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CHAPTER I

INTRODUCTION

One of the fundamental problems in numerical analysis is the approximation of a continuous real valued function f on an interval of real numbers [c,d]. This problem has been extensively studied using a wide variety of norms to measure the closeness of the approximation. Under suitable hypotheses for f, theorems such as Taylor's theorem and the Weierstrass approximation theorem provide motivation for considering polynomials as approximating functions.

With the advent of the modern high speed computer, polynomial approximation has become increasingly more important as a basic tool for approximating functions which are difficult to evaluate. The reason is that, in addition to the rich theory of polynomial approximation using various norms, polynomials can be efficiently and accurately evaluated by computers provided the degree of the polynomial is not too large. Hence it is natural to consider the problem of approximating f by polynomials of degree less than or equal to n. Instead of evaluating a function on a computer, the best that can be done in many cases is to evaluate an approximation of the function. Thus the approximate values are used instead of the actual functional values. It is then necessary to know the maximum error of the approximation.

The choice of the norm of approximation is virtually dictated to be the uniform norm, $\|f(x)\| = \max_{c\le x\le d} |f(x)|$, by practical considerations of efficiency and accuracy. For example, it is not unusual to evaluate an approximation a hundred thousand times within a short period of time on a modern computer, with the additional requirement that each evaluation have an error of not more than $\pm\varepsilon$ for each $x \in [c,d]$. Hence approximations must be of the lowest possible degree which meet the error requirement.

The practical motivation is strong to consider the classical problem of best polynomial approximation which is to find and characterize a polynomial P_n^* in the class, $\mathbb{P}_n$, of polynomials of degree less than or equal to n such that

$$\inf_{P\in\mathbb{P}_n}\left(\max_{c\le x\le d} |f(x) - P(x)|\right) = \max_{c\le x\le d} |f(x) - P_n^*(x)|.$$

The Weierstrass approximation theorem insures that $\{P_n^*\} \to f$ uniformly on $[c,d]$ as $n \to \infty$. However, the convergence can be very slow. The following theorem of Bernstein shows just how slow this convergence can be.

If $\{\alpha_n\}$ is any monotone decreasing nonnegative sequence such that $\alpha_n \to 0$ as $n \to \infty$ then there exists a continuous function f on $[c,d]$ such that

$$\max_{c\le x\le d} |f(x) - P_n^*(x)| = \alpha_n.$$

Chapter II gives a brief introduction to theory of varisolvent functions. The results presented in Chapters III and IV constitute the author's contribution to the problem of computing good approximations to a real valued function f in situations where f lacks derivatives on [c,d] or where f is otherwise difficult to approximate by polynomials of low degree. Methods of obtaining approximations involving a low number of parameters are developed which provide closer uniform norm approximations of large classes of functions than are possible using polynomials of the same degree.

Either the absence of derivatives of f or a wide domain may prevent polynomials from giving satisfactory approximations of sufficiently low degree, as may a number of other conditions.

In the case where the domain of f is very wide, the difficulty may be overcome by breaking the domain into several small intervals and approximating f on each subinterval separately. In the case where f lacks derivatives, it follows from Bernstein's theorem that the convergence of the polynomials P_n^* may be quite slow - too slow, in many cases, for practical computational use.

A second approach to obtaining "good" approximations of "low degree" is to enlarge the class of approximating functions. The motivation for this is to include approximating functions which more closely imitate f, than do polynomials, in the neighborhood of points where f has no derivative. For example, one might choose those approximating functions which can be evaluated with a finite number of additions, multiplications, and divisions. In this case the approximating functions are rational functions and the problem of best rational approximation has been extensively studied. Certain severe computational problems can occur, Rice [12], chapter 9, when practical application of the results of best rational approximation is attempted.

Chapters III and IV deal with two classes of nonlinear approximating functions which are useful in sitations where f lacks derivatives on [c,d]. The class of approximating functions considered in Chapter III was motivated by extending the class of approximating functions to those functions which may be evaluated by finitely many additions or multiplications and a single square root. Such approximations would be of the form $P(x,\sqrt{R(x)})$ where P is a polynomial in two variables and R is a polynomial in one variable. Here, as in the case of rational approximation, the approximating functions depend nonlinearly on the parameters which determine them.

More explicitly, these approximations can be written as functions of the form

$$\sum_{i=0}^{n}\left(\sum_{j=0}^{m_i} a_{ij}x^j\right)\left(\sqrt{\sum_{m=0}^{k} a_m x^m}\right)^i .$$

These can be more conveniently written as functions of the form

$$P_\ell(x) + Q_m(x)\sqrt{R_k(x)}$$

where $P_\ell(x) = \sum_{i=0}^{\ell} p_i x^i$, $Q_m(x) = \sum_{i=0}^{m} q_i x^i$, and $R_k(x) = \sum_{i=0}^{k} r_i x^i$. It is shown in Chapter III that this class of approximating functions does not admit a Tchebycheff type of theory, but that a subclass of these functions does admit such a theory.

Chapter 4 considers approximating functions of the form $\sum_{i=1}^{m} a_i \frac{1}{(x+t_i)}$ where $a_1,\cdots,a_m$ are real numbers and $t_1,\cdots,t_m$ are real numbers such that $x+t_i \neq 0$ on [c,d] for $i = 1,\cdots,m$. It is shown that to obtain an existence theorem, the above class of approximating functions must be expanded to functions of the form

$$\sum_{i=1}^{k} \sum_{j=1}^{m_i+1} a_{ij} \frac{1}{(x+t_i)^j} .$$

These approximating functions are, of course, rational functions. The methods used to analyze this subclass are different from those for general rational functions and the subclass may prove more useful in numerical application.

F(a,x) will denote a real valued approximating function defined on an interval of real numbers [c,d] where $a = (a_1,\cdots,a_n)$ is the n tuple of parameters which determines the approximating function. In general F(a,x) will depend nonlinearly on the parameters a.

The problem under consideration is, for a given real valued function f, continuous on [c,d], and a given set of parameters P, find, if possible, $a^* \in P$ such that

$$\text{(1.1)} \qquad \inf_{a \in P} \left(\max_{c \le x \le d} |F(a,x) - f(x)|\right) = \max_{c \le x \le d} |F(a^*,x) - f(x)| .$$

$F(a^*,x)$ satisfying (1.1) is called a best approximation to f(x) on [c,d]. When the interval of approximation is understood we say that $F(a^*,x)$ is a best approximation to f(x).

The notation $\|f(x)\|$ will be used to denote the uniform norm

$$\|f(x)\| = \max_{c \le x \le d} \|f(x)\| .$$

In the study of best approximation theory three major problem areas must be considered.

First, of course, is the theoretical existence of a best approximation. Many times this entails modifications to the class of permissible approximations to insure the existence of a best approximant.

Second, results which characterize a best approximation must be obtained. For example, if the class of approximating functions admits a Tchebycheff type theory, the characterization is that $F(a^*,x)$ is a best approximation to $f(x)$ if and only if $F(a^*,x) - f(x)$ alternates a certain number of times. Such results are essential in developing algorithms for approximating best approximations.

Third, the problem of existence of best approximations on finite subsets must be studied. Surprisingly, best approximations may fail to exist on finite subsets of $[c,d]$ even when best approximations are known to exist on $[c,d]$. Without existence of best approximations on finite subsets of $[c,d]$, numerical approximation of an existent best approximation may be very difficult or even impossible.

CHAPTER II

VARISOLVENT FUNCTIONS

A mathematical research technique which is sometimes used is to consider known results very carefully, and then discover the essential hypotheses used in obtaining the results. Then one studies abstract objects satisfying these hypotheses to obtain additional results, insight, and possibly, limits of the theory based on these hypotheses.

Rice [5,7,10] as well as the original investigators, Motzkin [4], and Tornheim [13] used this technique on the problem of Tchebycheff approximation with success. Rice's work resulted in what is known as varisolvent approximating functions. However, there are few known classes of varisolvent functions. Among these are weighted rational functions, rational trigonometric functions, and exponential functions.

The remainder of this chapter provides a brief introduction to the theory of varisolvent functions to facilitate reading of the remaining chapters.

The problem of Tchebycheff approximation is considered in the following setting. Let f be a real valued function defined on an interval of real numbers [c,d] and P be a set of parameters. Let $\{F(a,x) \mid a \in P\}$ be a class of real valued approximating functions defined on [c,d].

Problem. Find $a^* \in P$ such that

(2.1) $$\inf_{a \in P} \|F(a,x) - f(x)\| = \|F(a^*,x) - f(x)\|.$$

$F(a^*,x)$ satisfying (2.1) is a <u>best approximation to f on [c,d]</u>.

The error <u>$f(x) - F(a,x)$</u> is said to <u>alternate m times</u> on a set S if there are at least $m+1$ points

$$x_1 < x_2 < \cdots < x_{m+1}$$

in S such that

$$f(x_i) - F(a,x_i) = \pm (-1)^i \max_{x \in S} |f(x) - F(a,x)|$$

for $i = 1,2,\cdots,m+1$. Such a point set $\{x_1,\cdots,x_{m+1}\}$ is called a <u>critical point set</u> or an <u>extremal point set</u>.

Tchebycheff approximation is the study of classes of approximating functions $F(a,x)$ for which a best approximation $F(a^*,x)$ to f on [c,d] is characterized by $f(x) - F(a^*,x)$ alternating a certain number of times on [c,d].

The following definitions are used in defining varisolvent functions. The first definition involves the meaning of continuity of $F(a,x)$. The parameters $a \in P$ are n-tuples of real numbers which could be regarded as being in n dimensional Euclidean space and hence the norm for P would be the Euclidean norm. However, when it comes to relating $\|F(a_1,x) - F(a_2,x)\|$ to the closeness of a_1 to a_2 in P it is more convenient to use a norm N for P which is induced by another norm. This will be explained in more detail later.

Let P be a given parameter space. $F(a,x)$, defined on $P \times [c,d]$, is <u>continuous</u> at $(a_o,x_o) \in P \times [c,d]$ if for $\varepsilon > 0$ there is a $\delta > 0$ such that if $(a,x) \in P \times [c,d]$ and $N(a_o-a) + |x_o - x| < \delta$ then

$|F(a_o,x_o) - F(a,x)| < \varepsilon$.

The next definition generalizes the idea of a Tchebycheff set. _F(a,x) has property Z of degree m at $a^* \in P$_ if for $a \neq a^*$, $F(a^*,x) - F(a,x)$ has at most $m-1$ zeros on $[c,d]$.

The next definition considers interpolating with the functions $F(a,x)$ in neighborhoods of $F(a^*,x)$. _F(a,x) is locally solvent of degree m at $a^* \in P$_ if given a set $S = \{x_1 < \cdots < x_m\} \subseteq [c,d]$ and $\varepsilon > 0$ there is a $\delta > 0$ which depends on a^*, ε, and S such that if $\{y_1,\cdots,y_m\}$ is a set of m real numbers such that

$$|F(a^*,x_i) - y_i| < \delta$$

for $i = 1,\cdots,m$ then there is a solution $a \in P$ of

$$F(a,x_i) = y_i, \qquad i = 1,\cdots,m$$

and

$$\|F(a,x) - F(a^*,x)\| < \varepsilon.$$

F(a,x) is varisolvent of degree m at $a^* \in P$ if F(a,x) is both locally solvent and satisfies property Z of common degree m at $a^* \in P$. _F(a,x) is varisolvent on P_ if it is varisolvent at each $a \in P$. The degree of varisolvency of F(a,x) at $a \in P$ will be denoted by m(a).

Rice [5,7,10] studied the problem of Tchebycheff approximation under the framework of the above definitions. He obtained the following results.

A. If $\{f_j(x)\}$ is a uniformly bounded sequence of functions, each continuous on [c,d] and having property Z of common degree, then $\{f_j(x)\}$ has a pointwise convergent subsequence.

In the following results we assume that $f(x) \not\equiv F(a,x)$ on [c,d] for each $a \in P$.

B. If $F(a,x)$ is varisolvent of degree $m(a)$, then $F(a^*,x)$ is a best approximation to $f(x)$ on $[c,d]$ if and only if $f(x) - F(a^*,x)$ alternates at least $m(a^*)$ times on $[c,d]$.

C. If f is continuous on $[c,d]$ and $F(a,x)$ is varisolvent, then the best approximation $F(a^*,x)$ to $f(x)$ on $[c,d]$ is unique.

D. If $F(a,x)$ is varisolvent and $F(a^*,x)$ is the best approximation to $f(x)$ on $[c,d]$, then there exists a subset S of $m(a^*) + 1$ points in $[c,d]$ such that $F(a^*,x)$ is the best approximation to f on S and S is the subset which maximizes $|f(x) - F(a^*,x)|$ among all subsets of $m(a^*) + 1$ points in $[c,d]$.

The proofs of results B, C, and D closely resemble the proofs of the corresponding theorems when the approximating functions are linear combinations of Tchebycheff sets.

The result D is a generalization of the theorem on which the de la Vallée Poussin algorithm for calculating best approximations is based. The generalization of the de la Vallée Poussin algorithm is rather difficult for several reasons which will be discussed in a later chapter.

