THEORY OF MAXIMA AND MINIMA

AND APPLICATIONS

By

MARCUS HAROLD WHITFIELD, JR.

Bachelor of Arts
Tougaloo College
Tougaloo, Mississippi
1950

Master of Arts for Teachers'
Indiana University
Bloomington, Indiana
1957

Submitted to the faculty of the Graduate College
of the Oklahoma State University
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Thesis Approved:

W Ware Marsden
Thesis Adviser

Jeanne Agnew

Vernon Troxel

Milton E Berg

Robert D Morrison

D D Durham
Dean of the Graduate College

PREFACE

The purpose of this thesis is to develop some of the important aspects of the theory of maxima and minima including background, key concepts and applications. These concepts are presented in such a manner that a capable and mature undergraduate mathematics major should be able to comprehend them. This paper, if it can be classified as research, falls in the category of descriptive research. It involves an expository discussion of the stated topic with motivation, examples and proofs of propositions when appropriate. An effort has been made to present a unified and understandable treatment of the theory of maxima and minima. This effort includes a survey and review of many references, books and journals, touching upon each of the topics and subtopics. The material has been logically organized.

I wish to express my deep gratitude to Dr. Jeanne Agnew for the encouragement, guidance and assistance which she provided during the preparation of this thesis. I also wish to thank Dr. W. Ware Marsden for serving as committee chairman; Drs. Robert D. Morrison, Vernon Troxel, and Milton Berg for serving on my advisory committee. In addition, I wish to extend my thanks to Dr. Roy Deal for his help and to Dr. Richard H. Leftwich for his suggestions.

Last, but not least, I wish to express my gratitude to my wife, Josiepearl, and my children, Nadine and Joffria, for the sacrifices they made while assisting in the completion of my program of study.

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CHAPTER I

INTRODUCTION

In every human activity, man seeks the most efficient way of performing a given task, i.e., executing the task with the least amount of effort or determining the amount of effort necessary to achieve the greatest result in a given situation. Whenever, a man goes from one place to another, he seeks the shortest distance between those places. When he operates a business, he looks for ways and means of making the most profit and of accruing the least expense or cost. The wise consumer seeks to get the most for his money in quantity and quality of economic goods.

The problems expressed above can be discussed in relation to the theory of maximum and minimum. One might say that the theory of maximum and minimum concerns itself with the determination of optimal situations, i.e., choices for which functions have their greatest or least values. In some cases, we may be determining a curve or a surface for which a given quantity is greatest or least. Nature, as well as man, seems to follow the tendency of economizing certain magnitudes, of obtaining maximum effects with given means, or of spending minimal means for given effects.

Problems that seek to maximize or minimize a numerical function of a number of variables, with these variables subject to certain constraints, form a general class which may be called <u>optimization</u>

problems. Many optimization problems were first encountered in the physical sciences and geometry. The quest for solutions led to the application of differential calculus and to the development of the calculus of variations. Optimization techniques for dealing with these problems have been known for some time. However, in the last fifteen or twenty years, many new and important optimization problems have emerged in the field of economics and have received a great deal of attention. As a class, these problems may be referred to as programming problems. They are of much interest because of their applicability to practical problems in government, military and industrial operations, as well as to problems in economic theory. Although the concepts involved are still part of the theory of maximum and minimum, classical optimization techniques have been found to be of little assistance in some of these programming problems. Therefore, new methods had to be developed.

Linear programming deals with that class of programming problems for which all relations among the variables are linear. The relations must be linear both in the constraints and in the function to be optimized. The essential difference between the optimization problem and the programming problem lies in the nature of the constraints. In the optimization problem, the constraints must be equalities, and in the programming problem, the constraints are inequalities.

In this chapter, the nature and significance of the problem of maximum and minimum, the need for this study, and its scope and limitations are discussed.

Nature and Significance of the Problem

A variety of natural phenomena exhibit what is called the minimum principle. The principle is displayed where the amount of energy expended in performing a given action is the least required for its execution, where the path of a particle or wave in moving from one point to another is the shortest possible, where a motion is completed in the shortest possible time, and so on. A famous example of this economy of physical behavior was discovered by Heron of Alexandria. He found that the equality of the angles of incidence and reflection formed by a light ray meeting a plane mirror assures the shortest possible path of a ray in moving from its source to a reflected point by way of the mirror [1]. Sixteen hundred years later, Fermat showed that the minimum principle also defined the law of the refraction of light [2]. One finds many other instances of this minimum principle in mechanics, electrodynamics, relativity and quantum theory.

The minimum property and its inverse twin, the maximum property, find expression in certain simple statements of geometry; for example, "a straight line is the shortest distance between two points in the plane," or, of all closed curves of equal length the circle encloses the largest area. Many of these "self-evident" truths were also known to the ancients. The Phoenician Princess Dido obtained from a native North African chief a grant of as much land as she could enclose with an ox-hide. A clever girl, she cut the ox-hide into long thin strips, tied the ends together, and staked out a large and valuable territory on which she built Carthage [3]. Horatio, who made his reputation defending the bridge, was rewarded by a gift of as much land as he could plough around in a day, another illustration of an isoperimetric problem. The second law of thermodynamics provides a more modern (and a more discouraging) example of the maximum principle: the entropy (disorder) of the universe tends toward a maximum.

The search for maximum and minimum properties played an important part in the development of modern science. Fermat's discoveries in optics, James Bernoulli's work on the path of quickest descent were among the labors that led to the conviction that physical laws "are most adequately expressed in terms of a minimum principle that provides a natural access to a more or less complete solution of particular problems" [1].

Frequently it is found that nature acts in such a way as to minimize certain magnitudes. For example, soap film takes the shape of a surface of smallest area. Light always follows the shortest path, that is, the straight line, and, even when reflected or broken, follows a path which takes a minimum time. In mechanical systems one finds that the movements actually take place in a form which requires less effort in a certain sense than any other possible movement would use. There was a period, about 160 years ago, when physicists believed that the whole of physics might be deduced from certain minimizing principles, subject to calculus of variations, and these principles were interpreted as tendencies--economical tendencies, so to speak.

In this century Einstein's general theory of relativity has as one of its basic hypotheses such a minimal principle: that in the space-time world, however complicated its geometry be, light rays and bodies upon which no force acts move along shortest lines.

If one speaks of tendencies in nature or of economic principles of nature, then he does so in analogy to human tendencies and economic principles. A producer of any type of goods usually will wish to adopt a way of production which will require a minimum of cost, compared with other ways of equal results, or which, compared with other methods of equal cost, will promise a maximum return. For this reason the mathematical theory of economics involves to a large extent applications of calculus of variations. Such applications have been considered by G. C. Evans (University of California) and in particular by Charles F. Roos (New York City). A simple but interesting example, due to the economist H. Hotelling (Columbia University), concerns determining the most economical way of production in a mine. One may start with a great output and decrease the output later or he may increase the output in time or he may produce with a constant rate of output. Each way of production can be represented by a curve. If he has conjectures concerning the development of the price of the produced metal, then he may associate a number with each of those curves--possible profit. The problem is to find the way of production which will probably yield the greatest profit.

Need For The Study

Some problems concerning maxima and minima are studied in differential calculus, taught in the college undergraduate curriculum. Some, for example, may be formulated in the following way: Given a single curve, where is its lowest and where is the highest point? Or given a single surface, there is associated a certain number, namely, the height of the point on this surface above a horizontal plane. One is looking for those points at which this height is greatest or least. In elementary differential calculus one deals thus with maxima and minima of so-called functions of points, i.e., of numbers associated with points. In the calculus of variations, however, the problems are much more complicated. Here, for example, one deals with maxima and minima of so-called functions of curves, that is, of numbers associated with curves or of numbers associated with still more complicated geometric entities, like surfaces.

Since the treatment of the theory of maxima and minima is very limited in the undergraduate mathematics curriculum and since it has a high interest as a topic of pure analysis and finds immediate application to almost every branch of mathematics, there is certainly a need for an elementary, yet complete presentation of the theory of maxima and minima. With the mathematics programs being revised and upgraded gradually in nearly all of the independent undergraduate colleges, as well as in the universities, a course in this theory following the elementary course in calculus may well be in order in the near future. One might consider this possible change as part of the widespread current revolution in the mathematics curriculum. The Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America has made many recommendations for appropriate and justifiable changes in the undergraduate program of mathematics. There is a shifting of emphasis in mathematics courses as well as in topics of a given course. For example, logarithms are no longer emphasized as a tool for computing because this is now done by desk calculators or gigantic electronic computers. Instead, the logarithmic function is used to describe certain physical and social phenomena. Another illustration would be the trigonometric functions which were used chiefly in measuring distances and angles in surveying and navigation. Now these functions serve an even more important purpose in describing physical phenomena such as wave motions. On glancing through any modern day mathematics textbook on any level, but especially the college level, one who has been out of the undergraduate program during the last decade can see not only a great deal of new subject matter but a new point of view expressed in the traditional mathematics that still remains. Although the physical sciences, such as chemistry, physics, and astronomy have more or less always been mathematical in character, the modern biological science courses and even some of the social science courses, particularly, economics, are becoming more and more mathematical, demanding the use of higher mathematics--calculus and above. It has already been pointed out that the mathematical theory of economics depends largely upon the calculus of variations.

With the increasing enrollment in colleges accompanied by increased cost, there is an urgent need to maximize in some way the use of classrooms, facilities, and personnel owing to the pending shortage of funds. Perhaps, the theory of maxima and minima can be applied to this pressing problem.

Scope and Limitations

This thesis consists of two parts. Part I deals with the theory from the viewpoint of differential calculus: Functions of one variable, functions of several variables, and extremum problems with constraints (side conditions).

Part II deals with the theory of maxima and minima from the viewpoint of the calculus of variations relating to the simplest variational problem with fixed endpoints. For the most part, functionals depending on functions of one independent variable are considered in this report. However, a section is devoted to functionals depending on functions of two independent variables. Also, the isoperimetric problem and the functional depending on two functions of one variable with one finite subsidiary condition are considered briefly and illustrated.

CHAPTER II

FUNCTIONS OF ONE VARIABLE

In this chapter some basic definitions and theorems characterizing the function of one variable will be introduced and explained in detail. Also, some necessary and sufficient conditions for a function of one variable to have a maximum or minimum value will be treated.

Review of Some Basic Properties of a Function

A function may be thought of as a mapping from one nonempty set into another nonempty set. Every element of the first set is always mapped into some element of the second set. However, it is not always true that every element of the second set is used or is the correspondent of some element of the first set. In some functions two or more elements may be mapped into the same element of the second set, i.e., elements of the first set may not have distinct correspondents in the second set. A function may be represented by an algebraic expression, a table of data, a graph, or a general rule. If the graph of the function is a smooth curve without any breaks in it, then it is called a continuous function. Furthermore, if one considers a fixed element x_0 in the first set and another element x close to x_0, and forms the difference quotient,

$$\frac{f(x) - f(x_0)}{x - x_0}$$

where $f(x)$ and $f(x_0)$ are correspondents of x and x_0 respectively, and if he assumes as x approaches x_0

$$\frac{f(x) - f(x_0)}{x - x_0}$$

approaches some finite value, then he encounters another property, called the derivative of the function f at x_0. Not all functions possess derivatives at all points. In the remainder of this section, the general ideas expressed in this paragraph are stated precisely and are clarified.

Definition 2-1. A function (mapping) f from S to T is a correspondence which assigns to each element s in the set S a unique element t from the set T. The set S is called the domain of the function, and the image set, $f(S)$, of S under the mapping f is called the range of the function.

Note that there are three sets connected with the function: the first set, the domain; the second set T of which the range $f(S)$ is a subset; and the set of ordered pairs $(s, f(s))$ in which no two different pairs have the same first components. If $f(S) = T$, then the function f is said to be an onto function or a surjection. This means that every element t of the set T is the image of at least one element s in the set S under the mapping f. The definition of a function does not require this latter condition. In other words, a function may or may not be a surjection. Likewise, a function may or may not be one-to-one or an injection, that is, distinct elements in S have distinct images in T under the mapping. In other words, a function f is an injection if and only if $f(s_1) \neq f(s_2)$ implies $s_1 \neq s_2$.

Illustrative Example 2-1. Let S be the set of integers and T the set of real numbers. Define the function f as follows: $f(s) = 2s$. This function is an injection but not a surjection because there exists an element in T which is not the image of some element s in S, namely 1/2.

Illustrative Example 2-2. Let S be the set of real numbers and T the set of real numbers. Define the function g as follows: $g(s) = s^2 = t$. This function is neither an injection nor a surjection. Because $g(-2) = g(2)$ does not imply $-2 = 2$, g is not an injection. On the other hand, no negative real number is the image of some real number under this mapping. Consequently, g is not a surjection.

Definition 2-2. The function f is continuous at the number a if

(1) a is an accumulation point [See Definition 2-8] in the domain of f,

(2) the limit $\lim_{x \to a} f(x)$ exists, and

(3) $\lim_{x \to a} f(x) = f(a)$ [5].

This definition can be restated utilizing the concept of a neighborhood of a point which will be useful later on in this paper. Consider a point x and a positive real number δ. Then the open interval $(x - \delta, x + \delta)$ about x is said to be a neighborhood of x. Another notation for $(x - \delta, x + \delta)$ is $N_\delta(x)$, called a δ-neighborhood of x. If the radius is unimportant, $N_\delta(x)$ is written simply as $N(x)$. A function f is continuous at a if a is in the domain of f and if for every neighborhood $N_\epsilon[f(a)]$ of $f(a)$ there exists a neighborhood $N_\delta(a)$ of a such that $f(x)$ is in $N_\epsilon[f(a)]$ for every x in

$N_\delta(a)$. See Figure 2-1.

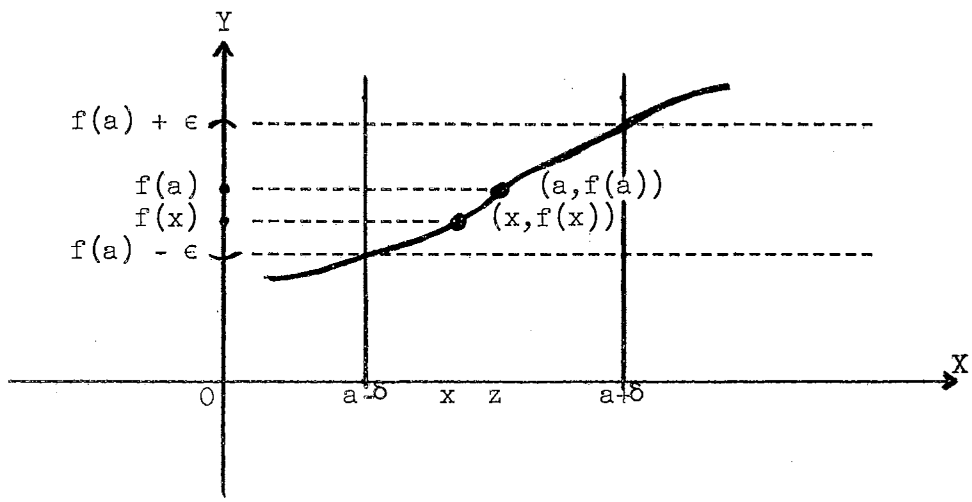

Figure 2-1.

If the function f is continuous at every point in some set A, then f is said to be continuous on the set A.

Illustrative Example 2-3. The function f defined by $f(x) = 1/x$ is continuous on the set of all nonzero real numbers. The function f is not continuous at 0 since $f(0)$ is undefined or all the conditions of the definition are not satisfied.

Intuitively speaking, a continuous function f is one whose graph, the set of points $(x,f(x))$, has no breaks in it.

In mathematical analysis, it is proved that the sum, the difference, or the product of two continuous functions is continuous. Also, if $f(x)$ and $g(x)$ are continuous functions and $g(x) \neq 0$ for all x, then $f(x)/g(x)$ is continuous.

Another concept of extreme importance in the development of the theory of maxima and minima is the derivative.

Definition 2-3. Let f be defined on the open interval (a,b) and assume that c belongs to (a,b). Then f is said to have a derivative at c whenever the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. This limit, denoted by $f'(c)$, is called the derivative of f at c [6].

The function f' is called the first derivative of f, f'' is the second derivative of f and first derivative of f', and so on. A function f which has a derivative at a point c (or at each point of an interval) is said to be differentiable at c (or on the interval). Also of interest is the fact that the differentiability of a function f at a point x implies the continuity of f at the same point x. However, the converse of the previous statement is not necessarily true, i.e., continuity of f does not imply the differentiability of f. For example, $f(x) = |x|$ is continuous at 0, but it is not differentiable at 0 since the $\lim_{x \to 0} [(|x| - |0|)/(x - 0)]$ does not exist.

Extrema of a Function

Consider a function f whose domain is the set S. If the range of f (the set of all values $f(x)$ such that $x \in S$) has a smallest or a largest value, then f has an extremum in S.

Definition 2-4. Let f be a real-valued function defined on a set S in E_1 (the real line). Then, f is said to have an absolute maximum on the set S if there exists a point a in S such that

$f(x) \leq f(a)$, for all x in S. If a is in S and if there is a neighborhood $N(a)$ such that $f(x) \leq f(a)$, for all x in $N(a) \cap S$, then f is said to have a relative maximum at the point a. Absolute minimum and relative minimum are similarly defined, using $f(x) \geq f(a)$ [6].

For example, if $f(x) = 3x^2 + 1$ and $S = [-1,5]$, then $f(5) = 76$ is the maximum value of f in S and $f(0) = 1$ is the minimum value of f in S. A function need not have a maximum or minimum value in a set S. For example, let $f(x) = 1/x$ and $S = (0,1]$, then f does not have a maximum value in $(0,1]$. However, it does have a minimum value $f(1) = 1$ in $(0,1]$. See Figure 2-2.

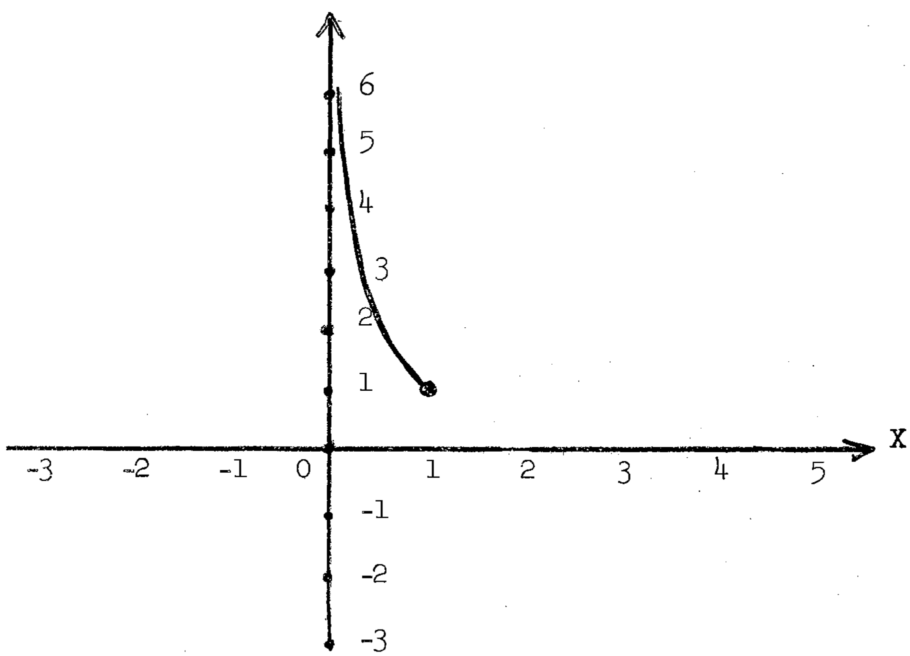

Figure 2-2.

The question as to whether or not a given function f has an extremum (maximum or minimum) can be answered by certain fundamental existence theorems. One such basic sufficient condition is that the function f be continuous on a closed and bounded set S. If a function is known to have an extremum f(c) in S and its derivative exists at each interior point of S, then $f'(c)$ necessarily vanishes. This result has great practical value in finding the extremum. A function f is increasing in an interval if for every pair of numbers x_1, x_2 in the interval with $x_1 < x_2$, $f(x_1) < f(x_2)$. If $x_1 < x_2$ implies $f(x_1) > f(x_2)$, f is decreasing in the interval. If a function f is defined on [a,b] such that f(a) = f(b) and if $f'(x)$ exists at every point x in (a,b), then there exists a number c in the domain of f' such that $f'(c) = 0$. Again suppose that f is defined on [a,b] and $f'(x)$ exists at each interior point of (a,b), then there exists a point x_0 in (a,b) such that $f(b) - f(a) = f'(x_0)(b - a)$. These ideas and theorems can be discussed more easily if some of the vocabulary of elementary topology is used. For this reason, a number of definitions follows:

Definition 2-5. Let S be a set in E_1 and assume x is in S. Then x is called an interior point of S if there is some neighborhood N(x) all of whose points belong to S [6].

Definition 2-6. Let S be a set in E_1. Then S is called an open set if every point of S is an interior point of S [6].

Definition 2-7. A set of points in E_1 is said to be bounded if it is a subset of some finite interval [6].

Definition 2-8. Let S be a set in E_1 and x a point in E_1, x not necessarily in S. Then x is called an accumulation point of

S, provided every neighborhood of x contains at least one point of S distinct from x [6].

Definition 2-9. A set is called closed if it contains all its accumulation point [6].

Definition 2-10. A set S in E_1 is said to be compact, if and only if, every open covering [6] of S contains a finite subcollection which also covers S [6].

In a Euclidean space compactness is closely connected closure and boundedness. It can be proved that a set S is compact if and only if it is closed and bounded.

Definition 2-11. Let A be a set of real numbers. If there is a real number x such that a belongs to A implies $a \leq x$; then x is called an upper bound for the set A, and the set A is said to be bounded above. Lower bound is similarly defined [6].

Definition 2-12. Let A be a set of real numbers bounded above. Suppose there is a real number x satisfying the following two conditions:

(i) x is an upper bound for A, and

(ii) if y is any upper bound for A, then $x \leq y$.

Such a number x is called a least upper bound (lub) or a supremum (sup) of the set A. The concept of greatest lower bound (glb), or infimum (inf) is similarly defined if A is bounded below [6].

It is proved that if a function f is continuous on a compact set S in E_1 with $f(S)$ in E_1, then $f(S)$ is a compact set. With these basic definitions and propositions from analysis one is now in a position to state and prove the fundamental theorem concerning the existence of an absolute maximum and an absolute minimum of a function which satisfies certain conditions.

Theorem 2-1. If a real-valued function f is continuous on a compact (closed and bounded) set S in E_1, then f has an absolute maximum and an absolute minimum on S.

This theorem requires f to be continuous at every point of S and S to be both closed and bounded. An example in which S is bounded but not closed is given by $f:(-2,2) \to E_1$ with $f(x) = x^3$. Here f has neither an absolute maximum nor an absolute minimum on $S = (-2,2)$. See Figure 2-3. On the other hand, an example in which S is closed but not bounded is given by $g:[0,\infty) \to E_1$ with $g(x) = x^2 + 1$. The function g does not have an absolute maximum on $S = [0,\infty)$. It does have a minimum $g(0) = 1$ on S. See Figure 2-4.

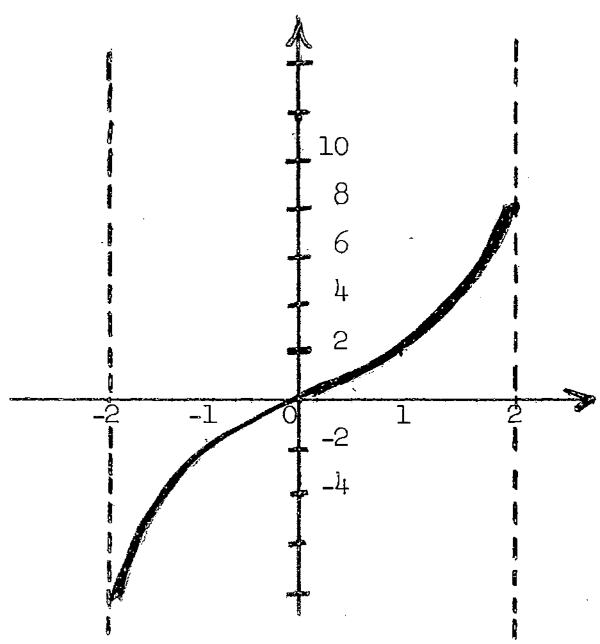

Figure 2-3.

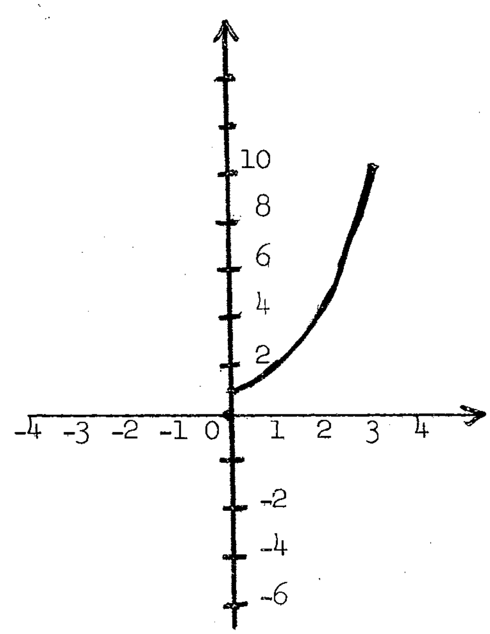

Figure 2-4.

