DIFFERENTIAL GAMES WITH SYSTEM UNCERTAINTY AND IMPERFECT INFORMATION

By

ROBERT B. ASHER

Bachelor of Science
Oklahoma State University
Stillwater, Oklahoma
1969

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1971

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Thesis Approved:

Henry R. Sebesta
Thesis Adviser

Charles M. Bacon

Craig S. Sims

Larry D. Zirkle

D. D. Durham
Dean of the Graduate College

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LIST OF SYMBOLS

ε	designation for the evader
$J_i(t)$	information set for the i-th player at time t
J	performance index
P	designation for the pursuer
R^i	real space of dimension i
t_f	final time
t_0	initial time
$x(t)$	state at time t
$\hat{x}(t)$	conditional mean of x(t)
$u(t)$	pursuer's control law at time t
U	admissible control set for the pursuer
$v(t)$	evader's control law at time t
V	admissible control set for the evader
V_i	performance cost to go as seen by the i-th player
Y_i	denotes the i-th player's measurement function
$\psi[x(t_f), t_f]$	denotes the terminal manifold
θ_i	denotes the i-th player's unknown parameter set
$E\{\}$	denotes the expected value operator
$\underset{\theta \mid I}{E}\{\cdot\}$	expected value of θ conditioned on I
$\cup$	union symbol
$\subset$	subset symbol
$\cap$	intersection symbol
$p(\cdot)$	probability density function

$a \times b$	Cartesian product of a and b
$\in$	element of symbol
$\mathcal{L}$	differential operator
$\int$	stochastic intergal
φ	null set

CHAPTER I

INTRODUCTION

Many problems in engineering and in socio-economic systems exist in which there are two or more entities with conflicting goals engaged in the process of control (for engineering systems) or in the process of decision making (for socio-economic systems). The goals conflict in that if one entity chooses the 'wrong' control law or makes a wrong decision the other entity stands to gain in some aspect if he chooses the right control law or makes the right decision. The first entity may lose in some aspect part of the goal he is trying to satisfy. The question arises as to how should each entity make a decision or choose a control law to insure that he realizes his goal in some sense. This choice must be irrespective of what decision or control variable his opponent chooses. That is, based upon the assumption that the entity's opponent will play his best decision or control law contrary to the first player's goals then what should the first entity choose as his control laws? The methodology for analysis of problems of this type comes from game theory. In specific, when the models of the entities may be modeled as differential equations, then the analysis methodology comes from differential game theory.

The study of dynamic games is a study of the process of decision making or the controlling of two or more entities with conflicting goals. The models of the entities are dynamic in the sense that the functional relationship representing the model evolves in time according to some functional rule. The relationship describing the time evolution of the models is a differential equation if the state space of the entity is a continuum. If the state space is a countable set (a set is countable if it is in the range space of the integers), then the time evolution of the models is a difference equation also referred to as a discrete-time equation. The dynamical game is considered to be a differential game if the functional rule for the time evolution of the states of the entities are differential equations. Similarly, the dynamic game is considered to be a multistage, difference or discrete-time game if the functional rule for the time evolution of the states of the entities are difference equations.

Each entity has certain variables called decision variables (in socio-economic systems) or control variables (in engineering systems) that he chooses in order to satisfy some particular goal. The control variables are, in general, constrained to be chosen from some admissible set.

The goals of the entities are assumed to be mathematically describable as a functional relationship between the states of the dynamic model and the decision or control variables of the entities. That is, if one entity makes a wrong decision or chooses a wrong control such that he loses some aspect of his goal, then the loss will be given directly to the other player assuming he has made the correct variable choice. Thus, the dynamic game is zero-sum.

Motivation for Differential Games

The process of decision making or control of two or more entities with conflicting goals occurs naturally in many problems in both engineering and socio-economic situations. For example, such problems may arise in the determination of the control laws required to control an anti-ballistic missile (ABM) in its pursuit and attempt to intercept an oncoming and maneuverable reentry vehicle (MaRV). The goal of the MaRV is to destroy its assigned target while the goal of the ABM is to intercept the MaRV. In order to achieve its goal, the reentry vehicle must choose its control law such that it avoids interception. Yet, it must carefully choose its control law such that it is able to recover from any perturbations of its state made to avoid the ABM and still reach the target with the required accuracy.

Another example is that of air-to-air combat where there are two or more aircraft engaged in dogfight situations. The goal of the attacker, for example, might be to minimize the distance between his aircraft and the evader's aircraft. The goal of the evading formations would be to maximize the distance between their aircraft and the attacker's aircraft. If the attacker (target) chooses his angle of attack, bank angle, and thrust rate (through his control stick and throttle) suboptimally, then the target (attacker) may take advantage of this in order to maximize (minimize) the distance between the aircraft. Another engineering example is that of min max controller design. That is, it is assumed that nature is playing against the dynamic system being designed. It is assumed that nature always acts to degrade the system performance. Systems designed in this manner are called 'worst' case controller designs. If nature does not play as the worst case, then the system performance will be better than that expected from the system design. This technique has been used for design of control systems by Salmon (83) and for the design of estimation algorithms by D'Appolito (27).

Another application is that of performance analysis of aircraft and missiles. That is, given that a fighter aircraft is in a design stage, what should its performance characteristics be in order to outperform its immediate or future threat aircraft assuming that the pilot in the threat aircraft plays optimally. One may use differential games for this type of application. Also, one of the questions that may be considered is the performance benefits of a thrust vector controlled (TVC) missile with thrust modulation control (TMC) over just a thrust vector controlled missile. One might formulate this problem as a differential game problem in which an intelligent target is choosing his control laws optimally in order to evade the missile. One may then compare the performance of a missile with TVC and TMC over that of a missile with just TVC.

In a socio-economic setting, many situations may arise where two or more corporations or two or more nations are in direct conflict. For example, the classical examples of game theory fall into that of two or more nations at war. Each nation is trying to choose its controlling variables such that it maximizes the other nation's losses while minimizing its own losses. Another example might be that of two or more businesses in direct competition to sell some particular good. The good might be such that it is known or at least assumed that the consumer market will over a given period of time buy a given amount of the particular item. An example is that of the automobile industry. Each particular corporation's profit structure might be represented as differential equations. This gives a time-varying representation of the profit structure to various factors such as labor disputes, advertising costs, manufacturing costs, sales of the automobiles, and any other major factor that may occur. Since it is assumed that a certain quantity of automobiles will be bought over the particular time period in question, then the sale of one automobile by a particular corporation represents a direct loss to the other corporation's profit structure. Thus, each corporation must carefully choose his advertising costs and manufacturing costs, and minimize labor disputes such that the sale of the corporation's automobiles are maximized (assuming a direct correlation between advertising costs and manufacturing costs to the profit structure and to the sale of automobiles, a highly complex structure but not an unrealistic assumption). The following explains the mathematical structure of differential games.

Dynamic Games

In order to establish the salient features of dynamic games, one must indicate under what conditions the dynamic models under consideration evolve. In a differential game, the state of the game at time t is described by the continuous vector function $x(t) \in R^n$ where x(t) evolves according to the functional relationship defined by the differential equation

$$\frac{dx(t)}{dt} = f[x(t), u(t), v(t), t] \tag{1}$$

with

$$x(t_0) = x_0 .$$

The variables contained in Equation (1) are defined in the following:

$x(t) \in R^n$ is a vector denoting the state of the game at time t,

$u(t) \in U$ where U is a subset of R^{m_1} and is a vector denoting the control variables of one of the players to be known as the pursuer,

$v(t) \in V$ where V is a subset of R^{m_2} and is a vector denoting the control variables of one of the players to be known as the evader,

$f \in R^n$ is the time derivative of the state of the game and is continuous with respect to x(t).

The goals of the players are assumed to be mathematically incorporated in a scalar functional known as the performance index, defined as

$$J = G(x(t_f), t_f) + \int_{t_0}^{t_f} Q(t(t), u(t), v(t), t)dt. \tag{2}$$

The assumption that the pursuer and the evader must choose his control from the set of allowable control actions U and V may be justified physically.

Solutions to differential game problems are conservative in nature in that it is assumed that a player's opponent is going to choose his control law contrary to the player's goals, and choose them in some optimal manner. Thus, the solution to the differential game will give conservative, worst case strategies. The objective of each player is to choose his control function u* or v* over the time interval (t_0, t_f) such that the following saddlepoint inequality is satisfied

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*). \tag{3}$$

If the pursuer plays u*, then the performance index will be no greater than J(u*, v*). Similarly, if the evader plays v*, then the performance index will be no less than J(u*, v*). This gives each player a guaranteed cost. However, if either player knows that the other player will play a solution suboptimal to the above strategies, then this information may be used to recompute an optimal strategy superior to the saddlepoint strategy. This will be illustrated in an example on page 11.

Unlike a conventional optimal control problem whereby one is trying to choose the best control to extremize the performance index in an unconservative manner, the problem of differential games is to conservatively choose your control law based on the assumption that your opponent will play optimally also.

In order to have a meaningful game solution, the control strategies must be in feedback form. This may easily be seen since if one player were to constrain his control to be open loop, then the other player's optimal strategy would be to play in any closed loop fashion to keep correcting his trajectory such that his goals are satisfied. Thus, each player must find his closed-loop, saddlepoint control strategies. In the terminology of Berkovitz (15), each player must choose his pure strategies (if they exist).

The game is said to be completed either when time evolves to a given point or when the state vector and final time enters a terminal manifold

$$\psi[x(t_f),\ t_f] = 0. \tag{4}$$

The dimensionality of the set of constraints, ψ, is less than or equal to the dimensionality of the state vector and includes a fixed final time constraint. Conditions under which a game might terminate are very complicated. In fact, the various problems that occur in game completion and in various surfaces that the game may transcend makes dynamic game problems very difficult to solve in the large. These concepts are explained in Isaacs (50).

Similarly, in a difference game, the state of the game evolves according to the difference equation

$$x(k+1) = f[x(k),\ u(k),\ v(k),\ k] \tag{5}$$

with

$$x(0) = x_0 \text{ and } k = 0,\ 1,\ \ldots,\ n.$$

The variables contained in Equation (5) are defined in the following

$x(k) \in R^n$ is a vector denoting the state of the game at time kT where T is the sampling period and is suppressed in the nomenclature,

$u(k) \in U$ where U is a subset of R^{m_1} and is a vector denoting the control variables of one of the players to be known as the pursuer,

$v(k) \in V$ where V is a subset of R^{m_2} and is a vector denoting the control variables of the other player to be known as the evader,

$f \in R^n$ is a functional representation of the transition to another point in the discrete space.

Again, the goals of the players are assumed to be mathematically represented as a scalar function known as the performance index and is defined as

$$J = G[x(N),\ N] + \sum_{i=0}^{N-1} Q[x(i),\ u(i),\ v(i),\ i]. \tag{6}$$

The players must again choose their control action from a set of allowable control actions. The objective of each player is to find the control sequence $\{u(i)\}_{i=0}^{N-1}$, where $\{\cdot\}$ denotes an ordered sequence, such that the saddlepoint condition occurs

$$J[u^*(i),\ v(i)] \leq J[u^*(i),\ v^*(i)] \leq J[u(i),\ v^*(i)]. \qquad (7)$$

One may justify the use of the discrete-time analog of the continuous-time physical model on the basis that in sophisticated controllers one usually uses a digital control loop. The computer is used to calculate the required control laws. These signals are converted from digital to analog signals and physically applied to the entity being controlled. Thus, since the computer is an inherent discrete-time device it is natural to solve the discrete-time approximation to the continuous physical system in order to obtain digital algorithms.

The following explains the motivation behind stochastic games.

Stochastic Differential Games

Many physical entities that must be very accurately controlled may be such that the system dynamical model may not be known to the accuracies required by a deterministic model. That is, there may be some residual modeling error that could possibly be treated as system noise.

Certain plant parameters may not be exactly known or the plant may be such that it is forced by a disturbance vector that is unknown. Another area of concern is that in order to use feedback signals to control the plant one must measure the state variables to be fed back. However, several of the variables may be measured by an inherently noisy measurement device. For example, one may use a tracking radar to measure range and angle information to a target. However, the angle information at short range is corrupted by scintillation noise (28) (33) (75) and dynamic lags in the tracker servo-system. The range information is corrupted by a similar phenomenon known as glint noise. Thus, one does not, in general, have perfect information about the state of a dynamical system.

The situation is that the players are taking imperfect measurements of the state of the game. The dynamical model of the game may not be perfectly known due to random disturbances. Each player must choose their feedback strategies based upon this imperfect information. This type of game is called a stochastic dynamic game.

The next section discusses a type of a stochastic differential game that is the basis for this dissertation.

Differential Games Under Uncertainty

Situations occur in differential games whereby one or both players may have uncertainty in their opponent's dynamics. For example, the combatants engaged in air-to-air combat may not know their opponent's maximum life coefficient. An actuator lag necessary to model the pertinent missile dynamics may not be known with certainty. In the problem of interception of a maneuvering reentry vehicle by an ABM, the defense is taking noisy measurements about the location of the reentry vehicle. The problem the defense has is that of determining the optimal control variables it should use in order to intercept the maneuvering reentry vehicle. However, in order to determine the set of control variables it must use, the defense has to know the dynamic model of the reentry vehicle. However, unless intelligence reports were exceedingly good, certain important parameters in the dynamic model of the reentry vehicle may be unknown. An example would be that of not knowing the minimum turn radius of the reentry vehicle or the ballistic coefficient. In the use of differential games to determine precise control necessary for intercept, it is important that one knows the dynamic characteristics of the adversary.

As pointed out by M. Ciletti and A. Starr (24), the assumption that each of the players has total knowledge of all the state variables and of the dynamic description of their opponent's system is very basic to realistic applications of dynamic game theory. That is, in many applications problems, as previously illustrated, one does not know certain physical parameters that may be modeled in the state equations. Also, in many applications, one may not be able to measure all the state variables perfectly. Thus, the above statement points out a basic deficiency in differential game theory. To date, no work has been accomplished in the problem with uncertainty in physical parameters. This is the basis for the dissertation.

In order to illustrate some of the pertinent aspects of the theory developed in the dissertation, a simple example will be considered. The purpose is to show that the player improves his performance by learning his unknown parameters. It is shown that, by playing nonconservatively, a player may improve his cost. That is, if he uses his information level as to his opponent's uncertainty, then he stands to improve his cost as compared to the cost if he plays conservatively.

The system dynamics are

$$\dot{x} = \theta x + au + bv \tag{8}$$

where u is the pursuer's control and v is the evader's control. The pursuer is trying to minimize and the evader is trying to maximize the following performance index:

$$J = \frac{1}{2} s x^2(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (r_1 u^2 + r_2 v^2)\,dt \tag{9}$$

where $r_1 > 0$. Several cases will be considered. The first case will be that of each player having perfect knowledge as to his opponent's dynamics. Next, the case where the evader has uncertainty as to the pursuer's dynamics is treated. The time interval (t_0, t_f) is partitioned into two subintervals (t_0, t_1) and (t_1, t_f) where $t_0 < t_1 < t_f$. The evader does not know the true value of θ over (t_0, t_1). However, at t_1 he learns the true value of θ and uses this new information to recompute his optimal strategy over (t_1, t_2). It is shown that he improves his performance by trying to learn his opponent's dynamics since his optimal strategy depends on knowledge of the pursuer's dynamics.

In this case, each player has the same information from which to choose his control strategies. Each player is trying to find the control strategies such that the following saddlepoint inequality is satisfied

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*). \tag{10}$$

The general problem of this type has been treated by Rhodes (80). This general problem is stated with dynamics:

$$\dot{x} = Fx + G_1 u + G_2 v \tag{11}$$

and performance index

$$J = \frac{1}{2} x^T(t_f) S(t_f)\, x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (u^T Q_1 u + v^T Q_2 v)\,dt \tag{12}$$

where the matrices Q_1 and Q_2 are symmetric, positive definite and negative definite, respectively. The matrix $S(t_f)$ is symmetric, positive definite. The solution for the strategies is

$$u(t) = -Q^{-1}(t)G_1^T(t)P(t)x(t) \tag{13}$$

$$v(t) = -Q_2^{-1}(t)G_2^T(t)P(t)x(t)$$

where P(t) is the solution to the matrix Riccati equation

$$\dot{P} + PF + F^T P - P[G_1 Q_1^{-1} G_1^T + G_2 Q_2^{-1} G_2^T]P = 0 \tag{14}$$

with boundary condition

$$P(t_f) = S(t_f).$$

The general solution will be applied to the specific example.

The solution to the previously posed problem may be written as

$$u^*(t) = -\frac{a}{r_1} P(t)x(t)$$

(15)

$$v^*(t) = -\frac{b}{r_2} P(t)x(t)$$

where P(t) is the solution to the Riccati equation

$$\dot{P}(t) = -2\theta P(t) + \left(\frac{a^2}{r_1} + \frac{b^2}{r_2}\right)P^2(t) \tag{16}$$

with boundary condition

$$P(t_f) = s.$$

Thus, each player's optimal strategy depends upon knowledge of the system eigenvalue θ. If each player used the above control strategies, the performance index would be equal to $J(u^*, v^*)$. Also, $J(u^*, v^*)$ is such that the following inequality is satisfied:

$$J(u^*, v) \leq J(u^*, v^{*}) \leq J(u, v^*). \tag{17}$$

Another important point to note is that each player has perfect knowledge about the game state $x(t)$. This assumption is not valid in general, but the discussion of this point will be deferred until later. Also, it has been assumed that each player knows the weighting of the performance index each player is using. In a problem solved by Ho et al. on optimal guidance laws, the weighting factors are assumed known by each player.

The next case to consider is that of the evader having uncertainty as to the value of the game dynamics over a subinterval of $[t_0, t_f)$. The evader has uncertain knowledge as to the game dynamics over the first time segment $[t_0, t_1)$ and perfect knowlege of θ over the time interval $[t_1, t_f)$ where $t_0 < t_1 < t_f$. The cost to go from t_0 to t_1 is given as

$$J_1 = \tfrac{1}{2}\int_{t_0}^{t_1} (r_1 u^2 + r_2 v^2)\,dt \tag{18}$$

and the cost over the second segment is given as

$$J_2 = \tfrac{1}{2} s x^2(t_f) + \tfrac{1}{2}\int_{t_1}^{t_f} (r_1 u^2 + r_2 v^2)\,dt. \tag{19}$$

The evader's best knowledge of the dynamics over $[t_0, t_1)$ are given as

$$\dot{x} = \theta_{\varepsilon} x + au + bv. \tag{20}$$

Thus, the strategy the evader will play is given by

$$v^{0}(t) = -\frac{b}{r_2} P_{\varepsilon}(t)x(t) \tag{21}$$

where $P_{\varepsilon}(t)$ is the solution to the Riccati equation

$$\dot{P}_g(t) = -2\theta_g P_g(t) + \left(\frac{a^2}{r_1} + \frac{b^2}{r_2}\right)P_g^2(t) \tag{22}$$

with

$$P_g(t_f) = s.$$

Since the control v° is not equal to v^*, the following inequality applies at t,

$$J_1(u^*, v^\circ) < J_1(u^*, v^*). \tag{23}$$

Thus, over the interval $[t_0, t_1)$, the evader loses some aspect of his goal. At t_1 he learns the true value of the game eigenvalue and recomputes his strategy to be played over $[t_1, t_f)$. This will be equal to v^*. Thus, the total cost J_T equals the cost over the interval $[t_0, t_1)$ plus the cost over $[t_1, t_f)$, i.e.,

$$J_T = J_1(u^*, v^\circ) + J_2(u^*, v^*). \tag{24}$$

However,

$$J_T < J_1(u^*, v^*) + J_2(u^*, v^*) = J(u^*, v^*). \tag{25}$$

Also, if the evader never learned the true eigenvalue and played v° over the total interval $[t_0, t_f)$, then the cost would be equal to the cost over $[t_0, t_1)$, i.e., $J_1(u^*, v^\circ)$ plus $J_2(u^*, v^\circ)$. However,

$$J_2(u^*, v^\circ) < J_2(u^*, v^*). \tag{26}$$

Thus,

$$J_1(u^*, v^\circ) + J_2(u^*, v^\circ) < J_T < J(u^*, v^*). \tag{27}$$

The evader will, thus, gain if he can learn the pursuer's dynamics, and will lose some degree of his goal if he does not know or learn the pursuer's dynamics.

If each player had uncertainty about each other's dynamics, then it is reasonable to expect that each player would base his game solution upon his best guess of his opponent's dynamics. Thus, the pursuer would use the strategy u_s where

$$u^s(t) = -\frac{a}{r_1} p_{s1}(t)x(t) \tag{28}$$

and the evader would use the strategy v^s where

$$v^s(t) = -\frac{b}{r_2} P_{s2}(t)x(t) \tag{29}$$

where P_{s_1} and P_{s_2} are the solutions to the following differential equations:

$$\begin{aligned} \dot{P}_{s_1} &= -2\theta_{s_1} P_{s_1} + \left(\frac{a^2}{r_1} + \frac{b^2}{r_2}\right) P_{s_1}^2 \\ \dot{P}_{s2} &= -2\theta_{s2} P_{s2} + \left(\frac{a^2}{r_1} + \frac{b^2}{r_2}\right) P_{s2}^2 \end{aligned} \tag{30}$$

with

$$P_{s1}(t_f) = P_{s_2}(t_f) = s.$$

The pursuer is in essence solving the differential game with the performance index

$$J_p(u, v, \theta_{s1}), \tag{31}$$

while the evader is solving the differential game with the performance index

$$J_E(u, v, \theta_{E2}). \tag{32}$$

If the pursuer has knowledge that the evader is playing his optimal strategy but that he has uncertainty in the knowledge of the pursuer's dynamics, then the pursuer may use this knowledge in order to better his strategy if he has perfect information. For example, if the pursuer realizes that the evader will use the parameter values θ_E, then he knows that v will be of the form

$$v = -r^{-1} b P_E(t, \theta_E) x(t) \tag{33}$$

where P_E is the solution to the differential equation

$$\dot{P}_E = -2\theta_E P_E + \left(\frac{a^2}{r_1} + \frac{b^2}{r_2}\right) P_E^2 \tag{34}$$

with

$$P_E(t_f) = s.$$

The pursuer may now form the new system equation

$$\dot{x} = \theta x + au - r_2^{-1} b^2 P_E x. \tag{35}$$

The performance index he may minimize is

$$J = \tfrac{1}{2} s x^2(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f} (r_1 u^2 + r_2^{-1} b^2 P_3^2 x^2)\,dt. \tag{36}$$

This is a linear quadratic problem with solution for u

$$u = -r_1^{-1} a P_p(t) x \tag{37}$$

where $P_p(t)$ is the solution to the differential equation

$$\dot{P}_p = -2\left(\theta - \frac{b^2}{r_2} P_E\right) P_p + \frac{a^2}{r_1} P_p^2 - \frac{b^2}{r_2} P_E^2 \tag{38}$$

with

$$P_p(t_f) = s.$$

Thus, if the pursuer plays nonconservatively, he stands to decrease the cost as seen by him as his optimal strategy as given by (37) and not by (15).

Another problem which necessitates total knowledge of the game dynamics will be illustrated. This problem will not be heuristically solved as was the previous example. However, the problem will be considered again in Chapter V after the basic theoretical results necessary to solve the problem have been developed.

The system dynamics of the pursuer is

$$\dot{x}_p = \theta_p x_p + u \tag{39}$$

where u is the pursuer's control. The system dynamics for the evader is

$$\dot{x}_E = \theta_E x_E + v \tag{40}$$

where v is the evader's control. The performance index is

$$J = \tfrac{1}{2}s(x_p(t_f) - x_E(t_f))^2 + \tfrac{1}{2}\int_{t_0}^{t_f}(r_1 u^2 + r_2 v^2)dt \tag{41}$$

where $r_1 > 0$ and $r_2 < 0$. The pursuer is trying to minimize J and the evader is trying to maximize J. The solution to the general problem as considered by Baron, Bryson, and Ho (20) will be given next.

One may follow the formulation as explained in reference (20) or in Chapter V and redefine a new state vector z(t) where

$$Z(t) = \Phi_p(t_f, t)x_p(t) - \Phi_E(t_f, t)x_E(t). \tag{42}$$

This new state vector represents the predicted terminal miss if each player uses no control over the interval $[t, t_f)$.

One may easily derive the fact that the general formulation applied to the specific problem of consideration yields

$$\dot{Z} = e^{\theta_p(t_f - t)}u(t) - e^{\theta_E(t_f - t)}v(t) \tag{43}$$

where

$$Z(t_0) = e^{\theta_p(t_f - t_0)}x_p(t_0) - e^{\theta_E(t_f - t_0)}x_E(t_0)$$

and

$$J = \tfrac{1}{2}sZ^2(t_f) + \int_{t_0}^{t_f} (r_1 u^2 + r_2 v^2)dt. \tag{44}$$

The solution for u and v is

$$\begin{aligned} u(t) &= -r_1^{-1} e^{\theta_p(t_f - t)} K^{-1}(t_f, t)Z(t) \\ v(t) &= -r_2^{-1} e^{\theta_E(t_f - t)} K^{-1}(t_f, t)Z(t) \end{aligned} \tag{45}$$

where

$$K(t_f, t) = \frac{1}{s} + M_p(t_f, t) - M_E(t_f, t) \tag{46}$$

and

$$M_p(t_f, t) = \frac{1}{r_1}\int_{t_0}^{t_f} e^{2\theta_p(t_f - t)}dt \tag{47}$$

$$M_E(t_f, t) = \frac{1}{r_2}\int_{t_0}^{t_f} e^{2\theta_E(t_f - t)}dt. \tag{48}$$

One may note that in order to solve the optimal problem each player must compute the value of the state at time t and K at time t. However, knowledge of the reduced state Z(t) and K(t) depends on knowledge of θ_p and θ_E. Thus, if an erroneous value of θ_p or θ_E were used, the strategies would be suboptimal.

Research Objectives and Results

The class of systems considered in the research are modeled by linear or nonlinear stochastic differential equations which are parameterized by a time invariant parameter vector, elements of which are known to the pursuer and unknown to the evader, and elements of which are known to the evader, but unknown to the pursuer. The system is forced by both the pursuer's control and the evader's control. The goals of each player directly conflict. Each player is trying to choose his control laws in order to extremize some performance index. The performance index is a mathematical measure of the player's goals. The players are assumed to have measurement subsystems that give either a perfect measurement of state or a noise corrupted measurement of state.

The objective of the research was to develop a sufficiency condition for the class of problem described and solve the linear quadratic problem for the above class of systems. Chapter II contains the results of a literature search in differential games.

The results of this dissertation are as follows. In Chapter III:

(a) A structure for the type strategies that may occur for differential games with the above uncertainty in the system dynamics is defined.

(b) A sufficiency condition for differential games under uncertainty and perfect information is developed and proved.

(c) The open-loop feedback strategies for the linear quadratic game under uncertainty and perfect information is solved for several types of strategies that occur.

In Chapter IV:

(a) A sufficiency condition for differential games under uncertainty and imperfect information is developed.

(b) The open-loop feedback strategies for the linear quadratic game under uncertainty and imperfect information is solved.

In Chapter V:

(a) The problem of interception first introduced by Ho et al. (20) is formulated and solved.

(b) The results are applied to a typical missile intercept problem.

CHAPTER II

LITERATURE REVIEW

Differential Games

Isaacs, in his book (50) and in a series of Rand reports, first developed the theory of differential games totally independent of what is commonly known as optimal control theory. His "main equation" derived independent of Hamilton-Jacobi theory is in fact the sufficiency condition based upon the Hamilton-Jacobi theory. One interesting point is that one may imbed optimal control theory in differential game theory. Thus, Isaacs in the fifties accomplished some very basic work that imbeds aspects of optimal control theory. In the book by Pontryagin (77), Kelendzheridze considered a deterministic minimum time pursuit-evasion problem. Other results in linear differential games were published by Pontryagin (78) and Gadzhiev (40). Berkovitz (15) treated differential game theory from a rigorous calculus of variations viewpoint. This was an extension of work published by Fleming (37). Other work in the time period included additional work by Fleming (38). Ho, Bryson, and Baron (20) considered the continuous time deterministic pursuit evasion game for linear systems, quadratic cost, and fixed final time. Survey papers which summarize the aspects of differential games include Athans (7), Ho (49), and Sarma and Ragade (86).

In 1965, the first result in stochastic differential games appeared in a paper by Ho (44). Speyer (91) in 1967 considered another formulation of a stochastic differential game with controllable parameters. Other papers that appeared were by Ciletti (22), Meschler (74), Wong (100), and Berkovitz (16). The paper by Wong was one of the first general aerospace applications of differential game theory. In 1968, Behn and Ho (13) treated the linear stochastic differential game with one player having imperfect measurements. Behn (14) also treated this problem in his dissertation. Rhodes (80), (81), (82) treated the linear dynamics, quadratic cost problem with imperfect measurements. Willman (99) also treated this problem, but the results were not as general as Rhodes; results. Shea (87) treated the differential and discrete linear game problem independent of the above papers. Other papers at this time were by Meier (71), and Salmon (85). Many papers began to appear at this time. Interest dictated the First International Conference on the Theory and Applications of Differential Games (48).

In 1970, many papers including several by Ciletti (23), (24) appeared in the important problem of differential games with information time lag. Another survey by Ho (46) pointed out a new concept called Generalized Control Theory (GCT) in which both optimal control and differential games were only subsets of GCT. Interest increased in trying to apply differential games. Several references are given in the preceding paper. Other application papers include the dissertations by McFarland (67) and Othling (76). Also, a report by Baron et al. (9) attempts to apply differential games to air-to-air combat. A report by Systems Control Inc. treats the ABM versus MaRV problem (84). Bernhard (18) treated a theoretical application problem. Another dissertation by Lin (62) considered the ABM versus MaRV problem.

In 1971, Merz (72) treated the homicidal chauffeur problem.

Leatham (59) and Baron (10) considered several subclasses in treating air-to-air combat as a differential game. Other dissertations included (41) (63) (93) (97).

Further interest was felt at the Air-to-Air Combat Analysis and Simulation Symposium at Kirtland Air Force Base in 1972 where several papers on differential games were presented.

Two interesting and important features of all the above papers are that the assumption that each player knows the game dynamics with certainty and that each player assumes his opponent knows the game dynamics with certainty.

Estimation Theory

In many control problems, one is faced with the problem of extracting estimates of the state of the system from noisy measurement data. The theory by which this may be accomplished in estimation theory. Weiner (98) solved the problem when the system and noise statistics are stationary by spectral factorization. In 1960, Kalman (53) solved the problem of a nonstationary discrete linear system. Kalman and Bucy (54) developed what is known as the Kalman filter for continuous, linear nonstationary systems. The method Kalman utilized was that of orthogonal projections. There has been many results and applications of linear and nonlinear filtering. Several books include that by Bucy (21), Meditch (68), and Jazwinski (52). Applications of filtering theory include many aerospace problems in orbit determination, navigation, and pointing and tracking problems (5) (6).

One of the more important results that is utilized in this dissertation is that of Bucy's Representation Tehorem (21) (also stated in

Chapter IV). This theorem is the basis of the filtering problem for nonlinear stochastic systems. The result represents the evaluation of the conditional probability density function.

Estimation Under Uncertainty

Estimation under uncertainty implies that one has uncertainty of key variables or parameters in the estimation problem. This may include uncertainty in system parameters or uncertainty in elements of covariance matrices necessary to solve the estimation problem. There has been an outflux of work in what is known as adaptive estimation. Key work in this area includes work by Mehra (69) (70), by Magill (65), Tapley (92), Martz (66), Jazwinski (51), Lainiotis (57), Hilborn (42) (43), and Sims (88). A good survey on adaptive filtering may be found in Mehra (70).

The main result used in this research is the Partition Theorem of Lainiotis (57).