For the following result E, the norm N for the parameter space P is defined as an induced norm. The closeness of $F(a_1,x)$ to $F(a_2,x)$ is measured by $\|F(a_1,x) - F(a_2,x)\| = \max_{c \le x \le d} |F(a_1,x) - F(a_2,x)|$. To relate the closeness of $F(a_1,x)$ to $F(a_2,x)$ to the closeness of a_1 to a_2 in P it is natural to define a norm N on P as the norm induced by the metric topology defined on P by requiring that an open ε-sphere about $a_o \in P =$

$$\{a \in P \,|\, \|F(a,x) - F(a_o,x)\| < \varepsilon\}.$$

Hence $N(a_1 - a_2) < \varepsilon$ if and only if

$$\|F(a_1,x) - F(a_2,x)\| < \varepsilon.$$

$F = \{F(a,x) \mid a \in P\}$ is closed under pointwise convergence if $\lim_{i\to\infty} F(a_i,x) = g(x)$ for each $x \in [c,d]$ and $|F(a_i,x)| < M$ for each $x \in [c,d]$ and $i = 1,2,\cdots$ implies there exists $a_0 \in P$ such that $g(x) = F(a_0,x)$.

E. If $F(a,x)$ is continuous and P is arcwise connected and F is closed under pointwise convergence then it is necessary and sufficient for $F(a^*,x)$ to be a best approximation to $f(x)$ on $[c,d]$ if $f(x) - F(a^*,x)$ alternates at least $m(a^*)$ times on $[c,d]$ if and only if $F(a,x)$ is varisolvent of degree $m(a^*)$.

Result E characterizes varisolvent functions and gives a partial answer to the extent of a Tchebycheff type of theory of approximation for nonlinear approximating functions $F(a,x)$.

CHAPTER III

APPROXIMATION BY $F(a,x) = \sqrt{P_k(x)}$

Existence Theorem

In choosing a class of approximating functions for a particular numerical application, special consideration must be given to the question of the existence of a best approximation. Since the parameter space P is usually defined indirectly by the form of the approximating function, the parameter space for some classes of approximating functions may not be compact. It follows that in such cases best approximations may not exist for certain functions f. This is illustrated by the class of approximating functions considered in Chapter IV. The selection of the form of the approximating function F(a,x) must insure that the associated parameter space P has no missing limit points. However, a general proof of existence may not be sufficient to guarantee existence from a computational point of view. For example, most computational schemes require computing best approximations on finite subsets of [c,d] and best approximations may not exist on finite subsets even though a best approximation exists on [c,d].

The following theorem gives sufficient conditions for the existence of a best approximation $F(a^*,x)$ to f(x) on [c,d]. In the following remarks we suppose that f is continuous on [c,d].

<u>THEOREM 3.1</u>: If (i) $F(a,x)$ is continuous on $[c,d]$, (ii) $F(a,x)$ satisfies property Z of degree $m(a)$, (iii) $\{F(a,x) \mid a \in P\}$ is closed under pointwise convergence, (iv) $m(a)$ is bounded on P, then there exists $a^* \in P$ such that $F(a^*,x)$ is a best approximation to $f(x)$ on $[c,d]$.

<u>PROOF</u>: Let $a^1 \in P$. Then $\|F(a^1,x) - f(x)\| < \infty$. Let $P^1 = \{a \in P \mid \|F(a,x) - f(x)\| \leq \|F(a^1,x) - f(x)\|\}$. It is clear that $P^1 \neq \phi$ and that the search for a^* may be restricted to P^1. There exists $M < \infty$ such that $|F(a,x)| < M$ for each $a \in P^1$ and $x \in [c,d]$. To see this, notice that for each $a \in P^1$ $\max_{c \leq x \leq d} |F(a,x) - f(x)| = \|F(a,x) - f(x)\| \leq \|F(a^1,x) - f(x)\|$. However, $\max_{c \leq x \leq d} |F(a,x) - f(x)| \geq |F(a,x) - f(x)| \geq |F(a,x)| - |f(x)|$ for each $x \in [c,d]$. Hence $|F(a,x)| \leq \|F(a^1,x) - f(x)\| + |f(x)|$ for each $x \in [c,d]$. Since f is continuous on $[c,d]$ it follows that $|F(a,x)| \leq M$ on $[c,d]$ for each $a \in P^1$. Let $\gamma = \inf_{a \in P} \|F(a,x) - f(x)\|$. It is clear that $\gamma \leq \|F(a^1,x) - f(x)\|$ and that $\gamma = \inf_{a \in P^1} \|F(a,x) - f(x)\|$. There exists $\{a_i\} \subseteq P^1$ such that

$$\lim_{i \to \infty} \|F(a_i,x) - f(x)\| = \gamma.$$

Since $m(a)$ is bounded on P, it is bounded on P^1, i.e., there exists J such that $m(a) \leq J$ for each $a \in P^1$. It follows from the definition of property Z that for each $a \in P^1$, $F(a,x)$ satisfies property Z of degree J. By result A of Chapter II $\{F(a_i,x)\}$ has a convergent subsequence, i.e., there exists $g(x)$ and $\{F(a_{i_k},x)\}$ such that $\lim_{k \to \infty} F(a_{i_k},x) = g(x)$ for each $x \in [c,d]$. Since $\{F(a,x) \mid a \in P\}$ is closed under pointwise convergence, there is $a^* \in P$ such that $g(x) = F(a^*,x)$. Finally,

$$\gamma = \lim_{k \to \infty} \|F(a_{i_k},x) - f(x)\| = \|F(a^*,x) - f(x)\|,$$

i.e., $F(a^*,x)$ is a best approximation to $f(x)$ on $[c,d]$.

Selection of $F(a,x) = \sqrt{P_k(x)}$

This section deals with the problem of Tchebycheff approximation by functions of the form

(3.1) $$F(a,x) = P_\ell(x) + Q_m(x)\sqrt{R_k(x)} \quad \text{where}$$

$P_\ell(x) = \sum_{i=0}^{\ell} p_i x^i$, $Q_m(x) = \sum_{i=0}^{m} q_i x^i$, and $R_k(x) = \sum_{i=0}^{k} r_i x^i$. Let

$(\ell+1) + (m+1) + (k+1) \leq n$ and $R_k(x) \geq 0$ for each $x \in [c,d]$.

LEMMA 3.1: $F(a,x)$ has at most $\max\{2\ell, 2m+k\}$ zeros on $[c,d]$.

PROOF: Let z be a zero of $F(a,x)$. Then $P_e^2(z) - Q_m^2(z)R_k(z) = 0$. Hence z is a zero of a polynomial of degree at most $\max\{2\ell, 2m+k\}$ and the lemma follows.

If approximation by the functions $F(a,x)$ of the form (3.1) is to admit a Tchebycheff type theory, i.e., if $F(a,x)$ is varisolvent, then the degree of local solvency and the degree of property Z must be the same. Meinardus [3] has shown that when $F(a,x)$ has continuous partial derivatives with respect to the parameters $a = (a_1, \cdots, a_n)$, the degree of local solvency, $m(a)$, of $F(a,x)$ is equal to the dimension, $d(a)$, of the linear space consisting of linear combinations of the functions

$$\left\{\frac{\partial F(a,x)}{\partial a_1}, \cdots, \frac{\partial F(a,x)}{\partial a_n}\right\}.$$

Lemma 3.2 establishes that $F(a,x)$ of the form (3.1) does not satisfy property Z of degree $d(a)$ and hence $F(a,x)$ is not varisolvent and approximation by $F(a,x)$ does not admit a Tchebycheff type theory.

LEMMA 3.2: $F(a,x)$ of the form (3.1) does not satisfy property Z of degree $d(a)$.

PROOF: Let $n = 4$. Then there are three cases for $F(a,x)$:

1) $F(a,x) = p_o + p_1x + m_o\sqrt{r_o}$

2) $F(a,x) = p_o + (m_o + m_1x)\sqrt{r_o}$

3) $F(a,x) = p_o + m_o\sqrt{r_o + r_1x}$.

In case 1) $d(a) = 2$, in case 2) $d(a) = 2$, and in case 3) $d(a) = 4$. For $F(a,x)$ to satisfy property Z of degree $d(a)$ one must show that for each parameter $a \in P$, $F(a,x) - F(b,x)$ has at most $d(a) - 1$ zeros on $[c,d]$ for each $b \in P$. The property is global in the sense that b is allowed to range throughout the entire parameter space P. Suppose a is chosen from case 1) and b is chosen from case 3), i.e., $F(a,x) = p_o + P_1(x) + m_o\sqrt{r_o}$ and $F(b,x) = \hat{p}_o + \hat{m}_o\sqrt{\hat{r}_o + \hat{r}_1x}$. Then $F(a,x) - F(b,x) =$ $p_o - \hat{p}_o + m_o\sqrt{r_o} + p_1x - \hat{m}_o\sqrt{\hat{r}_o + \hat{r}_1x}$. By Lemma 3.1 $F(a,x) - F(b,x)$ has at most 2 zeros, but $d(a) - 1 = 1$ in this case. Hence $F(a,x)$ does not satisfy property z of degree $d(a)$. We should remark that the bound furnished by Lemma 3.1 is the sharpest possible bound as it is easy to construct examples where this bound is attained. For example, consider $F(a,x) = x - \sqrt{x}$ on $[0,1]$. By Lemma 3.1 $F(a,x)$ has at most 2 zeros on $[0,1]$. These zeros are $x = 0$ and $x = 1$.

Lemma 3.2 raises the question; can $F(a,x)$ be modified in some way, compatible with the goals of Chapter I, so that $F(a,x)$ is varisolvent? Consider the subclass of functions of the form

(3.2) $$F(a,x) = Q_m(x)\sqrt{R_k(x)} \quad \text{where} \quad R_k(x) \geq 0$$

on $[c,d]$ and $(m+1) + (k+1) \leq n$.

Lemma 3.3 establishes that $F(a,x)$ of the form (3.2) does not satisfy property Z of degree $d(a)$ and hence $F(a,x)$ is not varisolvent and approximation by $F(a,x)$ does not admit a Tchebycheff type theory.

LEMMA 3.3: $F(a,x)$ of the form (3.2) does not satisfy property Z of degree $d(a)$.

PROOF: Let $n = 3$. Then there are two cases for $F(a,x)$:

1) $F(a,x) = (m_o + m_1 x)\sqrt{r_o}$

2) $F(a,x) = m_o\sqrt{r_o + r_1 x}$.

In case 1) $d(a) = 2$ and in case 2) $d(a) = 3$. Let a be chosen from case 1) and b be chosen from case 2). Then $F(a,x) - F(b,x) =$ $(m_o + m_1 x)\sqrt{r_o} - \hat{m}_o\sqrt{\hat{r}_o + \hat{r}_1 x}$. By Lemma 3.1 $F(a,x) - F(b,x)$ has at most 2 zeros, but $d(a) - 1 = 1$. Hence $F(a,x)$ does not satisfy property Z of degree $d(a)$.

The negative results of Lemmas 3.2 and 3.3 leave the subclass of approximating functions of the form $F(a,x) = \sqrt{P_k(x)}$ to be considered. It is established in the next section that these approximating functions do admit a Tchebycheff type theory.

Approximation by $F(a,x) = \sqrt{P_k(x)}$ on $[c,d]$

Let

(3.3) $$F(a,x) = \sqrt{P_k(x)} = \sqrt{a_o + a_1 x + \cdots + a_k x^k} \quad \text{where}$$

$P_k(x) \geq 0$ on $[c,d]$. It is clear that $F(a,x) \geq 0$ on $[c,d]$ for each parameter a so we make the assumption that $f(x) \geq 0$ on $[c,d]$. In agreement with earlier notation let $k + 1 \leq n$. The degree of $F(a,x)$, $m(a)$, is defined to be $m(a) = n$.

LEMMA 3.4: $F(a,x)$ of the form (3.3) satisfies property Z of degree $m(a)$ on $[c,d]$.

PROOF: $F(a,x) - F(b,x) = \sqrt{a_o + \cdots + a_k x^k} - \sqrt{b_o + \cdots + b_k x^k}$. Let z be a zero of $F(a,x) - F(b,x)$. Then $\sqrt{a_o + \cdots + a_k z^k} = \sqrt{b_o + \cdots + b_k z^k}$, or

$a_o + \cdots + a_k z^k - (b_o + \cdots + b_k z^k) = 0$. Hence z is a zero of a polynomial of degree at most k. But $k \leq n-1 = m(a) - 1$ and the result follows.

Theorem 3.2 gives existence of a best approximation $F(a^*,x)$ of the form (3.3).

THEOREM 3.2: There exists a best approximation $F(a^*,x)$ of the form (3.3) to f(x) on [c,d].