Also, suppose S is a closed and bounded set but f is not continuous, then f might fail to have an absolute maximum or an absolute minimum. For example, let $S = [-1,1]$ and $f(x) = 1/x$, $x \neq 0$, and $f(0) = 0$. See Figure 2-5.

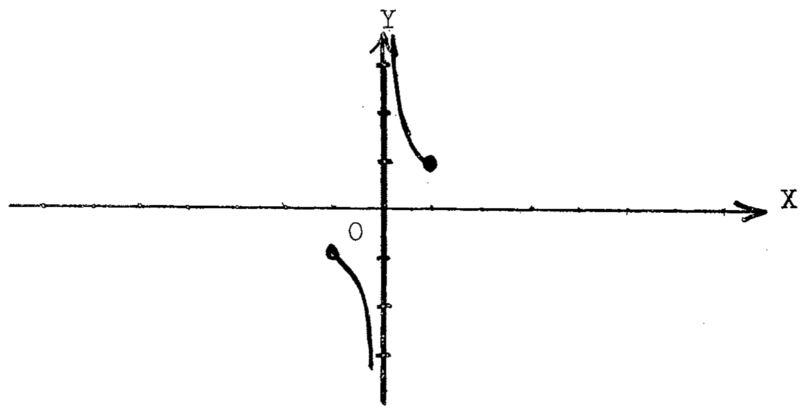

Figure 2-5.

At this point the proof of Theorem 2-1 is given.

Proof: Since S is a compact set and f is continuous on S, then $f(S)$ is a compact set. By definition of $\inf[f(S)]$ and by definition of $\sup[f(S)]$, $\inf[f(S)] \leq f(x) \leq \sup[f(S)]$ for all x in S. It needs to be shown that the compact set $f(S)$ of real numbers contains its inf and its sup. In accordance with Definition 2-4 they would be the minimum value and the maximum value. For finite sets this follows immediately.

Let $A = f(S)$ be a closed and bounded infinite set in E_1 and let $a = \sup(A)$. Then, if a does not belong to A, for every $\epsilon > 0$, there exists a point x in A such that $a - \epsilon < x < a$. This means that every neighborhood of a contains points of A distinct from a. Hence, a is an accumulation point of A. Since A is closed, this contradicts the assumption that a does not belong to A. Therefore, A contains its sup [6].

Let $b = \inf(A)$. Then, if b does not belong to A, for every $\epsilon > 0$, there exists a point y in A such that $b < y < b + \epsilon$. This implies that every neighborhood of b contains points of A distinct from b. Consequently, b is also an accumulation point of A. Again, since A is closed, this contradicts the assumption that b does not belong to A. Therefore, A contains its inf. This completes the proof.

Although theorem 2-1 is the fundamental theorem concerning the existence of maxima and minima, it by no means includes all cases that may be needed in applications. The reader might notice numerous variations that may be useful. One illustration is the following theorem:

Theorem 2-2. Let $f(x)$ be continuous in the interval $0 < x < \infty$, and suppose $f(x)$ approaches $+\infty$ both as x approaches 0 and as x approaches $+\infty$, then $f(x)$ has a minimum in that interval [7].

The reader may refer to Figure 2-6 for a further clarification of this theorem.

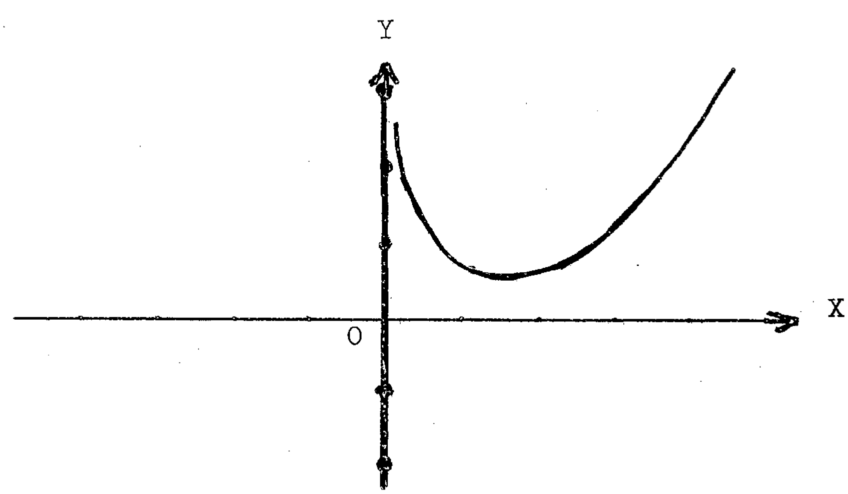

Figure 2-6.

The condition $f(x)$ approaches $+\infty$ as x approaches 0 implies by definition that if any number M is given, there exists a one-sided neighborhood $0 < x \leq x_1$ of $x = 0$ within which $f(x) > M$ holds. The condition $f(x)$ approaches $+\infty$ as x approaches $+\infty$ implies that if M is given, there exists x_2 such that whenever $x_2 \leq x < +\infty$, one has $f(x) > M$.

Proof of Theorem 2-2. To prove this theorem, one may choose any number x_0, $0 < x_0 < +\infty$, for instance, $x_0 = 1$ is a convenient choice. Then, one chooses x_1 so that he has

(1) $f(x) > f(x_0)$ whenever $0 < x \leq x_1$,

and he chooses x_2 so that

(2) $f(x) > f(x_0)$ whenever $x_2 \leq x < +\infty$.

It follows from (1) and (2) that $x_1 < x_0 < x_2$. By Theorem 2-1 the function $f(x)$ has a minimum in the interval $x_1 \leq x \leq x_2$, say, $f(X)$, where $x_1 \leq X \leq x_2$, one has then $f(x_0) \geq f(X)$ as well as the more general inequality

(3) $f(x) \geq f(X)$, $x_1 \leq x \leq x_2$.

Consequently, $f(X)$ is the minimum of $f(x)$ in the original interval $0 < x +\infty$, by (1), (2), and (3), so Theorem 2-2 is established [7].

Definition 2-13. A function f is said to be increasing at c if there exists a neighborhood N of c contained in the domain of f such that

(i) $f(x) < f(c)$ if $x < c$, and

(ii) $f(x) > f(c)$ if $x > c$

for every x in N. If there exists a neighborhood N of c such that

(i) $f(x) > f(c)$ if $x < c$, and

(ii) $f(x) < f(c)$ if $x > c$

for every x in N, then f is said to be decreasing [5].

The reader may refer to Figures 2-7 and 2-8.

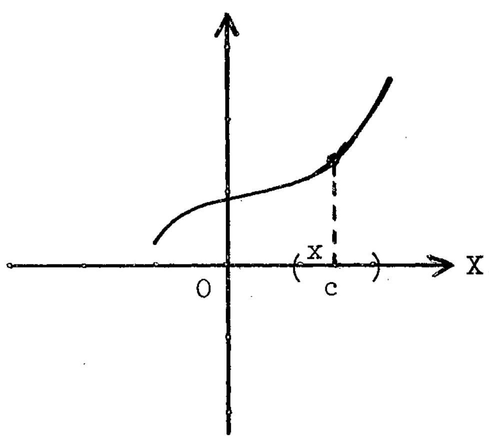

Figure 2-7.

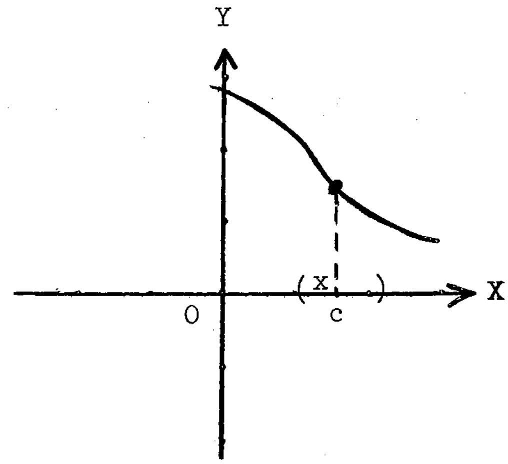

Figure 2-8.

Surely, if a function f is increasing or decreasing at the point c in its domain, then $f(c)$ is not an extremum of f in any neighborhood A of c. Let f be increasing at c and N a neighborhood of c such that

(4) $f(x) < f(c)$ if $x < c$ and $f(x) > f(c)$ if $x > c$, for every x in N, then (4) also holds for every x in $A \cap N$. Therefore, $f(c)$ is not an extremum of f in A, since $f(c)$ is not extremum of f in $A \cap N$.

The next theorem gives a technique of determining whether a differentiable function is increasing or decreasing at a given point c. This theorem is now stated without proof [5]. The proof is easy.

Theorem 2-3. If f is a function and c is a point in the domain of f then:

(1) f is increasing at c if $f'(c) > 0$, and

(2) f is decreasing at c if $f'(c) < 0$.

A few comments on this theorem are now in order. Recall that the derivative $f'(x)$ represents the slope of the tangent line to the curve $y = f(x)$ at the point $(x,f(x))$.

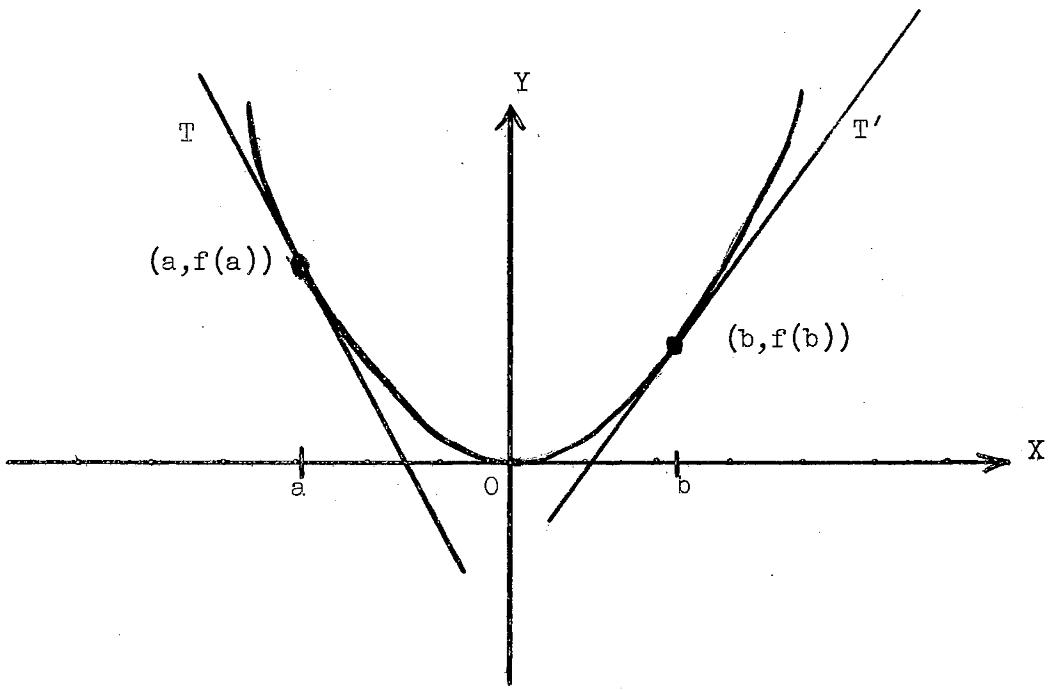

Figure 2-9.

In Figure 2-9, the tangent line T at $(a,f(a))$ has negative slope, and, since a line with negative slope is falling (as one traverses it from left to right), it is intuitively clear that the curve, which is closely approximated by T in the neighborhood of $(a,f(a))$, also is falling at $(a,f(a))$. Similarly, T' is rising and therefore the graph of f is rising at $(b,f(b))$. Consider the equation of the graph in Figure 2-9, $f(x) = x^2$, $f'(x) = 2x < 0$ if $x < 0$. Therefore, f is decreasing if $x < 0$. Since $f'(x) > 0$ if $x > 0$, f is increasing if

$x > 0$.

Illustrative Example 2-4. If $f(x) = x^3 + 3x^2$, find where the function f is increasing and where it is decreasing.

Solution: The first derivative is $f'(x) = 3x^2 + 6x = 3x(x + 2)$. Thus $f'(x) = 0$ if $x = 0$ or if $x = -2$, and $f'(x) \neq 0$ otherwise. If $x > 0$, then $f'(x) > 0$. Thus, f is increasing if $x > 0$. If $-2 < x < 0$, then $3x < 0$ and $x + 2 > 0$, hence $f'(x) = 3x(x + 2) < 0$, and f is decreasing if $-2 < x < 0$. If $x < -2$, then both $3x < 0$ and $x + 2 < 0$, therefore, $f'(x) > 0$ and f is increasing if $x < -2$. The graph of f is sketched in Figure 2-10 from this information.

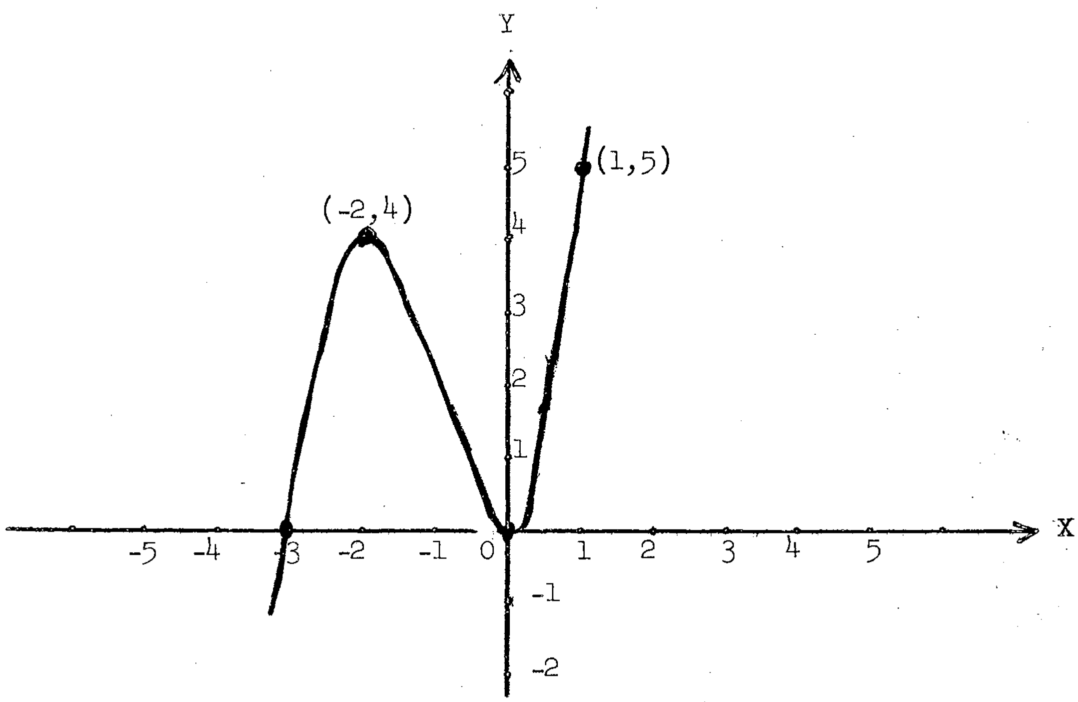

Figure 2-10.

A necessary condition for the existence of an extremum of a differentiable function f is expressed by the following theorem:

Theorem 2-4. If $f(c)$ is an extremum of a function f in some neighborhood of c contained in the domain of f and $f'(c)$ exists, then $f'(c) = 0$.

This number c is called a critical number of the function f.

The proof of this theorem is based on the Law of Trichotomy for the real number system, i.e., only one of the following relations can hold: $f'(c) > 0$, $f'(c) < 0$, or $f'(c) = 0$. By Theorem 2-3, if $f'(c) > 0$ in some neighborhood N of c, then f is increasing in N. Also, if $f'(c) < 0$ in some neighborhood N of c, then f is decreasing in N. Both of these latter two statements contradict the fact that $f(c)$ is an extremum of f in N. Therefore, $f'(c)$ must be zero. This proves the theorem.

This $f'(c)$ may not exist. In other words $f(c)$ can be an extremum of f in some neighborhood N of c without the existence of $f'(c)$. A function for which this is true is $f(x) = |x| + 1$ in the neighborhood of 0. Since this function f is not differentiable at 0, $f(0)$ does not exist but $f(0) = 1$ is a minimum value of this function. It is important to see that the converse of Theorem 2-4 is not true. That is, one cannot conclude from $f'(a) = 0$ that $f(a)$ is an extremum of f. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$. Hence $f'(0) = 0$. However, $f(0)$ is neither a maximum nor a minimum value of f, since $f(x) < 0$ if $x < 0$ and $f(x) > 0$ if $x > 0$. The graph of f is sketched in Figure 2-11 which shows that f is increasing at $x = 0$.

Let $y = f(x)$ be a function of x, continuous throughout the interval $[a,b]$ and $f(a) = f(b) = 0$. Suppose also that $f(x)$ has a derivative $f'(x)$ at each interior point $a < x < b$ of the interval.

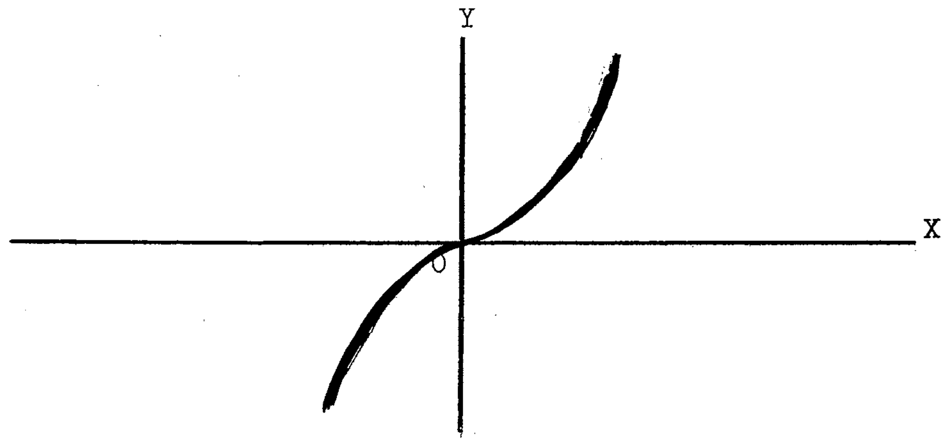

Figure 2-11.

The function will then be represented graphically by a continuous curve as in Figure 2-12. Geometric intuition shows at once that for at least one value of x between a and b the tangent is parallel to the x-axis (as at P), that is, the slope is zero. This illustrates Rolle's Theorem which is now stated and proved.

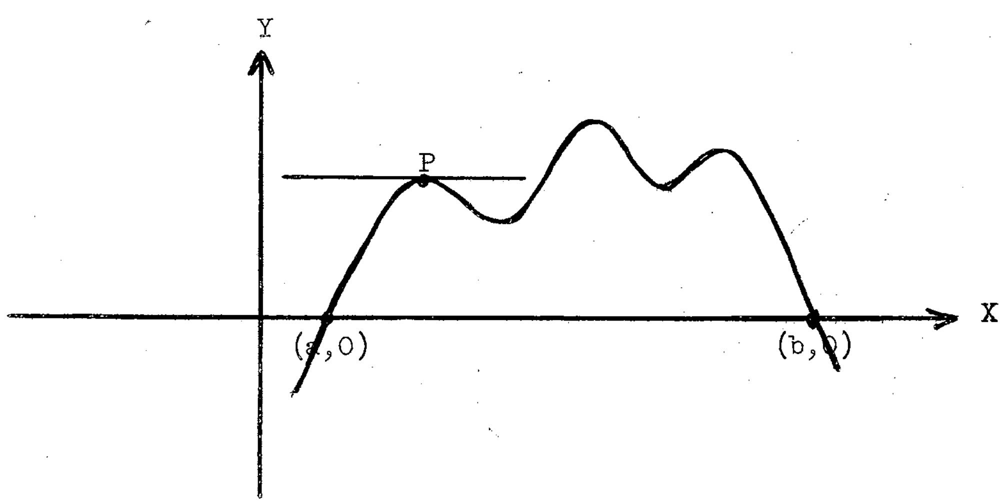

Figure 2-12.

Theorem 2-5 (ROLLE'S THEOREM).

(1) If f is a continuous function in the closed interval $[a,b]$,

(2) if $f(a) = f(b) = 0$, and

(3) if f has a derivative at every interior point of (a,b), then f' must vanish for at least one value of x between a and b, i.e., $f'(x_0) = 0$ for $a < x_0 < b$ [5].

Proof: If $f(x) = f(a)$ for every x in $[a,b]$, then f is a constant function and $f'(x) = 0$. This implies that every x in (a,b) is a critical point. Suppose $f(x) \neq f(a)$ for some x in (a,b), then either the maximum value if $f(x) > f(a)$ or the minimum value if $f(x) < f(a)$ of f occurs at a number c in (a,b). This number c is, by Theorem 2-4, a critical number of f.

Illustrative Example 2-5. Let $f(x) = 9 - x^2$ (Figure 2-13). Since $f(-3) = 0$ and $f(3) = 0$, f has a critical number between -3 and 3. One sees that 0 is the critical number in this case.

Suppose that in Rolle's Theorem one changes the condition that $f(a) = f(b)$ and let the other conditions remain in tact. Then, perhaps there exists a number c in (a,b) such that the tangent line T to the graph of f at $(c,f(c))$ is parallel to the secant line S on the points $(a,f(a))$ and $(b,f(b))$. See Figure 2-14. Since the tangent line T and the secant line S are parallel they must have equal slopes, i.e.,

$$f'(c) = [f(b) - f(a)]/[b - a].$$

The ideas expressed above are embodied in the following basic theorem, called the MEAN VALUE THEOREM.

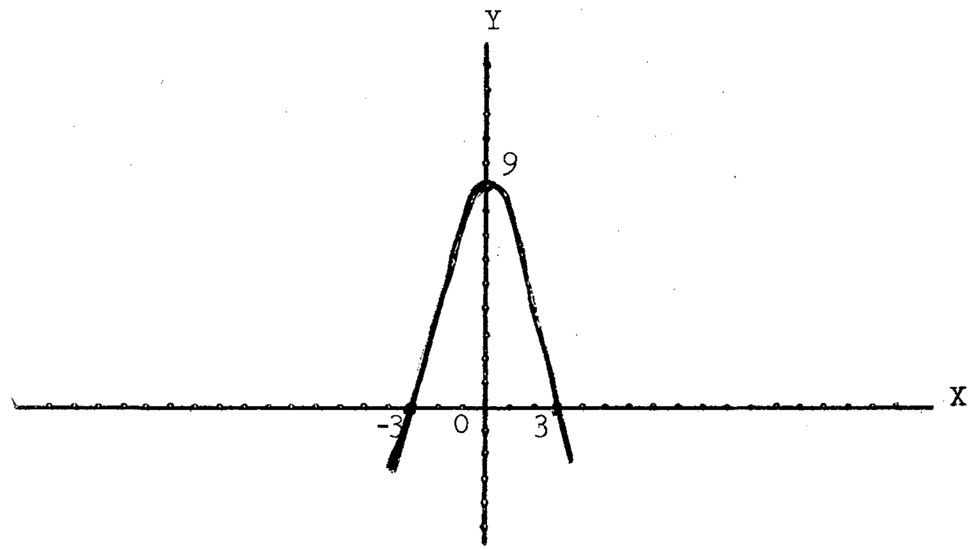

Figure 2-13.

Theorem 2-6. If f is a continuous function in a closed interval $[a,b]$ and if the open interval (a,b) is in the domain of f', then there exists a number c in (a,b) such that

$$f(b) - f(a) = (b - a)f'(c) \quad [5].$$

The proof of the above theorem given in the reference cited will be omitted here. However, this proof is based upon Rolle's Theorem.

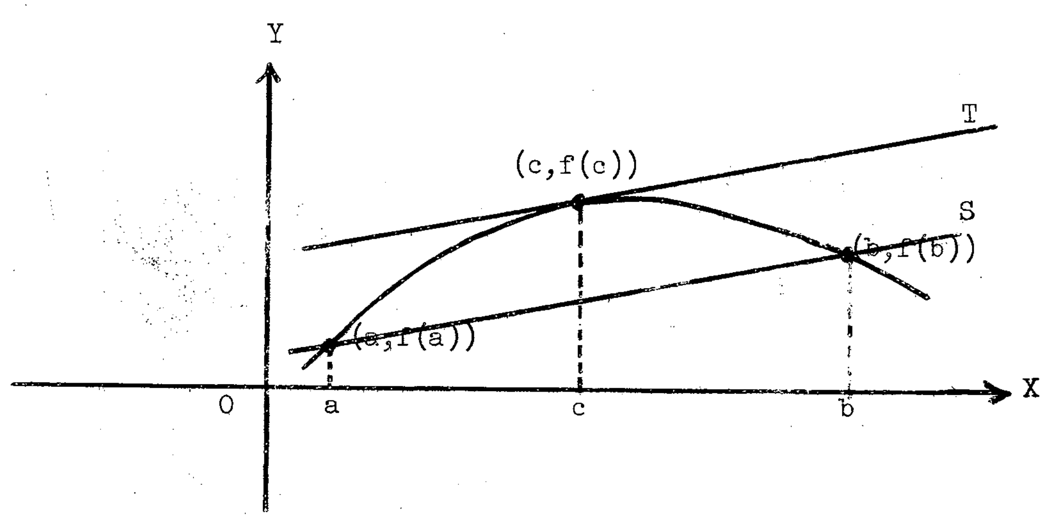

Figure 2-14.

