modulo the fixed space P of σ. Then the following are equivalent:

(i) σ is a big dilation.

(ii) σ is proper and $\tilde{\sigma}$ is a nontrivial radiation $\alpha\tilde{I}$ with $\alpha - 1$ in R^*.

(iii) σ is proper and $\sigma|Q$ is nontrivial radiation αI_Q with $\alpha - 1$ in R^*.

Proof: (i) implies (ii):

Suppose σ is a big dilation. Then σ is proper and $V = U \oplus W$ with $\sigma = (I_U) \oplus (\alpha I_W)$ with $W \neq 0$, $\alpha \neq 0$, $\alpha - 1$ in R^*. Let $\sim$ be the tilda mapping going with reduction modulo $P = U$. Let $\tilde{x}$ be in V/U. Then $\tilde{x} = \widetilde{u + w} = \tilde{u} + \tilde{w} = \tilde{w}$. So

$$\tilde{\sigma}(\tilde{x}) = \tilde{\sigma}(\tilde{w}) = \widetilde{\sigma(w)} = \widetilde{\alpha w} = \alpha\tilde{w} = \alpha\tilde{x}.$$

So $\tilde{\sigma}$ is a nontrivial radiation.

(ii) implies (iii):

Suppose σ is proper and $\tilde{\sigma}$ is a nontrivial radiation. Then $\tilde{\sigma}(\tilde{x}) = \alpha\tilde{x}$ for all $\tilde{x}$ in V/P, for some $\alpha \neq 1$, α in R^*. Then $\widetilde{\sigma(x)} = \widetilde{\alpha x}$ so $\sigma(x) - \alpha x$ is in P for all x in V. Since $\tilde{\sigma}$ is a nontrivial radiation,

$$\text{res}(\tilde{\sigma}) = m = \dim(V/P) = \dim(V) - \dim(P) = \dim(Q) = \text{res}(\sigma),$$

since $\alpha - 1$ is in R^*. So $Q \cap P = 0$.

Let x be in Q. Now $\sigma(Q) = Q$ so $\sigma(x)$ is in Q. Also αx is in Q. So $\sigma(x) - \alpha x$ is in Q. Thus $\sigma(x) - \alpha x$ is in $P \cap Q = 0$. Thus $\sigma(x) = \alpha x$. So $(\sigma|Q)$ is a nontrivial radiation.

(iii) implies (i):

Suppose σ is proper and $(\sigma|Q)$ is a nontrivial radiation $\alpha\tilde{I}$ with $\alpha - 1$ in R^*. Then $(\sigma - I)(x) = (\alpha - 1)x$ for all x in Q. Define $\tau: Q \to V$ by $\tau(x) = (\alpha - 1)^{-1}x$. Then we have the split exact sequence

$$0 \longrightarrow P \xrightarrow{\text{incl.}} V \underset{\underset{\tau}{\longleftarrow}}{\xrightarrow{\sigma - I}} Q \longrightarrow 0$$

since $(\sigma - I)\tau = I_Q$. So

$$V = P \oplus \tau Q = P \oplus (\alpha - 1)^{-1}Q = P \oplus Q.$$

Clearly, then σ is a big dilation.

In the last part of the above proof, it is necessary that $\alpha - 1$ be a unit. For suppose that $\alpha - 1$ is not a unit and not a zero-divisor. Let $\dim(V) = 2$ and let $\sigma: V \to V$ be defined by $\sigma(x,y) = (x + y, \alpha y)$. Then σ is in $GL(V)$, since $\sigma^{-1}(x,y) = (x - \alpha^{-1}y, \alpha^{-1}y)$. Suppose (x,y) is in P. Then $\sigma(x,y) = (x,y)$. So $x + y = x$ and hence $y = 0$. Thus $P = \{(x,0) | x \text{ is in } R\} = R(1,0)$. Suppose (x,y) is in Q. Then

$$\begin{aligned}(x,y) &= \sigma(z,w) - (z,w)\\ &= (z + w, \alpha w) - (z,w)\\ &= (w, (\alpha - 1)w)\end{aligned}$$

for some z and w in R. So $Q = \{(w, (\alpha - 1)w) | w \text{ is in } R\} = R(1,\alpha - 1)$. Then σ is proper and $P \cap Q = 0$ since $\alpha - 1$ is not a zero-divisor. Let $(w, (\alpha - 1)w)$ be in Q. Then

$$\begin{aligned}\sigma(w, (\alpha - 1)w) &= (w + (\alpha - 1)w, \alpha(\alpha - 1)w)\\ &= (\alpha w, \alpha(\alpha - 1)w)\\ &= \alpha(w, (\alpha - 1)w).\end{aligned}$$

So $\sigma|Q = \alpha I_Q$. But $P \oplus Q \neq V$ since $(0,1)$ is not in $P \oplus Q$.

<u>Theorem 1.7.5</u>: Let σ be proper in $GL(V)$ with $P \cap Q = 0$ and $(\sigma|Q)$ in $RL(Q)$. Let $r = \text{res}(\sigma) > 0$. Then σ is not the product of r transvections.

<u>Proof</u>: Suppose σ is the product of r transvections. Let $\sigma = \tau_{a_r,\rho_r} \cdots \tau_{a_2,\rho_2} \tau_{a_1,\rho_1}$. Now $Q \subset Ra_1 + Ra_2 + \cdots + Ra_r$ so $Q = Ra_1 \oplus Ra_2 \oplus \cdots \oplus Ra_r$. If $(\sigma|Q)$ is in $RL(Q)$, then $\sigma(x) = \alpha x$ for all x in Q, for some α in R^*. Then $\sigma(a_1) = \alpha a_1$ since a_1 is in Q.

$$\tau_{a_r,\rho_r} \cdots \tau_{a_2,\rho_2}\tau_{a_1,\rho_1}(a_1) = \tau_{a_r,\rho_r} \cdots \tau_{a_2,\rho_2}(a_1)$$

$$= \tau_{a_r,\rho_r}(a_1 + \sum_{i=2}^{r-1} c_i a_i), \ c_i \text{ in } R$$

$$= a_1 + \sum_{i=2}^{r-1} c_i a_i + \rho_r(a_1 + \sum_{i=2}^{r-1} c_i a_i)a_r.$$

So

$$(\alpha - 1)a_1 = \sum_{i=2}^{r-1} c_i a_i + \rho_r(a_1 + \sum_{i=2}^{r-1} c_i a_i)a_r.$$

But the a_i's are independent so $\alpha - 1 = 0$ thus $\alpha = 1$. So $\sigma = I$. Hence $\operatorname{res}(\sigma) = 0$, a contradiction.

CHAPTER 2

CENTERS AND COMMUTATORS

Introduction

In this chapter we simply note a few theorems on centers and centralizers of subgroups of PGL(V) and on the conjugacy classes of proper transvections.

2.1 Centers

If $n \geq 2$, then PSL(V) and SL(V) are not abelian.

Suppose PSL(V) is abelian. Let $\{b_1, b_2, \ldots, b_n\}$ be a basis of V and $\{\rho_1, \rho_2, \ldots, \rho_n\}$ the dual basis of V^*, where V^* is the dual space of V. Then

$$\begin{aligned}\tau_{b_1,\rho_2}\tau_{b_2,\rho_1}(b_1) &= \tau_{b_1,\rho_2}(b_1 + b_2) \\ &= b_1 + b_2 + \rho_2(b_1 + b_2)b_1 \\ &= 2b_1 + b_2\end{aligned}$$

and

$$\begin{aligned}\tau_{b_2,\rho_1}\tau_{b_1,\rho_2}(b_1) &= \tau_{b_2,\rho_1}(b_1) \\ &= b_1 + b_2.\end{aligned}$$

So $2b_1 + b_2 = \alpha(b_1 + b_2) = \alpha b_1 + \alpha b_2$ for some α in R^*. But this gives $\alpha = 1$ and $\alpha = 2$, which is a contradiction, so PSL(V), and hence SL(V), are nonabelian.

Theorem 2.1.1: (i) The centralizer of PSL(V) in PGL(V) is trivial.

(ii) The centralizer of SL(V) in GL(V) is RL(V).

(iii) PGL(V) and PSL(V) are centerless.

(iv) $Z(GL(V)) = RL(V)$, $Z(SL(V)) = SL(V) \cap RL(V)$.

Proof: (i) Let $\bar{\sigma}$ be in the centralizer of PSL(V) in PGL(V). Let L be any line in V and let τ be a transvection with residual line L. Then $\sigma\tau\sigma^{-1}$ has residual line $\sigma(L)$. But $\sigma\tau\sigma^{-1} = \tau$ so $\sigma(L) = L$. Thus σ fixes all lines. So σ is in RL(V) and $\bar{\sigma} = I$.

(ii) If σ is in the centralizer of SL(V) in GL(V), then $\bar{\sigma}$ is in the centralizer of PSL(V) in PGL(V), so $\bar{\sigma} = I$. Thus σ is in RL(V).

Now suppose σ is in RL(V). Then $\sigma = r_\alpha$ for some α in R^*. Let x be in V and σ_1 be in SL(V). Then $\sigma\sigma_1(x) = \alpha\sigma_1(x) = \sigma_1(\alpha x) = \sigma_1\sigma(x)$. So $\sigma\sigma_1 = \sigma_1\sigma$. So σ is in the centralizer of SL(V) in GL(V).

(iii) Let $\bar{\sigma}$ be in Z(PGL(V)) or Z(PSL(V)). Then $\bar{\sigma}$ is in the centralizer of PSL(V) in PGL(V), so $\bar{\sigma} = I$. Thus $Z(PGL(V)) = Z(PSL(V)) = I$.

(iv) Let σ be in Z(GL(V)). Then $\sigma\tau = \tau\sigma$ for all τ in GL(V). So $\sigma\tau = \tau\sigma$ for all τ in SL(V). Thus from (ii) above, σ is in RL(V). Now suppose σ is in RL(V), say $\sigma = r_\alpha$. Then $\sigma\tau(x) = \alpha\tau(x) = \tau(\alpha x) = \tau\sigma(x)$ for all τ in GL(V). So $Z(GL(V)) = RL(V)$.

From (ii) it follows that $Z(SL(V)) \subset RL(V)$. So $Z(SL(V)) \subset SL(V) \cap RL(V)$. It also follows that $SL(V) \cap RL(V) \subset Z(SL(V))$. So $Z(SL(V)) = SL(V) \cap RL(V)$.

2.2 Commutator Subgroups

Theorem 2.2.1: Any two nontrivial proper transvections of V are conjugate under GL(V). If $n \geq 3$, then they are conjugate under SL(V).

Proof: If $n = 1$, it is immediate, so assume $n \geq 2$. Let $\tau_{a,\rho}$, $\tau_{b,\phi}$ be two non-trivial proper transvections. Extend $\{a\}$ and $\{b\}$ to bases $\{a, a_2, \ldots, a_n\}$ and $\{b, b_2, \ldots, b_n\}$ of V, respectively, where $\rho(a_2) = \ldots = \rho(a_{n-1}) = 0$, $\rho(a_n) = 1$, $\phi(b_2) = \ldots = \phi(b_{n-1}) = 0$, and $\phi(b_n) = 1$.

Define $\sigma:V \to V$ by $\sigma(a) = b$ and $\sigma(a_i) = b_i$. Then $\rho\sigma^{-1} = \phi$. So $\sigma\tau_{a,\rho}\sigma^{-1} = \tau_{\sigma a,\rho\sigma^{-1}} = \tau_{b,\phi}$.

