2016

Conceptual Limitations in Curricular Presentations of Area Measurement: One Nation's Challenges

John P. Smith III
Michigan State University, jsmith@mwsu.edu

Lorraine Males
University of Nebraska-Lincoln, lmales2@unl.edu

Funda Gonulates
Michigan State University

Conceptual Limitations in Curricular Presentations of Area Measurement: One Nation's Challenges

John P. Smith III, Lorraine M. Males, and Funda Gonulates

Michigan State University

Corresponding author — John P. Smith III, jsmith@mwsu.edu, Department of Counseling, Educational Psychology, and Special Education, Michigan State University, 512 Erickson Hall, 620 Farm Lane, East Lansing, MI 48824

Abstract

Research has found that elementary students face five main challenges in learning area measurement: (1) conserving area as a quantity, (2) understanding area units, (3) structuring rectangular space into composite units, (4) understanding area formulas, and (5) distinguishing area and perimeter. How well do elementary mathematics curricula address these challenges? A detailed analysis of three U.S. elementary textbook series revealed systematic deficits. Each presented area measurement in strongly procedural terms using a shared sequence of procedures across grades. Key conceptual principles were infrequently expressed and often well after related procedures were introduced. Particularly weak support was given for understanding how the multiplication of lengths produces area measures. The results suggest that the content of written curricula contributes to students' weak learning of area measurement.

Published in *Mathematical Thinking and Learning* 18:4 (2016), pp 239–270.
doi 10.1080/10986065.2016.1219930

Introduction

Measurement is a fundamentally important domain of school mathematics. Because it provides tools for the quantification and control of physical quantities, measurement has been essential to the development and practice of science and the design of technologies (Crosby, 1997). Instruction in measurement begins early around the globe, typically in the first year of schooling, and continues into the middle grades. The initial focus on spatial measurement (length, area, and volume) involves the coordination of two dimensions of students' experience—continuous space and discrete number. Measurement is highly practical mathematics, and the need to measure is intuitive to children. But research has indicated that students' understanding of measurement is weaker than for other mathematical domains in numerous countries (Smith, Males, Dietiker, Lee, & Mosier, 2013; Thompson & Preston, 2004).

This article focuses on area measurement—specifically, how it is presented in elementary mathematics curricula commonly used in classrooms in the United States. In contrast to length, measuring area involves a shift from the use of physical tools (rulers) to numerical computations (formulas) (Bonotto, 2003; Hino, 2002; Kordaki & Potari, 2002; Lehrer, 2003; Zacharos, 2006). The focus on computation via formulas continues for volume and other quantities that students explore in later mathematics and science work (e.g., torque = weight × distance). So area measurement represents an important transition in the teaching and learning of measurement more generally. Area measure also frequently arises in everyday activity and plays a foundational role in more advanced mathematics, from fractions to calculus. For these reasons, understanding area measurement is an important goal in elementary mathematics.

Area is the quantity of two-dimensional (2D) space enclosed in shapes with closed boundaries, whether they lie on a plane or nonplanar surface. Using an area unit (a smaller segment of 2D space), continuous space can be partitioned into equal parts; area measures are the number of area units that fill the space. Although the amount of space enclosed remains constant, the numerical magnitude of area measures varies in proportion to the size of the unit. Larger area units produce smaller area measures because larger units fill more space than smaller units. Area is conserved under many transformations and partitioning operations. Identical replicas of area units can be used to *tile* (cover) the space enclosed. A single area unit can be *iterated* through the space to generate a virtual tiling. Many shapes can serve as units of area, but squares have special status. They tessellate the plane and fill the same space uniformly on both dimensions, thereby supporting the multiplication of lengths to produce area measures. Both individual and composite units can be iterated; rows and columns of square units are important composite units. "Spatial structuring," the ability to visualize and locate composite units in rectangular spaces (Battista, Clements, Arnoff, Battista, & Borrow, 1998), facilitates counting by composites and motivates the multiplicative relationships in area formulas. But the composition of one length unit with another to generate one square unit differs from the multiplicative relationships that are typically emphasized in the elementary grades—the replication of equal groups and scalar multiplication (Greer, 1992; Lehrer & Slovin, 2014; Nesher, 1988).

Many elementary students—in the United States and other countries (Smith et al., 2013; Zacharos, 2006)—do not learn measurement well, and area measurement presents particular challenges. US grade 8 students' performance in measurement is weaker than any other content area on the National Assessment of Educational Progress [NAEP] and the Third International Mathematics and Science Study [TIMSS] (Thompson & Preston, 2004). Where US students are reasonably successful solving simple routine problems, their performance drops sharply on tasks that are nonroutine or require explanation. For example, about half of grade 4 and 75% of grade 8 students found the area of a polygon drawn on a grid by counting square centimeters (Blume, Galindo, & Walcott, 2007). But only about 25% of grade 8 students successfully determined the surface area of a rectangular solid; less than 20% produced the correct number of squares that covered a given region; and fewer than 10% found the area of figure depicted on a geoboard and then constructed another with the same area (Blume et al., 2007).

Although weak learning has been reported in numerous countries, the factors responsible remain unclear, making it difficult to know where to focus intervention efforts. In this article, we examine the evidence that the content of curricular materials contributes to the problem of weak and shallow learning. We analyzed how three U.S elementary textbook series present area measurement, with specific attention to how they address the key learning challenges documented in the research literature. Textbooks exercise a strong influence on the mathematics lessons taught to students, in the United States and other countries (Brown & Edelson, 2003; Hino, 2002; Remillard, Harris, & Agodini, 2014). But it is also true that textbook lessons are read, interpreted, and enacted differently by teachers in classrooms with students whose backgrounds, engagement, and participation also differ. Since written texts do not determine how teachers teach area measurement (much less how students experience it), limitations in their content, no matter how significant, cannot be the single cause of weak student learning.

From our results, we will argue that curricular content likely *contributes to* students' struggles to learn area measurement, and in two ways: Directly via limitations in their opportunity to learn and indirectly via messages to teachers about the procedural nature of area measurement. But the evidence of curricular contributions to students' challenges remains correlational and therefore circumstantial, pending more direct tests of curricular impact. We return to this central issue in the Discussion. In addition, our focus on curricular content and its limitations should be taken to indicate that curriculum revision is the most efficacious approach to addressing the documented challenges that students face with area measurement. The depth of teachers' knowledge and understanding of the content and students' learning of that content will remain an important factor as well.

Our analysis complements a prior study of the presentation of length measurement in the same curricular materials (Smith et al., 2013). In that study we found that students' access to key conceptual principles such as unit iteration was limited in its frequency and timing relative to closely related procedures. The materials focused primarily on *how* length is measured with little attention to *why* those procedures work and what to do in nonstandard situations (e.g., measuring lengths with "broken" rulers). But 2D space and area measurement introduce new challenges for students, such as units of different shape and size, 2D shapes with multiple spatial attributes (e.g., perimeter and area), and numerical methods for finding area measures. Given these differences, a careful analysis of how curriculum materials present area measurement was warranted.

Research on students' learning challenges

Researchers have reported some challenges that students face in understanding area measurement that parallel results for length (e.g., conservation of quantity and the nature of units), where other challenges are particular to area—the spatial structuring of rectangular space, understanding multiplicative composition, and distinguishing area and perimeter. Research has also shown that elementary teachers struggle with the same challenges (Baturo & Nason, 1996; Simon & Blume, 1994), but for reasons of space, we focus on research on students' learning.

Conservation of the area quantity

The understanding that the amount of enclosed 2D space does not change when shapes are moved or partitioned develops gradually over time. Piaget's early studies assessed the area conservation in two different ways (Piaget, Inhelder, & Szeminska, 1960). Children were presented with two equal-sized rectangular "meadows" with "houses" spread around in different ways and asked whether the amount of remaining green space was the same.[1] Not all young children saw equal areas, even after confirming that the number of houses was the same. For most nonconservers, the spread-out houses took up more space. Some who asserted equivalence for a few houses (≤ 5) changed their minds when the number increased (≥ 10). In a second approach, children were presented with identical 2×3 rectangles, marked in square units. After they affirmed the areas were equal, one square was cut from one rectangle and re-attached to form a "staircase" of six square units. When asked if the resulting shapes had the same area, some children denied equivalence, even after counting six squares in each shape. Recent work with grade 1 to 3 students using a related task has produced similar results (Lehrer, Jenkins, & Osana, 1998). However, with a meaningful sequence of tasks and informed teacher questions, grade 2 students have also moved beyond visual appearance to use decomposition and recomposition to see that

1 The underlying issue concerns whether subtracting two equal areas from two larger equal areas leaves two equal areas remaining.

different sized rectangular shapes can have the same area (Lehrer et al., 1998). Overall, children's reasoning about area invariance is often influenced by visual appearance—a challenge that can continue even into the high school years (Kospentaris, Spyrou, & Lappas, 2011).

Understanding and using units

Area measurement moves beyond qualitative comparison when area units are used to determine *how much* more space is enclosed in one shape than another. Research has shown that students' understanding of area units also develops gradually, as they work to coordinate different conceptual properties. Some elementary students try to adapt length units and/or tools to measure area (e.g., measuring rectangular area by sequential placements of a ruler) (Lehrer et al., 1998). Kamii and Kysh (2006) found that middle school students often counted geoboard pegs rather the square spaces between them to find the area of shapes. Elementary students can choose to mix different "units" to cover shapes, even when a sufficient supply of identical units is available (Clements & Sarama, 2009; Lehrer et al., 1998). When collections of different shapes are available, students often select units that resemble the target space (e.g., triangular units for triangular shapes) (Heraud, 1987; Lehrer et al., 1998). In covering 2D spaces, elementary students tend to avoid overlapping the boundary, even at the cost of not filling the space enclosed. Achieving the complete coverage of 2D space with identical units can be challenging as students must choose between covering the space and overlapping a boundary.

Spatial structuring of rectangles

Placing and counting individual area units becomes cumbersome for larger shapes. To find the area of rectangular spaces without counting each square unit, students must mentally organize the array of squares into *composite units*, typically rows or columns. This organization speeds counting and helps to motivate the standard area formula for rectangles ("length × width = area"). But research has shown that the ability to visualize arrays and isolate composite units develops gradually (Battista et al., 1998; Lehrer, 2003; Outhred & Mitchelmore, 2000; Owens & Outhred, 1998). Simply showing students correctly drawn arrays and/or boundaries partitioned into equal length units does not immediately allow them to produce such arrays. Primary grade children often struggle to fill rectangular space with identical units; the square "units" they draw may quickly lose regularity. Older students who successfully draw square units along the boundary can fail to fill the interior space uniformly. With support from boundaries demarcated in length units, spatial structuring is easier for smaller rectangles (e.g., 2 × 3) than larger (e.g., 7 × 9), and success with small dimensions does not always "scale up" to larger dimensions.

Multiplicative composition

Even if students can structure rectangular space into composite units, they face the additional challenge of relating such arrays to the multiplication of length and width (Stephan & Clements, 2003). Understanding area measurement requires a new meaning for multiplication, *multiplicative composition* (Confrey, 2012; Lehrer, 2003; Watson, 2010). Substantial evidence indicates that many students learn the area formula for rectangles without understanding how or why it works (e.g., Baturo & Nason, 1996). Given the choice of covering rectangles with physical units and counting them and measuring and multiplying lengths, many students choose the former (Kordaki & Potari, 2002; Nunes, Light, & Mason, 1993). Students may overgeneralize "length times width" as the meaning of area (Schifter & Szymaszek, 2003); produce area measures by additively combining lengths, widths, or heights (Lehrer et al., 1998; Zacharos, 2006); or simply answer questions about areas with numbers without units (Clements & Sarama, 2009). Some researchers have suggested that an early focus on numerical computation makes understanding area formulas more difficult (Zacharos, 2006).

"Confusing" area and perimeter

Research has also shown that elementary and middle school students often fail to distinguish area from perimeter as different attributes of 2D shapes. Woodward and Byrd (1983) asked grade 8 students which of five different-sized rectangular "gardens" with the same perimeter had the "largest possible garden area." Only about a quarter selected the rectangle with the greatest area; about 60% responded that the gardens were "all the same size." Some have suggested this error results from a more general intuitive rule, “same A (perimeter), same B (area)” (Stavy & Tirosh, 2000). When Chappell and Thompson (1999) asked middle school students to draw a figure whose perimeter was 24 units, many drew figures whose areas were 24-square units. Very few could illustrate or explain how different figures could have the same area and different perimeters. Even college mathematics majors can make judgments about area based on the side length of shapes (Kospentaris et al., 2011).