Monotonic Functions

A function f is said to be increasing in an interval I contained in the domain of f if

$f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$ for all numbers x_1, x_2 in I.

If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for all numbers x_1, x_2 in I, then f is said to be strictly increasing in the interval I. Decreasing and strictly decreasing functions are similarly defined [5]. A function is said to be monotonic on I if it is increasing (or non-decreasing) or decreasing (or non-increasing) on I.

In connection with monotonic functions several theorems [5] which will be useful in later proofs are quoted now.

Theorem 2-7. If f is a function and I is an interval contained in the domain of f', then:

(1) f is strictly increasing in I if $f'(x) > 0$ for all x in I, and

(2) f is strictly decreasing in I if $f'(x) < 0$ for all x in I.

Theorem 2-8. If a function f is continuous in a closed interval $[a,b]$ and if the open interval (a,b) is contained in the domain of f', then:

(1) f is strictly increasing in $[a,b]$ if $f'(x) > 0$ for all x in (a,b), and

(2) f is strictly decreasing in $[a,b]$ if $f'(x) < 0$ for all x in (a,b).

Theorem 2-9. If a function f is continuous in $[a,b]$ and if

$f'(a) > 0$ and $f'(b) < 0$ [or $f'(a) < 0$ and $f'(b) > 0$], then f has a critical number c in (a,b).

Theorem 2-10. If a function f is continuous and has no critical number in an interval I contained in the domain f', then either $f'(x) > 0$ or $f'(x) < 0$ for all x in I. Therefore, f is strictly monotonic in I.

Tests for Relative Extrema of a Function

The tests discussed below apply to functions which are smooth enough to possess one or more derivatives. The graph of a function may have many "relative" maximum and minimum points. For example, Figure 2-15 displays a graph with several such points.

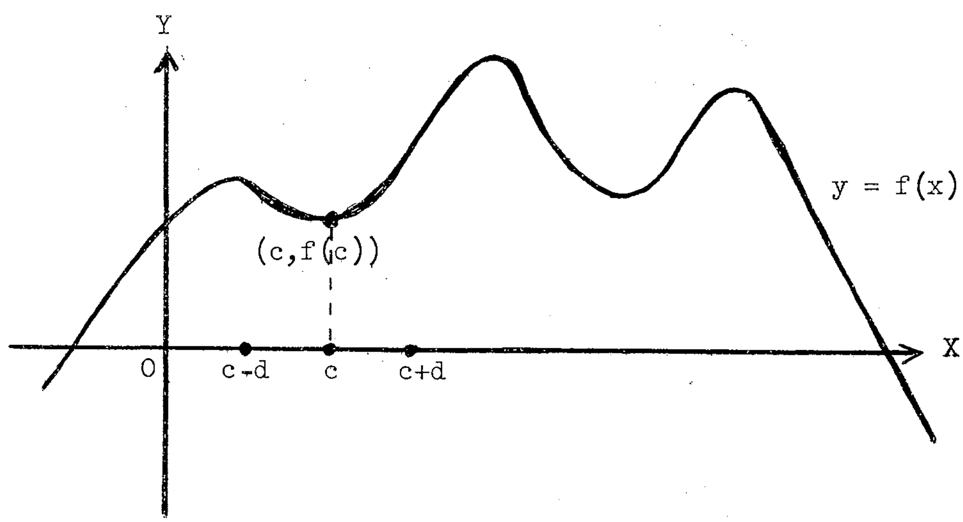

Figure 2-15.

Recall from definition 2-4 that the number $f(c)$ is called a relative extremum of a function f (Figure 2-15) if there exists a neighborhood $N(c)$ in the domain of f such that $f(c)$ is a maximum or minimum in $N(c)$. At this point, the First Derivative Test is now stated and proved.

Theorem 2-11 (FIRST DERIVATIVE TEST). Let f be a function and c a critical number of f. If a and b are numbers such that $a < c < b$, f' exists in $[a,b]$, and c is the only critical number of f in $[a,b]$, then:

(1) $f(c)$ is a relative maximum value of f if $f'(a) > 0$ and $f'(b) < 0$.

(2) $f(c)$ is a relative minimum value of f if $f'(a) < 0$ and $f'(b) > 0$.

(3) $f(c)$ is not a relative extremum otherwise [5].

Proof:

(1) Consider the interval $[a,c)$ in which there is no critical number. Then according to theorems 2-10 and 2-8(1) with $f'(a) > 0$, f is strictly increasing in $[a,c]$. Also since $f'(b) < 0$ and f has no critical number in $(c,b]$ by theorems 2-10 and 2-8(2), f is strictly decreasing in $[c,b]$. Hence, $f(c)$ is a relative maximum.

(2) Since f has no critical number in $[a,c)$ and $f'(a) < 0$ for every x in (a,c), f is strictly decreasing in $[a,c]$ by theorems 2-10 and 2-8(2). On the other hand, since f has no critical number in $(c,b]$ and $f'(b) > 0$ for every x in (c,b), f is strictly increasing in $[c,b]$ by theorems 2-10 and 2-8(1). Therefore, $f(c)$ is a relative minimum in $[a,b]$.

(3) If $f'(a) > 0$ for every x in (a,c) and f has no critical point in $[a,c)$, then f is strictly increasing in $[a,c]$.

Similarly, if $f'(b) > 0$ for every x in (c,b) and f has no critical number in $(c,b]$, then f is strictly increasing in $[c,b]$. Therefore, f has neither a maximum value nor a minimum value at c. Similarly, it can be shown that f is decreasing in $[a,b]$ if both $f'(a) < 0$ and $f'(b) < 0$. This completes the proof.

<u>Illustrative Example 2-6.</u> Let $f(x) = x^3 - 3x$ whose graph is given in Figure 2-16. Since $f(x) = x(x^2 - \sqrt{3}) = x(x + \ 3)(x - \sqrt{3})$, the x-intercepts of the graph are $-\sqrt{3}, 0, \sqrt{3}$ (See Figure 2-16).

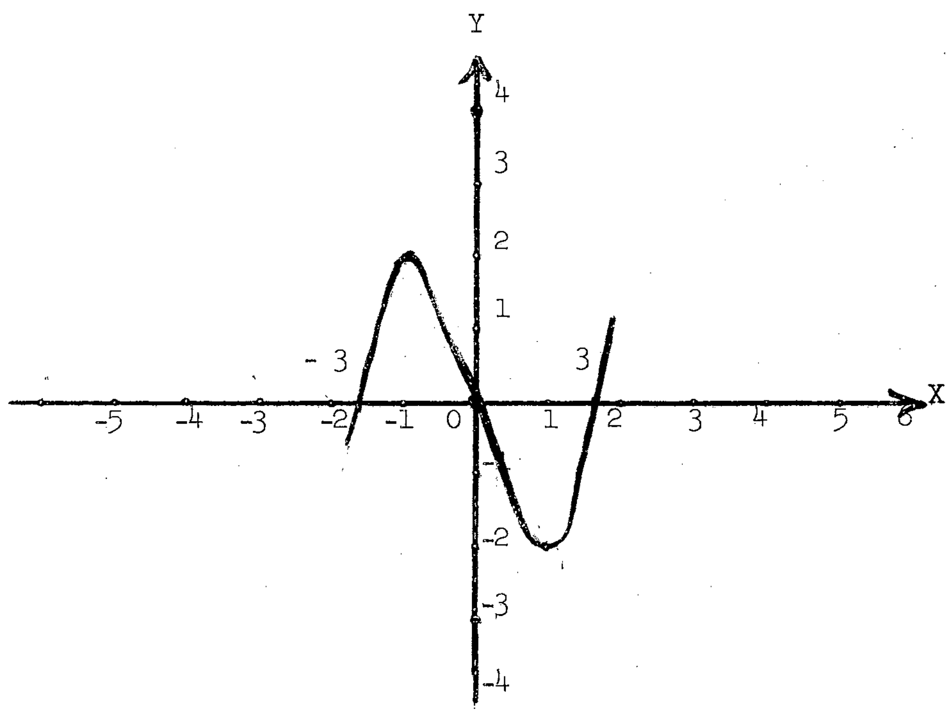

Figure 2-16.

Further information about the graph of f may be obtained from the derivative of f.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1)$, so $f'(x) = 0$ for $x = -1, 1$.

It can be shown that f' is positive to the right of $+1$ and to the left of -1 and is negative between -1 and 1. If $x > 1$, then

$(x + 1) > 0$ and $(x - 1) > 0$, so $f'(x) > 0$. If $-1 < x < 1$, then $x + 1 > 0$ and $x - 1 < 0$, so $f'(x) < 0$. If $x < -1$, then $x - 1 < 0$ and $x + 1 < 0$, so $f'(x) > 0$. So by theorem 2-3, the function f is increasing to the right of 1, increasing to the left of -1, and decreasing between these as indicated in the graph of f (Figure 2-16). By theorem 2-11, f has a relative minimum at $x = 1$ and a relative maximum at $x = -1$.

It must be observed that the First Derivative Test works only if the function is differentiable on the interval $a \leq c \leq b$.

The sign of the second derivative f'' reveals the concavity of the graph of a function f in somewhat the same manner that the sign of the first derivative reveals where the graph of f is rising and falling.

Definition 2-14. The graph of a function f is said to be concave upward at the point $(c,f(c))$ if $f'(c)$ exists and if there exists a deleted neighborhood $D(c)$ such that the graph of f in $D(c)$ is above the tangent line at $(c,f(c))$. The graph of a function f is said to be concave downward at the point $(c,f(c))$ if $f'(c)$ exists and there exists a deleted neighborhood $D(c)$ such that the graph of f in $D(c)$ is below the tangent line at the point $(c,f(c))$ [5].

In Figure 2-17, the graph of a function f which is concave upward and concave downward at the points $(a,f(a))$ and $(b,f(b))$ respectively is displayed.

Consider the equation of the tangent line to the curve at $(a,f(a))$, i.e., $y = f(a) + f'(a)(x - a)$. Let

$$g(x) = f(x) - [f(a) + f'(a)(x - a)]$$

be the vertical directed distance from the curve $y = f(x)$ to the tangent line $y = f(a) + f'(a)(x - a)$. Then, the sign of $g(x)$ will

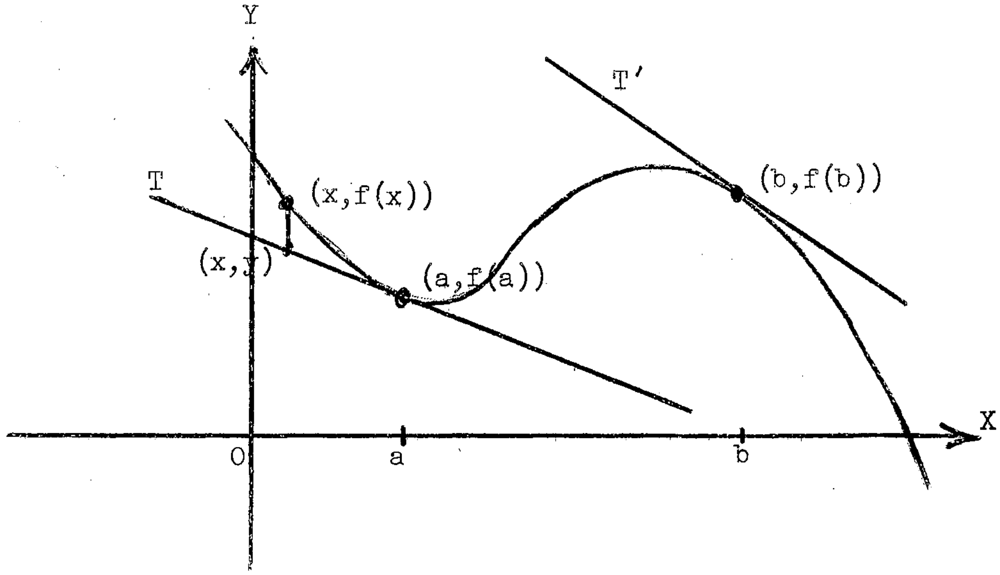

Figure 2-17.

determine if the tangent line is above or below the curve for every x in some deleted neighborhood of a. Thus the graph of f at $(a,f(a))$ is concave upward if $g(x) > 0$ for every x in some deleted neighborhood of a, and concave downward if $g(x) < 0$ for every x in some deleted neighborhood of a.

A test theorem for concavity based on the ideas expressed in the previous paragraph is now stated and proved.

Theorem 2-12 (TEST FOR CONCAVITY). If f is a function and c is a number such that the derivatives f', f'' are defined in some deleted neighborhood of a, then:

(1) The graph of f is concave upward at $(a,f(a))$ if $f''(a) > 0$.

(2) The graph of f is concave downward at $(a,f(a))$ if $f''(a) < 0$ [5].

Proof:

(1) If $f''(a) > 0$, then f' is increasing at a by theorem 2-3. By definition 2-13 there exists a neighborhood N of a such that $f'(x) < f'(a)$ if $x < a$ and $f'(x) > f'(a)$ if $x > a$ for every x in N. Applying the mean value theorem to

$g(x) = [f(x) - f(a)] - f'(a)(x - a)$, $g(x)$ can be written as

$g(x) = [f'(d) - f'(a)](x - a)$

for some number d between x and a. If x is in N and $x < a$, $x < d < a$, $f'(d) < f'(a)$, and $g(x) > 0$. If x is in N and $x > a$, then $a < d < x$, $f'(a) < f'(d)$, and once again $g(x) > 0$. Therefore, $g(x) > 0$ for every x in $N(x \neq a)$, and the graph is concave upward at $(a,f(a))$.

(2) If $f''(a) < 0$, then f' is decreasing at a. By the definition of a decreasing function there exists a neighborhood N of a such that $f'(x) > f'(a)$ if $x < a$ and $f'(x) < f'(a)$ if $x > a$. Since

$f(x) - f(a) = f'(d)(x - a)$, $g(x) = [f(x) - f(a)] - f'(a)(x - a)$

can be written as

$$g(x) = [f'(d) - f'(a)](x - a)$$

for some d between x and a. If x is in N and $x < a$, then $x < d < a$, $f'(d) > f'(a)$, $g(x) < 0$. If x is in N and $x > a$, then $a < d < x$, $f'(d) < f'(a)$, and once again $g(x) < 0$. Therefore, $g(x) < 0$ for every x in $N(x \neq a)$, and the graph is concave downward. This completes the proof.

The relationship between the direction of concavity and the nature of relative extrema is investigated next. Note that in Figure 2-18 at those points a, b where the tangent is horizontal, a relative maximum occurs where the curve is concave downward and a relative minimum where the curve is concave upward. Since the sign of the second derivative of a function determines the direction of its concavity, one should expect that there would be a test for relative extrema involving the second derivative. By virtue of this relationship, one gets the following useful corollary of theorem 2-12 called the SECOND DERIVATIVE TEST.

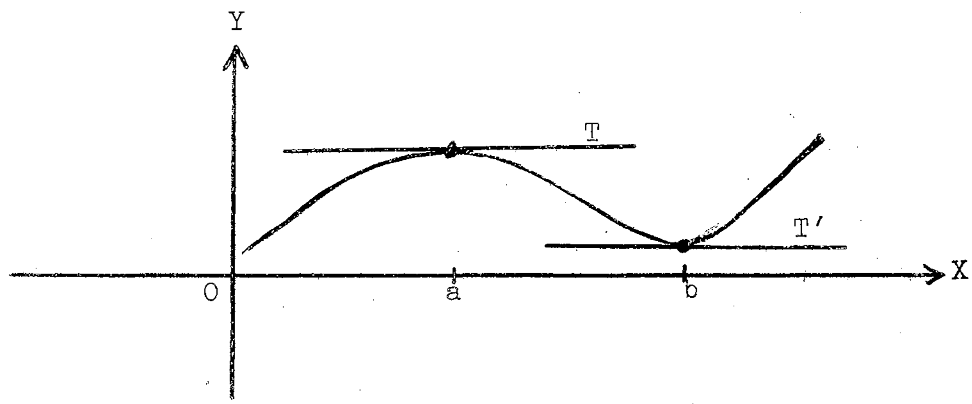

Figure 2-18.

Theorem 2-13 (SECOND DERIVATIVE TEST). If f is a function and c is a critical number of f such that f is twice differentiable in some neighborhood of c, then:

(1) $f(c)$ is a relative maximum value of f if $f'(c) = 0$ and $f''(c) < 0$.

(2) $f(c)$ is a relative minimum value of f if $f'(c) = 0$ and $f''(c) > 0$ [5].

These results follow readily from theorem 2-12 since the tangent line is now horizontal and the graph of f is below the tangent line in (1) and above the tangent line in (2) for some neighborhood of c.

Note that if c is critical number of f for which either

$f''(c) = 0$ or $f''(c)$ does not exist, then the second derivative test cannot be used. In this case, one should resort to the first derivative test. When applicable, the second derivative test is simpler to use.

Definition 2-15. The point $(c,f(c))$ is a point of INFLECTION of the graph of f if there exists a neighborhood (a,b) of c such that $f''(x) > 0$ for every x in (a,c) and $f''(x) < 0$ for every x in (c,b), or vice versa [5].

Intuitively, the point of inflection is the point P (See Figure 19) at which the curve changes from being concave upward to being concave downward (or vice versa) or the point at which the tangent line to the curve intersects the curve. This point can be quite helpful in plotting the graph of the curve.

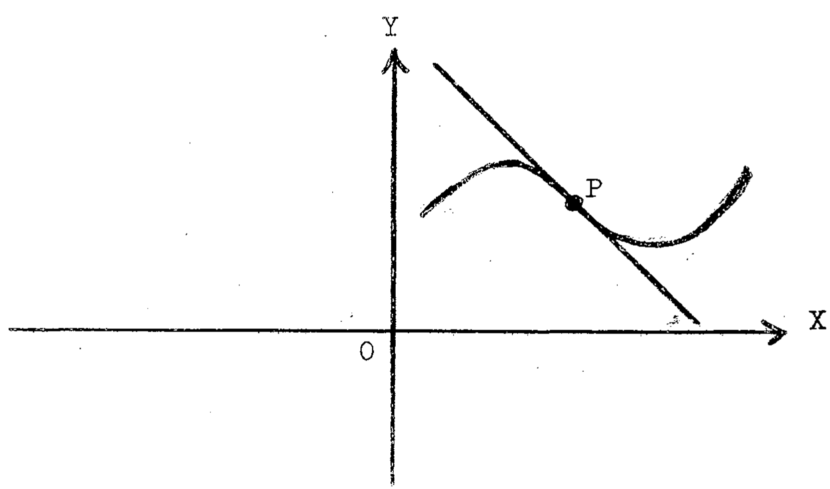

Figure 2-19.

If $(c,f(c))$ is a point of inflection of the graph of f and if $f''(c)$ exists, then necessarily $f''(c) = 0$. The points of inflection occur at the critical numbers of f'. However, it cannot be assumed that every critical number of f' will give a point of inflection of f.

The function $f(x) = x^4$ is a counter-example for $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Since $f'(0) = 0$ and $f''(x) > 0$ for every $x \neq 0$, 0 is a critical number of f', whereas $(0,0)$ is not a point of inflection of the graph of f. See Figure 2-20.

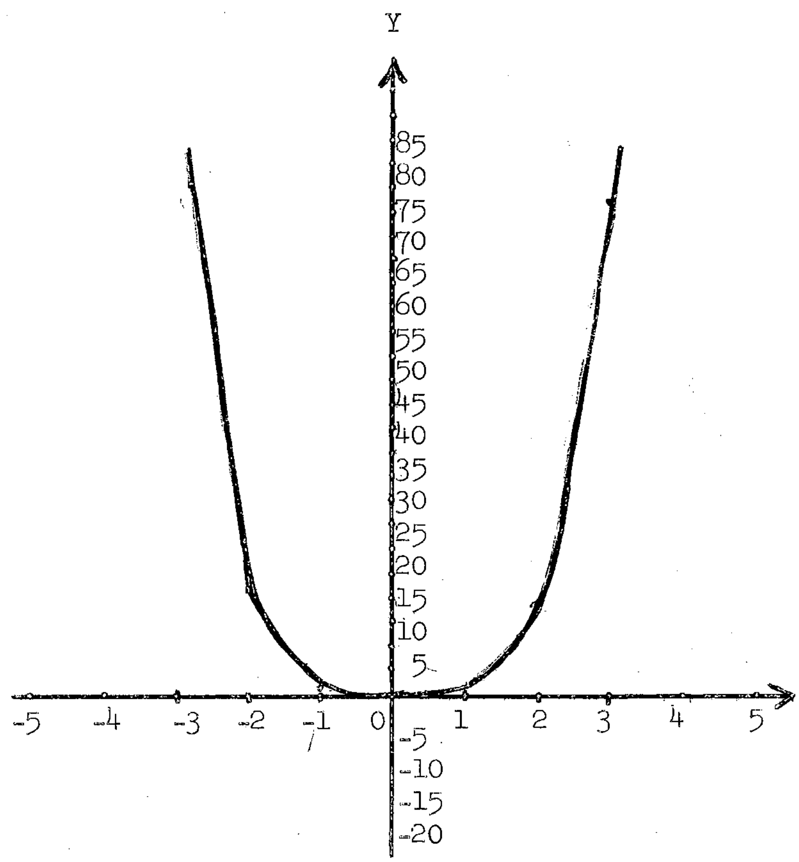

Figure 2-20.

A theorem which follows from the first derivative test will now be stated without proof.

Theorem 2-14. If f is a function and c is a number such that $f''(c) = 0$, then $(c,f(c))$ is a point of inflection of the graph of f provided there exists a neighborhood (a,b) of c such that

(1) $f''(x)$ exists and is nonzero for every x in $[a,b]$, $(x \neq c)$,

and

(2) $f''(a)$ and $f''(b)$ differ in sign [5].

This theorem is very useful in plotting the graph of a twice differentiable function of one variable.

The second derivative test is a sufficient condition for the existence of an extremum of a function f. This theorem can be generalized by means of Taylor's Theorem which may be thought of as an extension of the Mean Value Theorem since it reduces to that in the case $n = 1$. Taylor's Theorem is now stated without proof.

Theorem 2-15 (TAYLOR).

(1) Let f be a function and n be a nonnegative integer such that the nth derivative $f^{(n)}$ exists everywhere in the open interval (a,b).

(2) Assume $f^{(n-1)}$ is continuous on the closed interval $[a,b]$.

(3) Assume c belongs to $[a,b]$.

Then, for every x in $[a,b]$, $x \neq c$, there exists a point z in (x,c) such that

$$f(x) = f(c) + f'(c)(x - c) + f''(c)(x - c)^2/2 + f^{(3)}(c)(x - c)^3/6 + f^{(4)}(c)(x - c)^4/24 + \dots + f^{(n-1)}(c)(x - c)^{n-1}/(n - 1)! + f^{(n)}(z)(x - c)^n/n! \quad [6].$$

The next theorem is a generalization of the SECOND DERIVATIVE TEST.

Theorem 2-16. For some $n \geq 1$, let f have a continuous nth derivative in the open interval (a,b). Suppose also that for some interior point c in (a,b) one has

$$f'(c) = f''(c) = f^{(3)}(c) = \dots = f^{(n-1)}(c) = 0, \quad \text{but} \quad f^{(n)}(c) \neq 0.$$

Then for n even, f has a local (relative) minimum at c if

$f^{(n)}(c) > 0$, and a local maximum at c if $f^{(n)}(c) < 0$. If n is odd, there is neither a local maximum nor a local minimum [6].

Proof: By the definition of a derivative and since $f^{(n)}(c) \neq 0$, there exists a neighborhood $N(c)$ such that for every x in $N(c)$, the $f^{(n)}(x)$ will have the same sign as $f^{(n)}(c)$. Using Taylor's formula (Theorem 2-15), one can write for every x in $N(c)$ the following equation:

$$f(x) - f(c) = f^{(n)}(z)(x - c)^n/n! \text{ where } z \text{ belongs to } N(c).$$

If n is even, this equation implies $f(x) \geq f(c)$ whenever $f^{(n)}(c) > 0$ and $f(x) \leq f(c)$ whenever $f^{(n)}(c) < 0$ since $f^{(n)}(z)$ has the same sign as $f^{(n)}(c)$ in $N(c)$. Any even power of $(x - c)$ will be nonnegative. Consequently, the sign of $f(x) - f(c)$ will depend on the sign of $f^{(n)}(c)$ deciding whether $f(c)$ is a local maximum or a local minimum value of f at c.

If n is odd and $f^{(n)}(c) > 0$, then $f(x) > f(c)$ when $x > c$ but $f(x) < f(c)$ whenever $x < c$ implying f is increasing at c by definition. On the other hand if n is odd and $f^{(n)}(c) < 0$, then $f(x) > f(c)$ when $x < c$ but $f(x) < f(c)$ when $x > c$ implying f is decreasing at c by definition. Therefore, f can have no extremum if n is odd. Any odd power of $(x - c)$ is either positive or negative depending on whether $x > c$ or $x < c$ respectively.

Illustrative Example 2-7. Examine $f(x) = e^x + 2\cos(x) + e^{-x}$ for maximum and minimum values.