PROOF: The only hypothesis of theorem 3.1 that needs argument is that $\{F(a,x) | F(a,x)$ is of the form (3.3)$\}$ is closed under pointwise convergence. Let $F(a_i,x)$ be a uniformly bounded sequence converging pointwise to g(x) on [c,d]. Then $F^2(a_i,x)$ is a sequence of polynomials of degree $\leq k$ converging pointwise to $g^2(x)$ on [c,d]. Let $F^2(a_i,x) = a_{io} + \cdots + a_{ik}x^k$. Since $F(a_i,x)$ is uniformly bounded and hence $F^2(a_i,x)$ is uniformly bounded, the coefficients $a_{io},\cdots,a_{ik}$ are bounded sequences. Suppose this assertion is not true. Then there exists an integer j_o, $0 \leq j_o \leq k$, such that $\{a_{i,j_o}\}$ is not bounded. Let $x_o \in [c,d]$. If $F^2(a_i,x_o)$ is not bounded, then $F^2(a_i,x)$ is not uniformly bounded which is a contradiction. If $F^2(a_i,x_o)$ is a bounded sequence of real numbers, say $\{\alpha_i\}$, then it follows that $F^2(a_i,x_o) - \alpha_i = 0$ for $i = 1,2,\cdots$. Since a_{i,j_o} is unbounded, there exists an integer i_o such that $a_{i_o,j_o} \neq 0$. This is a contradiction since $F^2(a_{i_o},x_o) - \alpha_{i_o}$ is a nontrivial linear combination of the linearly independent functions $\{1,x,\cdots,x^k\}$ evaluated at $x = x_o$. Hence $\{a_{ij}\}$, $j = 0,\cdots,k$ are bounded sequences. It follows from the Bolzano-Weierstrass theorem that there exists a subsequence $\{a_{i_r,j}\}$ such that $\{a_{i_r,j}\} \to a_{oj}$, $j = 0,\cdots,k$, as $r \to \infty$. Then for each $x \in [c,d]$, $g^2(x) = \lim_{i\to\infty} F^2(a_i,x) = \lim_{i\to\infty}(a_{io} + \cdots + a_{ik}x^k) = \lim_{r\to\infty}(a_{i_r o} + \cdots + a_{i_r k}x^k) = \lim_{r\to\infty} a_{i_r o} + \cdots + \lim_{r\to\infty} a_{i_r k}\, x^k = a_{oo} + \cdots + a_{ok}x^k$. Hence g(x) is the square root of the polynomial

$a_{oo} + \cdots + a_{ok}x^k$ which is nonnegative an [c,d], i.e., there is a parameter a_o such that $g(x) = F(a_o,x)$ on [c,d].

A special subclass of varisolvent functions, namely, unisolvent functions (Motzkin [4]), is obtained when the degree of varisolvency of F(a,x) is constant for $a \in P$. Although we have not proved local solvency yet, we suspect, in view of the definition of the degree of F(a,x) of the form (3.3), that F(a,x) is unisolvent. This will be shown to be the case in Theorem 3.3.

F(a,x) is said to be <u>solvent of degree j</u> on [c,d] if given a set $\{x_1,\cdots,x_j\}$ of distinct points in [c,d] and a set of real numbers $\{y_1,\cdots,y_j\}$ there is a unique parameter $b \in P$ such that

$$F(b,x_i) = y_i \quad \text{for} \quad i = 1,\cdots,j.$$

Solvency is a generalization of the interpolation problem using the functions F(a,x).

F(a,x) is said to be <u>unisolvent of degree j on [c,d]</u> if F(a,x) is solvent of degree j on [c,d] and F(a,x) satisfies property Z of degree j on [c,d] for each $a \in P$.

<u>THEOREM 3.3</u>: F(a,x) of the form (3.3) is solvent of degree n.

<u>PROOF</u>: Let $\{x_1,\cdots,x_n\}$ be n distinct points in [c,d] and $\{y_1,\cdots,y_n\}$ be n nonnegative real numbers. Then $F^2(b,x_i) = y_i^2$, $i = 1,\cdots,n$ is a linear system. The coefficient matrix of this system is the well known Vandermonde matrix which has a nonzero determinant when $x_1,\cdots,x_n$ are distinct. Hence, there is a unique parameter b such that

$$F(b,x_i) = y_i, \quad i = 1,2,\cdots,n.$$

It follows that F(a,x) of the form (3.3) is a unisolvent (hence varisolvent) function. Hence the uniqueness and characterization results

B, C, and D of Chapter II hold for $F(a,x)$.

Approximation on Finite Subsets of [c,d]

The remainder of this chapter is concerned with the problem of approximating a function f on finite subsets of [c,d]. Result D of Chapter II characterizes the approximation problem on finite point sets but says nothing about the existence of best approximations on finite points sets. This difficult question must be handled individually for each class of approximating functions $F(a,x)$.

Existence of best approximations by varisolvent approximating functions on finite subsets of [c,d] can be established by showing that given a subset of $m(a) + 1$ points, $\{x_1,\cdots,x_{m(a)+1}\}$ with $x_1 < \cdots < x_{m(a)+1}$, in [c,d], where $m(a)$ is the degree of varisolvency, there is a parameter a and a real number d satisfying the equations

$$(3.4) \qquad F(a,x_i) - f(x_i) = (-1)^{i-1}d, \quad i = 1,\cdots,m(a)+1.$$

Solving the nonlinear system (3.4) for a varisolvent family of functions $F(a,x)$ is at best a difficult problem. It requires that one know, on an <u>a priori</u> basis, the degree of varisolvency at the solution of the system (3.4) in addition to the difficulties of solving a nonlinear system of equations. For unisolvent approximation the problem is simplified because the degree of varisolvency is constant as the parameter a ranges throughout P and is usually known in advance.

For $F(a,x)$ of the form (3.3) best approximations may not exist on finite subsets. For example, if $S = \{x_1,\cdots,x_{k+2}\}$ is a set of $k+2$ distinct points in [c,d] enumerated so that $x_1 < \cdots < x_{k+2}$ and f is zero on two consecutive points of S, then clearly f does not have a best approximation of the form (3.3) on S. This is because a solution F(a,x) of (3.4) would require that F(a,x) be less than zero at one of the two points which is impossible. However, it is not sufficient to require that f never be zero on consecutive points of S to insure the existence of best approximations on finite point sets. This is established in the example following Theorem 3.4. The problem of finding sufficient conditions which will insure the existence of best approximations by F(a,x) of the form (3.3) on finite subsets of [c,d] is, at the present, unsolved.

The following theorem provides a valuable computational test which may be applied without solving the system (3.4), to determine whether a best approximation F(a,x) of the form (3.3) exists on finite sets having $k+2$ points.

For F(a,x) of the form (3.3), the system (3.4) becomes

$$(3.5) \qquad \sqrt{a_o + a_1x_1 + \cdots + a_kx_i^k} - f(x_i) = (-1)^{i-1}d, \quad i = 1,\cdots,k+2.$$

Theorem 3.4 deals with the system obtained from (3.5) by transposing the term $f(x_i)$ and squaring (3.5).

THEOREM 3.4: Let $S = \{x_1,\cdots,x_{k+2}\}$ be a subset of distinct points enumerated so that $x_1 < \cdots < x_{k+2}$ and suppose that f is not zero at each point of S. Then there exists a unique solution to the system

$$(3.6) \qquad a_o + a_1x_i + \cdots + a_kx_i^k = (f(x_i) + (-1)^{i-1}d)^2, \quad i = 1,\cdots,k+2.$$

PROOF: Consider the first $k+1$ equations of the system (3.6).

$$(3.7) \qquad a_o + a_1x_i + \cdots + a_kx_i^k = (f(x_i) + (-1)^{i-1}d)^2, \quad i = 1,\cdots,k+1.$$

Suppose there is a real number d such that the system (3.7) has a solution. Solving (3.7) is an interpolation problem which is solvable under the assumption that $x_1,\cdots,x_{k+1}$ are distinct. Let

$$\ell_i(x) = \frac{(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_{k+1})}{(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_{k+1})} .$$

$\ell_i(x)$ is the i-th LaGrange interpolating function. Then $P(x) = (f(x_1)+d)^2\ell_1(x)+\cdots+(f(x_{k+1})+(-1)^k d)^2\ell_{k+1}(x)$ is the polynomial which interpolates $(f(x_1)+d)^2, \cdots, (f(x_{k+1})+(-1)^k d)^2$ at the points $x_1,\cdots,x_{k+1}$. A unique solution to the system (3.7) exists if there is a unique real number d such that

$$P(x_{k+2}) = (f(x_{k+2}) + (-1)^{k+1}d)^2. \tag{3.8}$$

Equation (3.8) is a quadratic in d. The coefficient of d^2 is

(3.9) $\ell_1(x_{k+2}) + \ell_2(x_{k+2}) + \cdots + \ell_{k+1}(x_{k+2}) - 1$. The coefficient of d is

$$2[f(x_1)\ell_1(x_{k+2}) - f(x_2)\ell_2(x_{k+2}) + \cdots + (-1)^k f(x_{k+1})\ell_{k+1}(x_{k+2}) + (-1)^k f(x_{k+2})].$$

The constant term is

$f^2(x_1)\ell_1(x_{k+2}) + f^2(x_2)\ell_2(x_{k+2}) + \cdots + f^2(x_{k+1})\ell_{k+1}(x_{k+2}) - f^2(x_{k+2})$. It is clear that for $k \geq 1$

$$\ell_1(x) + \ell_2(x) + \cdots + \ell_{k+1}(x)$$

is a polynomial of degree at most k which has the value 1 at the distinct points $x_1,\cdots,x_{k+1}$. Hence, this polynomial has value 1 for each x and in particular it is 1 at x_{k+2} and this proves that the coefficient of d^2, given by (3.9), is zero. Consider

$$q(x) = f(x_1)\ell_1(x) - f(x_2)\ell_2(x) + \cdots + (-1)^k f(x_{k+1})\ell_{k+1}(x) + (-1)^k f(x_{k+2}).$$

q is a polynomial of degree at most k having values $(-1)^{i-1}f(x_i) + (-1)^k f(x_{k+2})$ at the points x_i for $i = 1,\cdots,k+1$. There are two cases to consider. Case 1, k even. Case 2, k odd. In case 1 it follows that $0 < (-1)^k f(x_{k+2}) + (-1)^k f(x_{k+1})$ and that q is increasing for $x > x_{k+1}$. Hence $q(x_{k+2}) > 0$, and the coefficient of d, $2q(x_{k+2})$, is nonzero. In case 2 it follows that $0 \geq (-1)^k f(x_{k+2}) + (-1)^k f(x_{k+1})$ and that q is decreasing for $x > x_{k+1}$. Hence $q(x_{k+2}) < 0$ and again the coefficient of d, $2q(x_{k+2})$, is nonzero. It should be noted that the above argument holds when and only when q is not identically zero which occurs when and only when $f(x_1),\cdots,f(x_{k+2})$ are not all zero. This proves the existence of a unique solution to the system (3.6).

The solution of (3.6), given by Theorem 3.4, is the solution of the system

$$(3.5) \qquad \sqrt{a_o + a_1 x_i + \cdots + a_k x_i^k} - f(x_i) = (-1)^{i-1} d, \quad i = 1,\cdots,k+2.$$

whenever $f(x_i) + (-1)^{i-1} d \geq 0$ for $i = 1,\cdots,k+2$. Hence, Theorem 3.4 gives the existence of a best approximation on $S = \{x_1,\cdots,x_{k+2}\}$ whenever the solution of (3.6) satisfies

$$(3.10) \qquad f(x_i) + (-1)^{i-1} d \geq 0, \quad i = 1,\cdots,k+2.$$

The following example shows that even when $f(x_i) > 0$ for $i = 1,\cdots,k+2$, the solution of (3.6) may not be a solution of (3.5) and hence best approximations may fail to exist even when $f(x_i) > 0$ for $i = 1,\cdots,k+2$. Let $k = 2$, $S = \{1,2,3,4\}$, $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, and $f(4) = 10$. A simple calculation shows that $d = -\frac{9}{2}$ and hence $f(1) + d < 0$ and $f(3) + d < 0$. This shows that there does not exist a best approximation to f of the form $\sqrt{a_o + a_1 x + a_2 x^2}$ on S.

Let $S = \{x_1,\cdots,x_{k+2}\}$. Theorem 3.4 provides an easy computational test to determine whether the system (3.5) has a solution, and hence, whether a best approximation to f exists on S. Compute

$$(3.11)\quad d = \frac{-(f^2(x_1)\ell_1(x_{k+2}) + \cdots + f^2(x_{k+1})\ell_{k+1}(x_{k+2}) - f^2(x_{k+2}))}{2(f(x_1)\ell_1(x_{k+2}) + \cdots + (-1)^k f(x_{k+1})\ell_{k+1}(x_{k+2}) + (-1)^k f(x_{k+2}))}$$

Then verify whether $f(x_i) + (-1)^{i-1}d \geq 0$, $i = 1,\cdots,k+2$. d is the ratio of two interpolating polynomials evaluated at x_{k+2}. This can be efficiently evaluated using differences.

The following algorithm determines the best approximation of the form (3.3) on subsets $S = \{x_1,\cdots,x_m\}$ where $m > k+2$ under the hypothesis that a best approximation exists on each subset of S having $k+2$ points. If $S^1 \subseteq S$ is a subset having $k+2$ points denote the d obtained by solving the system (3.5) by d_{S^1}.