Stochastic Systems Under Uncertainty

In many dynamic problems, one may be concerned with controlling a partially unknown system. The system may contain parameters that are not totally known or even may be completely unknown. Many papers have been written in this area. For example, Sims and Asher (4) consider the problem whereby the control gain matrix contains uncertain parameters. Tse (94) treats a similar problem. Dajani (26) considers the problem with system uncertainty. This problem has also been considered by Lainiotes (56) and by Lee (60). The basic work in these two papers leads to suboptimal controls. The work is suboptimal for two reasons:

Their results do not take into consideration the dual control problem (Feldbaum (34)) in which the control is used for both identification and control objectives. Also, their results are suboptimal open-loop feedback strategies to the problem which solutions are shown in this dissertation.

CHAPTER III

SYSTEM UNCERTAINTY AND PERFECT INFORMATION GAME

Introduction

In this chapter, the problem of differential games with uncertain parameters contained within the system matrix and perfect measurements is solved. It is assumed that each player knows the basic structure of the game dynamics. However, the system matrix for the game is parameterized by elements of a time invariant parameter vector containing those parameters unknown to either player. This parameter vector, θ, may be partitioned into two subvectors, θ_P and θ_E. The subvector, θ_E, contains elements that are known to the pursuer but unknown to the evader. The subvector, θ_P, contains elements that are known to the evader but unknown to the pursuer. It is assumed in this chapter that each player has a measurement subsystem capable of giving perfect measurements of the system state.

Definitions are given to indicate the type strategies that may be found by solutions of the differential game under uncertainty and imperfect information. It is shown that, under these definitions, previously found strategies for differential games with imperfect information, Rhodes (80) and Ho (13) are a type of security strategy called a system security strategy in which it is assumed that each player assumes his opponent has perfect knowledge of the game dynamics.

The structure given by the definitions helps one to identify the types of strategies that are both previously given and given in this dissertation. Also, it is useful in visualizing some of the future problems to be solved in differential games under uncertainty and imperfect information.

A Hamilton-Jacobi equation formulation is developed for the general nonlinear problem. The Hamilton-Jacobi equation formulation is proved to be a sufficient condition for optimality. The sufficiency condition is used to develop the strategies for the linear dynamics, quadratic cost differential game with system uncertainty. In this game, the dynamics of the players are assumed modeled by linear differential equations with uncertainty in the system matrix for the state equations. Each player has, in general, different uncertainty in the state equations. This model uncertainty is assumed represented by a time invariant parameter vector, θ, as explained earlier. It is assumed that each player has an a priori probability density function relating his best knowledge of the parameter subvector unknown to him.

It is shown that the linear problem may be solved in a feedback form whereby the equations necessary to solve for the gain are integral, partial differential equations. Each player's strategy also includes use of his measurement vector for adaptation in order to estimate the unknown parameters. Also, each player's strategy includes a risk in that his control strategy depends upon knowledge of his opponent's uncertainty as to the game dynamics. The dual control aspect (see definition 1 on page 31) of choosing the control strategies for both identification and control objective is not considered. The strategies found may be classified as open-loop feedback strategies (see definition

2 on page 31) in that one solves the open-loop problem with measurements taken at the particular time of interest based upon the assumption that no more measurements will be taken. Thus, any control strategy used for identification will not be considered since the assumption is made that no more measurements will be taken to use for identification purposes. The strategies are recomputed as open-loop feedback strategies at each time of control application based upon the above assumption. The dual control aspect of the differential games problem may consist of each player both choosing his control to both identify his opponent's parameters and to cause his opponent's measurement and estimation subsystem to have excessive error. This will be left for future work. Thus, the assumption that each player uses his control input only for control objective is made. References on dual control include (34), (95), and (96).

In this chapter, it is assumed that the system is continuous and that the control is continuously applied to the system, but that the parameter estimation occurs at discrete instants of time. An information set formulation of the problem is made. Each player must find the function mapping the information set into the control space such that the performance index is extremized.

There are several insights to be pointed out in the formulation of the game under uncertainty. At the outset of the game, each player has essentially solved a differential game that is different than what his opponent has solved. This is evident since each player has different uncertainty as to the game dynamics. This means that the solution to the game is dependent upon the information sets of each player. Each player uses his measurement information to adapt upon and learn the parameter subvector unknown to him. This is reasonable as each player can obtain a cost favorable to him if information as to the true value of the parameter set is used to obtain a better estimate of the true values as was shown in Chapter I. Each player may solve for his strategy by using his best information as to the true value of the game dynamics. This would yield a somewhat nonconservative strategy in that if he were grossly in error then this strategy could yield a result very favorable to his opponent. He could take a very conservative estimate as to bounds on the unknown parameter set and solve for the parameter set that would give the worst case results. This would yield a very conservative strategy. However, this type of strategy would not easily allow for inclusion of available information obtained as the game progresses. That is, this would not allow for the use of measurements in order to learn the values of the unknown parameters. Thus, each player may use the approach placed forth in the research to obtain a more realistic, conservative strategy and to obtain non-conservative strategies that will allow a gain in desired performance.

The following lists the contributions of this chapter:

(a) a structure for the strategies for differential games under uncertainty and imperfect information;

(b) a sufficiency condition for differential games under uncertainty;

(c) the solution to the linear, quadratic game under uncertainty and perfect information;

(d) the open-loop feedback strategies for the stochastic control problem may be found by constraining the evader's controls to be zero. The solution extends those found by (56), (58), and (60). The solution to this problem is shown in Appendix C.

The following section considers a structure for differential games under uncertainty and imperfect information.

Game Structure

In this section, a structure for differential games with uncertainty and imperfect information is given. This structure gives several definitions which relate to the types of strategies that one may obtain for this type of game. It is important to classify the types of strategies for two basic reasons. The first is that previous work and the work considered in this dissertation are of a special class of the general problem. This may be identified from the structure. Secondly, it enables one to obtain insight into some of the areas for future research in games under uncertainty.

Definition 1: If the control input is used for both control objective and identification, then the strategies found are dual control strategies.

Definition 2: If each player solves for his strategies at each instant of time under the assumption that he may not obtain any more measurements of the state of the game, then the strategies found are open-loop feedback strategies (see (11)). Open-loop feedback strategies imply that neither player will try to impulsively control the system in order to instantaneously identify the system. This may be discussed further in that each player is trying to obtain a terminal miss according to his goals, but at the same time limit the energy expenditure. Thus, a high energy expenditure at the beginning may allow a player to identify the system, but may make his control cost too expensive. Also, if the control energy is constrained, he may exceed this constraint during the play of the game. If either player feels that the other player is going to do the above, then he may either choose a canceling control input or may increase the noise level by playing his control input as white noise. There is much research to be conducted into the dual control aspect of differential games under uncertainty. In effect there is a tri-control problem in which one is choosing control for control objective, for identification, and for decreasing his opponent's measurement subsystem's signal to noise ratio. Thus, the strategies found in this thesis are open-loop feedback strategies (OLFS).

Defintion 3: If the information set for each player includes the assumption that his opponent has no uncertainty as to the game dynamics, then the strategies found by mapping the information set into the control space are called system security strategies (SSS).

Definition 4: If this information set for each player includes the knowledge (or assumed knowledge) of the opponent's best knowledge of the game dynamics, then the strategies found by mapping the information set into the control space are called system risk strategies (SRS).

Definition 5: If the information set for each player includes the assumption that his opponent has a measurement subsystem that can obtain perfect measurements of the state of the game, then the strategies found by mapping the information set into the control space are called measurement security strategies (MSS).

Definition 6: If the information set for each player includes the knowledge of the opponent's error in his estimate of the state of the game, then the strategies found by mapping the information set into the control space are called measurement risk strategies (MRS).

Definition 7: If the strategies found by each player include the assumption that his opponent is playing measurement security, system security strategies, then the strategies are called opponent security strategies (OSS).

Definition 8: If the strategies found by each player include the assumption that his opponent is playing measurement security, system risk strategies, then the strategies are called opponent risk strategies (ORS).

The implication of the term security modifying a player's strategy is that no risk of making a wrong guess is taken by assuming the worst case of the player's opponent having perfect measurements or no uncertainty as to the game dynamics. Similarly, the contrary of the term security is the term risk modifying a player's strategy. The implication is that the player takes a risk by trying to include information as to his opponent's uncertainty or his opponent's imperfect measurements in order to calculate his strategy.

Thus, under the previous definitions, the previous work by Rhodes, Ho. etc. (980) (13) considered the measurement risk, system security strategies.

The problem to be solved is formulated in the next section.

Statement of the Problem

In this section, the general nonlinear problem is formulated. The dynamical description for the state of the game is given as the following stochastic differential equation:

$$dx(t) = f\{x(t),u(t),v(t),\theta,t\}dt + g\{x(t),u(t),v(t)\theta,t\}d\beta(t), \quad (1)$$

to be interpreted in the sense of Ito (52).

The variables are defined as follows:

$x(t)\in R^n$ is a vector denoting the state of the game at time t

$u(t)\in U$ where $U \subset R^{m1}$ is a vector denoting the control variables of the pursuer at time t

$v(t) \in V$ where $V \subset R^{m_2}$ is a vector denoting the control variables of the evader at time t

$\theta \in R^{p_1+p_2}$ is a time invariant parameter vector parameterizing the system dynamic matrix and which is partitioned as follows:

$$\theta^T = \{ \theta_p \; : \; \theta_E{}^T \}$$

where

$\theta_p \in R^{p_1}$ is a time invariant parameter vector known to the pursuer but unknown to the evader

$\theta_E \in R^{p_2}$ is a time invariant parameter vector known to the evader but unknown to the pursuer

$f \in R^n$ is a nonlinear system vector

g is a n X m matrix

$d\beta(t)$ is a m vector of zero-mean Brownian motion processes with

$$E\{ d\beta(t) d\beta^T(t) \} = w(t)dt. \tag{2}$$

The initial conditions are assumed known to both players. The initial condition is

$$x(t_0) = x_0$$

where

$$\|x(t_0)\| < \infty. \tag{3}$$

Each player has access to certain information sets that he uses to solve for his strategies. The sets contain the a priori information as to the uncertain parameter sets, any a priori information that he has as to his opponent's uncertainty, and in the perfect information case considered in this chapter, the information set contains the state trajectory of the game. The information set of the pursuer at time t is denoted by $I_p(t)$. Similarly, the information set of the evader at time t is denoted by $I_E(t)$. For every time $t \in [t_0, t_f)$, the information sets $I_p(t)$ and $I_E(t)$ of the pursuer and the evader are, respectively,

$$I_p(t) = p_{\theta_E}(\theta_E) \cup \theta_p \cup p_{E\,\theta_p}(\theta_p) \cup$$

$$p_{E\,\theta_E}(\theta_E) \cup (x(\tau), \tau \in [t_0, t]) \tag{4}$$

and

$$I_E(t) = p_{\theta_p}(\theta_p) \cup \theta_E \cup p_{p\,\theta_E}(\theta_E) \cup$$

$$p_{p\,\theta_p}(\theta_p) \cup (x(\tau), \tau \in [t_0, t]) \tag{5}$$

where

$p_{\theta_E}(\theta_E)$ is the probability density function representing the a priori information known by the pursuer about the unknown parameter vector θ_E,

$p_{\theta_p}(\theta_p)$ is the probability density function representing the a priori information known by the evader about the unknown parameter vector θ_p,

θ_p is the parameter vector known by the pursuer,

θ_E is the parameter vector known by the evader,

$p_{E\,\theta_p}(\theta_p)$ is the probability density function representing any knowledge the pursuer may have about the knowledge the evader possesses as to the parameter vector θ_p,

$p_{p\,\theta_E}(\theta_E)$ is the probability density function representing any knowledge the evader may have about the knowledge the pursuer possesses as to the parameter vector θ_E ,

$p_{E\,\theta_E}(\theta_E)$ is the probability density function representing any knowledge the pursuer has about the evader's knowledge of the pursuer's uncertainty of the parameter vector θ_E ,

$p_{p\,\theta_p}(\theta_p)$ is the probability density function representing any knowledge the evader has about the pursuer's knowledge of the evader's uncertainty of the parameter vector θ_p ,

$x(\tau), \tau\epsilon[t_0, t]$ is a functional that represents the state trajectory.

The significance of the inclusion of the parameter vector θ_p in I_p and the parameter θ_E in I_E is that it is assumed the pursuer and the evader, respectively, have perfect knowledge of these parameters. This could be easily weakened to a knowledge of an a priori probability density function. However, this will not be considered any further since the salient features of the game under uncertainty might be obscured. The significance of the probability density functions representing knowledge that the player's opponent possesses is that the game solution depends upon each player knowing his opponent's knowledge. The worst case or the most secure strategy would occur when the player assumes his opponent has perfect knowledge of the game dynamics. This would yield a security strategy that is most conservative for each player based upon his uncertainty.

An important point to note about the information sets is that $I_p(t) \cap I_E(t) \neq \varphi$ where φ is the null set. It is reasonable to expect that in many applications problems the intersection of the two information sets should be the null set. That is, if

$I_p(t) \cap I_E(t) = \varphi$, $\forall\, t \in [t_0, t_f)$, then each player must choose his strategies from totally different information, a very realistic situation. The perfect information problem insures that $I_p \cap I_E \neq \varphi$ since each player has access to the state trajectory.

The dynamics and information structure is given by Equations (1), (4), and (5). It is assumed that the goals of each player are adequately incorporated in the scalar function known as the performance index, i.e.,

$$J = E\left\{G(x(t_f),\ t_f) + \int_{t_0}^{t_f} Q(x(t),\ u(t),\ v(t),\ t)dt\right\} \tag{6}$$

where $E\{\cdot\}$ denotes the expectation over all random processes under the bracket.

It is assumed that each player chooses deterministic controls and does not randomize his control policy. The final time will be assumed fixed in the developments of this dissertation, i.e.,

$$t_f = \text{constant}. \tag{7}$$

Thus, the performance index in Equation (6) is a functional mapping the state space and control space into the reals, i.e.,

$$J : R^n \times U \times V \rightarrow R^1. \tag{8}$$

Each player must choose closed-loop control laws as was explained previously. Thus, the pursuer must at each time $t \in [t_0, t_f)$ find the function mapping the information set available to him at time t into the admissible control set such that the performance index is minimized, i.e.,

$$u^* : \{x(\tau),\ \tau \in [t_0,\ t]\} \times [t_0, t] \rightarrow U \subset R^{m_2}. \tag{9}$$

Similarly, the evader must each time $t \in [t_0, t_f)$ find the function mapping the information set available to him at time t into the admissible control set such that the performance index is maximized, i.e.,

$$v^* : \{x(\tau),\ \tau \in [t_0, t]\} \times [t_0, t] \rightarrow V \subset R. \tag{10}$$

The control strategies u* and v* are assumed to be the minimizing and maximizing control strategies, respectively.

The set of admissible controls u is assumed to be a subset of $L_2\{I, R^{m1}\}$ where $I = [t_0, t_f)$, and the set of admissible controls v is assumed to be a subset of $L_2\{J, R^{m2}\}$ where $J = [t_0, t_f)$.

Also, the admissible control set consists of control functions which are nonanticipating. That is, one may define the extension of a function f(s), $s \in [t_0, t)$ as

$$(\pi_t\ f)(s) = \begin{cases} f(s) , & t \leq s \leq t \\ f(t) , & t \leq s \leq t . \end{cases} \tag{11}$$

Thus, the admissible controls u and v are such that

$$\begin{aligned} u(t) &= \psi_p(t,\ \pi_t \mathcal{Z}(t)) \\ v(t) &= \psi_E(t,\ \pi_t \mathcal{Z}(t)) \end{aligned} \tag{12}$$

where $$\mathcal{Z}(t) = \{x(\tau),\ \tau \in [t_0, t]\}.$$

Each player wishes to choose his control strategies such that the following inequalities are satisfied:

$$\begin{aligned} E\{J(u^*, v^*) \mid I_p\} &\leq E\{J(u, v^*) \mid I_p\} \\ E\{J(u^*, v) \mid I_E\} &\leq E\{J(u^*, v^*) \mid I_E\} . \end{aligned} \tag{13}$$

These inequalities will now be explained.

Each player agrees on the basic performance index that he must satisfy. However, each player will not agree as to the value of the optimal cost since he is basing his optimal strategy on different information sets. Thus, the game is similar in nature to a non-zero sum differential game whereby each player is trying to extremize different performance indices. However, since the basic performance index is the same, it is not a non-zero sum game. The cost that the pursuer will calculate to be the optimal cost is $E\{J(u, v) \mid I_p\}$. The cost that the evader will calculate to be the game cost is given by $E\{J(u, v) \mid I_E\}$. The pursuer is trying to minimize the game cost while conservatively choosing his strategy such that he obtains a guaranteed bound on the cost. Similarly, the evader is trying to maximize the game cost while conservatively choosing his strategy such that he obtains a guaranteed bound on the cost.

Neither the pursuer nor the evader can find their opponent's worst case strategies since they do not in general have the totality of the game dynamics. If either player has access to the uncertainty his opponent has as to the game dynamics, then this may reflect into the player's choice of strategy. The pursuer will use the inequality

$$E\{J(u^*, v^{\circleddash}) \mid I_p\} \leq E\{J(u, v^{\circleddash}) \mid I_p\} \tag{14}$$

where

$v^{\circleddash}$ is the evader's maximizing strategy for the pursuer's best guess of the true game

to find his strategy u^*.

The evader will use the inequality

$$E\{J(u^o,v)|I_E\} \leq E\{J(u^o,v^*)|I_E\} \quad (15)$$

where

u^o is the pursuer's minimizing strategy for the evader's best guess of the true game

to find v*. One must note that the evader's (pursuer's) best strategy is not $v^o(u^o)$ since this strategy is based upon an erroneous game and not the true game.

Thus, there will be two performance cost surfaces. The first surface is due to the pursuer minimizing J conditioned on this information set I_P. The second surface is due to the evader maximizing I conditioned on his information set I_E. Each player is, thus, choosing his strategies based upon different cost surfaces. Thus, the game is very similar to a non-zero sum game. One may note that, in general, the basic definition of a Nash equilibrium strategy may not be applied since neither player has enough information, in general, to find the equivalent equilibrium point defined by

$$\begin{aligned} E\{J(u^*,v^*)|I_P\} &\leq E\{J(u,v^*)|I_P\} \\ E\{J(u^*,v)|I_E\} &\leq E\{J(u^*,v^*)|I_E\}. \end{aligned} \quad (16)$$

If enough information were available, say to a third player, then the above would define an equilibrium strategy.

In order to find the most conservative strategies based upon a particular game, each player may assume the worst case. That is, that their opponent has certain knowledge of all the game parameters. This will give them a conservative guaranteed cost based upon their information sets. However, in a similar manner as to the measurement risk, system security strategies one may take into account the opponent's uncertainty.

Sufficiency Condition

In this section, a Hamilton-Jacobi equation is derived for the general problem of differential games under uncertainty. The results are used later in order to find the optimal strategies of the problem posed in this chapter.

The first use of the Hamilton-Jacobi equation was made by Issacs (50). In his book, the Hamilton-Jacobi equation is called the main equation. The Hamilton-Jacobi approach has been used by Maguiraga (64) and by Kushner (55) for the problem of stochastic differential games under the assumption of certainty. Rhodes (80) proved a sufficiency condition for differential games with imperfect information that is similar to a Hamilton-Jacobi approach. Maguiraga considered the problem where the state equations of the game contained both control dependent and state dependent noise. Kushner considered the general problem of a stochastic nonlinear game state equation without random parameters. However, none of the above references consider the problem with uncertainty.

The general problem will be considered first. The problem of major concern of this dissertation will be worked as a special case of the general problem. The state of the game evolves according to Equation (1), repeated below for convenience,

$$dx(t) = f\{x(t), u(t), v(t), \theta, t\}dt + g\{x(t), u(t), v(t), \theta, t\}d\beta(t). \tag{17}$$

The cost function is scalar functional (see Equation (6)).

$$J = E\{G(x(t_f), t_f) \int_{t_0}^{t_f} Q(x(t), u(t), v(t), t)dt\} \tag{18}$$

where the expected value is over all random variables within the bracket. It is assumed that each player chooses deterministic control laws and does not randomize his control policy.

It is assumed that the final time is fixed. The players have access to certain information sets, $I_p(t)$ and $I_E(t)$, where the subscripts p and $\mathcal{E}$ denote the pursuer and evader, respectively. These sets are defined as

$$I_p(t) = \mathcal{B}_p \cup \mathcal{J}$$
$$I_E(t) = \mathcal{B}_E \cup \mathcal{J} \tag{19}$$

where $\mathcal{B}$ denotes the collection of the a priori information as to the parameter sets θ_p and θ_E and $\mathcal{J}$ denotes the state trajectory $(x(\tau), \tau \in [t_0, t])$ (see Equations (4) and (5)).

The pursuer and the evader wish to find their optimal strategies u* and v*, respectively, such that the following saddlepoint inequalities are satisfied $\forall\ t \in [t_0, t_f)$

$$E\{J(u^*(t), v^*(t)) \mid I_p(t)\} \leq E\{J(u(t), v^*(t)) \mid I_p(t)\}$$
$$E\{J(u^*(t), v(t)) \mid I_E(t)\} \leq E\{J(u^*(t), v^*(t)) \mid I_E(t)\}. \tag{20}$$

This reflects the fact that both the pursuer and the evader wish to choose their control strategies such that if their opponent plays optimally then the cost is bounded above or below by some acceptable value (depending upon whether the player is the pursuer or the evader). However, the solution to the above problem depends upon total knowledge of the game dynamics. It is the major topic of the dissertation to consider the problem whereby neither player has total knowledge of the game dynamics. Thus, an alternative problem to the one above must be solved.

The pursuer will find his strategy prior to the start of the game by considering the following inequality:

$$E\{J(u^*, v^\circ) \mid I_p(t)\} \leq E\{J(u(t), v^\circ(t) \mid I_p(t)\} \tag{21}$$

where $v^\circ(t)$ is the evader's optimal strategy based upon the pursuer's assumed game. This is not the strategy that the evader should play since it is for a game different than what the players are actually playing. Similarly, the evader will find his strategy prior to the start of the game by considering the inequality

$$E\{J(u^\circ(t), v(t)) \mid I_E(t)\} \leq E\{J(u^\circ(t), v^*(t)) \mid I_E(t)\}. \tag{22}$$

Again, $u^\circ(t)$ is the evader's best knowledge of the pursuer's optimal strategy. The Hamilton-Jacobi equations will now be derived. The derivation will first be carried out by considering the inequalities (20). This will be done in order to gain insight into this problem. However, the extension to the problem defined by inequalities (2) are easily made by consideration of the type of strategy each player is assuming his opponent is playing. The derivation follows.

The cost that will be incurred by the pursuer in order to terminate the game given that at time t the state of the game is known and given as $x(t)$ is defined as $V_p(x(t), t)$. It is assumed that $V_p(x(t), t)$ is twice continuously differentiable with respect to $x(t)$ and continuously differentiable with respect to t. The cost $V_p(x(t), t)$ is given as

$$V_p(x(t),\ t) = \min_{u \in U} E\{G(x(t_f),\ t_f) + \int_{t}^{t_f} Q(x(t),\ u(t),\ v^*(t),\ t)dt \mid I_p(t)\} \tag{23}$$

where $v^*(t)$ is the optimal strategy of the evader. Similarly, the cost incurred by the evader in order to terminate the game given that at time t the state of the game is known and given as $x(t)$ is defined as $V_E(x(t),\ t)$. It is assumed that $V_E(x(t),\ t)$ is twice continuously differentiable with respect to $x(t)$ and continuously differentiable with respect to t. The cost $V_E(x(t),\ t)$ is given as

$$V_E(x(t),\ t) = \max_{v \in V} E\{G(x(t_f),\ t_f) + \int_{t}^{t_f} Q(x(t),\ u^*(t),\ v(t),\ t)dt \mid I_E(t)\}. \tag{24}$$

The Principle of Optimality (8) allows one to write the costs as

$$V_p(x(t),\ t) = \min_{u \in U} E\{V_p(x(t) + \Delta x,\ t + \Delta t) + \int_{t}^{t+\Delta t} Q(x(t),\ u(t),\ v^*(t),\ t)dt \mid I_p(t)\} \tag{25}$$

$$V_E(x(t),\ t) = \max_{v \in V} E\{V_E(x(t) + \Delta x,\ t + \Delta t) + \int_{t}^{t+\Delta t} Q(x(t),\ u^*(t),\ v(t),\ t)dt \mid I_E(t)\} \tag{26}$$

by the mean value theorem. Since $V_p(x(t),\ t)$ and $V_E(x(t),\ t)$ are twice continuously differentiable in $x(t)$ and continuously differentiable in t, $V_p(x(t) + \Delta x(t),\ t + \Delta t)$ and $V_E(x(t) + \Delta x(t),\ t + \Delta t)$ may be expanded in a Taylor series about $x(t)$ and t. This yields the following expressions:

$$V_p(x(t) + \Delta x(t),\ t + \Delta t) = V_p(x(t),\ t) + \frac{\partial V_p}{\partial t}\Delta t + \frac{\partial V_p^T}{\partial x}\Delta x + \tag{27}$$

$$\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_p}{\partial x_i\,\partial x_j}\Delta x_i\;\Delta x_j + O(\Delta t^2)$$

and

$$V_E(x(t)+\Delta x(t),\ t+\Delta t) = V_E(x(t),\ t) + \frac{\partial V_E}{\partial t}\Delta t + \frac{\partial V_E^T}{\partial x}\Delta x + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_E}{\partial x_i\,\partial x_j}\Delta x_i\;\Delta x_j + O(\Delta t^2). \tag{28}$$

Thus, the cost $V_p(x(t),\ t)$ and $V_E(x(t),\ t)$ may be written as

$$V_p(x(t),\ t) = \min_{u\in U} E\{V_p(x(t),\ t) + \frac{\partial V_p}{\partial t}\Delta t + \frac{\partial V_p^T}{\partial x}\Delta x + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_p}{\partial x_i\,\partial x_j}\Delta x_i\;\Delta x_j + Q(x(t),\ u(t),\ v^*(t),\ t)\,\Delta t + O(\Delta t^2)\ |\ I_p(t)\} \tag{29}$$

and

$$V_E(x(t),\ t) = \max_{v\in V} E\{V_E(x(t),\ t) + \frac{\partial V_E}{\partial t}\Delta t + \frac{\partial V_E^T}{\partial x}\Delta x + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_E}{\partial x_i\,\partial x_j}\Delta x_i\;\Delta x_j + Q(x(t),\ u^*(t),\ v(t),\ t)\,\Delta t + O(\Delta t^2)\ |\ I_E(t)\}. \tag{30}$$

The use of the smoothing property of expectations allows one to write equations (29) and (30) as

$$V_p(x(t),t) = \min_{u\in U}\ \underset{\theta\,|\,I_p(t)}{E}\{E\{V_p(x(t),\ t) + \frac{\partial V_p}{\partial t}\Delta t + \frac{\partial V_p}{\partial x}\Delta x + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_p}{\partial x_i\,\partial x_j}\Delta x_i\;\Delta x_j + Q(x(t),\ u(t),\ v^*(t),\ t)\,\Delta t + O(\Delta t^2)\ |\ \theta,\ I_p(t)\}\} \tag{31}$$

and

$$V_E(x(t),\ t) = \max_{v \in V} \underset{\theta | I_E(t)}{E} \{E\{V_E(x(t),\ t) + \frac{\partial V_E}{\partial t} \Delta t + \frac{\partial V_p^T}{\partial x} \Delta x +$$

$$\tfrac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_E}{\partial x_i\ \partial x_j} \Delta x_i\ \Delta x_j + Q(x(t),\ u^*(t),\ v(t),\ t)\Delta t$$

$$+\ O(\Delta t^2)\ |\ \theta,\ I_E(t)\}\}. \tag{32}$$

The inner expectation may be distributed yielding

$$V_p(x(t),\ t) = \min_{u \in U} \underset{\theta | I_p(t)}{E} \{V_p(x(t),\ t) + \frac{\partial V_p}{\partial t} \Delta t + \frac{\partial V_p^T}{\partial x} E[\Delta x | \theta,\ I_p(t)] +$$

$$\tfrac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_p}{\partial x_i\ \partial x_j} E\{\Delta x_i\ \Delta x_j\ |\ \theta,\ I_p(t)\} + \tag{33}$$

$$Q(x(t),\ u(t),\ v^*(t),\ t)\Delta t + O(\Delta t^2)\}$$

and

$$V_E(x(t),\ t) = \max_{v \in V} \underset{\theta | I_E(t)}{E} \{V_E(x(t),\ t) + \frac{\partial V_E}{\partial t} \Delta t + \frac{\partial V_E^T}{\partial x} E[\Delta x | \theta,\ I_E(t)] +$$

$$\tfrac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_E}{\partial x_i\ \partial x_j} E\{\Delta x_i\ \Delta x_j\ | \theta,\ I_E(t)\} + \tag{34}$$

$$Q(x(t),\ u^*(t),\ v(t),\ t)\Delta t + O(\Delta t^2)\}.$$

One must notice that since the inner expectation is conditioned on x(t) and since the controls u(t) and v(t) are assumed deterministic, i.e., functions of t and x(t) where x(t) is known, several of the terms are deterministic. A similar argument occurs with the outer expectation; however, the distribution of the outer expectation will be deferred until later. The increment Δx must now be considered. The increment may be written as

$$\Delta x(t) = f[x(t),\ u(t),\ v(t),\ \theta,\ t]\ \Delta t + g[x(t),\ u(t),\ v(t),\theta,t]\ \Delta\beta(t). \tag{35}$$

The expectations over the increments may be obtained as follows

$$E[\Delta x(t)\mid \theta,\ I_p(t)] = f[x(t),\ u(t),\ v^*(t),\ \theta,\ t]\Delta t$$
$$E[\Delta x(t)\mid \theta,\ I_E(t)] = f[x(t),\ u^*(t),\ v(t),\ \theta,\ t]\Delta t \tag{36}$$

and

$$E[\Delta x(t)\Delta x^T(t)\mid \theta,\ I_p(t)] = g[x(t),\ u(t),\ v^*(t),\ \theta,\ t] \times w(t)g^T[x(t),\ u(t),\ v^*(t),\ \theta,\ t]\Delta t + O(\Delta t^2)$$
$$E[\Delta x(t)\Delta x^T(t)\mid \theta,\ I_E(t)] = g[x(t),\ u^*(t),\ v(t),\ \theta,\ t] \times w(t)g^T[x(t),\ u^*(t),\ v(t),\ \theta,\ t]\Delta t + O(\Delta t^2). \tag{37}$$

Thus, the expressions in Equations (33) and (34) may be rewritten as the following:

$$V_p(x(t),\ t) = \min_{u\in U}\ \underset{\theta\mid I_p(t)}{E}\Big\{V_p(x(t),\ t) + \frac{\partial V_p}{\partial t}\Delta t + \frac{\partial V_p^T}{\partial x} f\{x(t),\ u(t),\ v^*(t),\ \theta,\ t\}\Delta t + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_p}{\partial x_i\ \partial x_j} m_{ij}\Big|_{v^*}\Delta t + Q(x(t),\ u(t),\ v^*(t),\ t)\Delta t + O(\Delta t^2)\Big\} \tag{38}$$

and

$$V_E(x(t),\ t) = \max_{v\in V}\ \underset{\theta\mid I_E(t)}{E}\Big\{V_E(x(t),\ t) + \frac{\partial V_E}{\partial t}\Delta t + \frac{\partial V_E^T}{\partial x} f\{x(t),\ u^*(t),\ v(t),\ \theta,\ t\}\Delta t + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_E}{\partial x_i\ \partial x_j} m_{ij}\Big|_{u^*}\Delta t + \tag{39}$$

$$Q(x(t),\ u^*(t),\ v(t),\ t)\Delta t + O(\Delta t^2)\}$$

where m_{ij} corresponds to the i, j-th element of the following matrix:

$$\{m_{ij}\} = g[x(t),\ u(t),\ v(t),\ \theta,\ t]w(t)g^T[x(t),\ u(t),\ v(t),\ \theta,\ t]. \tag{40}$$

Both sides of Equations (38) and (39) may be divided by Δt and the limit taken as $\Delta t \to 0$. This yields the following partial differential equations for the cost for both the pursuer and the evader:

$$\min_{u \in U} \ \underset{\theta | I_p(t)}{E} \left\{ \frac{\partial V_p}{\partial t} + \frac{\partial V_p^T}{\partial x} f\{x(t),\ u(t),\ v^*(t),\ \theta,\ t\} + \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_p}{\partial x_i\ \partial x_j} m_{ij}\Big|_{v^*} + Q(x(t),\ u(t),\ v^*(t),\ t) \right\} = 0 \tag{41}$$

and

$$\max_{v \in V} \ \underset{\theta | I_E(t)}{E} \left\{ \frac{\partial V_E}{\partial t} + \frac{\partial V_E^T}{\partial x} f\{x(t),\ u^*(t),\ v(t),\ \theta,\ t\} + \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_E}{\partial x_i\ \partial x_j} m_{ij}\Big|_{u^*} + Q(x(t),\ u^*(t),\ v(t),\ t) \right\} = 0. \tag{42}$$

In order to obtain a shorthand notation for the Hamilton-Jacobi equation, one may define the modified differential generator as

$$\mathscr{L}_p = \sum_{i=1}^{n} f_i\{x(t),\ u(t),\ v^*(t),\ \theta,\ t\} \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}\Big|_{v^*} \frac{\partial^2}{\partial x_i\ \partial x_j} \tag{43}$$

and

$$\mathscr{L}_E = \sum_{i=1}^{n} f_i\{x(t),\ u^*(t),\ v(t),\ \theta,\ t\} \frac{\partial}{\partial x_i} \quad \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}\Big|_{u^*} \frac{\partial^2}{\partial x_i\ \partial x_j} \tag{44}$$

Thus, one may write the Hamilton-Jacobi equations as

$$\min_{u \in U} \underset{\theta | I_p}{E} \left\{ \frac{\partial V_p}{\partial t} + \mathcal{L}_p V_p + Q(x, u, v^*, t) \right\} = 0 \tag{45}$$

and

$$\max_{v \in V} \underset{\theta | I_E}{E} \left\{ \frac{\partial V_E}{\partial t} + \mathcal{L}_E V_E + Q(x, u^*, v, t) \right\} = 0 \tag{46}$$

where the boundary conditions are

$$V_p(x(t_f), t_f) = V_E(x(t_f), t_f) = G(x(t_f), t_f). \tag{47}$$

It will now be proved that the Hamilton-Jacobi equations are sufficient for an optimal strategy.

