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FOREWORD

The recent profusion of control-theoretic publications dealing with problems in finite-dimensional Euclidian spaces has inevitably given rise to attempts to extend these results to more general settings. The most natural avenue of extension is to Banach spaces of various kinds, especially Hilbert spaces.

Here we treat only the question of time-optimal controls for the canonical problem. Extensions to more general questions are simple, and the literature abounds with such extensions. In particular, see [1], [2] and [5].

Our first results are somewhat in the spirit of those given by Jacobs in [II]. We prove, in a Hilbert space setting, the existence of measurable controls bounded within a varying restraint set and the weak compactness of a class of functions.

In the following section we treat the question of a time-optimal control for a linear process in a separable and reflexive Banach space. Our work is based upon existence theorems found in Kato [3] and Kato and Tanabe [4]. It somewhat overlaps recent unpublished results of Friedman [2] and the recent publication of Lions [7]. We prove several corollaries concerning what one might term traditional control problems. In particular, we deal with a problem raised in Roxin [8].

Next, we discuss the same question for a nonlinear process, motivated by the paper of Lee and Markus [6]. The reader will no doubt notice that the hypotheses on the control function are quite strong, due to the absence of a Banach space form of the classic Carathéodory existence theorem for differential equations.

We have usually followed the notation most commonly used in the current literature on control. Elements of the various spaces considered are subscripted, and functions are superscripted. We usually use some form of the term "strong" as a modifier of some other term only when it is necessary to make a distinction from the weak form of the same modified term. However, even in cases where no confusion would be likely to result from the omission of some form of "strong", it is sometimes retained for purposes of emphasis. We never omit any form of "weak".

Where the context makes the meaning clear, we have omitted such phrases as

$$n \to \infty$$

All the functions hereafter considered are functions from a compact interval of the real line, generally denoted by I , to a Banach space of one sort or another. The measure on I is always the usual Lebesgue measure, and in some cases where no confusion with terms such as "strongly measurable" and "weakly measurable" is likely the modifier "Lebesgue" is omitted. This is frequently done when the function considered is from I to the real line. We never omit any form of "strong" or "weak" as a modifier of any form of the term "measurable".

The Hausdorff metric on the metric space of closed and bounded subsets of a metric space is symbolized by <u>dist</u>[,] , while the metric of the original space is symbolized by <u>dist</u>(,).

Derivatives are taken in the usual sense of the limit of the difference quotient.

The term "almost everywhere" is used in the sense of "except on a set of measure zero".

OPTIMIZATION PROBLEMS IN BANACH SPACE

CHAPTER I

THE BOCHNER INTEGRAL

In the past few decades several theories of integration have been developed that are somewhat divergent from that due to Lebesgue. We shall consider a theory due to S. Bochner.

The Bochner integral makes possible the integration of functions to a Banach space from an abstract set, S , on which a σ-finite measure has been defined on a σ-algebra of subsets, called the measurable subsets of S. The value of the integral of such a function is again an element of the Banach space, and many of the most useful and important properties of the Lebesgue integral can be extended almost word for word to this more general setting. As examples, we mention two such results. They are analogues of familiar results from Lebesgue theory, and will be used repeatedly in the sequel.

(1.1) $$\left\| (B)\int_S u(t) \right\| \leq (L)\int_S \| u(t) \|$$

(1.2) $(B)\int_S (au(t)+bv(t)) = a(B)\int_S u(t)+b(B)\int_S v(t)$, *where* a *and* b *are real*.

We now proceed to the precise definition of the Bochner integral that is most relevant to our control-theoretic setting. Let

$$u: I \to B,$$

where B is a Banach space and I is a closed and bounded interval of the real line. Then u is said to be *countably-valued* if the range of u is a countable subset of B and u assumes each non-zero value, u_k , on a Lebesgue measurable subset, E_k , of I.

In the setting in which we are interested, the countably valued function u is said to be *Bochner integrable* if the function

(1.3) $$\|u(\ )\| : I \to R :: t \to \|u(t)\|$$

has a finite Lebesgue integral over I, and the Bochner integral of u over I is defined by

(1.4) $$(B)\int_I u(t) = \sum_{k=1}^{\infty} u_k \text{meas}(E_k).$$

THEOREM 1.1. *If the function defined by* (1.3) *has a finite Lebesgue integral over* I , *then the limit in* (1.4) *exists*.

Proof. (Our proof follows [I, 79].) Note that

$$(L)\int_I \|u(t)\| = \sum_{k=1}^{\infty} \|u_k\| \text{meas}(E_k).$$

Hence, $\left\{\sum_{k=1}^{K} \|u_k\| \text{meas}(E_k),\ K=1,2,\ldots\right\}$ is a Cauchy sequence, and since

$$\left\|\sum_{k=1}^{K} u_k \text{meas}(E_k) - \sum_{k=1}^{K+L} u_k \text{meas}(E_k)\right\| \leq \sum_{k=K+1}^{K+L} \|u_k\| \text{meas}(E_k),$$

we see that $\left\{\sum_{k=1}^{K} u_k \text{meas}(E_k),\ K=1,2,\ldots\right\}$ is also a Cauchy sequence, and hence converges.

Q. E. D.

Next, a function u from I to B, not necessarily countably-valued, is said to be <u>Bochner integrable over</u> I iff there is a sequence $\{u^n\}$ of countably-valued Bochner integrable functions from I to B such that

(1.5) $\qquad u^n(t) \to u(t)$ a. e. <u>on</u> I;

(1.6) $\qquad (L)\int_I \|u^n(t)-u(t)\| \to 0.$

We define the Bochner integral of u by

(1.7) $\qquad (B)\int_I u^n(t) \to (B)\int_I u(t).$

THEOREM 1.2. <u>Given the above conditions on</u> $\{u^n\}$, <u>the limit in</u> (1.7) <u>exists</u>.

<u>Proof</u>. By definition (1.4),

$$(B)\int_I u^m(t) = \sum_{k=1}^{\infty} u_k^m \text{meas}(E_k^m), \quad m=1,\ 2,\ \ldots.$$

Consider two elements, u^m and u^n, of $\{u^n\}$, and then sequentialize the countable collection of Lebesgue measurable subsets of I, $\{E_{k,l}^{m,n} :: k,l = 1,\ 2,\ \ldots.\}$,

where $E_{k,l}^{m,n} = E_k^m \cap E_l^n$. Denote this sequentialized collection by

$$\{E_{j(k,l)} :: j(k,l) = 1,\ 2,\ \ldots\}.$$

Then,

$$(1.8) \qquad \sum_{j(k,l)=1}^{\infty} u_{k,l}^n \text{meas}(E_{j(k,l)}) = \sum_{l=1}^{\infty} u_l^n \text{meas}(E_l^n) = (B)\int_I u^n(t),$$

where $u_{k,l}^n$ is the functional value of u^n on $E_{j(k,l)}$.

A similar statement holds for u^m.

Also,

$$(1.9) \qquad \sum_{j(k,l)=1}^{\infty} \|u_{k,l}^n\| \text{meas}(E_{j(k,l)}) = \sum_{l=1}^{\infty} \|u_l^n\| \text{meas}(E_l^n)$$

$$= (L)\int_I \|u^n(t)\|,$$

and a similar statement holds for u^m. Also, the function $u^n - u^m$ is countably-valued and constant on each of the sets $E_{j(k,l)}$, and

$$(L)\int_I \|u^n(t)-u^m(t)\| = \sum_{j(k,l)=1}^{\infty} \|u^n_{k,l}-u^m_{k,l}\| \operatorname{meas}(E_{j(k,l)})$$

$$\leq \sum_{j(k,l)=1}^{\infty} \|u^n_{k,l}\| \operatorname{meas}(E_{j(k,l)}) + \sum_{j(k,l)=1}^{\infty} \|u^m_{k,l}\| \operatorname{meas}(E_{j(k,l)})$$

$$= (L)\int_I \|u^n(t)\| + (L)\int_I \|u^m(t)\| .$$

From the preceding statement and (1.8) and its analogue for m , we see that the limits

$$\sum_{j(k,l)=1}^{\infty} (u^n_{k,l}-u^m_{k,l}) \operatorname{meas}(E_{j(k,l)}) \quad \text{and}$$

$$\sum_{j(k,l)=1}^{\infty} \|u^n_{k,l}-u^m_{k,l}\| \operatorname{meas}(E_{j(k,l)}) \quad \text{both exist, and}$$

$$\sum_{j(k,l)=1}^{\infty} u^n_{k,l} \operatorname{meas}(E_{j(k,l)}) - \sum_{j(k,l)=1}^{\infty} u^m_{k,l} \operatorname{meas}(E_{j(k,l)})$$

$$= \sum_{j(k,l)=1}^{\infty} (u^n_{k,l}-u^m_{k,l}) \operatorname{meas}(E_{j(k,l)}).$$

Also, since

$$\Big\| \sum_{j(k,l)=1}^{K} u^n_{k,l} \operatorname{meas}(E_{j(k,l)}) - \sum_{j(k,l)=1}^{K} \operatorname{meas}(E_{j(k,l)}) \Big\|$$

$$= \Big\| \sum_{j(k,l)=1}^{K} (u^n_{k,l}-u^m_{k,l}) \operatorname{meas}(E_{j(k,l)}) \Big\|$$

$$\leq \sum_{j(k,l)=1}^{K} \|u^n_{k,l}-u^m_{k,l}\| \operatorname{meas}(E_{j(k,l)}) \quad \underline{\text{for all}}\ K,$$

we see that

$$(1.10)\qquad \left\|(B)\int_I u^n(t) - (B)\int_I u^m(t)\right\| = (L)\int_I \|u^n(t)-u^m(t)\| .$$

But since $\|u^n(t)-u^m(t)\| \leq \|u^n(t)-u(t)\| + \|u^m(t)-u(t)\|$,

we see that

$$(1.11)\qquad (L)\int_I \|u^n(t)-u^m(t)\| \leq (L)\int_I \|u^n(t)-u(t)\| + (L)\int_I \|u^m(t)-u(t)\| .$$

By condition (1.6) we see that the right side of (1.11) approaches zero as m and n become large. Hence the sequence $\left\{(B)\int_I u^n(t)\right\}$ is Cauchy and hence convergent, and we have shown that the limit in (1.7) exists.