ALGORITHM 3.1: Let $S = \{x_1,\cdots,x_m\} \subseteq [c,d]$ and $m > k+2$. Compute

$$\max_{\substack{S^1 \subseteq S \\ S^1 \text{ has } k+2 \text{ points}}} |d_{S^1}| = d^*.$$

The best approximation on a subset S^* of S having $k+2$ points such that $|d_{S^*}| = d^*$ is the best approximation on S.

The validity of Algorithm 3.1 follows directly from result D of Chapter II. Given a set S having $m > k+2$ points, Algorithm 3.1 requires computing d_{S^1} for $\binom{m}{k+2}$ subsets S^1 having $k+2$ points which is generally not practical when m is large.

The following algorithm for determining best approximations on finite subsets having more than $k+2$ points is more tractable.

ALGORITHM 3.2: Let $S = \{x_1,\cdots,x_m\} \subseteq [c,d]$ and $m > k+2$. Let $F(a_o,x)$ be an initial approximation satisfying the conditions

I: 1) $F(a_o,x) - f(x)$ assumes extreme values $d_1^o, -d_2^o, \cdots, (-1)^{k+1}d_{k+2}^o$

at k + 2 points in S, say $x_1^1,\cdots,x_{k+2}^1$ where $x_1^1 < \cdots < x_{k+2}^1$ and d_i^o all have the same sign for $i = 1,\cdots,k+2$.

2) There is at least one j, $1 \le j \le k+2$ such that

$$|d_j^o| = \max_{x \in S} |F(a_o,x) - f(x)|.$$

Then 1) Let $j = 1$.

2) Determine the approximation $F(a_j,x)$ and d_j such that

(3.12) $$F(a_j,x_i^j) = f(x_i^j) + (-1)^{i-1}d_j \quad \text{for} \quad i = 1,\cdots,k+2.$$

3) If $\max_{x \in S} |F(a_j,x) - f(x)| \le |d_j|$, then $F(a_j,x)$ is the best approximation to f on S and the algorithm is terminated. Otherwise determine $k+2$ points $\{x_1^{j+1},\cdots,x_{k+2}^{j+1}\} \subseteq S$, $x_1^{j+1} < \cdots < x_{k+2}^{j+1}$, such that $F(a_j,x_i^{j+1}) = f(x_i^{j+1}) + (-1)^{i-1}d_i^j$ for $i = 1,\cdots,k+2$ and $\min_{1 \le i < k+2} |d_i^j| \ge |d_j|$.

4. Let $j = j+1$ and go to step 2).

The following theorem establishes the convergence of Algorithm 3.2.

<u>THEOREM 3.5</u>: Let f be defined on [c,d] and $S = \{x_1,\cdots,x_m\}$, $m \ge k+2$ and $S \subseteq [c,d]$. If $F(a_o,x)$ is an initial approximation satisfying the conditions I, then Algorithm 3.2 will determine $F(a_S,x)$, the best approximation to f on S.

<u>PROOF</u>: We may assume $d_i^o > 0$ for $i = 1,\cdots,k+2$. Let $m_o = \min_{1 \le i \le k+2} d_i^o$.

If $\max_{x \in S} |F(a_o,x) - f(x)| \le m_o$ then all the d_i^o are equal for $i = 1,\cdots,k+2$ and $F(a_o,x) = F(a_S,x)$, the best approximation. Otherwise the k + 2 numbers $d_1^o,\cdots,d_{k+2}^o$ are not all equal. The next step in the algorithm is the determination of $F(a_1,x)$ and d_1 by solving the k + 2 equations

$F(a_1,x_i^1) = f(x_i^1) + (-1)^{i-1}d_1$ for $i = 1,\cdots,k+2$. $d_1 > m_o$. Assume $d_1 \le m_o$. Then $F(a_o,x_i^1) - F(a_1,x_i^1) = (-1)^{i-1}(d_i^o - d_1)$, $i = 1,\cdots,k+2$. But $d_i^o - d_1 \ge 0$ for $i = 1,\cdots,k+2$. It follows that $F(a_o,x) - F(a_1,x)$ has at least $k+1$ zeros. By Lemma 3.4 this is impossible unless $a_o = a_1$ in which case $F(a_o,x) \equiv F(a_1,x)$ and this is a contradiction since we have assumed the numbers, d_i^o, $i = 1,\cdots,k+2$, are not all equal. If $\max_{x \in S} |F(a_1,x) - f(x)| \le d_1$ then $F(a_1,x) = F(a_S,x)$, the best approximation to f on S, by result D of Chapter II. Otherwise, the next step in the algorithm is the determination of a subset $\{x_1^2,\cdots,x_{k+2}^2\} \subseteq S$ such that $F(a_1,x_i^2) = f(x_i^2) + (-1)^{i-1}d_i^1$, $i = 1,\cdots,k+2$, with $m_1 = \min_{1 \le i \le k+2} d_i^1 > d_1$. The next step in the algorithm is the determination of $F(a_2,x)$ and d_2 by solving the system

$$F(a_2,x_i^2) = f(x_i^2) + (-1)^{i-1}d_2, \quad i = 1,\cdots,k+2.$$

Since not all the numbers d_i^1, $i = 1,\cdots,k+2$, are equal it follows, as before, that $d_2 > m_1$. Continuing this procedure we must have, for some $j_o \ge 1$, that $\max |F(a_{j_o},x) - f(x)| \le d_{j_o}$, and hence $F(a_{j_o},x) = F(a_S,x)$, the best approximation on S, since if this were not so then there would exist an infinite monotone increasing sequence $m_o < d_1 \le m_1 < d_2 \le \cdots$. This is a contradiction since there exists only $\binom{m}{k+2}$ subsets of S having $k+2$ points, and hence there exist only finitely many numbers d_j.

Theorem 3.5 is valid, of course, only when best approximations exist on each of the finite subsets of S having $k+2$ points which is encountered by Algorithm 3.2. The subsets of S having $k+2$ points which are encountered by Algorithm 3.2 depend on f and the initial approximation $F(a_o,x)$. There is no known way to predict on an a priori basis which subsets of $k+2$

points will be encountered. A sufficient condition for the convergence of Algorithm 3.2, but stronger than really needed, is that best approximations exist on each subset of S having $k+2$ points.

When computing $F(a_S,x)$, it is not known in advance whether each of the systems (3.12) have solutions. In view of the monotonicity of $\{d_j\}$, Algorithm 3.2 can be employed without a priori knowledge of the existence of the solutions of the systems (3.12) since the numbers d_j can always be calculated as in Theorem 3.4 without actually solving the systems (3.12). If it should happen for some j_1 that $f(x_i^{j_1}) + (-1)^{i-1}d_{j_1}$ is not greater than or equal to zero for $i = 1,\cdots,k+2$, then the best approximation on $\{x_1^{j_1},\cdots,x_{k+2}^{j_1}\}$ does not exist. Since $\max_{x \in S} |F(a_S,x) - f(x)| \geq |d_{j_1}|$, it follows that the best approximation to f on S does not exist and the algorithm would be terminated. Hence Algorithm 3.2 not only computes the best approximation, $F(a_S,x)$, to f on S when it exists, but also successfully detects the cases when $F(a_S,x)$, the best approximation to f on S, does not exist!

Theorem 3.5 provides the basis for a practical computational procedure (practical on large scale digital computers) for the approximation of $F(a^*,x)$, the best approximation to $f(x)$ on $[c,d]$.

Let f be continuous on $[c,d]$. Let $\{S_i\}$ be a sequence of finite subsets of $[c,d]$ such that $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_i \subseteq \cdots \subseteq [c,d]$ and suppose that S_1 contains at least $k+2$ points. It is convenient to regard the sets S_i as partitions of $[c,d]$ and the condition that $S_i \subseteq S_{i+1}$ for each i then reduces to the requirement that S_{i+1} be a refinement of S_i for each i. Define the norm of S_i, denoted by $\|S_i\|$, in the usual manner as the length of the longest subinterval in the partition S_i. In addition, suppose that $\|S_i\| \to 0$ as $i \to \infty$.

Given an initial approximation $F(a_o,x)$ satisfying the conditions I on S_1, determine, by Algorithm 3.2, the best approximation, $F(a_{S_1},x)$, to $f(x)$ on S_1. Next, using $F(a_{S_1},x)$ as an initial approximation on S_2, determine, by Algorithm 3.2, the best approximation, $F(a_{S_2},x)$, to $f(x)$ on S_2. Continuing in this manner, generate a sequence $\{F(a_{S_i},x)\}$ of best approximations to $f(x)$ on S_i. We wish to analyze the behavior of this sequence as $i \to \infty$.

Let $D_k^{S_i}(f) = \max\limits_{x \in S_i} |F(a_{S_i},x) - f(x)|$ and $D_k(f) = \max\limits_{c \le x \le d} |F(a^*,x) - f(x)|$ where $F(a^*,x)$ is the best approximation to $f(x)$ on $[c,d]$. It is clear that

$$D_k^{S_1}(f) \le D_k^{S_2}(f) \le \cdots \le D_k(f).$$

Hence $\max\limits_{x \in S_i} |F(a_{S_i},x)| \le D_k^{S_i}(f) + \max\limits_{c \le x \le d} |f(x)| \le D_k(f) + \max\limits_{c \le x \le d} |f(x)|$. This means that $\{F(a_{S_i},x)\}$ is uniformly bounded on S_i if f is continuous on $[c,d]$, which is assumed. Then $\{F(a_{S_i},x)\}$ is uniformly bounded on any subset of k + 1 points of S_i and hence it follows from Theorem 3.3 that $\{F(a_{S_i},x)\}$ is uniformly bounded on $[c,d]$.

The following lemmas are helpful in the analysis of the convergence of $\{F(a_{S_i},x)\}$.

<u>LEMMA 3.5</u>: $F(a,x)$ of the form (3.3) has a continuous derivative on $[c,d]$, if $F(a,c) > 0$ and $F(a,d) > 0$.

<u>PROOF</u>: $F(a,x) = \sqrt{P_k(x)}$ where $P_k(x) \ge 0$ on $[c,d]$. By the hypothesis, the zeros of $F(a,x)$ are in (c,d). Let $x_o \in (c,d)$ be such that $P_k(x_o) = 0$. Then x_o is a zero of even multiplicity, say 2i, of $P_k(x)$.

$F'(z,x) = \dfrac{P_k'(x)}{2\sqrt{P_k(x)}}$. Hence $F'(a,x) = \dfrac{(x-x_o)^{2i-1}Q_{k-2i}(x)}{2(x-x_o)^{i}\sqrt{R_{k-2i}(x)}} =$

$\dfrac{(x-x_o)^{i-1}Q_{k-2i}(x)}{2\sqrt{R_{k-2i}(x)}}$ where Q_j and R_j are polynomials of degree at most j.

Hence each zero of the denominator of $F'(a,x)$ can be removed and it follows that $F'(a,x)$ is continuous on $[c,d]$.

LEMMA 3.6: Let $F(a,x)$ be of the form (3.3) and suppose that

$\max_{c\le x\le d} F(a,x) \le M$ and $\min_{c\le x\le d} F(a,x) \ge m > 0$. Then

$$\max_{c\le x\le d} |F'(a,x)| \le \frac{k^2M^2}{(d-c)m} .$$

PROOF: $|F'(a,x)| = \dfrac{|P_k'(x)|}{2\sqrt{P_k(x)}}$. Since $\sqrt{P_k(x)} \le M$ it follows that

$P_k(x) \le M^2$. By Markov's inequality $\max_{c\le x\le d} |P_k'(x)| \le \dfrac{2k^2M^2}{(d-c)}$. For each

$x \in [c,d]$, $|F'(a,x)| \le \dfrac{2k^2M^2}{(d-c)2\sqrt{P_k(x)}} \le \dfrac{k^2M^2}{(d-c)m}$. Hence

$\max_{c<x<d} |F'(a,x)| \le \dfrac{k^2M^2}{(d-c)m}$.

The following theorem details the behavior of the sequence $\{F(a_{S_i},x)\}$.

THEOREM 3.6: If the sequence of best approximations $\{F(a_{S_i},x)\}$, as previously defined, are uniformly bounded below by $m > 0$, then

$\max_{c\le x\le d} |F(a_{S_i},x) - f(x)| \le D_k(f) + \varepsilon_i$ where $\varepsilon_i \to 0$ as $i \to \infty$.

PROOF: Let $M_i = \max_{c\le x\le d} F(a_{S_i},x)$ and $m_i = \min_{c\le x\le d} F(a_{S_i},x)$. It was previously shown that there is $M < \infty$ such that $M_i \le M$ for $i = 1,2,\cdots$. Let $S_i = \{x_{1,j_i},\cdots,x_{j_i,j_i}\}$ where j_i is the number of points in S_i. Let

$x_o \in [c,d]$. Then there is an integer r such that $x_o \in [x_{r,j_i}, x_{r+1,j_i}]$.