Solution:
$$f(x) = e^x + 2\cos(x) + e^{-x},$$
$$f^{(1)}(x) = e^x - 2\sin(x) - e^{-x} = 0 \quad \text{for} \quad x = 0,$$
$$f^{(2)}(x) = e^x - 2\cos(x) + e^{-x} = 0 \quad \text{for} \quad x = 0,$$

$f^{(3)}(x) = e^x + 2 \sin (x) - e^{-x} = 0$ for $x = 0$,

$f^{(4)}(x) = e^x + 2 \cos (x) + e^{-x} = 4$ for $x = 0$.

Therefore, by Theorem 2-16, $f(0) = 4$ is a minimum value of f.

Problems

There are many interesting problems in geometry, the physical sciences, engineering, industry, business, economics and the other social sciences which are related to the theory of extrema. The remainder of this chapter will be devoted to the solution of a few problems of the types mentioned above.

Problem 2-1. A ladder is to reach over a fence 8 ft. high to a wall 1 ft. behind the fence. What is the length of the shortest ladder that can be used [5]?

Solution: In Figure 2-21, let L represent the length of the ladder, x the distance of the foot of the ladder from the fence, y the height of the top of the ladder, and A the angle of inclination of the ladder above the ground. Then

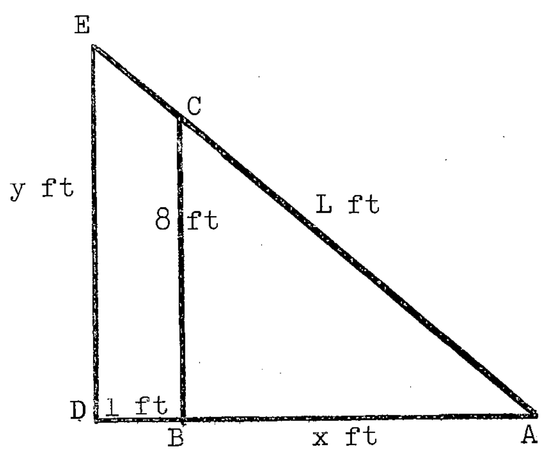

Figure 2-21.

$$x = 8 \cot A \tag{1}$$

$$L/(x + 1) = \sec A \tag{2}$$

$$L = x \sec A + \sec A \qquad (3)$$

$$L(A) = 8 \cot A \sec A + \sec A, \quad 0^\circ < A < 90^\circ \qquad (4)$$

Note that if $A = 0^\circ$, then the ladder would have to be of infinite length, and if $A = 90^\circ$, then the ladder would be parallel to the wall or it would be of infinite length touching the wall at the ideal point in the sense of projective geometry. So the angle A must be between 0° and 90°.

$$L(A) = 8 \csc A + \sec A, \quad 0^\circ < A < 90^\circ. \qquad (5)$$

Since $L(A)$ is continuous in the interval $0^\circ < A < 90^\circ$ and $L(A)$ approaches infinity both as A approaches 0° and as A approaches 90°, then, by Theorem 2-2, $L(A)$ has a minimum value in that interval.

The derivative of $L(A)$ exists at every interior point of $(0^\circ, 90^\circ)$. This fact along with the fact that $L(A)$ has a minimum value in $(0^\circ, 90^\circ)$ implies $L(A)$ has a critical number A_0 in $(0^\circ, 90^\circ)$ such that $L'(A_0) = 0$.

$$L'(A) = (-8 \csc A)(\cot A) + (\sec A)(\tan A) \qquad (6)$$

$$(8 \csc A)(\cot A) = \sec A \tan A \qquad (7)$$

$$[(\csc A)(\cot A)]/[(\sec A)(\tan A)] = .125 \qquad (8)$$

$$\cot^3 A = .125 \qquad (9)$$

$$\cot A = .5$$

$$A = \text{Arc} \cot (.5)$$

$$A = 63^\circ\ 26'$$

$$L(63^\circ\ 26') = 8 \csc 63^\circ\ 26' + \sec 63^\circ\ 26'$$

$$= 8\ (1.1180) + 2.2359$$

$$= 8.9440 + 2.2359$$

$= 11.1799$ ft., the shortest ladder.

Problem 2. A man in a motorboat 4 miles from the nearest point

P on the shore wishes to go to a point Q 10 miles from P along the straight shoreline. The motorboat can travel 18 miles per hour and a car, which can pick up the man at any point between P and Q, can travel 30 miles per hour. At what point should the man land so as to reach Q in the least amount of time [5]?

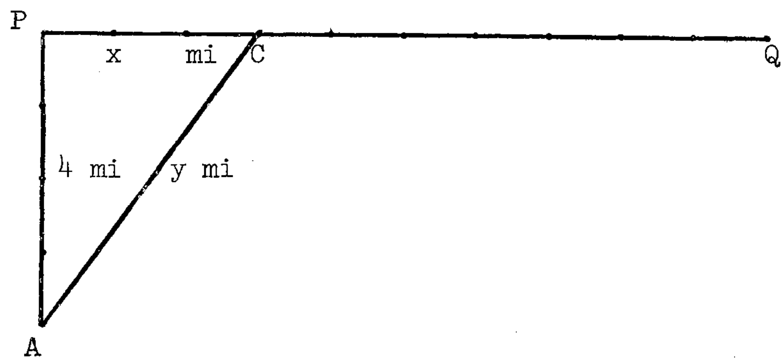

Figure 2-22.

Solution: In Figure 2-22.

$$y^2 = x^2 + 16 \tag{1}$$

$$y = \sqrt{(x^2 + 16)} \tag{2}$$

$$T(x) = [\sqrt{(x^2 + 16)}/18] + [(10 - x)/30],\ 0 \le x \le 10 \tag{3}$$

represents the total time equation. The function T is continuous in the closed and bounded interval [0,10] and therefore has a maximum and a minimum value on [0,10].

The derivative of the function T is

$$T'(x) = [x/18\sqrt{(x^2 + 16)}] - [1/30] \tag{4}$$

and T' exists at every interior point in [0,10]. Therefore, T has a critical number c such that $T'(c) = 0$ and $T(c)$ is an extremum.

$$T'(x) = [x/18\sqrt{(x^2 + 16)}] - [1/30] = 0$$

$$x/18\sqrt{(x^2 + 16)} = 1/30$$

$x = (3/5)\sqrt{(x^2 + 16)}$

$x^2 = (9/25)(x^2 + 16)$

$(16/25)x^2 = (9/25)(16)$

$x^2 = 9$

$x = 3$ mi. from P

$T(3) = 23/45$

$T(3) = .51$ hr. is the minimum time since $T(0) = .55 > T(3)$ and $T(10) = .59 > T(3)$.

Problem 3. A Boston lodge has asked the railroad company to run a special train to New York for its members. The railroad agrees to run the train if at least 200 people will go. The fare is to be 8 dollars per person if 200 go, and will decrease by 1 cent for everybody for each person over 200 that goes. What number of passengers will give the railroad maximum revenue [5]?

Solution: Let x be the number of persons over 200 to go. Then $(200 + x)$ = the total number of passengers,

$(800 - x)$ = the reduced fare per person, and

$(200 + x)(800 - x)$ = the total revenue.

Set $f(x) = 160000 + 600x - x^2, \quad 0 \le x < 800.$

The function f is known to be continuous on the whole real line and to have a maximum value in some interval of the real line. The derivative of f is $f'(x) = 600 - 2x$ for all x.

Since f has a maximum value and f' exists everywhere, then f has a critical number, i.e., $f'(x) = 600 - 2x = 0$ and $x = 300$. The second derivative test verifies the fact that f has a maximum value in that $f''(x) = -2$ and $f''(300) = -2 < 0$ which implies $f(300)$ is a maximum value of f.

Therefore, 500 passengers will give the railroad the maximum revenue.

<u>Problem 4</u>. A farmer erected a straight fence 100 feet long. In addition, he has 200 feet of fencing. He desires to use part or all of the fence already erected and the additional 200 feet of fencing to enclose a rectangular field. What should be the dimensions if the area is to be maximized?

<u>Solution</u>: Let x(in. feet) be the part of fence already standing that is needed. Then the side opposite this part will also be x feet. Thus $(200 - x)$ feet of fencing is left to be divided between the remaining two opposite sides of the field, $(100 - x/2)$ feet to each side. See Figure 2-23.

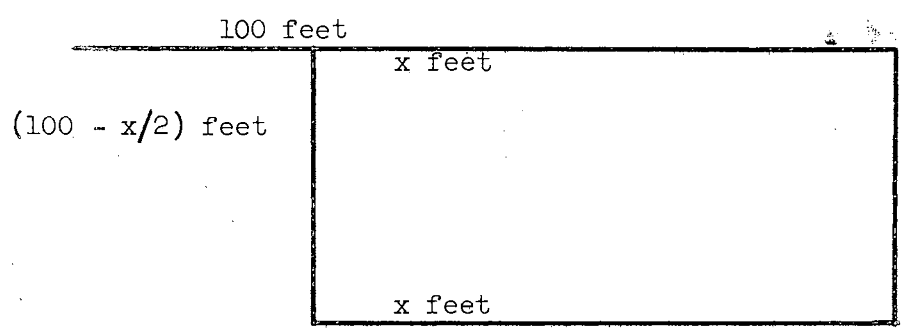

Figure 2-23.

Therefore, the area of the field is given by the function

$$A(x) = 100x - x^2/2.$$

The first derivative set equal to zero gives

$$D_x A(x) = 100 - x = 0.$$

The solution of this equation is $x = 100$. The second derivative test, i.e.,

$$D_x^2 A(x) = -1 < 0,$$

for all x, reveals that $A(100) = 5{,}000$ is the maximum value of the function. So the dimensions should be 50 feet x 100 feet.

Problem 5. A circular cylinder is to be made by rotating about the y-axis a rectangle two of whose sides lie along the x- and y-axes and which has one vertex at the origin and another on the curve $y = 1 - x$ in the first quadrant. Find the cylinder of greatest volume [7].

Solution: The volume of this cylinder is $V = \pi x^2 y$ where $x = 1 - y$. See Figure 2-24. Eliminate x by substitution.

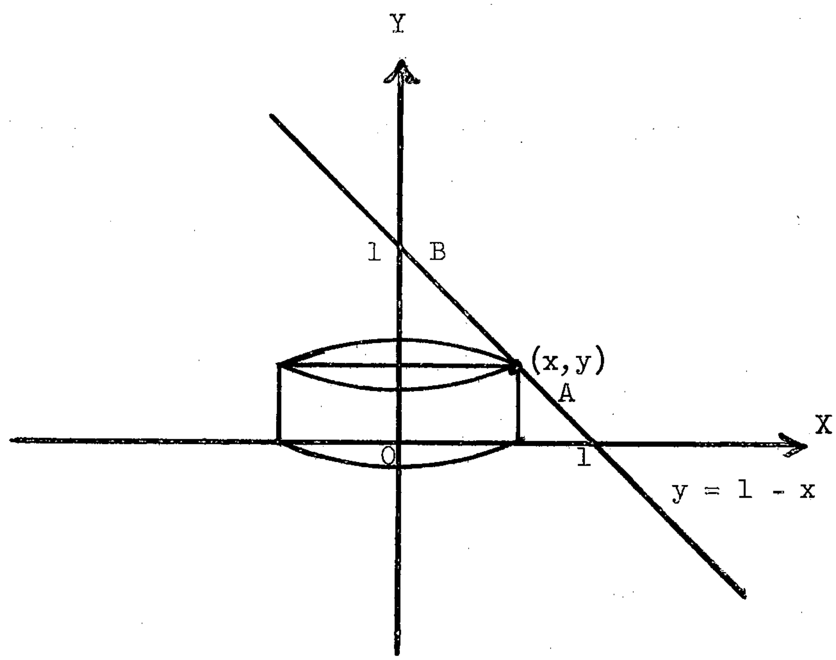

Figure 2-24.

Hence,

$$V(y) = \pi(y - 2y^2 + y^3).$$

Take the first derivative of $V(y)$ and set it equal to zero, i.e.,

$$3y^2 - 4y + 1 = 0.$$

The solution set consists of $y = 1/3, 1$. By the second derivative test (Theorem 2-13), the function $V(y)$ has a maximum value at $y = 1/3$, i.e., $V''(1/3) = -2\pi < 0$. Therefore, the cylinder of maximum volume has height $y = 1/3$ and radius $2/3$.

Problem 6 [15]. Suppose that the Airow and Zande Companies, with advertising budgets of a dollars and b dollars, respectively, are selling in a market having a total sales potential of Q units, and that there are no other firms competing in the market.

The Airow Company's expected sales volume has been found to be $aQ/(a + b)$. That is, the proportion of total possible sales which are obtained by the Airow Company is the ratio of the company's advertising expenditure to the total advertising expenditures by both firms.

The cost to the Airow Company of producing and selling N units (exclusive of advertising costs) is $C_1 + C_2N$, where C_1 represents fixed costs, and C_2 represents variable costs per unit. Therefore, if an expenditure of a dollars on advertising results in sales of $aQ/(a + b)$, the Airow Company's total costs are

$$C_1 + C_2[aQ/(a + b)] + a.$$

(1) Let P be the selling price per unit, which is assumed fixed and equal for both companies. Develop the expression for the profit of Airow as a function of its advertising expenditures.

(2) Determine the optimum level of advertising by

(a) Maximizing earnings

(b) Equating marginal revenue to marginal cost.

Are the two values equal? Explain.

(3) How does Airow's optimum action depend on Zande's decisions?

(4) Find a_0 when $b = 1{,}000{,}000$ dollars, $Q = 40{,}000$ units, $P = 200$ dollars, $C_2 = 100$ dollars, and $C_1 = 500{,}000$ dollars.

Solution:

(1) Let π represent the profit function. Then

$$\pi(a) = R(a) - C(a)$$

where $R(a) = P[aQ/(a + b)]$ is the revenue function and $C(a) = C_1 + C_2[aQ/(a + b)] + a$ is the cost function. Therefore,

$$\pi(a) = P[aQ/(a + b)] - C_1 - C_2[aQ/(a + b)] - a.$$

(2a) To determine the optimum level of advertising by maximizing earnings one proceeds as follows:

$$D_a\pi(a) = [PQb/(a + b)^2] - [C_2Qb/(a + b)^2] - 1.$$

Set $D_a\pi(a) = 0$ and solve for a.

$$[PQb/(a + b)^2] - [C_2Qb/(a + b)^2] - 1 = 0,$$

$$PQb - C_2Qb - (a + b)^2 = 0,$$

$$a^2 + 2ba - (PQb - C_2Qb - b^2) = 0,$$

$$a_0 = -b + [Qb(P - C_2)],$$

where one chooses the positive sign since $a > 0$. To verify that this value of a maximizes the function $\pi(a)$, one uses the second derivative test (Theorem 2-13), i.e.,

$$D_a^2\pi(a) = -[2PQb/(a + b)^3] + [2C_2Qb/(a + b)^3]$$

$$D_a^2\pi(a) = [-2PQb + 2C_2Qb][1/(a + b)^3].$$

Since P, Q, b, a are all positive quantities and $P > Q$, $[-2PQb + 2C_2Qb] < 0$ and $[1/(a + b)^3]$ for all a. Therefore,

$$D_a^2\pi(a) < 0.$$

Thus π has a maximum value at $a_0 = -b + Qb(P - C_2)$.

(2b) To determine the optimum level of advertising by equating marginal revenue to marginal cost one proceeds as follows:

$$D_a R = PQb/(a + b)^2 + 1 \quad \text{is the marginal revenue}$$

and

$$D_a C = C_2 Qb/(a + b)^2 + 1 \quad \text{is the marginal cost [15].}$$

Therefore,

$$PQb/(a + b)^2 = C_2 Qb/(a + b)^2 + 1,$$

$$PQb/(a + b)^2 - C_2 Qb/(a + b)^2 - 1 = 0,$$

$$a^2 + 2ab - (PQb - C_2 Qb - b^2) = 0,$$

which is the same quadratic equation in a obtained in (2a). Consequently, one gets the same value for a. An axiom of marginalism states that the optimum level of activity (i.e., that level where net profits are maximized) is where the marginal profit = 0. But this means, since marginal profit = marginal revenue - marginal cost, that marginal revenue = marginal cost [15].

(3) Airow's optimum action depends on Zande's advertising budget, i.e., $a = -b + \sqrt{Qb(P - C_2)}$.

(4) Find a_0 when b = 1,000,000 dollars, Q = 40,000 units, P = 200 dollars, C_2 = 100 dollars, C_1 = 500,000 dollars.

$a_0 = -1{,}000{,}000 + \sqrt{40{,}000(1{,}000{,}000)(100)}$

$a_0 = 1{,}000{,}000.$

The next problem has to do with finding the minimum cost lot size [16]. Inventory control is important for an efficiently functioning business firm. Although there is a growing variety of inventory control models, the central question in a typical inventory control problem is how to minimize the costs which are associated with obtaining and holding inventory. Such inventory optimization involves two kinds of costs: <u>setup</u> and inventory <u>carrying</u> costs. Setup cost represents the expense for setting up the machinery to produce a lot of units of a certain commodity. Setup cost is called reorder cost if the case involves ordering a lot from a supplier rather than producing it. Carrying cost is incurred because the produced lot of goods must be carried in stock until sold. Inventory carrying cost includes charges such as insurance, storage, interest, depreciation, obsolescence, and property taxes.

The inventory optimization problem arises from the fact that when setup cost is low, carrying cost tends to be high and vice versa. For instance, suppose the monthly demand for a commodity is uniform and fixed at 300 units. Then all 300 units could be produced by setting up the equipment once. In such a case the 300 units will be stocked at the beginning of each month and sold during the month. Since demand is uniform, the average monthly inventory will be 150 units. This inventory could be reduced by producing smaller than 300-unit lots each setup. For instance, producing a 150-unit lot every 15 days will reduce the monthly inventory to 75 units, a 100-unit lot every 10 days to 50 units, and so on. As inventory goes down, carrying cost will tend to fall, however, since smaller inventory requires more frequent production setups, these costs will tend to rise. In other words, large inventories are associated with low setup and high carrying costs, while small inventories tend to have the opposite effect. The problem is to find the minimum cost (setup plus carrying) lot size.

Minimum cost or optimum lot size formulae differ depending on the underlying assumptions of each inventory problem. The problem which follows is a case where the demand for the commodity is uniform and fixed each period with price predetermined.

Problem 7 [16]. A manufacturer needs 160 motors each of the 250 business days of the year for assembling an equivalent number of clothes dryers. He purchases these motors from another manufacturer at 20 dollars apiece. In order to finance this expense, he borrows from a bank on short-term basis at a simple interest rate of 5 per cent per annum. In addition, he has other inventory charges which amount to 1 dollar per motor per year. The 20-dollar price the manufacturer pays for each motor includes all shipping expenses except a fixed charge of 100 dollars per shipment.

(a) How many motors should the manufacturer order each time in order to minimize his annual inventory cost? What is this cost? How many orders does he have to place each year?

(b) Suppose that the demand for the manufacturer's clothes dryers has increased to the point that he plans to expand production to 1600 clothes dryers a business day. Although other costs remain the same, he has to provide now for special storage space for the entire shipment. He figures warehousing cost will be 1 dollar per motor. Under these new conditions, answer the questions in (a).

(c) With the demand of 1600 motors a business day, the supplier informs him that because of rising labor costs he can no longer absorb shipping costs. Such costs amount to 2 dollars per motor, but the fixed shipping charge per shipping has been reduced to 40 dollars per order. Answer the questions in (a) if other conditions remain as in (b).

Solution:

(a) Let x be the lot size. Then $x/2$ is the average inventory in stock throughout a period between reorders. On a short-term basis, the interest cost for each motor per year is $.05(20) = 1.00$. Therefore, the total inventory carrying cost is 2 dollars per motor per year or x dollars. For an annual demand of 40,000 motors the number of reorders per year is $(40{,}000/x)$. Since the cost per reorder is $(20x + 100)$, the total annual reorder cost is

$$(40{,}000/x)(20x + 100) = 800{,}000 + (4{,}000{,}000/x).$$

The total annual inventory cost is the sum of the carrying cost plus the reorder cost, i.e.,

$$C(x) = x + (800{,}000 + 4{,}000{,}000/x)$$

or

$$C(x) = (4{,}000{,}000/x) + 800{,}000 + x.$$

The first derivative is

$$D_xC(x) = 1 - 4{,}000{,}000/x^2.$$

When $D_xC(x) = 0$, one has

$$x^2 - 4{,}000{,}000 = 0$$

$$x = 2{,}000, \text{ the optimum lot size.}$$

The second derivative of $C(x)$ is

$$D_x^2C(x) = 8{,}000{,}000/x^3$$

and

$$D_x^2C(2{,}000) = 8{,}000{,}000/8{,}000{,}000{,}000$$
$$= (1/1{,}000) > 0$$

implying that $C(2{,}000)$ is a minimum value of $C(x)$ by the second derivative test (Theorem 2-13).

Hence the manufacturer should order 2,000 motors each time in order to minimize his annual inventory cost. The cost is $C(2{,}000) = 804{,}000$ dollars. He will have to place 20 orders each year.

(b) Let x be the lot size. Then $x/2$ is the average inventory in stock throughout a period between reorders. The carrying cost is now $(x/2)(1 + 1 + 1) = (3/2)x$. For an annual demand of 400,000 motors, the number of reorders is $(400{,}000/x)$. The total annual reorder cost is

$$(400{,}000/x)(20x + 100) = 8{,}000{,}000 + (40{,}000{,}000/x).$$

Therefore, the total annual inventory cost is given by the function

$$C(x) = (3/2)x + (8{,}000{,}000 + 40{,}000{,}000/x)$$

or

$$C(x) = 40{,}000{,}000/x + 8{,}000{,}000 + (3/2)x.$$

Taking the first derivative and setting the same equal to 0, one gets

$$D_x C(x) = 3/2 - (4 \times 10^7)/x^2 = 0.$$

Solving this equation, one gets

$$x^2 = (80/3)(10^6), \quad \text{or} \quad x = 5164.$$

Therefore, the manufacturer should order 5,164 motors each time, except the last time when he will only need to order 2,372 motors. The annual inventory cost will be

$$\begin{aligned} C(5{,}164) &= (3/2)(5{,}164) + 8{,}000{,}000 + (40{,}000{,}000/5{,}164) \\ &= 7646 + 8{,}000{,}000 + 7745.93 \\ &= 8{,}015{,}391.93 \text{ dollars.} \end{aligned}$$

He will need to place 77 orders of 5,164 motors each and 1 short order of 2,372 motors.

(c) Let x be the lot size. The inventory carrying cost is $(3/2)x$. The total annual reorder cost now is

$$(400{,}000/x)(22x + 40) = 8{,}800{,}000 + (16{,}000{,}000/x).$$

Hence the cost function is now

$$C(x) = (3/2)x + 8{,}800{,}000 + (16{,}000{,}000/x)$$

or

$$C(x) = (16{,}000{,}000/x) + 8{,}800{,}000 + (3/2)x.$$

The first derivative is

$$D_x C(x) = -(16{,}000{,}000/x^2) + 3/2.$$

Set $D_x C(x) = 0$ and solve for x, i.e., solve

$$(3/2)x^2 - 16{,}000{,}000 = 0.$$

The solution is

$x = 9796/3$ or 3265, the optimum lot size.

The cost is

$$C(9796/3) = \frac{16{,}000{,}000}{9796/3} + 8{,}800{,}000 + \frac{3}{2}(9796/3)$$

$$= 8{,}809{,}797.96 \text{ dollars.}$$

The manufacturer will have to place 122.5 orders or 122 complete orders and 1 short order.

The next problem relates to price, demand, and supply. A brief discussion on market equilibrium and monopoly is needed in order to clarify the terms and concepts which will be used.

A demand function shows the relationship between the quantity demanded X and the price p charged on a given market. In general, the higher the price the lower the demand, though there may be exceptions to this rule [17]. Collective or market-demand functions can be constructed by adding the individual-demand functions for all individuals in a market. The quantities demanded at a given price by all individuals are added. An approximation to a demand function, which may be linear, quadratic, and so on, can be derived, by statistical methods, from actual market data, that is, by using the prices and quantities recorded at various times on the market.

The individual-supply function of a firm, or of a private individual, shows the amount of a commodity that will be offered on the market at a given price. The collective or market-supply function is the sum of the amounts supplied by various individuals or firms at a given price [17]. Supply functions, which may be linear, quadratic, and so on, are derived from production theory and can also be obtained statistically from market data, that is, from the record of prices and quantities sold on a market at various dates. The statistically derived supply function is an approximation.

Under free competition no individual or firm can by itself influence the market price. There is free movement in and out of various industries. Market equilibrium exists under free competition if the quantity of a commodity demanded is equal to the quantity supplied. This fact determines the equilibrium price and the quantity exchanged [17].

The profit of the monopolist is $\pi = R - C$, where the total revenue $R = pX$. The function pX is the demand function and C is the total-cost function. A monopolist tries to maximize his profit by producing the amount and charging the price that will make his profit as large as possible.

The necessary condition for a maximum is $\pi' = R' - C' = 0$ or $R' = C'$, that is, marginal revenue equals marginal cost. For maximum profit, $\pi'' < 0$ or $R'' - C'' < 0$, from which we obtain $R'' < C''$. The last condition assures the stability of the situation. There is no incentive to the monopolist to produce more or less or charge a different price. One has a maximum rather than a minimum [17].

<u>Problem 8</u> [17]. The demand for steel in the United States is

estimated to be $p = 250 - 50X$ (Whitmann). The estimated average cost of making steel is $A = 182/X + 56$ (Yntema).

(a) Find R, C, C', R'.