Framing the analysis

This study analyzed the presentation of area measurement in elementary mathematics textbook materials to assess the evidence that curricular content may contribute to weak learning. In other words, we assessed students’ opportunity to learn (OTL) and understand area measurement. Theoretically, our approach to OTL was grounded in the “knowledge-in-pieces” perspective on knowing mathematics and science (diSessa, 1993) and informed by the insight that the “curriculum” experienced by students results from a series of transformations from broad standards to “real” lessons. Analytically, three basic issues framed our analysis: content/knowledge (what area measurement knowledge is presented?), expression (how is that content presented on the textbook page?), and timing and sequence (when and in what order is that content presented?). Although careful, content-specific studies may operationalize these three dimensions in the different ways, each is fundamental to curricular OTL.

Curriculum and its transformations

The term “curriculum” has been defined differently in analyses of mathematics learning and teaching. Some researchers have used the term synonymously with “instructional materials”; others have proposed meanings that are considerably broader to include students’ experience of mathematics in classrooms (Remillard & Heck, 2014). But there is broad agreement that curriculum undergoes a series of transformations in form and content from policy documents that specify the nature and sequence of what should be taught, to the “written” forms given to teachers for classroom instruction, to the “intended” and "enacted" forms where teachers are directly involved as interpreters (Ball & Cohen, 1996; Remillard & Heck, 2014; Stein, Remillard, & Smith, 2007). The intended curriculum consists of teachers' plans for using different resources (e.g., tasks and activities) provided by textbooks and other sources to structure their lessons; the enacted curriculum is the lessons that actually occur when plans become the joint activity of teachers and their students. This study focused on the "written" curriculum—the instructional materials that teachers use to construct and teach measurement lessons. Although teachers seek out and use other instructional materials, textbooks continue to play a strong role in shaping teachers' intended and enacted mathematics curricula, in the United States and other countries (Grouws, Smith, & Sztajn, 2004; Hino, 2002; Kaur, 2014; Polikoff, 2015; Remillard et al., 2014; Stein et al., 2007). The analysis of textbook content, as teachers' primary curricular resource, has practical merit. Although textbooks do not strictly determine teachers' actual lessons, the intended and enacted curriculum are difficult to study because of the sheer number and challenge of sampling among teachers and classrooms. The content of widely used mathematics textbooks not only shapes intended and actual lessons in many classrooms, it also communicates what is important mathematically (and what is not) to teachers, beyond the specific content of lessons (Stein & Kim, 2009). This communicative role is important for area measurement, given the evidence that elementary teachers' understandings are often conceptually weak and procedurally focused (Baturo & Nason, 1996; Berenson et al., 1997; Simon & Blume, 1994).

Conceptualizing curriculum as undergoing a series of transformations helps to identify other factors than written curricular content that contribute to shaping students' mathematical experience. These factors serve as reminders that written curricula shape, but do not determine students' OTL. National curriculum and assessment standards influence how teachers allot time to particular topics (e.g., area measurement). Their judgments of what will appear on high-stakes assessments can shape their choices about instructional time and attention. The placement of specific topics in textbooks may also influence how much attention is paid to that content. U.S. elementary textbooks that contain more content than can be taught in a year typically place measurement lessons toward the back of their materials (Smith et al., 2013), decreasing the likelihood they will be taught.

Other factors concern teachers' instructional practices. The focus of measurement lessons is often procedural (Lehrer, 2003; Stephan & Clements, 2003). Once area formulas are introduced, much more attention is given to numbers and calculation than to 2D space (Baturo & Nason, 1996; Murphy, 2012; Thompson, Philipp, Thompson, & Boyd, 1994). Descriptive terms for describing attributes of 2D spaces can create challenges for clear and productive discourse. Comparative terms, (e.g., "bigger" and "larger") are often used without clear reference to specific spatial attributes. References to the "base" and "height" of rectangles can refer to either distances or algebraic expressions (e.g., "B" or "H") (Herbel-Eisenmann & Otten, 2011). And as noted, teachers' knowledge of area measurement is often limited (Baturo & Nason, 1996; Simon & Blume, 1994; Woodward & Byrd, 1983), restricting their ability to pose productive questions, explain conceptual principles, and respond effectively to students' ideas.

Knowledge-in-pieces

In contrast to perspectives that frame students' developing knowledge in mathematics and science as theory-like (Vamvakoussi & Vosniadou, 2010; Vosniadou, Vamvakoussi, & Skopeliti, 2008), the knowledge-in-pieces perspective views students' knowledge as loosely structured collections of *many* different elements (diSessa, 1993, 1996; diSessa & Wagner, 2005). On this view, repeated contact with mathematical or scientific phenomena leads learners to construct diverse and often fragmented knowledge that is closely tied to specific contexts, either physical or numerical. The resulting knowledge system is a web of elements that are loosely related. The system as a whole may not be internally consistent, and individual elements may or may not be "true" from a disciplinary perspective.

Oriented by this perspective, we used the research literature, mathematical discussions of area and its measure, and textbook materials to develop an exhaustive list of all "atoms" of knowledge that could contribute to students' understanding of area. Our approach differed from many prior analyses of OTL that have targeted broad topics and problem types (Floden, 2002). For example, TIMSS analyzed measurement OTL in terms of two broad topics: (a) attributes and units and (b) tools, techniques, and formulas (Schmidt, McKnight, & Raizen, 1997). From the knowledge-in-pieces perspective, judgments about

OTL should not attempt to identify and track only "core" knowledge elements, because learners' inferences may compensate for individual deficits in the system. Rather, assessments of OTL should be based on broader patterns of presence or absence of many different elements.

Analytic frame

Content/knowledge

In identifying knowledge elements for area measurement, we began with the traditional distinction between mathematics concepts and procedures. *Procedures* designate sequences of steps that are sufficient to solve particular types of problems. By contrast, *concepts* (and conceptual knowledge) have often been vaguely defined (e.g., Hiebert & Lefvre's [1986, p. 3] characterization of conceptual knowledge as "knowledge that is rich in relationships."). We defined concepts as the basic principles that underlie and justify measurement procedures, systems, and tools. This definition emphasizes the important role that conceptual knowledge plays in constraining procedural knowledge. The analysis of conceptual knowledge in studies of OTL should be informed by research on particular conceptual issues that students find difficult to learn, as curricula should acknowledge and address these challenges. But these two knowledge types were not sufficient. All mathematical content areas have conventions governing how mathematics is represented and expressed. *Conventions* (e.g., measurement systems and abbreviations) are human choices about how to write and communicate mathematics; they are not strictly speaking conceptual.

Expression

Any body of mathematics content can be expressed and communicated differently in textbooks. Most fundamentally, mathematics can be expressed either directly to students on the printed page or indirectly to them via their teachers. Text of the former type contains direct OTL where the latter is indirect OTL. Content expressed to teachers only becomes available for students to learn when and if teachers choose to express it. This distinction proved important to our analysis because communication to students via teachers was prominent in the primary grades (K–2). With respect to the form of expression, mathematics textbooks, as in all subjects, state elements of content—whether concept, procedure, or convention. Central concerns are the completeness and correctness of these mathematical statements. Similarly, all mathematics textbooks present tasks of different types to students and teachers. These can differ widely in terms of cognitive demands (Stein, Grover, & Henningsen, 1996); many are routine exercises and relatively few "real" problems (Schoenfeld, 1992). Because it is insufficient simply to state procedures, mathematics textbooks typically include worked examples that illustrate how the general sequence of steps applies to a specific problem. Research has shown that worked examples can effectively support students' learning of procedures (Sweller & Cooper, 1985). For young children especially, the illustration of mathematical procedures often takes the form of teachers' demonstrations.

Timing and sequence

The presentation of significant mathematics content areas in curriculum typically spans multiple years of schooling and within years, often multiple units or collections of lessons. Textbook authors must decide where to start and how to segment and sequence content over time. As recent research on learning trajectories has shown in careful detail, the order in which mathematical content is presented to students matters (Maloney, Confrey, & Nguyen, 2014). Decisions about starting points and sequential development are made centrally in many countries, but until recently, different states in the United States have expressed their decisions in individual state curriculum standards. Analysts of OTL studying specific mathematical content areas must decide where to begin and end their analyses and determine what content to include (and what to leave out). Textbooks may introduce and develop mathematical ideas informally before they are named and defined (e.g., "size" of objects before "area"), and connections are often drawn between specific content areas and those closely related (e.g., using rectangular arrays to develop area measurement and numerical multiplication). One particular issue of sequence concerns the order and proximity of procedures and related concepts, as research has emphasized the importance of clear linkages between them (Baroody, Feil, & Johnson, 2007; National Research Council, 2001).

Study focus

Our study addressed two broad questions: (1) How do current mathematics curricula present area measurement in the elementary grades, and (2) How well do these presentations address the learning challenges documented in research? Characterizing the presentation of area measurement content involves issues of timing, sequence, and most of all content/knowledge. Embedded in the second question is the assumption is that mathematics curricula can and should address the specific learning challenges identified in the research literature. More specifically, we asked:

(1) How much curricular attention is given to area measurement in the elementary grades?

(2) How do the curricula distribute attention to area concepts, procedures, and conventions, and how do they express that content in text?

(3) In what order do the curricula present major procedures and related concepts for measuring area, and do their sequences differ significantly?

(4) How well do the curricula address the learning challenges that research has shown that students face in learning area measurement?

Answers to the first three questions provide the basis for answering the last and we think, crucial question for students' OTL.

Method

We analyzed all area measurement content in the grades K–4 materials of three U.S. elementary mathematics textbook series. In this section, we justify our choice of textbooks and grades and describe how we identified area content in those materials. We also describe our framework for coding that content and how we applied it to the textbook pages that contained area content. Figure 1 presents the main steps in this analytic process.

The choice of curricula and grades

Because many elementary mathematics textbooks are used in U.S. classrooms (Stein et al., 2007) and our analysis was quite detailed, we were forced to choose some series over others. We chose *Everyday Mathematics*, 3rd edition (The University of Chicago School Mathematics Project, 2007) (henceforth, *EM*), which was written in response to the National Council of Teachers of Mathematics' (1989) *Curriculum and Evaluation Standards for School Mathematics*. We also selected Scott-Foresman/Addison-Wesley's *Mathematics* (Charles, Warren, & Fennell, 2008) (*SFAW*) as an example of a publisher- developed curriculum. When we began our analysis, *EM* and *SFAW* both commanded large shares of the elementary mathematics textbook market in the United States (Reys & Reys, 2006). Our third series, *Saxon Math* (Larson, 2004) (*Saxon*), was used less widely, but its teacher-directed approach and structure differed from the other two (see also, Remillard et al., 2014). All three series began their presentation of area measurement informally in grades K and 1. Area received substantial attention in grades 2 and 3; and by grade 4 all curricula had introduced area formulas for basic geometric shapes. Where attention to area measurement continues into middle school in the United States, the analysis of its presentation in grades K– 4 seemed sufficient to assess the adequacy of students' access to the foundations of area and its measure.

Locating the area measurement content

We first located all textbook pages in the grade K–4 materials of the three series that contained area measurement content. The primary sources of these pages were the teacher's edition and the student's workbook for each grade. The teacher's editions contained snapshots of the student workbook pages proposed for use in each lesson.[2] Two coders examined every page for content that called for reasoning about area. All pages with at least one instance of such content were included. Disagreements between coders about that criterion were resolved in discussions among the entire research team. This

2 EM also included a Student Reference Book and a collection exercises, Minute Math, for grades 1 through 4. We coded the all pages of these materials that contained area content.

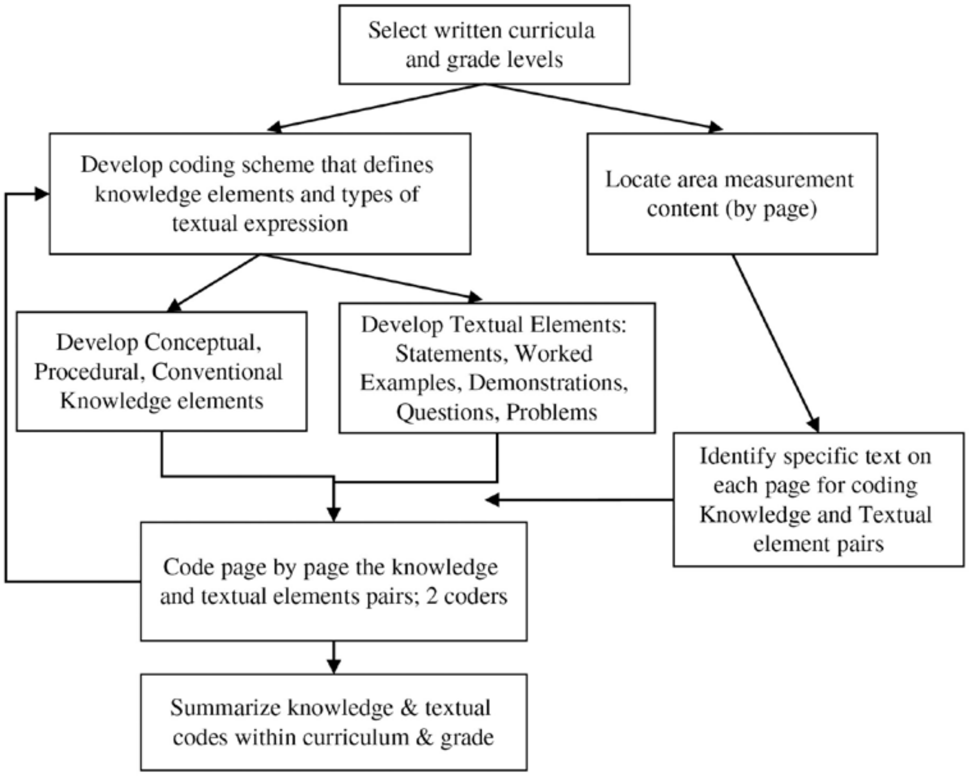

Figure 1. Major steps in the analysis.

process yielded many more textbook pages than would be the case if we had focused on area measurement lessons alone. For example, all three series distributed area-related tasks throughout their materials, and we included and coded all pages that contained at least one such task. We established procedures for deciding when instructional objects and representations made reasoning about area and its measure likely. As a result, we analyzed content that many mathematics educators might not see as "area," including partitioning shapes into equal-sized parts (fractions), reading and interpreting spinners (probability), and constructing and interpreting circle graphs (statistics). In these cases, we judged whether the content would lead students to reason about the relative size of parts and thus to judgments about area. This process produced a collection of pages that, as best we could determine, contained *every instance* of OTL for area measurement in each series.