By the mean value theorem there exists $U \in (x_{r,j_i}, x_{r+1,j_i})$ such that

$F(a_{S_i},x_o) - F(a_{S_i},x_{r,j_i}) = (x_o - x_{r,j_i})F'(a_{S_i},U)$. Hence

$|F(a_{S_i},x_o) - F(a_{S_i},x_{r,j_i})| \le \|S_i\| \cdot \max_{c\le x\le d} |F'(a_{S_i},x)| \le \|S_i\| \frac{k^2M_i^2}{(d-c)m_i} \le$

$\frac{k^2M^2}{(d-c)m} \cdot \|S_i\| = C \cdot \|S_i\|$. Finally, since $F(a_{S_i},x) - f(x)$ is continuous on $[c,d]$, there exists $x_i \in [c,d]$ such that

$$\max_{c\le x\le d} |F(a_{S_i},x) - f(x)| = |F(a_{S_i},x_i) - f(x_i)|.$$

We can find an integer r such that

$$x_i \in [x_{r,j_i}, x_{r+1,j_i}].$$

Hence $\max_{c\le x\le d} |F(a_{S_i},x) - f(x)| \le$

$|F(a_{S_i},x_i) - F(a_{S_i},x_{r,j_i})| + |F(a_{S_i},x_{r,j_i}) - f(x_{r,j_i})| + |f(x_{r,j_i}) - f(x_i)| \le$

$C \cdot \|S_i\| + D_k(f) + \omega_f(\|S_i\|) = D_k(f) + \varepsilon_i$ where $\varepsilon_i = \omega_f(\|S_i\|) + C\|S_i\|$ and ω_f is the modulus of continuity of f on $[c,d]$. Clearly $\varepsilon_i \to 0$ as $i \to \infty$.

The next theorem establishes the uniform convergence of $\{F(a_{S_i},x)\}$ to the best approximation, $F(a^*,x)$, for $f(x)$ on $[c,d]$.

<u>THEOREM 3.7</u>: If $\{a_m\}$ is a sequence of parameters such that $\max_{c\le x\le d} |F(a_m,x) - f(x)| \le D_k(f) + \varepsilon_m$ for $m = 1,2,\cdots$ where $\varepsilon_m \to 0$ as $m \to \infty$, then $F(a^*,x)$, the best approximation to $f(x)$ on $[c,d]$, is the uniform limit of $\{F(a_m,x)\}$ on $[c,d]$ as $m \to \infty$.

<u>PROOF</u>: We first note that $\{F(a_m,x)\}$ is uniformly bounded on $[c,d]$.

$|F(a_m,x)| = |F(a_m,x) - f(x) + f(x)| \le |F(a_m,x) - f(x)| + |f(x)| \le$

$D_k(f) + \varepsilon_m + \max_{c\le x\le d} |f(x)| \le M$ for each $x \in [c,d]$ and $m = 1,2,\cdots$. Let

$\{x_1,\cdots,x_{k+1}\}$ be $k+1$ distinct points in $[c,d]$. For i fixed, $\{F(a_m,x_i)\}$ is a bounded sequence of nonnegative real numbers. Hence it has a convergent subsequence $\{F(a_{m_r},x_i)\}$ such that $F(a_{m_r},x_i) \to B_i$ as $r \to \infty$ for $i = 1,\cdots,k+1$. It follows that $B_i \geq 0$ for $i = 1,\cdots,k+1$. Also $F^2(a_{m_r},x_i) \to B_i^2$ as $r \to \infty$ for $i = 1,\cdots,k+1$. $F^2(a_{m_r},x_i)$ is a polynomial, $P_{m_r}(x)$, of degree at most k. Let $P(x) = \sum_{i=1}^{k+1} B_i^2\, \ell_i(x)$ where

$$\ell_i(x) = \frac{(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_{k+1})}{(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_{k+1})}.$$

The subsequence $\{P_{m_r}(x)\}$ converges uniformly to $P(x)$ on $[c,d]$ since $P_{m_r}(x) - P(x) = \sum_{i=1}^{k+1} (P_{m_r}(x_i) - B_i^2)\ell_i(x)$. So $|P_{m_r}(x) - P(x)| \leq \sum_{i=1}^{k+1} |P_{m_r}(x_i) - B_i^2|\,|\ell_i(x)| \leq \left(\sum_{i=1}^{k+1} |P_{m_r}(x_i) - B_i^2|\right) M^*$ where $M^* = \max_{\substack{c\leq x\leq d\\ 1\leq i\leq k+1}} |\ell_i(x)|$. Hence

$$\max_{c\leq x\leq d} |P_{m_r}(x) - P(x)| \leq \left(\sum_{i=1}^{k+1} |P_{m_r}(x_i) - B_i^2|\right) M^*.$$

However for each i, $|P_{m_r}(x_i) - B_i^2| \to 0$ as $r \to \infty$. Hence $\{P_{m_r}(x)\}$ converges uniformly to $P(x)$ on $[c,d]$. Since $P_{m_r}(x) \geq 0$ on $[c,d]$ it follows that $P(x) \geq 0$ on $[c,d]$. Finally $\{F(a_{m_r},x)\}$ converges uniformly to $\sqrt{P(x)}$ on $[c,d]$. Since P is a polynomial of degree $\leq k$ we write $F(a^*,x) = \sqrt{P(x)}$. $F(a^*,x)$ is a best approximation to $f(x)$ on $[c,d]$. To see this note that $|F(a_{m_r},x) - f(x)| \leq D_k(f) + \varepsilon_{m_r}$ for $r = 1,2,\cdots$. Hence $|F(a^*,x) - f(x)| \leq D_k(f)$ for each $x \in [c,d]$ and $\max_{c\leq x\leq d} |F(a^*,x) - f(x)| \leq D_k(f)$. On the other hand $D_k(f) \leq \max_{c\leq x\leq d} |F(a^*,x) - f(x)|$ so that

$$\max_{c\leq x\leq d} |F(a^*,x) - f(x)| = D_k(f).$$

Since best approximations by $F(a,x)$ are unique it follows that $F(a^*,x)$ is the best approximation to $f(x)$ on $[c,d]$.

We next show that $P_m(x_i) \to B_i^2$ as $m \to \infty$ for $i = 1,\cdots,k+1$. Assume that this assertion is not true. Then there exists i_o, $1 \le i_o \le k+1$, such that $P_m(x_{i_o}) \not\to B_{i_o}^2$. So there must exist one subsequence $\{P_{m_r}(x_{i_o})\}$ such that

$$|P_{m_r}(x_{i_o}) - B_{i_o}^2| \ge \alpha > 0.$$

Thus we can find a subsequence $\{P_{m_j}(x_{i_o})\} \to B_{i_o}^{*2} \ne B_{i_o}^2$. As before we can find a subsequence of $\{P_{m_j}(x)\}$, say $\{P_{m_s}(x)\}$, such that $P_{m_s}(x_i) \to B_i^{*2}$ as $s \to \infty$ for $i = 1,\cdots,k+1$. Let $Q(x) = \sum_{i=1}^{k+1} B_i^{*2}\, \ell_i(x)$. We notice that Q is a polynomial of degree at most k. As before we may show that $\{P_{m_s}(x)\}$ converges uniformly to $Q(x)$ on $[c,d]$ and that $F(b,x) = \sqrt{Q(x)}$ is a best approximation to $f(x)$ on $[c,d]$. This is a contradiction since $F(a^*,x_{i_o}) = B_{i_o} \ne B_{i_o}^* = F(b,x_{i_o})$. Hence $\{P_m(x_i)\}$ converges to B_i^2 as $m \to \infty$. It follows, as before, that $\{P_m(x)\}$ converges uniformly to $P(x)$ on $[c,d]$ as $m \to \infty$ and hence $\{F(a_m,x)\}$ converges uniformly as $m \to \infty$ to $\sqrt{P(x)} = F(a^*,x)$, the best approximation to $f(x)$ on $[c,d]$.

Theorems 3.6 and 3.7 establish the convergence of $\{F(a_{S_i},x)\}$ to the best approximation $F(a^*,x)$. Sufficient conditions for these results to be applied in a practical way are that best approximations exist on each subset S_i and $\{F(a_{S_i},x)\}$ is uniformly bounded below away from zero. It is not known, at present, what the consequences, from a computational point of view, of nonexistence of best approximations on finite subsets are. Extensive numerical experimentation is needed to gain additional insight on this problem. It seems, however, that the above two hypotheses are related.

In Theorem 3.6, the rate of convergence is seen to depend on $\|S_i\|$.

The quantity ε_i of Theorem 3.6 is smallest when S_i is chosen to be a set of equally spaced points in [c,d] for each i. This is recommended when numerical computation of $F(a^*,x)$ is attempted.

In order to employ this computational procedure, an initial approximation $F(a_o,x)$ satisfying the conditions I on S_1 must be determined. $F(a_o,x)$ could be easily determined if it were possible to choose explicitly, on an a priori basis, a subset of $k + 2$ points $\{x_1,\cdots,x_{k+2}\}$ close enough to an extremal point set $\{x_1^*,\cdots,x_{k+2}^*\}$ on which the best approximation $F(a^*,x)$ alternates $k + 1$ times. $F(a_o,x)$ would then be determined as the solution of $F(a_o,x_i) - f(x_i) = (-1)^{i-1}d$, $i = 1,\cdots,k+2$. In general, such a selection is not possible.

It is sometimes possible to choose $F(a_o,x)$ as $F(a_{S_o},x)$, the best approximation to $f(x)$ on S_o, for special choices of S_o in two different ways. The first method involves choosing S_o to be a subset of $m > k+2$ equally spaced points in [c,d] where m is not very much larger than $k+2$. It is then computationally feasible to employ Algorithm 3.1 to determine $F(a_{S_o},x)$, the best approximation to $f(x)$ on S_o. If S_1 contains points sufficiently close to the extremal point set for $F(a_{S_o},x)$ on S_o then $F(a_{S_o},x)$ will satisfy conditions I on S_1 and may be used to start the computational procedure which generates $\{F(a_{S_i},x)\}$.

The second method is to choose $S_o = \{x_1,\cdots,x_{k+2}\}$ where the choice of x_i will be explained below. Once the points x_i are chosen, $F(a_o,x) = F(a_{S_o},x)$ is determined as the solution of $F(a_o,x_i) - f(x_i) = (-1)^{i-1}d$, $i = 1,\cdots,k+2$. To explain the choice of the points x_i, assume, without loss of generality, that $[c,d] = [-1,1]$. Suppose that $f(x) = \sqrt{P_{k+1}(x)}$ where P_{k+1} is a polynomial of degree at most $k+1$. Then $|F^2(a,x) - f^2(x)|$

is an approximation of $|F(a,x) - f(x)|$ which is reasonable when $|F(a,x) + f(x)|$ is near 1. Then $f^2(x) = b_{k+1}T_{k+1}(x) + \cdots + b_oT_o(x)$ where $T_k(x)$ is the Tchebycheff polynomial, $T_k(x) = \cos(k \text{ arc cos } x)$. Consider $F^2(a,x) = b_kT_k(x) + \cdots + b_oT_o(x)$. $|F^2(a,x) - f^2(x)| = |b_{k+1}T_{k+1}(x)| = |b_{k+1}| \cdot |T_{k+1}(x)| \leq |b_{k+1}|$. Moreover, at the k+ 2 points $x_i = \cos \frac{i\pi}{k+1}$, $i = 0, \cdots, k+1$, $F^2(a,x_i) - f^2(x_i) = b_{k+1}T_{k+1}(x_i) = (-1)^i b_{k+1}$. This means that $F^2(a,x_i)$ is the polynomial of best approximation to $f^2(x)$. More generally, if f is not a linear combination of a finite number of Tchebycheff polynomials, say $f^2(x) = b_oT_o(x) + b_1T_1(x) + \cdots$, the partial sum $P(x) = b_oT_o(x) + \cdots + b_kT_k(x)$ is not, in general, the polynomial of best approximation of degree at most k to $f^2(x)$. However, if the coefficients b_{k+1}, b_{k+2}, $\cdots$ are reasonably small, P(x) may be a good approximation of the best approximation, and hence the choice of $S_o = \{x_i \mid x_i = \cos \frac{i\pi}{k+1}, i = 0, \cdots, k+1\}$ may furnish $F(a_o,x) = F(a_{S_o},x)$ as a reasonable initial approximation.

CHAPTER IV

APPROXIMATION BY A SUBFAMILY OF RATIONAL FUNCTIONS

This chapter considers approximation by a subfamily of rational functions. More precisely, the approximating functions are members of the class

$$(4.1) \qquad \mathcal{F} = \{ \sum_{i=1}^{m} a_i \cdot \frac{1}{(x+t_i)} \mid x + t_i \neq 0 \text{ on } [c,d],\ a_i,\ t_i \text{ real numbers} \}.$$

Members of F will be denoted by F(a,x) where a denotes the 2m tuple of parameters $(a_1,\cdots,a_m,t_1,\cdots,t_m)$. Assume that the total number of parameters determining F(a,x) is less than or equal to n. If $a_i = 0$ for some i then t_i can be chosen arbitrarily and in such cases we choose t_i to be distinct from any of the other parameter values.

The practical motivation for considering approximation by the above class is that the approximating functions are combinations of functions that more closely imitate the lack of derivatives of the function which is being approximated than do polynomials. It is a well known fact that the maximum error of the polynomial of best approximation to f on [c,d] of degree $\leq n - 1$ is greater than or equal to the maximum error of the best rational approximation to f on [c,d] having at most n parameters, so that, in general, rational approximation produces smaller error than polynomial approximation. The class F is studied because of this and because the approximating functions are tractable from a numerical point of view. Evaluating $F(a,x) \in \mathcal{F}$ is similar to evaluating a polynomial.