Suppose $n \geq 3$. Let $u = \det(\sigma)$ which is in R^*. Define $\sigma_1:V \to V$ by $\sigma_1(a) = b$, $\sigma_1(a_i) = b_i$, for $i \geq 3$ and $\sigma_1(a_2) = u^{-1}b_2$. Then $\det(\sigma_1) = u^{-1}\det(\sigma) = 1$. So σ_1 is in SL(V). We still have $\rho\sigma^{-1} = \phi$ since $\rho\sigma_1^{-1}(b_2) = \rho(ua_2) = 0$ and $\phi(b_2) = 0$. So again $\sigma_1\tau_{a,\rho}\sigma_1^{-1} = \tau_{b,\phi}$.

Theorem 2.2.2: If L_1 and L_2 are any two lines in V, then the set of proper nontrivial transvections with residual line L_1 is conjugate to the set of proper nontrivial transvections with residual line L_2 under SL(V).

Proof: Let $L_1 = Rb_1$, $L_2 = Rb_2$. Extend $\{b_1\}$ and $\{b_2\}$ to bases $\{b_1, c_2, \ldots, c_n\}$ and $\{b_2, d_2, \ldots, d_n\}$ of V. Define $\sigma_1:V \to V$ by $\sigma_1(b_1) = b_2$ and $\sigma(c_i) = d_i$ for $i \geq 2$. Then let $u = \det(\sigma_1)$ which is in R^*. Define $\sigma_2:V \to V$ by $\sigma_2(b_1) = b_2$, $\sigma_2(c_i) = d_i$ for $i \geq 3$, and $\sigma_2(c_2) = u^{-1}d_2$. Then σ_2 is in SL(V) and $\sigma_2(L_1) = L_2$.

If $\tau_{b_1,\rho}$ is a proper transvection with residual line L_1, then $\sigma_2\,\tau_{b_1,\rho}\sigma_2^{-1} = \tau_{b_2,\rho\sigma_2^{-1}}$ is a proper transvection with residual line L_2.

CHAPTER 3

COLLINEAR TRANSFORMATIONS AND PROJECTIVE GEOMETRY

Introduction

The first part of this chapter is concerned with the Fundamental Theorem of Projective Geometry and its consequences which primarily deal with the automorphisms Φ_g and the contragredient isomorphism. We state the Fundamental Theorem of Projective Geometry without proof in Section 3.2. Later in the chapter we consider properties of full subgroups of GL(V) and PGL(V). We will define a subgroup G of GL(V) to be full if for every hyperplane H and every line L in H, there exists a transvection τ in G with fixed hyperplane H and residual line L.

3.1 Semilinear Algebra

A map $\sigma:V \to V$ is called <u>semilinear with automorphism</u> ϕ if $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(rx) = \phi(r)\sigma(x)$ for some automorphism $\phi:R \to R$. If, in addition, $\sigma:V \to V$ is a group automorphism, then σ is a <u>semilinear automorphism</u>.

<u>Theorem 3.1.1</u>: Let $\sigma:V \to V$ be a semilinear automorphism and W a subspace of V. Then

(i) $\sigma(W)$ is a subspace of V,

(ii) $\sigma^{-1}(W)$ is a subspace of V.

__Proof__: (i) Clearly $\sigma(W)$ is a subgroup of V. Let x be in W and r in R. Then $r\sigma(x) = \phi(\phi^{-1}(r))\sigma(x) = \sigma(\phi^{-1}(r)x)$ which is in $\sigma(W)$. So $\sigma(W)$ is a submodule of V. If $W \oplus U = V$, then clearly $\sigma(W) \oplus \sigma(U) = V$, so $\sigma(W)$ is a subspace of V.

(ii) Clearly σ^{-1} is also a semilinear automorphism.

Define a __collinear transformation__ σ of V onto V to be a semilinear bijection $\sigma:V \to V$. Then compositions and inverses of collinear transformations are again collinear transformations. From Theorem 3.1.1, we see that if σ is collinear, then σ is geometric. So if σ is collinear, then $\bar{\sigma}$ is a projectivity. If π is a projectivity and $\pi = \bar{\sigma}$ where σ is collinear, then π is called a __projective collinear transformation__. So if π is projective collinear, then π is projective geometric.

3.2 The Groups $\Gamma L(V)$ and $P\Gamma L(V)$

We state without proof the Fundamental Theorem of Projective Geometry, due to R. Sridharen and M. Ojanguran [4].

__Theorem__: Let V_1 and V_2 be free modules of finite dimension ≥ 3 over commutative rings R_1 and R_2, respectively. If $\sigma:P(V_1) \to P(V_2)$ is a projectivity, then there exists an isomorphism $\phi:R_1 \to R_2$ and a ϕ-semilinear automorphism $\Phi:V_1 \to V_2$ such that $\sigma = \bar{\Phi}$. If $\phi_i:R_1 \to R_2$, $i = 1, 2$, are isomorphisms and $\Phi_i:V_1 \to V_2$ are ϕ_i-semilinear isomorphisms such that $\bar{\Phi}_1 = \bar{\Phi}_2$, then there exists an r in R_2 such that $\Phi_1 = r\Phi_2$ and $\phi_1 = \phi_2$.

So if σ is a collinear transformation from V onto V, then by the Fundamental Theorem of Projective Geometry σ is a semilinear bijection with some $\phi:R \to R$ as its associated ring isomorphism. Thus ϕ is in

Aut(R). Clearly the set of all collinear transformations of V forms a group. In fact, this group is a subgroup of GG(V), the general geometric group of V. Denote this subgroup by $\Gamma L(V)$ and call it the <u>collinear group</u> of V. Clearly $GL(V) \subset \Gamma L(V)$ so GL(V) is a subgroup of $\Gamma L(V)$.

From each σ in $\Gamma L(V)$, we can form a projective collinear transformation $\bar{\sigma}$, where $\bar{\sigma}(U) = \{\sigma(x) | x \text{ is in } U\}$ for any subspace U. The set of all projective collinear transformations forms a group which we will call $P\Gamma L(V)$. Thus

$$P\Gamma L(V) = \{\bar{\sigma} | \sigma \text{ is in } \Gamma L(V)\}.$$

Call $P\Gamma L(V)$ the <u>projective collinear group</u> of V. Clearly $PGL(V) \subset P\Gamma L(V)$ so PGL(V) is a subgroup of $P\Gamma L(V)$.

<u>Theorem 3.2.1</u>: (i) RL(V) and GL(V) are normal subgroups of $\Gamma L(V)$.
(ii) PGL(V) is a normal subgroup of $P\Gamma L(V)$.

<u>Proof</u>: (i) Let ρ be in GL(V) and σ in $\Gamma L(V)$ with associated ring isomorphism ϕ. Then $\sigma\rho\sigma^{-1}(x + y) = \sigma\rho\sigma^{-1}(x) + \sigma\rho\sigma^{-1}(y)$ for all x and y in V and

$$\begin{aligned} \sigma\rho\sigma^{-1}(rx) &= \sigma\rho(\phi^{-1}(r)\sigma^{-1}(x)) \\ &= \sigma(\phi^{-1}(r)\rho\sigma^{-1}(x)) \\ &= \phi(\phi^{-1}(r))\sigma\rho\sigma^{-1}(x) \\ &= r(\sigma\rho\sigma^{-1}(x)) \text{ for all } r \text{ in } R \text{ and } x \text{ in } V. \end{aligned}$$

Thus $\sigma\rho\sigma^{-1}$ is linear, so $\sigma\rho\sigma^{-1}$ is in GL(V).

Let σ_r be in RL(V) and σ in $\Gamma L(V)$ as before. We know $\sigma\sigma_r\sigma^{-1}$ is in GL(V) so we need only show that $\sigma\sigma_r\sigma^{-1} = sI$ for some s in R. Let x be in V. Then

$$\sigma\sigma_r\sigma^{-1}(x) = \sigma(r\sigma^{-1}(x)) = \phi(r)(\sigma\sigma^{-1}(x)) = \phi(r)x$$

Thus $\sigma\sigma_r\sigma^{-1} = \phi(r)I = \sigma_{\phi(r)}$

(ii) Follows from (i).

Theorem 3.2.2: Let R be a local ring and let $\dim(V) \geq 2$. Let σ be in $\Gamma L(V)$ such that $\sigma(L) = L$ for all L in P(V). Then σ is in RL(V).

Proof: Let $\{b_1, \ldots ,b_n\}$ be a basis of V. Then $\sigma(b_i) = r_ib_i$ for all i. Now $b_1 + b_i$ is unimodular for $i \neq 1$, so $\sigma(b_1 + b_i) = s(b_1 + b_i)$. Thus $sb_1 + sb_i = \sigma(b_1 + b_i) = \sigma(b_1) + \sigma(b_i) = r_1b_1 + r_ib_i$. Therefore $r_1 = s = r_i$ for all i. Let $r = r_1$. Then $\sigma(b_i) = rb_i$ for all i.

Let b and c be any two unimodular elements of V. Extend b to a basis $\{b_1, b_2, \ldots ,b_n\}$ where $b_1 = b$. Then $c = a_1b_1 + a_2b_2 + \ldots + a_nb_n$. Now c is unimodular so one of the a_i's is a unit. Suppose a_i is a unit, $i \neq 1$. Then $b_i = a_i^{-1}c - a_i^{-1}a_1b_1 - \ldots -a_i^{-1}a_{i-1}b_{i-1} - a_i^{-1}a_{i+1}b_{i+1} - \ldots - a_i^{-1}a_nb_n$. So clearly $\{c, b_1, \ldots ,b_{i-1}, b_{i+1}, \ldots ,b_n\}$ spans V. Suppose $rc + r_1b_1 + \ldots + r_{i-1}b_{i-1} + r_{i+1}b_{i+1} + \ldots + r_nb_n = 0$. Then $rc = -(r_1b_1 + \ldots + r_{i-1}b_{i-1} + r_{i+1}b_{i+1} + \ldots + r_nb_n)$. Now

$$b_i = sc + s_1b_1 + \ldots + s_{i-1}b_{i-1} + s_{i+1}b_{i+1} + \ldots + s_nb_n.$$

$$\begin{aligned} rb_i &= rsc + rs_1b_1 + \ldots + rs_{i-1}b_{i-1} + rs_{i+1}b_{i+1} + \ldots + rs_nb_n \\ &= -s(r_1b_1 + \ldots + r_{i-1}b_{i-1} + r_{i+1}b_{i+1} + \ldots + r_nb_n) \\ &\quad + rs_1b_1 + \ldots + rs_{i-1}b_{i-1} + rs_{i+1}b_{i+1} + \ldots + rs_nb_n \\ &= (rs_1-sr_1)b_1 + \ldots + (rs_{i-1}-sr_{i-1})b_{i-1} + (rs_{i+1}-sr_{i+1})b_{i+1} \\ &\quad + \ldots + (rs_n - sr_n)b_n. \end{aligned}$$

Thus $r = 0$. Hence $r_1 = \ldots = r_n = 0$. So $\{c, b_1,\ldots,b_{i-1}, b_{i+1},\ldots b_n\}$ is a basis of V. So there exists some r in R^* such that $\sigma(c) = rc$ and $\sigma(b_1) = rb_1$.

Now suppose a_1 is a unit. Then a similar argument shows that $\{c, b_2, \ldots, b_n\}$ is a basis of V so there exists some s in R^* such that $\sigma(c) = sc$ and $\sigma(b_2) = sb_2$. But we know there exists some t in R^* such that $\sigma(b_1) = tb_1$ and $\sigma(b_2) = tb_2$. Thus $s = t$ so $\sigma(b_1) = sb_1$. Hence $\sigma(c) = sc$ for all unimodular c in V.