The uniqueness of the limit follows from the fact that any two sequences of functions with properties (1.5) and (1.6) can be combined into a single such sequence by alternating their terms.

Q. E. D.

Suppose the function u maps I into B. We say that u is <u>almost separably-valued</u> if there is a subset E_0 of I such that $\text{meas}(I-E_0)=0$, and the set $u(I-E_0)$ is separable.

We say that u is <u>weakly measurable</u> if for each b^* of B^* , the adjoint space of continuous linear functionals on B , the function

$$b^*u : I \to R :: t \to b^*[u(t)]$$

is Lebesgue measurable.

Finally, we say that u is strongly measurable if u is weakly measurable and almost separably-valued.

We say that a function u is Bochner p-integrable if u is strongly measurable and the function

$$\|u(\ )\|^p : I \to R :: t \to \|u(t)\|^p\ ,\ p \geq 1,$$

has a finite Lebesgue integral on I.

In the case where p=1 , it can be shown [I, 80] that this condition is equivalent to Bochner integrability as defined following Theorem 1.1. It is frequently useful to replace that definition by this characterization.

Finally, we define the norm of a Bochner p-integrable function by

$$\|u\|_p = [(L)\int_I \|u(t)\|^p]^{1/p}, \quad p \geq 1. \tag{1.12}$$

We denote the class of Bochner p-integrable functions from I to B by $B_p(I,B)$, and remark that it is shown in [I, 81] that $B_p(I,B)$ is a Banach space under the norm defined by (1.12).

It is useful to note that if the function u is Bochner p-integrable and also bounded, then u is Bochner p*-integrable, where $p^* \geq 1$.

CHAPTER II

EXISTENCE AND COMPACTNESS OF SETS OF FUNCTIONS

In the sequel we shall have need of a version of the Riesz theorem and related results. We first prove the following lemma.

LEMMA 2.1.1. Suppose X is a real Hilbert space [IV, 73]. Then one can define an inner product (u,v) on $B_2(I,X)$, where u and v are elements of $B_2(I,X)$, the set of Bochner square-integrable functions from I to X, by the statement

$$(2.1) \qquad (u,v) = (L)\int_I [u(t),v(t)] ,$$

where $[\ ,\ ]$ symbolizes the inner product on X.

Proof. Since $B_2(I,X)$ is a Banach space with the norm defined by (1.12) with $p=2$, we see that the function $u-v$ is an element of $B_2(I,X)$.
Hence the functions

$$\|u(\ )-v(\ )\|_2^2 : I \to R :: t \to \|u(t)-v(t)\|^2;$$

$$\|u(\ )\|_2^2 : I \to R :: t \to \|u(t)\|^2;$$

$$\|v(\ )\|_2^2 : I \to R :: t \to \|v(t)\|^2$$

are Lebesgue integrable. Then since

$$[u(t),v(t)] = \frac{\|u(t)\|^2 + \|v(t)\|^2 - \|u(t)-v(t)\|^2}{2},$$

the function

$$[u(\ ),v(\ )] : I \to R :: t \to [u(t),v(t)]$$

is Lebesgue integrable. Hence (u,v) is well defined by (2.1). The familiar properties of the inner product follow readily from the properties of the Lebesgue integral and the inner product on X. To illustrate,

$$(au,v) = (L)\int_I [au(t),v(t)] = (L)\int_I a[u(t),v(t)]$$

$$= a(L)\int_I [u(t),v(t)] = a(u,v).$$

The other properties follow similarly.

Q. E. D.

Suppose $\{x_n\}$ is a sequence of elements of a Banach space, X. We say that x_n <u>converges weakly</u> to a point x_o of X iff for each continuous linear functional x^* defined on X, $x^*(x_n)$ converges to $X^*(x_o)$.

THEOREM 2.1. *Suppose* $\{u^n\}$ *is a sequence of elements of* $B_2(I,X)$ *that converges weakly to* u^o *of* $B_2(I,X)$. *Then there is a subsequence of* $\{u^n\}$, *called* $\{u^{j(n)}\}$, *such that the sequence of arithmetic means of the elements of* $\{u^{j(n)}\}$ *converges strongly to* u^o . *That is, the sequence of functions* $\{v^k\}$, *defined by*

$$v^k = \frac{1}{k}\sum_{j(n)=1}^{k} u^{j(n)} , \quad k=1,\ 2,\ \ldots$$

has the property that

$$\|v^k-u^o\|_2 = (v^k-u^o,v^k-u^o)^{1/2} \longrightarrow 0.$$

Proof. It is enough to show that $(v^k-u^o,v^k-u^o)\longrightarrow 0$, and without loss of generality we take u^o equal to 0.

As defined in (2.1), the function

$$(\ ,v):B_2(I,X)\longrightarrow R::u\longrightarrow (u,v)$$

is continuous and linear, and so $(\ ,v)$ is an element of $B_2^*(I,X)$, the adjoint space to $B_2(I,X)$.

It is well known [IV, 124] that a weakly convergent sequence of elements of a Banach space is bounded in norm, so there is a constant M with the property that $\|u^n\|_2 \leq M$, $n=1,\ 2,\ \ldots$.

Let $j(1) = 1$, and take

$$S(1) = \{n:: |(u^n,u^{j(1)})| \leq 1,\ n > j(1)\} .$$

We see that $S(1)$ is not empty, since

$$(u^n, u^{j(1)}) \to (0, u^{j(1)}) = 0,$$

due to the weak convergence of u^n to 0.

Let $\min S(1) = j(2)$, and suppose, proceeding inductively, that

$$j(1) < j(2) < \ldots < j(k).$$

and define $S(k) = \{n:: |(u^n, u^{j(1)})| \leq 1/k, \ldots, |(u^n, u^{j(k)})| \leq 1/k;\ n > j(k)\}$.

We see that $S(k)$ is not empty just as in the case of $S(1)$.

Let $\min S(k) = j(k+1)$, and the sequence $u^{j(n)}$ is well defined. But then

$$\left(\frac{u^{j(1)}+u^{j(2)}+\ldots+u^{j(n)}}{n}, \frac{u^{j(1)}+u^{j(2)}+\ldots+u^{j(n)}}{n}\right)$$

$$= \frac{1}{n^2} \sum_{l,m=1}^{l,m=n} (u^{j(l)}, u^{j(m)}) = \frac{nM^2+2/1+4/2+\ldots+2(n-1)/(n-1)}{n^2}$$

$$\leq \frac{M^2+2}{n} \to 0$$

Q. E. D.

A set is said to be <u>weakly compact</u> if every sequence of elements of the set contains a weakly convergent subsequence. It is said to be <u>weakly compact in itself</u> if every sequence of elements of the set contains a subsequence that converges weakly to an element of the set.

THEOREM 2.2. *Let* $C\#(B)$ *denote the set of nonempty, convex, closed and bounded subsets of a complete inner product space* B. *Suppose that*

$$G: I \longrightarrow C\#(B),$$

and that $\bigcup\{G(t) :: t \in I\}$ *is bounded. Then the set of functions* $B_2(G)$, *where by definition the function* u *is an element of* $B_2(G)$ *iff*

$$u: I \longrightarrow B \ ;$$

$$u \in B_2(I,B) \ ;$$

$$u(t) \in G(t) \ \text{a.e. on } I \ ;$$

is weakly compact in itself.

Proof. If $p > 1$, then $B_p^*(I,B) = B_q(I,B^*)$, where $1/p + 1/q = 1$ [I,89]. Hence, since B is reflexive, we see that

$$B_p^{**}(I,B) = B_q^*(I,B^*) = B_p(I,B^{**}) = B_p(I,B).$$

Thus $B_2(I,B)$ is reflexive and so closed and bounded spheres of $B_2(I,B)$ are weakly compact in themselves [I,38]. Since the elements of $B_2(G)$ take their values almost everywhere in a bounded set, the set $B_2(G)$ is also bounded. Hence if $\{u^n\}$ is a sequence of elements of $B_2(G)$, there is a subsequence, also called $\{u^n\}$, and an element u^0 of $B_2(I,B)$ such that

$$u^n \to u^o(\underline{weak}).$$

We must show that u^o is an element of $B_2(G)$.

By Theorem 2.1 there is a subsequence, also called u^n, such that

$$\frac{1}{n}\sum_{k=1}^{n} u^k = v^n \to u^o(\underline{strong}).$$

That is,

$$[(L)\int_I \|v^n(t)-u^o(t)\|^2]^{1/2} = \|v^n-u^o\|_2 \to 0.$$

We now quote a result on real variables given in [III,86].

If a sequence of elements of L_2 *converges in norm to zero, there is a subsequence that converges to zero almost everywhere.*

Thus we can suppose that

$$\|v^n(t)-u^o(t)\|^2 \to 0 \ \underline{a.e.}\ \underline{on}\ I.$$

or that

(2.1a) $\qquad v^n(t) \to u^o(t)(\underline{strong})$ on E_o ,

where E_o is a subset of I , and $\text{meas}(I-E_o) = 0$.
For each u^n there is a subset, E_n , of I such that $\text{meas}(I-E_n) = 0$ and for each t in E_n , $u^n(t)$ is an element of $G(t)$.

By the convexity of each $G(t)$ and the definition of v^n , we see that for each t in $\bigcap_{n=1}^{\infty} E_n$, $v^n(t)$ is an

element of $G(t)$, where $n=1, 2, \ldots$.
From (2.1a) and the closure of each $G(t)$, we see that for each t in $\bigcap_{n=0}^{\infty} E_n$, $u^0(t)$ is an element of $G(t)$. Since

$\text{meas}(I-\bigcap_{n=0}^{\infty} E_n) = 0$,

$$u^0(t) \in G(t) \quad \underline{a}.\underline{e}. \ \underline{on} \ I.$$

Hence u^0 is an element of $B_2(G)$.