(b) Find the necessary condition for a maximum of steel profits assuming a monopoly in steel.

(c) Find the sufficient condition for maximum profits.

(d) Establish the quantity produced, price, total revenue, total cost, and profits under monopoly.

(e) Assume the same demand curve and a competitive supply curve (marginal-cost curve) for steel, $p = 56$, assuming the same cost curve under free competition and monpoly. Find the quantity produced and the price established under conditions of free competition.

(f) Plot the demand curve, average-cost curve, marginal-cost curve (same as supply curve), marginal-revenue curve. Demonstrate the price formation under monopoly and under free competition.

Solution:

(a) $R = pX = 250X - 50X^2$, total-revenue

$C = AX = 182 + 56X$, total-cost curve

$C' = 56$, marginal-cost curve

$R' = 250 - 100X$, marginal-revenue curve

(b) The profit function assuming a monopoly in steel is

$$\pi = R - C,$$

or

$$\pi = -50X^2 + 194X - 182.$$

A necessary condition for π to have a maximum is that $\pi' = 0$ by Theorem 2-4. Since

$$\pi = pX - AX,$$

then

$$\pi' = p - A = 0 \quad \text{or} \quad p = A.$$

(c) The sufficient condition for π to be maximized is that

$$\pi''(X) = -50 + 182/X^2 < 0.$$

(d) $-100X + 194 = 0,$

$X = 1.94,$ quantity produced,

$p = 250 - 50(1.94) = 153,$ price,

$R = 153(1.94) = 296.82,$ total revenue,

$C = 182 + 56(1.94) = 290.64,$ total cost,

$\pi = R - C = 296.82 - 290.64 = 6.18,$ profits.

(e) $p = 250 - 50X,$ demand curve,

$p = 56,$ competitive-supply curve.

To establish price under free competition, set demand function equal to supply function, solve for X, and then substitute the result in the demand equation to obtain the equilibrium price, p, i.e.,

$$250 - 50X = 56$$

$$X = 3.88$$

$$p = 250 - 50(3.88)$$

$$p = 56$$

(f) See Figure 2-25.

<u>Problem 9</u> [8]. A man can walk 200 yards per minute and can swim 100 yards per minute. To get from a point A on the edge of a circular pool 400 yards in diameter to a point B diametrically opposite he may walk around the edge, swim straight across, or walk part way around and swim the rest of the way in a straight line. How shall he proceed if he is to make the trip in the least time? greatest time?

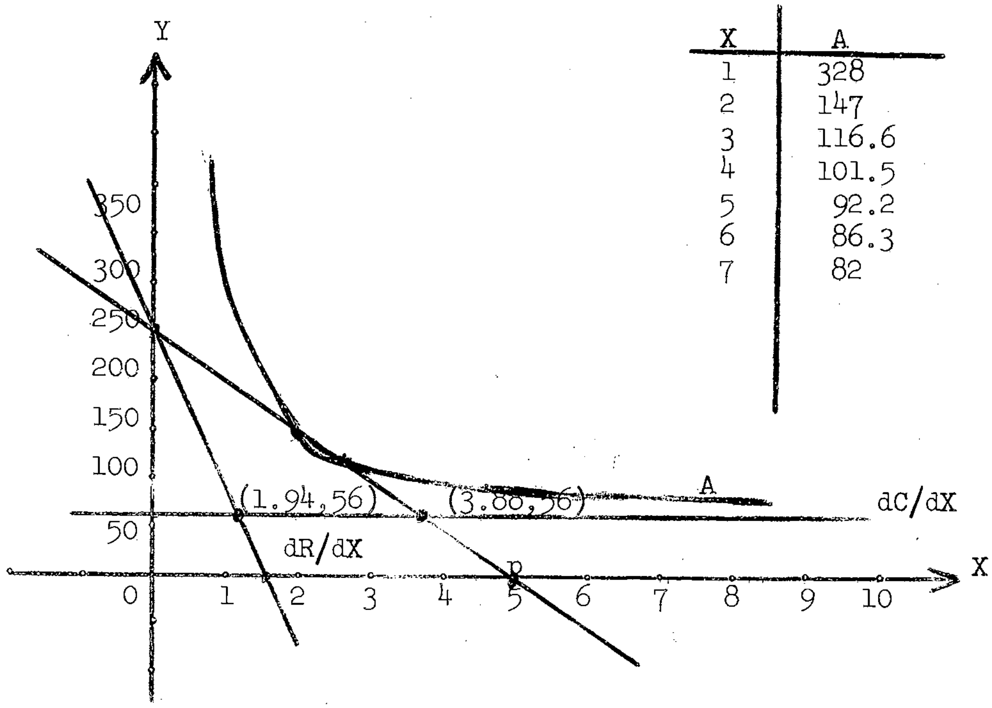

Figure 2-25.

Solution: In figure 2-26, AOB = 400 yards is the diameter. arc(AP) = 400θ, PB = $400\cos(\theta)$. Therefore, the total time function is

$$T(\theta) = \frac{400\theta}{200} + \frac{400\cos(\theta)}{100}.$$

or

$$T(\theta) = 2\theta + 4\cos(\theta),\ 0 \le \theta \le \pi/2.$$

Differentiating with respect to θ, one gets

$$D_\theta T(\theta) = 2 - 4\sin(\theta)$$

and the second derivative is

$$D_\theta^2 T(\theta) = -4\cos(\theta).$$

When $D_\theta T(\theta) = 0$, one obtains $\theta = \pi/6$ and $D_\theta^2 T(\pi/6) < 0$ which implies $T(\pi/6)$ is a maximum value at an interior point in $[0,\pi/2]$, by Theorem 2-13. The Second Derivative Test does not reveal that the function $T(\theta)$ has a minimum value at an interior point of $[0,\pi/2]$. However, since this function is continuous on a closed and bounded

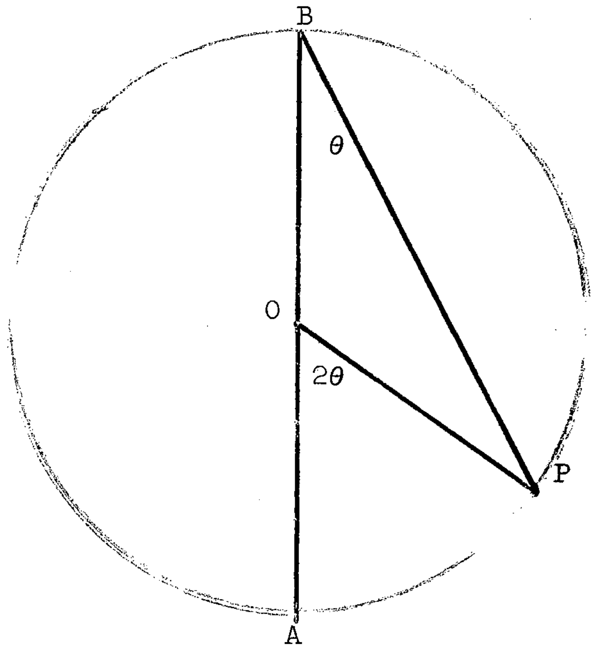

Figure 2-26.

interval $[0, \pi/2]$ of E_1, it has both an absolute maximum and an absolute minimum in this interval by Theorem 2-1. Therefore, the function must have a minimum at one of its end points. If one evaluates the function T at the endpoints, he gets $T(0) = 4.00$ and $T(\pi/2) = 3.14$. So the minimum of T is $T(\pi/2) = 3.14$. See Figure 2-27.

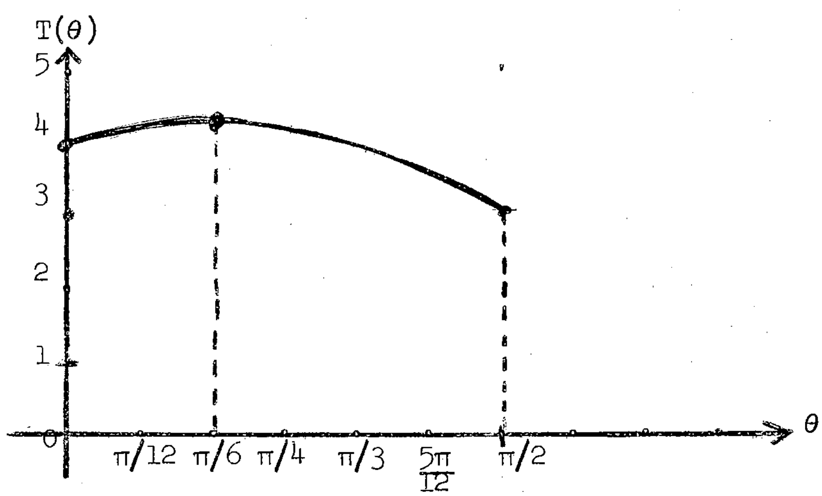

Figure 2-27.

Hence, the man should walk in order to make the trip in the least amount of time. To make the trip in the greatest amount of time, he should walk $400\pi/6$ yards around the edge of the pool and then swim 346.4 yards in a straight line from point P to point B.

CHAPTER III

FUNCTIONS OF SEVERAL VARIABLES

Most of what was said about functions, continuity, neighborhood of points, derivatives, etc., in connection with the extrema of functions of one variable can be generalized or extended to functions of several variables. For instance, a neighborhood of a point x_0 in E_1 was defined to be an open interval of the form $(x_0 - h, x_0 + h)$. However, such an interval can also be described by the inequality $|x - x_0| < h$. If $X = (p,q)$ and $X_0 = (p_0,q_0)$ are points in the plane, this inequality, with $|X - X_0|$ appropriately defined, still makes sense and, in fact, describes the interior of a circle with center at X_0 and radius h. Such a circle is called a two-dimensional neighborhood of a point in the plane. This concept may also be expressed as

$$(p - p_0)^2 + (q - q_0)^2 < h^2$$

for points in E_2. A point X_0 in E_3 (three-dimensional space) is defined as an ordered triple of real numbers (p_0,q_0,r_0) and a neighborhood of X_0 in E_3 can be expressed as

$$|X - X_0| < h \quad \text{or} \quad (p_1 - p_0)^2 + (q_1 - q_0)^2 + (r_1 - r_0)^2 < h^2.$$

In this manner, one can just as easily consider an ordered set of n real numbers $(p_1, p_2,\ldots,p_n)$ and refer to this as a point in a n-dimensional space. The distance between the points P and Q can be defined to be

$$|Q - P| = [(q_1 - p_1)^2 + (q_2 - p_2)^2 + \ldots + (q_n - p_n)^2]^{1/2}$$

and a neighborhood of P can be defined as $|Q - P| < h$. This discussion of the definition of neighborhood illustrates the way in which many concepts in one dimension can be readily extended to n dimensions. In this chapter some basic definitions and theorems characterizing functions of two, three, and n variables will be treated as they apply to the extrema of such functions. Necessary and sufficient conditions for a function of several variables to have an extremum will be developed. Illustrative examples and figures where possible will be used to clarify the theory. Some of the definitions and theorems will be stated for the two-dimensional case in such a way that they can be readily extended to three or more dimensions. Finally, Lagrange's method will be explained and illustrated.

Review of Some Basic Properties

A real-valued function f of n variables may be thought of as a mapping having its domain in E_n and its range in E_1 (reals). The concepts of neighborhood, interval, interior point, accumulation point, limit, and continuity in E_n are stated as extensions of their one-dimensional analogues. Differentiation of a real-valued function of several variables, in one sense, is achieved by treating it as a function of one variable at a time. This leads to the concept of a partial derivative. The existence of a differential is a basic property involved in extending the principal theorems of one-dimensional derivative theory to functions of several variables. The Mean Value Theorem is extended to a function of several variables. Higher order partial derivatives and differentials are used to extend Taylor's formula to a function of several variables.

Definition 3-1. A real-valued function in E_2 is the set f of ordered pairs $(\bar{a},b)$ such that no two different pairs have the same first components and where $\bar{a}$ itself is an ordered pair, i.e., $\bar{a} = (x,y)$, and b is a real number. The set D_f of all $\bar{a}$ (first components) is called the domain of f, and the set R_f of all $b = f(\bar{a})$ [second components] is called the range of f.

If each number-pair (x,y) in the domain D_f of the function f is associated with a point in the rectangular coordinate plane, then the domain D_f of f may be represented as a region in the plane.

Illustrative Example 3-1. Let $f(x,y) = (1 - x^2 - y^2)$. Then the domain D_f of f is the set of all number-pairs (x,y) such that $x^2 + y^2 \leqslant 1$ and D_f in this case is a circle and its interior, a region of the rectangular coordinate plane. The reader may refer to Figure 3-1.

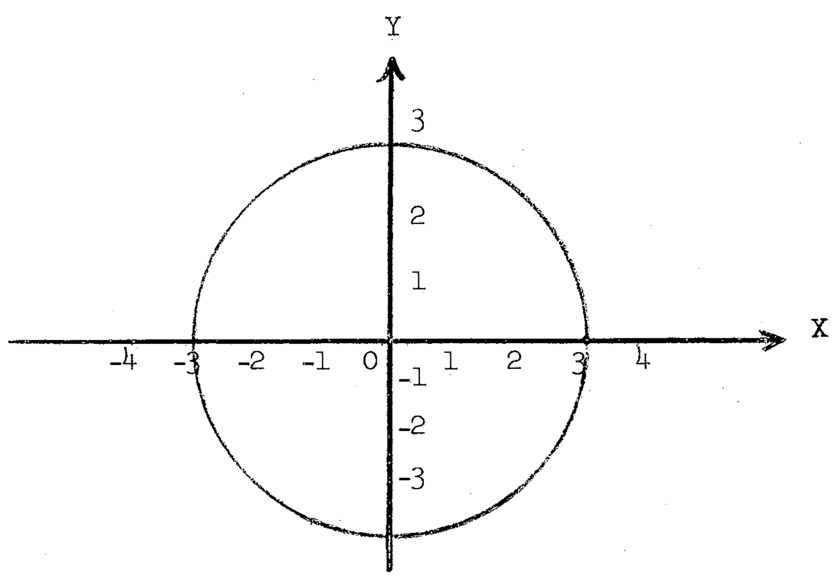

Figure 3-1.

Definition 3-1 for E_3 can be stated in similar manner simply by redefining $\bar{a}$ as (x,y,z). In E_3, the domain D_f of f may be represented as region in a rectangular coordinate space.

As a matter of notation, the letter x with the bar over it will represent a vector or a point in E_n, i.e., $\bar{x} = (x_1,x_2,\ldots,x_n)$, in the definitions which follow.

<u>Definition 3-2</u>. An open sphere of radius $r > 0$ having its center at the point $\bar{x}_0$ in E_n is the set of all $\bar{x}$ in E_n such that $|\bar{x} - \bar{x}_0| < r$. An open sphere with center at $\bar{x}_0$ is called a neighborhood of $\bar{x}_0$ and is denoted by $N(\bar{x}_0)$ or by $N(\bar{x}_0;r)$, if r is its radius. The open sphere with its center removed is called a deleted neighborhood of $\bar{x}_0$ and is denoted by $N'(\bar{x}_0)$ [6].

In E_2, this open sphere turns out to be an open circle. Instead of using circles in E_2, spheres in E_3 and n-dimensional spheres in E_n, we could use rectangles in E_2, rectangular parallelepipeds in E_3 and n-dimensional parallelepipeds in E_n as neighborhoods of points. The next definition generalizes the concept of a one-dimensional interval.

<u>Definition 3-3</u>. Let $\bar{a} = (a_1,a_2,\ldots,a_n)$ and $\bar{b} = (b_1,b_2,\ldots,b_n)$ be two distinct points in E_n such that $a_k < b_k$ for each $k = 1,2,\ldots,n$. The n-dimensional closed interval $[\bar{a},\bar{b}]$ is defined to be the set of points $(x_1,x_2,\ldots,x_n)$ such that $a_k \leq x_k \leq b_k$, $k = 1,2,\ldots,n$. If $a_k < b_k$ for every k, the n-dimensional open interval $(\bar{a},\bar{b})$ is the set of points $(x_1,x_2,\ldots,x_n)$ such that $a_k < x_k < b_k$, $k = 1,2,\ldots,n$ [6].

The open interval $(\bar{a},\bar{b})$ can be interpreted as the cartesian product $(\bar{a},\bar{b}) = (a_1,b_1)x(a_2,b_2)x\ldots x(a_n,b_n)$ of the n one-dimensional open intervals (a_k,b_k). In a similar manner, $[\bar{a},\bar{b}]$ can be expressed as a cartesian product of n one-dimensional closed intervals $[a_k,b_k]$.

With the concept of a neighborhood of a point $\bar{x}_0$ in E_n clarified, one can now generalize the definitions of interior and accumulation points in E_1 with little or no change in the wording of these definitions, except for dimension.

Definition 3-4. Let S be a set of points in E_n and assume $\bar{x}$ is in S. Then $\bar{x}$ is called an interior point of S if there exists a neighborhood $N(x)$ contained in S. The set S is said to be open if each of its points is an interior point [6].

Every open sphere or open interval is an example of an open set.

Definition 3-5. Assume S is contained in E_n, $\bar{x}$ is in E_n. Then $\bar{x}$ is called an accumulation point of S if every neighborhood $N(\bar{x})$ contains at least one point of S distinct from $\bar{x}$, i.e. the intersection of $N'(\bar{x})$ and S is not empty. A set S in E_n is said to be closed if it contains all of its accumulation points [6].

A closed sphere or a closed interval is a closed set.

Definition 3-6. A set S in E_n is said to be bounded if S lies entirely within some sphere or interval [6].

Consider a real-valued function defined on a set S in E_n. Let $\bar{a}$ be an accumulation point of S and let b belong to E_1. Then the limit $f(\bar{x}) = b$ as $\bar{x}$ approaches $\bar{a}$ if and only if: for every neighborhood $N(b)$ in E_1, there exists a neighborhood $N(\bar{a})$ in E_n such that

$$\bar{x} \text{ belongs to } N'(\bar{a}) \cap S \text{ implies } f(\bar{x}) \text{ is in } N(b).$$

It can easily be proved that if the limits of two functions f and g exist as $\bar{x}$ approaches $\bar{a}$, then the limits of their sum, difference, and product exist as $\bar{x}$ approaches $\bar{a}$. Furthermore, if $g(\bar{x}) \neq 0$ in some neighborhood of $N(\bar{a})$ then the limit of $(f/g)(\bar{x})$ exists as $\bar{x}$ approaches $\bar{a}$.

In the definition of the symbol $\lim_{\bar{x} \to \bar{a}} f(\bar{x}) = b$, it is not required that the function be defined at $\bar{x} = \bar{a}$. Moreover, if f is defined at $\bar{x} = \bar{a}$, the value of $f(\bar{a})$ need not be equal to b. In the event $f(\bar{a})$ is equal to b, then the function f is said to be continuous at $\bar{a}$.

Definition 3-7. Let f be defined on a set S in E_n with function values in E_1, and let $\bar{a}$ be an accumulation point of S. Then f is continuous at the point $\bar{a}$ if and only if

(i) f is defined at $\bar{a}$,

(ii) $\lim f(\bar{x}) = f(\bar{a})$ as $\bar{x} \to \bar{a}$.

If $\bar{a}$ is not an accumulation point of S, f is continuous at $\bar{a}$ if only (i) holds. If f is continuous at every point of S, then f is continuous on the set S [6]. It follows readily from the definitions that the sum, difference, or product of two continuous functions f and g is continuous. If $g(\bar{x}) \neq 0$ for all $\bar{x}$ in the intersection of D_f and D_g, then (f/g) is continuous in the intersection of D_f and D_g.

Two other concepts of extreme importance in the development of the theory of maxima and minima are the partial derivative and the differential of a function.

Definition 3-8. Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ be a point in E_n, and let $\bar{y} = (y_1, y_2, \ldots, y_n)$ be another point all of whose coordinates, except the kth coordinate, are the same as those of $\bar{x}$. Consider the limit

$$\text{limit } \frac{f(\bar{y}) - f(\bar{x})}{y_k - x_k} \quad \text{as} \quad y_k \to x_k.$$

When this limit exists, it is called the partial derivative of f with respect to the kth coordinate and is denoted by $D_k f(\bar{x})$ or $f_k(\bar{x})$ [6].

Although the concept of the partial derivative plays an important role in the theory of extrema, it is not a satisfactory extension of the one-dimensional concept of a derivative. For example, the mere existence of the partial derivatives of a function of several variables at a given point $(a_1, a_2, \ldots, a_n)$ does not imply continuity at that point. One is thus led to define the differential of a function. This concept permits us to extend the principal theorems of one-dimensional derivative theory to functions of several variables.

<u>Definition 3-9</u>. Let f be a real-valued function defined on an open set S in E_n, and assume $\bar{x}$ is in S. Then f has a differential at $\bar{x}$ if there exists another function g which satisfies the following conditions:

(a) g is a real-valued function of two n-dimensional variables, and the function values, denoted by $g(\bar{x};\bar{t})$, are defined for the given point $\bar{x}$ in S and for every point $\bar{t}$ in E_n.

(b) g is linear in the second variable, that is, for every pair of points $\bar{t}_1$ and $\bar{t}_2$ in E_n and for every pair of real numbers a_1 and a_2, we have

$$g(\bar{x};a_1\bar{t}_1 + a_2\bar{t}_2) = a_1 g(\bar{x};\bar{t}_1) + a_2 g(\bar{x};\bar{t}_2).$$

(c) For every $\epsilon > 0$, there exists a neighborhood $N(\bar{x})$ such that $\bar{y}$ is in $N'(\bar{x})$ implies

$$|f(\bar{y}) - f(\bar{x}) - g(\bar{x}; \bar{y} - \bar{x})| < \epsilon|\bar{y} - \bar{x}| \quad [6].$$

Frequently the symbol df is used instead of g and the symbols $dx_1, dx_2, \ldots, dx_n$ instead of $t_1, t_2, \ldots, t_n$ for the components of $\bar{t}$. In this notation one would write

$$df(\bar{x};d\bar{x}) = D_1f(\bar{x})dx_1 + \ldots + D_nf(\bar{x})dx_n$$

$$= \nabla f(\bar{x})\cdot d\bar{x}$$

where $\nabla f(\bar{x}) = (D_1f(\bar{x}), \ldots, D_nf(\bar{x}))$, a vector-valued function, is called the gradient of f. It is assumed that the n partial derivatives of f exist at the point $\bar{x}$ in E_n. A sufficient condition for the existence of the differential of f at a point $\bar{x}$ is that the n partial derivatives of f exist and be continuous at $\bar{x}$. Also, if a function f has a differential at $\bar{x}$, then f is continuous at $\bar{x}$.

<u>Definition 3-10</u>. If $\bar{x}$ and $\bar{y}$ are two distinct points in E_n, then by the line segment $L(\bar{x},\bar{y})$ joining $\bar{x}$ and $\bar{y}$, is meant the set

$$L(\bar{x},\bar{y}) = \{\bar{z} : \bar{z} = \theta\bar{x} + (1 - \theta)\bar{y},\ 0 < \theta < 1\}.$$

$L(\bar{x},\bar{y})$ is called an open line segment, and $L[\bar{x},\bar{y}]$ is called a closed line segment [6].

<u>Theorem 3-1</u> (MEAN VALUE THEOREM IN E_n). Assume that f has a differential at each point of an open set S in E_n. Let $\bar{x}$ and $\bar{y}$ be two points of S such that the line segment $L(\bar{x},\bar{y}) \subset S$. Then there exists a point $\bar{z}$ of $L(\bar{x},\bar{y})$ such that

$$f(\bar{y}) - f(\bar{x}) = \nabla f(\bar{z})\cdot(\bar{y} - \bar{x}) \quad [6].$$

In connection with differentials of higher order one needs to use partial derivatives of higher order. So at this point, these two concepts are defined and Taylor's formula for functions of several variables is stated.

Definition 3-11. Let f be defined on an open set in E_n, and assume that the partial derivative $D_k f$ exists in S. Then $D_{r,k}f$ will denote the partial derivative of $D_k f$ with respect to the rth variable, that is, $D_{r,k}f = D_r(D_k f)$ whenever this derivative exists. Higher-order partial derivatives are similarly defined [6].

Definition 3-12. Let f be a real-valued function defined on a subset of E_n. The second-order differential d^2f is a function of two n-dimensional variables defined for those points $\bar{x}$ in E_n where f has second-order partial derivatives and for every $\bar{t}$ in E_n by the equation

$$d^2f(\bar{x};\bar{t}) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j}f(\bar{x})t_j t_i, \text{ if } \bar{t} = (t_1, t_2, \ldots, t_n).$$

The third-order differential d^3f is defined by the equation

$$d^3f(\bar{x},\bar{t}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} D_{i,j,k}f(\bar{x})t_k t_j t_i,$$

and the mth order differential $d^m f$ is similarly defined when all mth order partials exist [6].

It should be noted that the higher-order differentials are not linear in the second variable. Also, d^2f is a quadratic form, d^3f a form of the third degree, and $d^m f$ a form of the mth degree.

Theorem 3-2 (TAYLOR'S FORMULA IN E_n). Let f have continuous partial derivatives of order m at each point of an open set S in E_n. If $\bar{a}$ is in S, $\bar{b}$ is in S, $\bar{a} \neq \bar{b}$, and if $L(\bar{a},\bar{b}) \subset S$, then there exists a point $\bar{z}$ on the line segment $L(\bar{a},\bar{b})$ such that

$$f(\overline{b}) - f(\overline{a}) = \sum_{k=1}^{m-1} (1/k!)d^k f(\overline{a};\overline{b} - \overline{a}) + (1/m!)d^m f(\overline{z};\overline{b} - \overline{a}) \quad [6].$$

Illustrative Example 3-2. Write out Taylor's Formula for $f(x,y) = e^x \sin(y)$ about the point $(0,\pi/2)$ where $m = 3$.