Coding the identified pages

To code this content objectively, we developed a coding scheme that identified knowledge elements for area and its measure and textual forms for expressing that knowledge on the textbook page.

Knowledge elements

Following our analytic frame, we identified and expressed many elements of Conceptual, Procedural, and Conventional knowledge; the complete list is given in the Appendix. We distinguished three subtypes of Conceptual knowledge: (1) general properties of quantities and their measures (including area), (2) principles specific to area measurement, and (3) principles of area measurement for specific shapes (e.g., rectangles and triangles). Principles of the first subtype (e.g., Unit-Measure Compensation—smaller units produce greater numerical area measures) constitute what Lehrer has called “the child’s theory of measure,” (Lehrer, Jaslow, & Curtis, 2003). We also distinguished three subtypes of Procedural knowledge. Premeasurement includes qualitative processes for judging the relative size of 2D objects and shapes; these procedures involve no numerical reasoning. Numerical measurement groups procedures for generating area measures of single 2D shapes. Reasoning with area measures includes procedures for generating areas of more complex shapes or reasoning with two or more area measures (e.g., finding the area of a complex shape by decomposition). As their number indicates (n = 50), Procedures included more than formulas for computing the area of specific shapes. Conventional knowledge elements included definitions of standard units (e.g., the actual size of a square centimeter), abbreviations for units, and numerical conversion rates (e.g., 1 square meter = 10,000 square centimeters).[3] In all, this framework identified 93 different knowledge elements (35 Conceptual, 50 Procedural, and 8 Conventional).

3 That each square meter contains 10,000 square centimeters is not arbitrary once a meter has been defined in terms of centimeters, but the chosen length of the standard meter is mathematically arbitrary.

Types of textual expression

How mathematical knowledge is expressed and engaged by teachers and students also influences students' OTL (Remillard et al., 2014). Informed by our framework, we identified five textual forms that each textbook series used to express area content: (1) Statements, (2) Questions, (3) Demonstrations, (4) Worked Examples, and (5) Problems. To separate direct from indirect OTL we also coded all content for its location in either teacher or student materials. Statements express elements of Conceptual, Procedural, or Conventional knowledge of area measurement. We distinguished Full Statements of knowledge elements from Partial Statements that omitted major content relative to ours. This distinction proved particularly important in coding Conceptual knowledge. Worked Examples present some area-related task and its solution. Demonstrations are the "enacted" analogs of Worked Examples; they are the spoken or drawn presentations of area reasoning, typically procedures for solving tasks, from teachers or teacher-designated students. By definition, Demonstrations appeared only in teacher materials and Worked Examples in student materials, where Statements, Questions, and Problems appeared in both. Questions and Problems both pose area-related tasks. Questions require little reasoning, may be answered by one student, or are under teacher direction, where Problems require more reasoning and/or activity, and most, if not all students are expected to respond and work independently of teacher direction.

Coding tasks and queries as either Questions or Problems required considerable discussion and in some cases, recoding. Our distinction was based on three equally weighted criteria: (1) Does the task or query ask more than simple recall or observation, (2) how many students are expected to respond, and (3) does the curricular context suggest that students will work relatively autonomously or under teacher's direct guidance? Queries and tasks were coded as Questions if (1) only simple recall or observation was required, (2) one student could answer, and (3) teacher direction was indicated. They were coded as Problems if (1) cognitive demand exceeded simple recall or observation, (2) responses from most, if not all students were expected, and (3) relatively autonomous student work was indicated. When the three criteria did not align, outcomes on two of three decided the coding. This definition of Problem admittedly set a low threshold for cognitive demand; it included what Schoenfeld (1985) has called "exercises" that are fundamentally different from "problems"—tasks that are genuinely "problematic" for students. As a result, many tasks we coded as Problems were quite routine. The challenge of reliably coding the cognitive demand of mathematical tasks in curricular analysis has been cited by other researchers (Charalambous, Delany, Hsu, & Mesa, 2010).We return to this important issue in the Discussion.

Applying the coding scheme to the data

Coding the textbook pages involved assigning a knowledge element and textual form to each instance of area measurement content, typically expressed in a single sentence, clause, or problem. The result was a list of ordered triads (knowledge element, textual element, frequency) for each textbook page, where frequency was the number of times a particular knowledge-textual element pair appeared on that page. Two members of the research team coded each page. Each pair did so independently and then compared their results and resolved any differences in discussion. When coding disagreements could not be resolved within the pair, they were presented, discussed, and resolved in meetings with the full research team. To minimize bias, coding pairs were systematically varied across grades, and the textbook pages were distributed evenly so that all pairs coded content from all curricula. The research team included the project director (a faculty member), ten mathematics education graduate students, and two undergraduate pre-service teachers. Each held at least a bachelor's degree in mathematics; most had experience teaching K–12 or collegiate mathematics.[4] We entered all final codes into a two-way spreadsheet table (knowledge elements by textual forms) and aggregated the frequencies of each knowledge element and textual form for each series and grade.

4 One graduate student had experience tutoring in a community college math lab; a second had no mathematics teaching experience.

The lesson-level analysis

We separately identified the lessons that directly concerned area measurement in grades 2 through 4 to roughly measure overall attention area in each series—an issue obscured in the more detailed analysis described previously. All curricula presented area measurement informally in grades K and 1 before explicitly defining area in grade 2, so we limited this analysis to lessons in grades 2, 3, and 4. The first author carried out the analysis; lessons with 50% or more area measurement content were counted as "area lessons."

Results

We first examine the amount of curricular attention given to area measurement in each series (question 1). Next, we examine how the textbooks distributed area-related content across the knowledge types and textual forms identified in our framework (question 2). Third, we present a sequence of four groupings of procedures and related concepts that was common to all three series, noting differences when they are relevant (question 3). Finally, to assess how the textbooks addressed the learning challenges identified in research, we report the frequency and placement of Conceptual knowledge elements that addressed those challenges (question 4). With the exception of question 1, we report our results in terms of absolute and relative frequencies of knowledge and textual elements, as they have direct implications for students' OTL. Very low frequencies for particular knowledge elements and groups of elements would suggest that students would have limited access to those ideas.

Area measurement lessons

Table 1 presents two measures of curricular attention to area measurement in grades 2 through 4. The first row for each series presents the percentage and frequency of all "area-related" lessons, using our more inclusive criterion for coding textbook content. "Area-related" lessons included those whose central content required reasoning about area measures (e.g., partitioning of 2D shapes) but were not

Table 1. Two measures of curricular attention to area measurement.

Curriculum	*Lesson Type*	*Grade 2*	*Grade 3*	*Grade 4*
EM	Area-related	8% (10)	9% (11)	9% (11)
	Area	3% (4)	3% (4)	5% (6)
	Total	123	121	119
SFAW	Area-related	6% (10)	8% (13)	7% (12)
	Area	1% (1)	1% (2)	1% (2)
	Total	159	162	161
Saxon	Area-related	12% (19)	8% (12)	7% (10)
	Area	4% (7)	1% (2)	2% (3)
	Total	160	160	135

framed as "area lessons." The middle row presents the same information for lessons that explicitly concerned "area." The bottom presents the total number of lessons in each series at each grade.

Area measurement received modest attention in all grades and series; few lessons explicitly addressed the topic. The greater number of "area-related" lessons in all series and grades was primarily due to the focus on equi-partitioning in support of learning fractions. This analysis also revealed *where* area measurement lessons were placed in the yearly sequence of lessons. Most area lessons appeared in the second half of the year. This was always true for *SFAW*'s lessons. Most of *EM*'s grade 2 and 4 area lessons also appeared in the second half of the year. In grade 3, however, this relation was reversed; three of four lessons appeared in the first half of the year. *Saxon* placed one area lesson early in its grade 2 and 3 materials, but the most appeared in the second half of their texts. In all three series, most lessons in the first half of the year focused on base-10 number and operations.

The presentation of area by knowledge type and textual expression

Our main analysis was designed to reveal which specific concepts, procedures, and conventions appeared in each curricular presentation of area measurement and how they were presented. For the broadest overview, we report how the curricula presented area content by knowledge type, textual expression, and placement in teacher or student materials.

Table 2. Conceptual, procedural, and conventional knowledge for area measurement.

		Conceptual	*Procedural*	*Conventional*	*Total*
EM	K	0	100	0	20
	1	2.8	97.2	0	144
	2	6.5	87.6	5.9	185
	3	6.0	91.8	2.2	730
	4	8.1	89.2	2.7	732
SFAW	K	2.1	97.9	0	292
	1	0.2	99.8	0	492
	2	3.4	96.6	0	298
	3	10.3	88.1	1.7	477
	4	4.6	94.1	1.3	1039
Saxon	K	0	100	0	233
	1	1.3	98.7	0	240
	2	1.2	91.8	7.0	497
	3	2.5	93.6	3.9	514
	4	1.7	89.8	8.6	888

Knowledge type

Table 2 presents the total number of coded instances of area measurement for each series and grade and their percentage by knowledge type.

The total number of instances of area content increased in each grade and textbook series, except for *SFAW* grades 1 to 2.[5] All three series gave very strong attention to area measurement procedures. Procedural content never fell below 85%, and for the most part was more than 90%. This procedural focus was appreciably higher for area than it was for length (Smith et al., 2013). Conceptual content appeared in all series in grades 1 through 4 but generally accounted for less than 10% of all area content. *EM* devoted modest attention to Conceptual knowledge in Grades 2 through 4; *SFAW*'s attention was greatest in grade 3 but very modest in other grades; *Saxon* gave very little attention to conceptual content in any grade. Conventional knowledge generally appeared less frequently than Conceptual knowledge. Its

5 SFAW's large grade 1 total (n = 492) was due to the large number of instances (n = 208) where students were asked to compare the size of two or more shapes or objects.

greater frequency in *EM*, grade 2 was due to the frequent definition of standard area units; *Saxon*'s increased frequency in grades 2 through 4 was due to its focus on naming arrays.

Textual expression

Each series used all five textual forms to express area content, and generally did so at each grade. All three used Problems and Questions most frequently; taken together, they accounted for more than 80% of all area content in all grades. Problems were the more dominant form—usually accounting for more than 50% of all instances in all grades. However, *Saxon* consistently used Questions more frequently (see Table 3 below). Statements never accounted for more than 11% of area content in any grade or series. *EM* used Statements more frequently than the others (between 4% and 11% in grades 1 through 4); only once did *SFAW*'s or *Saxon*'s use of Statements reach 5% (*SFAW*, grade 3). Demonstrations and Worked Examples were used infrequently, generally accounting for less than 10% of total content when combined. However, *Saxon* used Demonstrations and *SFAW* used Worked Examples more frequently at each grade. These results closely parallel our findings for length (Smith et al., 2013).

Knowledge type by textual expression

The interaction between knowledge type and textual expression may also affect students' OTL. Some pairings of knowledge type and textual expression were more common than others. Table 3 presents the distribution of area content across knowledge type and textual expression. Cells with zero frequency have been left blank.

Not surprisingly, Procedural knowledge was most frequently expressed in Problems. As indicated above, *Saxon* used Procedural Questions more frequently than the other curricula. Overall, Conceptual knowledge was expressed in Statements, Questions, and Problems; *EM* used more Statements frequently to express conceptual content, and *SFAW* used more Problems in grades 3 and 4. *EM* and *SFAW* used Statements most frequently to express Conventional knowledge, where *Saxon* used Problems, Questions, and Demonstrations.