Q. E. D.

A map G from a metric space X into the space of closed and bounded subsets of a metric space Y is said to be <u>upper</u> <u>semicontinuous</u> (hereafter abbreviated to <u>u</u>. <u>s</u>. <u>c</u>.) at the point x_0 of X iff

$$\bigcap_{n=1}^{\infty} [\overline{\bigcup\{G(x)::x \in S(x_0,1/n)\}}] \subset G(x_0) ,$$

where $S(x_0,1/n)$ denotes the sphere of radius $1/n$ centered at x_0 and the upper bar denotes closure. The concept of upper semicontinuity can be defined for more general spaces, but the above definition will suffice for our purposes.

The following theorem generalizes a result of Jacobs [II,31].

THEOREM 2.3. <u>Suppose the map</u> G <u>from</u> I <u>to</u> $C\#(H)$ <u>is u.s.c., where</u> $C\#(H)$ <u>denotes the collection of convex, closed and bounded subsets of a separable Hilbert space which has the real numbers as its scalar field</u>.

If $\overline{\bigcup\{G(t)::t\in I\}}$ *is compact in itself, then* $S(G)$ *is not empty, where the function* u *is an element of* $S(G)$ *iff*

$$u:I\longrightarrow H \text{ (strongly measurable)} ;$$
$$u(t)\in G(t) \text{ for each } t \text{ in } I .$$

Proof. Select a denumerable orthonormal basis $\{b_i\}$ for H. (For a discussion of the relevant definitions and a proof that such a basis must exist, see [III,78].) We now define a function u and show that it is an element of $S(G)$.

For each t in I there is an element x_0 of $G(t)$ such that $[\ ,b_1]$ is a minimum over $G(t)$. To show this, consider t in I. Since the map

$$(2.2) \qquad [\ ,b_1]:H\longrightarrow R::x\longrightarrow [x,b_1]$$

is continuous and linear, there is a real number m such

that $$m = \inf([x,b_1],x\in G(t)).$$

Take a sequence $\{x_n\}$ of elements of $G(t)$ such that $[x_n,b_1]\longrightarrow m$. Then since $G(t)$ is bounded, it is weakly compact [I,38]. Hence there is a point x_0 of H and a subsequence of $\{x_n\}$, also called $\{x_n\}$, such that x_n converges weakly to x_0.

By Theorem 2.1 there is a further subsequence, also called $\{x_n\}$, such that the arithmetic means of the elements of $\{x_n\}$ converge strongly to x_0. That is,

$$\frac{1}{n}\sum_{k=1}^{n} x_k = z_n \rightarrow x_o \text{ (\underline{strong})}.$$

Since each $G(t)$ is convex and each x_k is an element of $G(t)$, then each z_n is an element of $G(t)$. Since $G(t)$ is closed, x_o is an element of $G(t)$. Due to the definition of $\{x_n\}$, the weak convergence of x_n to x_o and the continuity and linearity of the map $[\ ,b_1]$,

$$[x_n,b_1] \rightarrow [x_o,b_1] = \underline{\inf}([x,b_1],x\in G(t)).$$

Since x_o is an element of $G(t)$, the infimum is actually a minimum.

Suppose that this does not uniquely define $u(t)$. That is, suppose there is another element of $G(t)$, say x_1 , such that $[x_o,b_1] = [x_1,b_1]$.
Then let

$$U_1(t) = \{y::y\in G(t) \text{ \underline{and} } [y,b_1] = \underline{\min}([x,b_1],x\in G(t))\}.$$

We now minimize the function $[\ ,b_2]$ over $U_1(t)$.

To see that this minimum exists, observe that $U_1(t)$ is bounded, since it is a subset of $G(t)$. Since $U_1(t)$ is the intersection of the closed set $G(t)$ and the pre-image of the closed subset of the reals consisting of the single point $\underline{\min}([x,b_1],x\in G(t))$ under the continuous map $[\ ,b_1]$, then $U_1(t)$ is also closed. To show that $U_1(t)$

is convex, consider two elements, x and y of $U_1(t)$ and two non-negative real real numbers, a and b, such that $a + b = 1$. Since

$$[ax+by,b_1] = a[x,b_1] + b[y,b_1] = (a+b)(\underline{\min}([x,b_1],x\epsilon G(t))),$$

and x and y are elements of $G(t)$, which is convex, we see that $ax + by$ is an element of $U_1(t)$. Hence $U_1(t)$ is convex.

Thus the minimum of $[x,b_2]$ over $U_1(t)$ exists by the same argument that showed the existence of the minimum of $[x,b_1]$ over $G(t)$.

Suppose that this does not uniquely define $u(t)$. That is, suppose that $U_2(t)$ consists of more than one point, where

$$U_2(t) = \{y::y\epsilon U_1(t) \ \underline{\text{and}}\ [y,b_2] = \underline{\min}([x,b_2],x\epsilon U_1(t))\}.$$

Then we can minimize $[x,b_3]$ over $U_2(t)$ by the above argument. This yields a set $U_3(t)$ of elements that minimize $[x,b_3]$ over $U_2(t)$. We proceed in this manner, minimizing $[x,b_{n+1}]$ over $U_n(t)$, where

$$(2.3) \quad U_n(t) = \{y::y\epsilon U_{n-1}(t) \ \underline{\text{and}}\ [y,b_n]$$

$$= \underline{\min}([x,b_n],x\epsilon U_{n-1}(t))\}.$$

If after a finite number of such steps we obtain a set $U_n(t)$ consisting of a single point, we define that point to be $u(t)$.

In the case where no $U_n(t)$ consists of a single point, we say that

$$u(t) = \bigcap_{n=0}^{\infty} U_n(t),$$

where we define $U_0(t)$ to be $G(t)$.

To show that the above intersection can contain at most one point, suppose that it contains two points, y and z. Then since y and z are both elements of $U_n(t)$ for all n, we see that

$$[y,b_{n+1}] = [z,b_{n+1}] = \underline{\min}([x,b_{n+1}], x \in U_n(t)) \quad , n=0,1,\ldots.$$

But then y and z have the same 'coordinates' and are hence equal.

To show that the intersection is not empty, select a sequence of points of H, $\{x_n\}$, such that each x_n is an element of $U_n(t)$. There is a subsequence of $\{x_n\}$ that converges weakly to a point x^* of H. Since each $U_{n+1}(t)$ is a subset of $U_n(t)$, we see that <u>for each</u> n we can find a subsequence of our original sequence that consists solely of elements of $U_n(t)$, and <u>still</u> converges weakly to x^*. But since convex, closed and bounded subsets of a reflexive Banach space are weakly compact in themselves [I,37-38], we see that x^* is an element of each $U_n(t)$. Hence the intersection is not empty, and so $u(t)$ is well defined.

Next we show that the function u so defined is strongly measurable.

Consider the coordinate map, c^k, where

(2.4) $$c^k : I \to R :: t \to [u(t), b_k], \quad k=1, 2, \ldots .$$

We show that each c^k is Lebesgue measurable. To this end, let $$F^1(s) = \{t :: t \in I \text{ and } [u(t), b_1] \leq s\}.$$

It is enough to show that $F^1(s)$ is closed. Suppose that $\{t^n\}$ is a sequence of elements of $F^1(s)$ and that $t^n \to t^o$. To show that t^o is an element of $F^1(s)$, consider the following special case of a theorem due to Jacobs [II,16].

(2.5) _Let_ $C(X)$ _be the collection of closed and bounded subsets of a metric space,_ X, _and_ I _be a compact interval of the real line. Suppose_

$$F : I \to C(X) \ (\underline{u}.\underline{s}.\underline{c}.),$$

and that $\{t^n\}$ _and_ $\{x^n\}$ _are sequences of elements of_ I _and_ X _respectively such that_

$$t^n \to t^o \ I ;$$

$$x^n \to x^o \ X ;$$

$$x^n \in F(t^n) \ \text{for all } n.$$

Then we conclude that x^o *is an element of* $F(t^o)$.

Since each $u(t^n)$ is an element of the compact set $\overline{\bigcup\{G(t)::t\in I\}}$, we see that there is a subsequence of $\{t^n\}$, also called $\{t^n\}$, and a point h_o of H such that

(2.5a) $$u(t^n) \longrightarrow h_o \text{ (\underline{strong})}.$$

By the definition of the function u, each $u(t)$ is an element of $G(t)$. Hence, by (2.5), (2.5a) and the convergence of t^n to t^o, we see that h_o is an element of $G(t^o)$. Hence, again due to the definition of the function u, $[u(t^o),b_1] \leqq [h_o,b_1]$.

Since for each n, $[u(t^n),b_1] \leqq s$, we see from (2.5a) and the continuity of the inner product map that

$$[u(t^n),b_1] \longrightarrow [h_o,b_1] \leqq s.$$

Hence $[u(t^o),b_1] \leqq s$, and we see that t^o is an element of $F^1(s)$. Hence $F^1(s)$ is closed, and so c^1 is measurable, where

$$c^1 : I \longrightarrow R :: t \longrightarrow [u(t),b_1].$$

We now show by induction that each c^k is measurable.

Suppose that we have closed subsets of I,

(2.6) $$J_1(=I),\ J_2,\ \ldots,\ J_k$$

such that the maps $c^1, c^2, \ldots, c^k$ are measurable on $J_1, J_2, \ldots, J_k$ respectively, and

(2.7) $J_i \supset J_{i+1}$, $i=1,2,\ldots,k-1$;

(2.8) $\text{meas}(J_i - J_{i+1}) < \dfrac{\varepsilon}{2^{i+2}}$, $i=1, 2, \ldots ,k-1$, $\varepsilon > 0$;

(2.9) c^i <u>is continuous on</u> J_{i+1} , $i = 1, 2, \ldots, k-1$.

Recall the following form of Lusin's Theorem [V,236]:

<u>Suppose</u> S <u>is a measurable subset of the reals and</u> f <u>is a measurable map from</u> S <u>to the reals. Then if</u> $\varepsilon > 0$, <u>there is a closed subset of</u> S, <u>call it</u> E, <u>such that</u> f <u>is continuous on</u> E, <u>and</u> $\text{meas}(S-E) < \varepsilon$.