Theorem 3.1: It is sufficient that there exists two scalar functions V_p and V_E where

$$V_p(x(t), t) : R^n \times [t_0, t] \rightarrow R^1 \tag{48}$$

and

$$V_E(x(t), t) : R^n \times [t_0, t] \rightarrow R^1 \tag{49}$$

in order to solve for the closed loop optimal strategies u* and v*. The functions V_p and V_E are twice continuously differentiable in x(t) and continuously differentiable in t. The functions are defined as the solutions to the following equations:

$$L_p = \frac{\partial V_p}{\partial t} + \mathcal{L}_p V_p + Q(x(t), u(t), v^*(t), t) \tag{50}$$

and

$$L_E = \frac{\partial V_E}{\partial t} + \mathcal{L}_E V_E + Q(x(t), u^*(t), v(t), t) \tag{51}$$

where the differential generators are as shown in Equations (43) and (44) and

$$|V_p| + |V_{p_t}| + |x||V_{p_x}| + |x|^2|V_{p_{xx}}| < C(1 + |x|^2)$$

$$|V_E| + |V_{E_t}| + |x||V_{E_x}| + |x|^2|V_{E_{xx}}| < C(1 + |x|^2).$$

The boundary conditions for the above equations are

$$V_p(x(t_f), t_f) = V_E(x(t_f), t_f) = G(x(t_f), t_f). \tag{52}$$

The functions L_p and L_E are such that

$$\min_{u \in U} E\{L_p(x(t), u(t), v^*(t), \theta, t) \mid I_p(t)\} = 0 \tag{53}$$

and

$$\max_{v \in V} E\{L_E(x(t), u^*(t), v(t), \theta, t) \mid I_E(t)\} = 0 \tag{54}$$

Proof: Consider any $u \in U$, then

$$E\{L_p(x(t), u(t), v^*(t), \theta, t) \mid I_p(t)\} \geq 0. \tag{55}$$

One may take an additional expectation conditioned on the information set $I_p(\tau)$, $\tau \leq t$. This yields

$$E\{E\{L_p\ (x(t), u(t), v^*(t), \theta, t) | I_p(t)\} \mid I_p(\tau)\} \geq 0. \tag{56}$$

The expected value operators may be interchanged

$$E\{E\{L_p(x(t), u(t), v^*(t), \theta, t) \mid I_p(\tau)\} | I_p(t)\} \geq 0. \tag{57}$$

Since $I_p(\tau) \subset I_p(t)$, this inequality may be rewritten as

$$E\{L_p(x(t), u(t), v^*(t), \theta, t) \mid I_p(\tau)\} \geq 0. \tag{58}$$

This may be integrated over $[\tau, t_f]$, i.e.,

$$\int_{\tau}^{t_f} E\{L_p(x(t), u(t), v^*(t), \theta, t) \mid I_p(\tau)\}dt \geq 0. \tag{59}$$

The integration and expectation may be interchanged

$$E\left\{ \int_{\tau}^{t_f} L_p(x(t),\ u(t),\ v^*(t),\ \theta,\ t)dt \mid I_p(\tau) \right\} \geq 0. \tag{60}$$

The definition of L_p may now be used to give

$$E\left\{ \int_{\tau}^{t_f} \left(\frac{\partial V_p}{\partial t} + \mathcal{L}_p V_p + Q(x(t),\ u(t),\ v^*(t),\ t) \right) dt \mid I_p(\tau) \right\} \geq 0 \tag{61}$$

or

$$E\left\{ \int_{\tau}^{t_f} \left(\frac{\partial V_p}{\partial t} + \mathcal{L}_p V_p \right) dt \mid I_p(\tau) \right\} \geq - E\left\{ \int_{\tau}^{t_f} Q(x(t),\ u(t),\ v^*(t),\ t)dt \mid I_p(\tau) \right\}. \tag{62}$$

The integrand of the lefthand side is the total derivative of V_p with respect to time. Thus, this equation may be written as

$$E\left\{ \int_{\tau}^{t_f} \left(\frac{dV_p}{dt} \right) dt \mid I_p(\tau) \right\} \geq -E\left\{ \int_{\tau}^{t_f} Q(x(t),\ u(t),\ v^*(t),\ t)dt \mid I_p(\tau) \right\}. \tag{63}$$

Thus,

$$E\left\{ (V_p(x(t_f),\ t_f) - V_p(x(\tau),\ \tau)) \mid I_p(\tau) \right\} \geq -E\left\{ \int_{\tau}^{t_f} Q(x(t),\ u(t),\ v^*(t),\ t)dt \mid I_p(\tau) \right\}. \tag{64}$$

However,

$$V_p(x(t_f),\ t_f) = G(x(t_f),\ t_f). \tag{65}$$

Thus,

$$E\{ (G(x(t_f),\ t_f) - V_p(x(\tau),\ \tau)) \mid I_p(\tau) \} \geq \tag{66}$$

$$-E\left\{\int_{\tau}^{t_f} Q(x(t),\ u(t),\ v^*(t),\ t)dt \mid I_p(\tau)\right\}.$$

$$E\left\{V_p(x(\tau),\ \tau) \mid I_p(\tau)\right\} \leq E\left\{G(x(t_f),\ t_f) + \int_{\tau}^{t_f} Q(x(t),\ u(t),\ v^*(t),\ t)dt \mid I_p(\tau)\right\}. \quad (67)$$

The second term is the expected value of the cost $J(u,\ v^*,\ \tau)$ where τ denotes that the cost is over the interval $[\tau,\ t_f]$. Since τ is arbitrary, τ can range over the interval $[t_0,\ t_f]$. Thus, Equation (67) may be written as

$$E\{V_p(x(\tau),\ \tau) \mid I_p(\tau)\} \leq E\{J(u,\ v^*,\ \tau) \mid I_p(\tau)\}. \quad (68)$$

If one used the optimal strategy u^*, then the inequality becomes an equality.

$$E\{V_p(x(\tau),\ \tau) \mid I_p(\tau)\} = E\{J(u^*,\ v^*,\ \tau) \mid I_p(\tau)\}. \quad (69)$$

Thus,

$$E\{J(u^*,\ v^*,\ \tau) \mid I_p(\tau)\} \leq E\{J(u,\ v^*,\ \tau) \mid I_p(\tau)\}, \quad (70)$$

$$\forall\ \tau \in [t_0,\ t_f].$$

Thus, the equations are proved for the pursuer. The equations will be proved for the evader. Consider and $v \in V$, then

$$E\{L_E(x(t),\ u^*(t),\ v(t),\ \theta,\ t) \mid I_E(t)\} \leq 0. \quad (71)$$

One may take an additional expectation conditioned on the information set $I_E(\tau)$, $\tau \leq t$. This yields

$$E\{E\{L_E(x(t),\ u^*(t),\ v(t),\ \theta,\ t) \mid I_E(\tau)\} \mid I_E(t)\} \leq 0. \quad (72)$$

Since $I_\varepsilon(\tau) \subset I_\varepsilon(t)$, the inequality may be rewritten as

$$E\{L_\varepsilon(x(t),\ u^*(t),\ v(t),\ \theta,\ t) \mid I_\varepsilon(\tau)\} \leq 0. \tag{73}$$

This may be integrated over $[\tau,\ t_f]$, i.e.,

$$\int_\tau^{t_f} E\{L_\varepsilon(x(t),\ u^*(t),\ v(t),\ \theta,\ t) \mid I_\varepsilon(\tau)\}\ dt \leq 0. \tag{74}$$

The integration and expectation may be interchanged

$$E\left\{\int_\tau^{t_f} L_\varepsilon(x(t),\ u^*(t),\ v(t),\ \theta,\ t)\ dt \mid I_\varepsilon(\tau)\right\} \leq 0. \tag{75}$$

The definition of L_ε may now be used to give

$$E\left\{\int_\tau^{t_f} \left(\frac{\partial V_\varepsilon}{\partial t} + \mathcal{L}_\varepsilon V_\varepsilon + Q(x(t),\ u^*(t),\ v(t),\ t)\right) dt \mid I_\varepsilon(\tau)\right\} \leq 0. \tag{76}$$

or

$$E\left\{\int_\tau^{t_f} \left(\frac{\partial V_\varepsilon}{\partial t} + \mathcal{L}_\varepsilon V_\varepsilon\right) dt \mid I_\varepsilon(\tau)\right\} \leq -E\left\{\int_\tau^{t_f} Q(x(t),\ u^*(t),\ v(t),t)dt \mid I_\varepsilon(\tau)\right\}. \tag{77}$$

The integrand of the left-hand side is the total derivative of V_ε with respect to time. Thus, this equation may be written as

$$E\left\{\int_\tau^{t_f} \left(\frac{dV_\varepsilon}{dt}\right) dt \mid I_\varepsilon(\tau)\right\} \leq -E\left\{\int_\tau^{t_f} Q(x(t),\ u^*(t),\ v(t),\ t)\ dt \mid I_\varepsilon(\tau)\right\}. \tag{78}$$

Thus,

$$E\left\{(V_\varepsilon(x(t_f),\ t_f) - V_\varepsilon(x(\tau),\ \tau)) \mid I_\varepsilon(\tau)\right\} \leq -E\left\{\int_\tau^{t_f} Q(x(t),\ u^*(t),\ v(t),\ t)\ dt \mid I_\varepsilon(\tau)\right\}. \tag{79}$$

However,

$$V_\varepsilon(x(t_f),\ t_f) = G(x(t_f),\ t_f). \tag{80}$$

Thus,

$$E\{(G(x(t_f), t_f) - V_E(x(\tau), \tau)) \mid I_E(\tau)\} \leq$$

$$-E\left\{\int_{\tau}^{t_f} Q(x(t), u^*(t), v(t), t)\, dt \mid I_E(\tau)\right\} \tag{81}$$

or

$$E\left\{V_E(x(\tau), \tau) \mid I_E(\tau)\right\} \geq E\left\{G(x(t_f), t_f) + \right.$$

$$\left.\int_{\tau}^{t_f} Q(x(t), u^*(t), v(t), t)\, dt \mid I_E(\tau)\right\}. \tag{82}$$

The second term is the expected value of the cost $J(u^*, v, \tau)$ where τ denotes that the cost is over the interval $[\tau, t_f]$. Since τ is arbitrary, τ can range over the interval $[t_0, t_f]$. Thus, Equation (82) may be written as

$$E\{V_E(x(\tau), \tau) \mid I_E(\tau)\} \geq E\{J(u^*, v, \tau) \mid I_E(\tau)\}. \tag{83}$$

If one used the optimal strategy v^*, then the inequality becomes an equality.

$$E\{V_E(x(\tau), \tau) \mid I_E(\tau)\} = E\{J(u^*, v^*, \tau) \mid I_E(\tau)\}. \tag{84}$$

Thus,

$$E\{J(u^*, v, \tau) \mid I_E(\tau)\} \leq E\{J(u^*(t), v^*(t)) \mid I_E(\tau)\},$$

$$\forall\ \tau \in [t_0, t_f]. \tag{85}$$

Sufficiency is thus proved.

Linear, Quadratic Problem

The problem of main concern in this chapter will now be explored. The system dynamics are assumed to be adequately represented as

$$\dot{x}(t) = F(t, \theta_p, \theta_E)x(t) + G_p(t)\,u(t) + G_E(t)\,v(t) + w(t) \tag{86}$$

where

$x(t) \in R^n$ is a vector denoting the state of the game at time t

$u(t) \in U$ where $U \subset R^{m1}$ is a vector denoting the control variables of the pursuer at time t

$v(t) \in V$ where $V \subset R^{m2}$ is a vector denoting the control variables of the evader at time t

$F(t, \theta_p, \theta_E)$ is a $n \times n$ matrix parameterized by θ_p and θ_E with continuous and bounded elements

$\theta_p \in R^{p1}$ is a time invariant parameter vector known to the pursuer but unknown to the evader

$\theta_E \in R^{p2}$ is a time invariant parameter vector known to the evader but unknown to the pursuer

$w(t) \in R^n$ is a vector of white noise inputs corrupting the system model, assumed Gaussian with known statistics

$$E\{w(t)\} = 0$$
$$E\{w(t)w^T(\tau)\} = W(t)\delta(t - \tau) \tag{87}$$

$G_p(t)$ is a $n \times m_1$ control gain matrix for the pursuer

$G_E(t)$ is a $n \times m_2$ control gain matrix for the evader.

The initial conditions are assumed known to both players. The initial condition is

$$x(t_0) = x_0.$$

The performance index is

$$J = \tfrac{1}{2}E\{x^T(t_f)S(t_f)\,x(t_f) + \int_{t_0}^{t_f} (x^T(t)Q(t)\,x(t) + u^T(t)\,R_p(t)\,u(t) + v^T(t)\,R_E(t)\,v(t))\,dt\} \tag{88}$$

where

$S(t_f)$ is a $n \times n$ positive semi-definite, symmetric matrix

$Q(t)$ is a $n \times n$ positive semi-definite, symmetric matrix

$R_p(t)$ is a $m_1 \times m_1$ positive definite, symmetric matrix

$R_E(t)$ is a $m_2 \times m_2$ negative definite, symmetric matrix.

The information structure is as given in Equations (4) and (5), i.e.,

$$I_p(t) = p_{\theta_E}(\theta_E) \cup \theta_p \cup p_{E\,\theta_p}(\theta_p) \cup p_{E\,\theta_E}(\theta_E) \cup$$
$$(x(\tau),\ \tau \in [t_0,\ t]) \qquad (89)$$

$$I_E(t) = p_{\theta_p}(\theta_p) \cup \theta_E \cup p_{p\,\theta_E}(\theta_E) \cup p_{p\,\theta_p}(\theta_p) \cup$$
$$(x(\tau),\ \tau \in [t_0,\ t]) \qquad (90)$$

where the variables are as defined on page 35. The first problem to be considered is the solution for the system security strategies. The information structure is given as follows for this problem.

$$I_p(t) = p_{\theta_E}(\theta_E) \cup \theta_p \cup (x(\tau),\ \tau \in [t_0,\ t]) \qquad (91)$$

and

$$I_E(t) = p_{\theta_p}(\theta_p) \cup \theta_E \cup (x(\tau),\ \tau \in [t_0,\ t]). \qquad (92)$$

Theorem 3.2: The measurement security, system security, opponent security strategies u* and v* for the pursuer and the evader, respectively, for the system defined in Equation (86), the cost in Equation (88), and the information structure as in Equations (89) and (90) are given as

$$u^*(t) = -R_p^{-1}(t)G_p^T(t)_{\theta_p} E_{,\theta_E \mid I_p(t)}\{P_p(t)\}x(t)$$

$$v^*(t) = -R_E^{-1}(t)G_E^T(t)\underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\}x(t) \tag{93}$$

where $P_p(t, \theta_p, \theta_E)$ is given as the solution to the integro-partial differential equation

$$\frac{\partial P_p(t)}{\partial t} = -P_p(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T P_p(t) + P_p(t)G_p(t)$$

$$\cdot R_p^{-1}(t)G_p^T(t)\underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\} + \underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)$$

$$\cdot P_p(t) + P_p(t)G_E(t)R_E^{-1}(t)G_E^T(t)T_E(t) + T_E(t)G_E(t)R_E^{-1}(t)$$

$$\cdot G_E^T(t)P_p(t) - \underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)\underset{\theta_p,\theta_E|I_p(t)}{E}\{$$

$$P_p(t)\} - T_E(t)G_E(t)R_E^{-1}(t)G_E^T(t)T_E(t) - Q(t) \tag{94}$$

and

$$\frac{\partial T_E(t)}{\partial t} = -T_E(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_E(t) + \tag{95}$$

$$T_E(t)\{G_p(t)R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T_E(t) - Q(t)$$

and where $P_E(t, \theta_p, \theta_E)$ is given as the solution to the integro-partial differential equation

$$\frac{\partial P_E(t)}{\partial t} = -P_E(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T P_E(t) +$$

$$P_E(t)G_E(t)R_E^{-1}(t)G_E^T(t)\underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\} +$$

$$\underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)P_E(t) +$$

$$P_E(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_p(t) + T_p(t)G_p(t)R_p^{-1}(t)G_p^T(t)P_E(t) -$$

$$T_p(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_p(t) - \underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\}G_E(t)R_E^{-1}(t)$$

$$\cdot\ G_E^T(t)\ {}_{\theta_p,\theta_E}E_{\theta_E}\,|\,I_E(t)\{P_E(t)\} - Q(t) \tag{96}$$

and

$$\frac{\partial T_p(t)}{\partial t} = -T_p(t)F(t,\ \theta_p,\ \theta_E) - F(t,\ \theta_p,\ \theta_E)^T T_p(t) + \tag{97}$$

$$T_p(t)\{G_p(t)R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T_p(t) - Q(t)$$

with boundary conditions

$$P_p(t_f) = P_E(t_f) = T_p(t_f) = T_E(t_f) = S(t_f)$$

The expected values used are the best values or estimates of the parameter values θ_p and θ_E since each player assumes the other player has perfect knowledge of the game dynamics but is playing a security strategy.

Proof: The Hamilton-Jacobi equations may be written as follows:

$$\min_{u \in U}\ {}_{\theta_p,\theta_E}E_{\theta_E}\,|\,I_p(t)\Big\{\frac{\partial V_p}{\partial t} + \frac{\partial V_p^T}{\partial x}\Big[F(t,\ \theta_p,\ \theta_E)x(t) + G_p(t)u(t) +$$

$$G_E(t)v^{*T}(t)\Big] + \tfrac{1}{2}(x^T(t)Q(t)x(t) + u^T(t)R_p(t)u(t) + \tag{98}$$

$$v^{*T}(t)R_E(t)v^*(t) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_p}{\partial x_i \partial x_j} m_{ij}\Big\} = 0$$

where

$$m_{ij} = \{W(t)\}_{ij}$$

and

$$\max_{v \in V}\ {}_{\theta_p,\theta_E}E_{\theta_E}\,|\,I_E(t)\Big\{\frac{\partial V_E}{\partial t} + \frac{\partial V_E^T}{\partial x}[F(t,\ \theta_p,\ \theta_E)x(t) + G_p(t)u^*(t) +$$

$$G_E(t)v(t)] + \tfrac{1}{2}(x^T(t)Q(t)x(t) + u^{*T}(t)R_p(t)u^*(t) + \tag{99}$$

$$v^T(t)R_E(t)v(t)) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_E}{\partial x_i \partial x_j} m_{ij}\Big\} = 0.$$

The minimization over $u \in U$ of the first equation yields

$$u(t) = -R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E}\left\{\frac{\partial V_p}{\partial x}\right\} \tag{100}$$

and maximization over $v \in V$ of the second equation yields

$$v(t) = -R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E | I_E(t)}{E}\left\{\frac{\partial V_E}{\partial x}\right\}. \tag{101}$$

Since R_p is positive definite, Equation (98) is minimized by u. Similarily, since R_E is negative definite, (99) is maximized by v. The strategies may be substituted into (98) and (99). This yields the following equations:

$$\underset{\theta_p,\theta_E | I_p(t)}{E}\left\{\frac{\partial V_p}{\partial t} + \frac{\partial V_p^T}{\partial x} F(t, \theta_p, \theta_E)x(t) - \frac{\partial V_p^T}{\partial x} G_p(t)R_p^{-1}(t)G_p^T(t)\right.$$

$$\cdot \underset{\theta_p,\theta_E | I_p(t)}{E}\left\{\frac{\partial V_p}{\partial x}\right\} - \frac{\partial V_p^T}{\partial x} G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E | I_E(t)}{E}\left\{\frac{\partial V_E}{\partial x}\right\} +$$

$$\tfrac{1}{2}x^T(t)Q(t)x(t) + \tfrac{1}{2}\underset{\theta_p,\theta_E | I_p(t)}{E}\left\{\frac{\partial V_p^T}{\partial x}\right\}G_p(t)R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E}\left\{\frac{\partial V_p}{\partial x}\right\} +$$

$$\tfrac{1}{2}\underset{\theta_p,\theta_E | I_E(t)}{E}\left\{\frac{\partial V_E^T}{\partial x}\right\}G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E | I_E(t)}{E}\left\{\frac{\partial V_E}{\partial x}\right\} +$$

$$\left.\tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_p}{\partial x_i \partial x_j} m_{ij}\right\} = 0 \tag{102}$$

and

$$\underset{\theta_p,\theta_E | I_E(t)}{E}\left\{\frac{\partial V_E}{\partial t} + \frac{\partial V_E^T}{\partial x} F(t, \theta_p, \theta_E)x(t) - \frac{\partial V_E^T}{\partial x} G_p(t)R_p^{-1}(t)G_p^T(t)\right.$$

$$\cdot \underset{\theta_p,\theta_E | I_p(t)}{E}\left\{\frac{\partial V_p}{\partial x}\right\} - \frac{\partial V_E^T}{\partial x} G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E | I_E(t)}{E}\left\{\frac{\partial V_E}{\partial x}\right\} +$$

$$\tfrac{1}{2}x^T(t)Q(t)x(t) + \tfrac{1}{2}\underset{\theta_p,\theta_E | I_p(t)}{E}\left\{\frac{\partial V_p^T}{\partial x}\right\}G_p(t)R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E}\left\{\frac{\partial V_p}{\partial x}\right\} +$$

$$\tfrac{1}{2}\underset{\theta_p,\theta_E\,|\,I_E(t)}{E}\left\{\frac{\partial V_E^T}{\partial x}\right\}G_E(t)R_E^{-1}(t)G_E^T(t)\underset{\theta_p,\theta_E\,|\,I_E(t)}{E}\left\{\frac{\partial V_E}{\partial x}\right\} +$$

$$\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_E}{\partial x_i \partial x_j}m_{ij}\Big\} = 0. \qquad (103)$$

One may note that the Hamilton-Jacobi equations are coupled partial differential equations. The solutions to the equations will be assumed to be

$$V_p(t) = \tfrac{1}{2}x^T P_p(t)x + A_p(t) \qquad (104)$$

$$V_E(t) = \tfrac{1}{2}x^T P_E(t)x + A_E(t).$$

The assumed solutions will be used in the above equations in order to determine the necessary equations for a solution.

The strategies may be written as

$$u(t) = -R_p^{-1}(t)G_p^T(t)\underset{\theta_p,\theta_E\,|\,I_p(t)}{E}\{P_p(t)\}x(t) \qquad (105)$$

$$v(t) = -R_E^{-1}(t)G_E^T(t)\underset{\theta_p,\theta_E\,|\,I_E}{E}\{P_E(t)\}x(t).$$

The use of the assumed solutions in the Hamilton-Jacobi equations yields the following equations:

$$\underset{\theta_p,\theta_E\,|\,I_p(t)}{E}\Big\{\tfrac{1}{2}x^T\Big(\frac{\partial P_p(t)}{\partial t} + P_p(t)F(t,\theta_p,\theta_E) + F^T(t,\theta_p,\theta_E)P_p(t) +$$

$$Q(t) - P_p(t)G_p(t)R_p^{-1}(t)G_p^T(t)\underset{\theta_p,\theta_E\,|\,I_p(t)}{E}\{P_p(t)\} - \underset{\theta_p,\theta_E\,|\,I_p(t)}{E}$$

$$\cdot\{P_p(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)P_p(t) - P_p(t)G_E(t)R_E^{-1}(t)G_E^T(t)\underset{\theta_p,\theta_E\,|\,I_E(t)}{E}\{$$

$$P_E(t)\} - \underset{\theta_p,\theta_E\,|\,I_E(t)}{E}\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)P_p(t) + \underset{\theta_p,\theta_E\,|\,I_p(t)}{E}\{Pp(t)\}\cdot$$

$$\cdot G_p(t)R_p^T(t)\,{}_{\theta_p,\theta_E|I_p(t)}E\{P_p(t)\} + {}_{\theta_p,\theta_E|I_E(t)}E\{P_E(t)\}G_E(t)R_E^{-1}(t)\cdot$$

$$\cdot G_E^T(t)\,{}_{\theta_p,\theta_E|I_E(t)}E\{P_E(t)\}\Big)x + \mathring{A}_p(t) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} P_{p_{ij}} m_{ij}\Big\} = 0 \qquad (106)$$

and

$${}_{\theta_p,\theta_E|I_E(t)}E\Big\{\tfrac{1}{2}x^T\Big(\frac{\partial P_E(t)}{\partial t} + P_E(t)F(t,\theta_p,\theta_E) + F^T(t,\theta_p,\theta_E)P_E(t) -$$

$$P_E(t)G_p(t)R_p^{-1}(t)G_p^T(t)\,{}_{\theta_p,\theta_E|I_p(t)}E\{P_p(t)\} - {}_{\theta_p,\theta_E|I_p(t)}E\{P_p(t)\}\cdot$$

$$\cdot G_p(t)R_p^{-1}(t)G_p^T(t)P_E(t) - P_E(t)G_E(t)R_E^{-1}(t)G_E^T(t)\,{}_{\theta_p,\theta_E|I_E(t)}E\{P_E(t)\} -$$

$${}_{\theta_p,\theta_E|I_E(t)}E\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)P_E(t) + Q(t) +$$

$${}_{\theta_p,\theta_E|I_p(t)}E\{P_p(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)\,{}_{\theta_p,\theta_E|I_p(t)}E\{P_p(t)\} +$$

$${}_{\theta_p,\theta_E|I_E(t)}E\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)\,{}_{\theta_p,\theta_E|I_E}E\{P_E(t)\}\Big)x +$$

$$\mathring{A}_E(t) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} P_{E_{ij}}\, m_{ij}\Big\} = 0. \qquad (107)$$

This yields the following equations which must be solved for P_p and P_E:

$$\frac{\partial P_p(t)}{\partial t} = -P_p(t)F(t,\theta_p,\theta_E) - F^T(t,\theta_p,\theta_E)P_p(t) + P_p(t)G_p(t)R_p^{-1}(t)\cdot$$

$$\cdot G_p^T(t)\,{}_{\theta_p,\theta_E|I_p(t)}E\{P_p(t)\} + {}_{\theta_p,\theta_E|I_p(t)}E\{P_p(t)\}G_p(t)R_p^{-1}(t)\cdot$$

$$\cdot G_p^T(t)P_p(t) + P_p(t)G_E(t)R_E^{-1}(t)G_E^T(t)\,{}_{\theta_p,\theta_E|I_E(t)}E\{P_E(t)\} +$$

$$\underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\} G_E(t) R_E^{-1}(t) G_E^T(t) P_p(t) - \underset{\theta_p,\theta_E}{E} \mid I_p(t)\{P_p(t)\}$$

$$\cdot G_p(t) R_p^{-1}(t) G_p^T(t) \underset{\theta_p,\theta_E}{E} \mid I_p(t)\{P_p(t)\} - \underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\}$$

$$\cdot G_E(t) R_E^{-1}(t) G_E^T(t) \underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\} - Q(t) \tag{108}$$

and

$$\frac{\partial P_E(t)}{\partial t} = -P_E(t)F(t, \theta_p, \theta_E) - F^T(t, \theta_p, \theta_E)P_E(t) + P_E(t)G_p(t)R_p^{-1}(t)$$

$$\cdot G_p^T(t) \underset{\theta_p,\theta_E}{E} \mid I_p(t)\{P_p(t)\} + \underset{\theta_p,\theta_E}{E} \mid I_p(t)\{P_p(t)\} G_p(t) R_p^{-1}(t) \cdot$$

$$\cdot G_p^T(t) P_E(t) + P_E(t) G_E(t) R_E^{-1}(t) G_E^T(t) \underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\} +$$

$$\underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\} G_E(t) R_E^{-1}(t) G_E^T(t) P_E(t) - \underset{\theta_p,\theta_E}{E} \mid I_p(t)$$

$$\cdot \{P_p(t)\} G_p(t) R_p^{-1}(t) G_p^T(t) \underset{\theta_p,\theta_E}{E} \mid I_p(t)\{P_p(t)\} - \tag{109}$$

$$\underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\} G_E(t) R_E^{-1}(t) G_E^T(t) \underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\} - Q(t)$$

with boundary conditions

$$P_p(t_f) = P_E(t_f) = S(t_f)$$

and

$$\mathring{A}_p(t) + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} P_{p_{ij}} \, m_{ij} = 0 \tag{110}$$

$$\mathring{A}_E(t) + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} P_{E_{ij}} \, m_{ij} = 0$$

with boundary conditions

$$A_p(t_f) = A_E(t_f) = 0.$$

The differential equations for A_p and A_E are uncoupled from the problem since they do not affect the control. The equations for P_p and P_E are functions of θ_p and θ_E. Thus, $P_p = P_p(t, \theta_p, \theta_E)$ and $P_E = P_E(t, \theta_p, \theta_E)$. However, the arguments have been suppressed.

One may note that the equations for P_p and P_E are coupled. Thus, the solution depends upon each player knowing his opponent's information set in order to solve for the strategies. In general, neither player has this information. Thus, only a third player with perfect intelligence can solve the game. Therefore, in order to obtain a strategy playable by each player, the players must use the knowledge within his information set to determine the required expected values. However, these strategies may be utilized to compare the strategies obtained by each player using the knowledge in his information set. The strategies may also be used to find playable strategies — ones that are based upon only each player's information sets.