Let b be unimodular and u be in R^*. Then ub is also unimodular. So

$$\phi(u)sb = \phi(u)\sigma(b) = \sigma(ub) = sub.$$

Therefore $s\phi(u) = su$. Hence $\phi(u) = u$ since s is in R^*. Therefore ϕ fixes all units of R. Let x be a nonunit. Then $1 - x$ is a unit, so $\phi(1 - x) = 1 - x$. But $\phi(1 - x) = \phi(1) - \phi(x) = 1 - \phi(x)$. Thus $\phi(x) = x$. Hence $\phi = I$. Therefore σ is linear and hence σ is in RL(V).

<u>Example</u>: Let k be a field and $R = k[X]$. Let $V = R$. Define σ in $\Gamma L(V)$ by $\sigma(a) = a$ whenever a is unimodular and define the associated ring isomorphism ϕ by $\phi(a) = a$ if a is in k and $\phi(X) = X + 1$. Then σ is not linear so σ is not in RL(V), but $\sigma(L) = L$ for all lines L of V.

<u>Theorem 3.2.3</u>: (i) The group of projectivities of V is equal to PGG(V) for $n \neq 2$.

(ii) $PGG(V) = P\Gamma L(V)$ for $n \geq 3$.

<u>Proof</u>: (i) We have $P\Gamma L(V) \subset PGG(V)$, which is contained in the group of projectivities. By the Fundamental Theorem of Projective Geometry, these are all equal if $n \geq 3$.

Let $n = 1$. Then there is only one line in V so the group $PGG(V) = \{\overline{I}\}$ and there is only one projectivity so the group of projectivities is equal to $\{\overline{I}\}$.

(ii) Follows from the above.

<u>Theorem 3.2.4</u>: Define $f:\Gamma L(V) \to \mathrm{Aut}(R)$ by $f(\sigma) = \phi_\sigma$ where ϕ_σ is the ring automorphism associated with σ. Then

(i) f is a surjective group morphism and $\ker(f) = GL(V)$,

(ii) $\Gamma L(V)/GL(V) \simeq \mathrm{Aut}(R)$.

<u>Proof</u>: (i) Let $\{b_1, \ldots, b_n\}$ be a basis of V. Let ϕ be in $\mathrm{Aut}(R)$. Define $\sigma:V \to V$ by $\sigma(\Sigma a_i b_i) = \Sigma\phi(a_i)b_i$. Clearly σ is in $\Gamma L(V)$ and ϕ is the ring automorphism associated with σ. So f is surjective. Let σ_1 and σ_2 be in $\Gamma L(V)$ with associated ring automorphisms ϕ_1 and ϕ_2, respectively. Then

$$\begin{aligned} \sigma_1\sigma_2(\Sigma a_i b_i) &= \sigma_1(\Sigma\phi_2(a_i)\sigma_2(b_i)) \\ &= \Sigma\phi_1\phi_2(a_i)\sigma_1\sigma_2(b_i). \end{aligned}$$

So $\phi_1\phi_2 = f(\sigma_1\sigma_2)$ is the ring automorphism associated with $\sigma_1\sigma_2$.

Clearly $GL(V) \subset \ker(f)$ and if $f(\sigma) = I$, then the associated ring automorphism is I. Hence σ is in $GL(V)$.

(ii) Follows directly from (i).

We have the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & 1 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & RL(V) & & & & & & \\
 & & \downarrow & & & & & & \\
1 & \to & GL(V) & \to & \Gamma L(V) & \to & \mathrm{Aut}(R) & \to & 1 \quad (\text{exact}) \\
 & & \downarrow & & \downarrow & & & & \\
 & & PGL(V) & \to & P\Gamma L(V) & & & & \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & & \\
 & & & & & & & & \\
 & & (\text{exact}) & & & & & &
\end{array}$$

3.3 The Automorphisms Φ_g

Let $g:V \to V$ be a collinear transformation. Let $\phi:R \to R$ be the associated ring automorphism. Define $\Phi_g:\Gamma L(V) \to \Gamma L(V)$ by $\Phi_g(\sigma) = g\sigma g^{-1}$. Clearly Φ_g is an automorphism. We have $\Phi_{g_1}\Phi_{g_2} = \Phi_{g_1 g_2}$ and $(\Phi_g)^{-1} = \Phi_{g^{-1}}$.

Theorem 3.3.1:

$$\Phi_g|GL(V): \quad GL(V) \to GL(V)$$

$$\Phi_g|SL(V): \quad SL(V) \to SL(V)$$

$$\Phi_g|RL(V): \quad RL(V) \to RL(V)$$

are all automorphisms.

Proof: An argument similar to that used in Theorem 3.2.1 shows that if σ is in GL(V), then $g\sigma g^{-1}$ is in GL(V). Similarly, if σ is in GL(V), then $g^{-1}\sigma g$ is in GL(V), so Φ_g is surjective since $\Phi_g(g^{-1}\sigma g) = \sigma$.

Let σ be in SL(V). Let $\{b_1, \ldots ,b_n\}$ be a basis of V. Let $\mathrm{mat}(\sigma) = A$ and $\mathrm{mat}(g) = B$ with respect to the basis. Then $\mathrm{mat}(g^{-1}) = (B^{\phi^{-1}}) = (B^{-1})^{\phi^{-1}}$. Now

$$\begin{aligned} \mathrm{mat}(g\sigma g^{-1}) &= \mathrm{mat}(g)(\mathrm{mat}(\sigma g^{-1}))^{\phi} \\ &= B(\mathrm{mat}(\sigma)\mathrm{mat}(g^{-1}))^{\phi} \\ &= B(A(B^{-1})^{\phi^{-1}})^{\phi} \\ &= BA^{\phi}B^{-1}. \end{aligned}$$

where $[a_{ij}]^{\phi} = [\phi(a_{ij})]$ for any matrix $[a_{ij}]$. So

$$\det(\mathrm{mat}(g\sigma g^{-1})) = \det(B)\det(A^{\phi})\det(B^{-1})$$
$$= \det(A^{\phi})$$
$$= \phi(\det(A))$$
$$= \phi(1)$$
$$= 1.$$

So $g\sigma g^{-1}$ is in SL(V).

Let σ_r be in RL(V). Then $\mathrm{mat}(\sigma_r) = rI$. Then

$$\mathrm{mat}(g\sigma_r g^{-1}) = \mathrm{mat}(g)(\mathrm{mat}(\sigma_r)\mathrm{mat}(g^{-1}))^{\phi}$$
$$= B(rI(B^{-1})^{\phi^{-1}})^{\phi}$$
$$= B(r(B^{-1})^{\phi^{-1}})^{\phi}$$
$$= B(\phi(r)B^{-1})$$
$$= \phi(r)I.$$

Thus $g\sigma_r g^{-1} = \sigma_{\phi(r)}$ which is in RL(V).

<u>Theorem 3.3.2</u>: Let σ be in GL(V) and g in ΓL(V). Then

(i) $\det(\Phi_g(\sigma)) = (\det(\sigma))^{\phi}$,

(ii) $P_{\Phi_g(\sigma)} = g(P)$,

(iii) $Q_{\Phi_g(\sigma)} = g(Q)$,

(iv) If σ is proper then $\Phi_g(\sigma)$ is proper and $\mathrm{res}(\Phi_g(\sigma)) = \mathrm{res}(\sigma)$.

<u>Proof</u>: (i) Let σ be in GL(V) and $\mathrm{mat}(\sigma) = A$. Then

$$\det(\Phi_g(\sigma)) = \det(g\sigma g^{-1})$$
$$= \det(B(A(B^{-1})^{\phi^{-1}})^{\phi})$$
$$= \det(BA^{\phi}B^{-1})$$
$$= \det(A^{\phi}) = (\det(A))^{\phi} = (\det(\sigma))^{\phi}.$$

(ii) Let v be in P, the fixed space of σ. Then $\sigma(v) = v$. Then $g\sigma g^{-1}(g(v)) = g\sigma(v) = g(v)$. So $g(P) \subset P_{\Phi_g(\sigma)}$.

Let v be in $P_{\Phi_g(\sigma)}$. Then $g\sigma g^{-1}(v) = v$. So $\sigma(g^{-1}(v)) = g^{-1}(v)$, thus $g^{-1}(v)$ is in P. Therefore v is in g(P). Thus $P_{\Phi_g(\sigma)} = g(P)$.

(iii) Let v be in Q. Then $v = \sigma(x) - x$. Let $x = g^{-1}(y)$ for some y in V. Then

$$g(v) = g\sigma(x) - g(x) = g\sigma g^{-1}(y) - y$$

which is in $Q_{\Phi_g\sigma}$. So $g(Q) \subset Q_{\Phi_g(\sigma)}$.

Let v be in $Q_{\Phi_g(\sigma)}$. Then $v = g\sigma g^{-1}(x) - x$ for some x in V. Then $g^{-1}(v) = \sigma(g^{-1}(x)) - g^{-1}(x)$ which is in Q. So v is in g(Q). Thus $Q_{\Phi_g(\sigma)} = g(Q)$.

(iv) Suppose σ is proper in GL(V). Then Q is a direct summand of V, $V = W \oplus Q$. Thus $V = g(V) = g(W) \oplus g(Q)$. So $\Phi_g(\sigma)$ is proper in GL(V) and $\text{res}(\Phi_g(\sigma)) = \text{res}(\sigma)$.

Let L be a line and H a hyperplane with $L \subset H$. Then $g(L) \subset g(H)$. So if τ is a transvection with line L and hyperplane H, then $\Phi_g(\tau)$ is a transvection with line g(L) and hyperplane g(H). Let $\tau = \tau_{a,\rho}$ be a transvection. Then

$$\begin{aligned}(\Phi_g(\tau))(x) &= g\,\tau_{a,\rho} g^{-1}(x)\\ &= g(g^{-1}(x) + a\rho(g^{-1}(x)))\\ &= x + g(a)\phi(\rho(g^{-1}(x)))\\ &= x + g(a)\phi\rho\, g^{-1}(x)\\ &= \tau_{ga,\phi\rho g^{-1}}(x).\end{aligned}$$

Thus $\Phi_g(\tau_{a,\rho}) = \tau_{ga,\phi\rho g^{-1}}$.

Now consider a projective collinear transformation $g:P(V) \to P(V)$. Define $\Phi_g(h) = ghg^{-1}$ for all h in $P\Gamma L(V)$. We have $\Phi_{g_1}\Phi_{g_2} = \Phi_{g_1g_2}$ and $\Phi_g^{-1} = \Phi_{g^{-1}}$, so Φ_g is a group isomorphism. Since g is in $P\Gamma L(V)$, there is an h in $\Gamma L(V)$ such that $g = \bar{h}$. So

$$\Phi_g(\bar{\sigma}) = g\bar{\sigma}g^{-1} = \bar{h}\bar{\sigma}\bar{h}^{-1} = \overline{h\sigma h^{-1}} = \overline{\Phi_h(\sigma)}.$$

Therefore we have isomorphisms

$$\Phi_g: \ PGL(V) \to PGL(V)$$

$$\Phi_g: \ PSL(V) \to PSL(V).$$

3.4 The Contragredient

Consider a semilinear mapping $\sigma:V \to V$ with ring automorphism $\phi:R \to R$. Then for each ρ in V^*, we have that $\phi^{-1}\rho\sigma$ is in V^*. Denote $\phi^{-1}\rho\sigma$ by $\sigma^t(\rho)$. Then we have a mapping $\sigma^t:V^* \to V^*$ defined by $\sigma^t:\rho \mapsto \phi^{-1}\rho\sigma$. If we denote $f(x)$ by $\langle x,f\rangle$ for x in V and f in V^*, then $\langle x,\sigma^t(\rho)\rangle^\phi = \langle\sigma(x),\rho\rangle$.