Given the closed subsets mentioned in (2.6), Lusin's Theorem shows that there is a closed subset J_{k+1} of I such that

(2.10) $J_k \supset J_{k+1}$;

(2.11) $\text{meas}(J_k - J_{k+1}) < \dfrac{\varepsilon}{2^{k+2}}$;

(2.12) c^k <u>is continuous on</u> J_{k+1}.

The maps $c^1, c^2, \ldots, c^k$ are certainly continuous on J_{k+1}. We now show that the map c^{k+1} is measurable on J_{k+1}. Let

$$F^{k+1}(s) = \{t :: t \in J_{k+1} \text{ and } [u(t), b_{k+1}] \leq s\}.$$

Suppose that $\{t^n\}$ is a sequence of elements of $F^{k+1}(s)$ and $t^n \to t^o$. We show that t^o is an element of $F^{k+1}(t)$, which is then seen to be closed. We will have then shown that c^{k+1} is measurable on J_{k+1}.

Since each $u(t^n)$ is an element of $\overline{\bigcup\{G(t)::t\in I\}}$, which is compact in itself, there is a subsequence of $\{t^n\}$, also called $\{t^n\}$, and a point h_o of H such that

$$(2.12a) \qquad u(t^n) \longrightarrow h_o \ (\underline{\text{strong}}).$$

By another application of (2.5), the theorem of Jacobs, h_o is an element of $G(t^o)$.

Since the maps $c^1, c^2, \ldots, c^k$ are continuous on J_{k+1}, we see that

$$(2.13) \qquad [u(t^n),b_i] \longrightarrow [u(t^o),b_i] \ , \ i = 1, 2, \ldots, k.$$

(Remember that $t^n \longrightarrow t^o$ and

$$c^i : I \longrightarrow R :: t \longrightarrow [u(t),b_i] \ , \ i = 1, 2, \ldots .)$$

Again, by the continuity of the inner product map and (2.12a), we have

$$(2.14) \qquad [u(t^n),b_i] \longrightarrow [h_o,b_i] \ , \ i = 1, 2, \ldots, k+1.$$

We now consider two cases: A and B.

A. $\qquad u(t^o) = h_o.$

In this case, since each t^n is an element of $F^{k+1}(s)$, we see that

(2.15) $[u(t^n), b_{k+1}] \leqq s$, $n = 1, 2, \ldots.$

So, by (2.15) and (2.14) for the case $i = k+1$, we see that $[u(t^o), b_{k+1}] = [h_o, b_{k+1}] \leqq s$,

and t^o is an element of $F^{k+1}(s)$.

B. $u(t^o) \neq h_o$.

By (2.13) and (2.14) we see that

(2.16) $[u(t^o), b_i] = [h_o, b_i]$, $i = 1, 2, \ldots, k$.

Thus both h_o and $u(t^o)$ are elements of $U_i(t^o)$, $i = 1, 2, \ldots, k$, where $U_i(t^o)$ is as defined by (2.3). Then by the definition of $u(t^o)$,

(2.17) $\underline{\min}([x, b_{k+1}], x \in U_k(t^o)) = [u(t^o), b_{k+1}] = [h_o, b_{k+1}]$

From (2.14) and the fact that each t^n is an element of $F^{k+1}(s)$,

$$[h_o, b_{k+1}] \leqq s,$$

and so from (2.14) we see that $[u(t^o), b_{k+1}] \leqq s$, and hence that c^{k+1} is measurable on J_{k+1}.

Thus we have established by induction the existence of a sequence $\{J_k\}$ of closed subsets of I , with $J_1 = I$, such that

(2.18) $J_k \supset J_{k+1}$, $k = 1, 2, \ldots$;

(2.19) $\text{meas}(J_k - J_{k+1}) < \frac{\varepsilon}{2^{k+2}}$, $k = 1, 2, \ldots$;

(2.20) c^k <u>is measurable on</u> J_k , $k = 1, 2, \ldots$.

From (2.18) each c^k is measurable on $\bigcap_{k=1}^{\infty} J_k$, and by (2.18) and (2.19) , we see that

$$\text{meas}(I - \bigcap_{k+1}^{\infty} J_k) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+2}} < \varepsilon$$

Since ε is arbitrary, each c^k is measurable almost everywhere on I. That is, each c^k is measurable on I , and we remind the reader that "measurable" has heretofore meant "Lebesgue measurable".

We must now show that each map b^k , where

(2.20a) $$b^k : I \to H :: t \to \sum_{i=1}^{k} [u(t), b_i] b_i ,$$

is weakly measurable.

It is known [IV,110] that each linear functional in H^* is of the form

(2.21) $$[\ ,h] : H \to R :: x \to [x,h] ,$$

where h is some point of H. But $\{b_k\}$ is a complete orthonormal basis, and so each h of H has a unique representation of the form

(2.22) $$h = \sum_{i=1}^{\infty} [h,b_i]b_i \ .$$

Hence, from (2.20a) and (2.22), we see that for all choices of k, t and h,

(2.23) $$[b^k(t),h] = \sum_{i=1}^{k} [u(t),b_i][h,b_i] \ .$$

Hence the function

(2.24) $$[b^k(\ ),h]:I \longrightarrow R::t \longrightarrow \sum_{i=1}^{k} [u(t),b_i][h,b_i]$$

is identical with the function

(2.25) $$\sum_{i=1}^{k} [h,b_i]c^i(\ ):I \longrightarrow R::t \longrightarrow \sum_{i=1}^{k} [h,b_i]c^i(t) \ ,$$

where we remind the reader that $c^i(t) = [u(t),b_i]$.

Then for each h in H, the function defined by (2.25) is Lebesgue measurable, being the linear sum of a finite number of Lebesgue measurable functions defined over a finite interval. Hence for each h in H, the function defined by (2.24) is Lebesgue measurable.

Hence the composition of any linear functional in H^* with any b^k as defined in (2.20a) is of the form

$$[\ ,h]b^k:I \longrightarrow R::t \longrightarrow [b^k(t),h] \ ,$$

which we have shown to be Lebesgue measurable. Hence each

b^k is weakly measurable. Since the range of each b^k is in H, which is by hypothesis separable, each b^k is strongly measurable.

Further, since $\{b_k\}$ is an orthonormal basis, for each t in I,

$$b^k(t) = \sum_{i=1}^{k}[u(t),b_i]b_i \to \sum_{i=1}^{\infty}[u(t),b_i]b_i = u(t) \quad \underline{\text{as}} \quad k \to \infty .$$

Hence the function u is the pointwise limit of the sequence $\{b^k\}$ of strongly measurable functions, and by Theorem 3.5.4.(3), [I,74], the function u is itself strongly measurable.

Q. E. D.

CORROLLARY 2.3.1 The function u defined in Theorem 2.3 is Bochner integrable.

Proof. Since $\cup\{G(t)::t \in I\}$ is bounded, there is a constant M such that for each t in I and for each k,

$$(2.26) \quad \|b^k(t)\| = \Big(\sum_{i=1}^{k}[u(t),b_i]^2\Big)^{1/2} \leq \Big(\sum_{i=1}^{\infty}[u(t),b_i]^2\Big)^{1/2}$$

$$= \|u(t)\| \leq M.$$

Since each c^k is Lebesgue measurable, where $c^k(t) = [u(t),b_k]$, we see that for each k, the function

$$(2.27) \qquad \|b^k(\ )\|:I \to R::t \to \|b^k(t)\|$$

is Lebesgue measurable, being defined as in (2.26) by a finite number of Lebesgue measurable functions. From (2.26) we see that for each k, the function defined by (2.27) is bounded by M, and so has finite Lebesgue integral. Hence each b^k is Bochner integrable on I, and so by Theorem 3.7.9, [I,83], a generalization of the classic dominated convergence theorem of Lebesgue, and the pointwise convergence of b^k to u on I, we see that

$$(B)\int_I b^k(t) \longrightarrow (B)\int_I u(t) \quad (\underline{\text{strong}}).$$

Q. E. D.

COROLLARY 2.3.2. *If* $p > 1$, *then* u *is Bochner* p-*integrable*.

Proof. Since $\|u(t)\|$ is bounded on I, the function

$$\|u(\ )\|^p : I \longrightarrow R :: t \longrightarrow \|u(t)\|^p$$

is bounded and has finite Lebesgue integral on I.

Q. E. D.

Suppose X and Y are metric spaces. A map F from X to $C\#(Y)$ is said to be *upper semicontinuous with respect to inclusion* (hereafter abbreviated to *u. s. c. i.*) at a point x_0 of X iff

(2.28) *for any* $\varepsilon > 0$ *there is a* $\delta(\varepsilon) > 0$ *such that if* *dist*$(x, x_0) < \delta(\varepsilon)$, *then*

$$F(x) \subset \bigcup \{ S_\varepsilon(z) :: z \in F(x_0) \} .$$

THEOREM 2.4 *Suppose that* H *is as in* Theorem 2.3 *and that*

$$G:I \to C\#(H) \ (\underline{u}.\underline{s}.\underline{c}.\underline{i}.) ,$$

and that each G(t) *is convex, closed and bounded. Then there is a map* u *from* I *to* H *that is strongly measurable and such that* u(t) *is an element of* G(t) *for each* t *in* I.

Proof. Our proof will lean heavily upon the methods of the preceding theorem. The map which we exhibit is the same. The hypotheses on each G(t) are the same, and so the map is still well defined.

We shall show only the measurability of c^1 on I, where

$$c^1:I \to R::t \to [u(t),b_1].$$

The measurability of each c^k and the strong measurability of u follow exactly as in the preceding theorem.

As in Theorem 2.3, let

$$F^1(s) = \{t::t \in I \ \underline{\text{and}} \ [u(t),b_1] \leq s\} .$$

We show that $F^1(s)$ is closed, and hence that c^1 is measurable.

Suppose that $\{t^n\}$ is a sequence of elements of $F^1(s)$, and that $t^n \to t^o$. We show that t^o is an element of $F^1(s)$.

Consider some $\varepsilon > 0$ and the $\delta(\varepsilon) > 0$ of the definition of u.s.c.i. above. By the fact that $t^n \to t^0$, there is an integer N such that if $n \geq N$, then

$|t^n - t^o| < \delta(\varepsilon)$. By the definition of $\delta(\varepsilon)$, we see that for each x^N in $G(t^N)$ there is an element y of $G(t^o)$ such that $\underline{dist}(x^N,y) < \varepsilon$. Since $u(t^N)$ is an element of $G(t^N)$, there is an element y^N of $G(t^o)$ such that $\underline{dist}(u(t^N),y^N) < \varepsilon$.