The problem may generate into what Ciletti (24) calls an "infinite well" problem in that if each player chose his strategies based upon counter, ..., counter intelligence, then the amount of information necessary may become infinite. In this theorem, it is assumed that each player is choosing his strategies based upon the assumption that his opponent is playing worst case, i.e., with no attempt to use intelligence in order to independently optimize the game, but is playing with perfect game information.

The required strategies are measurement security, system security, opponent security strategies. Thus, each player assumes his opponent has perfect information, but does not try to independently optimize by use of intelligence.

The pursuer solves the game by assuming the evader is solving the game with no uncertainty. That is, he assumes his Hamilton-Jacobi equation is coupled with the following equation (derived in Appendix B):

$$\frac{\partial V_E}{\partial t} + \frac{\partial V_E^T}{\partial x}\{F(t, \theta_p, \theta_E)x(t) + G_p(t)u(t) + G_E(t)v(t)\} + \tag{111}$$
$$\tfrac{1}{2}(x^T(t)Q(t)x(t) + u^T(t)R_p(t)u(t) + v^T(t)R_E(t)v(t)) = 0$$

with the solution

$$V_E = \tfrac{1}{2}x^T T_E(t)x \tag{112}$$

and

$$u(t) = -R_p^{-1}(t)G_p^T(t)T_E(t)x(t)$$
$$v(t) = -R_E^{-1}(t)G_E^T(t)T_E(t)x(t). \tag{113}$$

This yields the following equations for the pursuer in order to determine P_p:

$$\frac{\partial P_p(t)}{\partial t} = -P_p(t)F(t, \theta_p, \theta_E) - F^T(t, \theta_p, \theta_E)P_p(t) +$$
$$P_p(t)G_p(t)R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E}\{P_p(t)\} +$$
$$\underset{\theta_p,\theta_E | I_p(t)}{E}\{P_p(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)P_p(t) + P_p(t)G_E(t)R_E^{-1}(t)G_E^T(t) \cdot$$
$$\cdot T_E(t) + T_E(t)G_E(t)R_E^{-1}(t)G_E^T(t)P_p(t) - \underset{\theta_p,\theta_E | I_p(t)}{E}\{P_p(t)\}$$
$$\cdot G_p(t)R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E}\{P_p(t)\} - T_E(t)G_E(t)R_E^{-1}(t) \cdot$$
$$\cdot T_E(t) - Q(t) \tag{114}$$

and

$$\frac{\partial T_E(t)}{\partial t} = -T_E(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_E(t) +$$
$$T_E(t)\{G_p(t)R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\} T_E(t) - Q(t). \tag{115}$$

A similar derivation holds for the evader. Thus, the evader solves the following equations to determine $P_E(t)$:

$$\frac{\partial P_E(t)}{\partial t} = -P_E(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T P_E(t) + P_E(t)G_p(t)R_p^{-1}(t) \cdot G_p^T(t)T_p(t) + T_p(t)G_p(t)R_p^{-1}(t)G_p^T(P_E(t) + P_E(t)G_E(t)R_E^{-1}(t) \cdot G_E^T(t) \underset{\theta_p, \theta_E | I_E(t)}{E}\{P_E(t)\} + \underset{\theta_p, \theta_E | I_E(t)}{E}\{P_E(t)\} G_E(t) \cdot R_E^{-1}(t)G_E^T(t)P_E(t) - T_p(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_p(t) - \underset{\theta_p, \theta_E | I_E(t)}{E}\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p, \theta_E | I_E(t)}{E}\{P_E(t)\} - Q(t) \tag{116}$$

and

$$\frac{\partial T_p(t)}{\partial t} = -T_p(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_p(t) + T_p(t)\{G_p(t)R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T_p(t) - Q(t) \tag{117}$$

with boundary conditions

$$P_p(t_f) = P_E(t_f) = T_p(t_f) = T_E(t_f) = S(t_f).$$

The above uses the result developed in Appendix B, i.e., Equation (9) that gives the form of the Riccati equation if each player has no uncertainty in the game dynamics. This gives each player his opponent's security strategy.

Theorem 3.3: The measurement security, system security, opponent security strategies u* and v* for the pursuer and the evader, respectively, for the problem posed in Theorem 3.2 and under the assumption that θ_p and θ_E have a discrete parameter range are given as

$$u^*(t) = -R_p^{-1}(t)G_p^T(t)\left(\sum_{i=1}^{p_2} P_r(\theta_{E_i} | I_p(t))P_{p_i}(t, \theta_p, \theta_{E_i})\right)x(t) \tag{118}$$

$$v^*(t) = -R_E^{-1}(t)G_E^T(t)\left(\sum_{i=1}^{p_1} P_r(\theta_{p_i} | I_E(t))P_{E_i}(t, \theta_{p_i}, \theta_E)\right)x(t)$$

where $P_{p_i}(t, \theta_p, \theta_{E_i})$ is the solution of the following equation evaluated at the i^{th} parameter value:

$$\begin{aligned}\dot{P}_{p_i}(t) = {} & -P_{p_i}(t)F(t, \theta_p, \theta_{E_i}) - F(t, \theta_p, \theta_{E_i})P_{p_i}(t) + P_{p_i}(t)G_p(t) \\ & \cdot R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} | I_p(t))P_{p_j}(t) + \left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} | I_p(t)) \right.\right. \\ & \left.\cdot P_{p_j}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t)P_{p_i}(t) + P_{p_i}(t)G_E(t)R_E^{-1}(t)G_E^T(t) \\ & \cdot T_{E_i}(t) + T_{E_i}(t)G_E(t)R_E^{-1}(t)G_E^T(t)P_{p_i}(t) \\ & \left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} | I_p(t))P_{p_j}(t)\right)G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2} P_r \cdot \right. \\ & \left.(\theta_{E_j} | I_p(t))P_{p_j}(t)\right) - T_{E_i}(t)G_E(t)R_E^{-1}(t)G_E^T(t)T_{E_i}(t) - Q(t)\end{aligned} \tag{119}$$

and

$$\begin{aligned}\dot{T}_{E_i}(t) = {} & -T_{E_i}(t)F(t, \theta_p, \theta_{E_i}) - F(t, \theta_p, \theta_{E_i})^T T_E(t) + T_{E_i}(t)\{G_p(t)R_p^{-1}(t) \\ & \cdot G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T_{E_i}(t) - Q(t), \quad i = 1, 2, \ldots, p_2\end{aligned} \tag{120}$$

and where $P_{E_i}(t, \theta_{p_i}, \theta_E)$ is the solution of the following equation evaluated at the i^{th} parameter value:

$$\begin{aligned}\dot{P}_{E_i}(t) = {} & -P_{E_i}(t)F(t, \theta_{p_i}, \theta_E) - F(t, \theta_{p_i}, \theta_E)^T P_{E_i}(t) + \\ & P_{E_i}(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_{p_i}(t) - T_{p_i}(t)G_p(t)R_p^{-1}(t)G_p^T(P_{E_i}(t) + \\ & P_{E_i}(t)G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} | I_E(t))P_{E_j}(t)\right) +\end{aligned}$$

$$\left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_E(t))P_{E_j}(t)\right) G_E(t)R_E^{-1}(t)G_E^T(t)P_{E_i}(t) -$$

$$T_{p_i}(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_{p_i}(t) - \left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_E(t))P_{E_j}(t)\right)$$

$$\cdot G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_E(t))P_{E_j}(t)\right) - Q(t) \qquad (121)$$

and

$$\dot{T}_{p_i}(t) = -T_{p_i}(t)F(t, \theta_{p_i}, \theta_E) - F(t, \theta_{p_i}, \theta_E)^T T_{p_i}(t) +$$
$$T_{p_i}(t)\{G_p(t)R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\} T_{p_i} - Q(t) \qquad (122)$$

with boundary conditions

$$P_p(t_f) = P_E(t_f) = T_p(t_f) = T_E(t_f) = S(t_f).$$

Proof: The proof follows that of Theorem 3.2 with the use of the definition of the expected value operator over a discrete parameter range.

The next theorem considers the problem of the measurement security, system risk, opponent security strategies. In these strategies, a risk is taken in order to use any knowledge of the player's opponent's uncertainty as to the game dynamics. The information structure for this game is as follows:

$$I_p(t) = p_{\theta_E}(\theta_E) \cup \theta_p \cup p_{E\,\theta_p}(\theta_p) \cup$$
$$(x(\tau), \tau \in [t_0, t]) \qquad (123)$$

and

$$I_E(t) = p_{\theta_p}(\theta_p) \cup \theta_E \cup p_{p\,\theta_E}(\theta_E) \cup$$
$$(x(\tau), \tau \in [t_0, t]). \qquad (124)$$

Theorem 3.4: The measurement security, system risk, opponent security strategies for the system defined in Equation (86), the cost in Equation (88) is given as

$$u^*(t) = -R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E \mid I_p(t)}{E}\{P_p(t)\}x(t) \tag{125}$$

$$v^*(t) = -R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E \mid I_E(t)}{E}\{P_E(t)\}x(t) \tag{126}$$

where $P_p(t)$ is the solution to the integro-partial differential equations given as follows:

$$\frac{\partial P_p(t)}{\partial t} = -P_p(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T P_p(t) + P_p(t)G_p(t)R_p^{-1}(t) \cdot G_p^T(t) \underset{\theta_p,\theta_E \mid I_p(t)}{E}\{P_p(t)\} + \underset{\theta_p,\theta_E \mid I_p(t)}{E}\{P_p(t)\}G_p(t)R_p^{-1}(t) \cdot G_p^T(t)P_p(t) + P_p(t)G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E \mid I_p(t)}{E'}\{T_E(t)\} + \underset{\theta_p,\theta_E \mid I_p(t)}{E'}\{T_E(t)\}G_E(t)R_E^{-1}G_E^T(t)P_p(t) - \underset{\theta_p,\theta_E \mid I_p(t)}{E}\{P_p(t)\} \cdot G_p(t)R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E \mid I_p(t)}{E}\{P_p(t)\} - \underset{\theta_p,\theta_E \mid I_p(t)}{E'}\{T_E(t)\} \cdot G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E \mid I_p(t)}{E'}\{T_E(t)\} - Q(t) \tag{127}$$

and

$$\frac{\partial T_E(t)}{\partial t} = -T_E(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_E(t) + T_E(t)G_p(t)R_p^{-1}(t) \cdot G_p^T(t)T_p(t) + T_p(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_E(t) + T_E(t)G_E(t)R_E^{-1}(t) \cdot G_E^T(t) \underset{\theta_p,\theta_E \mid I_p(t)}{E'}\{T_E(t)\} + \underset{\theta_p,\theta_E \mid I_p(t)}{E'}\{T_E(t)\}G_E(t)R_E^{-1}(t) \cdot G_E^T(t)T_E(t) - T_p(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_p(t) - \underset{\theta_p,\theta_E \mid I_p(t)}{E'}\{T_E(t)\} \cdot G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E \mid I_p(t)}{E'}\{T_E(t)\} - Q(t) \tag{128}$$

and

$$\frac{\partial T_p(t)}{\partial t} = -T_p(t)F(t,\ \theta_p,\ \theta_E) - F(t,\ \theta_p,\ \theta_E)^T T_p(t) + T_p(t)\{G_p(t)R_p^{-1}(t) \cdot G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T_p(t) - Q(t) \tag{129}$$

with boundary conditions

$$P_p(t_f) = P_E(t_f) = T_p(t_f) = T_E(t_f) = S(t_f).$$

The matrix $P_E(t)$ is given as the solution to the following integro-partial differential equation:

$$\begin{aligned}
\frac{\partial P_E(t)}{\partial t} = {} & -P_E(t)F(t,\ \theta_p,\ \theta_E) - F(t,\ \theta_p,\ \theta_E)^T P_E(t) + P_E(t)G_p(t)R_p^{-1}(t) \\
& \cdot G_p^T(t)\ E'_{\theta_p,\theta_E | I_E(t)}\{T_p'(t)\} + E'_{\theta_p,\theta_E | I_E(t)}\{T_p'(t)\}G_p(t)R_p^{-1}(t) \\
& \cdot G_p^T(t)P_E(t) + P_E(t)G_E(t)R_E^{-1}(t)G_E^T(t)\ E_{\theta_p,\theta_E | I_E(t)}\{P_E(t)\} + \\
& E_{\theta_p,\theta_E | I_E(t)}\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)P_E(t) - \\
& E'_{\theta_p,\theta_E | I_E(t)}\{T_p'(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)\ E'_{\theta_p,\theta_E | I_E(t)}\{T_p'(t)\} - \\
& E_{\theta_p^-,\theta_E | I_E(t)}\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)\ E_{\theta_p,\theta_E | I_E(t)}\{P_E(t)\} - \\
& Q(t)
\end{aligned} \tag{130}$$

and

$$\begin{aligned}
\frac{\partial T_p'(t)}{\partial t} = {} & -T_p'(t)F(t,\ \theta_p,\ \theta_E) - F(t,\ \theta_p,\ \theta_E)^T T_p'(t) + T_p'(t)G_p(t) \\
& \cdot R_p^{-1}(t)G_p^T(t)\ E_{\theta_p,\theta_E | I_E(t)}\{T_p'(t)\} + E_{\theta_p,\theta_E | I_E(t)}\{T_p'(t)\}G_p(t) \\
& \cdot R_p^{-1}(t)G_p^T(t)T_p'(t) + T_p'(t)G_E(t)R_E^{-1}(t)G_E^T(t)T_E'(t) +
\end{aligned}$$

$$T_E'(t)G_E(t)R_E^{-1}(t)G_E^T(t)T_p'(t) - \underset{\theta_p,\theta_E \mid I_E(t)}{E}\{T_p'(t)\}G_p(t)$$

$$\cdot R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E \mid I_E(t)}{E}\{T_p'(t)\} - T_E'(t)G_E(t)R_E^{-1}(t)G_E^T(t)$$

$$\cdot T_E'(t) - Q(t) \qquad (131)$$

where

$$\frac{\partial T_E'(t)}{\partial t} = -T_E'(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_E'(t) + T_E'(t)\{G_p(t)$$

$$\cdot R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T_E'(t) - Q(t) \qquad (132)$$

with boundary conditions

$$P_E(t_f) = T_p'(t_f) = T_E'(t_f) = S(t_f).$$

The expected value operators will now be explained. The operator E{ } denotes the expected value for the player's best estimate of the parameters based upon his observation functional. The operator E′{ } denotes the expected value over what the player feels his opponent possesses as the best estimates of the parameters.

Proof: Each player is solving his Hamilton-Jacobi equation for his control based upon the assumption that he has the coupling Hamilton-Jacobi equation his opponent is using to solve for his controls. Since the strategies are system risk, opponent security strategies, the strategies for each player assume that he knows the uncertainty as to the unknown parameter of his opponent and that his opponent is playing a system security strategy. Thus, the coupling equation is of the form as in Theorem 3.2 except that all expected values are conditioned on the player's information set. Thus, the proof is the same as in Theorem 3.2 with the above considerations.

Theorem 3.5: The measurement security, system risk, opponent security strategies for the system defined in Equation (86), the cost in Equation (88) and the information structure as in Theorem 3.4, and under the assumption that θ_p and θ_E have a discrete parameter range are given as

$$u^*(t) = -R_p^{-1}(t)G_p^T(t)\left(\sum_{i=1}^{p_2} P_r(\theta_{E_i} \mid I_p(t))P_{p_i}(t)\right)x(t)$$
$$v^*(t) = -R_E^{-1}(t)G_E^T(t)\left(\sum_{i=1}^{p_1} P_r(\theta_{p_i} \mid I_E(t))P_{E_i}(t)\right)x(t) \tag{133}$$

where $P_{p_i}(t)$ is the solution to the following differential equation

$$\begin{aligned}\dot{P}_{p_i}(t) = {} & -P_{p_i}(t)F(t, \theta_p, \theta_{E_i}) - F(t, \theta_p, \theta_{E_i})^T P_{p_i}(t) + \\ & P_{p_i}(t)G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} \mid I_p(t))P_{p_j}(t)\right) + \\ & \left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} \mid I_p(t))P_{p_j}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t)P_{p_i}(t) + \\ & P_{p_i}(t)G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{k=1}^{p_1} P_r(\theta_{p_k} \mid I_p(t))T_{E_{ik}}(t)\right) + \\ & \left(\sum_{k=1}^{p_2} P_r(\theta_{p_k} \mid I_p(t))T_{E_{ik}}(t)\right) G_E(t)R_E^{-1}(t)G_E^T(t)P_{p_i}(t) - \\ & \left(\sum_{j=1}^{p_1} P_r(\theta_{E_j} \mid I_p(t))P_{p_j}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} \mid I_p(t))\right. \\ & \left.\cdot P_{p_j}(t)\right) - \left(\sum_{k=1}^{p_1} P_r(\theta_{p_k} \mid I_p(t))T_{E_{ik}}(t)\right) G_E(t)R_E^{-1}(t)G_E^T(t) \\ & \cdot\left(\sum_{k=1}^{p_1} P_r(\theta_{p_k} \mid I_p(t))T_{E_{ik}}(t)\right) - Q(t)\end{aligned} \tag{134}$$

and

$$\dot{T}_{E_{ik}}(t) = -T_{E_{ik}}(t)F(t, \theta_{p_k}, \theta_{E_i}) - F(t, \theta_{p_k}, \theta_{E_i})^T T_{E_{ik}}(t) +$$
$$T_{E_{ik}}(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_{p_{ik}}(t) + T_{p_{ik}}(t)G_p(t)R_p^{-1}(t)G_p^T(t)$$

$$\cdot T_{E_{1k}}(t) + T_{E_{1k}}(t)G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_p(t))\right.$$

$$\left.\cdot T_{E_{1j}}(t)\right) + \left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_p(t))T_{E_{1j}}(t)\right) G_E(t)R_E^{-1}(t)G_E^T(t)T_{E_{1k}}(t) -$$

$$T_{p_{1k}}(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_{p_{1k}}(t) - \left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_p(t))T_{E_{1j}}(t)\right)$$

$$\cdot G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_p(t))T_{E_{1j}}(t)\right) - Q(t) \qquad (135)$$

where

$$\dot{T}_{p_{1k}}(t) = -T_{p_{1k}}(t)F(t, \theta_{p_k}, \theta_{E_1}) - F(t, \theta_{p_k}, \theta_{E_1})^T T_{p_{1k}}(t) + \qquad (136)$$

$$T_{p_{1k}}(t)\{G_p(t)R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T_{p_{1k}}(t) - Q(t),$$

$$i = 1, 2, \ldots, p_2$$
$$k = 1, 2, \ldots, p_1$$

with boundary conditions

$$P_{p_1}(t_f) = T_{E_{1k}}(t_f) = T_{p_{1k}}(t_f) = S(t_f).$$

The matrix $P_{E_1}(t)$ is the solution to the following differential equation:

$$\dot{P}_{E_1}(t) = -P_{E_1}(t)F(t, \theta_{p_1}, \theta_E) - F(t, \theta_{p_1}, \theta_E)^T P_{E_1}(t) +$$

$$P_{E_1}(t)G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{k=1}^{p_2} P_r(\theta_{E_k} \mid I_E(t))T'_{p_{1k}}(t)\right) +$$

$$\left(\sum_{k=1}^{p_2} P_r(\theta_{E_k} \mid I_E(t))T'_{p_{1k}}(t)G_p(t)R_p^{-1}(t)G_p^T(t)P_{E_1}(t)\right. + \qquad (137)$$

$$P_{E_1}(t)G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_E(t))P_{E_j}(t)\right) +$$

$$\left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_E(t))P_{E_j}(t)\right) G_E(t)R_E^{-1}(t)G_E^T(t)P_{E_1}(t) -$$

$$\left(\sum_{k=1}^{p_2} P_r(\theta_{E_k} \mid I_E(t))T'_{p_{1k}}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{k=1}^{p_2} P_r(\theta_{E_k} \mid\right.$$

$$\left.\cdot I_E(t)T'_{p_{1k}}(t)\right) - \left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_E(t))P_{E_j}(t)\right) G_E(t)R_E^{-1}(t)$$

$$\cdot G_E^T(t)\left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_E(t))P_{E_j}(t)\right) - Q(t)$$

and

$$\mathring{T}'_{p_{ik}}(t) = -T'_{p_{ik}}(t)F(t, \theta_{p_i}, \theta_{E_k}) - F(t, \theta_{p_i}, \theta_{E_k})^T T'_{p_{ik}}(t) +$$

$$T'_{p_{ik}}(t)G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} | I_E(t))\ T'_{p_{ij}}(t)\right.$$

$$\left(\sum_{j=1}^{n} P_r(\theta_{E_j} | I_E(t))T'_{p_{ij}}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t)T'_{p_{ik}}(t) +$$

$$T'_{p_{ik}}(t)G_E(t)R_E^{-1}(t)G_E^T(t)T'_{E_{ik}}(t) + T'_{E_{ik}}(t)G_E(t)R_E^{-1}(t)$$

$$\cdot G_E^T(t)T'_{p_{ik}}(t) - \left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} | I_E(t))T'_{p_{ij}}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t)$$

$$\cdot\left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} | I_E(t))T'_{p_{ij}}(t)\right) - T'_{E_{ik}}(t)G_E(t)R_E^{-1}(t)$$

$$\cdot G_E^T(t)T'_{E_{ik}}(t) - Q(t) \qquad (138)$$

where

$$\mathring{T}'_{E_{ik}}(t) = -T'_{E_{ik}}(t)F(t, \theta_{p_i}, \theta_{E_k}) - F(t, \theta_{p_i}, \theta_{E_k})^T T'_{E_{ik}}(t) + \qquad (139)$$

$$T'_{E_{ik}}(t)\{G_p(t)R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T'_{E_{ik}} - Q(t)$$

$$i = 1, 2, \ldots, p_1$$

$$k = 1, 2, \ldots, p_2$$

with boundary conditions

$$P_{p_i}(t_f) = T'_{E_{ik}}(t_f) = T'_{p_{ik}}(t_f) = S(t_f).$$

The interpretation of the adaptive feature on parameters known to each player is that each player is trying to reconstruct his opponent's best estimate of the parameters unknown to him.

Proof: The use is made of Theorem 3.4 and the definitions of the expected value operation over a discrete range.

Theorem 3.6: The measurement security, system risk, opponent risk strategies u* and v* for the pursuer and the evader, respectively, for the system defined in Equation (86), the cost in Equation (88), and the information structure as in Equations (89) and (90) are given as

$$u^*(t) = -R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E}\{P_p(t)\}x(t) \tag{140}$$

$$v^*(t) = -R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E | I_E(t)}{E}\{P_E(t)\}x(t)$$

where $P_p(t)$ is given as the solution to the following integro, partial differential equation:

$$\begin{aligned}\frac{\partial P_p(t)}{\partial t} = {} & -P_p(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T P_p(t) + P_p(t)G_p(t)R_p^{-1}(t) \\ & \cdot G_p^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E}\{P_p(t)\} + \underset{\theta_p,\theta_E | I_p(t)}{E}\{P_p(t)\}G_p(t) \\ & \cdot R_p^{-1}(t)G_p^T(t)P_p(t) + P_p(t)G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E'}\{T_E(t)\} + \\ & \underset{\theta_p,\theta_E}{E'}\{T_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)P_p(t) - \underset{\theta_p,\theta_E | I_p(t)}{E}\{P_p(t)\}G_p(t) \\ & \cdot R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E}\{P_p(t)\} - \underset{\theta_p,\theta_E | I_p(t)}{E'}\{T_E(t)\}G_E(t) \\ & \cdot R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E'}\{T_E(t)\} - Q(t)\end{aligned} \tag{141}$$

and

$$\begin{aligned}\frac{\partial T_E(t)}{\partial t} = {} & -T_E(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_E(t) + \\ & T_E(t)G_p(t)R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E''}\{T_p(t)\} + \\ & \underset{\theta_p,\theta_E | I_p(t)}{E''}\{T_p(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)T_E(t) + T_E(t)G_E(t)R_E^{-1}(t) \\ & \cdot G_E^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E'}\{T_E(t)\} + \underset{\theta_p,\theta_E | I_p(t)}{E'}\{T_E(t)\}G_E(t)R_E^{-1}(t) \\ & \cdot G_E^T(t)T_E(t) - \underset{\theta_p,\theta_E | I_p(t)}{E''}\{T_p(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)\end{aligned}$$

$$\cdot {}_{\theta_p,\theta_E}E''_{\theta_E} \big| I_p(t)\{T_p(t)\} - {}_{\theta_p,\theta_E}E'_{\theta_E} \big| I_p(t)\{T_E(t)\}G_E(t)R_E^{-1}(t)G_E^T \cdot$$

$$\cdot {}_{\theta_p,\theta_E}E'_{\theta_E} \big| I_p(t)\{T_E(t)\} - Q(t) \qquad (142)$$

where

$$\frac{\partial T_p(t)}{\partial t} = -T_p(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_p(t) + T_p(t)G_p(t) \cdot$$

$$\cdot R_p^{-1}(t)G_p^T(t) \; {}_{\theta_p,\theta_E}E''_{\theta_E} \big| I_p(t)\{T_p(t)\} + {}_{\theta_p,\theta_E}E''_{\theta_E} \big| I_p(t)\{T_p(t)\}G_p(t) \cdot$$

$$\cdot R_p^{-1}(t)G_p^T(t)T_p(t) + T_p(t)G_E(t)R_E^{-1}(t)G_E^T(t)T_E'(t) + T_E'(t) \cdot$$

$$\cdot G_E(t)R_E^{-1}(t)G_E^T(t)T_p(t) - {}_{\theta_p,\theta_E}E''_{\theta_E} \big| I_p(t)\{T_p(t)\}G_p(t)R_p^{-1}(t)$$

$$G_p^T(t) \; {}_{\theta_p,\theta_E}E''_{\theta_E} \big| I_p(t)\{T_p(t)\} - T_E'(t)G_E(t)R_E^{-1}(t)G_E^T(t)T_E'(t) -$$

$$Q(t) \qquad (143)$$

and

$$\frac{\partial T_E'(t)}{\partial t} = -T_E'(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_E'(t) + \qquad (144)$$

$$T_E'(t)\{G_p(t)R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T_E'(t) - Q(t)$$

with boundary conditions

$$P_p(t_f) = T_E(t_f) = T_p(t_f) = T_E'(t_f) = S(t_f).$$

Similarly, the matrix $P_E(t)$ is obtained as the solution to the integro, partial differential equations:

$$\frac{\partial P_E(t)}{\partial t} = -P_E(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T P_E(t) + P_E(t)G_p(t) \cdot$$

$$\cdot R_p^{-1}(t)G_p^T(t) \; {}_{\theta_p,\theta_E}E'_{\theta_E} \big| I_E(t)\{T_p{*}(t)\} + {}_{\theta_p,\theta_E}E'_{\theta_E} \big| I_E\{T_p{*}(t)\}G_p(t) \cdot$$

$$\cdot R_p^{-1}(t)G_p^T(t)P_E(t) + P_E(t)G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\} +$$

$$\underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)P_E(t) - \tag{145}$$

$$\underset{\theta_p,\theta_E}{E'} \mid I_E(t)\{T_p^*(t)\}G_p(t)R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E}{E'} \mid I_E(t)\{T_p^*(t)\} -$$

$$\underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E}{E} \mid I_E(t)\{P_E(t)\} - Q(t);$$

and

$$\frac{\partial T_p^*(t)}{\partial t} = -T_p^*(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_p^*(t) + T_p^*(t)G_p(t)$$

$$\cdot R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\theta_E}{E'} \mid I_E(t)\{T_p^*(t)\} + \underset{\theta_p,\theta_E}{E'} \mid I_E(t)\{T_p^*(t)\}G_p(t)$$

$$\cdot R_p^{-1}(t)G_p^T(t)T_p^*(t) + T_p^*(t)G_E(t)R_E^{-1}(t)G_E^T(t)$$

$$\cdot \underset{\theta_p,\theta_E}{E} \mid I_E(t)\{T_E^*(t)\} + \underset{\theta_p,\theta_E}{E} \mid I_E(t)\{T_E^*(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)$$

$$\cdot T_p^*(t) - \underset{\theta_p,\theta_E}{E'} \mid I_E(t)\{T_p^*(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)$$

$$\cdot \underset{\theta_p,\theta_E}{E'} \mid I_E(t)\{T_p^*(t)\} - \underset{\theta_p,\theta_E}{E} \mid I_E(t)\{T_E^*(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)$$

$$\cdot \underset{\theta_p,\theta_E}{E} \mid I_E(t)\{T_E^*(t)\} - Q(t) \tag{146}$$

where

$$\frac{\partial T_E^*(t)}{\partial t} = -T_E^*(t)F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_E^*(t) + T_E^*(t)G_p(t)$$

$$\cdot R_p^{-1}G_p^T(t)T_p^{**}(t) + T_p^{**}(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_E^*(t) +$$

$$T_E^*(t)G_E(t)R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\theta_E}{E''} \mid I_E(t)\{T_E^*(t)\} +$$

$$\underset{\theta_p,\theta_E}{E''} \mid I_E(t)\{T_E^*(t)\}G_E(t)R_E^{-1}(t)G_E(t)T_E^*(t) - T_p^{**}(t)G_p(t)$$

$$\cdot R_p^{-1}(t)G_p^T(t)T_p^{**}(t) - \underset{\theta_p,\theta_E}{E''} \mid I_E(t)\{T_E^*(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)$$

$$\cdot \; E''_{\theta_p, \theta_E \mid I_E} \{ T_E^*(t) \} - Q(t) \tag{147}$$

where

$$\frac{\partial T_p^{**}(t)}{\partial t} = -T_p^{**}(t)\, F(t, \theta_p, \theta_E) - F(t, \theta_p, \theta_E)^T T_p^{**}(t) + T_p^{**}(t) \{ G_p(t) R_p^{-1}(t) G_p^T(t) + G_E(t) R_E^{-1}(t) G_E^T(t) \} T_p^{**}(t) - Q(t) \tag{148}$$

with boundary conditions

$$P_E(t_f) = T_p^*(t_f) = T_E^*(t_f) = T_p^{**}(t_f) = S(t_f).$$

The expected value operators will be explained. The operator $E\{\}$ denotes the expected value for the best estimate of the parameters. The operator $E'\{\}$ is over what the player feels his opponent possesses as the best estimate of the parameters. This takes into account the uncertainty the opponent has. The operator $E''\{\}$ denotes the expected value over what the player feels his opponent's information is as to the player's uncertainty.

Proof: Each player is solving his Hamilton-Jacobi equation for his control based upon the assumption that he has the coupling Hamilton-Jacobi equation his opponent is using to solve for his control. Since the strategies are opponent risk, each player realizes his opponent has the same assumption and is independently trying to optimize. Thus, each player uses three Hamilton-Jacobi equations representing the information known to him. Therefore, proof follows from previous procedures.

The equations over a discrete parameter range may be established as in previous theorems.