Table 3. Distribution of area content by knowledge type, textual type, curriculum, and grade.

		Conceptual					*Procedural*					*Conventional*				
		P	*Q*	*S*	*D*	*W*	*P*	*Q*	*S*	*D*	*W*	*P*	*Q*	*S*	*D*	*W*
EM	K						90	10								
	1	1		2			78	15	2	1	1					
	2	2		6			78	1	3	2	2		1	5	1	
	3	2	1	4			55	25	3	5	3			2		
	4	5		4		1	70	1	5	2	7			2		
SFAW	K		1				56	32		7	3					
	1						74	19	1	3	3					
	2	1	1	3			78	6	2	3	7					
	3	7	1	3	1		40	35	1	2	8			2		
	4	3	1	3			78	2	1	2	9		1	1		
Saxon	K						38	54		8						
	1		1				69	15	1	11	2					
	2			1			50	28	3	11			5		2	
	3	1		1			39	39	1	14			3			1
	4		1	1			53	18	2	17		3	1	1	4	

Placement in teacher and student materials

Given the strong procedural focus in all three series, the question of where conceptual content was presented becomes central, given its relatively infrequent appearance. To address this question, we first consider the distribution of area content across teacher and student materials. Figure 2 presents that distribution for the combined percentage of Problems, Questions, and Statements.[6]

Generally, each series shifted from expressing area content in teacher materials in the early grades, especially grade K, to student materials in later grades. This trend was clearest for *EM*. *Saxon* followed the same general trend except at grade 1, but then with rough parity (teacher vs. student materials) in grades 3 and 4. *SFAW* presented more area content in its teacher materials at grade K, but only at that grade. The proportion of area content in its student materials shifted abruptly in grade 1. Overall, these results indicate a shift from teacher presentation in the primary grades to more direct presentation to students in later grades.

6 Worked Examples, by definition, appeared only in student materials, where Demonstrations (by the teacher or teacher-designated student) appeared only in teacher materials.

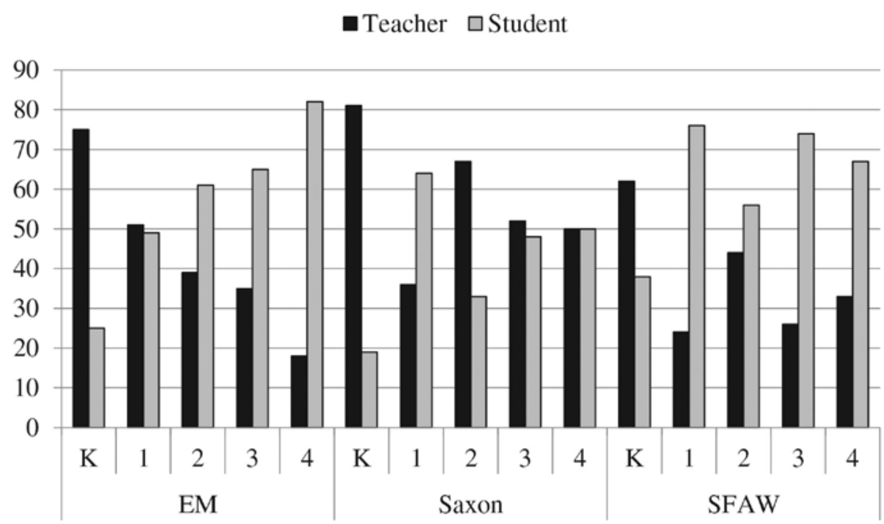

Figure 2. Area measurement Problems, Questions, and Statements (combined) in teacher and student materials by curricula and grade.

To return to the issue of the expression of Conceptual knowledge, most Conceptual Problems appeared in student materials, and most Conceptual Questions and Statements appeared in teacher materials.[7] But significantly, Statements appeared more frequently in teacher materials in most curriculum-by-grade cases, and this was particularly true for Conceptual Statements. In 6 of the 11 textbook-grades that included Conceptual Statements, *all* such Statements appeared in teacher materials. Overall, Conceptual Statements appeared twice as frequently in teacher materials. Placing so much conceptual content in teacher materials means that students' access to that content depends on their teachers' expression of those ideas. Should teachers choose not to (and we have seen reasons why some may not appreciate the importance of conceptual content), their students would likely have no access to it—at least in their classrooms. This danger is exacerbated by the large number of Conceptual elements (n = 35) and their generally sparse appearance overall (Table 2).

That the curricular materials typically did not provide students with direct access to statements of conceptual principles is not to suggest that simply stating those principles would be sufficient support

7 This result reflects, at least in part, how we defined and coded Problems and Questions.

for students to learn them. From a curricular perspective, their placement primarily in teacher materials seems relevant for students' opportunity to learn—hence our report of this result. From a broader learning perspective, we recognize that how teachers see and draw students' attention to conceptual principles that are implicit in their reasoning about area also can have a significant, if not greater impact on students' learning.

The sequence of area measurement procedures and related concepts

We now examine the specific procedures and concepts emphasized in each series' presentation of area. This analysis highlights specific problems in how conceptual content related to key procedures was introduced. Generally, we found a sequence of four conceptually related groups of procedures in all curricula, with modest differences in timing and focus. The first group consisted of procedures involving only the *qualitative judgment* of relative size among objects and shapes ("pre-measurement" in our knowledge framework). The second involved procedures for physically (or mentally) placing and enumerating *units* to generate area measures. The third group concerned the construction, interpretation, and enumeration of *rectangular arrays*—that provide conceptual grounding for area formulas, and the fourth included *formulas* for basic geometric shapes (e.g., rectangles, squares, parallelograms, and triangles) that generate area measures from length measures.

Table 4 presents the relative frequency of each procedural group in each series and grade. The *Other* column lists the percentage of procedural content *not* included in the four groups. As before, cells with zero values have been left blank.

The four groups described collectively accounted for more than 70% of the procedural content at all grades and curricula, except for *EM* grade 4.[8] In 10 of the 15 textbook grades, they accounted for more than 80% of all procedural content. So not only did these series focus

8 In grades K through 2, the most frequent "other" procedure was Partition in Half with Support; it accounted for more than half of all "other" procedural codes for those grades in all curricula. In grades 3 and 4, the most common "other" procedures involved reading and making circle graphs and spinners (in probability lessons), solving problems involving sums and differences of area, along with Partition In Half with Support.

Table 4. The distribution of four groups of area procedures by curriculum and grade.

		Procedural Group				
Curr	*Gr*	*Qual. Judgment*	*Units*	*Arrays*	*Formulas*	*Other*
EM	K	100				
	1	52.9	21.4			25.7
	2	21.0	63.6	0.6		14.8
	3	18.8	31.6	29.4	2.7	17.5
	4	9.5	21.9	13.0	16.5	39.1
SFAW	K	61.5	28.0			10.5
	1	85.7	7.9			6.3
	2	82.3	13.2			4.5
	3	32.3	31.0	5.2	3.8	27.9
	4	7.4	11.2	34.7	24.3	22.5
Saxon	K	63.1	36.9			
	1	76.3	13.9			9.7
	2	52.9	20.2	15.6	2.4	9.0
	3	40.1	23.7	10.4	5.1	20.6
	4	45.9	5.5	21.3	12.4	14.8

heavily on procedures, there was a strong shared structure among the specific procedures they presented. We now describe those procedures in more detail, explain how they are conceptually related, and discuss the conceptual content that was intimately related to each group.

Qualitative judgment

As we also found for length (Smith et al., 2013), all curricula began their presentation of area by asking students to make judgments of the relative size of 2D shapes. From early work to order shapes and objects by size, qualitative judgment extended in later grades to partitioning shapes into equal sized parts. As Table 4 shows, Qualitative Judgment procedures appeared in the grades in all series; they were significant content in *Saxon* at every grade and in *SFAW* for all but grade 4. Visual comparative judgment, the procedure for judging relative size of objects or shapes, accounted for more than 50% of all procedural content in grades K and 1 in all three series and in grade 2, for significant content in *SFAW* (55%) and *Saxon* (27%). All series frequently asked students to partition shapes into two equal parts (Partition in half),[9] but gave only modest attention to partitioning into three and six parts in later grades. *SFAW* and *Saxon* introduced Partition in half in grade K; all three series addressed it in grade 1; and *SFAW* and *Saxon* continued to do so in grades 2 and 3. All series presented Visual comparative judgment and Partition in half primarily in Problems and Questions.

The ability to make qualitative judgments of relative size (order and equality) depends on understanding that the area of shapes and objects does not change when they are moved or divided into parts. If, for example, students think that changing the orientation of shapes (e.g., rotating a rectangle from a horizontal to a vertical position) can change their 2D size relative to other shapes, those students' ability to order those shapes from least to greatest area becomes unstable. We coded for Conservation under motion and Conservation under partitioning and found these conceptual principles were rarely mentioned. *Saxon* addressed only the latter and did so once (in grade 4); *SFAW* mentioned both (four total instances); *EM* mentioned both but focused on Conservation under partitioning (16 of its 18 instances). Strikingly, all of *EM*'s 18 instances appeared in grades 3 and 4, well after students were asked to make judgments of relative size. As we have indicated earlier, noting the number and placement of statements of these principles is not a suggestion that more frequent, timely, and direct statements would be sufficient to support students' learning, only that their frequency and placement in the examined materials did not seem optimal.

Placing and enumerating area units

Procedures for placing and/or enumerating physical units, nonstandard and standard, were numerous and diverse. This group included procedures for covering regions with space-filling units, without counting them (covering only) and procedures for covering regions

9 The curricula often provided visual support for partitioning in half via folding actions and dotted partition lines, especially in the early grades. We separated instances where support was provided from those where it was not and only counted the latter in this procedural group, as only those instances required qualitative judgments of area.

with replicas of the same unit and counting them.[10] Due to the conceptual importance of unit iteration in measurement, we distinguished instances where a *sufficient* supply of units was provided from those where the supply was *insufficient*, as only the latter requires *iteration*. This group also included procedures for *drawing* regions of a given area by placing units on paper and tracing a boundary around them and procedures for counting the number of square units enclosed in regions presented on square grids. Finally, the group included procedures for estimating the area of shapes, since research has shown that estimation often involves selecting and mentally iterating units (Joram, Subrahmanyam, & Gelman, 1998).

Given their number and diversity, it is not surprising that Units procedures appeared in every textbook series and grade (except for *EM*, grade K). *SFAW* and *Saxon* presented Unit procedures in grade K, primarily covering procedures. By grade 1, all three series did so, sometimes with different objects, sometimes with the same but with frequency (< 10% of all procedures). But only *EM* (in grade 2) addressed covering regions with *insufficient* units. Measuring area by covering and counting appeared in grades 1 and 2, with modest frequency. Drawing regions of some given area was infrequent in all series and grades (< 2% of all procedures). Counting units enclosed in regions drawn on grids was more common, especially at grades 2 and 3.[11] All three series worked on estimation in either grade 2 or 3 (> 10% of all procedures). Not surprisingly, Units procedures were primarily presented in Problems in all curricula.

In our framework, two different types of Conceptual knowledge underlie and justify Units procedures. Three were definitional in nature—Definition of area, Meaning of area measure, and Area is not perimeter; five others expressed properties of area units—No ruler for area, Area measure requires area units, Standard and non-standard units, Unit iteration, and Unit-measure compensation. Our initial definition of area was, "Area quantifies the space enclosed in a region." But this definition, which presents area as a continuous quantity, did not fit well with some definitions in the textbooks that presented area as discrete quantity (e.g., area is the number of square

10 We distinguished procedures for covering with different units (e.g., all pattern blocks) from procedures for covering with the same unit.

11 This work expanded to include regions with fractional units, primarily in grade 4.

Table 5. Distribution of definitional knowledge elements by curriculum and grade.

Knowledge Element	*EM*				*Saxon*				*SFAW*			
	1	*2*	*3*	*4*	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*
Definition of Area		5	9	8		2				5	6	2
Meaning of Area Measure	1		2	2			3	1			5	7
Area is not Perimeter		4	1	1					1	2	6	2

Note. No instances of any element appeared in grade K in any curriculum.

units that fill the space). Given this mismatch, we added a separate knowledge element, Meaning of area measure, to capture this discrete conception—area measures as counts of square units enclosed. We considered Area is not perimeter as definitional because it distinguished two attributes of simple closed curves—the space enclosed and the length of the boundary—that many students struggle to hold separate. Table 5 shows the frequency of these knowledge elements by series and grade.

All three series defined area in grade 2; *EM* and *SFAW* continued to do so in grades 3 and 4. They primarily addressed the Meaning of area measure in grades 3 and 4, after defining area in continuous terms. The continuous nature of our definition fit well with *EM*'s and *Saxon*'s presentations. *SFAW*'s definitions often mixed continuous and discrete elements (e.g., "the area of the shape is the number of square units inside the shape"). Table 5 shows two important patterns in how these curriculum materials defined area: All began with a continuous definition, and most definitional content appeared *after* students began to cover and count area units. *EM* and *SFAW* explicitly distinguished perimeter and area; *Saxon* never addressed the issue.