Since H is a separable Hilbert space and $\{b_k\}$ is a complete orthonormal basis, we see that

$$(2.29) \quad [u(t^N)-y^N,u(t^N)-y^N)] = \underline{dist}^2(u(t^N),y^N) = \sum_{i-1}^{\infty}[u(t^N)-y^N,b_i]^2 .$$

From (2.29) and the inequality immediately preceding, we see that

$$(2.30) \qquad [u(t^N)-y^N,b_1]^2 < \varepsilon^2.$$

From the linearity of the inner product in its first argument, (2.30) and elementary algebra, we see that

$$(2.31) \qquad -\varepsilon \leq [u(t^N),b_1]-y^N,b_1] \leq \varepsilon .$$

Since y^N is an element of $G(t^o)$, then by the definition of $u(t^o)$,

$$(2.32) \qquad [u(t^o),b_1] \leq [y^N,b_1] .$$

Since t^N is an element of $F^1(s)$, we see that

(2.33) $$[u(t^N),b_1] \leq s.$$

From (2.31), (2.32) and (2.33) we see that

$$[u(t^0),b_1] = s+\varepsilon .$$

Since ε is arbitrary,

$$[u(t^o),b_1] \leq s ,$$

and $F^1(s)$ is closed.

Q. E. D.

Theorems 2.3 and 2.4 are motivated by the well known lemma of Filippov [11]. By strengthening the hypothesis on the map G of Theorem 2.3 from u. s. c. to u. s. c. i. (Filippov originally hypothesized u. s. c. i.), we are able to eliminate the quite restrictive hypothesis of compactness of the set in which the controls take their values. Indeed, this set need not even be bounded.

CHAPTER III

THE EQUATION OF EVOLUTION AND TIME-OPTIMAL CONTROLS

Let X be a separable and reflexive Banach space. Assume that the linear operator A from X into itself either is independent of t or satisfies the following concitions:

(i) <u>For each</u> t <u>in an interval</u> $[0,T]$ <u>the operator</u> $A(t)$ <u>is closed and the domain of</u> $A(t)$ <u>is dense in</u> X <u>and independent of</u> t;

(ii) <u>for</u> $\operatorname{Re}(\lambda) \leqq 0$, $(\lambda I - A(t))^{-1}$ exists;

(iii) <u>for</u> $\operatorname{Re}(\lambda) \leqq 0$, $\|(\lambda I - A(t))^{-1}\| \leqq \frac{c}{1+|\lambda|}$, <u>where</u> c <u>is a real constant</u>;

(iv) $\|(A(t)-A(s))A(r)^{-1}\| \leqq k|t-s|^{a}$, where $k>0$ and $a>0$.

Then by the results given in [3,211], [4,117] and [11,363], if

$$u:[0,T] \longrightarrow X::t \longrightarrow u(t)$$

in such a manner that

(3.0) $\|u(t)-u(s)\| \leqq C|t-s|^{b}$, <u>where</u> $C>0$ <u>and</u> $b>0$,

there is a unique differentiable solution to

$$(3.1) \qquad \frac{dx(t)}{dt} + A(t)x(t) = u(t) \ , \ t \in [0,T]$$

of the form

$$(3.2) \qquad x^u(t) = S(t,0)x_o + (B)\int_0^t S(t,s)u(s) \ , \ t\in[0,T] \ .$$

where the linear operators $\{S(t,s)::(t,s) \in [0,T]x[0,T]\}$ where $s \leq t$, from X into itself all have norm less than or equal to one and are all continuous in t and s simultaneously.

It should be emphasized that $\{S(t,s)::(t,s)\in[0,T]x[0,T]\}$ depends only upon $\{A(t)::t \in [0,T]\}$ in a very complicated manner which it is not feasible to discuss here, but which is developed in [3,210].

If the function u fails to satisfy the very strong condition given in (3.0), which is called <u>Hölder continuity</u>, the function x^u, defined in (3.2), may fail to be differentiable, and so of course may fail to satisfy (3.1). However, if u is merely bounded and strongly measurable, then the function x^u is still well defined, and is called the <u>weak solution</u> to (3.1).

THEOREM 3.1. <u>Let U_M be the set of strongly measurable functions that satisfy the following conditions</u>:

$$(3.3) \qquad u:[0,T] \to X \ ;$$

(3.4) $u(t) \in U$ *a.e.*, *where* U *is a convex, closed and bounded subset of* X;

(3.5) $(L)\int_0^T \|u(t)\|^p \leq M$, *where* $p > 1$. (*That is*, $\|u\|_p = M^{1/p}$.);

(3.6) $x^u(0) = x_0$ *and* $x^u(T^u) = x_1$, *where* $0 \leq T^u \leq T$.

We say that the control u *directs its response* x^u *from* x_0 *to* x_1 *in time* T^u.

If U_M *is not empty, it contains a time-optimal control.*

Proof. We begin the proof by establishing the following lemma.

LEMMA 3.1.1. *If* u *maps* $[0,T]$ *into* X, *and* u *is strongly measurable and bounded, then for almost all* s *in* $[0,T]$,

$$\frac{1}{\varepsilon}(B)\int_s^{s+\varepsilon} u(t) \longrightarrow u(s) \quad \text{as} \quad \varepsilon \longrightarrow 0^+.$$

Proof of the Lemma. If the function f from $[0,T]$ to the real numbers is bounded and Lebesgue measurable, then it is well known that

$$\frac{1}{\varepsilon}(L)\int_s^{s+\varepsilon} f(t) \longrightarrow f(s) \quad \text{as} \quad \varepsilon \longrightarrow 0^+$$

for almost all s in $[0,T]$.

Since X is separable, let $\{x_n\}$ be a countable dense subset of X. Since u is strongly measurable and bounded, the function

$$\|u(\ )-x_n\|:[0,T]\longrightarrow R::t\longrightarrow\|u(t)-x_n\|$$

is bounded and Lebesgue measurable for each x_n.

Applying the above quoted result, we see that

$$(3.7)\quad \frac{1}{\varepsilon}(L)\int_s^{s+\varepsilon}\|u(t)-x_n\|\longrightarrow\|u(s)-x_n\| \text{ \underline{as} } \varepsilon\longrightarrow 0^+ \text{ \underline{for} } s\in E_n,\ n=1,\ 2,\ \ldots,$$

where E_n is a subset of $[0,T]$ and $\underline{\text{meas}}([0,T]-E_n) = 0$, $n=1, 2, \ldots$.

Then $\underline{\text{meas}}([0,T]-\bigcap_{n=1}^{\infty}E_n) = 0$. Select s in $\bigcap_{n=1}^{\infty}E_n$ and then select some x_n from $\{x_n\}$ such that $\|u(s)-x_n\|<\delta$, where $\delta>0$. We must also demand that $s\neq T$ in order that integration over the range $[s,s+\varepsilon]$ be defined for sufficiently small positive ε.

Then by properties (1.1) and (1.2) of the Bochner integral and the properties of the norm of a Banach space,

$$0 \leq \left\| u(s) - \frac{1}{\varepsilon}(B)\int_s^{s+\varepsilon} u(t) \right\| = \frac{1}{\varepsilon}\left\| (B)\int_s^{s+\varepsilon} u(s) - (B)\int_s^{s+\varepsilon} u(t) \right\| =$$

$$\frac{1}{\varepsilon}\left\| (B)\int_s^{s+\varepsilon} (u(t)-u(s)) \right\| \leq \frac{1}{\varepsilon}(L)\int_s^{s+\varepsilon} \| u(s)-u(t) \| \leq$$

$$\frac{1}{\varepsilon}(L)\int_s^{s+\varepsilon} \| u(s)-x_n \| + \frac{1}{\varepsilon}(L)\int_s^{s+\varepsilon} \| u(t)-x_n \| \leq \delta + \frac{1}{\varepsilon}(L)\int_s^{s+\varepsilon} \| u(t)-x_n \| \ .$$

Since we see by (3.7) that the second expression in the last term of the above inequality approaches $\| u(s)-x_n \|$ as ε approaches zero positively, we have that

$$0 \leq \varlimsup_{\varepsilon \to 0+} \left\| u(s) - \frac{1}{\varepsilon}(B)\int_s^{s+\varepsilon} u(t) \right\| \leq 2\delta \ .$$

Since is arbitrary,

$$\varlimsup_{\varepsilon \to 0+} \left\| u(s) - \frac{1}{\varepsilon}(B)\int_s^{s+\varepsilon} u(t) \right\| = 0 \ .$$

Hence

$$\frac{1}{\varepsilon}(B)\int_s^{s+\varepsilon} u(t) \to u(s) \ \underline{\text{as}} \ \varepsilon \to 0^+ \ .$$

The lemma is established, since s was any point of $\bigcap_{n=1}^{\infty} E_n$.

Now, let $\{u^n\}$ be a sequence of elements of U_M such that

(3.8) $$T^n \to T^o = \inf_{u \in U_M}(T^u),$$

where for simplicity of notation we replace T^{u^m} by T^m.

Since U_M is bounded as a subset of $B_p([0,T],X)$ in the norm defined by (1.12), there is an element u^o of $B_p([0,T],X)$ and a subsequence of $\{u^n\}$, also called $\{u^n\}$, such that

(3.9) $$u^n \to u^o \text{ (\underline{weak})}.$$

We shall show that u^o is an element of U_M and that u^o is time-optimal.

As is well known, there is a family $\{f^i\}$ of linear functionals on X and a set $\{c_i\}$ of real numbers such that x is an element of the convex set U iff $f^i(x) \leq c_i$ for all i. (Due to the separability of X, these sets may be denumerable, but this is irrelevant to our proof.)

In what follows, $\varepsilon > 0$ and $0 \leq s < s+\varepsilon \leq T$.