Theorem 3.7: The measurement security, system risk, opponent risk strategies u* and v* for the pursuer and the evader, respectively, for the system defined in Equation (86), the cost in Equation (88), and the information structure as in Equations (89) and (90) and under the assumption that θ_p and θ_E have a discrete parameter range are given as

$$u^*(t) = -R_p^{-1}(t)G_p^T(t)\left(\sum_{i=1}^{p_2} P_r(\theta_{E_i} \mid I_p(t))P_{p_i}(t)\right)x(t)$$
$$v^*(t) = -R_E^{-1}(t)G_E^T(t)\left(\sum_{i=1}^{p_1} P_r(\theta_{p_i} \mid I_E(t))P_{E_i}(t)\right)x(t) \tag{149}$$

where $P_{p_i}(t)$ is given as the solution to the following differential equation

$$\begin{aligned}\dot{P}_{p_i}(t) = &-P_{p_i}(t)F(t, \theta_p, \theta_{E_i}) - F(t, \theta_p, \theta_{E_i})^T P_{p_i}(t) + \\ &P_{p_i}(t)G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} \mid I_p(t)P_{p_j}(t)\right) \\ &\left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} \mid I_p(t))P_{p_j}(t)\right) G_p(t)R_p^{-1}(t)G_p(t)P_{p_i}(t) + \\ &P_{p_i}(t)G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{k=1}^{p_1} P_r'(\theta_{p_k} \mid I_p(t))T_{E_{ik}}(t)\right) + \\ &\left(\sum_{k=1}^{p_1} P_r'(\theta_{p_k} \mid I_p(t))T_{E_{ik}}(t)\right) G_E(t)R_E^{-1}(t)G_E^T(t)P_{p_i}(t) - \\ &\left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} \mid I_p(t))P_{p_j}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2} P_r(\theta_{E_j} \mid I_p(t)) \cdot \right. \\ &\left. \cdot P_{p_j}(t)\right) - \left(\sum_{k=1}^{p_1} P_r'(\theta_{p_k} \mid I_p(t))T_{E_{ik}}(t)\right) G_E(t)R_E^{-1}(t) \cdot \\ &\cdot G_E^T(t)\left(\sum_{k=1}^{p_1} P_r'(\theta_{p_k} \mid I_p(t))T_{E_{ik}}(t)\right) - Q(t)\end{aligned} \tag{150}$$

and

$$\dot{T}_{E_{ik}}(t) = -T_{E_{ik}}(t)F(t, \theta_{p_k}, \theta_{E_i}) - F(t, \theta_{p_k}, \theta_{E_i})^T T_{E_{ik}}(t) +$$

$$T_{E_{ik}}(t)G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2}P_r''(\theta_{E_j}|I_p(t))T_{p_{ij}}(t)\right) + \left(\sum_{j=1}^{p_2}P''(\theta_{E_j}|I_p(t))T_{p_{ij}}(t)\right)G_p(t)R_p^{-1}(t)G_p^T(t)T_{E_{ik}}(t) + T_{E_{ik}}(t)G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{j=1}^{p_1}P_r'(\theta_{p_j}|I_p(t))T_{E_{ij}}(t)\right) + \left(\sum_{j=1}^{p_1}P_r'(\theta_{p_j}|I_p(t))T_{E_{ij}}(t)\right)G_E(t)R_E^{-1}(t)G_E^T(t)T_{E_{ik}}'(t) - \left(\sum_{j=1}^{p_2}P_r''(\theta_{E_j}|I_p(t))T_{p_{ij}}(t)\right)G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2}P_r''(\theta_{E_j}| \cdot I_p(t))T_{p_{ij}}(t)\right) - \left(\sum_{j=1}^{p_1}P_r'(\theta_{p_j}|I_p(t)T_{E_{ij}}(t)\right)G_E(t)R_E^{-1}(t) \cdot G_E^T(t)\left(\sum_{j=1}^{p_1}P_r'(\theta_{p_j}|I_p(t))T_{E_{ij}}(t)\right) - Q(t) \qquad (151)$$

where

$$\mathring{T}_{p_{ik}}(t) = -T_{p_{ik}}(t)F(t,\ \theta_{p_k},\ \theta_{E_i}) - F(t,\ \theta_{p_k},\ \theta_{E_i})^T T_{p_{ik}}(t) + T_{p_{ik}}(t)G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2}P_r''(\theta_{E_j}|I_p(t))T_{p_{ij}}(t)\right) + \left(\sum_{j=1}^{p_2}P_r''(\theta_{E_j}|I_p(t))T_{p_{ij}}(t)\right)G_p(t)R_p^{-1}(t)G_p^T(t)T_{p_{ik}}(t) + T_{p_{ik}}(t)G_E(t)R_E^{-1}(t)G_E^T(t)T_{E_{ik}}'(t) + T_{E_{ik}}'(t)G_E(t)R_E^{-1}(t) \cdot G_E^T(t)T_{p_{ik}}(t) - \left(\sum_{j=1}^{p_2}P_r''(\theta_{E_j}|I_p(t))T_{p_{ij}}(t)\right)G_p(t)R_p^{-1}(t) \cdot G_p^T(t)\left(\sum_{j=1}^{p_2}P_r''(\theta_{E_j}|I_p(t))T_{p_{ij}}(t)\right) - T_{E_{ik}}'(t)G_E(t)R_E^{-1}(t) \cdot G_E^T(t)T_{E_{ik}}'(t) - Q(t) \qquad (152)$$

and

$$\mathring{T}_{E_{ik}}'(t) = -T_{E_{ik}}'(t)F(t,\ \theta_{p_k},\ \theta_{E_i}) - F(t,\ \theta_{p_k},\ \theta_{E_i})^T T_{E_{ik}}'(t) + T_{E_{ik}}'(t)\{G_p(t) \cdot R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T_{E_{ik}}'(t) - Q(t) \qquad (153)$$

$i = 1, 2, \ldots, p_2$
$k = 1, 2, \ldots, p_1$

with boundary conditions

$$P_{p_1}(t_f) = T_{E_{1k}}(t_f) = T_{p_{1k}}(t_f) = T''_{E_{1k}}(t_f) = S(t_f).$$

The probability $P_r(\theta_{E_j} | I_p(t))$ denotes the probability of the parameter θ_{E_j} assuming the a priori information available by the pursuer as to the best estimates of the true parameter values. The probability $P_r'(\theta_{p_k} | I_p(t))$ is the probability of the parameter θ_{p_k} as known by the evader. The pursuer uses his a priori information as to the evader's parameter uncertainty in order to calculate this value. The probability $P_r''(\theta_{E_j} | I_p(t))$ is the probability of the parameter θ_{E_j} which contains the knowledge the pursuer has as to the evader's knowledge of the pursuer's uncertainty. The pursuer uses his a priori knowledge of the evader's knowledge of the pursuer's uncertainty in order to calculate this value.

The matrix $P_{E_1}(t)$ is given as the solution to the following differential equation:

$$\begin{aligned}
\dot{P}_{E_1}(t) = {} & -P_{E_1}(t)F(t, \theta_{p_1}, \theta_E) - F(t, \theta_{p_1}, \theta_E)^T P_{E_1}(t) + \\
& P_{E_1}(t)G_p(t)R_p^{-1}(t)G_p^T(t) \left(\sum_{k=1}^{p_2} P_r'(\theta_{E_k} | I_E(t))T^*_{p_{1k}}(t)\right) + \\
& \left(\sum_{k=1}^{p_2} P_r'(\theta_{E_k} | I_E(t))T^*_{p_{1k}}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t)P_{E_1}(t) + \\
& P_{E_1}(t)G_E(t)R_E^{-1}(t)G_E^T(t) \left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} | I_E(t))P_{E_j}(t)\right) + \\
& \left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} | I_E(t))P_{E_j}(t)\right) G_E(t)R_E^{-1}(t)G_E^T(t)P_{E_1}(t) - \\
& \left(\sum_{k=1}^{p_2} P_r'(\theta_{E_k} | I_E(t))T^*_{p_{1k}}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t) \left(\sum_{k=1}^{p_2} P_r'(\theta_{E_k} | \right. \\
& \left. \cdot I_E(t))T^*_{p_{1k}}(t)\right) - \left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} | I_E(t))P_{E_j}(t)\right) G_E(t)R_E^{-1}(t) \cdot
\end{aligned} \tag{154}$$

$$\cdot G_E^T(t)\left(\sum_{j=1}^{p_1} P_r(\theta_{p_j} \mid I_E(t)) P_{E_j}(t)\right) - Q(t)$$

and

$$\begin{aligned}\dot{T}_{p_{ik}}^*(t) = {} & -T_{p_{ik}}^*(t)F(t,\ \theta_{p_i},\ \theta_{E_k}) - F(t,\ \theta_{p_i},\ \theta_{E_k})^T T_{p_{ik}}^*(t) + \\ & T_{p_{ik}}^*(t)G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2} P_r'(\theta_{E_j} \mid I_E(t))T_{p_{ij}}^*(t)\right) + \\ & \left(\sum_{j=1}^{p_2} P_r'(\theta_{E_j} \mid I_E(t))T_{p_{ij}}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t)T_{p_{ik}}^*(t) + \\ & T_{p_{ik}}^*(t)G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{j=1}^{p_1} P_r''(\theta_{p_j} \mid I_E(t))T_{E_{ij}}^*(t)\right) + \\ & \left(\sum_{j=1}^{p_1} P_r''(\theta_{p_j} \mid I_E(t))T_{E_{ij}}^*(t)\right) G_E(t)R_E^{-1}(t)G_E^T(t)T_{p_{ik}}^*(t) - \\ & \left(\sum_{j=1}^{p_2} P_r'(\theta_{E_j} \mid I_E(t))T_{p_{ij}}(t)\right) G_p(t)R_p^{-1}(t)G_p^T(t)\left(\sum_{j=1}^{p_2} P_r'(\theta_{E_j} \mid \right. \\ & \left. \cdot I_E(t))T_{p_{ij}}(t)\right) - \left(\sum_{j=1}^{p_1} P_r''(\theta_{p_j} \mid I_E(t))T_{E_{ij}}^*(t)\right) G_E(t)\ \cdot \\ & \cdot R_E^{-1}(t)G_E^T(t)\left(\sum_{j=1}^{n} P_r''(\theta_{p_j} \mid I_E(t))T_{E_{ij}}^*(t)\right) - Q(t)\end{aligned} \tag{155}$$

where

$$\begin{aligned}\dot{T}_{E_{ij}}^*(t) = {} & -T_{E_{ij}}^*(t)F(t,\ \theta_{p_i},\ \theta_{E_j}) - F(t,\ \theta_{p_i},\ \theta_{E_j})^T T_{E_{ij}}^*(t) + \\ & T_{E_{ij}}^*(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_{p_{ij}}^{**}(t) + T_{p_{ij}}^{**}(t)G_p(t)R_p^{-1}(t)\ \cdot \\ & \cdot G_p^T(t)T_{E_{ij}}^*(t) + T_{E_{ij}}^*(t)G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{k=1}^{p_1} P_r''(\theta_{p_k} \mid I_E(t))\ \cdot \right. \\ & \left. \cdot T_{E_{ik}}^*(t)\right) + \left(\sum_{k=1}^{p_1} P_r''(\theta_{p_k} \mid I_E(t)T_{E_{ik}}^*(t)\right) G_E(t)R_E^{-1}(t)G_E^T(t)\ \cdot \\ & \cdot T_{E_{ij}}^*(t) - T_{p_{ij}}^{**}(t)G_p(t)R_p^{-1}(t)G_p^T(t)T_{p_{ij}}^{**}(t) - \\ & \left(\sum_{k=1}^{p_1} P_r''(\theta_{p_k} \mid I_E(t))T_{E_{ik}}^*(t)\right) G_E(t)R_E^{-1}(t)G_E^T(t)\left(\sum_{k=1}^{p_1} P_r''(\theta_{p_k} \mid \right. \\ & \left. I_E(t))T_{E_{ik}}^*(t)\right) - Q(t)\end{aligned} \tag{156}$$

where

$$\dot{T}_{p_{ik}}^{**}(t) = -T_{p_{ik}}^{**}(t)F(t,\ \theta_{p_i},\ \theta_{E_k}) - F(t,\ \theta_{p_i},\ \theta_{E_k})^T T_{p_{ik}}^{**}(t) +$$

$$T_{p_{ik}}^{**}(t)\{G_p(t)R_p^{-1}(t)G_p^T(t) + G_E(t)R_E^{-1}(t)G_E^T(t)\}T_{p_{ik}}^{**}(t) -$$

$$Q(t), \qquad i = 1, 2, \ldots, p_2 \qquad k = 1, 2, \ldots, p_1 \tag{157}$$

with boundary conditions

$$P_{E_i}(t_f) = T_{p_{ik}}^{*}(t_f) = T_{E_{ij}}^{*}(t_f) = T_{p_{ik}}^{**}(t_f) = S(t_f).$$

The probability $P_r(\theta_{p_j} | I_E(t))$ denotes the probability of the parameter θ_{p_j}, denoting the best information available by the evader as to the best estimates of the true parameter values. The probability $P_r'(\theta_{E_k} | I_E(t))$ is the probability of the parameter θ_{E_k} as known by the evader. The evader uses his a priori information as to the pursuer's parameter uncertainty in order to calculate the value. The probability $P_r''(\theta_{p_k} | I_E(t))$ is the probability of the parameter θ_{p_k} which contains the knowledge the evader has as to the pursuer's knowledge of the evader's uncertainty. The evader uses his a priori knowledge of the pursuer's knowledge of the evader's uncertainty in order to calculate this value.

Proof: The proof follows from Theorem 3.6 and the definition of the conditional expectation over a discrete range.

Parameter Estimation

The expected value operators $\underset{\theta_p,\theta_E | I_p(t)}{E}\{\cdot\}$ and $\underset{\theta_p,\theta_E | I_E(t)}{E}\{\cdot\}$ will be discussed. The expected value operators may be written as

$$\underset{\theta_p,\theta_E | I_p(t)}{E}\{\cdot\} = \int_D (\cdot) p(\theta_p, \theta_E | I_p(t)) d\theta_p \, d\theta_E$$
$$(158)$$
$$\underset{\theta_p,\theta_E | I_E(t)}{E}\{\cdot\} = \int_D (\cdot) p(\theta_p, \theta_E | I_E(t)) d\theta_p \, d\theta_E$$

where D denotes the domain of the parameters. It is assumed that the parameters θ_p and θ_E are independent. Thus, Equation (158) may be rewritten as

$$\underset{\theta_p,\theta_E | I_p(t)}{E}\{\cdot\} = \int_D (\cdot) p(\theta_p | I_p(t)) p(\theta_E | I_p(t)) d\theta_p \, d\theta_E$$
$$(159)$$
$$\underset{\theta_p,\theta_E | I_E(t)}{E}\{\cdot\} = \int_D (\cdot) p(\theta_p | I_E(t)) p(\theta_E | I_E(t)) d\theta_p \, d\theta_E \; .$$

A result will be given for the probability density functions $P(\theta_p | \mathcal{X})$ where $\mathcal{X} = (x(\tau), \; \tau \in [t_0, \; t])$.

At this point, an assumption that need not be made for the problem in Chapter IV will be made. This is that the time interval over which the game is played, i.e., $[t_0, \; t_f]$ is partitioned into a finite number of time intervals

$$\text{part}[t_0, \; t_f] = t_0 = t_1 < t_2 < \ldots < t_n < t_f .$$

It is assumed that the control strategies previously found are continuously applied to the system but that adaption for learning the unknown parameters occurs at the time t_i. That is, the conditional probability density functions are updated at times t_1, t_2, ..., t_n. The updated density functions are then used over the next interval to compute the control strategies. Thus, the probability density functions may be written as $p(\theta_p | \{x(i)\}_{i=1}^{k})$ and $p(\theta_E \; \{x(i)\}_{i=1}^{k})$ where $\{x(i)\}_{i=1}^{k}$ denotes the ordered sequence of states at the k^{th} time instant for updating. The use of the nomenclature O_n will be made in order to denote the ordered sequence $\{x(i)\}_{i=1}^{n}$.

The objective is to find the conditional density function $P(\theta|O_n)$. The system equation

$$\frac{dx(t)}{dt} = F(t,\ \theta_p,\ \theta_E)x(t) + G_p(t)u(t) + G_E(t)v(t) + w(t) \tag{160}$$

may be solved and the solution given as (where i, i + 1, represent the corresponding sampling instant

$$x(i+1) = \Phi(i+1,\ i,\ \theta_p,\ \theta_E)x(i) + \int_i^{i+1} \Phi(i+1,\ \tau,\ \theta_p,\ \theta_E)\{G_p(\tau)u(\tau) + G_E(\tau)v(\tau) + w(\tau)\}d\tau. \tag{161}$$

This may be rewritten as

$$x(i+1) = \Phi(i+1,\ i,\ \theta_p,\ \theta_E)x(i) + \int_i^{i+1} \Phi(i+1,\ \tau,\ \theta_p,\ \theta_E)\{G_p(\tau)u(\tau) + G_E(\tau)v(\tau)\}d\tau + \bar{w}(i) \tag{162}$$

where

$$\bar{w}(i) = \int_i^{i+1} \Phi(i+1,\ \tau,\ \theta_p,\ \theta_E)w(\tau)d\tau. \tag{163}$$

The variance of $\bar{w}(i)$ is given as $V_{\bar{w}}(i+1)$ where

$$V_{\bar{w}}(i+1) = \int_i^{i+1} \Phi(i+1,\ \tau,\ \theta_p,\ \theta_E)W(\tau)\Phi^T(i+1,\ \tau,\ \theta_p,\ \theta_E)d\tau \tag{164}$$

and $W(\tau)$ is the variance of $w(\tau)$. It may be shown that $V_{\bar{w}}$ is the solution to the following matrix differential equation at the $(i+1)^{st}$ sampling time, i.e.,

$$\dot{V}_{\bar{W}} = F V_{\bar{W}} + V_{\bar{W}} F^T + W \tag{165}$$

$$V_{\bar{W}}(t_1) = 0$$

One may note that $V_{\bar{W}}$ is a function of θ_p and θ_E.

The conditional expectation is dependent upon knowledge of the conditional probability density function $P(\theta|O_{n-1})$ where O_{n-1} denotes the ordered sequence $\{x(i)\}_{i=0}^{n-1}$. The conditional probability density function will now be developed.

Lemma 1.1: A sequential equation for the evaluation of the above probability density function is given as

$$p(\theta|O_n) = \frac{p(\theta|O_{n-1})p(x(n)|x(n-1),\ u(n-1),\ v(n-1),\ \theta)}{\int_R p(\theta|O_{n-1})(p(x(n)|x(n-1),\ u(n-1),\ v(n-1),\ \theta)\, d\theta} \tag{166}$$

Proof: Application of Baye's Rule

The probability density function $p(x(n)|x(n-1),\ u(n-1),\ v(n-1),\ \theta)$ is Gaussian with mean $\mu_{x_n} = \Phi(n,\ n-1,\ \theta)x(n-1) + \bar{G}_p(n-1)u(n-1) + \bar{G}_E(n-1)v(n-1)$ and variance $V_{\bar{W}}(i)$.

At the initial stage, the probability density function $P(\theta|x(0))$ may be written as

$$p(\theta|x(t_0)) = p(\theta)$$

where $P(\theta)$ is the a priori probability density function for the parameter θ.

Since the density function $p(x(n),\ x(n-1),\ u(n-1),\ v(n-1),\ \theta)$ is Gaussian, it may be written as

$$p(x(n)|x(n-1),\ u(n-1),\ v(n-1),\ \theta) =$$

$$\frac{1}{\sqrt{(2\pi)^n |V_{\bar{w}}(n|\theta)|}} \exp\{-\tfrac{1}{2}(x(n) - \mu_{x_n})^T (V_{\bar{w}}(n|\theta))^{-1}(x(n) - \mu_{x_n})\} \tag{167}$$

where the nomenclature $V_{\bar{w}}(n|\theta)$ includes the possibility that the variance may be a function of θ and, thus, must be conditioned upon θ. This allows one to write the required density function as the following sequential equation:

$$p(\theta|O_n) = \frac{\exp\{-\tfrac{1}{2}Q(n|\theta)\}p(\theta|O_{n-1})}{\sqrt{|V_{\bar{w}}(n)\theta)|} \int_R \frac{1}{\sqrt{V_{\bar{w}}(n|\theta)|}} \exp\{-\tfrac{1}{2}Q(n|\theta)\}p(\theta|O_{n-1})\, d\theta}. \tag{168}$$

If it is known that the range of θ is discrete where N denotes the total number of values of θ in the range, then the a priori probability density may be written as

$$p(\theta) = \sum_{i=1}^{N} p(\theta_i)\ \delta(\theta - \theta_i). \tag{169}$$

The a posteriori probability for each parameter in the range of θ may be written as

$$P_r(\theta_i|O_n) = \frac{\exp\{-\tfrac{1}{2}Q(n|\theta)\}P_r(\theta_i|O_{n-1})}{\sqrt{|V_{\bar{w}}(n|\theta_i)|} \sum_{j=1}^{N} \frac{1}{\sqrt{|V_{\bar{w}}(n|\theta_j)|}} \exp\{-\tfrac{1}{2}Q(n|\theta_j)\}P_r(\theta_j|O_{n-1})} \tag{170}$$

$$i = 1,\ 2,\ \ldots,\ N$$

where the initial probability is $P_r(\theta_i)$, $i = 1, \ldots, N$.

In order to relate the mathematics to a physical process at this point, one may consider the problem of precision pointing and tracking (6).

One of the quantities required to track a target is the target's acceleration. It has been suggested by Singer (89) that an appropriate model for acceleration in a tracking filter would be given by

$$\dot{a} = \frac{1}{\tau_s} a + w \tag{171}$$

where w is a white noise input. If the target were maneuvering rapidly the time constant, τ_s, would be very large indicating little correlation from time instant to time instant. If the target was unaware, then it would be very small indicating a large correlation from time instant to time instant. Thus, one might choose an appropriate quantization for θ and apply the above equations in order to adapt upon the correct value for $\frac{1}{\tau_s}$.

As the adaptation proceeds, the probability corresponding to the correct value of the parameter will converge to one while the remaining probabilities will converge to zero.

One may notice that the computation of the mean is given by an expression necessitating knowledge of the opponent's control. Each player must use the information sets to obtain their opponent's control strategies. These will, in general, be in error since each player has uncertainty as to their opponent's optimal strategies since he does not have knowledge of the game dynamics and uncertain knowledge of the opponent's information. This type of problem has arisen in preceding papers by Ho et al. and Rhodes in the treatment of the imperfect measurement game. If the opponent's control is in error, then one would expect the estimator to be biased.

Example

The following example problem will be considered. The system dynamics are given as

$$\dot{x} = \theta_T x + au + bv + cw \tag{172}$$

where

θ_T is a system eigenvalue perfectly known by the pursuer but unknown by the evader

w is white noise, zero mean and unity variance.

The performance index is given as

$$J = E\left\{\tfrac{1}{2}sx^2(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f} (r_p u^2 + r_E v^2)\,dt\right\}. \tag{173}$$

The system parameters are

$$\begin{array}{rlrl} \theta_T &= -2.0 & r_p &= 2 \\ a &= 2.0 & r_E &= -3 \\ b &= 1.0 & s &= 10 \\ c &= 2.0 & t_f &= 2.5 \text{ secs.} \\ x(0) &= 5.0 & & \end{array} \tag{174}$$

Each player must find his measurement security, system security, opponent security strategy.

The evader has an initial probability function for his best guess of the system eigenvalue of $\{1, .5, -2.0\}$ where each element has probability $\frac{1}{3}$.

The strategy the pursuer plays is

$$u(t) = -\frac{a}{r_p} P_p(t)x(t) \tag{175}$$

where

$$\dot{P}_p(t) = -2\theta_T P_p(t) + \left\{ \frac{a^2}{r_p} + \frac{b^2}{r_E} \right\} P_p^2(t). \tag{176}$$

The strategy the evader plays is

$$v(t) = -\frac{b}{r_E}\left(\sum_{i=1}^{3} P_r(\theta_i \mid O_{n-1}) P_E(t, \theta_i)\right) x(t) \tag{177}$$

where

$$\dot{P}_{E_i}(t) = -2\theta_i P_{E_i}(t) + \frac{2a^2}{r_p} P_{E_i}(t) T_{p_i}(t) +$$

$$\frac{2b^2}{r_E} P_{E_i}(t) \left(\sum_{j=1}^{3} P_r(\theta_j \mid O_{n-1}) P_{E_j}(t)\right) - \frac{a^2}{r_p} T_{p_i}^2(t) -$$

$$\frac{b^2}{r_E}\left(\sum_{j=1}^{3} P_r(\theta_j \mid O_{n-1}) P_{E_j}(t)\right)^2 \tag{178}$$

and

$$\dot{T}_{p_i}(t) = -2\theta_i T_{p_i}(t) + \left\{\frac{a^2}{r_p} + \frac{b^2}{r_E}\right\} T_{p_i}^2(t) \tag{179}$$

with

$$P_p(t_f) = P_{E_i}(t_f) = T_{p_i}(t_f) = s.$$

The probability of the i^{th} parameter conditioned upon the measurement sequence is given as

$$P_r(\theta_i \mid O_n) = \frac{\exp\{-\tfrac{1}{2}Q(n \mid \theta)\} P_r(\theta_i \mid O_{n-1})}{\sqrt{|V_{\bar{w}}(n \mid \theta_i)|} \sum_{j=1}^{3} \frac{1}{\sqrt{|V_{\bar{w}}(n \mid \theta_j)|}} \exp\{-\tfrac{1}{2}Q(n \mid \theta_j)\} P_r(\theta_j \mid O_{n-1})} \tag{180}$$

where

$$Q(n \mid \theta_i) = (x(n) - \mu_{x_{n_i}})^T W^{-1}(n-1)(x(n) - \mu_{x_{n_i}}) \tag{181}$$

and

$$\mu_{x_{n_i}} = \Phi(i+1,\ i,\ \theta_i)\mu_{x_{n-1_i}} + \int_i^{i+1} \Phi(i+1,\ \tau,\ \theta_i)\{au(\tau) +$$

$$b\,v\,(\tau)\}d\,\tau \qquad (182)$$

$$\Phi(i+1,\ \tau,\ \theta_i) = e^{-\theta_i\{t_{i+1} - \tau\}}.$$

The evader uses his control v(i) to calculate $\mu_{x_{n_i}}$. However, he does not have knowledge of u(i). Thus, he must use his best knowledge of the control that the pursuer will use in the equation for $\mu_{x_{n_i}}$.

In this example, it is assumed that the evader updates his parameter values at a rate of ten per second.

The probability of the ith parameter conditioned on the measurement update sequence may be found from Equation (170) with the required mean $\mu_{x_{n_i}}$ found from

$$\mu_{x_{n_j}} = \Phi(i+1,\ i,\ \theta_j)\mu_{x_{n-1_j}} + \int_i^{i+1} \Phi(i+1,\ i,\ \theta_j)\{au(\tau) +$$

$$b\,v\,(\tau)\}d\,\tau \qquad (183)$$

and

$$\Phi(i+1,\ \tau,\ \theta_i) = e^{-\theta_i\{t_{i+1} - \tau\}}. \qquad (184)$$

The evader uses his control v(i) to calculate $\mu_{x_{n_i}}$. However, he does not have knowledge of u(i). Thus, he must use his best knowledge of the control that the pursuer will use in the equation for $p_r(x|O_n)$.

The results by using the contributions of this research are compared with the results for differential games following the derivations of references (56), (58), and (60). The resulting equations for the evader's control strategies using this derivation method may be easily found to be

$$v(t) = -\frac{b}{r_E}\left(\sum_{i=1}^{3} P_r(\theta_i | O_{n-1}) P_E^*(t, \theta_i)\right) x(t) \tag{185}$$

where

$$\dot{P}_E^* = -2\theta_i P_E^* + \left(\frac{a^2}{r_p} + \frac{b^2}{r_E}\right) P_E^{*2} . \tag{186}$$

The average performance index (average over 30 runs) using the results of this research was -0.02990. The average performance index (average over 30 runs) using the above results was -0.08864. Thus, the pursuer was able to gain in cost when the evader used the suboptimal strategies as based on the equations for the evader's control found by strategies derived by using the method of Lainiotis. The strategies based on the work of this research give superior performance.

CHAPTER IV

SYSTEM UNCERTAINTY AND IMPERFECT INFORMATION GAME

Introduction

In this chapter, the problem of differential games under uncertainty in the system matrix and imperfect measurements is solved. As in Chapter III, the system matrix for the game is parameterized by elements of a time invariant parameter vector θ. The parameter vector θ may be partitioned into two subvectors θ_p and θ_E where θ_p is known to the pursuer but unknown to the evader and where θ_E is known to the evader but unknown to the pursuer. Each player has a measurement subsystem that takes imperfect measurements of the state of the game. The measurement equation is a linear transformation of the state plus additive noise. The strategies found in this chapter are measurement security strategies. That is, each player assumes that his opponent has a measurement subsystem that is capable of taking perfect measurements of the state of the game. The assumption that the parameters be updated at discrete instants of time may be weakened such that adaptation occurs continuously. A sufficiency condition is developed and used to solve for the strategies for the linear quadratic problem. It is shown that for the open-loop feedback strategies the control and estimation separate. The strategies for the linear quadratic problem include the equations developed in Chapter III in order to calculate the necessary control gains and equations in order to obtain the conditional mean of the state estimate. The conditional mean may be generated by use of Bucy's Representation Theorem and Lainiotis' partition theorem.

The contributions of this chapter are as follows:

(a) Extension of the sufficiency condition developed in Chapter III to include the measurement functional.

(b) Solution to the linear quadratic game under uncertainty and imperfect information.

(c) The open-loop feedback strategies for the stochastic control with imperfect information and uncertainty may be found by constraining the evader's control to be zero. The solution extends those in (56), (58), and (60). The results may be found in Appendix C.

Statement of the Problem

The dynamical equations representing the system models of each player are assumed to be adequately represented by the following differential equation:

$$\frac{dx(t)}{dt} = F(t, \theta_p, \theta_E)x(t) + G_p(t)u(t) + G_E(t)v(t) + w(t) \tag{1}$$

where the subscripts p and E denote the pursuer and the evader, respectively. The variables in Equation (1) are defined as follows:

$x(t) \in R^n$ is a vector denoting the state of the game at time t,

$u(t) \in U$ where $U \subset R^{m_1}$ is a vector denoting the control variables of the pursuer at time t,

$v(t) \in V$ where $V \subset R^{m_2}$ is a vector denoting the control variables of the evader at time t,

$F(t, \theta_p, \theta_E)$ is an $n \times n$ matrix parameterized by θ_p and θ_E with continuous and bounded elements,

$\theta_p \epsilon R^{p_1}$ is a time invariant parameter vector known to the pursuer but unknown to the evader,

$\theta_E \epsilon R^{p_2}$ is a time invariant parameter vector known to the evader but unknown to the pursuer,

$w(t) \epsilon R^m$ is a vector of white noise inputs corrupting the system model, assumed Gaussian with known statistics

$$E\{w(t)\} = 0$$
$$E\{w(t)w^T(\tau)\} = W(t)\delta(t-\tau) \tag{2}$$

$G_p(t)$ is a $n \times m_1$ control gain matrix for the pursuer,

$G_E(t)$ is a $n \times m_2$ control gain matrix for the evader.

The initial conditions are assumed to be non-Gaussian with a priori probability density $p_{x_0}(x(t_0)|\theta_p, \theta_E)$. This probability density function is assumed known to both players prior to the start of the game.