Let $\{b_1, \ldots ,b_n\}$ be a basis of V and $\{b_1^*, \ldots ,b_n^*\}$ the corresponding dual basis of V^*. Let ρ be in V^* and $\rho = \sum_{i=1}^n a_ib_i^*$. Then

$$\begin{aligned}
\sigma^t(\rho) &= \sigma^t(\Sigma a_ib_i^*) \\
&= \phi^{-1}(\Sigma a_ib_i^*)\sigma \\
&= \phi^{-1}(\Sigma a_ib_i^*\sigma) \\
&= \Sigma(\phi^{-1}(a_i))(\phi^{-1}b_i^*\sigma) \\
&= \Sigma(\phi^{-1}(a_i))(\sigma^t(b_i^*)).
\end{aligned}$$

Thus σ^t is semilinear with associated ring automorphism ϕ^{-1}. Suppose $\sigma^t = 0$. Then $\phi^{-1}\rho\sigma = 0$ for all ρ in V^*. So $\rho\sigma = 0$ for all ρ in V^*. Thus $b_i^*\sigma = 0$ for $i = 1, \dots, n$. Suppose there exists x in V such that $\sigma(x) \neq 0$. Let $\sigma(x) = \Sigma a_i b_i$. Then some $a_i \neq 0$, say a_j. But $0 = b_j^*\sigma(x) = b_j^*(\Sigma a_i b_i) = a_j$. Thus $\sigma(x) = 0$ for all x in V. So $\sigma = 0$.

Suppose $\sigma_1^t = \sigma_2^t$. Then $\phi^{-1}\rho\sigma_1 = \phi^{-1}\rho\sigma_2$ for all ρ in V^*. So $\rho\sigma_1 = \rho\sigma_2$ for all ρ in V^*. Then $b_i^*\sigma_1 = b_i^*\sigma_2$ for $i = 1, \dots, n$. Let x be in V and $\sigma_1(x) = \Sigma a_i b_i$, $\sigma_2(x) = \Sigma a_i' b_i$. Then

$$a_j = b_j^*(\Sigma a_i b_i) = b_j^*\sigma_1(x) = b_j^*\sigma_2(x) = b_j^*(\Sigma a_i' b_i) = a_j'$$

for $j = 1, \dots, n$. Thus $\sigma_1(x) = \sigma_2(x)$. Therefore $\sigma_1 = \sigma_2$.

Let $\sigma_1 : V \to V$ and $\sigma_2 : V \to V$ be semilinear with ring automorphisms ϕ_1 and ϕ_2, respectively. Then $\sigma_2\sigma_1$ is semilinear with ring automorphism $\phi_2\phi_1$. Let ρ_1 be in V^*. Then

$$\begin{aligned}(\sigma_2\sigma_1)^t(\rho_1) &= (\phi_2\phi_1)^{-1}\rho_1\sigma_2\sigma_1 \\ &= \phi_1^{-1}\phi_2^{-1}\rho_1\sigma_2\sigma_1 \\ &= \phi_1^{-1}(\phi_2^{-1}\rho_1\sigma_2)\sigma_1 \\ &= \phi_1^{-1}(\sigma_2^t\rho_1)\sigma_1 \\ &= \sigma_1^t\sigma_2^t\rho_1.\end{aligned}$$

Thus $(\sigma_2\sigma_1)^t = \sigma_1^t\sigma_2^t$.

Let $\{b_1, \dots, b_n\}$ be a basis for V and $\{b_1^*, \dots, b_n^*\}$ the corresponding dual basis for V^*. Let A be the matrix of σ with respect to $\{b_1, \dots b_n\}$ and C be the matrix of σ^t with respect to $\{b_1^*, \dots, b_n^*\}$. Then

$$\begin{aligned} a_{ij} &= b_i^*(\sum_k a_{kj}b_k) \\ &= b_i^*(\sigma(b_j)) \\ &= \phi(\phi^{-1}b_i^*\sigma)(b_j) \\ &= \phi\sigma^t(b_i^*)(b_j) \\ &= \phi(\sum_k c_{ki}b_k^*)(b_j) = \phi(c_{ji}) \end{aligned}$$

Thus $A^t = C^\phi$. Therefore $\det(A) = \phi\det(C)$. So σ is bijective if and only if σ^t is bijective. So if σ is bijective, then $(\sigma^t)^{-1}$ and $(\sigma^{-1})^t$ are also bijective. Note that

$$(\sigma^t)((\sigma^{-1})^t) = (\sigma^{-1}\sigma)^t = I^t = I \text{ so } (\sigma^t)^{-1} = (\sigma^{-1})^t.$$

Let $\sigma: V \to V$ be a collinear transformation. Then define the <u>contragredient</u> of σ to be $\check{\sigma} = (\sigma^t)^{-1}$. Then $\check{\sigma}: V^* \to V^*$ is a semilinear bijection with ring automorphism ϕ. So $\check{\sigma}$ is also a collinear transformation.

Let $\sigma_1: V \to V$ and $\sigma_2: V \to V$ be collinear transformations. Then

$$\begin{aligned} (\widecheck{\sigma_1\sigma_2}) &= ((\sigma_1\sigma_2)^t)^{-1} \\ &= (\sigma_2^t\sigma_1^t)^{-1} \\ &= (\sigma_1^t)^{-1}(\sigma_2^t)^{-1} \\ &= \check{\sigma}_1\check{\sigma}_2 \end{aligned}$$

and

$$(\widecheck{\sigma^{-1}}) = ((\sigma^{-1})^t)^{-1} = ((\sigma^t)^{-1})^{-1} = (\check{\sigma})^{-1}.$$

So we have the map $\check{}: \Gamma L(V) \to \Gamma L(V^*)$. This is a homomorphism by the above.

Suppose $\check{\sigma} = I$. Then $(\sigma^t)^{-1} = I$, so $\sigma^t = I$. Thus $\sigma = I$. So $\check{}$ is injective.

Let μ be in $\Gamma L(V^*)$ and $\mu(b_i^*) = \sum_j a_{ji}b_j^*$, and let ϕ be its ring automorphism. Let $f = \sum_i d_ib_i^*$ be in V^* and $v = \sum_i e_ib_i$ be in V. Then

$$\begin{aligned}
\phi f\sigma^{-1}(v) &= \phi f\sigma^{-1}(\sum_j e_jb_j) \\
&= \phi f(\sum_j \phi^{-1}(e_j)\sigma^{-1}(b_j)) \\
&= \phi f(\sum_j \phi^{-1}(e_j)(\sum_k \phi^{-1}(a_{jk})b_k)) \\
&= \phi f(\sum_k \sum_j \phi^{-1}(a_{jk})\phi^{-1}(e_j)b_k) \\
&= \phi(\sum_k \sum_j \phi^{-1}(a_{jk})\phi^{-1}(e_j)f(b_k)) \\
&= \phi(\sum_k \sum_j \phi^{-1}(a_{jk})\phi^{-1}(e_j)(\sum_i d_ib_i^*(b_k))) \\
&= \phi(\sum_k \sum_j \phi^{-1}(a_{jk})\phi^{-1}(e_j)d_k) \\
&= \sum_k \sum_j a_{jk}e_j\phi(d_k)
\end{aligned}$$

and

$$\begin{aligned}
(\mu(f))(v) &= (\mu(\sum_k d_kb_k^*))(v) \\
&= (\sum_k \phi(d_k)\mu(b_k^*))(v) \\
&= [\sum_k \phi(d_k)(\sum_i a_{ik}b_i^*)](v) \\
&= (\sum_k \sum_i \phi(d_k)a_{ik}b_i^*)(v) \\
&= \sum_k \sum_i \phi(d_k)a_{ik}b_i^*(v) \\
&= \sum_k \sum_i \phi(d_k)a_{ik}b_i^*(\sum_j e_jb_j) \\
&= \sum_k \sum_j \sum_i \phi(d_k)a_{ik}e_jb_i^*(b_j) \\
&= \sum_k \sum_j \phi(d_k)a_{jk}e_j.
\end{aligned}$$

So $\phi f\sigma^{-1} = \mu(f)$ for all f in $\Gamma L(V^*)$. Therefore $\check{\sigma} = \mu$. So $\check{}$ is surjective. Thus $\check{}$ is a semilinear isomorphism.

Let σ be in GL(V). Then $\text{mat}(\check{\sigma}) = ((\text{mat}(\sigma))^t)^{-1}$ so $\det(\check{\sigma})$ is in R^*. Let $\Sigma a_i b_i^*$ be in V^*. Then

$$\check{\sigma}(\Sigma a_i b_i^*) = (\Sigma a_i b_i^*)\sigma^{-1}$$
$$= \Sigma a_i b_i^* \sigma^{-1}$$
$$= \Sigma a_i \check{\sigma}(b_i^*).$$

Thus $\check{\sigma}$ is in $GL(V^*)$.

If σ is in SL(V), then $\det(\sigma) = 1$. So $\det(\check{\sigma}) = \det(\sigma)^{-1} = 1$. Thus $\check{\sigma}$ is in $SL(V^*)$.

Let σ be in RL(V). Then $\text{mat}(\sigma) = rI$ for some r in R^*. Then $\text{mat}(\check{\sigma}) = ((\text{mat}(\sigma))^t)^{-1} = r^{-1}I$. So $\check{\sigma}$ is in $RL(V^*)$.

In addition, the maps

$$\check{} : GL(V) \to GL(V^*)$$
$$\check{} : SL(V) \to SL(V^*)$$
$$\check{} : RL(V) \to RL(V^*)$$

are isomorphisms.

Let σ be in $\Gamma L(V)$ and let U be a subspace of V. Define

$$U^\circ = \{f \text{ in } V^* \mid f(u) = 0 \text{ for all u in V}\}.$$

Thus U° is a subspace of V^* because if $V = U \oplus W$ then $V^* = U^\circ \oplus W^\circ$.

Let f be in $\check{\sigma}(U^\circ)$. Then $f = \check{\sigma}(g)$ for some g in U°, so $f = \phi g\sigma^{-1}$. Then $f(x) = \phi g\sigma^{-1}(x) = 0$ for all x in $\sigma(U)$. So f is in $(\sigma(U))^\circ$.

Now suppose f is in $(\sigma(U))^\circ$. Then $f(x) = 0$ for all x in $\sigma(U)$. So $f\sigma\,(u) = 0$ for all u in U. Let $g = (\check{\sigma^{-1}})(f) = \phi^{-1}f\sigma$. Then $g(u) = \phi^{-1}f\sigma(u) = 0$ for all u in U, so g is in U°. Also

$$\check{\sigma}(g) = \phi g \sigma^{-1} = \phi\phi^{-1} f \sigma\sigma^{-1} = f.$$

So f is in $\check{\sigma}(U^o)$. Thus $\check{\sigma}(U^o) = (\sigma(U))^o$. Call $\check{}$: $\Gamma L(V) \to \Gamma L(V^*)$ the <u>contragredient isomorphism</u>.