In contrast, the presentation of the conceptual properties of area units was quite sparse in all three series. Table 6 presents the frequencies of these five elements; only *SFAW* presented this content in grade K.

Although research has shown that some students think that rulers can be used to measure area (Lehrer et al., 1998), no textbook explicitly stated that no ruler-like tool exists for measuring area. Of the other four elements, *Saxon* addressed only two (*Unit iteration* and *Unit-measure compensation*) but not at grade 2 when area was defined. *EM* and *SFAW* addressed each of the four but infrequently and for *SFAW*, generally not in grade 2. *SFAW* addressed

Table 6. Distribution of conceptual elements for area units by curriculum and grade.

Knowledge Element	*EM*				*Saxon*					*SFAW*			
	1	*2*	*3*	*4*	*1*	*2*	*3*	*4*	*K*	*1*	*2*	*3*	*4*
No Ruler for Area													
Area Measure Requires Area Units		3	2	3							3	1	2
Standard vs. Nonstandard Units	1	2										1	1
Unit Iteration	1	2	2	3			2		2	1			
Unit-Measure Compensation	1		1	2	3				1			3	

Unit iteration in grades K and 1, before area was defined. Overall, 11 of the 13 instances of Unit iteration were Statements, five of which were Partial Statements that reminded students to avoid gaps or overlaps between units. *SFAW* mentioned Unit-measure compensation four times in five grades, when it did so 73 times in its presentation of length in grades K–3 (Smith et al., 2013). *EM*'s attention to these ideas was most frequent and distributed across grades; it addressed three of four elements in grade 2 when area was defined. Consistent with the general pattern reported, Statements of area properties appeared far more often in teacher materials (n = 19) than student materials (n = 6).

These results show limited attention to the conceptual properties of area units in all three textbook series. The attention that was given was not located in grade 2 when area was defined; it appeared either earlier or later. So consideration of the meaning of area was separated from consideration of the properties of area units. Generally speaking, students' access to statements of the properties of area units was indirect; it depended largely on teachers' voicing those properties. Given their importance, attention to Unit iteration and Unit-measure compensation was very limited.

Arrays

Attention to the geometric properties of rectangular arrays can support students' understanding of area formulas, as well as the operation of multiplication. We identified four procedures, one each for making, interpreting, and enumerating arrays and one for determining the area of rectangles from the row and column structure.[12] All three series included procedures for dealing with arrays, but with different timing. *Saxon* introduced them early, in grade 2; *EM* devoted almost 30% of its grade 3 procedural content to arrays; *SFAW* did not address the topic seriously until grade 4. Each made, interpreted, and enumerated arrays of *contiguous* elements (typically squares) in support of area measurement, as well as *noncontiguous* elements (e.g., dots) that related more directly to whole number multiplication, with roughly equal attention to each type. *EM* gave the greatest attention to using the row and column structure of rectangles, but still infrequently (≤ 3% of all procedures at any grade); *Saxon* never did. All three series worked with arrays, primarily in grades 3 and 4, but gave very little attention to the composite unit structure of those arrays.

Three Conceptual elements are directly related to array procedures, Definition of arrays, Arrays and computation, and Spatial structuring of rectangles. The first defined arrays as patterns of objects organized in rows and columns. The second concerned how the row and column structure supports enumeration by skip-counting or multiplication. Informed by Battista and colleagues' (1998) analysis, Spatial structuring stated that rectangles can be filled with iterated composite units, where the number of squares in rows and columns match the number of length units of the respective sides, thus explicitly linking length units and area units. All three curricula defined arrays, but only *EM* located its four definitions in grade 3 when arrays were introduced. *Saxon* and *SFAW* placed 20 of their 21 definitions in grade 4 after work on arrays began. Arrays and computation received little attention. *EM* and *SFAW* addressed it in the grade that emphasized work on arrays; *Saxon* never did.

Spatial structuring of rectangles appeared only three times across textbook series and grades, twice in *EM* (once each in grades 3 and 4) and once in *SFAW* (grade 4). Both *EM* instances were Statements (one Full and one Partial) accompanied by illustrations of rectangles that were partially filled with square units. One diagram showed a single square unit falling into place in a partially filled row in a rectangular

12 We placed this procedure in the arrays group because it involved reading the geometric structure of rectangular arrays and eventually counting of squares, not multiplying lengths.

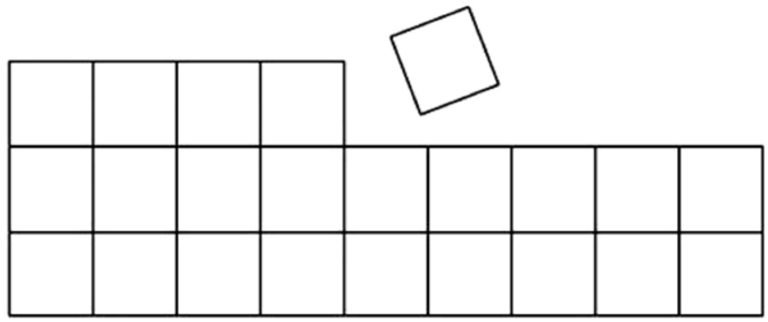

Figure 3. An example of spatial structuring of rectangles, EM, grade 3, Teacher Lesson Guide, p. 213).

array (Figure 3). The text directed teachers to explain that finding the area of the floor involved deciding how many rows of squares were needed. This Statement, with the accompanying diagram, combined the filling of rectangular space with single units, rows as composite units, area as the measure of that filled space, and implicitly at least, how rows and columns of squares related to side lengths. The singularity of this example indicates the challenge that curriculum developers face in attempting to represent the motion of area units in print materials. The one *SFAW* instance in grade 4, a Teacher Statement, drew students' attention to the correspondence between the number of squares in a rectangle's rows and columns and the length of its sides, but was silent on the iteration of composite units to fill the space.

Overall, conceptual support for understanding arrays and linking them to area measurement was limited. Two series defined arrays after students were asked to draw and interpret them—continuing the pattern of late appearance of conceptual content relative to procedures. Attention to the linkage between array structure and computation was limited and late, and support for visualizing the structure of rectangular space in composite units was virtually nonexistent.

Area formulas

Procedures for computing area measures from measures of length first appeared in grade 3 in all three series and became significant in grade 4, when all three devoted more than 10% of their procedural content to work with formulas. The exception was *Saxon*'s brief introduction of the formula for rectangles in grade 2. All curricula gave the greatest attention to the Formula for rectangles; it accounted for 80% (412 of 514) of all formula procedures. The Formula for squares received very modest attention in all curricula in grades 3 and 4. Only *EM* developed formulas for other shapes (parallelograms, right triangles, and general triangles) and began work on the area of circles by approximation in grade 4. Across textbooks and grades, the Formula for rectangles was frequently stated but was even more frequently expressed in Problems and Questions. By grade 4, numerous Problems were multistep, where the computation of rectangular area served some other mathematical purpose.

Given the extensive evidence that students struggle to understand how the multiplication of lengths produces area measures (e.g., Clements & Sarama, 2009; Hino, 2002; Zacharos, 2006), we separately examined each grade 3 and 4 lesson that introduced the area formula for rectangles. All of those lessons moved quickly from *counting* squares, in individual or composite units, to *multiplying lengths*—with the objectives of establishing that multiplying lengths produces the same result as counting squares and the former is more efficient. No lesson provided a rationale for *why* the multiplication of length and width produces a count of square units. In particular, none pointed to the one-to-one correspondence between the number of length units that made up the rectangle's length and the number of squares in each row.

None of the three textbook series seriously addressed the conceptual issues that underlie how the multiplication of lengths produces area measures. The Spatial structuring of rectangles was scarcely mentioned. No series explicitly addressed the fact that the multiplication of two lengths produces a new quantity (an area), not another length. We also extended this search to Multiplicative composition involving area—that the multiplicative composition of an area and any other quantity produces a new quantity (e.g., volume) that is not an area. Again, we found no instances. Overall, conceptual support for understanding *why* the formulas for area "work" was virtually nonexistent.

Curricular attention to learning challenges

Our last research question asked how the curricula addressed the five main challenges that research has shown students face in learning to measure area: (a) conservation of area as a stable quantity, (b) understanding area units, (c) spatially structuring rectangular arrays, (d) understanding the multiplicative composition of area measures from lengths, and (e) distinguishing area from perimeter. Here we review our results concerning the frequency and placement of Conceptual knowledge elements that explicitly address these issues. Where work with related procedures (e.g., covering regions) could provide relevant experience for students in grapple with those challenges, the impact of such experiences is difficult to assess. Existing research evidence discussed earlier suggests that students can learn to apply procedures without knowing why they work.

For the first challenge, all curricula asked students in the primary grades to reason about the relative "size" of shapes and objects, but did not define "size." Area was explicitly defined as a continuous quantity in grade 2 in all three series, but the stability of that quantity under transformations was not addressed frequently or when area was defined. The two series (*EM* and *SFAW*) that addressed conservation under motion and partitioning did so in grades 3 and 4, well after area was defined and area units were deployed.

For the second challenge, two Conceptual elements, Unit iteration and Unit-measure compensation, express most of the key properties of area units. Unit iteration combined three key properties: (a) Units must be identical, (b) their placement must exhaust the space to be measured, and (c) they cannot overlap the boundary or leave gaps. Given its importance, Unit iteration appeared infrequently—less often for area than for length (Smith et al., 2013), and as with length, often with the single focus of avoiding gaps and overlaps. Unit-measure compensation appeared far less often for area than it did for length (n = 11 vs. n = 104 total instances in all series) and often well after students were asked to cover and count with units. Overall, the conceptual content that addressed the nature of area units was sparse and located after students began to cover and count physical units.

Third, where the ability to impose the regular row-by-column structure of square units onto rectangular space is central to understanding area measurement (Battista et al., 1998), we found weak attention to this challenge. Two-dimensional arrays were defined in all three series, but with little attention to either the iteration of composite units (rows or columns) or the relationship between the side lengths and the number of squares in a row (or column). So little support was provided to develop the imagery of filling rectangular space with rows (or columns) of squares. At the end of our analysis, we also searched for situations that asked students to complete partially filled-arrays, since these could also support the development of students' abilities to visualize completed arrays. But we found no instances. Overall, these curriculum materials provided little support for developing students' cognitive ability to structure rectangular space. Fourth, there was no curricular attention to Multiplicative composition in producing area measures from lengths. That principle was never mentioned, and activities that could draw students' attention to the relationship between length and area units were not part of lessons introducing the area formula of rectangles.

With the last challenge, *EM* and *SFAW* gave some attention to distinguishing area from perimeter in two-dimensional shapes. Both addressed the issue with Statements or Problems in grade 2 close to where they defined area and again in grades 3 and 4. *Saxon* did not address the issue at any grade. Whether the few statements that distinguished these quantities and the tasks that called on that distinction constitute sufficient support for students is unclear. But in contrast to other challenges discussed, *EM* and *SFAW* authors did make explicit efforts to address this challenge.

Finally and more generally, we found it striking that the three series never drew teachers' or students' attention to the strong set of shared conceptual principles that underlie both length and area measurement. When key conceptual principles (e.g., Unit iteration and Unit-measure compensation) were discussed, they appeared *de novo*, with no reference to their identical role in structuring length measurement. Put somewhat differently, area measurement was presented as if it had no relation to length measurement (or the measure of any other quantity).

In sum, explicit and timely support for addressing key challenges for learning area measurement was evident on for one issue (distinguishing area from perimeter) and there only in two series. In some cases, those challenges did not seem to have drawn authors' attention; in others, the attention was given but late relative to related work with procedures. Key Conceptual elements that directly addressed these issues appeared infrequently and without connection to their role in length measurement.

Discussion

These results show that all three textbook series devoted quite modest attention to area measurement through grade 4, as generally, fewer than 5% of each year's lessons directly addressed the topic. All three focused very strongly on procedures, even more so than for length (Smith et al., 2013). They presented a similar sequence of procedures and related concepts that began with qualitative comparison, followed by covering shapes with area units and counting them, exploring 2D arrays, and finally introducing formulas for different geometric shapes, beginning with rectangles. In general, these materials did not sufficiently articulate the conceptual knowledge that would support students' efforts to work through known learning challenges. There was weak attention to the conservation of area, the properties of area units, and to distinguishing perimeter and area, and essentially no support for developing students' ability to structure rectangular space or understand area formulas. When conceptual principles relevant to these issues were expressed, they typically appeared well after students were expected to use the procedures they justify. Statements of conceptual principles were often left to teachers to express rather than made directly to students. No series explicitly linked to and built on prior work on length measurement to explore and develop area measurement.