Each player has access to certain observations or measurements of the state of the game. These measurements are taken to be linear transformations of the state of the game plus additive measurement noise. The measurement equation available to the pursuer is

$$y_p(t) = H_p(t)x(t) + \eta_p(t) \tag{3}$$

where

$y_p(t) \epsilon R^{q_1}$ is a vector of measurements made by the pursuer of the pursuer's state,

$H_p(t)$ is a $q_1 \times n$ measurement matrix with continuous and bounded elements,

$\eta_p(t) \in R^{q_1}$ is a vector of white noise inputs corrupting the measurement model, assumed Gaussian with known statistics

$$E\{\eta_p(t)\} = 0$$
$$E\{\eta_p(t)\eta_p^T(\tau)\} = N_p(t)\delta(t-\tau). \tag{4}$$

The measurement equation available to the evader is

$$y_E(t) = H_E(t)x(t) + \eta_E(t) \tag{5}$$

where

$y_E(t) \in R^{q_2}$ is a vector of measurements made by the pursuer of the pursuer's state,

$H_E(t)$ is a $q_2 \times n$ measurement matrix with continuous and bounded elements,

$\eta_E(t) \in R^{q_2}$ is a vector of white noise inputs corrupting the measurement model, assumed Gaussian with known statistics

$$E\{\eta_E(t)\} = 0$$
$$E\{\eta_E(t)\eta_E^T(\tau)\} = N_E(t)\delta(t-\tau). \tag{6}$$

Each player has access to certain information sets that he uses to solve for his strategies. The sets contain the a priori information as to the uncertain parameter sets, any a priori information that he has as to his opponent's uncertainty, the measurement functional, and implicit in the information set is the assumption that the player's opponent has perfect measurements of the state of the game. The information set of the pursuer at time t is denoted by $I_p(t)$. Similarly, the information set of the evader at time t is denoted by $I_E(t)$. For every time $t \epsilon [t_0, t_f)$ the information sets $I_p(t)$ and $I_E(t)$ of the pursuer and the evader, respectively, are

$$I_p(t) = p_{x_0}(x(t_0)|\theta_p, \theta_E) \cup p_{\theta_E}(\theta_E) \cup \theta_p \cup p_{E_{\theta_p}}(\theta_p) \cup p_{E_{\theta_E}}(\theta_E) \cup (y_p(\tau), \tau \epsilon [t_0, t]) \tag{7}$$

and

$$I_E(t) = p_{x_0}(x(t_0)|\theta_p, \theta_E) \cup p_{\theta_p}(\theta_p) \cup \theta_E \cup p_{p_{\theta_E}}(\theta_E) \cup p_{p_{\theta_p}}(\theta_p) \cup (y_E(\tau), \tau \epsilon [t_0, t])$$

where

$p_{x_0}(x(t_0)|\theta_p, \theta_E)$ is the a priori probability density function for the initial conditions,

$p_{\theta_E}(\theta_E)$ is the probability density function representing the a priori information known by the pursuer about the unknown parameter vector θ_E,

$p_{\theta_p}(\theta_p)$ is the probability density function representing the a priori information known by the evader about the unknown parameter vector θ_p,

θ_p is the parameter vector known by the pursuer,

θ_E is the parameter vector known by the evader,

$p_{E\,\theta_p}(\theta_p)$ is the probability density function representing any knowledge the pursuer may have about the knowledge the evader possesses as to the parameter vector θ_p,

$p_{p\,\theta_E}(\theta_E)$ is the probability density function representing any knowledge the evader may have about the knowledge the pursuer possesses as to the parameter vector θ_E,

$p_{E\,\theta_E}(\theta_E)$ is the probability density function representing any knowledge the pursuer has about the evader's knowledge of the pursuer's uncertainty of the parameter vector θ_E,

$p_{p\,\theta_p}(\theta_p)$ is the probability density function representing any knowledge the evader has about the pursuer's knowledge of the evader's uncertainty of the parameter vector θ_p,

$(y_p(\tau), \tau \in [t_0, t])$ is the infinite dimensional measurement functional of the pursuer,

$(y_E(\tau), \tau \in [t_0, t])$ is the infinite dimensional measurement functional of the evader.

The dynamics and the information structure of the game is given by Equations (1) and (7). It is assumed that the goals of each player are adequately incorporated in the scalar functional known as the performance index, i.e.,

$$J = E\{\tfrac{1}{2}x^T(t_f)S(t_f)x(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}(x^T(t)Q(t)x(t) + u^T(t)R_p(t)u(t) + v^T(t)R_E(t)v(t))dt\} \tag{8}$$

where $E\{\cdot\}$ denotes the expectation over all random processes under the bracket and where

$S(t_f)$ is a positive semi-definite, symmetric matrix,

$Q(t)$ is a positive semi-definite, symmetric matrix,

$R_p(t)$ is a positive definite, symmetric matrix,

$R_E(t)$ is a negative definite, symmetric matrix.

Thus, the performance index in Equation (8) is a functional mapping the state space and control space into the reals, i.e.,

$$J: R^n \times U \times V \rightarrow R^1 \tag{9}$$

Each player must choose closed-loop control laws as was previously explained. Thus, the pursuer must at each time $t \in [t_0\, t_f)$ find the function mapping the information set available to him at time t into the admissible control set such that the performance index is minimized, i.e.,

$$u^*: \{I_p(t)\} \times [t_0,\ t] \rightarrow U \subset R^{m_1}. \tag{10}$$

Similarly, the evader must at each time $t \in [t_0,\ t_f)$ find the function mapping the information set available to him at time t into the admissible control set such that the performance index is maximized, i.e.,

$$v^*: \{I_E(t)\} \times [t_0,\ t] \rightarrow V \subset R^{m_2}. \tag{11}$$

The control strategies u* and v* are assumed to be the minimizing and maximizing control strategies, respectively. The set of admissible controls U is assumed to be a subset of $L_2\{I, R^{m_1}\}$ where $I = [t_0, t_f]$, and the set of admissible controls V is assumed to be a subset of $L_2\{I, R^{m_2}\}$ where $I = [t_0, t_f]$.

Each player wishes to choose his control strategies such that the following inequalities are satisfied

$$E\{J(u^*, v^*) \mid I_p\} \leq E\{J(u, v^*) \mid I_p\}$$
$$E\{J(u^*, v) \mid I_E\} \leq E\{J(u^*, v^*) \mid I_E\}. \tag{12}$$

These inequalities were discussed in Chapter III.

A sufficiency condition based upon the results obtained in Chapter III will be shown now.

Sufficiency Condition

Theorem 4.1: It is sufficient that there exists two scalar functions V_p and V_E where

$$V_p(x(t), t) = R^n \times [t_0, t] \tag{13}$$

and

$$V_E(x(t), t) = R^n \times [t_0, t] \tag{14}$$

in order to solve for the closed-loop optimal strategies u* and v*. The functions V_p and V_E are twice continuously differentiable in x(t) and continuously differentiable in t. The functions are defined as the solutions to the following equations

$$L_p = \frac{\partial V_p}{\partial t} + \mathcal{L}_p V_p + Q(x(t), u(t), v^*(t), t) \tag{15}$$

$$L_E = \frac{\partial V_E}{\partial t} + \mathcal{L}_E V_E + G(x(t),\ u^*(t),\ v(t),\ t) \tag{16}$$

where the differential generators are as shown in Equations (50) and (51) of Chapter III and

$$|V_p| + |V_{p_T}| + |x|\,|V_{p_x}| + |x|^2\,|V_{p_{xx}}| < C(1+|x|^2)$$

$$|V_E'| + |V_{E_T}'| + |x|\,|V_{E_x}'| + |x|^2\,|V_{E_{xx}}'| < C(1+|x|^2)\ .$$

The boundary conditions for the above equations are

$$V_p(x(t_f),\ t_f) = V_E'(x(t_f),\ t_f) = G(x(t_f),\ t_f). \tag{17}$$

The functions L_p and L_E are such that

$$\min_{u \in U} E\{L_p(x(t),\ u(t),\ v^*(t),\ \theta,\ t) | I_p(t)\} = 0 \tag{18}$$

and

$$\max_{v \in V} E\{L_E(x(t),\ u^T(t),\ v(t),\ \theta,\ t) | I_E(t)\} = 0. \tag{19}$$

Proof: The proof follows from Theorem 1 of Chapter III where the information sets have been redefined as in Equation (7) of this chapter.

Linear Quadratic Problem

The following lemma will be used to solve the problem outlined in the previous section.

Definition: A statistic $g(t, Y(t))$ a function of the data $(Y(t) = (y_i(\tau),\ \tau \in [t_0, t]),\ i = P, E)$ at time t will be called equivalent to the distribution $p(A|Y(t))$ if the distribution depends on the data only through $g[t, Y(t)]$, that is

$$p(A|Y(t)) = p(A|g[t,\ Y(t)]). \tag{20}$$

The above definition denotes that if there exists statistics(sufficient statistics) satisfying the above conditions, then the distribution may be replaced by these statistics.

Lemma 4.1: The conditional mean $\hat{x}(t|\theta)$ and the distribution $p(\theta|Y)$ are statistics which are equivalent to the distribution $p(x(t)|Y(t))$ for the system as defined in Equation (1).

Proof: The use of Bayes rule allows one to write $p(x(t)|Y(t))$ as

$$p(x(t)|Y(t)) = p(x(t)|Y(t), \theta)p(\theta|Y). \tag{21}$$

For the linear problem under consideration it is well known that the statistic $\hat{x}(t|\theta)$ where $\hat{x}(t|\theta)$ is the expected value of $x(t)$ conditioned on $Y(t)$ and θ is equivalent to $p(x(t)|Y(t), \theta)$. This is the conditional mean generated by the Kalman filter which is valid if θ is known. Thus, $p(x(t)|Y(t))$ may be written as

$$p(x(t)|Y(t)) = p(x(t)|\hat{x}(t|\theta), p(\theta|Y)). \tag{22}$$

Lemma 4.2: Let x be a random variable with mean $\hat{x}$ and variance P. Then,

$$E\{x^T Qx\} = \hat{x}^T Q\hat{x} + \mathrm{tr}\, QP . \tag{23}$$

Proof: One may write

$$\begin{aligned} E\{x^T Qx\} = {} & E\{(x - \hat{x})^T Q(x - \hat{x})\} + E\{\hat{x}^T Qx\} + \\ & E\{x^T Q\hat{x}\} - E\{\hat{x}^T Q\hat{x}\}. \end{aligned} \tag{24}$$

Since

$$E\{x\} = \hat{x},$$

this may be rewritten as

$$E\{x^T Qx\} = E\{(x - \hat{x})^T Q(x - \hat{x})\} + \hat{x}^T Q\hat{x}. \tag{25}$$

Also,

$$(x-\hat{x})^T Q(x-\hat{x}) = \text{tr}\,\{(x-\hat{x})^T Q(x-\hat{x})\}$$
$$= \text{tr}\,\{Q(x-\hat{x})(x-\hat{x})^T\}. \tag{26}$$

One may take the expectation and obtain

$$E\{(x-\hat{x})^T Q(x-\hat{x})\} = E\{\text{tr}\,[Q(x-\hat{x})(x-\hat{x})^T]\} \tag{27}$$
$$= \text{tr}\,\{QE\{(x-\hat{x})(x-\hat{x})^T\}\}$$
$$= \text{tr}\ QP.$$

Thus,

$$E\{x^T Qx\} = \hat{x}^T Q\hat{x} + \text{tr}\ QP. \tag{28}$$

The problem outlined in the previous section will now be solved.

Theorem 4.2: The measurement security, system security, opponent security, strategies u* and v* for the pursuer and the evader, respectively, for the system defined in Equation (1), the cost in Equation (8), the measurement subsystems in Equations (3) and (5), and the information structure as in Equation (7) are given as

$$u^*(t) = -R_p^{-1}(t)G_p^T(t) \underset{\theta_p,\,\theta_E|I_p(t)}{E}\{P_p(t)\} \underset{\theta_E|I_p(t)}{E}\{\hat{x}(t|\theta_E)\}$$
$$v^*(t) = -R_E^{-1}(t)G_E^T(t) \underset{\theta_p,\,\theta_E|I_E(t)}{E}\{P_E(t)\} \underset{\theta_p|I_E(t)}{E}\{\hat{x}(t)|\theta_p)\} \tag{29}$$

where $P_p(t)$ and $P_E(t)$ are given as the solutions to Equations (94), (95), (96), and (97) of Chapter III. The conditional mean denoting the best estimate of the state of the game given the measurement functional is given as

$$\hat{x}_P(t) = \underset{\theta_E | I_{P(t)}}{E} \{\hat{x}(t)|\theta_E)\} \tag{30}$$

and

$$\hat{x}_E(t) = \underset{\theta_P | I_{E(t)}}{E} \{\hat{x}(t|\theta_P)\} \tag{31}$$

where $\hat{x}(t)\theta_i)$, $i = P, E$ is the mean conditioned upon both the measurement functional and θ_i.

Proof: The Hamilton-Jacobi equations may be written as

$$\min_{u \in U} E\{\frac{\partial V_P}{\partial t} + \frac{\partial V_P^T}{\partial x}[F(t, \theta_P, \theta_E)x(t) + G_P(t)u(t) +$$

$$G_E(t)v^*(t)] + \tfrac{1}{2}(x^T(t)Q(t)x(t) + u^T(t)R_P(t)u(t) + \tag{32}$$

$$v^{*T}(t)R_E(t)v^*(t)) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_P}{\partial x_i \partial x_j} m_{ij} | I_P(t)\} = 0$$

where $m_{ij} = \{W(t)\}_{ij}$

and

$$\max_{v \in V} E\{\frac{\partial V_E}{\partial t} + \frac{\partial V_E^T}{\partial x}[F(t, \theta_P, \theta_E)x(t) + G_P(t)u^*(t) +$$

$$G_E(t)v(t)] + \tfrac{1}{2}(x^T(t)Q(t)x(t) + u^{*T}(t)R_P(t)u^*(t) + \tag{33}$$

$$v^T(t)R_E(t)v(t)) + \tfrac{1}{2}\sum_{i=1}\sum_{j=1}\frac{\partial^2 V_E}{\partial x_i \partial x_j} m_{ij} | I_E(t)\} = 0.$$

These equations may be rewritten as

$$\min_{u \in U} E\{\frac{\partial V_p}{\partial t} + \frac{\partial V_p^T}{\partial x}[F(t, \theta_p, \theta_E)x(t) + G_p(t)u(t) +$$

$$G_E(t)v^*(t)] + \tfrac{1}{2}(x^T(t)Q(t)x(t) + u^T(t)R_p(t)u(t) + \tag{34}$$

$$v^{*T}(t)R_E(t)v^*(t)) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_p}{\partial x_i \partial x_j} m_{ij} | Y_p(t), I_p(t) - Y_p(t)\} = 0$$

and

$$\max_{v \in V} E\{\frac{\partial V_E}{\partial t} + \frac{\partial V_E^T}{\partial x}[F(t, \theta_p, \theta_E)x(t) + G_p(t)u^*(t) +$$

$$G_E(t)v(t)] + \tfrac{1}{2}(x^T(t)Q(t)x(t) + u^{*T}(t)R_p(t)u^*(t) + \tag{35}$$

$$v^T(t)R_E(t)v(t)) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_E}{\partial x_i \partial x_j} m_{ij} | Y_E(t), I_E(t) - Y_E(t)\} = 0$$

where

$$Y_p(t) = (y_p(\tau), \tau \epsilon [t_0, t))$$

$$Y_E(t) = (y_E(\tau), \tau \epsilon [t_0, t)). \tag{36}$$

The use of Lemma 4.1 allows one to write the equations as

$$\min_{u \in U} E\{\frac{\partial V_p}{\partial t} + \frac{\partial V_p^T}{\partial x}[F(t, \theta_p, \theta_E)x(t) + G_p(t)u(t) +$$

$$G_E(t)v^*(t)] + \tfrac{1}{2}(x^T(t)Q(t)x(t) + u^T(t)R_p(t)u(t) +$$

$$v^{*T}(t)R_E(t)v^*(t)) + \tfrac{1}{2}\sum_{i=1}\sum_{j=1}\frac{\partial^2 V_p}{\partial x_i \partial x_j} m_{ij} | \hat{x}_p(t|\theta_E), p(\theta_E | Y_p), \tag{37}$$

$$I_p(t) - Y_p(t)\} = 0$$

and

$$\max_{v \in V} E\{\frac{\partial V_E}{\partial t} + \frac{\partial V_E^T}{\partial x}[F(t, \theta_p, \theta_E)x(t) + G_p(t)u^*(t) +$$

$$G_E(t)v(t)] + \tfrac{1}{2}(x^T(t)Q(t)x(t) + u^{*T}(t)R_p(t)u^*(t) +$$

$$v^T(t)R_E(t)v(t)) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_E}{\partial x_i \partial x_j} m_{ij} \,|\hat{x}_E(t|\theta_p),\ p(\theta_p|Y_E), \tag{38}$$

$$I_E(t) - Y_E(t)\} = 0.$$

Minimization and maximization yields

$$u^*(t) = -R_p^{-1}(t)G_p^T(t)\ E\{\frac{\partial V_p}{\partial x} \mid \hat{x}_p(t|\theta_E),\ p(\theta_E|Y_p),$$

$$I_p(t) - Y_p(t)\} \tag{39}$$

$$v^*(t) = -R_E^{-1}(t)G_E^T(t)\ E\{\frac{\partial V_p}{\partial x} \mid \hat{x}_E(t|\theta_p),\ p(\theta_p|Y_E),$$

$$I_E(t) - Y_E(t)\}.$$

For convenience the following nomenclature will be used

$$A_p = \{\hat{x}_p(t|\theta_E),\ P(\theta_E|Y_p),\ I_p(t) - Y_p(t)\}$$

$$A_E = \{\hat{x}_E(t|\theta_p),\ P(\theta_p|Y_E),\ I_E(t) - Y_E(t)\} \tag{40}$$

Thus, the equations may be written as

$$\min_{u\in U} E\{\frac{\partial V_p}{\partial t} + \frac{\partial V_p}{\partial x}^T [F(t,\ \theta_p,\ \theta_E)x(t) - G_p(t)R_p^{-1}(t)\ G_p^T(t)\ \cdot$$

$$E\{\frac{\partial V_p}{\partial x} \mid A_p\} - G_E(t)\ R_E^{-1}(t)G_E^T(t)E\{\frac{\partial V_E}{\partial x} \mid A_E\} + \tag{41}$$

$$\tfrac{1}{2}(x^T(t)Q(t)x(t) + E\{\frac{\partial V_p}{\partial x}^T |A_p\}G_p(t)R_p^{-1}(t)G_p^T(t)$$

$$\cdot E\{\frac{\partial V_p}{\partial x} \mid A_p\} + E\{\frac{\partial V_E}{\partial x}^T \mid A_E\}G_E(t)R_E^{-1}(t)G_E^T(t)E\{\frac{\partial V_E}{\partial x} \mid A_E\} +$$

$$\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V_p}{\partial x_i \partial x_j} m_{ij} \,|A_p\} = 0$$

and

$$\max_{v \in V} E\{\frac{\partial V_E}{\partial t} + \frac{\partial V_E}{\partial x}^T [F(t, \theta_p, \theta_E)x(t) - G_E(t)R_E^{-1}(t)G_E^T(t)$$

$$\cdot E\{\frac{\partial V_E}{\partial x}|A_E\} - G_p(t)R_p^{-1}(t)G_p^T(t)E\{\frac{\partial V_p}{\partial x}|A_p\} + \tag{42}$$

$$½(x^T(t)Q(t)x(t) + E\{\frac{\partial V_p}{\partial x}^T|A_p\}G_p(t)R_p^{-1}(t)G_p^T(t)E\{\frac{\partial V_p}{\partial x}|A_p\}$$

$$+ E\{\frac{\partial V_E}{\partial x}^T|A_E\}G_E(t)R_E^{-1}(t)G_E^T(t)E\{\frac{\partial V_E}{\partial x}|A_E\} +$$

$$½ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_E}{\partial x_i \partial x_j} m_{ij} |A_E\} = 0 .$$

The Hamilton-Jacobi equations may be rewritten as

$$E\{\frac{\partial V_p}{\partial t} + \frac{\partial V_p}{\partial x}^T [F \hat{x}_p(t) - G_p R_p^{-1} G_p^T E\{\frac{\partial V_p}{\partial x}|A_p\} -$$

$$G_E R_E^{-1} G_E^T E\{\frac{\partial V_E}{\partial x}|A_E\}] + ½(\hat{x}_p(t)Q\hat{x}_p(t) + \text{tr } QP_{pp} + \tag{43}$$

$$E\{\frac{\partial V_p}{\partial x}^T|A_p\}G_p R_p^{-1} G_p^T E\{\frac{\partial V_p}{\partial x}|A_p\} + E\{\frac{\partial V_E}{\partial x}^T|A_E\}G_E R_E^{-1}$$

$$\cdot G_E^T E\{\frac{\partial V_E}{\partial x}|A_E\}) + ½ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_p}{\partial x_i \partial x_j} m_{ij} |A_p\} = 0$$

and

$$E\{\frac{\partial V_E}{\partial t} + \frac{\partial V_E}{\partial x}^T [F \hat{x}_E(t) - G_p R_p^{-1} G_p^T E\{\frac{\partial V_p}{\partial x}|A_p\} -$$

$$G_E R_E^{-1} G_E^T E\{\frac{\partial V_E}{\partial x}|A_E\}] + (½\hat{x}_E^T(t)Q\hat{x}_E(t) + \text{tr } QP_{EE} + \tag{44}$$

$$E\{\frac{\partial V_p}{\partial x}^T|A_p\}G_p R_p^{-1} G_p^T E\{\frac{\partial V_p}{\partial x}|A_p\} + E\{\frac{\partial V_E}{\partial x}^T|A_E\}G_E$$

$$\cdot R_E^{-1} G_E^T E\{\frac{\partial V_E}{\partial x}|A_E\}) + ½ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_E}{\partial x_i \partial x_j} m_{ij} |A_E\} = 0$$

where

$$\hat{x}_p(t) = \underset{\theta_E | Y_p}{E} \{\hat{x}_p(t|\theta_E)\}$$

$$\hat{x}_E(t) = \underset{\theta_p | Y_E}{E} \{\hat{x}_E(t|\theta_p)\}. \tag{45}$$

The solution to the Hamilton-Jacobi equations will be taken in the form

$$V_p(t) = \tfrac{1}{2} \underset{\theta_E | Y_p}{E} \{\hat{x}_p^T(t|\theta_E)\} P_p(t) \underset{\theta_E | Y_p}{E} \{\hat{x}_p(t|\theta_E)\} + A_p(t)$$

$$V_E(t) = \tfrac{1}{2} \underset{\theta_p | Y_E}{E} \{\hat{x}_E^T(t|\theta_p)\} P_E(t) \underset{\theta_p | Y_E}{E} \{\hat{x}_E(t|\theta_p)\} + A_E(t). \tag{46}$$

This yields the optimal open-loop feedback strategies

$$u^*(t) = -R_p^{-1}(t) G_p^T(t) \underset{\theta_p, \theta_E | I_p(t)}{E} \{P_p(t)\} \underset{\theta_E | Y_p}{E} \{\hat{x}_p(t|\theta_E)\}$$

$$v^*(t) = -R_E^{-1}(t) Q_E^T(t) \underset{\theta_p, \theta_E | I_E(t)}{E} \{P_E(t)\} \underset{\theta_p | Y_E}{E} \{\hat{x}_E(t|\theta_p)\}. \tag{47}$$

In a similar manner as in Chapter III, the above Hamilton-Jacobi equations will be coupled. Thus, each player will make the assumption that his opponent has a measurement subsystem that takes perfect measurements and that his opponent is playing the security strategies as outlined in the statement of the theorem. Since each player does not have a measurement subsystem taking perfect measurements, he uses his best estimate of the state as his best knowledge of his opponent's measurements. One may use the assumption and the assumed solution form into the Hamilton-Jacobi equations as in Chapter III. The equations may be placed into the following form.

The Hamilton-Jacobi equations may be rewritten as follows with the above assumptions

$$E\{\tfrac{1}{2}\hat{x}_p^T(t)\ (\frac{\partial P_p(t)}{\partial t} + P_p(t)F(t,\ \theta_p,\ \theta_E) +$$

$$F(t,\ \theta_p,\ \theta_E)^T P_p(t) + Q(t) - P_p(t)G_p(t)R_p^{-1}(t)G_p^T(t) \tag{48}$$

$$\underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\} - \underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)P_p(t) -$$

$$P_p(t)G_E(t)R_E^{-1}(t)G_E^T(t)\underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\} -$$

$$\underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)P_p(t) + \underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\}G_p(t)$$

$$\cdot R_p^{-1}(t)G_p^T(t)\underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\} + \underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\}G_E(t)R_E^{-1}(t)$$

$$\cdot G_E^T(t)\underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\})\hat{x}_p(t) + \text{tr } QP + \dot{A}_p(t) +$$

$$\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} P_{p_{ij}} m_{ij}\} = 0$$

and

$$E\{\tfrac{1}{2}\hat{x}_E^T(t)\ (\frac{\partial P_E(t)}{\partial t} + P_E(t)F(t,\ \theta_p,\ \theta_E) +$$

$$F(t,\ \theta_p,\ \theta_E)^T P_E(t) - P_E(t)G_p(t)R_p^{-1}(t)G_p^T(t)\underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\} - \tag{49}$$

$$\underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\}G_p(t)R_p^{-1}(t)G_p^T(t)P_E(t) - P_E(t)G_E(t)R_E^{-1}(t)$$

$$\cdot G_E^T(t)\underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\} - \underset{\theta_p\theta_E|I_E(t)}{E}\{P_E(t)\}G_E(t)R_E^{-1}(t)G_E^T(t)P_E(t) +$$

$$Q(t) + \underset{\theta_p,\theta_E | I_p(t)}{E} \{P_p(t)\} G_p(t) R_p^{-1}(t) G_p^T(t) \underset{\theta_p,\theta_E | I_p(t)}{E} \{P_p(t)\} +$$

$$\underset{\theta_p,\theta_E | I_E(t)}{E} \{P_E(t)\} G_E(t) R_E^{-1}(t) G_E^T(t) \underset{\theta_p,\theta_E | I_E(t)}{E} \{P_E(t)\}) x_E(t) +$$

$$tr\ QP + \dot{A}_E(t) + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} P_{E_{ij}} m_{ij} \} = 0 .$$

Thus, the strategies consist of solving the same equations for the control gains as in the perfect information problem except that the expected values used are conditioned on the measurement functional and not on the state trajectory. The best estimate of the state is used for feedback. Thus, separation under the assumptions occur.

It may be established that the strategies for the imperfect information problem are the same as the perfect information problem except that the controller is cascaded with an estimator. Thus, one may use the same equations as in the theorems of Chapter III for the control gains. In order to eliminate redundancy, the equivalent theorems will not be proved. The particular security strategy may be found by use of the control gains for the particular security strategy in Chapter III with a state estimator used for state feedback.

The type strategies considered in this chapter are measurement security strategies. The subclass of measurement security strategies considered in this chapter are both system security strategies and system risk strategies. Since each player has uncertainty in the game dynamics in general the error in the estimate of the game state is not realizable by each player. It is recognized that in specific problems where one player has no uncertainty as to the game dynamics the player may be able to realize his opponent's error in his estimate of the game dynamics. In general, the player runs a very definite risk by use of a measurement risk strategy. Therefore, the measurement security strategy will give the player a conservative and more realistic strategy. This may be easily seen since each player does not have perfect knowledge of the game dynamics. Thus, he runs a risk of his state estimator not performing adequately versus that of an optimal Kalman filter. Any attempt on his part to reconstruct his opponent's state estimator is extremely nonconservative as this reconstructed estimator may be in general at best inaccurate and be at worst totally erroneous. There are three basic reasons. The players, in general, only have a probabilistic representation of his opponents uncertainty and, thus, runs a risk because of this. Each player must utilize his opponent's control law in both his estimator and his opponent's estimator. Since his estimator will be biased because of an erroneous opponent's control law caused by his uncertainty, any further uncertainty in his opponent's control law may be compounded by trying to reconstruct his opponent's estimation error. Since he does not have his opponent's measurement subsystem, he certainly cannot reconstruct his measurement functional.

The required parameter and state estimation will be discussed.

Parameter and State Estimation

The expected value operators $\underset{\theta_p,\theta_E|I_p(t)}{E}\{\cdot\}$ and $\underset{\theta_p\theta_E|I_E(t)}{E}\{\cdot\}$ will be discussed. The expected value operators may be written as in Equation (159) in Chapter III, i.e.,

$$\underset{\theta_p,\theta_E|I_p(t)}{E}\{\cdot\} = \int_D (\cdot)p(\theta_p,\theta_E|I_p(t))d\theta_p d\theta_E$$
$$\underset{\theta_p,\theta_E|I_E(t)}{E}\{\cdot\} = \int_D (\cdot)p(\theta_p,\theta_E|I_E(t))d\theta_p d\theta_E . \tag{50}$$

As in Chapter III, the above equations may be rewritten as

$$\underset{\theta_p,\theta_E|I_p(t)}{E}\{\cdot\} = \int_D (\cdot)p(\theta_p|I_p(t))p(\theta_E|I_p(t))d\theta_p d\theta_E$$

$$\underset{\theta_p, \theta_E | I_E(t)}{E}\{\cdot\} = \int_D (\cdot) p(\theta_p | I_E(t)) p(\theta_E | I_E(t)) d\theta_p d\theta_E .$$

The probability density functions for $p(\theta|Y)$ where Y is the measurement functional ($Y \subset I$) will be developed as in reference (57). This will allow one to obtain the necessary equations for parameter adaptation.

Also to be discussed in this section is the estimator in order to obtain the best estimate of the state x(t). This estimate is necessary since the measurement subsystems of each player do not necessarily observe each state component directly but through a linear transformation which is also driven by additive white noise.

Theorem 4.3: (Lainiotis Partition Theorem). The minimum variance estimate of the state of the system

$$\dot{x}(t) = F(t,\theta)x(t) + G(t)w(t) \tag{52}$$

where

x(t) is an n-vector denoting the state of the system,

θ is an p-vector of unknown time-invariant parameters with a priori probability density $P(\theta)$,

$F(t, \theta)$ is an $n \times n$ system matrix parameterized by θ,

w(t) is an m-vector of white noise inputs with zero mean and variance,

$$E\{w(t)w^T(\tau)\} = W(t)\delta(t-\tau) \tag{53}$$

which is observed by the measurement process

$$y(t) = H(t)x(t) + \eta(t) \tag{54}$$

where

y(t) is an q-vector of measurements,

H(t) is an $q \times n$ measurement matrix,

$\eta(t)$ is an q vector of white noise inputs with zero mean and variance,

$$E\{\eta(t)\eta^T(\tau)\} = R(t)\delta(t-\tau) \tag{55}$$

is given by

$$\hat{x}(t|O_t) = \int \hat{x}(t|O_t, \theta)p(\theta|O_t)d\theta$$

where

O_t represents the measurement functional, i.e.,

$$O_t = \{y(\tau),\ \tau \epsilon [t_0,\ t)\}$$

$\hat{x}(t|O_t,\ \theta)$ is the minimum variance estimate of the state given the parameter vector θ and the measurement functional,

$p(\theta|O_t)$ is the probability density of the parameter vector θ conditioned on the measurement functional.

The conditional state error covariance matrix is given as

$$\eta(t) = \int \{P(t|\theta) + [\hat{x}(t|\theta) - \hat{x}(t)][\hat{x}(t|\theta) - \hat{x}(t)]^T p(\theta|O_t)\} d\theta$$

where

$P(t|\theta)$ is the error variance for the Kalman filter conditioned on knowledge of θ.

Proof: The minimum variance estimate x(t O) may be written as

$$\hat{x}(t|O_t) = E\{x(t)|O_t\}. \tag{57}$$

The use of the smoothing properties of expectations allows one to write $\hat{x}(t|O_t)$ as

$$\begin{aligned} \hat{x}(t|O_t) &= \underset{\theta}{E}\{E\{x(t)|O_t,\ \theta\}|O_t\} \\ &= \int \hat{x}(t|O_t,\ \theta) p(\theta|O_t) d\theta \end{aligned} \tag{58}$$

where

$$\hat{x}(t|O_t,\ \theta) = E\{x(t)|O_t,\ \theta\} \tag{59}$$

which is the well known Kalman filter estimate of a linear system. This will be discussed later. The probability density function $p(\theta|O_t)$ is required. This will now be derived by use of Bucy's Representation Theorem (Lemma A.1 in Appendix A).