<u>Theorem 3.4.1</u>: Let $\check{}$ be the contragredient isomorphism of V and let σ be in GL(V). Then

(i) the residual module of $\check{\sigma}$ is P^o,

(ii) the fixed module of $\check{\sigma}$ is Q^o,

(iii) $\text{res}(\sigma) = \text{res}(\check{\sigma})$ if σ is proper,

(iv) the isomorphism $\check{}$ carries the proper transvections with spaces $L \subset H$ onto the set of proper transvections with spaces $H^o \subset L^o$,

(v) $\check{\tau}_{a,\rho} = \tau_{\rho,-\tilde{a}}$ where $-\tilde{a}$ in V^{**} is defined by $\tilde{a}(f) = f(a)$.

<u>Proof</u>: (i) Let f be in Q_{σ^t}. Then $f = \sigma^t(g) - g = g\sigma - g$ for some g in $GL(V^*)$. Let x be in P. Then

$$f(x) = g\sigma(x) - g(x) = g(x) - g(x) = 0.$$

So f is in P^o.

Let f be in P^o. Then $f(x) = 0$ for all x in P. Define $g: Q \to V$ by $g(x) = f(y)$ for all x in Q where $x = \sigma(y) - y$. Then g is well-defined since if $\sigma(y) - y = \sigma(z) - z$, then $\sigma(y - z) = y - z$ so $y - z$ is in P. Thus $f(y) - f(z) = f(y - z) = 0$. Then $g(\sigma(x) - x) = f(x)$ for all x in V. So

$$f = g(\sigma - I) = g\sigma - g = \sigma^t g - g = (\sigma^t - I)g.$$

Thus f is in Q_{σ^t}. Hence $P^o = Q_{\sigma^t} = Q_{(\sigma^t)^{-1}} = Q_{\check{\sigma}}$.

(ii) Let f be in Q^o. Then

$$(\sigma^t(f) - f)(x) = (f\sigma - f)(x) = f(\sigma(x) - x) = 0$$

since f is in Q^o. So $\sigma^t(f) = f$. Thus f is in P_{σ^t}. Now suppose f is in P_{σ^t}. Then $\sigma^t(f) = f$. So

$$f(\sigma(x) - x) = f\sigma(x) - f(x) = (\sigma^t(f) - f)(x) = 0.$$

So f is in Q°. Thus $Q^\circ = P_{\sigma^t}$. Since $P_{\sigma^t} = P_{(\sigma^t)^{-1}}$, we have $Q^\circ = P_{\check{\sigma}}$.

(iii) Now

$$\begin{aligned} \mathrm{res}(\check{\sigma}) &= \dim(Q_{\check{\sigma}}) \\ &= n - \dim(P_{\check{\sigma}}) \\ &= n - \dim(Q^\circ) \\ &= n - (n - \dim(Q)) \\ &= \dim(Q) \\ &= \mathrm{res}(\sigma). \end{aligned}$$

(iv) Let $\tau: V \to V$ be a proper transvection with spaces $L \subset H$. Then $\check{\tau}$ has spaces $H^\circ \subset L^\circ$ where H° is a line and L° is a hyperplane. Thus $\check{\tau}$ is a proper transvection in $GL(V^*)$.

Suppose $\check{\tau}$ in $GL(V^*)$ is a proper transvection with spaces $H^\circ \subset L^\circ$. Then τ has spaces $L \subset H$. So $\check{}$ maps the set of proper transvections with spaces $L \subset H$ onto the set of proper transvections with spaces $H^\circ \subset L^\circ$.

(v) We have

$$\begin{aligned} (\tau_{a,\rho}^t(\phi))(x) &= (\phi\tau_{a,\rho})(x) \\ &= \phi(x + \rho(x)a) \\ &= \phi(x) + \rho(x)\phi(a) \\ &= \phi(x) + \rho(x)\tilde{a}(\phi) \\ &= (\phi + \tilde{a}(\phi)\rho)(x) \\ &= (\tau_{\rho,\tilde{a}}(\phi))(x). \end{aligned}$$

Therefore $\tau^t_{a,\rho} = \tau_{\rho,\tilde{a}}$, so

$$\check{\tau}_{a,\rho} = (\tau^t_{a,\rho})^{-1} = (\tau_{\rho,\tilde{a}})^{-1} = \tau_{\rho,-\tilde{a}}.$$

Also, $\check{}$ maps RL(V) to RL(V^*) so we can define the map $\check{}$:PGL(V) → PGL(V^*) by $(\check{\bar{\sigma}}) = (\bar{\check{\sigma}})$. Call this map the <u>projective contragredient isomorphism</u>. This induces $\check{}$:PSL(V) → PSL(V^*).

If R is a local ring then we may also define the map $\check{}$:PΓL(V) → PΓL(V^*) in the same way.

<u>3.5 Unipotent Transformations</u>

We use $C_A(X)$ to denote the centralizer in A of a nonempty subset X of a group A. Thus $C_A(X)$ is a subgroup of A, and

(i) $X_1 \subset X_2$ implies that $C_A(X_1) \supset C_A(X_2)$ and

(ii) $X \subset C_A(C_A(X))$.

If ϕ is a one-to-one group homomorphism, then $\phi(C_A(X)) = C_{\phi(A)}(\phi(X))$. If A = GL(V), we will simply write $C_V(X)$ instead of $C_{GL(V)}(X)$.

<u>Theorem 3.5.1</u>: If n = 2, V = L ⊕ K, where L and K are lines, and τ_L and τ_K are proper transvections with residual lines L and K, respectively, then $\tau_L\tau_K\tau_L^{-1}\tau_K^{-1}$ is an element of GL(V) - RL(V) with residual space V.

<u>Proof</u>: Let $J = \tau_L(K) \neq K$. Let L = Ra and K = Rb. Then $\tau_L(b) - b$ is in L, so $\tau_L(b) - b = ra$, so $\tau_L(b) = ra + b$ where r is in R^*. Thus $J = R\tau_L(b) = R(ra + b)$.

Suppose x is in $J \cap L$. Then x is in $R(ra + b) \cap Ra$ so $x = s(ra + b) = ta$ for some s and t in R. But then s = 0, so x = 0.

Thus $J \cap L = 0$. Hence $V = J \oplus L$.

Now $\tau_J = \tau_L \tau_K \tau_L^{-1}$ is a proper transvection with residual line $\tau_L(K) = J$. Let x be in $J \cap K = R(ra + b) \cap Rb$. Then $x = s(ra + b) = tb$ for some s and t in R, so $sra = 0$. Thus $sr = 0$ but r is in R^*, so $s = 0$. Thus $x = 0$. Thus $V = J \oplus K$. Hence, by Theorem 1.4.3, the residual space of $\tau_L \tau_K \tau_L^{-1} \tau_K^{-1}$ is $J \oplus K = V$.

In addition, $\tau_L \tau_K \tau_L^{-1} \tau_K^{-1}$ is not in RL(V) since

$$\tau_L \tau_K \tau_L^{-1} \tau_K^{-1}(\tau_K(J)) = \tau_L \tau_K \tau_L^{-1}(J) = \tau_L \tau_K(K) = \tau_L(K) = J \neq \tau_K(J).$$

<u>Theorem 3.5.2</u>: Let $n = 2$, L and K be lines with $V = L \oplus K$ and let τ_L, τ_K, and T_K be proper transvections with residual lines L, K, and K, respectively, $\tau_K \neq T_K$. Then $\tau_L \tau_K \tau_L^{-1} \tau_K^{-1}$ and $\tau_L T_K \tau_L^{-1} T_K^{-1}$ do not commute.

<u>Proof</u>: Let $\{b_1, b_2\}$ be a basis of V with $L = Rb_1$ and $K = Rb_2$. Then there exist α, β, γ in R such that

$$\text{mat}(\tau_L) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \ \text{mat}(\tau_K) = \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}, \ \text{mat}(T_K) = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$$

with $\alpha, \beta, \gamma \neq 0$ and $\beta \neq \gamma$. So

$$\text{mat}(\tau_L \tau_K \tau_L^{-1} \tau_K^{-1}) = \begin{bmatrix} 1 + \alpha\beta + \alpha^2\beta^2 & -\alpha^2\beta \\ \alpha\beta^2 & 1 - \alpha\beta \end{bmatrix}$$

and

$$\text{mat}(\tau_L T_K \tau_L^{-1} T_K^{-1}) = \begin{bmatrix} 1 + \alpha\beta + \alpha^2\gamma^2 & -\alpha^2\gamma \\ \alpha\gamma^2 & 1 - \alpha\gamma \end{bmatrix}$$

which commute if and only if $\alpha^2\beta\gamma(\beta - \gamma) = 0$. But one can easily check that α, β, and γ must be units since the transvections are proper. Thus

$\beta = \gamma$. Since $\tau_K \neq T_K$, this is a contradiction. So $\tau_L \tau_K \tau_L^{-1} \tau_K^{-1}$ and $\tau_L T_K \tau_L^{-1} T_K^{-1}$ do not commute.

Let σ be in $GL(V)$. If there exists $k > 0$ such that $(\sigma - I)^k = 0$, then call σ unipotent.

If σ is unipotent and U is a submodule of V for which $\sigma(U) = U$, then $\sigma|U$ is unipotent.

Theorem 3.5.3: Suppose n is not a zero-divisor. Let τ be a proper transvection in $GL(V)$. If τ commutes projectively with σ in $GL(V)$, then τ and σ commute in $GL(V)$.

Proof: If τ and σ commute projectively, then $\sigma\tau\sigma^{-1}\tau^{-1} = \alpha I$ for some α in R^*. So $1 = \det(\sigma\tau\sigma^{-1}\tau^{-1}) = \alpha^n$. Now $\sigma\tau\sigma^{-1} = \alpha\tau$. So

$$\sigma\tau^n\sigma^{-1} = (\sigma\tau\sigma^{-1})^n = (\alpha\tau)^n = \alpha^n\tau^n = \tau^n.$$

Let $\tau = \tau_{a,\rho}$, where a is unimodular in V. Then

$$\sigma\tau^n\sigma^{-1} = \sigma\tau_{na,\rho}\sigma^{-1} = \tau_{n\sigma(a),\rho\sigma^{-1}},$$

so $\tau_{n\sigma(a),\rho\sigma^{-1}} = \tau_{na,\rho}$. Thus

$$\tau_{n\sigma(a),\rho\sigma^{-1}}(a) = \tau_{na,\rho}(a)$$

$$a + \rho(\sigma^{-1}(a))(n\sigma(a)) = a + \rho(a)(na)$$

$$n\rho(\sigma^{-1}(a))\sigma(a) = 0.$$

Then $\rho(\sigma^{-1}(a)) = 0$ since $\sigma(a)$ is unimodular and n is not a zero-divisor. Now

$$\sigma\tau\sigma^{-1}(a) = \alpha\tau(a)$$

$$a + \rho(\sigma^{-1}(a))\sigma(a) = \alpha a$$

$$a = \alpha a.$$

Thus $\alpha = 1$ since a is unimodular. So τ and σ commute.

Theorem 3.5.4: If char(R) = p, where p is an odd prime, then σ in GL(V) is unipotent if and only if $\sigma^{p^n} = I$ for some $n \geq 0$.

Proof: Suppose σ in GL(V) is unipotent. Then there exists $k > 0$ such that $(\sigma - I)^k = 0$. Choose n large enough that $p^n \geq k$. Then

$$\begin{aligned} 0 &= (\sigma - I)^{p^n} \\ &= \sigma^{p^n} - \binom{p^n}{1}\sigma^{p^n-1} + \binom{p^n}{2}\sigma^{p^n-2} - \dots - I \\ &= \sigma^{p^n} - I \end{aligned}$$

since p divides $\binom{p^n}{t}$ for $1 \leq t \leq p^n - 1$. So $\sigma^{p^n} = I$.