These limitations correlate well with the research showing that U.S. students' understanding of area is limited to routine tasks and conceptually shallow (Blume et al., 2007). Although curricular deficits are not the single cause of weak learning, the observed patterns of sparse and late attention to conceptual principles and their alignment to known learning challenges suggests that curricular content likely *contributes* to the problem of weak learning of area measurement—directly via limited student OTL and indirectly by failing to educate and direct teachers. This stance toward the contribution of curricular content to student learning aligns with the positions expressed by other researchers who have carried out similar analyses on different mathematical topics (Charalambous et al., 2010; Ding & Li, 2010; Li, Deng, Capraro, & Capraro, 2008). However, the evidence presented in this study remains correlational—if quite consistently so—and therefore circumstantial. More compelling evidence testing for curricular effects on student learning is needed. Such evidence could come from studies comparing the character of students' knowledge and understanding after experience with quite different curricular presentations of area and its measure. Unfortunately, it seems unlikely that such studies will be possible in the United States, due to the current absence of fundamentally different presentations of this content.

Is this problem limited to the United States?

Given the evidence indicating that the content of one country's elementary mathematics textbooks contributes to its students' difficulties learning area measurement, one important question concerns the generality of the linkage. Is this linkage specific to the U.S. context, perhaps because of particularities of its curriculum materials, or is there reason to think that curriculum materials may contribute to students' difficulties with area measurement in other countries as well? At present, we have only a partial answer to this question. We know that textbook content shapes the mathematics lessons that teachers present to their students in many countries other than the United States, whether they follow their textbooks to the letter or design and use parallel replacement materials (Hino, 2002; Kaur, 2014). We also know that researchers in numerous countries have identified weaknesses in their students' understanding of area, especially with respect to computational formulas (Nunes et al., 1993; Owens & Outhred, 1998; Zacharos, 2006). But we know little about the nature of curricular presentation of area measurement in different countries and how they align with their students' learning successes and struggles.

Where there is evidence that area is introduced at different ages/grades in different countries (e.g., in grade 4 in Japan [Hino, 2002]), the central question of *how* area measurement is presented, particularly with respect to common learning challenges, remains open. Given the wide recognition that area is an important topic in elementary mathematics, some cross-national curricular analysis is warranted. Since research has shown that many interrelated ideas are involved in understanding area measurement, we suggest that such studies be conducted in a relatively fine-grained manner and hope our knowledge and textual frameworks may be useful resources for such work. As other have argued (Li et al., 2008), these curricular presentations could be compared to national profiles of student performance in those countries to assess the evidence that curricular content aligns with those profiles (or not).

Implications for curriculum design

These results have implications for the design of elementary mathematics curricula that could more effectively support the teaching and learning of area measurement. First, although the optimal level of curricular attention to conceptual principles is difficult to establish, the level of attention in the materials we analyzed is inadequate, in two important respects. Important concepts (e.g., Unit iteration, Spatial structuring, and Multiplicative composition) were presented so infrequently that they may be effectively invisible in teachers' enacted lessons and students' experiences. Their infrequent expression would be exacerbated in the classrooms of teachers who do not understand area measurement well themselves, as they would not appreciate the role that conceptual knowledge plays in justifying and constraining procedures. They would certainly not be in a position to state and explain their importance to students.

In addition, when conceptual principles (e.g., Conservation under partitioning) are addressed well after the procedures justified by those principles are introduced and used, connections between conceptual and procedural knowledge are more difficult to grasp, by students and teachers. This pattern of "curricular distance" seems contrary to the position that clear linkages between procedural and conceptual knowledge are important in teaching and learning mathematics (Baroody et al., 2007; National Research Council, 2001). For example, instead of introducing the procedure for covering regions with nonstandard units first and Unit iteration later, the task of placing area units to cover 2D shapes should be seen as the best context for considering how identical units, exhausting space, and respect for boundaries inform "correct" placement. Addressing relevant conceptual principles when procedures are first introduced could support the productive disposition toward mathematics generally that procedures can and should be justified.

Second, the conceptual separation of area measurement from length measurement in curricular materials should be reconsidered, because it is mathematically problematic and educationally harmful. The absence of explicit connections between the two suggests that they are mathematically unrelated. Failing to highlight the shared conceptual core also means that teachers lose the opportunity to augment their own understandings and are also much less likely to build on students' prior understandings of length measure in their area lessons. Positive transfer from length to area measure is much more likely if curricular materials draw explicit attention to the tight conceptual connections between them. In this way, students are more likely to learn something about measurement per se, over and above learning how to measure a particular spatial quantity.

Third, more attention should be given to supporting the development of students' visualization skills. Examining the *motion* of units, individual and composite, and of line segments sweeping through 2D space may help students to spatially structure rectangular space and understand how the multiplication of lengths produces area measures. When we looked separately for explicit representations of motion in the textbooks, we found only a few isolated attempts. Here digital materials have promise as motion can be represented directly. But even within the confines of the paper medium, greater curricular attention can and should be given to support students' 2D visualization of space and space filling. Students could be asked to iterate rows or columns as composite units and complete partially filled rectangular arrays (see, e.g., Battista, 2012; Outhred & Mitchelmore, 2000). More attention could also be given to relating square area units to length units. As one example, *EM* depicted a paint roller "sweeping out" rectangular space in its *Teacher Reference Manual* for grades 3 to 5 but never included this representation in their teacher lesson guides or student materials. Since there is some evidence that sweeping line segments through 2D space can support students' understanding of how multiplication generates area measures (Kobiela, Lehrer, & Pfaff, 2010), more curricular attention to this relationship is warranted.

Limitations

One clear limitation of this study is its sample of three textbook series used in one country. In the United States, where so many different textbooks are used (Stein et al., 2007), nontrivial differences are present among at least some of them (Stein & Kim, 2009). Where the analysis could have included other texts, from the United States or other English-speaking countries, our analysis was quite costly in time and human effort. To address this limitation to some degree, we analyzed some other US elementary textbooks in a more cursory manner and found many more similarities than differences in content and structure. That said, these additional efforts do not remove the need to examine the curriculum/learning relationship in an international framework; they reinforce it.

Second, this analysis and the analytic frame that produced it sheds little light on the crucial issue of the cognitive demand of tasks related to area measurement. Where we collectively found that most of our coded Problems appeared most often to be exercises, not problems, for students at that grade level, we could not find principled ground for declaring that any given task was (or was not) "problematic" for students who were asked to solve it. What our analytic method revealed was simply whether an appropriate solution procedure had been introduced prior to the posing of the task and how much prior. We acknowledge that the problem of assessing the cognitive demand of mathematical tasks across students is challenging, given that "problems" are relationships between tasks and students' orientations and capabilities, and we see evidence of that challenge in other curricular analyses. For example, Stein and Kim (2009) have reported differences in cognitive demand in mathematical tasks in two U.S. elementary mathematics curricula, one of which was *EM*, but they did not describe their coding methods in any detail.

Third, it is worth restating that curricular analyses cannot replace studies of how area lessons are actually enacted by teachers, as the "enacted curriculum" more directly shapes students' experience and learning. For that reason, careful studies of how elementary teachers present and develop the topic of area measurement are needed (with complementary attention to how they understand it). One important focus for studies of instructional practice is how teachers present (express, delete, and/or supplement) the conceptual content present in their written lessons, including what they attend to in their curriculum materials, how they interpret what they notice, and how they make subsequent instructional decisions (Brown & Edelson, 2003; Choppin, 2011).

From the perspective of the enacted curriculum, a challenge that could be brought against this analysis is that teachers may address deficits in their assigned textbooks when they teach their area lessons. If true, the importance of improving curricular content becomes less compelling. But this scenario seems unlikely on a wide scale. Some teachers with strong understanding the content and their students' learning may be in a position to search our and use high-quality supplementary materials. But many elementary teachers, in the United States as well as other countries, lack both the knowledge and the time to carry out such repairs, even if high-quality supplementary materials were available. Most materials available on the Internet and accessed by U.S. teachers provide only practice on routine tasks. Addressing the problem of teaching and learning area measurement effectively will likely require both enriched curriculum materials and support for elementary teachers to understand and use those materials well.

Acknowledgments — We acknowledge the significant contributions of Kuo-Liang Chang, Dan Clark, Hanna Figueras, Patrick Greeley, Jia He, Leslie Dietiker, KoSze Lee, Aaron Mosier, Alexandria Musselman, and Matthew Pahl to the development of our coding scheme and to the coding of textbook content. The second and third authors presented parts of this analysis at the 2011 Meeting of the International Group for the Psychology of Mathematics Education (PME 35) in Ankara, Turkey.

Funding — The research reported in this article was carried out by the *Strengthening Tomorrow's Education in Measurement* (STEM) project team at Michigan State University with support from the National Science Foundation (REC-0634043 and DRL-0909745). The opinions expressed here are the authors' and do not necessarily reflect the views of the Foundation.

References

Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is—or might be—the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher*, *25*, 6–14.

Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education*, *38*, 115–131.

Battista, M. T. (2012). *Cognition-based assessment & teaching of geometric measurement: Building on students' reasoning*. Portsmouth, NH: Heinemann.

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education*, *29*, 503–532. doi:10.2307/749731

Baturo, A., & Nason, R. (1996). Student teachers' subject matter knowledge within the domain of area measurement. *Educational Studies in Mathematics*, *31*, 235–268. doi:10.1007/BF00376322

Berenson, S., Van Der Valk, T., Oldham, E., Runesson, U., Moreira, C. Q., & Broekman, H. (1997). An international study to investigate prospective teachers' content knowledge of the area concept. *European Journal of Teacher Education*, *20*, 137–150. doi:10.1080/0261976970200203

Blume, G. W., Galindo, E., & Walcott, C. (2007). Performance in measurement and geometry from the viewpoint of *principles and standards for school mathematics*. In P. Kloosterman & F. K. Lester (Eds.), 2003 *Mathematics assessment of the national assessment of educational progress* (pp. 95–138). Reston, VA: National Council of Teachers of Mathematics.

Bonotto, C. (2003). About students' understanding and learning of the concept of surface area. In D. H. Clements & G.

Bright (Eds.), *Learning and teaching measurement: 2003 Yearbook* (pp. 157–167). Reston, VA: National Council of Teachers of Mathematics.

Brown, M., & Edelson, D. (2003). *Teaching as design: Can we better understand the ways in which teachers use materials so we can better design materials to support their changes in practice? (Design brief)*. Evanston, IL: Center for Learning Technologies in Urban Schools.

Chappell, M. F., & Thompson, D. R. (1999). Perimeter or area? Which measure is it? *Mathematics Teaching in the Middle School*, *5*, 20–23.

Charalambous, C. Y., Delaney, C., Hsu, H., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical Thinking and Learning*, *12*, 117–151. doi:10.1080/10986060903460070

Charles, R., Crown, W., & Fennell, F. (2008). *Mathematics, Michigan edition*. Glenview, IL: Pearson Education Inc.

Choppin, J. (2011). Learned adaptations: Teachers' understanding and use of curriculum resources. *Journal of Mathematics Teacher Education*, *14*, 331–353. doi:10.1007/s10857-011-9170-3

Clements, D. H., & Sarama, J. (2009). *Teaching and learning early math: The learning trajectories approach*. New York, NY: Routledge.

Confrey, J. (2012, November). Articulating a learning sciences foundation for learning trajectories in the CCSSM. *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for Psychology in Mathematics Education (PME-NA), 1–22. Kalamazoo, MI.*

Crosby, A. W. (1997). *The measure of reality: Quantification and western society, 1250–1600*. Cambridge, UK: Cambridge University Press.

Ding, M., & Li, X. (2010). A comparative analysis of the distributive property in US and Chinese elementary mathematics textbooks. *Cognition & Instruction*, *28*, 46–180. doi:10.1080/07370001003638553

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition & Instruction*, *10*, 105–225. doi:10.1080/07370008.1985.9649008

diSessa, A. A. (1996). What do “just plain folks” know about physics? In D. R. Olson, & N. Torrance (Eds.), *The handbook of education and human development: New models of learning, teaching, and schooling* (pp. 709–730). Oxford, UK: Blackwell Publishers, Ltd.

diSessa, A. A., & Wagner, J. F. (2005). What coordination has to say about transfer. In J. Mestre (Ed.), *Transfer of learning from a modern multi-disciplinary perspective* (pp. 121–154). Greenwich, CT: Information Age.

Floden, R. E. (2002). The measurement of opportunity to learn. In A. C. Porter & A. Gamoran (Eds.), *Methodological advances in cross-national surveys of educational assessment* (pp. 231–266). Washington, DC: National Academy Press.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). New York, NY: MacMillan.

Grouws, D. A., Smith, M. S., & Sztajn, P. (2004). The preparation and teaching practices of United States mathematics teachers: Grades 4 and 8. In P. Kloosterman & F. K. Lester (Eds.), *Results and interpretations of the 1990 through 2000 mathematics assessments of the national assessment of educational progress* (pp. 221–267). Reston, VA: National Council of Teachers of Mathematics.