One may apply Baye's rule to obtain

$$p(\theta|O_t) = \frac{p(x(t),\ \theta|O_t)}{p(x(t)|O_t,\ \theta)} \cdot \tag{60}$$

Now, one may define a new state vector x_a by augmenting the state vector x with θ, i.e.,

$$x_a(t) = \begin{bmatrix} x(t) \\ --- \\ \theta \end{bmatrix}. \tag{61}$$

Bucy's Representation Theorem (see Appendix A) may now be used to obtain

$$p(x_a(t)|O_t) = \frac{E^{O_t}\{\exp H^{O}|x_a(t)\}p\{x_a(t)\}}{E^{O_t}\{\exp H^{O}\}} \tag{62}$$

where $p\{x_a(t)\} = p\{x(t)|\theta\}p(\theta)$

and

$$\begin{aligned} H^{O} &= \int_{t_0}^{t} x^T(\sigma)H^t(\sigma)R^{-1}(\sigma)y(\sigma)d\sigma - \\ &\quad ½\int_{t_0}^{t} ||H(\sigma)||^2_{R^{-1}(\sigma)}d\sigma - \\ &= \int_{t_0}^{t} x^T(\sigma)H^T(\sigma)R^{-1}(\sigma)y(\sigma)d\sigma - \\ &\quad ½\int_{t_0}^{t} ||H(\sigma)x(\sigma)||^2_{R^{-1}(\sigma)}\ d\sigma . \end{aligned} \tag{63}$$

The Representation Theorem may be applied again to obtain $p(x(t)|O_t,\ \theta)$, i.e.,

$$p(x(t)|O_t,\ \theta) = p(x_a(t)|\ O_t,\ \theta) = \frac{E^{O_t}\{\exp H^O|x_a(t)\}p\{x(t)\}}{E^{O_t}\{\exp H^O|\theta\}} . \tag{64}$$

Thus, Equation (60) may be rewritten as

$$p(\theta\ O_t) = \frac{E^{O_t}\{\exp H^O|\theta\}p(\theta)}{E^{O_t}\{\exp H^O\}} = \frac{E^{O_t}\{\exp H^O|\theta\}p(\theta)}{\int E^{O_t}\{\exp H^O|\theta\}p(\theta)d\theta} \tag{65}$$

where

$$E^{O_t}\{\exp H^O|\theta\} = \exp\{\int_{t_0}^{t}\hat{x}^T(\sigma|O_\sigma,\ \theta)H^T(\sigma)R^{-1}(\sigma)y(\sigma)d\sigma - \frac{1}{2}\int_{t_0}^{t}||H(\sigma)\hat{x}(\sigma|O_\sigma,\ \theta)||^2_{R^{-1}(\sigma)}d\sigma . \tag{66}$$

Thus,

$$p(\theta|O_t) = \frac{\exp\{\int_{t_0}^{t}\hat{x}^T(\sigma|O_\sigma,\ \theta)H^T(\sigma)R^{-1}(\sigma)y(\sigma)d\sigma) - \frac{1}{2}\int_{t_0}^{t}||H(\sigma)\hat{x}(\sigma|\ O_\sigma,\ \theta)||^2_{R^{-1}(\sigma)}d\sigma\}p(\theta)}{\int\exp\{\int_{t_0}^{t}\hat{x}^T(\sigma|O_\sigma,\ \theta)H^T(\sigma)R^{-1}(\sigma)y(\sigma)d\sigma - \frac{1}{2}\int_{t_0}^{t}||H(\sigma)\hat{x}(\sigma|O_\sigma,\ \theta)||^2_{R^{-1}(\sigma)}d\sigma\}p(\theta)d\theta} \tag{67}$$

the required result.

Theorem 4.4: (Lainiotis Partition Theorem over a discrete parameter range.)

If the parameters are defined over a discrete range, then the conditional probability $P_r(\theta_i|O_t)$ for every parameter index i is given as

$$P_r(\theta_i|O_t) = \frac{\exp\left\{\int_{t_0}^{t} \hat{x}^T(\sigma|O_\sigma, \theta_i)H^t(\sigma)R^{-1}(\sigma)y(\sigma)d\sigma - \frac{1}{2}\int_{t_0}^{t}||H(\sigma)\hat{x}(\sigma|O_\sigma, \theta_i)||^2_{R^{-1}(\sigma)}d\sigma\right\}p_r(\theta_i)}{\sum_{j=1}^{n}\exp\left\{\int_{t_0}^{t}\hat{x}^T(\sigma|O_\sigma, \theta_j)R^{-1}(\sigma)y(\sigma)d\sigma - \frac{1}{2}\int_{t_0}^{t}||H(\sigma)\hat{x}(\sigma|O_\sigma, \theta_j)||^2_{R^{-1}(\sigma)}d\sigma\right\}p(\theta_j)}. \quad (68)$$

Proof: Theorem 4.3 is used with the following substitution and performing the required integrations

$$p(\theta) = \sum_{j=1}^{n} p_r(\theta_i)\delta(\theta - \theta_i). \quad (69)$$

Lemma 4.5: (Kalman Filter)

The minimum variance estimate of the state of the system

$$\dot{x}(t) = F(t, \theta)x(t) + G(t)w(t) \quad (70)$$

where

$x(t)$ is an n-vector denoting the state of the system,

θ is an P-vector of <u>known</u>, <u>deterministic</u> parameters,

$F(t, \theta)$ is an $n \times n$ system matrix,

$w(t)$ is an m-vector of white noise inputs with zero-mean and variance

$$E\{w(t)w^T(\tau)\} = \bar{Q}(t)\delta(t-\tau) \tag{71}$$

which is observed by the measurement process

$$y(t) = H(t)x(t) + v(t) \tag{72}$$

where

$y(t)$ is an q-vector of measurements,

$H(t)$ is an $q \times n$ measurement matrix,

$v(t)$ is an q-vector of white noise inputs with zero mean and variance

$$E\{v(t)v^T(\tau)\} = R(t)\cdot\delta(t-\tau) \tag{73}$$

is given as the solution to the Differential equation

$$\dot{\hat{x}}(t|\theta, O_t) = F(t, \theta)\hat{x}(t|\theta, O_t) + K(t)\{y(t) - H(t)\hat{x}(t|O_t, \theta)\} \tag{74}$$

where

$K(t)$ is given by

$$K(t) = P(t)H^T(t)R^{-1}(t) \tag{75}$$

and

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)\bar{Q}(t)G^T(t) \tag{76}$$

where

$P(t_o) = P_o$

$\hat{x}(t_o) = E\{x(t_o)|\theta\}$.

Proof: This is the well known Kalman filter and the proof may be found in several references (68) (52) (21).

Example

The example problem in Chapter III will be studied again except that instead of the evader having a measurement subsystem capable of obtaining perfect measurements of the system state, the measurement subsystem is noisy, i.e.,

$$y = x + v \tag{77}$$

where

v is white noise with zero mean and variance

$$\mathrm{cov}(v) = 10.0 \ . \tag{78}$$

The pursuer has perfect information of the state. Each player plays a measurement security, system security, opponent security strategy. The pursuer's strategy is as shown in Equations (175) and (176) in Chapter Chapter III. The evader's strategy is

$$v = -\frac{b}{r_E}\Big(\sum_{i=1}^{3} P_r(\theta_i \mid O_{n-1}) P_E'(t, \theta_i)\Big)\Big(\sum_{i=1}^{3} P_r(\theta_i \mid O_{n-1})\hat{x}(t \mid \theta_i)\Big) \tag{79}$$

where $P_E'(t, \theta_i)$ is as in Equations (178) and (179), $Pr(\theta_i \mid O_{n-1})$ may be determined by using the Partition Theorem. The variable $\hat{x}(t \mid \theta_i)$ is the Kalman filter best estimate conditioned on the parameter θ_i.

The results of the research are compared with the strategies obtained by using the derivation method in the work in (58) (60) and extended to the differential game, i.e.,

$$v(t) = -\frac{b}{r_E}\left(\sum_{i=1} P_r(\theta_i \mid O_{n-1}) P_E^*(t, \theta_i)\hat{x}(t \mid \theta_i)\right) \tag{80}$$

where P_E^* is the solution to Equation (186) in Chapter III.

The performance index using the results of research yielded a performance index average over 200 runs of 1.77 while the results using Equation (80) yielded an average of 1.47. The results of this research lead to improved performance index.

Again, reference (60) did not claim optimal results. However, again it leads to an appealing engineering solution.

A typical optimal sample run is shown in Figures 1-4. In Figure 1 the state trajectory is plotted versus time. Figure 2 shows the evader's control law and Figure 3 shows the pursuer's control law. Figure 4 contains a plot of the parameter probabilities versus time. One may note that the adaptation takes place in ~0.5 sec.

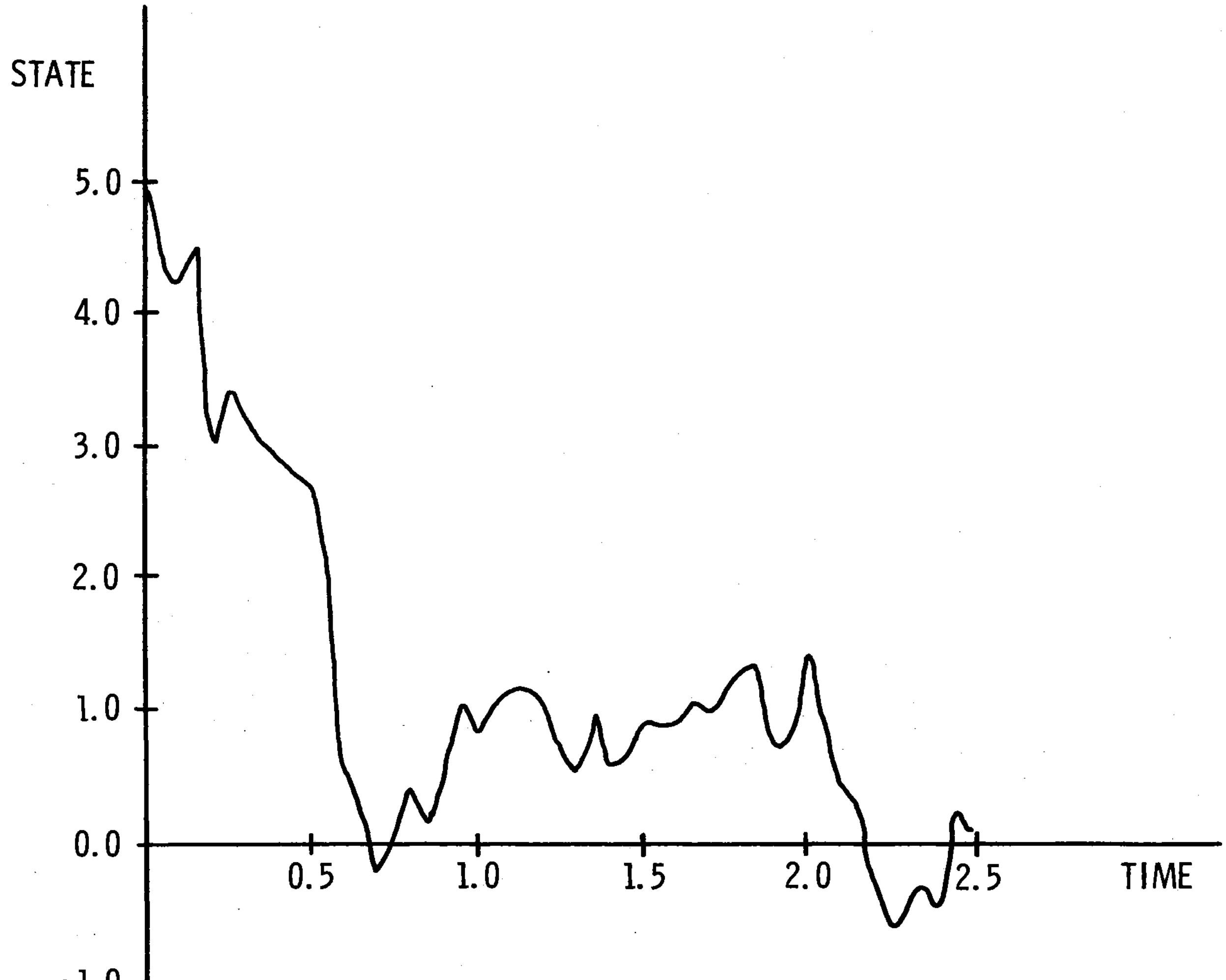

Figure 1. State Trajectory of System

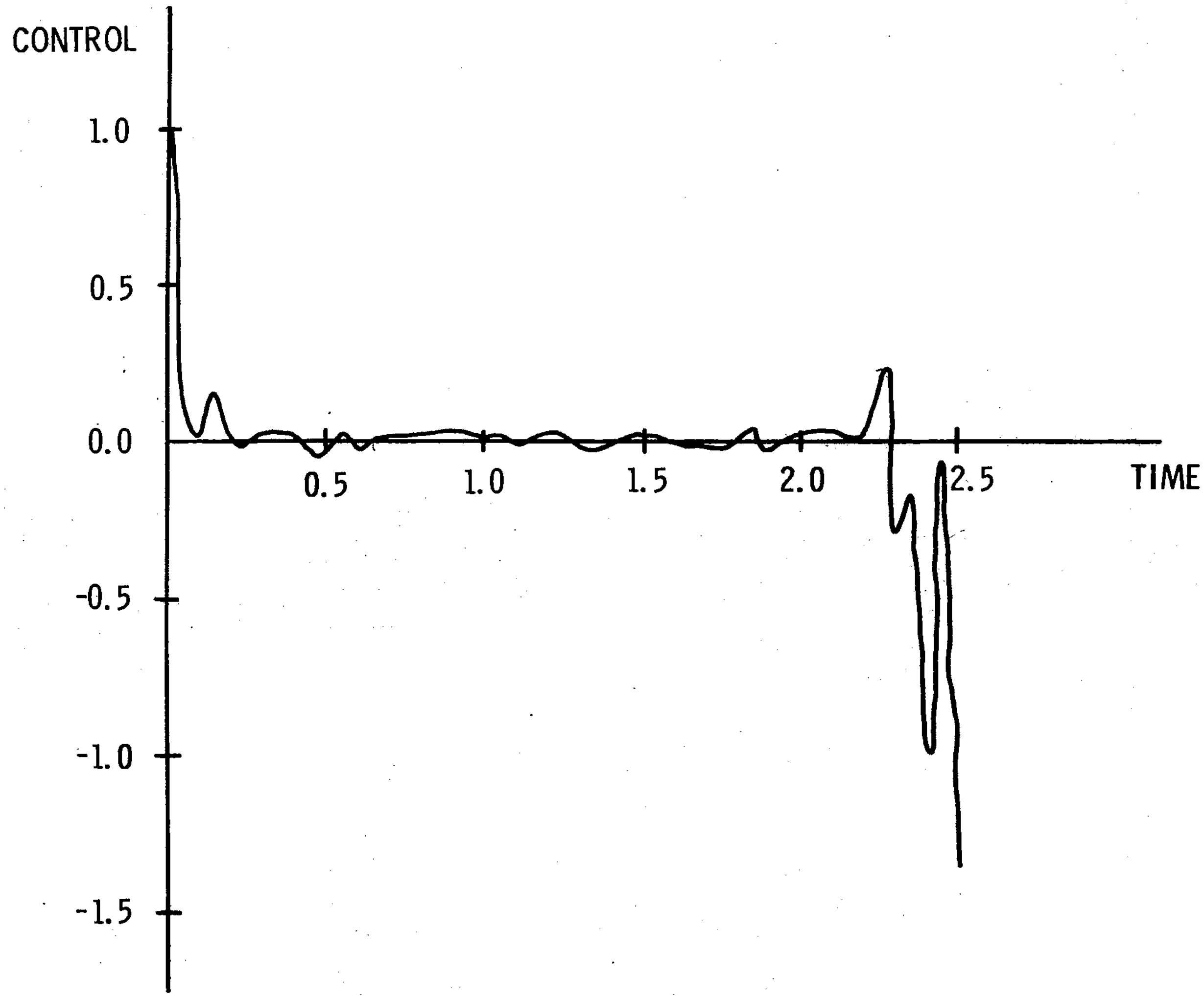

Figure 2. Evader's Control Trajectory

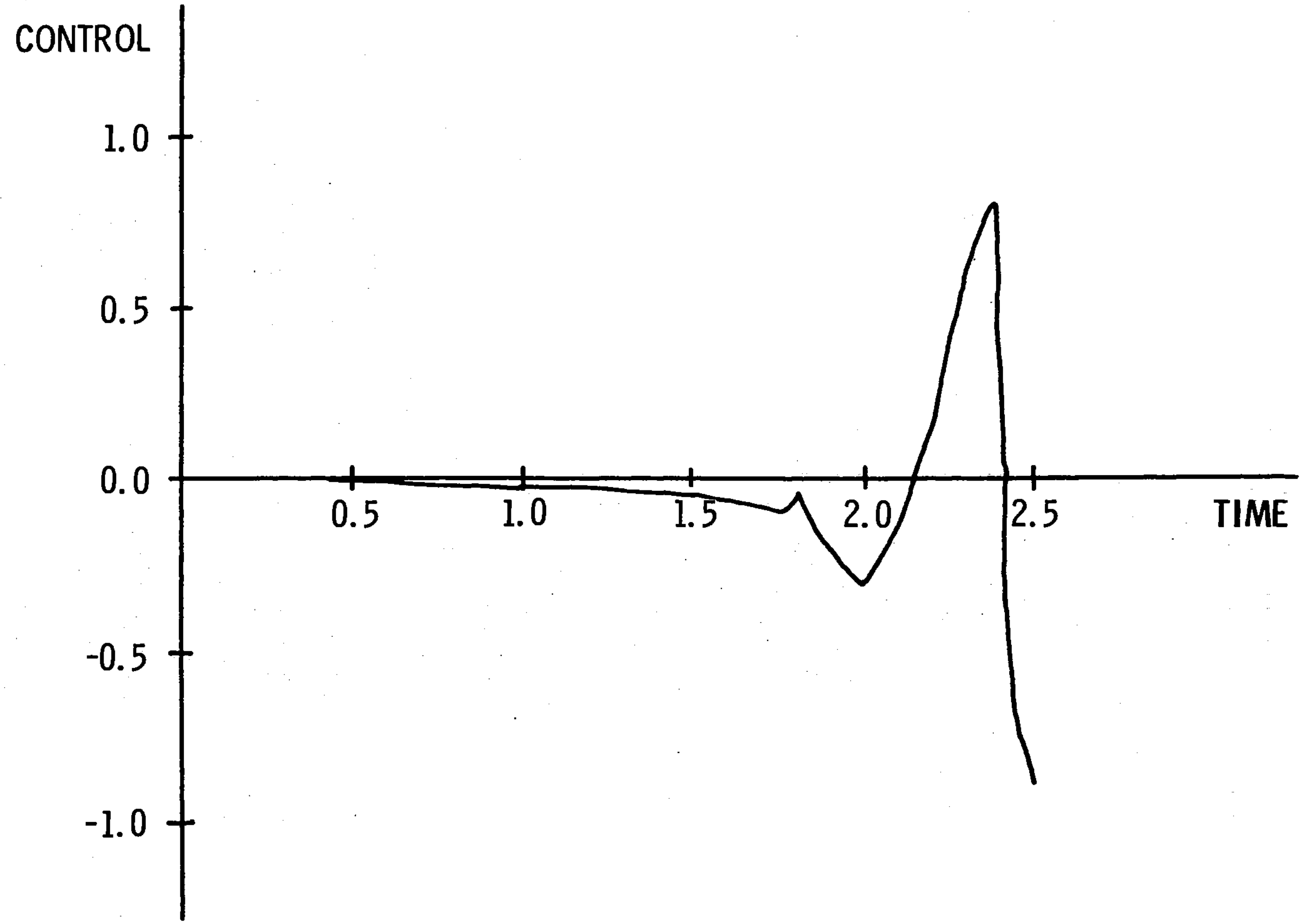

Figure 3. Pursuer's Control Trajectory

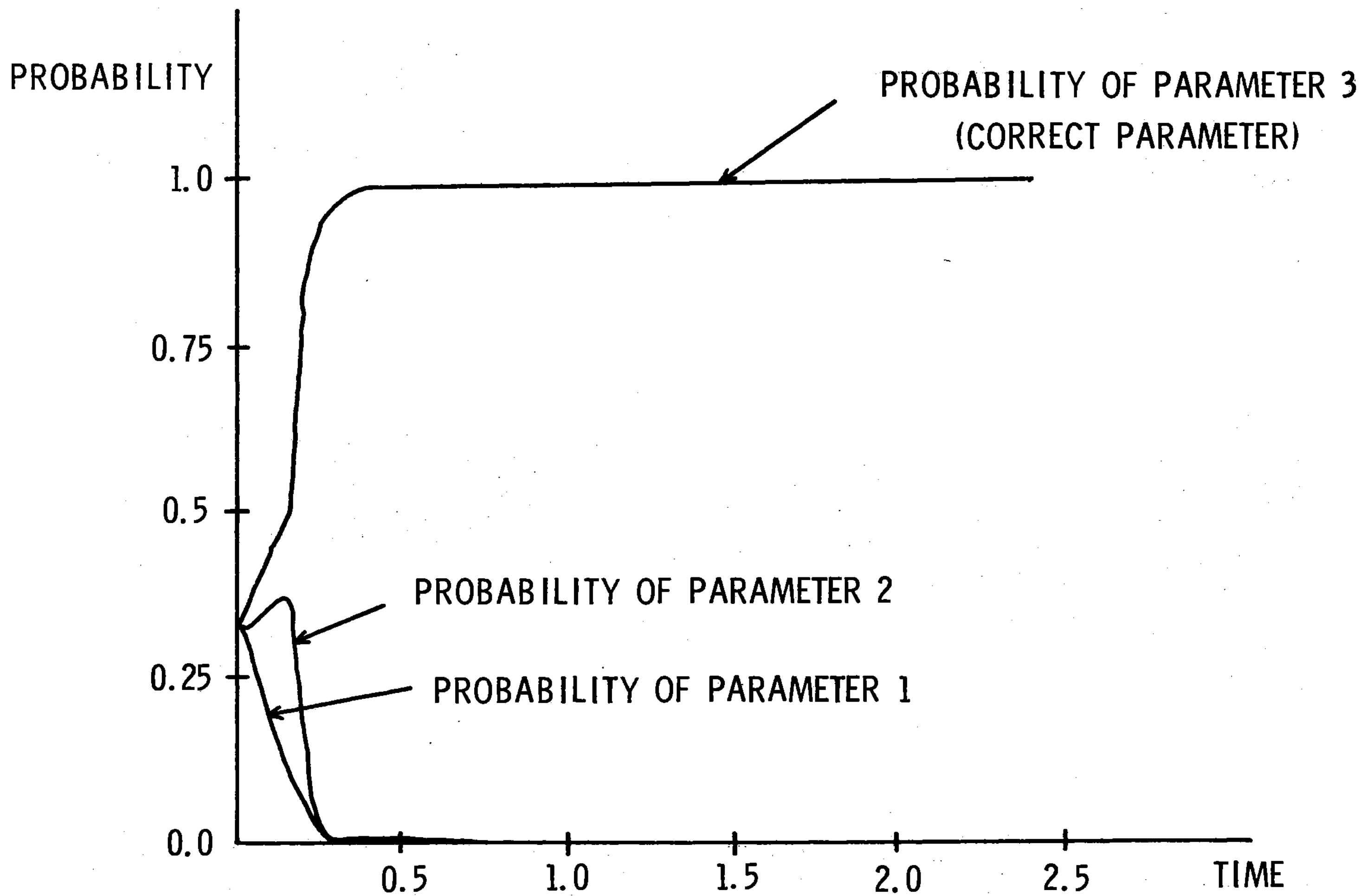

Figure 4. Parameter Probabilities

CHAPTER V

APPLICATIONS

Introduction

This chapter will address itself to the use of the methodology of solution for differential games under uncertainty to investigate an area of proven and potential application. The area is that of target interception. The uses of target interception theory include that of missile guidance, air-to-air combat with aircraft, and the uncooperative interception of vehicles in space. The problem of precision pointing of a laser beam while tracking a target might be considered that of target interception. In the previous problems, it is necessary to find control laws for a vehicle to insure that the vehicle will be as close as possible to the target at some time. In the pointing and tracking problem, one needs to find control laws for the pointing and tracking control system to insure that the beam illuminates the specified point on the accelerating target for a length of time.

In the development of guidance laws for target intercept as above, one is faced with the fact that if the models of the interceptor and the target were depicted as the full six degree of freedom equations, then the equations would be both extremely difficult to work with and it would be impossible to find feedback forms for the guidance laws. Thus, in the past, very simple engagement models were used in order to depict the kinematics of the interceptor-target geometry. Several references for this area are (1), (2), (39), (25), and (31).

In the engagement models, it was assumed, except for reference (25), that the interceptor control acceleration did not lag behind the commanded acceleration. This is, of course, erroneous due to the fact that one has lags due to computation time, actuator lags, vehicle responsiveness, and smoothing time of measurements of the target motion. In the development of new intercept laws in this chapter, the following assumptions will be made. First, the control acceleration lags the commanded acceleration by a first-order lag. This applies to both interceptor and target. Secondly, neither player knows the time constants of the lags associated with his opponent. Thirdly, it is assumed that each player has a direct measurement of the relative range vector and velocity vector corrupted with noise. It will also be assumed that the pertinent dynamics of the two vehicles be described by linear equations. The interceptor will play measurement security, system security, opponent security strategies, and the target will play measurement security, system security, opponent security strategies.

Statement of the Problem

A general problem of interception will be stated and a solution proposed in this section. In the next section, the results will be used more explicitly in the examples. The states of the pursuer and the evader are assumed to be described by the following differential equations

$$\begin{aligned} \dot{x}_p(t) &= F_p(t, \theta_p)x_p(t) + G_p(t)u(t) \\ \dot{x}_E(t) &= F_E(t, \theta_E)x_E(t) + G_E(t)v(t) \end{aligned} \qquad (1)$$

where the definitions of the variables are as in the previous chapters and dim x_p = dim x_E.

The performance index is the terminal miss

$$\frac{a}{2}||x_p(t_f) - x_E(t_f)||^2_{A^TA} \tag{2}$$

where the matrix A is of the form $[I \vdots 0]$. In order to limit the control the following inequality constraints will be used (it is evident that each player will use all control available to him -- thus, the equality will be used), i.e.,

$$\begin{aligned} \int_{t_0}^{t_f} ||u||^2_{R_p} dt &= E_p \\ \int_{t_0}^{t_f} ||v||^2_{R_E} dt &= E_E \ . \end{aligned} \tag{3}$$

The constraints may be adjoined to the performance index. Thus,

$$J = \frac{a}{2}||x_p(t_f) - x_E(t_f)||^2_{A^TA} + \tfrac{1}{2}\int_{t_0}^{t_f} (||u||^2_{R_p} + ||v||^2_{R_E})dt. \tag{4}$$

Except for the consideration of the parameter sets θ_p and θ_E at this point the problem formulation follows that of reference (20). A new state vector will be defined, i.e.,

$$Z(t, \theta_p, \theta_E) = A\{\Phi_p(t_f, t, \theta_p)x_p(t) - \Phi_E(t_f, t, \theta_E)x_E(t)\} \tag{5}$$

where $\Phi_p(t_f, t, \theta_p)$ is the transition matrix for the pursuer's dynamics and $\Phi_E(t_f, t, \theta_E)$ is the transition matrix for the evader's dynamics. It may be shown that the new state equations become

$$\dot{Z}(t, \theta_p, \theta_E) = \overline{G}_p(t, \theta_p)u(t) + \overline{G}_E(t, \theta_E)v(t) \tag{6}$$

where

$$\bar{G}_p(t, \theta_p) = A\Phi_p(t_f, t, \theta_p)G_p(t)$$
$$\bar{G}_E(t, \theta_E) = -A\Phi_p(t_f, t, \theta_E)G_E(t) \tag{7}$$

and

$$Z(t_0) = A[\Phi_p(t_f, t_0, \theta_p)x_p(t_0) - \Phi_E(t_f, t_0, \theta_E)x_E(t_0)\} \tag{8}$$

The performance index may be rewritten as

$$J = \frac{a}{2}\|Z(t_f)\|^2 + \tfrac{1}{2}\int_{t_0}^{t_f}(\|u\|^2_{R_p} + \|v\|^2_{R_E})dt. \tag{9}$$

The pursuer has the information set

$$I_p(t) = p_{\theta_E}(\theta_E) \cup \theta_p \cup y_{p_1}(\tau),$$
$$\tau_E([t_0, t]) \cup (y_{E_1}(\tau), \tau \in [t_0, t]). \tag{10}$$

where y_{p_2}, y_{E_2} are linear measurements of $x_p(t)$ and $x_E(t)$ assumed corrupted by additive noise.

The evader has the information set

$$I_E(t) = p_{\theta_E}(\theta_p) \cup \theta_E \cup (y_{p_2}(\tau),$$
$$\tau_E([t_0, t]) \cup y_{E_2}(\tau), \tau \in [t_0, t]). \tag{11}$$

where y_{p_1}, y_{E_1} are a linear measurements of $x_p(t)$ and $x_E(t)$ assumed corrupted by additive noise.