The class of approximating functions (4.1) does not have best approximations for certain continuous functions because the class is not closed. To see this consider the following example. Let $f(x) = |x-\frac{1}{2}|$ on [0,1]. Let m = 1. It is clear that there are sequences $\{F(a_i,x)\}$ such that $\{F(a_i,x)\}$ converges pointwise to the constant function $\frac{1}{4}$ on [0,1]. If the above class of functions are varisolvent then the degree of F(a,x) will be the number of nonzero parameters which is two in this case. It follows that the best approximation must alternate at least two times on [0,1] so that the constant function $\frac{1}{4}$ is the best approximation. Clearly, however, $\frac{1}{4}$ is not a member of the class $\mathcal{F}$.

To remedy this situation the class of admissible approximations must be enlarged so that it includes such missing limit points to ensure the existence of best approximations.

Clearly, the nonzero constant functions are missing as pointwise limits of functions of the form $F(a,x) = \frac{a}{x+t}$. More generally, if m > 1, $\mathcal{F}$ contains the function

$$(4.2) \qquad F(a,x) = \frac{1}{(t_1-t_2)}\cdot\frac{1}{(x+t_1)} - \frac{1}{(t_1-t_2)}\cdot\frac{1}{(x+t_2)}$$

if $t_1 \neq t_2$. Since $\frac{1}{x+t}$ has a derivative with respect to t, any sequence $\{F(a_i,x)\}$ of the form (4.2) with parameters t_1, t_2^i such that $\{t_2^i\} \to t_1$ as $i \to \infty$ will have the function $-\frac{1}{(x+t_1)^2}$ as a pointwise limit on [c,d]. These derivatives must be included in the class of approximating functions. Similarly, $\mathcal{F}$ contains the $1,2,\cdots,(m-1)$ divided differences of $\frac{1}{(x+t)}$ and hence the class of approximating functions must include the

$1,2,\cdots,(m-1)$ derivatives of $\frac{1}{x+t}$ with respect to t, as they are the pointwise limits of sequences of functions in $\mathcal{F}$.

The derivatives of $\frac{1}{(x+t)}$ (except for a constant multiple) are the functions $\frac{1}{(x+t)^2}, \frac{1}{(x+t)^3}, \cdots$. This provides the motivation to extend the class of admissible approximating functions to the class

$$\mathcal{F}^* = \{\sum_{i=1}^{k} \sum_{j=1}^{m_i+1} a_{ij} \cdot \frac{1}{(x+t_i)^j} | t_i \in T,\ a_{ij} \text{ real numbers, } \sum_{i=1}^{k} m_i + 1 \leq m\}$$

where T is a set of real numbers such that $x + t \neq 0$ for $x \in [c,d]$ and $t \in T$. $F(a,x)$ will denote members of $\mathcal{F}^*$ where a is the at most 2m tuple of parameters $(a_{11},\cdots,a_{1m_1+1},\cdots,a_{k1},\cdots,a_{km_k+1},t_1,\cdots,t_k)$.

THEOREM 4.1: If T is compact then $\mathcal{F}^*$ is closed under pointwise convergence.

PROOF: Let $\{F(a_r,x)\}$ be a uniformly bounded sequence in $\mathcal{F}^*$ converging pointwise to $g(x)$ on $[c,d]$. Let $F(a_r,x) = \sum_{i=1}^{k} \sum_{j=1}^{m_i+1} a_{ij}^{(r)} \frac{1}{(x+t_i^{(r)})^j}$.

Since the functions $\left\{\frac{1}{(x+t_i)^j} \middle| i = 1,\cdots,k,\ j = 1,\cdots,m_i+1\right\}$ are linearly independent for distinct t_i, it follows, as in the proof of Theorem 3.2, that

$$\{a_{ij}^{(r)}\}\ i = 1,\cdots,k,\ j = 1,\cdots,m_i+1$$

are bounded sequences of real numbers. This, together with the compactness of T, implies the existence of a subsequence of parameters a_{r_s} such that $a_{ij}^{(r_s)} \to a_{ij}^{(o)}$, $i = 1,\cdots,k$, $j = 1,\cdots,m_i+1$ and $t_i^{(r_s)} \to t_i^{(o)}$, $i = 1,\cdots,k$ as $s \to \infty$. Hence for each $x \in [c,d]$, $g(x) = \lim_{r\to\infty} F(a_r,x) =$

$$\lim_{s\to\infty} F(a_{r_s},x) = \sum_{i=1}^{k} \sum_{j=1}^{m_i+1} \lim_{s\to\infty} a_{ij}^{(r_s)} \frac{1}{(x+\lim_{s\to\infty} t_i^{(r_s)})^j} = \sum_{i=1}^{k} \sum_{j=1}^{m_i+1} a_{ij}^{(o)} \frac{1}{(x+t_i^{(o)})^j}.$$

This shows that $g(x) = F(a_o,x)$ which finishes the proof.

The <u>degree</u>, $m(a)$, of $F(a,x)$ is defined by $m(a) = m + \sum_{i=1}^{k} (m_i + 1)$.

Note that $m(a) \le 2m$.

<u>THEOREM 4.2</u>: $F(a,x) \in \mathcal{F}^*$ has property Z of degree $m(a)$.

<u>PROOF</u>: Let $a \ne a^*$. Then $F(a,x) - F(a^*,x) =$

$\sum_{i=1}^{k} \sum_{j=1}^{m_i+1} a_{ij} \frac{1}{(x+t_i)^j} - \sum_{i=1}^{k^*} \sum_{j=1}^{m_i^*+1} a_{ij}^* \frac{1}{(x+t_i^*)^j}$. This difference is a rational function. After adding the terms in $F(a,x) - F(a^*,x)$ it is seen that the degree of the numerator is

$$\sum_{i=1}^{k^*} (m_i^*+1) + \sum_{i=1}^{k} (m_i+1) - 1 \le m + \sum_{i=1}^{k} (m_i+1) - 1 = m(a) - 1$$

and hence the numerator has at most $m(a) - 1$ zeros on $[c,d]$. It follows that $F(a,x) - F(a^*,x)$ has at most $m(a) - 1$ zeros on $[c,d]$.

<u>COROLLARY 4.1</u>: Let f be continuous on $[c,d]$. Then there exists a best approximation $F(a^*,x) \in \mathcal{F}^*$ to $f(x)$ on $[c,d]$.

<u>PROOF</u>: Theorems 4.1 and 4.2 insure that the hypothesis of Theorem 3.1 is satisfied and the Corollary follows.

The next theorem establishes that $F(a,x)$ is locally solvent of degree $m(a)$.

<u>THEOREM 4.3</u>: $F(a^*,x) \in \mathcal{F}^*$ is locally solvent of degree $m(a^*)$ on $[c,d]$ for each parameter a^*.

<u>PROOF</u>: Let $x_1 < x_2 < \cdots < x_{m(a^*)}$ be $m(a^*)$ points in $[c,d]$ and $\varepsilon > 0$ be given. Consider the system

$$(4.3) \qquad F(a,x_\ell) = F(a^*,x_\ell) + y_\ell, \quad \ell = 1,\cdots,m(a^*).$$

To show that $F(a^*,x)$ is locally solvent of degree $m(a^*)$ we must show that there is a $\delta > 0$ such that if $(y_1^2 + \cdots + y_{m(a^*)}^2)^{\frac{1}{2}} < \delta$ (4.3) has a unique solution $F(a,x)$ with

$$(4.4) \qquad \|F(a,x) - F(a^*,x)\| < \varepsilon.$$

If (4.3) has a solution satisfying (4.4) then it is unique. To see this suppose that $F(a,x)$ and $F(b,x)$ are solutions of (4.3) satisfying (4.4) and $a \neq b$. Then $F(a,x_\ell) - F(b,x_\ell) = 0$, $\ell = 1,\cdots,m(a^*)$. Since ε is arbitrarily small and $\|F(a,x) - F(a^*,x)\| < \varepsilon$, it follows that $m(a) = m(a^*)$. However, $F(a,x)$ satisfies property Z of degree $m(a)$ and $F(a,x) - F(b,x)$ has at least $m(a)$ zeros and hence $a = b$ which is a contradiction. Let

$$F(a_\ell,x) = \sum_{i=1}^{k^\ell} \sum_{j=1}^{m_i^\ell+1} a_{ij}^\ell \frac{1}{(x+t_i^\ell)^j} \text{ and } F(a^*,x) = \sum_{i=1}^{k^*} \sum_{j=1}^{m_i^*+1} a_{ij}^* \frac{1}{(x+t_i^*)^j}.$$

Suppose that $\{F(a_\ell,x)\}$ is a sequence converging to a solution of (4.3) satisfying (4.4). Then it is clear that some $\{t_i^\ell\} \to t_i^*$ as $\ell \to \infty$ for each i. Since t_i^* are distinct, there are at least k^* parameters t_i in the solution of (4.3) satisfying (4.4). There is a notational problem in writing an expression for the solution $F(a,x)$ of (4.3) making an association of the parameters of $F(a^*,x)$ with the parameters of $F(a,x)$.

Let $F(a,x) = \sum_{i=1}^{k^*} F_i(a,x) + R(a,x)$ where each t_i in $F_i(a,x)$ satisfies $t_i \to t_i^*$ as $\varepsilon \to 0$ and $R(a,x) = \sum_{i=k^*+1}^{k} \sum_{j=1}^{m_i+1} a_{ij} \frac{1}{(x+t_i)^j}$ and the coefficients $a_{ij} \to 0$ as $\varepsilon \to 0$ for $i = k^*+1,\cdots,k$, $j = 1,\cdots,m_i+1$. Consider the expressions

$$(4.5) \qquad F_i(a,x) - \sum_{j=1}^{m_i^*+1} a_{ij}^* \frac{1}{(x+t_i^*)^j}.$$

Since each t_i in $F_i(a,x)$ satisfies $t_i \to t_i^*$ as $\varepsilon \to 0$ any one of them may be used in the expression (4.5) so that (4.5) becomes

$$(4.6) \qquad \sum_{j=1}^{m_i+1} a_{ij} \frac{1}{(x+t_i)^j} - \sum_{j=1}^{m_i^*+1} a_{ij}^* \frac{1}{(x+t_i^*)^j} .$$

and $m_i+1 > m_i^*+1$ for the same reasoning used to establish that there are at least k^* parameters t_i in the solution of (4.3). Rewrite (4.6) as

$$(4.7) \qquad \sum_{j=1}^{m_i^*+1} a_{ij} \frac{1}{(x+t_i)^j} - a_{ij}^* \frac{1}{(x+t_i^*)^j} + \sum_{j=m_i^*+2}^{m_i+1} a_{ij} \frac{1}{(x+t_i)^j} .$$

By Taylor's theorem $\frac{1}{(x+t_i)^j} = \frac{1}{(x+t_i^*)^j} - \frac{j(t-t^*)}{(x+t_i^*)^{j+1}} + o(t_i-t_i^*)$ where $o(z)$ denotes a continuous function such that $o(z) \to 0$ as $|z| \to 0$.

Hence (4.7) becomes

$$(4.8) \qquad \left[\sum_{j=1}^{m_i^*+1} a_{ij}\left(\frac{1}{(x+t_i^*)^j} - \frac{j(t_i-t_i^*)}{(x+t_i^*)^{j+1}}\right) - a_{ij}^* \frac{1}{(x+t_i^*)^j}\right]$$

$$+ \sum_{j=m_i^*+2}^{m_i+1} a_{ij}\left(\frac{1}{(x+t_i^*)} - \frac{(j)(t_i-t_i^*)}{(x+t_i^*)^{j+1}}\right) + o(t_i-t_i^*).$$

After rewriting, (4.8) becomes

$$(4.9) \qquad \sum_{j=1}^{m_i^*+1} (a_{ij} - a_{ij}^*) \frac{1}{(x+t_i^*)^j} - \sum_{j=1}^{m_i^*+1} a_{ij}^* \frac{(j)(t_i-t_i^*)}{(x+t_i^*)^{j+1}}$$

$$- \sum_{j=1}^{m_i^*+1} (a_{ij}-a_{ij}^*) \frac{(j)(t_i-t_i^*)}{(x+t_i^*)^{j+1}} + \sum_{j=m_i^*+2}^{m_i+1} a_{ij}\left(\frac{1}{(x+t_i^*)} - \frac{(j)(t_i-t_i^*)}{(x+t_i^*)^{j+1}}\right)$$

$$+ o(t_i-t_i^*).$$

Let $\delta a_{ij} = a_{ij} - a_{ij}^*$ and $\delta t_i = t_i - t_i^*$. Then there are m_i+2 unknowns in

(4.9), namely the δa_{ij}, δt_i, and a_{ij}. Hence the system (4.3), which can be written as

$$(4.10)\quad \sum_{i=1}^{k^*}\left(F_i(a,x_\ell) - \sum_{j=1}^{m_i^*+1} a_{ij}^* \frac{1}{(x_\ell+t_i^*)^j}\right) + R(a,x_\ell) = y_\ell,\ \ell = 1,\cdots,m(a^*),$$

has $\sum_{i=1}^{k^*}(m_i+2) + \sum_{i=k^*+1}^{k}(m_i+1) + k - k^* = m(a^*)$ unknowns. The $k - k^*$ parameters t_i in $R(a,x)$ can be chosen arbitrarily, but distinct from the parameters t_i^*. Rice [12] showed that if X, Y are n-dimensional vectors, Q a nonsingular $n \times n$ matrix, and $o(X)$ is a continuous vector valued function such that $\|o(X)\| \to 0$ as $\|X\| \to 0$ where $\|\cdot\|$ denotes the Euclidean norm, then there exists $\delta > 0$ which depends on Q such that if $\|Y\| < \delta$ there is a solution to

$$(4.11)\qquad QX = Y + X\,o(X).$$

When the expression (4.9) is substituted in (4.10) the system (4.3), after rearranging terms, is of the form (4.11). Since $F(a^*,x)$ has at most $\sum_{i=1}^{k^*}(m_i^*+1) - 1$ zeros on $[c,d]$ it follows that the matrix Q is nonsingular. Hence for δ sufficiently small the system (4.3) has a solution satisfying (4.4).