Heraud, B. (1987). Conceptions of area units by 8–9 year old children. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th International Conference of the Psychology of Mathematics Education* (Vol. 3, pp. 299–304). Paris, France: CNRS-Paris.

Herbel-Eisenmann, B. A., & Otten, S. (2011). Mapping mathematics in classroom discourse. *Journal for Research in Mathematics Education*, *42*, 451–485. doi:10.5951/jresematheduc.42.5.0451

Hiebert, J. & Lefvre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J.

Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Lawrence Erlbaum.

Hino, K. (2002). Acquiring new use of multiplication through classroom teaching: An exploratory study. *Journal of Mathematical Behavior*, *20*, 477–502. doi:10.1016/S0732-3123(02)00085-8

Joram, E., Subrahmanyam, K., & Gelman, R. (1998). Measurement estimation: Learning to map the route from number to quantity and back. *Review of Educational Research*, *68*, 413–449. doi:10.3102/00346543068004413

Kamii, C., & Kysh, J. (2006). The difficulty of “length × width”: Is a square the unit of measurement? *Journal of Mathematical Behavior*, *25*, 105–115. doi:10.1016/j.jmathb.2006.02.001

Kaur, B. (2014). Enactment of school mathematics curriculum in Singapore: Whither research! *ZDM*, *46*, 829–836. doi:10.1007/s11858-014-0619-6

Kobiela, M., Lehrer, R., & Pfaff, E. (2010, April). *Students’ developing conceptions of area via partitioning and sweeping*. Paper presented at the Annual Meeting of the American Education Research Association, Denver, Colorado.

Kordaki, M., & Potari, D. (2002). The effect of area measurement tools on student strategies: The role of a computer microworld. *International Journal for Computers for Mathematics Learning*, *7*, 65–100. doi:10.1023/A:1016051411284

Kospentaris, G., Spyrou, P., & Lappas, D. (2011). Exploring students' strategies in area conservation geometry tasks. *Educational Studies in Mathematics*, *77*, 105–127. doi:10.1007/s10649-011-9303-8

Larson, N. (2004). *Saxon Math*. Austin, TX: Saxon Publishers.

Lehrer, R. (2003). Developing understanding of measurement. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 179–192). Reston, VA: National Council of Teachers of Mathematics.

Lehrer, R., Jaslow, L., & Curtis, C. (2003). Developing an understanding of measurement in the elementary grades. In D. H. Clements & G. Bright (Eds.), *Learning and teaching measurement: 2003 yearbook* (pp. 100–121). Reston, VA: National Council of Teachers of Mathematics.

Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137–167). Mahwah, NJ: Lawrence Erlbaum.

Lehrer, R., & Slovin, H. (2014). *Developing essential understanding of geometry and measurement in grades 3–5*. Reston, VA: National Council of Teachers of Mathematics.

Li, X., Ding, M., Capraro, M. M., & Capraro, R. M. (2008). Sources of differences in children's understandings of mathematical equality: Comparative analysis of teacher guides and student texts in China and the United States. *Cognition & Instruction*, *26*, 195–217. doi:10.1080/07370000801980845

Maloney, A. P., Confrey, J., & Nguyen, K. H. (2014). *Learning over time: Learning trajectories in mathematics education*. Charlotte, NC: Information Age.

Murphy, C. (2012). The role of subject knowledge in primary student prospective teachers' approaches to teaching topic of area. *Journal of Mathematics Teacher Education*, *15*, 187–206. doi:10.1007/s10857-011-9194-8

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Nesher, P. (1988). Multiplicative school word problems: Theoretical approaches and empirical findings. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 19–40). Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Lawrence Erlbaum Associates.

Nunes, T., Light, P., & Mason, J. (1993). Tools for thought: The measurement of length and area. *Learning and Instruction*, *3*, 39–54. doi:10.1016/S0959-4752(09)80004-2

Outhred, L. N., & Mitchelmore, M. C. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education*, *31*, 144–167. doi:10.2307/749749

Owens, K., & Outhred, L. (1998). Covering shapes with tiles: Primary students' visualization and drawing. *Mathematics Education Research Journal*, *10*, 28–41. doi:10.1007/BF03217056

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry*. New York, NY: Basic Books.

Polikoff, M. S. (2015). How well aligned are textbooks to the common core standards in mathematics? *American Educational Research Journal*, *53*, 1185–1211. doi:10.3102/0002831215584435

Remillard, J. T., Harris, B., & Agodini, R. (2014). The influence of curriculum material design on opportunities for student learning. *ZDM*, *46*, 735–749. doi:10.1007/s11858-014-0585-z

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM*, *46*, 705–718. doi:10.1007/s11858-014-0600-4

Reys, B. J., & Reys, R. E. (2006). The development and publication of elementary mathematics textbooks: Let the buyer beware! *Phi Delta Kappan*, *87*, 377–383. doi:10.1177/003172170608700510

Schifter, D., & Szymaszek, J. (2003). Structuring a rectangle: Teachers write to learn about their students' thinking. In D. Clements, & G. Bright (Eds.), *Learning and Teaching Measurement: 2003 Yearbook* (pp. 143–156). Reston, VA: National Council of Teachers of Mathematics.

Schmidt, W. H., McKnight, C. C., & Raizen, S. A. (1997). *A splintered vision: An investigation of U.S. science and mathematics education*. Dordrecht, The Netherlands: Kluwer.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making. In D. A. Grouws (Ed.), *Handbook of Research in Teaching and Learning Mathematics* (pp. 334–370). New York, NY: Macmillan.

Simon, M. A., & Blume, G. W. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education*, *25*, 472–494. doi:10.2307/749486

Smith, J. P., Males, L. M., Dietiker, L. C., Lee, K., & Mosier, A. (2013). Curricular treatments of length measurement in the United States: Do they address known learning challenges? *Cognition & Instruction*, *31*, 388–433. doi:10.1080/07370008.2013.828728

Stavy, R., & Tirosh, D. (2000). *How students (mis-)understand science and mathematics: Intuitive rules*. New York, NY: Teachers College Press.

Stein, M. K., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*, 455–488. doi:10.3102/00028312033002455

Stein, M. K., & Kim, G. (2009). The role of mathematics curriculum materials in large-scale urban reform: An analysis of demands and opportunities for teacher learning. In J. T. Remillard, B. A. Herbbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 37–55). New York, NY: Routledge.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester Jr. (Ed.), *Second Handbook of Research on*

Mathematics Teaching and Learning (pp. 319–369). Reston, VA: National Council of Teachers of Mathematics.

Stephan, M., & Clements, D. (2003). Linear and area measurement in Prekindergarten to grade 2. In D. H. Clements & G. Bright (Eds.), *Learning and teaching measurement: 2003 Yearbook* (pp. 3–16). Reston, VA: National Council of Teachers of Mathematics.

Sweller, J., & Cooper, G. A. (1985). The use of worked examples as a substitute for problem solving in learning algebra. *Cognition and Instruction*, *2*, 59–89. doi:10.1207/s1532690xci0201_3

The University of Chicago School Mathematics Project. (2007). *Everyday mathematics* (3rd ed.). Chicago, IL: Wright Group/McGraw Hill.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. (1994). Calculational and conceptual orientations in teaching mathematics. In D. B. Aichele (Ed.), *1994 yearbook of the national council of teachers of mathematics* (pp. 79–92). Reston, VA: National Council of Teachers of Mathematics.

Thompson, T. D., & Preston, R. V. (2004). Measurement in the middle grades: Insights from NAEP and TIMSS. *Mathematics Teaching in the Middle School*, *9*, 514–519.

Vamvakoussi, X., & Vosniadou, S. (2010). How many *decimals* are there between two *fractions*? Secondary school students' understanding of rational numbers and their notation. *Cognition & Instruction*, *28*, 181–209. doi:10.1080/07370001003676603

Vosniadou, S., Vamvakoussi, X., & Skopeliti, I. (2008). The framework theory approach to conceptual change. In S.

Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 3–34). Mahwah, NJ: Lawrence Erlbaum.

Watson, A. (2010). Key understandings in learning mathematics. *Scottish Mathematics Council Journal*, *40*, 14–16.

Woodward, E., & Byrd, F. (1983). Area: Included topic, neglected concept. *School Science and Mathematics*, *83*, 343–347.

Zacharos, K. (2006). Prevailing educational practices for area measurement and students' failure in measuring areas. *The Journal of Mathematical Behavior*, *25*, 224–239. doi:10.1016/j.jmathb.2006.09.003

Appendix: Area measurement knowledge elements

Conceptual Knowledge (35 elements)

General properties of quantities & their measures

Positive values. Area takes on only positive values (> 0).

Single values. Given an area unit, all regions have a single (positive) area measure.

Greater measure means larger. The greater the area measure, the larger the region.

Order/equality: If one region is contained within another or can be transformed by partitioning and/or moving to be contained within another, the area of the first region is smaller than the area of the second. If the two regions exactly coincide, their areas are equal.

Conservation under partitioning. The area of a region is invariant under partitioning into subset regions. Area is invariant under different choices of area units.

Conservation under motion. Area is invariant under changes in location and orientation of the region.

Transitivity. The comparison of areas is transitive. For example, if area A is greater than area B, and area B is greater than area C, then area A is greater than area C. (also true for = , <, ≥, ≤.)

Trichotomy. Given any two areas, the first is greater than the second, the second is greater than the first, or they are equal.

Unit conversion. One unit of area is equivalent to some number of any other different units of area.

Unit-measure compensation. Larger units of area produce smaller measures of area; smaller units of area produce larger measures.

Unit iteration. Measures of area are produced by iterating an area unit (repeatedly adjoining) to exhaust the region and then enumerating the number of iterations (e.g., by counting). Iterated units may not overlap or leave gaps.

Additive composition. Two areas combined additively make another area (e.g., the sum of two areas is a larger area and the difference of two areas is a smaller area).

Numerical sums/differences. The sum (or difference) of two area measures can be completed numerically if and only if the addend areas (areas to be subtracted) have the same unit of measure or the unit of measure of one area is converted to the other.

Multiplicative composition. An area combined multiplicatively with any other quantity (including another area) makes a quantity that is not an area. An area multiplicatively combined with a length makes a volume.

Expressing one area in terms of another. Any area can be expressed as a number of units of another area.

Multiplicative comparison. An area compared multiplicatively with another area makes a ratio, which is a scalar (a pure number, not a measure).

Scalar product. The product of a scalar and an area measure is another area measure.

Conservation of order relations under positive scalar multiplication. The order of two or more areas is invariant under multiplication by any positive scalar.

Equal area does not imply congruence. Two regions that have equal areas are not necessarily congruent.

Numerical quotients. The numerical quotient of two area measures is a scalar or "pure" number.

Measurement involves error. All measures of area produced by humans, either directly or via computation from lengths, include some error.

Measurement of area specifically

Meaning/definition of area. Area quantifies the space enclosed in a region.

Varieties of regions. All regions have area, whether they are simple closed curves, closed curves with "cut-outs," or closed curves with intersections.

Meaning of area measure. A measure of area is the number of area units that additively compose that area.

Area measure requires units of area. Only units of area, standard or non-standard, produce measures of area.

Standard and nonstandard units of area. Area can be measured in either standard or nonstandard units. Where non-standard units differ across objects and cultures, standard units are fixed and accepted across cultures and situations. Specifically, a standard area unit has two defining characteristics: (1) it is a square, whose (2) side length is a standard unit of length.

No ruler for area. There is no tool, analogous to a ruler, which directly measures area.

Tools for measuring area. There are tools that support the evaluation of area though the area value cannot be "read off" the tool in a manner analogous to rulers. Transparent grids can be superimposed on regions and the number of whole and fractional units that cover the region can be counted and summed.

Dynamic representations. Area can be represented dynamically by creating two line segments with a common endpoint and independently varying the locations of the other endpoints. Measurement of area of specific shapes

Spatial structure of rectangles. The space enclosed in any rectangle is composed of a grid of iterated units that can be seen either as a series of rows equal in length to one side of the rectangle or a series of columns equal in length to the adjacent side.

Definition of (rectangular) arrays. An array is a set of objects or units arranged in a rectangular pattern, that is, into horizontal rows with equal numbers in each row. The elements in an array may be contiguous or not.

Arrays and computation. The rectangular structure of arrays supports the enumeration of the array in ways that avoid counting all elements one at a time, including skip-counting by the number of elements in a row or column and multiplying the number of rows by the number of elements in each row.

Triangles of equal area. All triangles with the same base length and height have the same area. The height of a triangle is the perpendicular distance from the base (any side) to the opposite vertex.

The composition of regular polygons. Regular polygons can be decomposed into N isosceles triangles whose vertices are the two endpoints of a side and the center of the polygon.