The Hamilton-Jacobi equations may be written as

$$\min_{u \in U_{\theta_p, \theta_E}} E_{|I_p(t)} \{\frac{\partial V_p}{\partial t} + \frac{\partial V_p^T}{\partial z}(\bar{G}_p(t, \theta_p)u(t) + \bar{G}_E(t, \theta_E)v(t)) +$$
$$\tfrac{1}{2}(u^T(t)R_p(t)u(t) + v^{*T}(t)R_E(t)v^*(t))\} = 0 \tag{12}$$

$$\max_{v \in V_{\theta_p, \theta_E}} E_{|I_E(t)} \{\frac{\partial V_E}{\partial t} + \frac{\partial V_E^T}{\partial z}(\bar{G}_p(t, \theta_p)u(t) + \bar{G}_E(t, \theta_E)v(t)) +$$
$$\tfrac{1}{2}(u^{*T}(t)R_p(t)u^*(t) + v^T(t)R_E(t)v(t))\} = 0. \tag{13}$$

Extremization yields

$$u^*(t) = -R_p^{-1}(t) \underset{\theta_p, \theta_E | I_{p(t)}}{E} \{\overline{G}_p(t, \theta_p) \frac{\partial V_p}{\partial Z}\}$$

$$v^*(t) = -R_E^{-1}(t) \underset{\theta_p, \theta_E | I_{E(t)}}{E} \{\overline{G}_E(t, \theta_E) \frac{\partial V_E}{\partial Z}\} \tag{14}$$

or

$$u^*(t) = -R_p^{-1}(t)\overline{G}_p(t, \theta_p) \underset{\theta_p, \theta_E | I_{E(t)}}{E} \{\frac{\partial V_p}{\partial Z}\}$$

$$v^*(t) = -R_E^{-1}(t)\overline{G}_E(t, \theta_E) \underset{\theta_p, \theta_E | I_{E(t)}}{E} \{\frac{\partial V_E}{\partial Z}\} . \tag{15}$$

The solution will be taken in the form

$$V_p(t) = \tfrac{1}{2} \underset{\theta_E | I_p}{E} \{Z(t, \theta_p, \theta_E)\}^T P_p(t) \underset{\theta_E | I_p}{E} \{Z(t, \theta_p, \theta_E)\} + A_p(t)$$

$$V_E(t) = \tfrac{1}{2} \underset{\theta_p | I_E}{E} \{Z(t, \theta_p, \theta_E)\}^T P_E(t) \underset{\theta_p | I_E}{E} \{Z(t, \theta_p, \theta_E)\} + A_E(t) . \tag{16}$$

Thus,

$$u^*(t) = -R_p^{-1}(t)\overline{G}_p(t, \theta_p) \underset{\theta_p, \theta_E | I_{p(t)}}{E} \{P_p(t)\} \underset{\theta_E | I_p}{E} \{Z(t, \theta_p, \theta_E)\}$$

$$v^*(t) = -R_E^{-1}(t)\overline{G}_E(t, \theta_E) \underset{\theta_p, \theta_E | I_{E(t)}}{E} \{P_E(t)\} \underset{\theta_p | I_E}{E} \{Z(t, \theta_p, \theta_E)\} . \tag{17}$$

The above may be rewritten as

$$u^*(t) = -R_p^{-1}(t)\overline{G}_p(t, \theta_p) \underset{\theta_p, \theta_E | I_{p(t)}}{E} \{P_p(t)\}\overline{Z}_p(t) \tag{18}$$

$$v^*(t) = -R_E^{-1}(t)\overline{G}_E(t,\ \theta_E)\ \underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\}\hat{Z}_E(t)$$

where

$$\hat{Z}_p(t) = \underset{\theta_E|I_p}{E}\{Z(t,\ \theta_p,\ \theta_E)\}$$

$$\hat{Z}_E(t) = \underset{\theta_p|I_E}{E}\{Z(t,\ \theta_p,\ \theta_E)\} \tag{19}$$

The Hamilton-Jacobi equations may be written as in Chapter IV. This yields the following equations for the gains. The gains for the pursuer are

$$\frac{\partial P_p(t)}{\partial t} = P_p(t)\overline{G}_p(t)R_p^{-1}(t)\overline{G}_p^T(t)\ \underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\} + \underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\}\overline{G}_p(t)$$

$$\cdot R_p^{-1}(t)\overline{G}_p^T(t)P_p(t) + P_p(t)\overline{G}_E(t)R_E^{-1}(t)\overline{G}_E^T(t)T_E(t) + \tag{20}$$

$$\cdot T_E(t)\overline{G}_E(t)R_E^{-1}(t)\overline{G}_E^T(t)P_p(t) - \underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\}\overline{G}_p(t)R_p^{-1}(t)$$

$$\overline{G}_p^T(t)\ \underset{\theta_p,\theta_E|I_p(t)}{E}\{P_p(t)\} - T_E(t)\overline{G}_E(t)R_E^{-1}(t)\overline{G}_E^T(t)T_E(t)$$

and

$$\frac{\partial T_E(t)}{\partial t} = T_E(t)\{\overline{G}_p(t)R_p^{-1}(t)\overline{G}_p^T(t) + \overline{G}_E(t)R_E^{-1}(t)\overline{G}_E^T(t)\}T_E(t). \tag{21}$$

The gains for the evader are

$$\frac{\partial P_E(t)}{\partial t} = P_E(t)\overline{G}_E(t)R_E^{-1}(t)\overline{G}_E^T(t)\ \underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\} + \underset{\theta_p,\theta_E|I_E(t)}{E}\{P_E(t)\}\overline{G}_E(t)$$

$$\cdot R_E^{-1}(t)\overline{G}_E^T(t)P_E(t) + P_E(t)\overline{G}_p(t)R_p^{-1}(t)\overline{G}_p^T(t)T_p(t) + \tag{22}$$

$$T_p(t)\overline{G}_p(t)R_p^{-1}(t)\overline{G}_p^T(t)P_E(t) - T_p(t)\overline{G}_p(t)R_p^{-1}(t)\overline{G}_p^T(t)T_p(t) -$$

$$\underset{\theta_p,\theta_E \mid I_E(t)}{E}\{P_E(t)\}\overline{G}_E(t)R_E^{-1}(t)\overline{G}_E^T(t)\underset{\theta_p,\theta_E \mid I_E(t)}{E}\{P_E(t)\}$$

and

$$\frac{\partial T_p(t)}{\partial t} = T_p(t)\{\overline{G}_p(t)R_p^{-1}(t)\overline{G}_p^T(t) + \overline{G}_E(t)R_E^{-1}(t)\overline{G}_E^T(t)\}T_p(t). \qquad (23)$$

The boundary conditions are

$$P_E(t_f) = P_p(t_f) = T_p(t_f) = a\,I\,.$$

Interception Problem

The problem to be considered is an interception problem in space. This problem was first treated by reference (20). However, the assumption was made that the control acceleration equaled the commanded acceleration. This is not true, in general, due to actuator lags and engine lags. Thus, the assumption that the control acceleration lags the commanded acceleration by a first order lag will be made. It is assumed that each player is observing the relative range and the relative velocities between the two vehicles. The state equations are as follows

$$\begin{aligned}
\dot{r}_p &= v_p \\
\dot{v}_p &= a_p \\
\dot{a}_p &= F_p\, a_p + u \\
\dot{r}_E &= v_E \\
\dot{v}_E &= a_E \\
\dot{a}_E &= F_E\, a_E + v
\end{aligned} \qquad (24)$$

where

r_i, $i = P, E$, is the position vector of a body in three dimensions,

v_i, $i = P, E$, is the velocity vector of a body in three dimensions,

a_i, $i = P, E$, is the control acceleration,

f_i, $i = P, E$, is the gravitational force per unit mass,

F_i, $i = P, E$, is a diagonal matrix of time constants representing the first order lag model.

It will be assumed that $f_p \approx f_E$. That is, the positions in space are near enough such that the differences between the two gravitational forces are negligible. The performance index to be considered is

$$J = \frac{a}{2}\left|\left|r_p(t_f) - r_e(t_f)\right|\right|^2 + \tfrac{1}{2}\int_{t_0}^{t_f} (c_p^{-1}u^T u + c_E^{-1} v^T v)\,dt. \tag{25}$$

The state equations for the pursuer may be written in matrix form as

$$\begin{bmatrix} \dot{r}_p \\ \dot{v}_p \\ \dot{a} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & F_p \end{bmatrix} \begin{bmatrix} r_p \\ v_p \\ a_p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} u \tag{26}$$

where I is a 3 X 3 identity matrix and

$$F_p = \begin{bmatrix} -1/T_1 & 0 & 0 \\ 0 & -1/T_2 & 0 \\ 0 & 0 & -1/T_3 \end{bmatrix}. \tag{27}$$

The state equations for the evader may be written in matrix form as

$$\begin{bmatrix} \dot{r}_E \\ \dot{v}_E \\ \dot{a}_E \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & F_E \end{bmatrix} \begin{bmatrix} r_E \\ v_E \\ a_E \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} v \tag{28}$$

where

$$F_E' = \begin{bmatrix} -1/\tau_1 & 0 & 0 \\ 0 & -1/\tau_2 & 0 \\ 0 & 0 & -1/\tau_3 \end{bmatrix} . \tag{29}$$

The transition matrix for each player may be written as

$$\Phi_p(t, t_0) = \exp\{A_p(t - t_0)\} \tag{30}$$

and

$$\Phi_E(t, t_0) = \exp\{A_E(t - t_0)\} \tag{31}$$

where

$$A_p = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & F_p \end{bmatrix}$$

$$A_E' = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & F_E' \end{bmatrix} . \tag{32}$$

Thus, since

$$\Phi_p(t_f, t_0) = \Phi_p(t_f, t_0)\Phi_p(t, t_0)^{-1} \tag{33}$$

and

$$\Phi_E'(t_f, t) = \Phi_E'(t_f, t_0)\Phi_E'(t, t_0)^{-1} \tag{34}$$

then

$$\begin{aligned} \Phi_p(t_f, t) &= \exp\{A_p(t_f - t_0)\}\exp\{-A_p(t - t_0)\} \\ \Phi_E'(t_f, t) &= \exp\{A_E'(t_f - t_0)\}\exp\{-A_E'(t - t_0)\} \end{aligned} \tag{35}$$

or

$$\begin{aligned} \Phi_p(t_f, t) &= \exp\{A_p(t_f - t)\} \\ \Phi_E'(t_f, t) &= \exp\{A_E'(t_f - t)\} . \end{aligned} \tag{36}$$

Thus, the matrices necessary to solve the auxiliary problem are as follows

$A = [I \vdots 0]$ where I is a 3 X 3 identity matrix and 0 is a 3 X 6 null matrix, (37)

$$G_p = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad G_E' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad R_p = c_p^{-1}, \text{ and } R_E = c_E^{-1} .$$

The transition matrix for the pursuer may be found as

$$\exp\{A_p \tau\} = \begin{bmatrix} I & I & B_1 \\ 0 & I & B_2 \\ 0 & 0 & B \end{bmatrix} \tag{38}$$

where

$$B_1 = \begin{bmatrix} \frac{1}{b_{11}^2}\{\exp(b_{11}\tau) - b_{11}\tau - 1\} & 0 & 0 \\ 0 & \frac{1}{b_{22}^2}\{\exp(b_{22}\tau) - b_{22}\tau - 1\} & 0 \\ 0 & 0 & \frac{1}{b_{33}^2}\{\exp(b_{33}\tau) - b_{33}\tau - 1\} \end{bmatrix} \quad (39)$$

$$B_2 \begin{bmatrix} \frac{1}{b_{11}}\{\exp(b_{11}\tau) - 1\} & 0 & 0 \\ 0 & \frac{1}{b_{22}}\{\exp(b_{22}\tau) - 1\} & 0 \\ 0 & 0 & \frac{1}{b_{33}}\{\exp(b_{33}\tau) - 1\} \end{bmatrix} \quad (40)$$

$$B_3 = \begin{bmatrix} \exp b_{11}\tau & 0 & 0 \\ 0 & \exp b_{22}\tau & 0 \\ 0 & 0 & \exp b_{33}\tau \end{bmatrix} \quad (41)$$

and the b_{ii} are the ii^{th} element of F_p. The transition matrix for the evader may be found as

$$\exp A_E \tau = \begin{bmatrix} I & I & B_1^E \\ 0 & I & B_2^E \\ 0 & 0 & B_3^E \end{bmatrix} \quad (42)$$

where the B_i^E matrices are of the same form as the B_i matrices except that the elements b_{ii} are the ii^{th} element of F_E.

The pseudo control gains $\overline{G}_p$ and $\overline{G}_E$ may be written as

$$\overline{G}_p(t_f, t) = B_1^p(t_f, t)$$
$$\overline{G}_E(t_f, t) = B_1^E(t_f, t). \quad (43)$$

The new pseudo state vector may be written as

$$Z(t) = [I \quad I \quad B_1] \begin{bmatrix} r_p \\ v_p \\ a_p \end{bmatrix} - [I \quad I \quad B_1^E] \begin{bmatrix} r_E \\ v_E \\ a_E \end{bmatrix}$$

$$= r_p + v_p + B_1 a_p - r_E - v_E - B_1^E a_E \tag{44}$$

$$= (r_p - r_E) + (v_p - v_E) + (B_1 a_p - B_1^E a_E).$$

Thus, the control law for the pursuer is

$$u(t) = -c_p B_1^T \underset{\theta_p, \theta_E}{E} \{P_p(t)\,|\,I_{p(t)}\} \underset{\theta_E}{E} \{Z(t)\,|\,Y_p\} \tag{45}$$

and the control law for the evader is

$$v(t) = -c_E B_1^{E^T} \underset{\theta_p, \theta_E}{E} \{P_E(t)\,|\,I_{E(t)}\} \underset{\theta_p}{E} \{Z(t)\,|\,Y_E\}. \tag{46}$$

The gain equations may be written as

$$\frac{\partial P_p}{\partial t} = P_p B_1 c_p B_1^T E\{P_p\} + E\{P_p\} B_1 c_p B_1^T P_p + P_p B_1^E c_E B_1^{E^T} T_E + T_E B_1^E c_E B_1^{E^T} P_p - E\{P_p\} B_1 c_p B_1^T E\{P_p\} - T_E B_1^E c_E B_1^{E^T} T_E \tag{47}$$

where

$$\frac{\partial T_E}{\partial t} = T_E \{B_1 c_p B_1^T + B_1^E c_E B_1^{E^T}\} T_E \tag{48}$$

and

$$\frac{\partial P_E}{\partial t} = P_E B_1^E c_E B_1^{E^T} E\{P_E\} + E\{P_E\} B_1^E c_E B_1^{E^T} P_E +$$

$$P_E B_1 c_p B_1^T T_p + T_p B_1 c_p B_1^T P_E - T_p B_1 c_p B_1^T T_p - \tag{49}$$

$$E\{P_E\} B_1^E c_E B_1^{E^T} E\{P_E\}$$

where

$$\frac{\partial T_p}{\partial t} = T_p \{B_1 c_p B_1^T + B_1^E c_E B_1^{E^T}\} T_p \tag{50}$$

with boundary conditions

$$P_p(t_f) = P_E(t_f) = T_E(t_f) = T_p(t_f) = a\,I.$$

The gain equations may be rewritten as

$$\frac{\partial P_p}{\partial t} = P_p D_p E\{P_p\} + E\{P_p\} D_p P_p + P_p D_E T_E +$$

$$T_E D_E P_p - E\{P_p\} D_p E\{P_p\} - T_E D_E T_E \tag{51}$$

where

$$\frac{\partial T_E}{\partial t} = T_E \{D_E + D_p\} T_E \tag{52}$$

and

$$\frac{\partial P_E}{\partial t} = P_E D_E E\{P_E\} + E\{P_E\} D_E P_E + P_E D_p T_p +$$

$$T_p D_p P_E - T_p D_p T_p - E\{P_E\} D_E E\{P_E\} \tag{53}$$

where

$$\frac{\partial T_p}{\partial t} = T_p \{D_E + D_p\} T_p \tag{54}$$

with

$$D_p = \begin{bmatrix} \tau_1\{\exp(-\frac{(t_f-t)}{\tau_1}) - \frac{(t_f-t)}{\tau_1} - 1\}^2 & 0 & 0 \\ 0 & \tau_2\{\exp(-\frac{(t_f-t)}{\tau_2}) - \frac{(t_f-t)}{\tau_2} - 1\}^2 & 0 \\ 0 & 0 & \tau\,\{\exp(-\frac{(t_f-t)}{\tau}) - \frac{(t_f-t)}{\tau} - 1\}^2 \end{bmatrix} \quad (55)$$

and

$$D_E = \begin{bmatrix} \tau\,\{\exp(-\frac{(t_f-t)}{\tau}) - \frac{(t_f-t)}{\tau} - 1\}^2 & 0 & 0 \\ 0 & \tau\,\{\exp(-\frac{(t_f-t)}{\tau} - \frac{(t_f-t)}{\tau} - 1\}^2 & 0 \\ 0 & 0 & \tau\,\{\exp(-\frac{(t_f-t)}{\tau}) - \frac{(t_f-t)}{\tau} - 1\}^2 \end{bmatrix} \quad (56)$$

and boundary conditions

$$P_p(t_f) = P_E(t_f) = T_p(t_f) = T_E(t_f) = a\,I.$$

One may note that each player requires knowledge of the other player's control acceleration. This may be obtained by obtaining an estimate of the opponent's acceleration. Several references that treat the problem include Asher (6), Singer (89), and Fitzgerald (35 (36). Also, one needs knowledge of the time to go, i.e., $t_f - t$. This may be approximated by dividing the range by the range rate.

Thus, it is shown that the problem of differential games under uncertainty does arise in a missile guidance problem.

The results shown in this chapter may be applied to a realistic guidance problem.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

The problem of differential games under uncertainty has been considered in this research. The state equations are assumed modeled by differential equations in which the equations are parameterized by a time invariant parameter vector. The parameter vector contains elements both known and unknown to each player. It has been shown that each player's strategies depend on the information contained in his information set. That is, the level of intelligence will dictate whether the player has to play certain security strategies or risk strategies. The types of strategies that may occur in differential games under uncertainty and imperfect information are developed in several structure definitions. Past work in differential games with imperfect information is shown to be imbedded in the structure definitions. The definitions classify the type strategies that may occur and, thus, classify the type strategies found in this research. Also, one may use the definitions to indicate future work necessary in differential games under uncertainty.

An optimality condition has been developed and proved to be sufficient for the general nonlinear dynamics problem. The condition uses an expectation conditioned on an information set for each player. It is shown that only a third player with an information set that contains both the pursuer's and the evader's information sets can solve the game without including the assumption as to the type strategy that his opponent will play. One may include this assumption in his information set.

The linear, quadratic problem for both perfect and imperfect measurements is treated and solved for the optimal open-loop feedback strategies. A separation between control and state estimation occurs. The stochastic control analog to this problem is solved and shown in Appendix C. This extends previous work by several authors in that this appendix gives the optimal open-loop feedback controls whereas the other authors only have suboptimal controls.

An example problem of target intercept originally treated by reference (20) is extended and shown to fall within the theory developed in this research. The optimal results are obtained.

One of the pertinent aspects of the solution to differential game under uncertainty is that each player must adapt and learn the parameters unknown to him. If he does not, then he stands to lose some aspect of his goal.

There are several areas to be explored in differential games under uncertainty, i.e.,

(a) the problem of unknown parameters in the performance index (may be solved by state augmentation if the parameters are identifiable),

(b) extensions to non-zero sum differential games,

(c) treatment of the problem whereby the parameters are true-varying,

(d) finding the dual control strategies

 (1) optimal strategies

(2) implementation schemes that are suboptimal,

(e) study of the matrix differential equations for the control gains to determine separability conditions,

(f) determining for several classical differential game problems the effect of an unknown parameter on the location of various surfaces such as barriers,

(g) application of the theory in order to gain insight into several problem areas such as aircraft performance, and

(h) study of the relative effect of parameter identifiability on the solution results.

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APPENDIX A

BUCY'S REPRESENTATION THEOREM

Bucy's Representation Theorem

The following theorem is the basis for work in filtering theory. The solution to the conditional probability density yields all information necessary for the nonlinear estimation problem.

Lemma A.1: (Bucy's Representation Theorem (21)) Consider the nonlinear system

$$dx = f(x, t)dt + \sigma(x, t)d\beta(t) \tag{1}$$

with

$$E\{(\beta(t) - \beta(t_0))(\beta(s) - \beta(t_0))^T\} = \int_{t_0}^{\min(t,s)} Q(\tau)d\tau$$

and

$$x(t_0) = c,$$

and where the system is observed through

$$dz = h(x, t)dt + dv(t) \tag{2}$$

where

$$E\{v(t)v^T(t)\} = \int_{t_0}^{t} R(s)ds.$$

Suppose there exists on the interval (t_0, t) a unique continuous sample function Markov process $x(t)$, a solution of Equation (1) having all its joint probability distributions absolutely continuous with respect to Lebesgue measure. Further, assume that

$$E\{\exp [\sup_{s \in (t_0,t)} ||h(x(s), s)||^2_{R_s^{-1}} (t - t_0)]\} < \infty.$$

Then the conditional distribution of $x(t)$ given O_t (where O_t is the minimal σ - field determined by $z(s)$ for $s \in (t_0, t))$ has a density $p(x|O_t)$ which satisfies

$$p(x|O_t) = \frac{E^{O_t}\{\exp^{H_O}|x(t)\} = \mu_t(x)}{E^{O_t}\{e \quad \}}$$

where

$$H^O = -\tfrac{1}{2} \int_{t_0}^{t} ||h(x(s), s)||^2_{R_s^{-1}} ds + \int_{t_0}^{t} h^T(x(s), s) R_s^{-1} d z(s)$$

and $\mu_t(x)$ is the density function of $x(t)$.

Proof: See reference (21).

APPENDIX B

SOLUTION OF THE PERFECT INFORMATION LINEAR QUADRATIC DIFFERENTIAL GAME

Solution of the Perfect Information Linear

Quadratic Differential Game

The solution to the linear quadratic differential game with perfect information will be shown in this appendix. The system equations are

$$\dot{x}(t) = F(t)x(t) + G_p(t)u(t) + G_E(t)v(t) \tag{1}$$

where

$x \epsilon R^n$ is the state vector,

$u \epsilon R^{m_1}$ is the pursuer's control vector,

$v \epsilon R^{m_2}$ is the evader's control vector.

The performance index is

$$J = \tfrac{1}{2} x^T(t_f)S(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}\{x^T(t)Q(t)x(t) + u^T(t)R_p(t)u(t) + v^T(t)R_E(t)v(t)\}dt \tag{2}$$

where

Q and S are n X n symmetric, positive semi-definite matrices,

R_p is a m_1 X m_1 symmetric, positive definite matrix,

R_E is a m_2 X m_2 symmetric, negative definite matrix.

The Hamilton-Jacobi equation may be written as

$$\min_{u \epsilon U} \max_{v \epsilon V} \{\frac{\partial V^*}{\partial t} + \frac{\partial V^{*T}}{\partial x}[Fx + G_p u + G_E v] + \tfrac{1}{2}(x^T Qx + u^T R_p u + v^T R_E v)\} = 0 \tag{3}$$

Extremization over u and v yields

$$u(t) = -R_p^{-1}(t)G_p^T(t)\frac{\partial V^*(t)}{\partial x}$$

$$v(t) = -R_E^{-1}(t)G_E^T(t)\frac{\partial V^*(t)}{\partial x} . \tag{4}$$

Substitution of u and v into the Hamilton-Jacobi equation yields

$$\frac{\partial V^*}{\partial t} + \frac{\partial V^{*T}}{\partial x}\left[Fx - G_p R_p^{-1} G_p^T \frac{\partial V^*}{\partial x} - G_E R_E^{-1} G_E^T \frac{\partial V^*}{\partial x}\right] + \tag{5}$$
$$\tfrac{1}{2}(x^T Qx + \frac{\partial V^{*T}}{\partial x} G_p R_p^{-1} G_p^T \frac{\partial V}{\partial x} + \frac{\partial V^{*T}}{\partial x} G_E R_E^{-1} G_E^T \frac{\partial V^*}{\partial x} = 0$$

or

$$\frac{\partial V^*}{\partial t} + \frac{\partial V^{*T}}{\partial x} Fx + \tfrac{1}{2}\, x^T Qx - \tfrac{1}{2}\frac{\partial V^{*T}}{\partial x} G_p R_p^{-1} G_p^T \frac{\partial V^*}{\partial x} - \tag{6}$$
$$\tfrac{1}{2}\frac{\partial V^{*T}}{\partial x} G_E R_E^{-1} G_E^T \frac{\partial V^*}{\partial x} = 0 .$$

The solution to the Hamilton-Jacobi equation will be taken to be

$$V^*(t) = \tfrac{1}{2} x^T P(t) x. \tag{7}$$

This may be substituted into the Hamilton-Jacobi equation. This yields

$$\tfrac{1}{2} x^T (\dot{P} + PF + F^T P - P(G_p R_p^{-1} G_p^T + G_E R_E^{-1} G_E^T)P + Q)x = 0. \tag{8}$$

Since x is arbitrary

$$\dot{P} = -PF - F^T P + P(G_p R_p^{-1} G_p^T + G_E R_E^{-1} G_E^T)P - Q. \tag{9}$$

At $t = t_f$

$$V^*(t_f) = \tfrac{1}{2} x^T S x . \tag{10}$$

Thus, the boundary condition for the matrix differential equation is

$$P(t_f) = S.$$

The control strategies may be written as

$$
\begin{aligned}
u(t) &= -R_P^{-1}(t)G_P^T(t)P(t)\mathbf{x}(t) \\
v(t) &= -R_E^{-1}G_E^T(t)P(t)x(t).
\end{aligned}
\qquad (11)
$$

APPENDIX C

OPTIMAL OPEN-LOOP STOCHASTIC CONTROL FOR SYSTEMS WITH UNCERTAINTY

Optimal Open-Loop Stochastic Control for Systems With Uncertainty

In this appendix, the problem of the optimal control of systems with parameter uncertainty will be considered. In particular, a specialization of the results given in the dissertation will be shown for the stochastic control (one sided differential game) problem. These results extend previous results found by Lainiotis (56) and Lee (60).

The system dynamics are

$$\dot{x}(t) = F(t, \theta)\, x(t) + G(t)\, u(t) + w(t) \tag{1}$$

where

$x \in R^n$ is the state of the system,

$u \in U \subset R^m$ is the control input,

$w \in R^n$ is a vector of white noise inputs with zero-mean and variance

$$E\{w(t)w^T(\tau)\} = W(t)\delta(t-\tau), \tag{2}$$

$\theta \in R$ is a time invariant unknown parameter vector.

The system is observed through

$$y(t) = H(t)x(t) + v(t) \tag{3}$$

where

$y \in R^q$ is the measurement vector,

$v \in R^q$ is a vector of white noise inputs corrupting the measurements with zero-mean and variance

$$E\{v(t)v^T(\tau)\} = R(t)\delta(t-\tau). \tag{4}$$

The control, $u \in L_E\{I, R^m\}$ where $I = [t_0, t_f)$, is required to minimize the performance index

$$J = E\{\tfrac{1}{2} x^T(t_f) S x(t_f) + \tfrac{1}{2} \int_{t_0}^{t_f} (x^T(t)A(t)x(t) + u^T(t)B(t)u(t))dt\}. \qquad (5)$$

One may write the Hamilton-Jacobi equation as

$$\min_{u \in U} E\{\frac{\partial V}{\partial t} + \frac{\partial V^T}{\partial x}[F(t, \theta)x + G u] + \tfrac{1}{2}(x^T A x + u^T B u) + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j} m_{ij} \,|\, \psi\} = 0 \qquad (6)$$

where Y is the observation functional $Y = \{y(\tau), \tau \in [t_0, t)\}$ and $m_{ij} = \{Q\}_{ij}$.

Sufficient statistics for Y are $\hat{x}(t|\theta)$ and $p(\theta|Y)$. These can also be written as $\hat{x}(t)$ and $p(\theta|Y)$ where

$$\hat{x}(t) = \int \hat{x}(t|\theta) p(\theta|Y) dt.$$

Thus, the Hamilton-Jacobi equation may be written as

$$\min_{u \in U} E\{\frac{\partial V}{\partial t} + \frac{\partial V^T}{\partial x}[F(t, \theta)x + G u] + \tfrac{1}{2}(x^T A x + u^T B u) + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j} m_{ij} \,|\, \hat{x}(t), p(\theta|Y)\} = 0. \qquad (7)$$

Minimization over $u \in U$ yields

$$u(t) = -B^{-1}(t)A^T(t) E\{\frac{\partial V}{\partial x} \,|\, \hat{x}(t), p(\theta|Y\}. \qquad (8)$$

This equation may be substituted for u in Equation (7). This yields

$$E\{\frac{\partial V}{\partial t} + \frac{\partial V^T}{\partial x}[F(t, \theta)x - GB^{-1}G^T E\{\frac{\partial V}{\partial x}|\hat{x}, p(\theta|Y)\}] + \tfrac{1}{2}x^T A x + \tfrac{1}{2}E\{\frac{\partial V^T}{\partial x}|\hat{x}, p(\theta|Y)\} GB^{-1}G^T E\{\frac{\partial V}{\partial x}|\hat{x}, p(\theta|Y)\} + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j}\{Q\}_{ij} \,|\, \hat{x}, p(\theta|Y)\} = 0. \qquad (9)$$

The use of Lemma 2 in Chapter IV and the properties of conditional expectations allows one to write this equation as

$$E\{\frac{\partial V}{\partial t} + \frac{\partial V^T}{\partial \hat{x}}[F(t,\theta)\hat{x} - GB^{-1}G^T E\{\frac{\partial V}{\partial \hat{x}} \mid \hat{x}, p(\theta|Y)\}] + \tfrac{1}{2}\hat{x}^T A\hat{x} + \mathrm{tr}AP$$

$$+ \tfrac{1}{2}E\{\frac{\partial V^T}{\partial \hat{x}} \mid \hat{x}, p(\theta|Y)\}GB^{-1}G^T E\{\frac{\partial V}{\partial \hat{x}} \mid \hat{x}, p(\theta|Y)\} + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V}{\partial \hat{x}_i \partial \hat{x}_j}\{Q\}_{ij} \mid \hat{x},$$

$$p(\theta|Y)\} = 0. \tag{10}$$

The solution to the Hamilton-Jacobi equation will be taken as

$$V = \tfrac{1}{2}\underset{\theta|Y}{E}\{\hat{x}^T(t|\theta)\}P(t)\underset{\theta|Y}{E}\{\hat{x}(t|\theta)\} + A(t)$$

$$= \tfrac{1}{2}\hat{x}^T P(t)\hat{x} + c(t). \tag{11}$$

Substitution into Equation (10) yields

$$E\{\tfrac{1}{2}\hat{x}^T(\dot{P} + PF + F^T P - PGB^{-1}G^T \underset{\theta|Y}{E}\{P\} + A - \underset{\theta|Y}{E}\{P\}GB^{-1}G^T P$$

$$+ \underset{\theta|Y}{E}\{P\}GB^{-1}G^T \underset{\theta|Y}{E}\{P\}\hat{x} + \dot{c} + \mathrm{tr}\,AP + \tfrac{1}{2}\sum_{i=1}\sum_{j=1}P_{ij}m_{ij}\} = 0 \tag{12}$$

where P_{ij} is the $i\,j^{th}$ element of P.

This implies

$$\dot{P}(t) = -P(t)F(t) - F(t)^T P(t) + P(t)G(t)B^{-1}(t)$$

$$\cdot G^T(t)\underset{\theta|Y}{E}\{P(t)\} - \underset{\theta|Y}{E}\{P(t)\}G(t)B^{-1}(t)G^T(t)$$

$$\cdot \underset{\theta|Y}{E}\{P(t)\} - A(t) + \underset{\theta|Y}{E}\{P(t)\}G(t)B^{-1}(t)G^T(t)P(t)$$

with boundary condition

$$P(t_f) = S. \tag{14}$$

The control law is

$$u(t) = -B^{-1}(t)G^T(t) \underset{\theta|\psi}{E}\{P(t)\}\hat{x}(t).$$

VITA

Robert B. Asher

Candidate for the Degree of

Doctor of Philosophy

Thesis: DIFFERENTIAL GAMES WITH SYSTEM UNCERTAINTY AND IMPERFECT INFORMATION

Major Field: Mechanical Engineering

Biographical:

Personal Data: Born in Chicago, Illinois, June 15, 1941, the son of Ben and Edwina R. Asher.

Education: Graduated from Calumet High School, Chicago, Illinois in June, 1959; received the Bachelor of Science degree with a major in Mechanical and Aerospace Engineering from Oklahoma State University in May, 1969; completed the requirements for the Degree of Master of Science in May, 1971; completed the requirements for the Degree of Doctor of Philosophy in July, 1974.

Professional Experience: Design Draftsman, General Scientific Corporation, 1960; Machine Designer, Brown Engineering and Machine Company, 1961-1963; Computer Programmer, U. S. Air Force, 1963-1967; Student, 1967-1970; Staff Engineer, Air Force Avionics Laboratory, 1971-1972; Technical Manager, Air-to-Air Analysis Group, Air Force Avionics Laboratory, 1973-present; also currently Captain, U. S. Air Force.

Professional Organizations: Institute of Electrical and Electronics Engineers and Phi Kappa Phi National Honorary Fraternity.

Presentations and Publications:

"Necessary Conditions for Optimal Control of Systems with State Dependent Time Delay." <u>Proceedings of the First Graduate Research Conference on Applied Mechanics</u>, March, 1969.

"Optimal Control of Systems with State Dependent Time Delay." <u>Proceedings of the Third Annual Houston Conference on Computers and System Science</u>, April, 1971.

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"Performance Evaluation of Suboptimal Filters." Air Force Avionics Laboratory Tech Memo; also accepted for publication in the IEEE Transactions on Aerospace and Electronic Systems.

"Optimal Guidance of Finite Bandwidth Missile Systems Against Maneuvering Targets." Proceedings of the 1974 Joint Automatic Control Conference; also accepted for publication in the ASME Journal of Dynamic Systems, Measurement, and Control.

"Kalman Filtering for Precision Pointing and Tracking Problems." Presented at the American Ordnance Association Conference on Impact of Sensors on Avionics, November, 1972.

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