Conversely, $\sigma^{p^n} = I$ implies that $(\sigma - I)^{p^n} = 0$.

Let Σ be in PΓL(V), $\Sigma = \overline{\sigma}$ where σ is in GL(V). We call Σ a projective unipotent transformation if σ is a unipotent transformation. Then σ is called a unipotent representative of Σ.

Let σ be a proper transvection with residual line L. Then $(\sigma - I)(V) = L$. But $(\sigma - I)(V) \subset P$ so $(\sigma - I)^2(V) = 0$. Thus all proper transvections are unipotent and all projective transvections are unipotent.

We say that two elements σ_1 and σ_2 in ΓL(V) commute projectively if $\overline{\sigma}_1$ and $\overline{\sigma}_2$ commute. Certainly commutability implies projective commutability.

3.6 Full Groups

We say that a subgroup G of ΓL(V) is full of transvections if $n \geq 2$ and for each hyperplane H of V and each line $L \subset H$, there is at least one proper transvection σ in G with P = H and Q = L. Similarly, a subgroup Δ of $P\Gamma L(V)$ is said to be full of projective transvections if $n \geq 2$ and for each hyperplane H of V and each $L \subset H$, there is at least one proper projective transvection σ in Δ with P = H and Q = L.

If $n \geq 2$, then clearly SL(V) and PSL(V) are full of transvections and projective transvections, respectively.

Throughout the rest of the chapter, $G \subset \Gamma L(V)$ will denote a group full of transvections, $\Delta \subset P\Gamma L(V)$ will denote a group full of projective transvections and $\Lambda : \Delta_1 \to \Delta_2$ will denote a group isomorphism. We say that Λ preserves the projective transvection σ_1 in Δ_1 if $\Lambda\sigma_1$ is a projective transvection in Δ_2 and it preserves the projective transvection σ_2 in Δ_2 if $\Lambda^{-1}\sigma_2$ is a projective transvection in Δ_1. It preserves projective transvections if it preserves all projective transvections in Δ_1 and Δ_2,

Theorem 3.6.1: If G and Δ are full, then $\check{G}$ and $\check{\Delta}$ are also full.

Proof: Let H be a hyperplane in V^* and L a line, $L \subset H$. Then L° is a hyperplane in V and H° a line in V with $H^\circ \subset L^\circ$. So there exists a proper transvection σ in G such that $P = L^\circ$ and $Q = H^\circ$. Thus by Theorem 3.4.1, $\check{\sigma}$ is a proper transvection in $\check{G}$ with residual space $(L^\circ)^\circ = L$ and fixed space $(H^\circ)^\circ = H$. Similarly, $\check{\Delta}$ is full.

Define $DG = [G,G]$ and $D\Delta = [\Delta,\Delta]$.

Theorem 3.6.2: If $n \geq 3$, then DG and $D\Delta$ are full.

Proof: Let H be a hyperplane of V and $L \subset H$ a line. Let $\{b_1, \ldots, b_n\}$ be a basis of V such that $L = Rb_1$ and $H = Rb_1 \oplus \ldots \oplus Rb_{n-1}$. Let $\{b_1^*, \ldots, b_n^*\}$ be the corresponding dual basis. Now G is full so there exists α, β in R^* such that $\tau_{b_1,\alpha b_2^*}$ is in G and $\tau_{b_2,\beta b_n^*}$ is in G.

Now $[\tau_{b_1,\alpha b_2^*}, \tau_{b_2,\beta b_n^*}] = \tau_{b_1,\alpha\beta b_n^*}$.

The proper transvection $\tau_{b_1,\alpha\beta b_n^*}$ has fixed space $\ker(\alpha\beta b_n^*) = H$ and residual line $Rb_1 = L$. Thus DG is full.

Now Δ is full of projective transvections. So $P^{-1}\Delta$ is full of transvections. Hence $DP^{-1}\Delta$ is full. So $P(DP^{-1}\Delta) = D\Delta$ is full of proper projective transvections.

<u>Theorem 3.6.3</u>: Let P_o and Q_o be subspaces of V such that $P_o \cap Q_o$ is also a subspace and $\dim(P_o) + \dim(Q_o) = n$. If $\dim(Q_o) = 1$, then assume $P_o \cap Q_o \neq 0$. Then there is a product σ of $\dim(Q_o)$ proper transvections in G such that $P = P_o$ and $Q = Q_o$.

<u>Proof</u>: If $\dim(Q_o) = 0$, then we are done.

If $\dim(Q_o) = 1$, then $P_o \cap Q_o = Q_o$, so choose a proper transvection σ in G with $P = P_o$, and $Q = Q_o$.

Let $\dim(Q_o) = 2$. Let $V = P_o \oplus M = Q_o \oplus N$ for subspaces M and N of V. Then $P_o = (P_o \cap Q_o) \oplus (P_o \cap N)$ and $Q_o = (P_o \cap Q_o) \oplus (Q_o \cap M)$. Now $P_o \cap Q_o$, $P_o \cap N$, $Q_o \cap M$ and $M \cap N$ are all subspaces since they are direct summands of subspaces. Clearly $(P_o \cap N) \cap (Q_o \cap M) = 0$. Let

$\{b_1, \ldots, b_i\}$ be a basis of $P_o \cap Q_o$,

$\{c_1, \ldots, c_j\}$ be a basis of $P_o \cap N$,

$\{d_1, \ldots, d_k\}$ be a basis of $Q_o \cap M$, and

$\{e_1, \ldots, e_\ell\}$ be a basis of $M \cap N$.

Then

$$V = Rb_1 \oplus \ldots \oplus Rb_i \oplus Rc_1 \oplus \ldots \oplus Rc_j \oplus Rd_1 \oplus \ldots \oplus Rd_k,$$

$$P_o = Rb_1 \oplus \ldots \oplus Rb_i \oplus Rc_1 \oplus \ldots \oplus Rc_j,$$

and $Q_o = Rb_1 \oplus \ldots \oplus Rb_i \oplus Rd_1 \oplus \ldots \oplus Rd_k$.

Case I: $i = 0$. Then $P_o = R_1c_1 \oplus \ldots \oplus Rc_{n-2}$ and $Q_o = Rd_1 \oplus Rd_2$. Let

$$L_1 = Rd_1,\ H_1 = Rd_1 \oplus Rc_1 \oplus \ldots \oplus Rc_{n-2}$$

$$L_2 = Rd_2,\ H_2 = Rd_2 \oplus Rc_1 \oplus \ldots \oplus Rc_{n-2}.$$

Then $Q_o = L_1 \oplus L_2$, $P_o = H_1 \cap H_2$, $L_1 \subset H_1$, $L_2 \subset H_2$.

Case II: $i = 1$. Then

$$P_o = Rb_1 \oplus Rc_1 \oplus \ldots \oplus Rc_{n-3}$$

$$Q_o = Rb_1 \oplus Rd_1$$

$$M \cap N = Re_1.$$

Let

$$L_1 = Rb_1,\ H_1 = Rb_1 \oplus Rc_1 \oplus \ldots \oplus Rc_{n-3} \oplus Re_1$$

$$L_2 = Rd_1,\ H_2 = Rb_1 \oplus Rc_1 \oplus \ldots \oplus Rc_{n-3} \oplus Rd_1$$

Then $Q_o = L_1 \oplus L_2$, $P_o = H_1 \cap H_2$, $L_1 \subset H_1$, $L_2 \subset H_2$.

Case III: $i = 2$. Then

$$P_o = Rb_1 \oplus Rb_2 \oplus Rc_1 \oplus \ldots \oplus Rc_{n-4}$$

$$Q_o = Rb_1 \oplus Rb_2$$

$$M \cap N = Re_1 \oplus Re_2.$$

Let

$$L_1 = Rb_1,\ H_1 = Rb_1 \oplus Rb_2 \oplus Rc_1 \oplus \ldots \oplus Rc_{n-4} \oplus Re_1$$

$$L_2 = Rb_2,\ H_2 = Rb_1 \oplus Rb_2 \oplus Rc_1 \oplus \ldots \oplus Rc_{n-4} \oplus Re_2$$

Then $Q_o = L_1 \oplus L_2$, $P_o = H_1 \cap H_2$, $L_1 \subset H_1$, $L_2 \subset H_2$.

In each case we have $Q_o = L_1 \oplus L_2$, $P_o = H_1 \cap H_2$, $L_1 \subset H_1$, $L_2 \subset H_2$. So choose a proper transvection σ_1 in G with $P_1 = H_1$ and $Q_1 = L_1$ and a proper transvection σ_2 in G with $P_2 = H_2$, $Q_2 = L_2$. Let $\sigma = \sigma_1\sigma_2$ which is in G. Since $Q_1 \cap Q_2 = 0$, then $P = H_1 \cap H_2 = P_o$. Since $V = P_1 + P_2$, then $Q = L_1 + L_2 = Q_o$.

We now proceed by induction on $\dim(Q_o)$. Suppose $\dim(Q_o) \geq 3$ and $\dim(P_o) \leq n - 3$. Let $\{b_1, \ldots ,b_n\}$ be a basis of V where $\{b_1, \ldots ,b_k\}$ is a basis of P_o. Let $P_1 = P_o \oplus Rb_{k+1} \oplus \ldots \oplus Rb_{n-1}$ and $P_2 = P_o \oplus Rb_n$. Then $\dim(P_2) = \dim(P_o) + 1$ and $P_o = P_1 \cap P_2$. Also $V = P_1 + P_2$. Now $P_o \cap Q_o \subset P_1 \cap Q_o$ so $P_1 \cap Q_o$ contains a line. Let Q_1 be a line in $P_1 \cap Q_o$. Then choose Q_2 such that $Q_o = Q_1 \oplus Q_2$. Now G is full so there exists a proper transvection σ_1 in G with residual space Q_1 and fixed space P_1. By induction, there exists a product σ_2 of $\dim(Q_2)$ proper transvections in G with residual space Q_2 and fixed space P_2. Let $\sigma = \sigma_1\sigma_2$. Now $V = P_1 + P_2$ and $Q_1 \cap Q_2 = 0$ so by Theorem 1.4.3, $P = P_1 \cap P_2 = P_o$ and $Q = Q_1 + Q_2 = Q_o$.

<u>Theorem 3.6.4</u>: If $n \geq 2$, then there is a σ in DG such that $Q = V$.

<u>Proof</u>: If $n \geq 3$, then DG is full. In Theorem 3.6.3 let $P_o = 0$, $Q_o = V$. Then there exists σ is DG such that $Q = V$ and $P = 0$.

So assume $n = 2$. Let $V = L \oplus K$. In G there are proper transvections σ_1 and σ_2 with $Q_1 = L$, $Q_2 = K$. Then by Theorem 3.5.1, $\sigma = \sigma_1\sigma_2$ has residual space V.

<u>Theorem 3.6.5</u>: Suppose $n \geq 2$ and that G is full. Then the centralizer of G in GL(V) is RL(V).