Let

(3.10) $$u(\varepsilon,n,s) = \frac{1}{\varepsilon}(B)\int_s^{s+\varepsilon} u^n(t), \quad n = 0, 1, \ldots .$$

From [I,80],

(3.11) $$f^i(u(\varepsilon,n,s)) = \frac{1}{\varepsilon}(L)\int_s^{s+\varepsilon} f^i(u^n(t)) \quad \text{\underline{for all} } i, \text{ \underline{and}}$$

$n = 1, 2, \ldots$. Since each u^n, $n = 1, 2, \ldots$ takes its values almost everywhere in U, then for almost all t in $[0,T]$, all i, and $n = 1, 2, \ldots$, $f^i(u^n(t)) \leq c_i$. Hence from (3.11),

(3.12) $f^i(u(\varepsilon,n,s)) \leq c_i$ <u>for all</u> i, <u>and</u> $n = 1, 2, \ldots.$

From (3.9) we see that for all i,

(3.13) $f^i(u(\varepsilon,n,s)) \longrightarrow f^i(u(\varepsilon,0,s))$ <u>as</u> $n \to \infty$.

From Lemma 3.1.1 and (3.10) we see that

(3.14) $u(\varepsilon,0,s) \longrightarrow u^0(s)$ <u>a.e.</u> <u>on</u> $[0,T]$, <u>as</u> $\varepsilon \to 0^+$.

From (3.12) and (3.13), we see that for all i,

(3.15) $$f^i(u(\varepsilon,0,s)) \leq c_i \ .$$

Hence for each i, for any positive ε, and for any s in $[0,T]$, $u(\varepsilon,0,s)$ is an element of U.

From (3.14) and the closure of U, we see that for almost all s in $[0,T]$, $u^0(s)$ is an element of U. Hence condition (3.4) is satisfied.

From the reflexivity of $B_p([0,T],X)$ and conditions (3.5) and (3.9) it follows that $\|u^0\|_p \leq M^{1/p}$. Hence condition (3.5) is satisfied for u^0.

Consider now the responses to the controls

$\{u^n : : n=0, 1, \ldots\}$, where

(3.16) $x^n(t) = S(t,s)x_o + (B)\int_0^t S(t,s)u^n(s)$, $n = 0, 1, \ldots.$

Since u^n is an element of U_M, $n = 1, 2, \ldots$, it follows that

(3.17) $x^n(0) = x_o$ <u>and</u> $x^n(T^n) = x_1$, $n = 1, 2, \ldots.$

The function x^o as defined by (3.16) is the weak solution of (3.1) corresponding to the control u^o, and we must show that $x^o(0) = x_o$ and $x^o(T^o) = x_1$. Since $S(0,0) = I$ [4,210], it is clear from (3.16) that $x^o(0) = x_o$. To show that $x^o(T^o) = x_1$, we will show that

(3.18) $$x^n(T^n) \rightarrow x^o(T^o) \quad (\underline{weak}).$$

Since $x^n(T^n) = x_1$, $n = 1, 2, \ldots.$, it follows that

(3.19) $x^n(T^n) \rightarrow x_1$ (<u>strong</u>) <u>and</u> a fortiori (<u>weak</u>).

Since the weak limit is unique [IV,121], it will follow from (3.18) and (3.19) that $x^o(T^o) = x_1$. We now proceed to prove (3.18).

Since S is continuous is t and s simultaneously [3,210], $T^n \rightarrow T^o$ and $[0,T]x[0,T]$ is compact, it follows that

(3.20) $S(T^n,s) \rightarrow S(T^o,s)$ (<u>strongly</u> <u>and</u> <u>uniformly</u> in s).

Since the functions u^n, $n=1, 2, \ldots$ take their values almost everywhere in U, they are uniformly bounded almost everywhere. Hence it follows from (3.20) that

$$(3.21) \qquad (B)\int_0^{T^o}(S(T^n,s)-S(T^o,s))u^n(s) \longrightarrow 0 \quad (\underline{\text{strong}}).$$

Let f be a continuous linear functional on X and suppose that u and v are elements of $B_p([0,T],X)$ and that a and b are real numbers. Due to the linearity of the Bochner integral and the functional f,

$$f[(B)\int_0^{T^o}S(T^o,s)(au(s)+bv(s))] =$$

$$af[(B)\int_0^{T^o}S(T^o,s)u(s)] + bf[(B)\int_0^{T^o}S(T^o,s)v(s)].$$

Since S is bounded [3,210], we see that the map F, where

$$(3.22) \quad F:B_p([0,T],X) \longrightarrow R::u \longrightarrow f[(B)\int_0^{T^o}S(T^o,s)u(s)].$$

is a continuous linear functional on $B_p([0,T],X)$. Hence from (3.9),

$$(3.23) \quad F(u^n) = f[(B)\int_0^{T^o}S(T^o,s)u^n(s)] \longrightarrow f[(B)\int_0^{T^o}S(T^o,s)u^o(s)]$$

$$= F(u^o).$$

Since f was arbitrary in X^*, it follows from (3.23) that

(3.24) $$(B)\int_0^{T^o} S(T^o,s)u^n(s) \longrightarrow (B)\int_0^{T^o} S(T^o,s)u^o(s) \quad (\underline{\text{weak}}).$$

We write $(B)\int_0^{T^o} S(T^o,s)u^n(s)$ in the form

(3.25) $$(B)\int_0^{T^o} [S(T^n,s)-S(T^o,s)]u^n(s) + (B)\int_{T^o}^{T^n} S(T^n,s)u^n(s) + (B)\int_0^{T^o} S(T^o,s)u^n(s),$$

We have shown that the first integral in (3.25) converges strongly to zero. (See (3.21).) The second integral converges strongly to zero due to the fact that $T^n \longrightarrow T^o$ and S and u^n, $n = 1, 2, \ldots$ are all uniformly bounded. Hence it follows from (3.24) and the statement immediately above that

(3.26) $$(B)\int_0^{T^n} S(T^n,s)u^n(s) \longrightarrow (B)\int_0^{T^o} S(T^o,s)u^o(s) \quad (\underline{\text{weak}}).$$

It then follows from (3.20) that

(3.27) $$S(T^n,0)x_0 \longrightarrow S(T^o,0)x_0 \quad (\underline{\text{strong}}).$$

Hence it follows from (3.16), (3.26) and (3.27) that

$$x^n(T^n) \longrightarrow x^o(T^o) \quad (\underline{\text{weak}}).$$

Q. E. D.

Several interesting corollaries are easily obtained.

Corollary 3.1.1. *By making M sufficiently large, condition* (3.5) *may be omitted.*

Proof. Since U is bounded, let $M = \sup_{x \in U} T\|x\|^p$.

This corollary is proved in [2,10].

Q. E. D.

The *set attainable* from x_o, called R_o, is defined by

(3.28) $R_o = \{z :: z \in X$ *and some* u *of* U_M *directs its response from* x_o *to* z *in time less than or equal to* $T.\}$

A set F is *weakly closed* iff for any sequence $\{x_n\}$ of elements of F,

(3.29) $$x_n \to x_o \text{ (weak) implies } x_o \in F.$$

Corollary 3.1.2 *The set attainable from* x_o *is closed and weakly closed.*

Proof. Since strong convergence implies weak convergence, it follows that if a set is weakly closed, it is strongly closed. We show that R_o is weakly closed. To this end, suppose that $\{z_n\}$ is a sequence of elements of R_o and that

(3.30) $$z_n \to z_o \text{ (weak)}.$$

Note that for each n, $z_n = x^n(T^n)$, where $0 = T^n = T$ and each x^n is the response to some control u^n

of U_M. As in Theorem 3.1, there is a subsequence of $\{z_n\}$, also called $\{z_n\}$, an element u^o of U_M, and a real number T^o of $[0,T]$ such that

(3.31) $$u^n \longrightarrow u^o \ (\underline{\text{weak}}) \ \underline{\text{and}} \ T^n \longrightarrow T^o,$$

where for each n, the control u^n directs its response from x_o to z_n in time T^n.

By the proof of Theorem 3.1, and the definition of the subsequence $\{z_n\}$,

(3.32) $$x^n(T^n) \longrightarrow x^o(T^o) \ (\underline{\text{weak}}).$$

By (3.30), (3.32) and the uniqueness of the weak limit, $z_o = x^o(T^o)$, and so z_o is an element of R_o.

Q. E. D.

Corollary 3.1.3. _The set attainable from_ x_o _is bounded_.

Proof. Suppose u is an element of U_M. Then its response x^u has the form given by (3.2). Since each S has norm less than or equal to one [3,210] and

$\|U\| = \sup_{x \in U} \|x\| < +\infty$, we see that

$$\|x(t)\| \leq \|S(t,0)x_o\| + \left\|(B)\int_0^t S(t,s)u(s)\right\| \leq \|x_o\| + T\|U\|$$

for all t in $[0,T]$.

Q. E. D.

We say that z is an element of $R(t)$, the _set attainable from_ x_o _at time_ t iff there is a control u

of U_M, where the constant M is as in Theorem 3.1, that directs its response from x_o to z in time t, where $0 \leq t \leq T$.

It is clear that for each t in $[0,T]$, $R(t)$ is bounded, for $R(t)$ is a subset of R_o. To show that each $R(t)$ is weakly closed and hence closed, suppose that $\{z_n\}$ is a sequence of elements of $R(t)$ and that

$$(3.33) \qquad z_n \longrightarrow z_o \quad (\underline{\text{weak}}).$$

As in the proof of Theorem 3.1, there is an element u^o of U_M and a subsequence of $\{z_n\}$ also called $\{z_n\}$, such that

$$(3.34) \qquad u^n \longrightarrow u^o \ (\underline{\text{weak}}) \ \underline{\text{and}} \quad x^n(t) \longrightarrow x^o(t) \quad (\underline{\text{weak}}).$$

Since for each n, $z_n = x^n(t)$, it follows from (3.33), (3.34) and the uniqueness of the weak limit that $z_o = x^o(t)$. Hence z_o is an element of $R(t)$, which is seen to be weakly closed.

Remark 3.1.1. Since each $S(t,s)$ is a bounded linear operator and Bochner integration is a linear operation, $R(t)$ is convex for each t in $[0,T]$. Since each $R(t)$ is also closed and bounded and X is separable, each $R(t)$ is weakly compact in itself [I,38].