COROLLARY 4.2: Results B, C, and D of Chapter II hold for $F(a,x)$.

The problem of best approximation on finite subsets of $[c,d]$ is considered next. Result D characterizes best approximations on finite subsets of $[c,d]$, provided these best approximations exist. Suppose $S = \{x_1,\cdots,x_\ell\} \subseteq [c,d]$ where $\ell \geq 2m + 1$. A sufficient condition for existence of a best approximation $F(a_S,x)$ to $f(x)$ on S is that for any subset

$$S = \{z_1,\cdots,z_{m(a)+1}\} \subseteq S,$$

$z_1 < \cdots < z_{(m(a)+1}$ there exists a unique solution to the system

$$(4.12) \qquad F(a,z_i) - f(z_i) = (-1)^{i-1}d, \quad i = 1,\cdots,m(a) + 1.$$

The problem of solving the nonlinear system (4.12) is one of considerable theoretical and computational difficulty. Aside from the difficulties of solving the system (4.12) is the problem that the degree, $m(a_S)$, of the best approximation $F(a_S,x)$ to $f(x)$ is not known in advance. Moreover, even if $m(a_S)$ were known in advance there would still be the additional problem of determining the exact form of the best approximation $F(a_S,x)$. These problems must be dealt with before computational procedures similar to those in Chapter III for determining the best approximation $F(a_S,x)$ can be developed.

A more promising approach is to consider the problem of determining the best approximation $F(a^*,x)$ to $f(x)$ on $[c,d]$ as a programming problem where one of the methods of descent might be applied. This could also be done for the problem of determining the best approximation $F(a_S,x)$ to $f(x)$ on $S = \{x_1,\cdots,x_\ell\}$.

For example, consider the graph of the function $\|F(a,x) - f(x)\| = d$ as a ranges throughout the parameter space. The problem is to find the parameter a^* which minimizes d. Let

$$B = \{(a,d) \mid \|F(a,x) - f(x)\| \leq d\}.$$

Since it is assumed that the number of parameters which determine $F(a,x)$ is less than or equal to n we may regard B as a subset of (n+1) dimensional Euclidean space E_{n+1}. It is clear that when $F(a,x)$ depends linearly on the parameters a that B is a convex subset of E_{n+1}. However, $F(a,x) \in \mathcal{F}^*$

does not depend linearly on the parameters a so that, in general, B is not convex. Hence a descent scheme for determining the "lowest point" in B may converge to a point which is not the absolute minimum.

In general terms the method of descent is as follows. Given an estimate (a_i,d_i) of the minimum point (a^*,d^*) in B, determine a direction which is down and then go in that direction a certain distance to obtain a new estimate (a_{i+1},d_{i+1}) for which $d_{i+1} < d_i$. There are various well known methods for determining a direction and how far to go in a given direction. The problem of lack of convexity may be overcome by choosing a good enough initial estimate. The methods suggested in Chapter III for this apply in this situation also.

The method of descent is suggested here because the problem of solving nonlinear systems of unknown dimension is circumvented and because the descent procedures are less sensitive to changes in the form of F(a,x).

More work is needed on the problem of computing the best approximation $F(a^*,x)$ to f(x) on [c,d] and this is a problem which will benefit considerably from practical numerical experimentation.

The difficulty in dealing with F(a,x) on finite subsets comes from the nonlinearity of F(a,x) in the parameters $t_1,\cdots,t_k$. From a practical point of view, the numerical analyst may be able to make a reasonable selection of the parameters $t_1,\cdots,t_k$ and the integers $m_1,\cdots,m_k$, such that $\sum_{i=1}^{k} m_i+1 \le m$, based on knowledge of the function f which is being approximated. Thus we are led to consider the subclass of $\mathcal{F}^*$

$$\mathcal{S} = \left\{ \sum_{i=1}^{k} \sum_{j=1}^{m_i+1} a_{ij} \frac{1}{(x+t_i)^j} \,\middle|\, a_{ij} \text{ real numbers} \right\} \text{ and } t_1,\cdots,t_k \text{ are fixed}$$

real numbers such that $x + t_i \neq 0$, $x \in [c,d]$, $i = 1,\cdots,k$ and $m_1,\cdots,m_k$ are fixed integers such that $\sum_{i=1}^{k} m_i + 1 \leq m$. $F(a,x)$ will denote members of $\mathcal{J}$ where a is the at most m tuple of parameters $(a_{11},\cdots,a_{1m_1+1},\cdots,a_{k1},\cdots,a_{km_k+1})$.

The class $\mathcal{J}$ is more tractable than $\mathcal{S}^*$ because $F(a,x) \in \mathcal{J}$ depends linearly on the parameters a.

Define the degree, $m(a)$, of $F(a,x) \in \mathcal{J}$ to be $m(a) = \sum_{i=1}^{k} m_i + 1$. Notice that $m(a)$ is constant as a ranges throughout the parameter space P.

THEOREM 4.4: $F(a,x) \in \mathcal{J}$ has property Z of degree $m(a)$.

PROOF: Let $a \neq a^*$. Then $F(a,x) - F(a^*,x) =$

$$\sum_{i=1}^{k} \sum_{j=1}^{m_i+1} a_{ij} \frac{1}{(x+t_i)^j} - \sum_{i=1}^{k} \sum_{j=1}^{m_i+1} a^*_{ij} \frac{1}{(x+t_i)^j} = \sum_{i=1}^{k} \sum_{j=1}^{m_i+1} (a_{ij}-a^*_{ij}) \frac{1}{(x+t_i)^j} .$$

After adding the terms in $F(a,x) - F(a^*,x)$ it is seen that the degree of the numerator is $\sum_{i=1}^{k} (m_i+1) - 1 = m(a) - 1$ and hence the numerator has at most $m(a) - 1$ zeros on $[c,d]$. It follows that $F(a,x) - F(a^*,x)$ has at most $m(a) - 1$ zeros on $[c,d]$.

THEOREM 4.5: $\mathcal{J}$ is closed under pointwise convergence.

PROOF: Let $\{F(a_r,x)\}$ be a uniformly bounded sequence in $\mathcal{J}$ converging pointwise to $g(x)$ on $[c,d]$. Let $F(a_r,x) = \sum_{i=1}^{k} \sum_{j=1}^{m_i+1} a^{(r)}_{ij} \frac{1}{(x+t_i)^j}$. Since the functions $\{\frac{1}{(x+t_i)^j} \mid i = 1,\cdots,k,\ j = 1,\cdots,m_i+1\}$ are linearly independent for distinct t_i, it follows, as in the proof of Theorem 3.2, that $\{a^{(r)}_{ij}\}$ $i = 1,\cdots,k$, $j = 1,\cdots,m_i+1$ are bounded sequences of real numbers. This implies the existence of a subsequence of parameters a_{r_s} such that $a^{(r_s)}_{ij} \to a^{(o)}_{ij}$, $i = 1,\cdots,k$, $j = 1,\cdots,m_i+1$ as $s \to \infty$. Hence for each $x \in [c,d]$,

$$g(x) = \lim_{r \to \infty} F(a_r,x) = \lim_{s \to \infty} F(a_{r_s},x) = \sum_{i=1}^{k} \sum_{j=1}^{m_i+1} \lim_{s \to \infty} a_{ij}^{(r_s)} \frac{1}{(x+t_i)^j} =$$

$\sum_{i=1}^{k} \sum_{j=1}^{m_i+1} a_{ij}^{(o)} \frac{1}{(x+t_i)^j}$. This shows that there is a parameter a_o such that $g(x) = F(a_o,x)$ on [c,d] which finishes the proof.

COROLLARY 4.3: Let f be continuous on [c,d].

Then there exists a best approximation $F(a^*,x) \in \mathfrak{F}$ to $f(x)$ on [c,d].

PROOF: Theorems 4.4 and 4.5 insure that the hypothesis of Theorem 3.1 is satisfied and the corollary follows.

Since m(a) is constant for each parameter a, $F(a,x) \in \mathfrak{F}$ is unisolvent. This is established in the following theorem.

THEOREM 4.6: $F(a,x) \in \mathfrak{F}$ is solvent of degree m(a).

PROOF: Let $\{x_1,\cdots,x_{m(a)}\}$ be m(a) distinct points in [c,d] and $\{y_1,\cdots,y_{m(a)}\}$ be m(a) real numbers. The system

(4.13) $$F(a,x_i) = y_i, \quad i = 1,\cdots,m(a)$$

is a linear system in the parameters a_{ij}. Writing (4.13) in matrix form we obtain

(4.14) $$\begin{bmatrix} \frac{1}{(x_1+t_1)} & \cdots & \frac{1}{(x_1+t_1)^{m_1+1}} & \cdots & \frac{1}{(x_1+t_k)} & \cdots & \frac{1}{(x_1+t_k)^{m_k+1}} \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \frac{1}{(x_{m(a)}+t_1)} & \cdots & \frac{1}{(x_{m(a)}+t_1)^{m_1+1}} & \cdots & \frac{1}{(x_{m(a)}+t_k)} & \cdots & \frac{1}{(x_{m(a)}+t_k)^{m_k+1}} \end{bmatrix} \begin{bmatrix} a_{11} \\ \vdots \\ a_{1m_1+1} \\ \vdots \\ a_{k1} \\ \vdots \\ a_{km_k+1} \end{bmatrix} = \begin{bmatrix} y_1 \\ \\ \\ \\ \\ \\ y_{m(a)} \end{bmatrix}$$

Since $\{\frac{1}{(x+t_i)^j} \mid i = 1,\cdots,k,\ j = 1,\cdots,m_i+1\}$ are linearly independent and $x_1,\cdots,x_{m(a)}$ are distinct, it follows that the coefficient matrix of (4.14) is nonsingular. Hence the system (4.13) has a unique solution and this finishes the proof.

Theorems 4.4 and 4.6 show that $F(a,x) \in \mathcal{J}$ is unisolvent (hence varisolvent) of degree $m(a)$. Hence the results B, C, and D of Chapter II hold for $F(a,x) \in \mathcal{J}$.

The remainder of this chapter deals with the problem of best approximation on finite subsets of $[c,d]$ by $F(a,x) \in \mathcal{J}$.

THEOREM 4.7: If S is any finite subset of $[c,d]$ having at least $m(a)+1$ points, then there exists a best approximation $F(a_S,x) \in \mathcal{J}$ to f on S.

PROOF: Let $U = \{x_1,\cdots,x_{m(a)+1}\}$ be an arbitrary subset of S having $m(a)+1$ points and suppose that the points in U are enumerated so that $x_1 < x_2 < \cdots < x_{m(a)+1}$. Consider the system

$$F(a,x_i) - f(x_i) = (-1)^{i-1}d, \quad i = 1,\cdots,m(a)+1. \tag{4.15}$$

The system (4.15) is linear in the parameters a and d. The coefficient matrix of the system 4.15 is

$$\begin{bmatrix} \frac{1}{x_1+t_1} & \cdots & \frac{1}{(x_1+t_k)^{m_k+1}} & 1 \\ \vdots & & & \vdots \\ \frac{1}{x_{m(a)+1}+t_1} & \cdots & \frac{1}{(x_{m(a)+1}+t_k)^{m_k+1}} & (-1)^{m(a)} \end{bmatrix}. \tag{4.16}$$

The matrix (4.16) is nonsingular. To see this, note that the first $m(a)$ columns of (4.16) are linearly independent and the last column of (4.16) is independent of the first $m(a)$ columns since $F(a,x)$ satisfies property Z of degree $m(a)$ and hence can change sign at most $m(a)-1$ times. Hence there exists a unique parameter a_U and a unique real number d_U satisfying the system (4.15). By result D of Chapter II, $F(a_U,x)$ is the best approximation to f on U. Since U is an arbitrary subset of S having $m(a)+1$ points, it follows from result D of Chapter II that the best approximation, $F(a_S,x)$, to f on S exists.