Proof: Let $\{b_1, \ldots, b_n\}$ be a basis of V and $\{b_1^*, \ldots, b_n^*\}$ the dual basis of V^*. Then G contains a proper transvection $\tau_{b_j, \alpha b_i^*}$ for some α in R^* for $i \neq j$. Let σ be in $C_V(G)$.

$$\sigma(\tau_{b_j,\alpha b_i^*}(x)) = \tau_{b_j,\alpha b_i^*}(\sigma(x))$$

$$\sigma(x + \alpha b_i^*(x)b_j) = \sigma(x) + \alpha b_i^*(\sigma(x))b_j$$

$$\sigma(x) + \alpha b_i^*(x)\sigma(b_j) = \sigma(x) + \alpha b_i^*(\sigma(x))b_j$$

So $b_i^*(x)\sigma(b_j) = b_i^*(\sigma(x))b_j$ for all x in V.

Let $\sigma(b_k) = \Sigma a_{\ell k} b_\ell$ for each k. Then

$$b_i^*(b_k)\sigma(b_j) = b_i^*(\sigma(b_k))b_j$$

$$\delta_{ik}\sigma(b_j) = a_{ik}b_j.$$

Thus $a_{ik} = 0$ if $i \neq k$. So $\sigma(b_k) = a_{kk}b_k$. Then $\delta_{ik}\sigma(b_j) = a_{ik}b_j$ implies $a_{jj}b_j = a_{ii}b_j$ so $a_{jj} = a_{ii}$ for all i. Thus σ is in RL(V). Clearly RL(V) is contained in the centralizer of G. Thus the centralizer of G in GL(V) is RL(V).

Theorem 3.6.6: If $n \geq 3$ and σ is a proper unipotent element of $C_V(DG)$, then $\sigma = I$.

Proof: If $n \geq 3$ then DG is full. By Theorem 3.6.5 $C_V(DG) = RL(V)$. So $\sigma = \alpha I$. Also $(\sigma - I)^k = 0$, so $0 = (\alpha I - I)^k = ((\alpha - 1)I)^k = (\alpha - 1)^k I$ and $(\alpha - 1)^k = 0$. But σ is a proper radiation so either $(\alpha - 1)$ is in R^* or $\alpha - 1 = 0$. So $\alpha - 1 = 0$, hence $\alpha = 1$. Thus $\sigma = I$.

Theorem 3.6.7: Let $n \geq 3$. Then for each hyperplane H in V and each line $L \subset H$, there are at least two distinct proper transvections in G, and at least two distinct proper projective transvections in Δ, with residual line L and fixed hyperplane H.

Proof: Let $\tau_{a,\rho}$ be in G with L = Ra and H = ker(ρ). Then $\tau_{a,\rho}^2 = \tau_{2a,\rho}$ is in G and $\tau_{2a,\rho}$ is distinct from $\tau_{a,\rho}$.

The above result applies to $P^{-1}\Delta$ and hence to Δ.

Theorem 3.6.8: If $G \subset GL(V)$ and $n \geq 3$, then $C_V(DG) = RL(V)$.

Proof: By Theorem 3.6.2, DG is full, so by Theorem 3.6.5, $C_V(DG) = RL(V)$.

3.7 The Group CDC(σ)

Let Δ be a full subgroup of $P\Gamma L(V)$. Then $P^{-1}\Delta \cap \Gamma L(V)$ is a subgroup of $\Gamma L(V)$ which is full, so let $G = P^{-1}\Delta \cap \Gamma L(V)$. If $\Delta \subset PGL(V)$, then $G \subset GL(V)$. Throughout this section we will assume that $\Delta \subset PGL(V)$, $G = P^{-1}\Delta \cap \Gamma L(V)$, so $G \subset GL(V)$.

Since $\check{}$ is an isomorphism, we also have $\check{\Delta} \subset PGL(V^*)$, $\check{G} = P^{-1}\check{\Delta} \cap \Gamma L(V^*)$, and $\check{G} \subset GL(V^*)$.

We will use C to denote the centralizer C_Δ, C_G, $C_{\check{\Delta}}$, $C_{\check{G}}$ when we are working with Δ, G, $\check{\Delta}$, $\check{G}$, respectively.

Theorem 3.7.1: For any σ in G, $\overline{C(\sigma)} \subset C(\overline{\sigma})$ and $\overline{DC(\sigma)} \subset DC(\overline{\sigma})$. Further, if σ commutes with an element of G whenever it commutes projectively with the element, then $\overline{C(\sigma)} = C(\overline{\sigma})$, $\overline{DC(\sigma)} = DC(\overline{\sigma})$. In particular, if τ is a proper transvection, then $\overline{C(\tau)} = C(\overline{\tau})$ and $\overline{DC(\tau)} = DC(\overline{\tau})$.

Proof: Let $\overline{\tau}$ be in $\overline{C(\sigma)}$, τ in $C(\sigma)$. Then $\tau\sigma = \sigma\tau$ so $\overline{\tau}\overline{\sigma} = \overline{\sigma}\overline{\tau}$. Thus $\overline{\tau}$ is in $C(\overline{\sigma})$. Hence $\overline{C(\sigma)} \subset C(\overline{\sigma})$.

Let $\overline{\tau_1\tau_2\tau_1^{-1}\tau_2^{-1}}$ be in $\overline{DC(\sigma)}$, $\tau_1\tau_2\tau_1^{-1}\tau_2^{-1}$ in DC(σ), and τ_1, τ_2 in C(σ). Then $\overline{\tau_1}$, $\overline{\tau_2}$ is in $C(\overline{\sigma})$ so $\overline{\tau_1\tau_2\tau_1^{-1}\tau_2^{-1}} = \overline{\tau}_1\overline{\tau}_2\overline{\tau}_1^{-1}\overline{\tau}_2^{-1}$ is in

$DC(\bar{\sigma})$. Hence $\overline{DC(\sigma)} \subset DC(\bar{\sigma})$.

Suppose σ commutes with an element of G whenever it commutes projectively with it. Let $\bar{\tau}$ be in $C(\bar{\sigma})$. Then $\bar{\tau}\bar{\sigma} = \bar{\sigma}\bar{\tau}$ so $\tau\sigma = \sigma\tau$. Then τ is in $C(\sigma)$. Thus $\bar{\tau}$ is in $\overline{C(\sigma)}$. Hence $\overline{C(\sigma)} = C(\bar{\sigma})$.

Let $\bar{\tau}_1\bar{\tau}_2\bar{\tau}_1^{-1}\bar{\tau}_2^{-1}$ be in $DC(\bar{\sigma})$, $\bar{\tau}_1,\bar{\tau}_2$ in $C(\bar{\sigma})$. Then τ_1, τ_2 is in $C(\sigma)$, so $\tau_1\tau_2\tau_1^{-1}\tau_2^{-1}$ is in $DC(\sigma)$. Thus $\bar{\tau}_1\bar{\tau}_2\bar{\tau}_1^{-1}\bar{\tau}_2^{-1} = \overline{\tau_1\tau_2\tau_1^{-1}\tau_2^{-1}}$ is in $\overline{DC(\sigma)}$. Hence $\overline{DC(\sigma)} = DC(\bar{\sigma})$.

For any two subspaces U and W of V, define

$$G(U,W) = \{\sigma \text{ in } G \mid Q \subset U \text{ and } P \supset W\}$$

$$\Delta(U,W) = \overline{G(U,W)}.$$

By Theorem 1.5.2, $G(U,W)$ and $\Delta(U,W)$ are subgroups of G and Δ, respectively. Let σ be in $G(U,W)$, $\Sigma = \bar{\sigma}$, u be in U and w be in W. Then $\sigma(u) - u$ is in $Q \subset U$ so $\sigma(u)$ is in U. So $\sigma(U) \subset U$. Now σ^{-1} is in $G(U,W)$ so $\sigma^{-1}(U) \subset U$. Thus $\sigma(U) = U$. Hence $\Sigma(U) = U$ since $\Sigma = \bar{\sigma}$. Clearly $\sigma(W) = W$ and $\Sigma(W) = W$ since $W \subset P$.

If H is a hyperplane in V and L is a line with $L \subset H$, then $G(L,H)$ contains the set of all proper transvections in G with residual line L and fixed space H, plus I.

<u>Theorem 3.7.2</u>: If U and W are subspaces of V, then

$$\widecheck{G(U,W)} = \check{G}(W^{\circ},U^{\circ})$$

and

$$\widecheck{\Delta(U,W)} = \check{\Delta}(W^{\circ},U^{\circ})$$

<u>Proof</u>: Let $\check{\sigma}$ be in $\widecheck{G(U,W)}$ with σ in $G(U,W)$. Then $P_{\check{\sigma}} = Q^{\circ} \supset U^{\circ}$ and $Q_{\check{\sigma}} = P^{\circ} \subset W^{\circ}$. So $\check{\sigma}$ is in $\check{G}(W^{\circ},U^{\circ})$, and conversely. So $\widecheck{G(U,W)} = \check{G}(W^{\circ},U^{\circ})$. Then

$$\begin{aligned}\widecheck{\Delta(U,W)} &= \widecheck{PG(U,W)}\\ &= P(\widecheck{G(U,W)})\\ &= P(\check{G}(W^{\circ},U^{\circ}))\\ &= P(\check{G})(W^{\circ},U^{\circ})\\ &= \widecheck{PG}(W^{\circ},U^{\circ})\\ &= \check{\Delta}(W^{\circ},U^{\circ}).\end{aligned}$$

<u>Theorem 3.7.3</u>: Let R be a ring without nontrivial nilpotents. Let σ_1 and σ_2 be nontrivial proper transvections in G. Then the following statements are equivalent:

(i) $P_1 = P_2$ and $Q_1 = Q_2$.

(ii) $C(\sigma_1) = C(\sigma_2)$.

(iii) $C(\overline{\sigma_1}) = C(\overline{\sigma_2})$.

<u>Proof</u>: (i) Suppose $P_1 = P_2$ and $Q_1 = Q_2$. Let Σ be in $C(\sigma_1)$, $\sigma_1 = \tau_{a,\rho}$ and $\sigma_2 = \tau_{\alpha a,\rho}$. Then $\tau_{a,\rho} = \Sigma\tau_{a,\rho}\Sigma^{-1} = \tau_{\Sigma a,\rho\Sigma^{-1}}$. Thus by Theorem 1.5.2, $\Sigma(a) = \lambda a$ and $\rho\Sigma^{-1} = \lambda^{-1}\rho$ for some λ in R^*. So

$$\Sigma\tau_{\alpha a,\rho}\Sigma^{-1} = \tau_{\alpha\Sigma a,\rho\Sigma^{-1}} = \tau_{\alpha\lambda a,\lambda^{-1}\rho} = \tau_{\alpha a,\rho}.$$

Thus Σ is in $C(\sigma_2)$. Hence $C(\sigma_1) \subset C(\sigma_2)$. Similarly, $C(\sigma_2) \subset C(\sigma_1)$. So $C(\sigma_1) = C(\sigma_2)$.

(ii) By Theorem 3.7.1, if $C(\sigma_1) = C(\sigma_2)$, then $C(\overline{\sigma_1}) = \overline{C(\sigma_1)} = \overline{C(\sigma_2)}$ $= C(\overline{\sigma_2})$.

(iii) Suppose $C(\overline{\sigma_1}) = C(\overline{\sigma_2})$. Suppose $P_1 \neq P_2$. Then there exists a line L with $L \subset P_2$ but $L \not\subset P_1$. The group G is full, so choose σ_3 in G with $P_3 = P_2$ and $Q_3 = L$. Then $Q_3 \subset P_2$ and $Q_2 \subset P_3$ so by Theorem 1.5.9, σ_3 is in $C(\sigma_2)$. Thus $\overline{\sigma_3}$ is in $C(\overline{\sigma_2})$ so $\overline{\sigma_3}$ is in $C(\overline{\sigma_1})$. Hence σ_3 is in

$C(\sigma_1)$ by Theorem 3.5.3. So by Theorem 1.5.9, $L \subset P_1$, which is a contradiction. So $P_1 = P_2$.