THEOREM 3.2. _For each constant_ M, _as defined in Theorem_ 3.1, _the map_

(3.35) $$R:[0,T] \longrightarrow C\#(X)::t \longrightarrow R(t)$$

is continuous.

Proof. We have seen that R is well defined. To show that R is continuous, we use the previously noted fact that $C\#(X)$ is a metric space under the Hausdorff metric, where

(3.36) $$\underline{dist}[A,B] = \underset{a\in A,\, b\in B}{\text{supremum}} \left[\inf_{b'\in B} \underline{dist}(a,b'),\ \inf_{a'\in A} \underline{dist}(a',b) \right].$$

We show that if $\{t^n\}$ is a sequence of elements of $[0,T]$, then

(3.37) $$t^n \longrightarrow t^o \ \underline{\text{implies}}\ R(t^n) \longrightarrow R(t^o).$$

To establish (3.37) it is enough to show that for every $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that when $|t^n - t^o| < \delta(\varepsilon)$, then

(3.38) if $z_n \in R(t^n)$, there is a $z_o^n \in R(t^o)$ such that $\underline{dist}(z_n, z_o^n) \leq \varepsilon/2$;

(3.39) if $z_o \in R(t^o)$, there is a $z_n^o \in R(t^n)$ such that $\underline{dist}(z_o, z_n^o) \leq \varepsilon/2$.

Suppose z_o is an element of $R(t^o)$. Then some control u^o of U_M has a response x^o such that

(3.40) $$z_o = x^o(t^o) = S(t^o,0)x_o + (B)\int_0^{t^o} S(t^o,s)u^o(s).$$

From (3.40), we see that z_n^o is an element of $R(t^n)$, where

(3.41) $$z_n^o = x^o(t^n) = S(t^n,0)x_o + (B)\int_0^{t^n} S(t^n,s)u^o(s).$$

From (3.40), (3.41), the triangle property of the norm, the uniform boundedness of $S(t,s)$, the previously noted properties of the Bochner integral and the boundedness of the restraint set U, we see that

(3.42) $$\underline{\text{dist}}(z_n^o,z_o) = \|z_n^o-z_o\| \leq \|S(t^n,0)-S(t^o,0)\|\cdot\|x_o\| +$$

$$\left\|(B)\int_0^{t^o}[S(t^o,s)-S(t^n,s)]u^o(s)\right\| + \left\|(B)\int_{t^o}^{t^n}S(t^n,s)u^o(s)\right\| \leq$$

$$\|S(t^n,0)-S(t^o,0)\|\cdot\|x_o\| + (t^o)\underset{s\in[0,T]}{\underline{\text{maximum}}}\|S(t^o,s)-S(t^n,s)\|\cdot\|U\| +$$

$$|t^n-t^o|\cdot\|U\|,$$

where $\|U\| = \underset{x\in U}{\underline{\sup}}\|x\| < +\infty$.

From (3.20) and (3.42) we see that

(3.43) $$\underline{\text{dist}}(z_n^o,z_o) \longrightarrow 0 \ \underline{\text{as}} \ t^n \longrightarrow t^o.$$

Now, suppose that z_n is an element of $R(t^n)$. There is a control u^n with a response x^n such that

$$(3.44)\quad z_n = x^n(t^n) = S(t^n,0)x_o + (B)\int_0^{t^n} S(t^n,s)u^n(s) \ .$$

Then z_o^n is an element of $R(t^o)$, where

$$(3.45)\quad z_o^n = x^n(t^o) = S(t^o,0)x_o + (B)\int_0^{t^o} S(t^o,s)u^n(s) \ .$$

By the same argument with which we established (3.43), we see that

$$(3.46)\qquad \underline{\text{dist}}(z_o^n, z_n) \longrightarrow 0 \ \ \underline{\text{as}} \ \ t^n \longrightarrow t^o.$$

It follows from (3.43) and (3.46) that for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that when $|t^n - t^o| < \delta(\varepsilon)$, conditions (3.38) and (3.39) hold. Thus we have established (3.37).

Q. E. D.

CHAPTER IV

TIME-OPTIMAL CONTROL OF A NONLINEAR PROCESS

Based upon a well known paper of Lee and Markus [6], we consider a control problem in a Banach space setting. Our results depend upon a theorem which extends the method of successive approximations to a class of differential equations in a Banach space. The theorem we use is found in [I,67], and we quote:

THEOREM 4.1. *Suppose that* B *is a Banach space and that*

$$f : R \times B \to B :: (t,x) \to f(t,x) \ ,$$

where f *is continuous in each variable separately for*

(4.1) $$|t-t_0| \leq a, \quad \|x-x_0\| \leq b,$$

and f *satisfies*

(4.2) $$\|f(t,x)\| \leq M;$$

(4.3) $$\|f(t,x_1)-f(t,x_2)\| \leq c\,\|x_1-x_2\|$$

for t, x, x_1, *and* x_2 *in the regions indicated in* (4.1). *Here* a, b, M, *and* c *are positive constants and*

(4.4) $$aM \leq b.$$

Then there is exactly one continuously differentiable function x _such that_

(4.5) $$\frac{dx(t)}{dt} = f(t,x(t)) \quad \underline{\text{and}} \quad x(t_0) = x_0,$$

where $|t-t_0| \leq a$.

The solution is of the form $x(t) = x_0 + (B)\int_{t_0}^{t} f(t,x(t))$.

In preparation for the use of this theorem, suppose that B_1 and B_2 are Banach spaces and that

(4.6) $$f:[0,T]xB_1xB_2 \rightarrow B_1::(t,x,u) \rightarrow f(t,x,u) ,$$

where f is continuous in t and u simultaneously and also Lipschitzian in x. (See condition (4.3).)

Suppose also that

(4.7) $$\|f(t,x,u)\| \leq \mu(t,u),$$

where the function μ is real-valued and upper semicontinuous.

Consider a function u such that

(4.8) $$u:[0,T] \rightarrow B_2 \quad (\underline{\text{continuous}}).$$

The range of u is compact in B_2, and so μ is bounded on $[0,T]x\text{Range}(u)$. Hence f is bounded on $[0,T]xB_1x\text{Range}(u)$

To apply Theorem 4.1 to the function defined in (4.6), note that

$$(4.9) \qquad f:[0,T]xB_1 \longrightarrow B_1::(t,x) \longrightarrow f(t,x,u(t)).$$

Then f as defined in (4.9) is continuous in t and Lipschitzian in x. Due to condition (4.7) f is bounded in norm by some constant M. Since f is defined for each x in B_1, the inequality in (4.4) is automatically satisfied, where $T/2 = a$.

Hence by Theorem 4.1 there is a unique continuously differentiable solution to the differential equation

$$(4.10) \qquad \frac{dx(t)}{dt} = f(t,x,u(t))$$

of the form

$$(4.11) \quad x^u(t) = x_o + (B)\int_{t_o^u}^{t} f(t,x^u(t),u(t)), \qquad 0 \leqq t_o^u \leqq T.$$

We call the solution x^u to (4.10) the <u>response</u> to the <u>control</u> u.

Next, we suppose that

$$(4.12) \qquad K:[0,T] \longrightarrow C(B_1) \quad (\underline{\text{upper semicontinuous}}),$$

where $C(B_1)$ is the space of closed and bounded subsets of B_1. A control u is said to direct its response x^u from x_o to the <u>target set</u> $K(t_1^u)$ if

$$(4.13) \qquad x^u(t_o^u) = x_o;$$

(4.14) $$x(t_1^u) \in K(t_1^u),$$

where

(4.15) $$0 \leqq t_0^u \leqq t_1^u \leqq T.$$

We call $(t_1^u - t_0^u)$ a time of flight.

THEOREM 4.2. *Let $\{u\}$ be a closed set of equicontinuous controls which take their values within C_2, a subset of B_2 which is compact in itself. Then of the controls which direct their responses from x_0 to the varying target set there is a control u^o, also an element of $\{u\}$, that has minimum time of flight. That is,*

(4.16) $$(t_1^o - t_0^o) = m = \inf\{(t_1^u - t_0^u) :: u \text{ directs } x^u \text{ from } x_0 \text{ to } K(t_1^u)\}.$$

Proof. Since $(t_1^u - t_0^u) \geqq 0$ for each u, the infimum in (4.16) surely exists. Hence there is a sequence $\{u^n\}$ of elements of $\{u\}$ such that

(4.17) $$(t_1^n - t_0^n) \longrightarrow m.$$

Since each element of $\{u^n\}$ takes its values in C_2, the sequence is bounded. As elements of $\{u\}$, the elements of $\{u^n\}$ are equicontinuous. Hence by Ascoli's Theorem [III,39] and the compactness of $[0,T]$ there is a function u^o that is equicontinuous with $\{u\}$ and a subsequence of $\{u^n\}$, also called $\{u^n\}$, such that

(4.18) $$(t_1^n - t_0^n) \longrightarrow m;$$

(4.19) $$t_1^n \longrightarrow t_1^o;$$

(4.20) $$t_0^n \longrightarrow t_0^o;$$

(4.21) $$u^n(t) \longrightarrow u^o(t) \quad (\underline{\text{uniformly on}} \ [0,T]).$$

We show that u^o is the desired optimal control by showing that u^o has time of flight $(t_1^o - t_0^o)$.

Suppose that $\{x^n\}$ is the sequence of responses to the sequence of controls $\{u^n\}$. Each x^n is of the form

(4.22) $$x^n(t) = x_o + (B)\int_{t_o^n}^{t} f(t,x^n(t),u^n(t)), \quad 0 \leq t \leq T.$$

Since f is bounded in norm due to (4.7) and the compactness of $[0,T]xC_2$, we see from (4.22) that the elements of $\{x^n\}$ are equilipschitzian and _a fortiori_ equicontinuous. We see that they are also uniformly bounded, and so by Ascoli's Theorem there is a function x^o and a subsequence of $\{x^n\}$, also denoted by $\{x^n\}$, such that

(4.23) $$x^n(t) \longrightarrow x^o(t) \quad (\underline{\text{uniformly on}} \ [0,T]).$$

The statements (4.18), (4.19), (4.20) and (4.21) remain true for the subsequence of u^n corresponding to the subsequence $\{x^n\}$. We will show that x^o is the response to u^o

and that u^o directs x^o from x_o to $K(t_1^o)$ in time $(t_1^o - t_o^o)$. This will complete our proof.