It follows from Theorem 4.7 that best approximations on subsets of $[c,d]$ having $m(a)+1$ points always exist. Hence if $S = \{x_1,\cdots,x_\ell\}$, $\ell > m(a)+1$, the best approximation, $F(a_S,x)$, to f on S may be determined by computing $F(a_U,x)$ and d_U, the best approximation to f on U, for each $U \subseteq S$ such that U has $m(a)+1$ points and choosing $a_S = a_{U^*}$ where U^* is such that $d_{U^*} = \max_{\substack{U \subseteq S \\ U \text{ has } m(a)+1 \text{ points}}} d_U$.

Algorithm 3.2 may be modified to determine the best approximation $F(a_S,x) \in \mathcal{F}$ to f on S as follows.

<u>ALGORITHM 4.1</u>: Let $S = \{x_1,\cdots,x_\ell\} \subseteq [c,d]$ and $\ell > m(a)+1$. Let $F(a_o,x)$ be an initial approximation satisfying the conditions

I: 1) $F(a_o,x) - f(x)$ assumes extreme values $d_1^o, -d_2^o, \cdots, (-1)^{m(a)} d_{m(a)+1}^o$ at $m(a)+1$ points in S, say $x_1^1,\cdots,x_{m(a)+1}^1$, where $x_1^1 < \cdots < x_{m(a)+1}^1$ and d_i^o all have the same sign for $i = 1,\cdots,m(a)+1$.

2) There is at least one j, $1 \le j \le m(a)+1$, such that $|d_j^o| = \max_{x \in S} |F(a_o,x) - f(x)|$.

Then 1) Let $j = 1$.

2) Determine the approximation $F(a_j,x)$ and d_j such that

$$F(a_j,x_i^j) - f(x_i^j) = (-1)^{i-1} d_j \quad \text{for } i = 1,\cdots,m(a)+1$$

3) If $\max_{x \in S} |F(a_j,x) - f(x)| \le |d_j|$, then $F(a_j,x)$ is the best approximation to f on S and the algorithm is terminated. Otherwise,

determine $m(a)+1$ points $\{x_1^{j+1},\cdots,x_{m(a)+1}^{j+1}\} \subseteq S$, $x_1^{j+1} < \cdots < x_{m(a)+1}^{j+1}$, such that $F(a_j,x_i^{j+1}) - f(x_i^{j+1}) = (-1)^{i-1}d_i^j$ for $i = 1,\cdots,m(a)+1$ and $\min_{1\le i\le m(a)+1} |d_i^j| \ge |d_j|$.

4) Let $j = j+1$ and go to step 2).

<u>THEOREM 4.8</u>: Let f be defined on $[c,d]$ and $S = \{x_1,\cdots,x_\ell\}$, $\ell \ge m(a)+1$, be a subset of $[c,d]$. If $F(a_0,x)$ is an initial approximation satisfying the conditions I, then Algorithm 4.1 will determine the best approximation $F(a_S,x) \in \mathfrak{J}$ to f on S.

<u>PROOF</u>: Since $F(a,x)$ satisfies property Z of degree $m(a)$ the proof is the same as the proof of Theorem 3.5.

Theorem 4.8 provides the basis for a computational procedure which will approximate $F(a^*,x) \in \mathfrak{J}$, the best approximation to $f(x)$ on $[c,d]$. Let S_i be a sequence of partitions of $[c,d]$ such that S_{i+1} is a refinement of S_i for $i = 1,2,\cdots$ and $\|S_i\| \to 0$ as $i \to \infty$. Determine the sequence of best approximations $F(a_{S_i},x)$ to $f(x)$ on S_i using Algorithm 4.1.

Let $D_{S_i}(f) = \max_{x\in S_i} |F(a_{S_i},x) - f(x)|$ and $D(f) = \max_{c\le x\le d} |F(a^*,x) - f(x)|$ where $F(a^*,x)$ is the best approximation to $f(x)$ on $[c,d]$. It follows, as in Chapter III, that $\{F(a_{S_i},x)\}$ is uniformly bounded on $[c,d]$.

The next lemma is helpful in establishing the convergence of $\{F(a_{S_i},x)\}$.

<u>LEMMA 4.1:</u> Let $F(a,x) \in \mathfrak{J}$ and suppose that $\max_{c\le x\le d} |F(a,x)| \le M$. Then there exists a constant γ which does not depend on $F(a,x)$ such that $\max_{c\le x\le d} |F'(a,x)| \le \gamma\cdot M$.

<u>PROOF</u>: Let $F(a,x) \in \mathfrak{J}$. Then $F(a,x) = \dfrac{P(x)}{Q(x)}$ where

$Q(x) = (x+t_1)^{m_1+1} \cdots (x+t_k)^{m_k+1}$. Let $m_1 = \min_{c\le x\le d} |Q(x)|$ and

$m_2 = \max_{c\le x\le d} |Q(x)|$. Suppose $\max_{c\le x\le d} |F(a,x)| \le M$. Since

$\dfrac{\max_{c\le x\le d} |P(x)|}{m_2} \le \max_{c\le x\le d} |F(a,x)|$, it follows that $\max_{c\le x\le d} |P(x)| \le m_2 M$. Hence,

by Markov's inequality we have

$$|F'(a,x)| = \left|\frac{Q(x)P'(x) - P(x)Q'(x)}{Q^2(x)}\right| \le \frac{|Q(x)||P'(x)| + |P(x)||Q'(x)|}{Q^2(x)} \le$$

$$\frac{m_2\cdot 2\cdot (m(a)-1)^2\cdot m_2\cdot M \quad + \quad m_2\cdot M\cdot 2\cdot (m(a))^2\cdot m_2}{(d-c)m_1^2} = \frac{2\cdot m_2^2((m(a))^2 + (m(a)-1)^2)}{(d-c)m_1^2}\cdot M =$$

$\gamma\cdot M$ for each $x \in [c,d]$. Then $\max_{c\le x\le d} |F'(a,x)| \le \gamma\cdot M$ and γ does not depend on $F(a,x)$.

<u>THEOREM 4.9</u>: If $\{F(a_{S_i},x)\}$ is the sequence of best approximations to f on S_i, as previously defined, then

$$\max_{c\le x\le d} |F(a_{S_i},x) - f(x)| \le D(f) + \varepsilon_i \quad \text{where}$$

$\varepsilon_i \to 0$ as $i \to \infty$.

<u>PROOF</u>: Let $M_i = \max_{c\le x\le d} |F(a_{S_i},x)|$. Since $\{F(a_{S_i},x)\}$ is uniformly bounded on $[c,d]$, there exists $M < \infty$ such that $M_i \le M$ for $i = 1,2,\cdots$. Let $S_i = \{x_{1,j_i},\cdots,x_{j_i,j_i}\}$ where j_i is the number of points in S_i. Let

$x_o \in [c,d]$. Then there is an integer r such that $x_o \in [x_{r,j_i}, x_{r+1,j_i}]$. By the mean value theorem there exists $U \in (x_{r,j_i}, x_{r+1,j_i})$ such that

$F(a_{S_i},x_o) - F(a_{S_i},x_{r,j_i}) = (x_o - x_{r,j_i})F'(a_{S_i},U)$. Hence

$$|F(a_{S_i},x_o) - F(a_{S_i},x_{r,j_i})| \le \|S_i\|\cdot \max_{c\le x\le d} |F'(a_{S_i},x)| \le \|S_i\|\cdot\gamma\cdot M_i \le M\cdot\gamma\cdot\|S_i\|.$$

Finally, since $F(a_{S_i},x) - f(x)$ is continuous on $[c,d]$, there exists $x_i \in [c,d]$ such that

$$\max_{c\le x\le d} |F(a_{S_i},x) - f(x)| = |F(a_{S_i},x_i) - f(x_i)|.$$

We can find an integer r such that

$$x_i \in [x_{r,j_i}, x_{r+1,j_i}].$$

Hence $\max_{c\le x\le d} |F(a_{S_i},x) - f(x)| \le$

$|F(a_{S_i},x_i) - F(a_{S_i},x_{r,j_i})| + |F(a_{S_i},x_{r,j_i}) - f(x_{r,j_i})| + |f(x_{r,j_i}) - f(x_i)| \le$

$M\cdot\gamma\cdot\|S_i\| + D(f) + \omega_f(\|S_i\|) = D(f) + \varepsilon_i$ where $\varepsilon_i = M\cdot\gamma\cdot\|S_i\| + \omega_f(\|S_i\|)$ and ω_f is the modulus of continuity of f on $[c,d]$. Clearly $\varepsilon_i \to 0$ as $i \to \infty$.

The next theorem establishes the uniform convergence of $\{F(a_{S_i},x)\}$ to the best approximation, $F(a^*,x)$, to $f(x)$ on $[c,d]$.

<u>THEOREM 4.10:</u> If $\{F(a_i,x)\}$ is a sequence in $\mathfrak{F}$ such that $\max_{c\le x\le d} |F(a_i,x) - f(x)| \le D(f) + \varepsilon_i$ for $i = 1,2,\cdots$ where $\varepsilon_i \to 0$ as $i \to \infty$, then $F(a^*,x)$, the best approximation to $f(x)$ on $[c,d]$, is the uniform limit of $\{F(a_i,x)\}$ on $[c,d]$ as $i \to \infty$.

<u>PROOF</u>: It follows, as in the proof of Theorem 3.7, that $\{F(a_i,x)\}$ is uniformly bounded on $[c,d]$. Let $\{x_1,\cdots,x_{m(a)}\}$ be $m(a)$ distinct points in $[c,d]$. For j fixed, $\{F(a_i,x_j)\}$ is a bounded sequence of real numbers. Hence there exists a convergent subsequence $\{F(a_{i_r},x)\}$ such that $F(a_{i_r},x_j) \to B_j$ as $r \to \infty$ for $j = 1,\cdots,m(a)$. By Theorem 4.6 there exists a unique parameter a_o such that $F(a_o,x_j) = B_j$ for $j = 1,\cdots,m(a)$. The subsequence $F(a_{i_r},x)$ converges uniformly to $F(a_o,x)$. Let $\varepsilon > 0$ be given. Since $F(a,x)$ is varisolvent there exists $\delta > 0$ such that if

$$|F(a_o,x_j) - F(a_{i_r},x_j)| < \delta, \quad j = 1,\cdots,m(a).$$

There is a solution of

$$(4.17) \qquad F(a,x_j) = F(a_{i_r},x_j), \quad j = 1,\cdots,m(a)$$

and

$$(4.18) \qquad \max_{c\le x\le d} |F(a,x) - F(a_o,x)| < \varepsilon.$$

However, the unique solution of (4.17) is given by the parameter $a = a_{i_r}$. Hence if $\varepsilon > 0$ is given then there is an integer R such that $r \geq R$ implies $|F(a_o,x_j) - F(a_{i_r},x_j)| < \delta$, $j = 1,\cdots,m(a)$. By (4.18) $\max_{c\le x\le d} |F(a_{i_r},x) - F(a_o,x)| < \varepsilon$ for $r \geq R$ and this proves the uniform convergence of $F(a_{i_r},x)$ to $F(a_o,x)$ on [c,d]. We next show that $F(a_o,x)$ is a best approximation to f(x) on [c,d]. To see this note that $\max_{c\le x\le d} |F(a_{i_r},x) - f(x)| \leq D(f) + \varepsilon_{i_r}$ for $r = 1,2,\cdots$. Hence $\max_{c\le x\le d} |F(a_o,x) - f(x)| \leq D(f)$. On the other hand $D(f) \leq \max_{c\le x\le d} |F(a_o,x) - f(x)|$ so that $\max_{c\le x\le d} |F(a_o,x) - f(x)| = D(f)$. Since best approximations are unique, $F(a_o,x) = F(a^*,x)$ for each $x \in [c,d]$.

$F(a_i,x_j) \to B_j$ as $i \to \infty$ for $j = 1,\cdots,m(a)$. Assume that this assertion is not true. Then there exists j_o, $1 \leq j_o \leq m(a)$, such that $F(a_i,x_{jo}) \not\to B_{jo}$. So there must exist one subsequence $\{F(a_{i_r},x)\}$ such that

$$|F(a_{i_r},x_{jo}) - B_{jo}| \geq \alpha > 0.$$

Thus there exists a subsequence $\{F(a_{i_s},x_{jo})\} \to B_{jo}^* \neq B_{jo}$. As before there exists a subsequence of $\{F(a_{i_s},x)\}$, say $\{F(a_{i_n},x)\}$, such that $F(a_{i_n},x_j) \to B_j^*$ as $n \to \infty$ for $j = 1,\cdots,m(a)$. Let F(b,x) be the solution of

$$F(a,x_j) = B_j^*, \; j = 1,\cdots,m(a).$$

It follows, as before, that $\{F(a_{i_n},x)\}$ converges uniformly to F(b,x) on [c,d] and that F(b,x) is a best approximation to f(x) on [c,d]. This is a contradiction since $F(a^*,x_{j_o}) = B_{j_o} \neq B^*_{j_o} = F(b,x_{j_o})$. Hence $F(a_i,x_j) \to B_j$ as $i \to \infty$ for $j = 1,\cdots,m(a)$. It follows, as before, that $\{F(a_i,x)\}$ converges uniformly to $F(a^*,x)$ on $[c,d]$ as $i \to \infty$.

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