Solution:

$$f(x,y) = e^x \sin(y),\ f(0,\pi/2) = 1,$$

$$f_x(x,y) = e^x \sin(y),\ f_x(0,\pi/2) = 1,$$

$$f_{xx}(x,y) = e^x \sin(y),\ f_{xx}(0,\pi/2) = 1,$$

$$f_{xxx}(x,y) = e^x \sin(y),\ f_{xxx}(0,\pi/2) = 1,$$

$$f_y(x,y) = e^x \cos(y),\ f_y(0,\pi/2) = 0,$$

$$f_{yy}(x,y) = -e^x \sin(y),\ f_{yy}(0,\pi/2) = -1,$$

$$f_{yyy}(x,y) = -e^x \cos(y),\ f_{yyy}(0,\pi/2) = 0,$$

$$f_{xy}(x,y) = e^x \cos(y),\ f_{xy}(0,\pi/2) = 0,$$

$$f_{xxy}(x,y) = e^x \cos(y),\ f_{xxy}(0,\pi/2) = 0,$$

$$f_{xyy}(x,y) = -e^x \sin(y),\ f_{xyy}(0,\pi/2) = -1.$$

$$f(x,y) - f(0,\pi/2) = f_x(0,\pi/2)x + f_y(0,\pi/2)(y - \pi/2) + (1/2)[f_{xx}(0,\pi/2)x^2 + 2f_{xy}(0,\pi/2)x(y - \pi/2) + f_{yy}(0,\pi/2)(y - \pi/2)^2] + (1/3)[f_{xxx}(x_1,y_1)x^3 + 3f_{xxy}(x_1,y_1)x^2(y - \pi/2) + 3f_{xyy}(x_1,y_1)x(y-\pi/2)^2 + f_{yyy}(x_1,y_1)(y - \pi/2)^3],\ 0 < x_1 < x,\ \pi/2 < y_1 < y.$$

By substitution, one gets

$$f(x,y) - 1 = x + (1/2)[x^2 - (y - \pi/2)^2] + (1/6)[f_{xxx}(x_1,y_1)x^3 + 3f_{xxy}(x_1,y_1)x^2(y - \pi/2) + 3f_{xyy}(x_1,y_1)x(y - \pi/2)^2 + f_{yyy}(x_1,y_1)(y - \pi/2)^3],\ 0 < x_1 < x,\ \pi/2 < y_1 < y.$$

Functions of Two Variables

The definition of the extrema of a function of two or more variables can be given as an extension of the definition of the one-dimensional case (Definition 2-4). In this section attention is centered on functions of two variables. A necessary condition for a differentiable function f to have an extremum is established. Some theorems giving sufficient conditions for a function f to have extrema are stated and proved. A saddle point of a function is defined and a condition for the same is stated without proof. The principle of least squares is demonstrated. Following each basic theorem is an illutrative example.

Definition 3-13. Let f be a real-valued function defined on a set S in E_n. Then f is said to have an absolute maximum on the set S if there exists a point $\bar{a}$ in S such that

$$f(\bar{x}) \leq f(\bar{a}), \quad \text{for all } \bar{x} \text{ in } S.$$

If $\bar{a}$ belongs to S and if there is a neighborhood $N(\bar{a})$ such that

$$f(\bar{x}) \leq f(\bar{a}), \quad \text{for all } \bar{x} \text{ in } N(\bar{a}) \cap S,$$

then f is said to have a relative maximum at the point $\bar{a}$.

Absolute minimum and relative minimum are similarly defined, using $f(\bar{x}) \geq f(\bar{a})$ [6].

In E_2, the definition can be simply stated as follows:

Definition 3-14. A function $f(x,y)$ has an absolute maximum at a point (a,b) of a region R if and only if

$$f(a,b) \geq f(x,y)$$

for all (x,y) in R. A function $f(x,y)$ has a relative maximum at a point (a,b) of a region R if and only if there exists a positive number δ such that

$$f(a,b) \geq f(x,y)$$

for all (x,y) of R at which

$$0 < (x - a)^2 + (y - b)^2 < {}^2 \quad [8].$$

In a similar manner, the definitions for absolute or relative minimum of $f(x,y)$ can be stated using the inequality $f(a,b) \leq f(x,y)$.

Illustrative Example 3-3. Let the function $f(x,y) = x^2 + y^2$ be defined on the region S as shown in Figure 3-2 in E_2. Then $f(S) = [0,8]$ implying $f(2,2) = f(2,-2) = 8$ is the absolute maximum value of f and $f(0,0) = 0$ is the absolute minimum value of f.

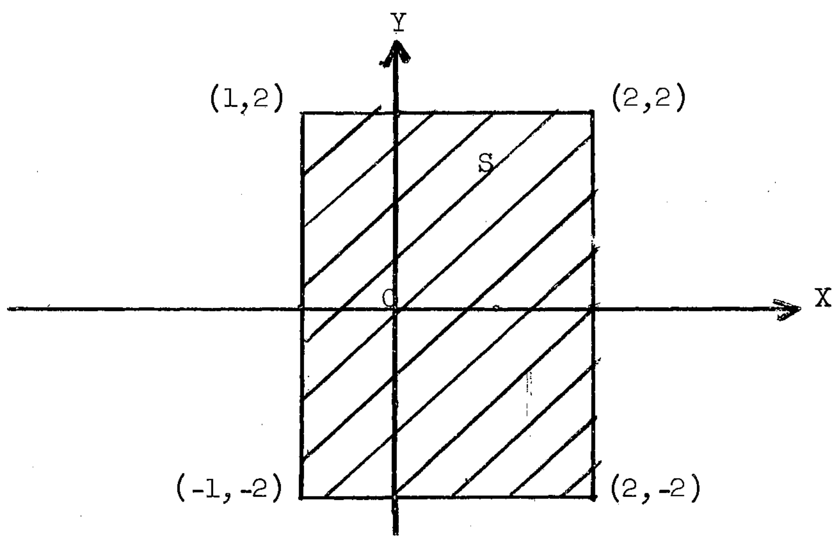

Figure 3-2.

A fundamental theorem for the existence of both absolute maximum and absolute minimum values of a function of several variables, analogous to the theorem for the one-dimensional case (Theorem 2-1) is now stated.

<u>Theorem 3-3</u>. Let f be a real-valued function which is continuous on a closed and bounded set S in E_2. Then f has an absolute maximum and an absolute minimum on S [6].

The proof of this theorem is identical to the proof of Theorem 2-1. Therefore, it is omitted here.

<u>Definition 3-15</u>. A point $\bar{x}_0$ is a critical point in E_2 if $df(\bar{x}_0) = 0$ [9].

If f is a differentiable function on S, one need look only among the critical points for relative extrema.

<u>Theorem 3-4</u>. If the function f has a relative extremum at $\bar{x}_0$ and f is differentiable at $\bar{x}_0$, then $\bar{x}_0$ is a critical point [9].

<u>Proof</u>: Given a direction v, let $F(t) = f(\bar{x}_0 + t\bar{v})$ for every t in some open subset of E_1 containing 0. Then F has a relative extremum at 0, and consequently by Theorem 2-4 $F'(0) = 0$. But $F'(0) = df(\bar{x}_0)\cdot v$ is the derivative at $\bar{x}_0$ in the direction $\bar{v}$. Hence, $df(\bar{x}_0)\cdot\bar{v} = 0$ for every $\bar{v}$, which implies that $df(\bar{x}_0) = 0$.

<u>Theorem 3-5 (Necessary Condition)</u>. Let f have finite partial derivatives $D_k f(\bar{x})$, $k = 1,2,\ldots,n$, at each point $\bar{x}$ of an open set S in E_n. If f has a relative maximum or a relative minimum at the point $\bar{x}_0$ in S, then $D_k f(\bar{x}_0) = 0$ for each $k = 1,\ldots,n$ [6].

<u>Proof</u>: Assume that the partial derivatives $D_k f(\bar{x})$ where $k = 1,2,\ldots,n$, are continuous at each point $\bar{x}$ of the open set S. Then f has a differential at $\bar{x}$ [6], and the gradient vector $\nabla f(\bar{x})$ exists. This implies

$$df(\bar{x};\bar{t}) = \nabla f(\bar{x})\cdot\bar{t}, \text{ if } \bar{t} \text{ is in } E_n \quad [6].$$

Therefore, if f has a relative extremum at an interior point $\bar{x}$ of an open set S in E_n, and if $\nabla f(\bar{x})$ exists, then $\nabla f(\bar{x}) = \bar{0}$. Since $\nabla f(x) = [D_1 f(\bar{x}), \ldots, D_n f(x)] = \bar{0}$, then

$$D_1 f(x) = D_2 f(x) = \ldots = D_n f(x) = 0.$$

This completes the proof.

Note that the formula $df(\bar{x};\bar{t}) = \nabla f(\bar{x}) \cdot \bar{t}$ bears a strong resemblance to the equation $df(x;t) = f'(x)t$, which holds in the one-dimensional case. This suggests that the gradient vector ∇f plays the same role in E_n as the derivative f' in E_1.

The writer now gives some definitions which will simplify the statement of several other important theorems.

Definition 3-16. A domain is an open set, of which any two of its points can be joined by a broken line having a finite number of segments, all of which the points belong to the set. A region is either a domain or a domain plus some or all of its boundary. If it contains all of its boundary, it is a closed region [9].

Definition 3-17. If f is a continuous function, then f is said to be a function of class C^0. If the partial derivatives $f_1(\bar{x})$, $f_2(\bar{x})$, ..., $f_n(\bar{x})$ exist for all $\bar{x}$ in D and f_1, f_2, ..., f_n are continuous, then f is a function of class C^1. If all of the qth order derivatives of f exist at every $\bar{x}$ in D and each $f_{i_1 \ldots i_q}$ is a continuous function, then f is a function of class C^q [9].

Theorem 3-6. Let

1. $f(x,y)$ belongs to C^1 in a bounded region R consisting of a domain D and boundary curve T, and

2. $f(a,b) > f(x,y)$ for some (a,b) in D and all (x,y) in T. Then there exists a point (x_0,y_0) in D such that

A. $f(x,y) \leq f(x_0,y_0)$ for all (x,y) in R, and

B. $f_1(x_0,y_0) = f_2(x_0,y_0) = 0$ [8].

Proof: Since $f(x,y)$ is continuous in the closed region R, it has a maximum there, by Theorem 3-1. This occurs at some (x_0,y_0) in D by virtue of hypothesis 2. Conclusion B follows from Theorem 3-5. This ends the proof of the theorem.

By reversing the inequality sign in hypothesis 2, one gets a minimum value of f in conclusion A.

Illustrative Example 3-4. Show that the function

$$f(x,y) = x^4 + y_4 - 2x^2 + 4,\ x^2 + y^2 < \infty$$

has an absolute minimum and find the same [8].

Solution: To establish hypothesis 2 of Theorem 3-6, polar coordinates are introduced as follows:

$$f[r\cos(\theta), r\sin(\theta)] = r^4(\cos^4\theta + \sin^4\theta) - 2r^2\cos^2\theta + 8r^2\sin^2\theta + 4.$$

On the circle $r = r_0$, the first term is at least $r_0{}^4/4$, the second term is greater than $-2r_0{}^2$ and the third term is greater than zero. Thus $f(x,y)$ on a circle $r = r_0$ is always greater than $(r_0{}^4/4) - 2r_0{}^2 + 4$. In particular if $r_0 = 4$, $f(x,y)$ is greater than 36 for all (x,y) on this circle. Since $f(0,0) = 4$ Theorem 3-6 shows that an absolute minimum exists in the region bounded by the circle $r_0 = 4$.

To find it, the first partial derivatives are set equal to zero as follows:

$$f_1(x,y) = 4x^3 - 4x = 0 \quad \text{or} \quad x^3 - x = 0, \quad \text{and}$$
$$f_2(x,y) = 4y^3 + 16y = 0 \quad \text{or} \quad y^3 + 4y = 0.$$

The solution set of this system consists of $(0,0)$, $(0,2i)$, $(0,-2i)$, $(1,0)$, $(1,2i)$, $(1,-2i)$, $(-1,0)$, $(-1, 2i)$, and $(-1,-2i)$. Since $(0,2i)$, $(0,-2i)$, $(1,2i)$, $(1,-2i)$, $(-1,2i)$, and $(-1,-2i)$ are imaginary solutions or points one only needs to try the points $(0,0)$, $(1,0)$, and $(-1,0)$ in E_2. There exists a minimum $f(-1,0) = f(1,0) = 3$ at the points $(-1,0)$ and $(1,0)$.

The writer will now establish sufficient conditions for relative maxima and minima. To make this discussion easier the appropriate theory of definite quadratic forms in two variables is introduced and used.

Definition 3-18. A real-valued function defined on E_2 by an equation of the type

$$F(x_1,x_2) = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij}x_ix_j,$$

where $a_{ij} = a_{ji}$ are real numbers, is called a quadratic form [6].

This form $F(x_1,x_2)$ can also be expressed as follows:

$$F(x_1,x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2. \quad (1)$$

A quadratic form F is called positive definite if, for every vector $(x_1,x_2) \neq (0,0)$,

$$F(x_1,x_2) > 0.$$

It is called negative definite if, for every vector $(x_1,x_2) \neq (0,0)$,

$$F(x_1,x_2) < 0.$$

Now the question as to when F is positive definite or negative definite is considered. First (1) is rewritten as

$$F(x_1,x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2.$$

Complete the square in x_1 as follows:

$$\begin{aligned} F(x_1,x_2) &= a_{11}[x_1^2 + (2a_{12}x_2/a_{11})x_1 + (a_{22}x_2^2/a_{11})] \\ &= a_{11}[x_1^2 + (2a_{12}x_2/a_{11})x_1 + (a_{12}^2x_2^2/a_{11}^2) - \\ &\quad (a_{12}^2x_2^2/a_{11}^2) + (a_{22}x_2^2/a_{11})] \\ &= a_{11}[(x_1 + a_{12}x_2/a_{11})^2 + (a_{22}x_2^2/a_{11}) - (a_{12}^2x_2^2/a_{11}^2)] \\ &= a_{11}[(1/a_{11}^2)(a_{11}x_1 + a_{12}x_2)^2 + (1/a_{11}^2)(a_{11}a_{22} - a_{12}^2)x_2^2]. \\ &= (1/a_{11})[(a_{11}x_1 + a_{12}x_2)^2 + (a_{11}a_{22} - a_{12}^2)x_2^2]. \qquad (2) \end{aligned}$$

In (2), it is observed that F is the product of $(1/a_{11})$ and the sum (or difference) of two squares. Therefore, if $a_{11} > 0$, $a_{11}a_{22} - a_{12}^2 > 0$, and $(x_1,x_2) \neq (0,0)$, i.e., F is positive definite, $F(x_1,x_2) > 0$. Also, if $a_{11} < 0$, $a_{11}a_{22} - a_{12}^2 > 0$, and $(x_1,x_2) \neq (0,0)$, $F(x_1,x_2) < 0$, i.e., F is negative definite. The statement $a_{11}a_{22} - a_{12}^2 > 0$ is now written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0.$$

The foregoing discussion establishes the following lemmas:

<u>Lemma</u> <u>3-1</u>.

1. $F(x_1,x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$

is a quadratic form in x_1, x_2, and

2. $a_{11} > 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$

imply $F(x_1,x_2)$ is positive definite.

Lemma 3-2.

1. $F(x_1,x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$

is a quadratic form in x_1, x_2, and

2. $a_{11} < 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$

imply $F(x_1,x_2)$ is negative definite.

Definition 3-18 of a quadratic form in two variables x_1 and x_2 as well as lemmas (3-1) and (3-2) can readily be extended to quadratic forms in three or more variables. Such extensions will be considered in the next section.

Sufficient conditions for a relative maximum at a point are now given.

Theorem 3-7 (SUFFICIENT CONDITIONS).

1. $f(x,y)$ belongs to C^2,
2. $f_1 = f_2 = 0$ at (x_0,y_0),
3. $f_{11}f_{22} - f_{12}^2 > 0$ at (x_0,y_0),
4. $f_{11} < 0$ at (x_0y_0)

imply $f(x,y)$ has a relative maximum at (x_0,y_0) [8].

Proof: The plan of this proof is to show that Δf is a negative definite quadratic form in some neighborhood of (x_0,y_0). By Theorem 3-2 (Taylor's formula with remainder) the following equation can be written:

$$\Delta f = f(x_0 + h,\ y_0 + k) - f(x_0,y_0)$$

$$= (1/2) \sum_{i=1}^{2} \sum_{j=1}^{2} f_{ij}(x_0 + \theta h,\ y_0 + \theta k)$$

where $0 < \theta < 1$. Or

$$\Delta f = (1/2)[f_{11}(x_0 + \theta h,\ y_0 + \theta k)(x - x_0)^2 +$$
$$2f_{12}(x_0 + \theta h,\ y_0 + \theta k)(x - x_0)(y - y_0) +$$
$$f_{22}(x_0 + \theta h,\ y_0 + \theta k)(y - y_0)^2]$$

clearly shows that d^2f, a function of $(x - x_0)$ and $(y - y_0)$ with $(x_0 + \theta h,\ y_0 + \theta k)$ fixed, is a quadratic form in two variables. By hypothesis 1 it is clear that inequalities 3 and 4 also hold in some neighborhood of $(x_0,\ y_0)$. If the point $(x_0 + h,\ y_0 + k)$ is in this neighborhood, the coefficients of the quadratic form Δf will satisfy the conditions of Lemma 3-2, so that $\Delta f < 0$ throughout the neighborhood, except at $h = k = 0$, where $\Delta f = 0$. Hence, f has a relative maximum at $(x_0,\ y_0)$. This completes the proof.

To apply this theorem for a minimum, one has only to reverse the inequality in hypothesis 4. This would make Δf positive giving $f(x,y)$ a minimum at (x_0,y_0) by Definition 3-14.

If $f_{11}f_{22} - f_{12}^2 = 0$, there may be a relative maximum, a relative minimum, or neither at (x_0,y_0), i.e.,

$$\Delta f = (1/2f_{11})[(f_{11}h + f_{12}k)^2] < 0 \quad \text{if} \quad f_{11} < 0,$$

$$\Delta f = (1/2f_{11})[(f_{11}h + f_{12}k)^2] > 0 \quad \text{if} \quad f_{11} > 0, \quad \text{and}$$

$$\Delta f = (1/2f_{11})[(f_{11}h + f_{12}k)^2] = 0 \quad \text{if} \quad f_{11}h + f_{12}k = 0.$$

In other words, $f_{11}f_{22} - f_{12}^2 = 0$ is inconclusive because the sign of $f_{11}f_{22} - f_{12}^2$ at $(x_0 + h,y_0 + k)$ is not known.

Illustrative Example 3-5. Classify and find the extreme values (if any) of the function

$$f(x,y) = x^2 + y^2 + x + y + xy \quad [6].$$

Solution: The first partial derivatives are

$$f_1(x,y) = 2x + 1 + y$$

$$f_2(x,y) = 2y + 1 + x.$$

Solving the system of equations

$$2x + y = -1$$

$$x + 26 = -1,$$

the solution set consisting of $(-1/3,-1/3)$ is obtained. The second partial derivatives are

$$f_{11}(x,y) = 2,\ f_{12}(x,y) = 1,\ f_{22}(x,y) = 2.$$

$f_{11}f_{22} - f_{12}^2 = 3 > 0$ at $(-1/3,-1/3)$ and $f_{11} = 2 > 0$ at $(-1/3,-1/3)$.

Therefore, $f(-1/3,-1/3) = -1/3$ is an absolute minimum.

Illustrative Example 3-6. Classify and find the extreme values of the function

$$f(x,y) = y^2 + x^2y + x^4.$$

Solution: Take the first partial derivatives and set them equal to zero, i.e.,

$$f_1(x,y) = 2xy + 4x^3 = 0$$

$$f_2(x,y) = 2y + x^2 = 0.$$

The solution of the system of equations above is $(0,0)$. Now one needs to determine the second order partial derivatives and apply Theorem 3-7.

$f_{11}(x,y) = 2y + 12x^2,$ $\qquad f_{22}(x,y) = 2$

$f_{12}(x,y) = 2x,$ $\qquad f_{21}(x,y) = 2x.$

At $(x,y) = (0,0)$, $f_{11}(x,y) = 0$ and $f_{11}(x,y)f_{22}(x,y) - f_{12}^2(x,y) = 0$. Therefore, the test is inconclusive.

The increment of f can be written as follows:

$$\begin{aligned}\Delta f &= f(x,y) - f(0,0)\\ &= y^2 + x^2y + x^4\\ &= y^2 + x^2y + (x^4/4) - (x^4/4) + x^4\\ &= [y + (x^2/2)]^2 + [3x^4/4].\end{aligned}$$

This is the sum of two positive terms and therefore is positive definite, which implies that f has a relative minimum at $(0,0)$.

A function $f(x,y)$ has a <u>saddle point</u> at (x_0,y_0) if $f_1(x_0,y_0) = f_2(x_0,y_0) = 0$ and if the difference

$$\Delta f = f(x_0 + h, y_0 + k) - f(x_0,y_0)$$

has both positive and negative values in every neighborhood of (x_0,y_0). See Figure 3-3.

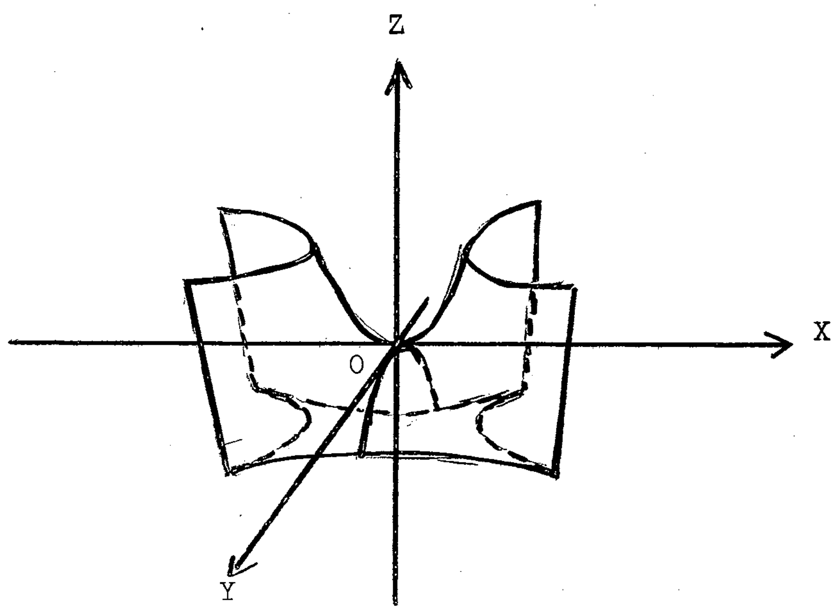

Figure 3-3.

One way to think of a saddle point is to picture f as a function of x

alone with a maximum at $x = x_0$, and then picture f as a function of y alone with a minimum at $y = y_0$. The surface $z = xy$ (the familiar hyperbolic paraboloid) has a saddle point at the origin as shown in Figure 3-3.

The theorem to follow gives conditions for a function f to have a saddle point.

<u>Theorem 3-8</u>.

1. $f(x,y)$ belongs to C^2
2. $f_1 = f_2 = 0$ at (x_0,y_0)
3. $f_{12}^2 - f_{11}f_{22} > 0$ at (x_0,y_0)

imply $f(x,y)$ has a saddle-point at (x_0,y_0) [8].

The proof of this theorem which is similar to that of Theorem 3-7 will be omitted here. The same may be found in the reference listed above.

<u>Illustrative Example 3-7</u>. Test the function for relative maxima, relative minima, and saddle-points

$$f(x,y) = x^3 - y^3 + 3x^2 + 3y^2 - 9x \quad [8].$$

<u>Solution:</u> Solving the system of equations

$$f_1(x,y) = 3x^2 + 6x - 9 = 0$$

$$f_2(x,y) = -3y^2 + 6y = 0$$

the solution set $\{(1,0), (1,2), (-3,0), (-3,2)\}$ is obtained. Now the second partial derivatives of $f(x,y)$ are as follows:

$$f_{11}(x,y) = 6x + 6,\ f_{12}(x,y) = 0,\ f_{22}(x,y) = -6y + 6.$$

See Table 3-1 for rest of problem.

TABLE I

TABULAR SOLUTION OF ILLUSTRATIVE EXAMPLE 3-7

Point in E_2	f_{11}	f_{12}	f_{22}	$f_{11}f_{22} - f_{12}^2$	Extrema or Saddle-points of $f(x,y)$
(1,0)	$12 > 0$	0	6	$72 > 0$	Min., -5
(1,2)	$12 > 0$	0	-6	$-72 < 0$	S.P., (1,2,-1)
(-3,0)	$-12 < 0$	0	6	$-72 < 0$	S.P., (-3,0,27)
(-3,2)	$-12 < 0$	0	-6	$72 > 0$	Max., 31

Another interesting application of this theory is the Principle of Least Squares. Let it be required to fit a straight line $y = ax + b$ to the data consisting of the points $(x_1,y_1),\ldots,(x_n,y_n)$--a line which comes nearest to fitting the linear distribution of the points. That is, determine constants a and b so that

$$f(a,b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2$$

will be a minimum. See Figure 3-4.