We have seen that u^o is equicontinuous with $\{u\}$. From (4.21) and the fact that each u^n takes its values in C_2, which is compact in itself and hence closed, we see that u^o takes its values in C_2. Hence $u^o \in \{u\}$.

Since

$$\|x^n(t_o^n) - x^o(t_o^o)\| \leq \|x^n(t_o^n) - x^o(t_o^n)\| + \|x^o(t_o^n) - x^o(t_o^o)\|,$$

it follows from (4.20), (4.23) and the continuity of x^o that

$$(4.25) \qquad x^n(t_o^n) \to x^o(t_o^o).$$

From (4.25) and the fact that for each n, $x^n(t_o^n) = x_o$, it follows that

$$(4.26) \qquad x^o(t_o^o) = x_o.$$

From the inequality

$$\|x^n(t_1^n) - x^o(t_1^o)\| \leq \|x^n(t_1^n) - x^o(t_1^n)\| + \|x^o(t_1^n) - x^o(t_1^o)\|,$$

conditions (4.19) and (4.23) and the continuity of x^o, it follows that

$$(4.27) \qquad x^n(t_1^n) \to x^o(t_1^o).$$

Since for each n we know that $x^n(t_1^n)$ is an element of $K(t_1^n)$, it follows from (4.19), (4.27), the upper semicontinuity of K and the theorem of Jacobs [II,16] quoted in the preceding section that

$$(4.28) \qquad x^{o}(t_1^{o}) \in K(t_1^{o}).$$

From (4.21), (4.23) and the fact that f is Lipschitzian in its second argument and continuous in its third argument we see that

$$f(t,x^{n}(t),u^{n}(t)) \longrightarrow f(t,x^{o}(t),u^{o}(t)).$$

Hence from the boundedness of f and the generalized Lebesgue theorem [I,83] we see that

$$(4.29) \quad (B)\int_{t_o^o}^{t} f(t,x^{n}(t),u^{n}(t)) \longrightarrow (B)\int_{t_o^o}^{t} f(t,x^{o}(s),u^{o}(s))$$

<u>on</u> [0,T].

From (4.20) and the boundedness of f we see that

$$(4.30) \qquad (B)\int_{t_o^n}^{t_o^o} f(t,x^{n}(t),u^{n}(t)) \longrightarrow 0.$$

From (4.29) and (4.30) it follows that

$$(4.31) \quad (B)\int_{t_o^n}^{t} f(t,x^{n}(s),u^{n}(s)) \longrightarrow (B)\int_{t_o^o}^{t} f(t,x^{o}(s),u^{n}(s))$$

<u>on</u> [0,T].

From (4.22) and (4.31) it follows that

$$(4.32)\quad x^n(t) \longrightarrow x_0 + (B)\int_{t_0^0}^{t} f(t,x^0(s),u^0(s)) \quad \underline{\text{on}} \quad [0,T].$$

From (4.23) and (4.32) we see that

$$(4.33)\quad x^0(t) = x_0 + (B)\int_{t_0^0}^{t} f(t,x^0(s),u^0(s)) \quad \underline{\text{on}} \quad [0,T].$$

Since u^0 and x^0 are continuous and f is simultaneously continuous in its first and third arguments and Lipschitzian in its second argument, it follows that the function

$$(4.34)\qquad f:[0,T] \longrightarrow B_1::t \longrightarrow f(t,x^0(t),u^0(t))$$

is continuous. Hence we see from this and (4.33) that

$$(4.35)\qquad \frac{dx^0(t)}{dt} = f(t,x^0(t),u^0(t)).$$

Thus we see that x^0 is the response to u^0. It follows from (4.26) and (4.28) that u^0 has time of flight $(t_1^0 - t_0^0)$.

Q. E. D.

We define the <u>set attainable from</u> x_0 by

$$(4.36)\quad A_0 = \{z::z\in B_1 \ \underline{\text{and there is a control}}\ u\ \underline{\text{of}}\ \{u\}\ \underline{\text{that directs its response from}}\ x_0\ \text{to}\ z.\}$$

Corollary 4.1.1. <u>The attainable set is compact in itself</u>.

Proof. Suppose $\{z_n\}$ is a sequence of elements of A_o. Then by the proof of Theorem 4.1 there is a subsequence of $\{z_n\}$, also called $\{z_n\}$, a control u^o of $\{u\}$ with response x^o such that

$$z_n = x^n(t_1^n) \longrightarrow x^o(t_1^o). \tag{4.37}$$

Hence $x^o(t_1^o)$ is an element of A_o, which we see is compact in itself.

Q. E. D.

CHAPTER V

EXAMPLES

The reader has doubtless noticed that the compactness hypothesis in Theorem 2.3 is quite restrictive. One is tempted to replace it by demanding only that $\bigcup\{G(t)::t \in I\}$ be bounded and then using the fact that its closed convex hull would be compact in itself. [I,38]

This would be a simple matter if one could replace the strong convergence of x_n to x_o in (2.5) by weak convergence. The following counterexample shows that this is impossible.

Let the space of (2.5) be l_2, and let $I = [0,1]$. For each t in $[0,1]$ suppose that the constant (and a fortiori continuous) map G be

$$G(t) = \{x:: \|x\| = 1, \quad x \in l_2\}.$$

Let x_n be defined by:

$$a_n^n = 1;$$

$$a_n^i = 0, \quad i \neq n.$$

For each n, let $t^n = 1/n$. Then we see that

$$t^n \longrightarrow 0;$$

$$x_n \longrightarrow 0 \ (\underline{weak});$$

$$x_n \in G(t^n) \ \underline{for}\ \underline{each}\ t^n.$$

Hence the hypotheses of (2.5) are satisfied with the exception of that of strong convergence. But it is clear that $0 \notin G(0)$.

Since the map G is continuous and the space considered is quite highly structured, it would seem unlikely that the hypothesis of weak convergence could be retained by means of any very realistic strengthening of any of the other hypotheses.

A better counterexample in which the map G is u.s.c. and each $G(t)$ is closed, bounded and convex has been obtained. However, it has not been included.

I am indebted to Dr. Ewing for the following example, which relates a standard problem in variational theory to control theory.

If we read Theorem 3.1 down to the familiar real-value setting, we have a simple form of the Mayer problem,

$$(5.1) \qquad t^1(u,x) = \underline{global}\ \underline{minimum}\ \underline{on}\ C,$$

where C is the class of all pairs

$u:[t^o,t^1] \longrightarrow R$, $x:[t^o,t^1] \longrightarrow R$ such that u is measurable and x is A. C. on $[t^o, t^1]$, and

(5.2) $$|u(t)| \leq M \text{ on } [0,T]$$

such that

(5.3) $\dot{x}(t) = f[t,x(t),u(t)] = A(t)x(t) + u(t)$ a.e. on $[t^o, t^1]$

where A( ) is a continuous scalar function and such that

(5.4) $$t^o = 0 \qquad t^1 \leq T = \underline{\text{constant}}$$

$$x(t^o) = 0 \qquad x(t^1) = x_1 = \underline{\text{constant}}.$$

Theorem <u>If</u> C <u>is not empty, then there is an element</u> (u^o, x^o) <u>of</u> C <u>such that</u>

$$t^1(u^o,x^o) = \underset{(u,x)\in C}{\text{infimum}}\, t^1(u,x).$$

<u>Proof</u>. Every (u,x) of C can be extended to [0,T] by setting $u(t) = u(t^1)$ and $x(t) = x(t^1)$ for $t\in(t^1,T]$.

Define

(5.5) $$z(t) = \int_0^t u(s) \quad \underline{\text{for}} \quad t\ [0,T].$$

We see by (5.2) that z is lipschitzian with constant M.

Let $\{(u^n,x^n)\}$ be a sequence of elements of C such that

$$t^1(u^n,x^n) \longrightarrow \underset{(u,x)\in C}{\underline{\text{infimum}}}\ t^1(u,x),$$

and let $\{z^n\}$ correspond via (5.5) to $\{u^n\}$.

The functions in $\{z^n\}$ being equi-lipschitzian are _a fortiori_ equicontinuous and are equally bounded. Hence by Ascoli's Theorem we can suppose the sequence so chosen that it converges uniformly to z^o, where

$$z^o:[0,T] \longrightarrow R,$$

and z^o satisfies the same Lipschitz condition as the z^n.

Moreover, $\dot{z}^o(t)$ exists and is $\leq M$ a. e. on $[0,T]$. Hence we can define

$$u^o(t) = \begin{cases} \dot{z}^o(t) & \text{when it exists} \\ \text{an arbitrary real number in } [0,T] & \text{elsewhere,} \end{cases}$$

and $u^o:[0,T] \longrightarrow R$ is an admissible component of $(u,x)\in C$.

Observe that

(5.6) $$x^n(t) = \int_0^t [\exp\int_r^t A(s)]u^n(r) \quad \underline{\text{for}} \quad t\in[0,T].$$

Since $x^n(0) = 0$ and $x^n[t^1(u^n,x^n)] = x_1$

$$= \text{constant by } (5.4),$$

it follows from (5.6) (after an integration by parts) that $x^o(0) = 0$ and $x^o[t^1(u^o,x^o)] = x_1$. Hence the pair

(u^o, x^o) furnishes the desired minimum.

Extensions to more general Mayer problems and to some forms of the Bolza Problem should be easy.

Notice that in (5.6) the term $\exp\int_r^t A(s)$ plays the role (with a slight change of notation) of the operator $S(t,x)$ of Theorem 3.1.

Since we are treating the case of a real variable, the rather elaborate hypotheses of Theorem 3.1, particularly the highly restrictive uniform Hölder condition (iv), are not necessary.

The foregoing example, with Theorem 3.1, illustrates the difficulty of obtaining the Banach space analogues of familiar theorems of real variables.

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