Suppose $Q_1 \neq Q_2$. Then there exists a hyperplane, H, such that $Q_2 \subset H$, but $Q_1 \not\subset H$, by Theorem 1.1.4. The group G is full so choose σ_3 in G with $P_3 = H$ and $Q_3 = Q_2$. Then $Q_3 \subset P_2$ and $Q_2 \subset P_3$ so σ_3 is in $C(\sigma_2)$. Then $\overline{\sigma_3}$ is in $C(\overline{\sigma}_2) = C(\overline{\sigma_1})$. So σ_3 is in $C(\sigma_1)$ by Theorem 3.5.3. Then by Theorem 1.5.9, $Q_1 \subset P_3 = H$, which is a contradiction. Hence $Q_1 = Q_2$.

<u>Theorem 3.7.4</u>: Let R be a ring without nilpotents. If $n \geq 3$ and σ is a nontrivial proper transvection in G, then $G(Q,P) \cap DC(\sigma) \neq I$.

<u>Proof</u>: Choose a basis $\{b_1, \ldots, b_n\}$ of V with the dual basis $\{b_1^*, \ldots, b_n^*\}$ of V^* such that $\sigma = \tau_{b_1, b_n^*}$. Then $Q = Rb_1$ and $P = \ker(b_n^*)$. The group G is full of proper transvections so there exist α and β in R^* such that $\tau_{b_1, \alpha b_2^*}$ and $\tau_{b_2, \beta b_n^*}$ are in G. Also $\tau_{b_1, \alpha b_2^*}$ is in $C(\sigma)$ and $\tau_{b_2, \beta b_n^*}$ is in $C(\sigma)$ by Theorem 1.5.9. Now

$$\begin{aligned}[\tau_{b_1,\alpha b_2^*}, \tau_{b_2,\beta b_n^*}] &= \tau_{b_1,\alpha b_2^*}\, \tau_{\tau_{b_2,\beta b_n^*}(b_1), -\alpha b_2 \tau_{-b_2,\beta b_n^*}} \\ &= \tau_{b_1,\alpha b_2^*}\, \tau_{b_1, -\alpha b_2^* + \alpha\beta b_n^*} \\ &= \tau_{b_1, \alpha\beta b_n^*}\end{aligned}$$

which is in $G(Q,P) \cap DC(\sigma)$.

<u>Theorem 3.7.5</u>: If $n \geq 3$ and H is a hyperplane of V and L is a line in H, then there is a nontrivial proper transvection τ in G with spaces

$L \subset H$ such that τ is in $DC(\tau)$.

Proof: Let σ be a proper transvection in G with spaces $L \subset H$. By Theorem 3.7.4 there exists a proper transvection τ in G with spaces $L \subset H$ and τ in $DC(\sigma)$. By Theorem 3.7.3, $C(\sigma) = C(\tau)$, so τ is in $DC(\tau)$.

Theorem 3.7.6: Let R be a ring without nontrivial nilpotents. If $n \geq 4$ and σ is a nontrivial proper transvection in G, then $G(L,P) \cap DC(\sigma) \neq I$ for all lines L in P.

Proof: Fix a line K in P with $K \not\subset Q + L$. This is possible since $n \geq 4$. By Theorem 1.1.4 there exists a hyperplane, M, containing $Q + L$ but not K. Let τ_L be a proper transvection in G with residual line L and fixed hyperplane M and let τ_K be a proper transvection in G with residual line K and fixed hyperplane P. By Theorem 1.5.9, τ_L and τ_K are in $C(\sigma)$.

Let $\Sigma = \tau_L \tau_K \tau_L^{-1} \tau_K^{-1}$. Then Σ is in $DC(\sigma)$. Now $\tau_L K \neq K$ since $K \not\subset M$, hence $\tau_L \tau_K \tau_L^{-1}$ is a proper transvection with residual line $\tau_L K$ distinct from K and fixed hyperplane $\tau_L P = P$. Thus $\Sigma = (\tau_L \tau_K \tau_L^{-1})\tau_K^{-1}$ having fixed hyperplane P, is a nontrivial proper transvection. Similarly, $\Sigma = \tau_L(\tau_K \tau_L^{-1} \tau_K^{-1})$ has residual line L. Hence Σ is in $G(L,P) \cap DC(\sigma)$.

Theorem 3.7.7: If $n \geq 2$ and σ is a nontrivial proper transvection in G, then the nontrivial proper elements of $G(Q,P)$ are contained in $CC(\sigma)$ and if Σ is in $CC(\sigma)$, then $\alpha\Sigma$ is in $G(Q,P)$ for some α in R^*.

Proof: If Σ is a typical nontrivial proper element of $G(Q,P)$, then by Theorem 3.7.3 $C(\Sigma) = C(\sigma)$. Hence Σ is in $CC(\Sigma) = CC(\sigma)$.

Suppose $n \geq 3$. Let Σ be in $CC(\sigma)$. For each line $L \subset P$ there exists a proper transvection in G with residual line L and fixed hyperplane P. This proper transvection is in $C(\sigma)$, hence Σ commutes with it. Hence $\Sigma L = L$ for all lines $L \subset P$. So $\Sigma|P$ is in RL(P). Thus $\Sigma|P = \alpha I_P$ for some α in R^*. So $(\alpha^{-1}\Sigma)|P = I_P$. Thus $P_{\alpha^{-1}\Sigma} \supset P$, where $P_{\alpha^{-1}\Sigma}$ is the fixed space of $\alpha^{-1}\Sigma$.

Now $\check{\Sigma}$ is in $CC(\check{\sigma})$ with $\check{\Sigma}$ in $\check{G}$. For each line $L \subset Q^o$, there exists a proper transvection in $\check{G}$ with residual line L and fixed hyperplane Q^o. This proper transvection is in $C(\check{\sigma})$, hence Σ commutes with it. So $\check{\Sigma}(L) = L$ for all lines $L \subset Q^o$. So $\check{\Sigma}|Q^o$ is in $RL(Q^o)$. Thus $\check{\Sigma}|Q^o = \beta I_{Q^o}$ for some β in R^*. So $(\beta^{-1}\check{\Sigma})|Q^o = I_{Q^o}$. Thus $P_{\beta^{-1}\check{\Sigma}} \supset Q^o$. So

$$Q \supset (P_{\beta^{-1}\check{\Sigma}})^o = (P_{\widecheck{(\beta^{-1}\Sigma)}})^o = (Q^o_{\beta^{-1}\Sigma})^o \supset Q_{\beta^{-1}\Sigma}.$$

Let $Q = Rb_1$ and choose b_2 in P independent of b_1. Then $\alpha\Sigma(b_2) = b_2$, so $\beta\Sigma(b_2) = \beta\alpha^{-1}b_2$. Now $\beta\Sigma(b_2) - b_2$ is in Q, so $\beta\Sigma(b_2) - b_2 = rb_1$, for some r in R. But

$$\beta\Sigma(b_2) - b_2 = \beta\alpha^{-1}b_2 - b_2 = (\beta\alpha^{-1} - 1)b_2.$$

So $\beta\alpha^{-1} - 1 = 0$, which implies that $\alpha = \beta$. So $\alpha\Sigma$ is in G(Q,P). Thus $\overline{\Sigma}$ is in $\Delta(Q,P)$.

Now suppose n = 2. Let Σ be in G with Σ in $CC(\sigma)$. Then Σ is in $C(\sigma)$. Choose a basis of V in which σ has matrix $\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$. Let Σ have matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc = u$ in R^*. Then

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} = u^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= u^{-1} \begin{bmatrix} d & d\lambda - b \\ -c & -c\lambda + a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= u^{-1} \begin{bmatrix} ad + cd\lambda - bc & db + d^2\lambda - bd \\ -ac - c^2\lambda + ac & -bc - cd\lambda + ad \end{bmatrix}$$

$$= u^{-1} \begin{bmatrix} u + \lambda cd & \lambda d^2 \\ -\lambda c^2 & u - \lambda cd \end{bmatrix}$$

$$= \begin{bmatrix} 1 + u^{-1}\lambda cd & u^{-1}\lambda d^2 \\ -u^{-1}\lambda c^2 & 1 - u^{-1}\lambda cd \end{bmatrix}$$

Now $u^{-1}\lambda$ is in R^* since σ is proper, so $cd = 0$, $d^2 = u$, and $c^2 = 0$. Thus d is in R^* and $c = 0$. So $ad = u$, which gives $ud = ad^2 = ua$. So $a = d$. Thus $\text{mat}(\Sigma) = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. So $\Sigma = a\Sigma_1$ where $\text{mat}(\Sigma_1) = \begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix}$. So $a^{-1}\Sigma$ is in $G(Q,P)$.

<u>Theorem 3.7.8</u>: Let σ be a nontrivial proper transvection in G. If $n \geq 3$, then the nontrivial proper elements of $G(Q,P)$ are contained in $CDC(\sigma)$.

<u>Proof</u>: The result follows from $CC(\sigma) \subset CDC(\sigma)$.

<u>Theorem 3.7.9</u>: Assume $n \geq 4$. Let σ be a proper element of G with $\text{res}(\sigma) \leq n - 3$ and suppose that $P \cap Q = 0$. Then $CDC(\sigma) \subset G(Q,P)$.

<u>Proof</u>: For each hyperplane H of P and each line L in H, fix a proper transvection $\tau_{L,H}$ in G with residual line L and fixed hyperplane $Q \oplus H$. Choose two such proper transvections, by Theorem 3.6.7, $\tau_{L,H}$ and $\tau_{L',H}$.

Clearly $\tau_{L,H}(P) = P$ and $\tau_{L,H}(Q) = Q$, so $(\tau_{L,H}|P)$ is a proper transvection with spaces $L \subset H$ and $(\tau_{L,H}|Q) = I$. Similarly $(\tau_{L',H}|P)$ is a proper transvection with spaces $L \subset H$ and $(\tau_{L',H}|Q) = I$.

Let G_P denote the subgroup of $GL(P)$ generated by all $\tau_{L,H}$, one for each L in each H. Then G_P is full. Then $I_Q \oplus G_P \subset C(\sigma)$. So $I_Q \oplus DG_P = D(I_Q \oplus G_P) \subset DC(\sigma)$. Thus $CDC(\sigma) \subset C(I_Q \oplus DG_P)$

Let Σ be in $CDC(\sigma)$. Then Σ commutes with each element of $I_Q \oplus DG_P$. Now DG_P is full so for each line L and each hyperplane H containing L, there is a proper transvection in DG_P with residual line L and fixed hyperplane H. But Σ will commute with any such proper transvection. Thus $\Sigma(L) = L$ for all $L \subset P$. Hence $\Sigma(P) = P$ and by duality, $\Sigma(Q) = Q$. Hence Σ is in $G(Q,P)$.

Theorem 3.7.10: Let $n \geq 4$ and σ be in G with $\text{res}(\sigma) \leq n - 3$. Suppose that $Q \cap P = 0$. Then every proper unipotent transformation in $CDC(\sigma)$ is a proper transvection in $G(Q,P)$.

Proof: This follows from Theorem 3.7.9 since $CDC(\sigma) \subset G(Q,P)$ and all proper elements of $G(Q,P)$ are proper transvections.

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