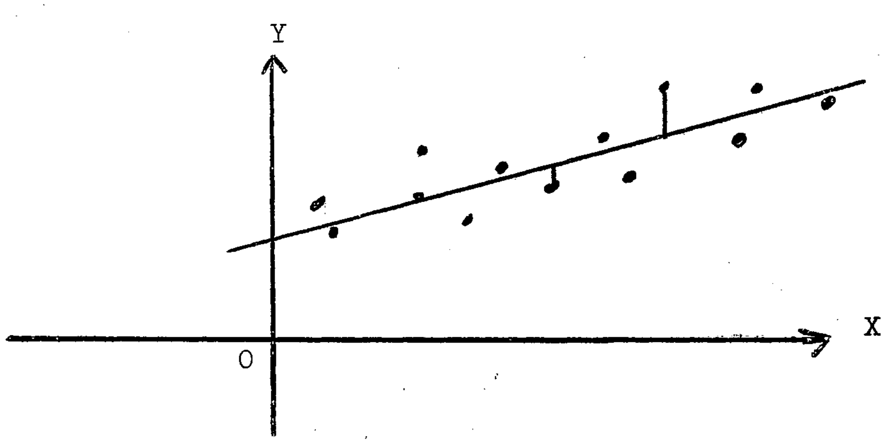

Figure 3-4.

Set the first partial derivatives equal to 0 and find the second order partial derivatives as follows:

$$f_1(a,b) = 2 \sum_{i=1}^{n} (ax_i + b - y_i)x_i = 0 \qquad (3)$$

$$f_2(a,b) = 2 \sum_{i=1}^{n} (ax_i + b - y_i) = 0$$

$$f_{11}(a,b) = 2 \sum_{i=1}^{n} x_i^2$$

$$f_{12}(a,b) = 2 \sum_{i=1}^{n} x_i$$

$$f_{22}(a,b) = 2 \sum_{i=1}^{n} 1 = 2n.$$

Since

$$(\sum_{i=1}^{n} 1)(\sum_{i=1}^{n} x_i^2) - (\sum_{i=1}^{n} x_i)^2 = (1/2)\sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2 > 0 \qquad (5)$$

[8], f has a relative minimum value by Theorem 3-7. By Theorem 3-6, this relative minimum is unique which makes it absolute [8].

Solve equations (3) and (4) simultaneously, i.e.,

$$(\sum_{i=1}^{n} x_1^2)a + (\sum_{i=1}^{n} x_i)b = \sum_{i=1}^{n} x_i y_i$$

$$(\sum_{i=1}^{n} x_i)a + (\sum_{i=1}^{n} 1)b = \sum_{i=1}^{n} y_i.$$

getting

$$b = \sum_{i=1}^{n} y_i/n) - (a \sum_{i=1}^{n} x_i/n)$$

$$b = \bar{y} - a\bar{x} \tag{7}$$

$$a = \frac{\sum_{i=1}^{n} x_i y_i - (\sum_{i=1}^{n} x_i)^2/n}{\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2/n} \tag{8}$$

Substituting these values for a and b in the equation

$$y = ax + b$$

one obtains what is called the regression equation, the estimating equation, or the prediction equation. These labels are used interchangeably. The process of passing a straight line or curve through a set of points (data) in an effort to describe the trend of events is called curve fitting. Determinants may be used to advantage in writing down the equation for the line or curve. For a straight line the determinantal equation [8] is

$$\begin{vmatrix} x & y & 1 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & \sum_{i=1}^{n} 1 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} x_i \end{vmatrix} = 0. \tag{9}$$

<u>Illustrative Example 3-8</u>. Pass a line through the following points by least squares: (-2,0), (-1,0), (0,1), (1,3), (2,2). Plot the line and the given points [8].

Solution: $n = 5, \quad \sum_{i=1}^{5} x_i = 0, \quad \sum_{i=1}^{5} y_i = 6, \quad \sum_{i=1}^{5} x_i^2 = 10, \quad \sum_{i=1}^{5} x_i y_i = 7$

$\bar{x} = 0, \qquad \bar{y} = 1.2$

Substituting in formulas (7) and (8), one gets

$$a = [7 - (0)(1.2)]/[10 - 0] = .7, \; b = 1.2 - 0.7(0) = 1.2$$

Therefore, $y = .7x + 1.2$. Equation (9), the determinantal form of the equation of a line, gives the same result,

$$\begin{vmatrix} x & y & 1 \\ 0 & 6 & 5 \\ 10 & 7 & 0 \end{vmatrix} = 0$$

which expands into

$$y = .7x + 1.2.$$

For the graph of the points and line, see Figure 3-5.

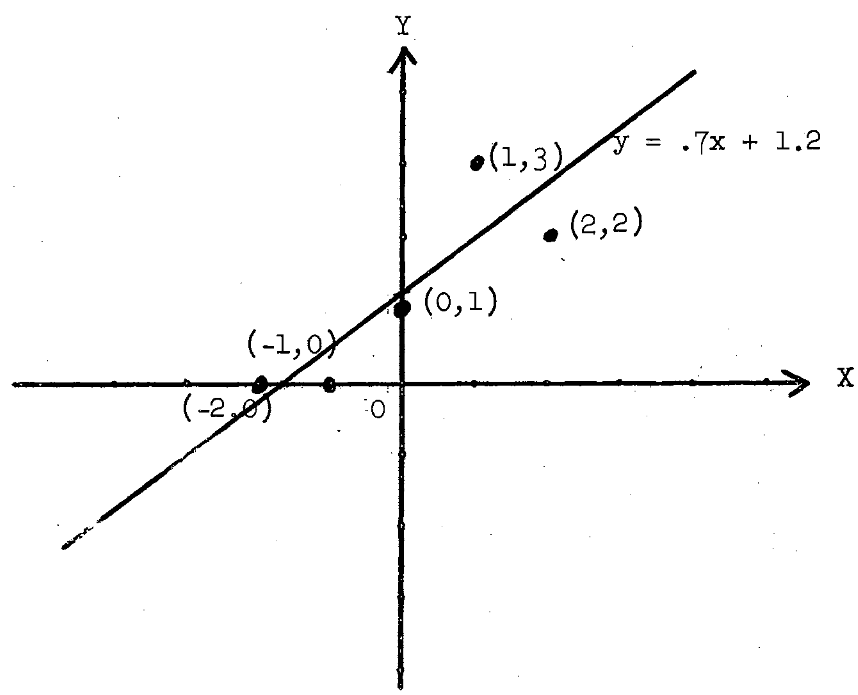

Figure 3-5.

Functions of Three or More Variables

A theorem analogous to Theorem 3-6 for functions of three variables can be easily developed and proved as an extension of the two-dimensional analogue. To extend Theorem 3-7, a corresponding extension of the theory of definite quadratic forms in two variables is needed. This is done and then the corresponding theorems for the n-dimensional case are stated.

The definition which follows is an extension of Definition 3-18.

<u>Definition 3-19</u>. A real-valued function defined on E_3 by an equation of the type

$$F(x_1,x_2,x_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}x_ix_j,$$

where $a_{ij} = a_{ji}$ are real numbers, is called a quadratic form [6].

$F(x_1,x_2,x_3)$ can also be expressed as follows:

$$\begin{aligned} F(x_1,x_2,x_3) = {} & a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \\ & a_{21}x_2x_1 + a_{22}x_2^2 + a_{23}x_2x_3 + \\ & a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3^3. \end{aligned} \tag{10}$$

It is positive definite if $F(x_1,x_2,x_3) > 0$, except when $x_1 = x_2 = x_3 = 0$. $F(0,0,0) = 0$. It is positive semidefinite if $F \geq 0$, the equality holding for certain values of x_1,x_2,x_3, not all zero [8]. If $F(x_1,x_2,x_3) < 0$ for $\overline{x} \neq \overline{0}$, then F is negative definite.

<u>Illustrative Example 3-9</u>. Let $F = x_1^2 + x_2^2 + x_3^2$. Then F is positive definite. Let $G = x_1^2 + x_3^2$. Then $G(x_1,x_2,x_3)$ is positive semidefinite. For instance, $G(0,1,0)$ is a case in which equality

holds.

Let $a_{11} = A$, $a_{12} = B$, and $a_{22} = C$, then the quadratic form

$$Ax_1^2 + 2Bx_1x_2 + Cx_2^2$$

is positive definite if

$$A > 0, \quad \begin{vmatrix} A & B \\ B & C \end{vmatrix} > 0. \tag{11}$$

by Lemma 3-1.

At this point, an analogous result for quadratic forms in three variables is developed. Of course, these quadratic forms can be extended to $n(>3)$ variables [6].

Theorem 3-9. The quadratic form in three variables is positive definite, if and only if,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0 \quad [8]. \tag{12}$$

Proof: Only the sufficiency of condition (12) is proved here. Let Δ represent the three-rowed determinant in (12) and A_{ij} the co-factor of its element a_{ij}. By use of the formula for the product of two determinants, one has

$$\Delta \begin{vmatrix} 1 & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{vmatrix} \tag{13}$$

$$\begin{vmatrix} a_{11} & a_{12}A_{22} + a_{13}A_{32} & a_{12}A_{23} + a_{13}A_{33} \\ a_{21} & a_{22}A_{22} + a_{23}A_{32} & a_{22}A_{23} + a_{23}A_{33} \\ a_{31} & a_{32}A_{22} + a_{33}A_{32} & a_{32}A_{23} + a_{33}A_{33} \end{vmatrix} \tag{14}$$

Interchanging the rows and columns [11] this determinant may be written as

$$= \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12}A_{22} + a_{13}A_{32} & a_{22}A_{22} + a_{23}A_{32} & a_{32}A_{22} + a_{33}A_{32} \\ a_{12}A_{23} + a_{13}A_{33} & a_{22}A_{23} + a_{23}A_{33} & a_{32}A_{23} + a_{33}A_{33} \end{vmatrix}. \qquad (15)$$

Now

$$\Delta = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{13} \quad [11].$$

Therefore,

$$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{32} \quad \text{or} \quad a_{22}A_{22} + a_{23}A_{32} = \Delta - a_{21}A_{21}, \quad (16a)$$

$$\Delta = a_{31}A_{31} + a_{32}A_{23} + a_{33}A_{33} \quad \text{or} \quad a_{32}A_{23} + a_{33}A_{33} = \Delta - a_{31}A_{31}. \quad (16b)$$

The sum of the products of the elements of one row of a determinant by the cofactors of the corresponding elements of a different row of the determinant is zero [12]. Consequently,

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{32} = 0 \quad \text{or} \quad a_{12}A_{22} + a_{13}A_{32} = -a_{11}A_{21}. \qquad (17a)$$

$$a_{11}A_{31} + a_{12}A_{23} + a_{13}A_{33} = 0 \quad \text{or} \quad a_{12}A_{23} + a_{13}A_{33} = -a_{11}A_{31}. \qquad (17b)$$

$$a_{21}A_{31} + a_{23}A_{23} + a_{23}A_{33} = 0 \quad \text{or} \quad a_{22}A_{23} + a_{23}A_{33} = -a_{21}A_{31}. \qquad (17c)$$

$$a_{31}A_{31} + a_{32}A_{22} + a_{33}A_{32} = 0 \quad \text{or} \quad a_{32}A_{22} + a_{33}A_{32} = -a_{31}A_{31}. \qquad (17d)$$

Substituting (16a), (16b), (17a), (17b), (17c), and (17d) in (15) one gets

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ -a_{11}A_{21} & \Delta - a_{21}A_{21} & -a_{31}A_{21} \\ -a_{11}A_{31} & -a_{21}A_{31} & \Delta - a_{31}A_{31} \end{vmatrix} \qquad (18)$$

The value of a determinant is unchanged if one adds to some fixed row (or column) a fixed multiple of another row (or column) [11]. Thus one may write (18) as

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = a_{11}\Delta^2. \tag{19}$$

$$\text{Hence, } \Delta \begin{vmatrix} 1 & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{vmatrix} = a_{11}\Delta^2 \quad \text{or}$$

$$\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = a_{11}\Delta. \tag{20}$$

Now the terms in x_1^2 and in x_1 of the quadratic form (10) are collected as follows:

$$F = a_{11}x_1^2 + (a_{12}x_2 + a_{13}x_3 + a_{21}x_2 + a_{31}x_3)x_1 +$$
$$(a_{22}x_2^2 + a_{23}x_2x_3 + a_{32}x_3x_2 + a_{33}x_3{}^2) \quad \text{or}$$

$$F = a_{11}x_1^2 + 2(a_{12}x_2 + a_{13}x_3)x_1 + (a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2). \tag{21}$$

Let $A = a_{11}$, $B = a_{12}x_2 + a_{13}x_3$, $C = a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2$ in (21). Thus (21) can be written as

$$F = Ax_1^2 + 2Bx_1 + C \tag{22}$$

It will be shown that $AC - B^2 > 0$ unless $x_2 = x_3 = 0$, and this will prove $F > 0$ by (11). If $x_2 = x_3 = 0$, $F = a_{11}x_1^2$, and this is positive unless x_1 is also zero since $a_{11} > 0$, so that F is positive definite.

In $AC - B^2$ collect the terms in x_2^2, in x_2x_3, and in x_3^2 as follows:

$$AC - B^2 = a_{11}a_{22}x_2^2 + 2a_{11}a_{23}x_2x_3 + a_{11}a_{33}x_3^2 - a_{12}^2x_2^2 - 2a_{12}a_{13}x_2x_3 -$$
$$a_{13}^2x_3^2,$$

$$AC - B^2 = (a_{11}a_{22} - a_{12}^2)x_2^2 + 2(a_{11}a_{23} - a_{12}a_{13})x_2x_3 + (a_{11}a_{33} - a_{13}^2)x_3^2 \tag{23}$$

Since A_{ij} is the cofactor of the element a_{ij} in the determinant

$$A_{33} = a_{11}a_{22} - a_{12}^2, \; A_{23} = a_{11}a_{23} - a_{12}a_{13}, \; A_{22} = a_{11}a_{33} - a_{13}^2 \tag{24}$$

Therefore, substituting equations (24) in (23), one gets

$$AC - B^2 = A_{33}x_2^2 - 2A_{23}x_2x_3 + A_{22}x_3^2. \tag{25}$$

To show that this is always positive, unless $x_2 = x_3 = 0$, (11) is used again. One needs

$$A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0.$$

But these facts follow at once by hypothesis. This completes the proof.

A distinction needs to be made between quadratic forms in two variables and forms in more than two variables. The former is positive semidefinite if, and only if, the sign $>$ is replaced by $\geq$ in (11) (not both $>$). If a corresponding change is made in inequalities (12), a necessary but not a sufficient condition for (10) to be positive semidefinite is obtained. For, suppose all $a_{ij} = 1$, except $a_{33} = 0$. Then

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{22} \end{vmatrix} = 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

$$F = (x_1 + x_2 + x_3)^2 - x_3^2$$

$$F(1,1,2) = -4 < 0 \quad [8].$$

Quadratic forms enter into the theory of functions of several variables by way of the second-order differential

$$d^2f(\bar{x};\bar{t}) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j}f(\bar{x})t_i t_j,$$

which is a quadratic form in $\bar{t}$ when $\bar{x}$ is held fixed. If all the derivatives $D_{i,j}$ are continuous at $\bar{x}$, then the mixed partials $D_{i,j}f(\bar{x})$ and $D_{j,i}f(\bar{x})$ are equal and d^2f is a symmetric form [6].

With this theory of definite quadratic forms in three variables, a sufficient condition for a function f of three variables to have an extremum can now be established.

Theorem 3-10.

1. $f(x,y,z)$ belongs to C^2,
2. $f_1 = f_2 = f_3 = 0$ at (x_0,y_0,z_0)
3. $f_{11} > 0$, $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0$, $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} > 0$ at (x_0,y_0,z_0),

imply that $f(x,y,z)$ has a relative minimum at (x_0,y_0,z_0) [8].

Proof: By Taylor's theorem,

$$\Delta f = f(x_0 + h_1,\ y_0 + h_2,\ z_0 + h_3) - f(x_0,y_0,z_0)$$

$$= (1/2)\sum_{i=1}^{3}\sum_{j=1}^{3} f_{ij}(x_0 + \theta h_1,\ y_0 + \theta h_2,\ z_0 + \theta h_3)h_i h_j$$

where $0 < \theta < 1$. Or

$$\Delta f = (1/2)[f_{11}(x_0 + \theta h_1,\ y_0 + \theta h_2,\ z_0 + \theta h_3)(x - x_0)^2 +$$

$$f_{22}(x_0 + \theta h_1,\ y_0 + \theta h_2,\ z_0 + \theta h_3)(y - y_0)^2 +$$

$$f_{33}(x_0 + \theta h_1,\ y_0 + \theta h_2,\ z_0 + \theta h_3)(z - z_0)^2 +$$

$$2f_{12}(x_0 + \theta h_1,\ y_0 + \theta h_2,\ z_0 + \theta h_3)(x - x_0)(y - y_0) +$$
$$2f_{13}(x_0 + \theta h_1,\ y_0 + \theta h_2,\ z_0 + \theta h_3)(x - x_0)(z - z_0) +$$
$$2f_{23}(x_0 + \theta h_1,\ y_0 + \theta h_2,\ z_0 + \theta h_3)(y - y_0)(z - z_0)]$$

clearly shows that d^2f, a function of $(x - x_0)$, $(y - y_0)$, and $(z - z_0)$ with $(x_0 + \theta h_1, y_0 + \theta h_2, z_0 + \theta h_3)$ fixed, is a quadratic form in three variables. By hypothesis 1 it is clear that inequalities 3 also hold in some neighborhood of (x_0,y_0,z_0). If the point $(x_0 + h_1, y_0 + h_2, z_0 + h_3)$ is in this neighborhood, the coefficients of the quadratic form Δf will satisfy the conditions of Theorem 3-9, so that $\Delta f > 0$ throughout the neighborhood, except at $h_1 = h_2 = h_3 = 0$, where $\Delta f = 0$. Hence, f has a relative minimum at (x_0,y_0,z_0). This completes the proof.

A set of sufficient conditions for a relative maximum of a function f is obtained by reversing the first and third inequalities in Theorem 3-10 (Hypothesis 3).

Illustrative Example 3-10. Let

$$f(x,y,z) = x^2 - 2y^2 + 3z^2 - xy + 5xz + yz + x - 2y + 3z.$$

Test the function for extreme values.

Solution: Take the first partial derivatives and set them equal to zero, i.e.,

$$f_1 = 2x - y + 5z + 1 = 0$$
$$f_2 = -4y - x + z - 2 = 0$$
$$f_3 = 6z + 5x + y + 3 = 0.$$

The solution of this system is $(-5/17,-8/17,-3/17)$. The second partial derivatives of f at $(-5/17,-8/17,-3/17)$ are as follows:

$$f_{11} = 2,\ f_{11}(-5/17,\ -8/17,\ -3/17) = 2$$

$f_{12} = -1,\ f_{12}(-5/17,\ -8/17,\ -3/17) = -1$

$f_{13} = 5,\ f_{13}(-5/17,\ -8/17,\ -3/17) = 5$

$f_{21} = -1,\ f_{21}(-5/17,\ -8/17,\ -3/17) = -1$

$f_{22} = -4,\ f_{22}(-5/17,\ -8/17,\ -3/17) = -4$

$f_{23} = 1,\ f_{23}(-5/17,\ -8/17,\ -3/17) = 1$

$f_{31} = 5,\ f_{31}(-5/17,\ -8/17,\ -3/17) = 5$

$f_{32} = 1,\ f_{32}(-5/17,\ -8/17,\ -3/17) = 1$

$f_{33} = 6,\ f_{33}(-5/17,\ -8/17,\ -3/17) = 6.$

Conditions 3 become for $(-5/17,\ -8/17,\ -3/17)$

$$f_{11} = 2 > 0,\quad \begin{vmatrix} 2 & -1 \\ -1 & -4 \end{vmatrix} = -9 < 0,\quad \begin{vmatrix} 2 & -1 & 5 \\ -1 & -4 & 1 \\ 5 & 1 & 6 \end{vmatrix} = 34 > 0.$$

Since the signs in conditions 3 are alternately positive and negative, then f has a relative maximum at $(-5/17,\ -8/17,\ -3/17)$ according to Theorem 3-10.

Illustrative Example 3-11. Let

$$f(x,y,z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz.$$

Classify and find the extreme values of f.

Solution:

$$f_1 = 2x - y + 2z = 0$$

$$f_2 = -x + 2y + z = 0$$

$$f_3 = 2x + y + 6z = 0$$

This system of linear equations has only one solution, $(0,0,0)$, since its coefficient determinant is not zero. At $(0,0,0)$ conditions 3 become

$$f_{11} = 2 > 0, \quad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0, \quad \begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{vmatrix} = 4 > 0.$$

Therefore, $f(x,y,z)$ has a relative minimum at $(0,0,0)$, namely, $f(0,0,0) = 0$.

Finally, in this section the definition of a quadratic form in n variables is stated as well as the conditions for positive and negative definiteness, and the extension of Theorem 3-10 to n variables. The proofs of these theorems are essentially the same as those of theorems 3-9 and 3-10. For this reason, they will be omitted here.

Definition 3-20. A real-valued function F defined on E_n by an equation of the type

$$F(\bar{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_i x_j,$$

where $\bar{x} = (x_1,\ldots,x_n)$ in E_n and the a_{ij} are real numbers, is called a quadratic form. The form F is called symmetric if $a_{ij} = a_{ji}$ for all i and j, positive definite if $\bar{x} \neq \bar{0}$ implies $F(\bar{x}) > 0$ and negative definite if $\bar{x} \neq \bar{0}$ implies $F(x) < 0$ [6].

Theorem 3-11. Let $\Delta = \det[a_{ij}]$ and let Δ_{n-k} denote the determinant with $n-k$ rows, obtained by deleting the last k rows and columns of Δ. Also, put $\Delta_0 = 1$. A necessary and sufficient condition for a symmetric form F to be positive definite is that the $n+1$ numbers $\Delta_0, \Delta_1, \ldots, \Delta_n$ be positive. The form is negative definite if, and only if, the same $n+1$ numbers are alternately positive and negative [13].

The extension of Theorem 3-10 is now stated as follows:

Theorem 3-12.

1. f belongs to C^2 on an open set S in E_n
2. $f_1 = f_2 = \dots = f_n = 0$ at $\bar{x}_0$ in S
3. $\Delta = \det[f_{ij}(\bar{x}_0)] \neq 0$
4. $\Delta_0 = 1 > 0, \Delta_1 > 0, \dots, \Delta_n > 0$ at $\bar{x}_0$

implies f has a relative minimum at $\bar{x}_0$. If in hypothesis 4 the numbers $\Delta_0, \Delta_1, \dots, \Delta_n$ are alternately positive and negative, then f has a relative maximum at $\bar{x}_0$ [6].

Illustrative Example 3-12. Let

$$f(w,x,y,z) = w^2 + 2x^2 - y^2 + 3z^2 + wx + 5wy + 7wz - xy + 2xz - yz.$$

Examine f for relative extrema.

Solution: Take the first partial derivatives and set them equal to zero, i.e.,

$$\begin{aligned} f_1 &= 2w + x + 5y + 7z = 0 \\ f_2 &= w + 4x - y + 2z = 0 \\ f_3 &= 5w - x - 2y - z = 0 \\ f_4 &= 7w + 2x - y + 6z = 0 \end{aligned} \qquad (26)$$

The system (26) has the unique solution $(0,0,0,0)$ since its coefficient determinant is nonzero. Conditions 4 in Theorem 3-12 become for $(w_0, x_0, y_0, z_0) = (0,0,0,0)$

$$\Delta_0 = 1,\ \Delta_1 = f_{11} = 2 > 0,\quad \Delta_2 = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7 > 0,\quad \Delta_3 = \begin{vmatrix} 2 & 1 & 5 \\ 1 & 4 & -1 \\ 5 & -1 & -2 \end{vmatrix} = -126 < 0,$$

$$\Delta = \begin{vmatrix} 2 & 1 & 5 & 7 \\ 1 & 4 & -1 & 2 \\ 5 & -1 & -2 & -1 \\ 7 & 2 & -1 & 6 \end{vmatrix} = -388 < 0.$$

Theorem 3-12 is inconclusive in this case. To decide whether this function has relative extrema, it is necessary to express $f(x,y,z,w) - f(0,0,0,0)$ as the sum or difference of squares. This is done as follows:

$$\begin{aligned} f(x,y,z,w) - f(0,0,0,0) &= w^2 + 2x^2 - y^2 + 3z^2 + wx + 5wy + 7wz - \\ &\quad xy + 2xz - yz \\ &= [w + (x/2) + (5y/2) + (7z/2)]^2 + (7x^2/4) - \\ &\quad (29y^2/4) - (37z^2/4) - xy + 2xz - yz \\ &= [w + (x/2) + (5y/2) + (7z/2)]^2 + (7/4)[x - \\ &\quad (2y/7) + (4z/7)]^2 - (207y^2/28) - \\ &\quad (275z^2/28) - yz \\ &= [w + (x/2) + (5y/2) + (7z/2)]^2 + \\ &\quad (7/4)[x - (2y/7) + (4z/7)]^2 - \\ &\quad (207/28)[y + (14z/207)]^2 - (56729z^2/5796) \end{aligned}$$

which can assume both positive and negative values in any neighborhood of $(0,0,0,0)$. Consequently, this quadratic form is indefinite. This implies that the function $f(x,y,z,w)$ has neither a relative maximum nor a relative minimum.