Area and probability with spinners. Given a circular spinner composed of two or more sectors, the relative frequency/likelihood of an outcome is dependent on the areas of those sectors. The greater the area, the greater the relative frequency/likelihood.

Procedural Knowledge (50 elements)

Pre-measurement of area

Cover region with different shapes. Use different shapes to cover a region without leaving gaps or overlapping the boundary of the region. The collection must include two or more different shapes. Options: The region has no internal partitioning lines (no support) or partitioning lines inside the region match at least some of the shapes (support provided).

Cover region with same shape. Use a collection of the same shapes to cover a region without leaving gaps or overlapping the boundary of the region. Same options as just above.

Make visual comparative judgment. Given two or more regions, make a visual comparative judgment (judgment by sight) about which regions are congruent or which region has the greatest, least, equal, or approximately equal area.

Comparison by superposition. To compare the areas of two regions, make a physical copy of one region (e.g., on paper), superimpose it on another, and make a judgment about which is greater, lesser, or equal based on the visible excess.

Indirect comparison. To compare the areas of two regions, find a third region whose area is smaller than (or equal to) the first (by either direct or visual comparison) and compare it to the second (by direct or visual comparison). If the area of the third region is larger than (or equal to) the area of the second, the first region has greater (or equal) area than the second.

Draw a region whose area is larger or smaller than another. Given one region, draw another that completely contains (or is contained by) the first or that is larger or smaller by visual judgment. Numerical measurement of area

Measure area by covering with sufficient non-standard units. Select a physical object whose area is less than the region's area and can "cover." Cover the region with those objects. Count the number of objects; the area is the number of objects.

Measure area by covering without sufficient non-standard units. Select a physical object whose area is less than the region's area and can "cover." Cover part of the region with a series of those objects (until they are used up) and count them.

Mark the part of the region covered and number of objects used. Repeat in the uncovered part of the region until the entire region has been covered. Count the total number of objects that cover all parts of the region.

Make or draw a region of × nonstandard units. Place × nonstandard units of area adjacent to each other without leaving gaps or overlapping the objects. If drawing is required, draw a boundary around the objects that totally encloses them with no additional space.

Read an array. Given an array, determine how many rows in the array and how many elements in each row. Options: The elements in the array are square or rectangular and aligned uniformly without gaps or overlaps (contiguous), or are either not square or rectangular, not aligned uniformly, or are presented with gaps between them (noncontiguous).

Make an array. From sufficient information on the number of rows and columns (or the number of elements in each row and column), construct an array with that structure. Same options as just above.

Enumerate an array. Given an array, determine the total number of objects or units in it, either by counting all the objects, skip-counting, or multiplying. Options: The elements in the array are square or rectangular and aligned uniformly without gaps or overlaps (contiguous), or are either not square or rectangular, not aligned uniformly, or are presented with gaps between them (non-contiguous).

Measure area; general. Use this knowledge element only when students are asked to measure the area of a region, but the text gives no detail about how. Options: The space occupied by the region is not structured (no support), or the space is structured (e.g., by a grid or pattern of equally-spaced dots [support]).

Measure area by covering with sufficient standard units. Select an object that is a standard unit of area, cover the region with those objects, and count the number used.

Measure area on a grid by counting whole units. To determine the area of regions drawn on square grids, count the number of grid units that cover the region.

Measure area on a grid by counting whole & fractional units. To determine the area of regions drawn on square grids, count the number of whole grid units in the interior of the region, count the number or half- or partial units, and combine the half- or partial units into a number of whole grid units. The area is the sum of these units, either in whole grid units or in whole + partial grid units.

Measure the area of a rectangle on a grid by counting rows or columns. To determine the area of a rectangle or square drawn on a grid, count the number of squares in a horizontal row and the number of rows (or the number of squares in a vertical column and the number of columns) and multiply the two numbers.

Measure a length in the service of finding the area of a polygon. Measure a side of or height in a polygon to use that length to find its area. Options: When space has not been structured, measure with a ruler (ruler), on a grid, count the number of length intervals between endpoints (grid), or on dot paper, count the number of length intervals between endpoints (dots).

Measure a length in the service of finding the area of a circle. Measure the radius or diameter of a circle in order to use that length measure to find the area of the circle. Same options as just above.

Compute the area of a rectangle. If the length of the longer side of the rectangle is × units and the length of the shorter side is Y units, multiply the two numbers. If the length and width are measured in the same length unit, the area unit is that unit^2.

Compute the area of a square. If the length of the side is × units, multiply × by itself. The area is × * × unit^2.

Compute the area of a right triangle. Multiply the lengths of two perpendicular sides of the triangle and divide that product by two (L1 × L2 ÷ 2). If the sides are measured in the same length unit, the area is measured in that unit^2.

Compute the area of any triangle (by the standard formula). Multiply the length of any side by the height/altitude to that side and divide by two (S × H ÷ 2). That area is measured in units of [side unit × height unit]. If the side and the height are measured in the same unit, the area is measured in that unit^2.

Compute the area of any triangle (by Heron's formula). Compute the semi-perimeter (S) of the triangle (S = 1/2 of A + B + C, where A, B, and C are the lengths of the sides of the triangle, all measured in the same length unit). The area of the triangle is the square root of the product of S, S— A, S—B, and S—C, measured in unit^2.

Compute the area of a parallelogram. Multiply the length of either side by the height/altitude to that side (B × H). If the side and the height are measured in the same unit, the area is measured in that unit^2.

Compute the area of a trapezoid. Add the lengths of the parallel sides, multiply by the height, and divide that product by two ([B1 + B2] × H ÷ 2). If the bases and the height are measured in the same unit, the area is measured in that unit^2.

Compute the area of a regular polygon of N sides (N > 4). Multiply the area of the triangle whose vertices are the two endpoints of a side and the center of the polygon by N. If the bases and the apothem are measured in the same unit, the area is measured in that unit^2.

Compute the area of a circle. Multiply the square of the radius by π. The area of the circle is that numerical product, measured in square units of the radius length.

Draw a region of × standard units. Draw a region whose area is × standard area units without measuring any lengths. Options: Place units on blank paper, or draw a boundary connecting dots or along grid lines.

Draw a region of × whole and Y fractional standard units. Draw a region whose area is × whole and Y fractional standard area units without measuring any lengths. Same options as just above.

Draw a region of × whole or × whole and Y fractional units, given a fractional unit. Given a fractional part of a unit of area, draw the regions whose area is equal to a given number of whole units or a number of whole and fractional units. Options: The fractional unit is drawn (only) but cannot be iterated (no support), or the fractional unit can be iterated and/or the space is structured (e.g., as a grid) to support the solution (support).

Visual estimation of area of a single region. Use a unit of area to estimate the area of a given region by mentally placing/iterating and counting the number of units required.

Partition region by halving. To partition a region into two halves of equal area, divide it in the middle. Options: The region is drawn on blank background (no support), or on blank background but folding is encouraged (support, folding), or is located on dot or grid paper or geoboard (other support).

Partition region into an odd number of parts. To partition a region into three, five, or any larger odd/prime number of parts, divide it into two unequal regions and partition those parts as needed. Same options as just above.

Partition region into odd × even number of parts. To partition a region into a number of parts that is the product of an even and odd number (e.g., 6, 10, 12, etc.), either first divide the region in half and then partition each half into the required odd number of parts or first partition into an odd number of parts (N ÷ 2) and then partition each part in half. Same options as just above.

Partition region, general. Use this knowledge element only when students are asked to partition a region, but the text gives no detail about how many parts.

Unit conversion: To convert an area measure from one unit to another, multiply the given area by a ratio of the two area units. The "new" unit of area must be the numerator of the ratio. Reasoning with area measures

Compute the area of a sector of a circle. Given the arc length of the sector, find the ratio of the arc length to the circumference of the circle. Multiply that ratio by the area of the circle. Alternatively, given the central angle of the sector, find the ratio of the central angle to 360 degrees and multiply that ratio by the area of the circle.

Measure an area by partitioning, moving, recomposing, and computing. To find the area of an irregular region, divide, move, and recompose it as a more convenient shape (e.g., a rectangle) and apply the computational procedure appropriate for that shape.

Measure an area by decomposing and computing. To find the area of a complex region, partition it into a series of more convenient shapes, compute the area of each using the procedure appropriate for that shape, and add the resulting areas.

Measure an area by enclosure, computing, and subtraction. To find the area of a complex region, construct a region that entirely encloses the initial region and compute its area. Then compute the area of the enclosing region that is not part of the initial region, and subtract that area from the area of the enclosing region.

Find the sum or difference of areas. To find the total area of a collection of non-intersecting regions, add the areas of each region. To find the difference between (or to compare) the areas of two non-intersecting regions, subtract the smaller area from the larger. To find the area of a cut-out region, subtract the area of the inside region from the area of the outside region. Options: Relevant text is given in words only (words), in numbers only (numbers), in words and area is represented with area units (units), or in words and area is represented via pictures of regions (pictures).

Find the scalar multiple or quotient of an area. To find a scalar multiple of a given area, multiply the area by the scalar. To find the scalar quotient of a given area (i.e., to divide the area of a given region into a specified number of parts of equal area), divided the area by the scalar. Same options as just above.

Find the multiplicative comparison of two areas. Given two areas given in the same area units, divide one area number by the other to produce the appropriate scalar.

Visual estimation of area of one region via comparison with another. Given two regions—one contained in the other, one overlapping the other, or one beside another—when the area of one region is known or easily computable, use the area of the known region to estimate the smaller or larger area of the other region.

Estimate the area of a complex region. If a complex region (one with a geometrically inconvenient shape) cannot be decomposed into subregions with convenient shapes, decompose the region approximately into convenient sub-regions, estimate the areas of each, and add the resulting areas.

Read/interpret a spinner. Given a circular spinner that is divided into sectors where the issue concerns the frequency or likelihood of the spinner landing in one sector or a set of sectors (combined) after one or many spins, quantify the area of the relevant sector (or sectors). Options: When non-numerical judgments only are requested (greater/greatest or less/least), make a visual comparative judgment (qualitative); when numerical quantification of target frequency of an outcome to total frequency is requested, visually match the sector to the fraction it represents (part/ whole); when numerical quantification of the frequency of one outcome to another is requested, visually match the sectors to the fractions they represent (part/part).

Construct a spinner to given specifications. Given information specifying the relative frequency of outcomes (qualitative or quantitative), partition the circle into sectors so that the areas of the sectors satisfy those specifications. Options: No sector lines or divisions of the circle are provided (no support), or the circle has been partitioned into sectors or divisions of the circle are provided on the circle itself (support).

Read a circle graph. Given a circle divided into sectors, quantify the area of the relevant sector (or sectors). Options: When a non-numerical judgment is requested, make a visual comparative judgment (qualitative); when a numerical judgment of part to whole is requested, visually match the sector to the fraction it represents (part/whole); when a numerical judgment of part to part is requested, visually match the sectors to the fractions they represent (part/part).

Construct a circle graph to given specifications. Given information specifying the relative frequency of outcomes (qualitative or quantitative), partition the circle into sectors so that the areas of the sectors satisfy those specifications. Options: No sector lines or divisions of the circle are provided (no support), or the circle is partitioned into sectors or divisions on the circle itself are provided (support).

Conventional Knowledge (8 elements)

Systems of standardized area measure. There are two widely used systems of area measure: The customary or English system and the metric system. The commonly used area units in the customary or English system are square inch, square foot, square yard, acre, and square mile, and in the metric system square millimeter, square centimeter, square decimeter, square meter, and square kilometer.

Definitions of standard area units. Standard units of area can be defined by actual size drawings (e.g., "This [square] is an square inch"), scale drawings, or verbal statements.

Definitions of nonstandard area units. Nonstandard units of area can be defined by actual size drawings, scale drawings, or verbal statements.

Conventions for abbreviating area units. There are standard ways of indicating area units that do not require writing the full name of the unit (e.g., square inch = sq in; in^2 and square centimeter = sq cm; cm^2).

Conventions for abbreviating for area-related terms. There are standard ways of indicating measures related to the area with single letters (e.g., A/a for area and B/b for base).

Label an array. Given an array, write the number of rows to the left or right of the array and the number of columns above or below the array. Options: Elements in the array are square or rectangular and aligned uniformly without gaps or overlaps (contiguous), or elements in the array are either not square or rectangular, not aligned uniformly, or are presented with gaps between them (non-contiguous).

Conventions for naming rectangles by side lengths. Rectangles are often identified by the lengths of their nonequal sides in one of two ways: "A × B" or "A by B.

Numerical conversion ratios for area units. The conversion ratios for units of area are derivative of their corresponding conversion ratios for length. Given that one unit of length measure equals some number of units of another length measure, one square unit of area measure equals the square of that number of units of the second length measure (e.g., 1 square foot = 144 square inches (12 × 12) and 1 square meter = 10,000 square centimeters (100 × 100)).