<u>Illustrative Example 3-13</u>. Let

$$f(x,y,z) = x^2 + 2y^4 + xz + z^2.$$

Examine f for relative extrema.

<u>Solution:</u> The first partial derivatives set equal to zero give

$$f_1(x,y,z) = 2x + z = 0$$

$$f_2(x,y,z) = 8y^3 = 0$$

$$f_3(x,y,z) = x + 2z = 0$$

The solution of this system is $(0,0,0)$. The second partial derivatives are as follows:

$$f_{11}(x,y,z) = 2 \qquad f_{21}(x,y,z) = 0 \qquad f_{31}(x,y,z) = 1$$

$$f_{12}(x,y,z) = 0 \qquad f_{22}(x,y,z) = 24y^2 \qquad f_{32}(x,y,z) = 0$$

$$f_{13}(x,y,z) = 1 \qquad f_{23}(x,y,z) = 0 \qquad f_{33}(x,y,z) = 2.$$

At $(0,0,0)$,

$$f_{11} = 2 \geq 0, \quad \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} \geq 0, \quad \begin{vmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{vmatrix} \geq 0$$

imply that the matrix

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}$$

is semi-definite. Therefore, the test is inconclusive in this case [21].

If Taylor's formula with remainder is applied directly to the function f about the origin, an increment of f can be written as follows:

$$\begin{aligned} \Delta f &= f(x,y,z) - f(0,0,0) \\ &= \frac{1}{2}[f_{11}(\theta h_1, \theta h_2, \theta h_3)x^2 + 2f_{12}(\theta h_1, \theta h_2, \theta h_3)xy + \\ &\quad 2f_{13}(\theta h_1, \theta h_2, \theta h_3)xz + f_{22}(\theta h_1, \theta h_2, \theta h_3)y^2 + \\ &\quad 2f_{23}(\theta h_1, \theta h_2, \theta h_3)yz + f_{33}(\theta h_1, \theta h_2, \theta h_3)z^2]. \end{aligned}$$

Or

$$\Delta f = \frac{1}{2}[2x^2 + 2xz + 24\theta^2 h_2^2 y^2 + 2z^2],$$

$$\Delta f = x^2 + xz + 12\theta^2 h_2^2 y^2 + z^2.$$

If the square in x is completed, one gets

$$\begin{aligned} \Delta f &= (x + z/2)^2 + (1/4)(3z^2 + 48\theta^2 h_2^2 y^2), \quad \text{or} \\ &= (1/4)[(2x + z)^2 + 48\theta^2 h_2^2 y^2 + 3z^2]. \end{aligned}$$

Therefore, for all $(x,y,z) \neq (0,0,0)$ in some neighborhood of $(0,0,0)$, Δf is positive definite. This implies that f has a relative minimum at $(0,0,0)$ by definition.

Lagrange's Multipliers

Consider a function $f(x_1, \ldots, x_n)$ where the variables are not independent but are connected by one or more relations. These relations are called subsidiary conditions, restrictions, or constraints. To find the extrema of such a function no new theory is needed. However, the formal procedure can be freed of any consideration of which variables are to be regarded as independent by the introduction of extraneous parameters, known as Lagrange's Multipliers. In this section, the method in the case of one relation between two variables is illustrated and then the results are summarized in a basic theorem. The cases of one relation among three variables and of two relations among three variables will be discussed. Finally, it will be indicated how the technique can be generalized to m relations among n variables with $m < n$.

First the case of one relation between two variables is considered. Suppose it is desired to maximize a function

$$u = f(x,y) \tag{1}$$

where x and y are connected by an equation

$$g(x,y) = 0. \tag{2}$$

Let f,g belong to C^1, $g_1^2 + g_2^2 > 0$ in a region of the xy plane. If g_2 is not zero, equation (2) may be solved for y by the implicit function theorem [6] and substituted in equation (1), thus regarding x as the independent variable. A necessary condition for a maximum (or minimum) is thus seen to be

$$\frac{du}{dx} = f_1 - f_2(g_1/g_2) = 0$$

This is obtained as follows:

$$u = f(x,y),\ g(x,y) = 0,\ \frac{dy}{dx} = \frac{-\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} = -(g_1/g_2)$$

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{du}{dx} = f_1 - f_2(g_1/g_2) = 0.$$

Now

$$f_1 g_2 - f_2 g_1 = \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} = \frac{\partial(f,g)}{\partial(x,y)}\ .$$

Therefore, the points desired will be included among the simultaneous solutions of the equations

$$\frac{\partial(f,g)}{\partial(x,y)} = 0,\ g(x,y) = 0. \tag{3}$$

One could also assume $g_1 \neq 0$ and take y as the independent variable. This will lead to the same pair of equations (3).

To solve this problem by the method of Lagrange, the Lagrange multiplier λ is introduced, forming the function

$$V = f(x,y) + \lambda g(x,y).$$

Now one treats x and y as though they were independent variables and set

$$\frac{\partial V}{\partial x} = f_1 + \lambda g_1 = 0 \tag{4}$$

$$\frac{\partial V}{\partial y} = f_2 + \lambda g_2 = 0 \tag{5}$$

Using the method of substitution, one can solve one of these equations for λ and substitute in the other equation. Combining the result with equation (2) one arrives anew at equations (3). Thus, instead of solving the two equations (1), (2) for x and y, one must now solve the three equations (2), (4), (5) for x, y, and λ. This gives the same pairs (x,y). This result is embodied in the following theorem:

Theorem 3-13.

1. $f(x,y)$, $g(x,y)$ belongs to C^1 in a domain D
2. $g_1^2 + g_2^2 > 0$ in D

then the set of points (x,y) on the curve $g(x,y) = 0$, where $f(x,y)$ has maxima or minima, is included in the set of simultaneous solutions (x,y,λ) of the equations

$$f_1(x,y) + \lambda g_1(x,y) = 0$$
$$f_2(x,y) + \lambda g_2(x,y) = 0$$
$$g(x,y) = 0 \quad [8].$$

Illustrative Example 3-12. Find the shortest distance from the point $(1,0)$ to the parabola $y^2 = 4x$. One must minimize the function

$$u = (x - 1)^2 + y^2$$

with the constraint

$$y^2 = 4x.$$

See Figure 3-6.

Consider the domain D as the entire xy-plane. Then

$$V = (x - 1)^2 + y^2 + \lambda(y^2 - 4x).$$

Differentiating one gets

$$\frac{\partial V}{\partial x} = 2(x - 1) - 4\lambda = 0$$

$$\frac{\partial V}{\partial y} = 2y + 2\lambda y = 0$$

$$\frac{\partial V}{\partial \lambda} = y^2 - 4x = 0$$

From the second equation either $y = 0$ or $y = -1$. The latter must be rejected since it would lead to $x = -1$ (See Figure 3-6). Hence, the only real solution is $x = 0$, $y = 0$, $\lambda = -1/2$, and the minimum distance from $(1,0)$ to the curve $y^2 = 4x$ is unity.

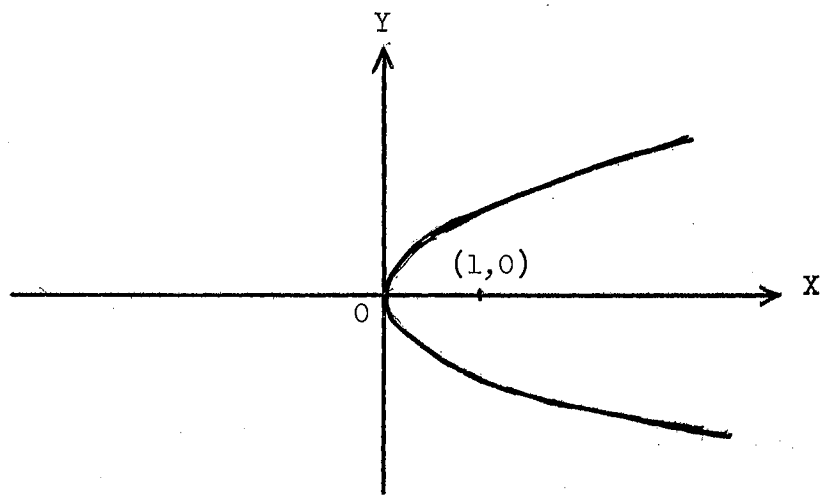

Figure 3-6.

The case of one relation among three variables is now considered. Let

$$u = f(x,y,z)$$
$$g(x,y,z) = 0$$
$$g_1^2 + g_2^2 + g_3^2 > 0.$$

It is seen by elimination that the desired extrema will lie among the simultaneous solutions of one of the three systems:

$$g = 0 \qquad\qquad g = 0 \qquad\qquad g = 0$$

$$\frac{\partial(f,g)}{\partial(x,y)} = 0 \qquad \frac{\partial(f,g)}{\partial(y,x)} = 0 \qquad \frac{\partial(f,g)}{\partial(z,x)} = 0$$

$$\frac{\partial(f,g)}{\partial(x,z)} = 0 \qquad \frac{\partial(f,g)}{\partial(y,z)} = 0 \qquad \frac{\partial(f,g)}{\partial(z,y)} = 0$$

according as it is g_1, g_2, or g_3 which is different from zero [8].

Looking for extrema of the function of three variables x, y, z, i.e.,

$$V = f(x,y,z) + \lambda g(x,y,z)$$

leads to the system

$$g = 0$$
$$f_1 + \lambda g_1 = 0$$
$$f_2 + \lambda g_2 = 0$$
$$f_3 + \lambda g_3 = 0$$

Consequently, one can solve at least one of the equations for λ and thus arrive at one of the above systems.

Illustrative Example 3-13. **Find the rectangular parallelepiped of maximum volume inscribed in a sphere,**

$$x^2 + y^2 + z^2 - 1 = 0.$$

Solution: Since the inscribed parallelepiped will have its center at the center of the sphere, x,y,z will be the half lengths of the three sides. Then one needs to maximize the function $u = xyz$ subject to the constraint $x^2 + y^2 + z^2 - 1 = 0$.

Form the function

$$V = xyz + \lambda(x^2 + y^2 + z^2 - 1)$$

Differentiating, one gets

$$\begin{aligned} yz + 2\lambda x &= 0 \\ xz + 2\lambda y &= 0 \\ xy + 2\lambda z &= 0 \\ x^2 + y^2 + z^2 - 1 &= 1. \end{aligned} \tag{6}$$

From the first three equations in system (6) one gets

$$\lambda = \frac{-yz}{2x}, \quad \lambda = \frac{-xz}{2y}, \quad \lambda = \frac{-xy}{2z} \tag{7}$$

Dividing the first equation by the second equation, the first by the third, and the second by the third in (7) one gets

$$1 = y^2/x^2, \qquad 1 = z^2/x^2, \qquad 1 = z^2/y^2$$

which implies $x = y$, $x = z$, and $y = z$ or $x = y = z$. The numbers x,y,z must be determined so that the fourth equation of system (6) is also satisfied. Using x to represent x,y,z and substituting in the last equation of (6), one gets

$$3x^2 - 1 = 0 \quad \text{or} \quad x = \sqrt{1/3}$$

since x,y,z must all be positive. Now $\lambda = \dfrac{-1}{2\sqrt{3}}$. Therefore, $u = xyz$ has a maximum at $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, i.e., $u = 1/3\sqrt{3}$.

Hence, the rectangular parallelepiped is a cube with edge $2/\sqrt{3}$.

The case of two relations among three variables is now considered where

$$u = f(x,y,z)$$

$$g(x,y,z) = 0$$

$$h(x,y,z) = 0$$

$$\left[\frac{\partial(g,h)}{\partial(x,y)}\right]^2 + \left[\frac{\partial(g,h)}{\partial(y,z)}\right]^2 + \left[\frac{\partial(g,h)}{\partial(z,x)}\right]^2 > 0. \tag{8}$$

There is now a single independent variable which must be chosen in accordance with the Jacobian which is not zero. All three cases lead to the system

$$g = h = \frac{\partial(f,g,h)}{\partial(x,y,z)} = 0. \tag{9}$$

In this case the Lagrange method introduces two parameters λ and μ giving the function

$$F = u(x,y,z) + \lambda g(x,y,z) + \mu h(x,y,z).$$

By differentiating, this function leads to five equations in five unknowns

$$f_1 + \lambda g_1 + \mu h_1 = 0$$
$$f_2 + \lambda g_2 + \mu h_2 = 0$$
$$f_3 + \lambda g_3 + \mu h_3 = 0$$
$$g = 0$$
$$h = 0.$$

Under conditions (8) this system reduces to system (9) when λ and μ are eliminated.

Another way to illustrate Lagrange's method is as follows: Consider the problem of maximizing $w = f(x,y,z)$, where $g(x,y,z) = 0$ and $h(x,y,z) = 0$ are given. The equations $g = 0$ and $h = 0$ describe two surfaces in space and the problem is thus one of maximizing $f(x,y,z)$ as (x,y,z) varies on the curve of intersection of these surfaces. At a maximum point the derivative of f along the curve, i.e., the directional derivative along the tangent to the curve, must be 0. This directional derivative is the component of the vector ∇f along the tangent. It follows that ∇f must lie in a plane normal to the curve at the point. This plane also contains the vectors ∇g and ∇h, i.e., the vectors ∇f, ∇g, and ∇h are coplanar at the point. Hence there must exist scalars λ_1 and λ_2 such that

$$\nabla f + \lambda_1 \nabla g + \lambda_2 \nabla h = \overline{0} \tag{10}$$

at the critical point. This is equivalent to three scalar equations:

$$f_1 + \lambda_1 g_1 + \lambda_2 h_1 = 0$$
$$f_2 + \lambda_1 g_2 + \lambda_2 h_2 = 0$$
$$f_3 + \lambda_1 g_3 + \lambda_2 h_3 = 0.$$

These three equations, together with the equations $g(x,y,z) = 0$,

$h(x,y,z) = 0$ serve as five equations in five unknowns $x,y,z,\lambda_1,\lambda_2$. By solving them for x,y,z, one locates the critical points on the curve. The critical points can be tested by using the second directional derivative [14]. Of course, it has been assumed here that the surfaces $g = 0$, $h = 0$ do actually intersect in a curve and that ∇g and ∇h are linearly independent.

The method described here can be applied to functions of n variables with $m(m < n)$ constraints by introducing m multipliers. That is, if the function $u = f(x_1,\ldots,x_n)$ is subject to the following constraints $g_1(x_1,\ldots,x_n) = 0$, $\ldots$, $g_m(x_1,\ldots,x_n) = 0$, then a new function is formed as follows:

$$F = f(x_1,\ldots,x_n) + \lambda_1 g_1(x_1,\ldots,x_n) + \ldots + \lambda_m g_m(x_1,\ldots,x_n)$$

Then one proceeds as in the simpler cases above. Of course, the same basic assumptions regarding the actual intersection of $g_1,\ldots,g_m$ and the linear independence of ∇g_i where $i = 1, 2, \ldots, m$ are made.

The method of Lagrange gives the critical points but these must be tested to determine whether they are maximum points, minimum points, or neither.

Problems

There is a variety of problems which relate to the theory discussed in this chapter. In this section we shall select samples from the fields of number theory, business (profits, joint-production), and economics (utility function).

Problem 3-1 [8]. The sum of three positive numbers is unity. What is the maximum value of their product?

Solution: Let x, y, z be the three positive (real) numbers. Then one must maximize the function xyz subject to the condition

$$x + y + z = 1 \tag{1}$$

Eliminate z and consider the function

$$f(x,y) = xy(1 - x - y),\ x > 0,\ y > 0,\ 0 < x + y < 1.$$

Let R be chosen as the region in the first quadrant bounded by the following three curves

$$x = 0,\ y = 0,\ x + y = 1.$$

Choose $a = b = 5/36$, $f(a,b) = 25/1944$. Then, $f(x,y) = 0$ on the whole boundary of R and $f(x,y) > 0$ for all (x,y) not on the boundary of R. See Figure 3-7.

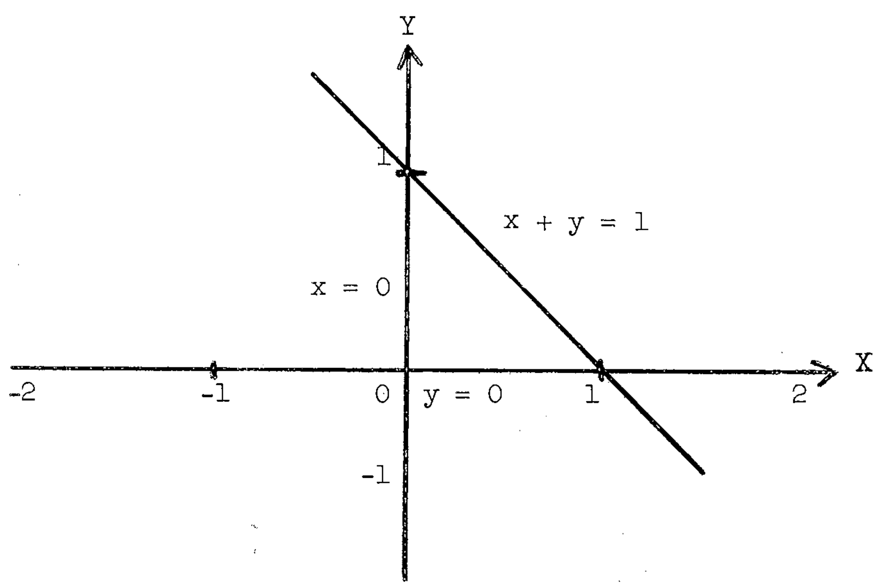

Figure 3-7.

Since $f(x,y)$ belongs to C^1 in R and $f(\frac{5}{36}, \frac{5}{36}) > f(x,y)$ for $(\frac{5}{36}, \frac{5}{36})$ in D and all (x,y) in T, there exists a point (x_0,y_0) in D such that

A. $f(x,y) \leq f(x_0,y_0)$ for all (x,y) in R, and

B. $f_1(x_0,y_0) = f_2(x_0,y_0) = 0$, by Theorem 3-5.

To find this absolute maximum in R, set the first partial derivatives equal to zero as follows:

$$f_1(x,y) = y - 2xy - y^2 = 0$$

$$f_2(x,y) = x - x^2 - 2xy = 0.$$

Solving this system of equations

$$y - 2xy - y^2 = 0$$

$$x - 2xy - x^2 = 0$$

one gets $x = (1 - y)/2$ from the first equation in the system, substituting this result in the second equation of the system, one gets a quadratic equation in y of which the solution set is $\{1/3, 1\}$, and x is 0 or 1/3. Since the absolute maximum of $f(x,y)$ occurs at a point in the domain and is a positive number, $x_0 = 1/3$ and $y_0 = 1/3$. Hence, the maximum value of f is $f(1/3,1/3) = 1/27$.

Problem 3-2 [15]. The total profit per acre on a wheat ranch has been found to be related to the expenditure per acre for (a) labor, and (b) soil conditioners and fertilizers. If x represents the dollars per acre spent on labor, and y represents the dollars per acre spent on soil improvement, the following relationship holds:

$$\text{Profit,}\quad P = 48x + 60y + 10xy - 10x^2 - 6y^2.$$

Determine the optimum expenditure levels.

Solution: Take the first partial derivatives getting

$$P_x = 48 + 10y - 20x$$

$$P_y = 60 + 10x - 12y.$$

Solve the system $P_x = 0$ and $P_y = 0$. The solution is found to be

$x = 8.4$ and $y = 12$. The second derivatives are

$$P_{xx} = -20,\ P_{yy} = -12,\ P_{xy} = 10.$$

The second derivative test, i.e.,

$$[P_{xy}(8.4,\ 12)]^2 - P_{xx}(8.4,\ 12) \times P_{yy}(8.4,\ 12) = 100 - 240 < 0$$

and

$$P_{xx}(8.4,\ 12) < 0$$

implies that P has relative maximum at $(8.4, 12)$, by Theorem 3-7. Thus the maximum profit is $P(8.4, 12) = 561.60$ dollars.

Problem 3-3 [15]. Refer to Problem 3-2. Suppose that, due to financial constraints, it had been decided that a maximum of 15 dollars per acre would be spent on labor and soil improvement combined. What would be the optimum allocation of expenditure? Interpret the significance of the sign of the Lagrange multiplier.

Solution: Form the function

$$F(x,y,\lambda) = 48x + 60y + 10xy - 10x^2 - 6y^2 + \lambda(x + y - 15).$$

The first partial derivatives are as follows:

$$F_1 = 48 + 10y - 20x + \lambda$$

$$F_2 = 60 + 10x - 12y + \lambda$$

$$F_3 = x + y - 15.$$

The solution of the system of equations $F_1 = 0, F_2 = 0, F_3 = 0,$ is the set $\{(159/26,\ 231/26,\ -189/13)\}$, i.e., $x = 159/26$, $y = 231/26$, $\lambda = -189/13$.

As was done in the previous problem, it can be shown that $P(159/26, 231/26) = 522.30$ dollars is a maximum value by Theorem 3-7. The function $F(x,y,\lambda)$ consists of two parts:

Unrestricted earnings

\+ Decrease in earnings due to combined labor and soil improvement restriction.

Since $\lambda = -14.5$ is negative, the combined restriction on labor and soil improvement limits the earnings to 522.30 dollars as compared with the unrestricted earnings of 561.60 dollars.

Problem 3-4 [17]. The demand curves for 2 commodities are $p_A = a - bX_A$ and $p_B = c - dX_B$. The joint-cost function is

$$C = mX_A^2 + nX_B^2 + qX_AX_B,$$

where a, b, c, d, m, n, and q are constants.

(a) Find the necessary and sufficient conditions for maximum profits.

(b) What are the prices?

Solution:

(a) The profit function is

$$\pi = aX_A - bX_A^2 + cX_B - dX_A^2 - mX_B^2 - qX_AX_B.$$

Take the first partial derivatives and set the same equal to zero getting the necessary condition for maximum profits, i.e.,

$$\pi_1 = a - 2bX_A - 2nX_A - qX_B = 0$$

$$\pi_2 = c - 2dX_B - 2nX_B - qX_A = 0$$

by Theorem 3-5. The second partial derivatives are

$$\pi_{11} = -2b - 2m, \pi_{22} = -2d - 2n, \pi_{12} = -q$$

By Theorem 3-7, the sufficient conditions for maximum profits are that

1) $-2b - 2m < 0$, and

2) $(-q)^2 - (-2b - 2m)(-2d -2n) < 0$, or

$$q^2 - 4(bd + bn + dm + mn) < 0$$

(b) Solve the system

$$(2b + 2m)X_A + qX_B = a$$

$$qX_A + (2d + 2n)X_B = c$$

by Cramer's rule getting

$$X_A = \begin{vmatrix} a & q \\ c & 2d+2n \end{vmatrix} \Big/ \begin{vmatrix} 2b+2m & q \\ q & 2d+2n \end{vmatrix}, \quad \text{and}$$

$$X_B = \begin{vmatrix} 2b+2m & a \\ q & c \end{vmatrix} \Big/ \begin{vmatrix} 2b+2m & q \\ q & 2d+2n \end{vmatrix}.$$

Substitute these values in the respective demand equations and solve for the prices.

Problem 3-5 [17]. The demand curves for 3 commodities are $p_A = 10 - 3X_A$, $p_B = 20 - 5X_B$, and $p_C = 60 - 7X_C$. The joint-cost function is

$$C = 10 + 5X_A = 2X_B + 6X_C.$$

(a) Find the necessary conditions for maximum profits.

(b) Determine the prices, total cost, and profits.

Solution:

(a) The profit function is

$$\pi = 10X_A - 3X_A^2 + 20X_B - 5X_B^2 + 60X_C - 7X_C^2 - 10 - 5X_A - 2X_B - 6X_C$$

or

$$\pi = 5X_A - 3X_A^2 + 18X_B - 5X_B^2 + 54X_C - 7X_C^2 - 10.$$

The necessary conditions for maximum profits are that

$$\pi_1 = 5 - 6X_A = 0$$

$$\pi_2 = 18 - 10X_B = 0$$

$\pi_3 = 54 - 14X_C = 0$

(b) To find the prices, one solves the above system of equations and substitutes in the corresponding demand equations, i.e.,

$X_A = 5/6,\ X_B = 9/5,\ X_C = 27/7$

$p_A = 10 - 3(5/6) = 15/2$

$p_B = 20 - 5(9/5) = 11$

$p_C = 60 - 7(27/7) = 33$

The total cost is

$$C = 10 + 5(5/6) + 2(9/5) + 6(27/7) = 8591/210 = 40.91.$$

The profit is

$$\pi = 5(5/6) - 3(5/6)^2 + 18(9/5) - 5(9/5)^2 + 54(27/7) - 7(27/7)^2 - 10$$

$$\pi = 11.24.$$

A brief discussion on utility theory is now given before introducing the next problem. There is a function $U = f(x,y)$, which (in a sense) indicates the satisfaction derived by the individual from varying combinations of the amounts of the commodities X and Y. The amounts of these commodities are x and y and their prices are p_x and p_y. Assume that the individual spends all his money on just these two commodities X and Y and that his income I is also given. Then the so-called budget equation is

$$p_x x + p_y y = I,$$

where p_x, p_y, and I are constants.

The individual will try to maximize U by choosing appropriate amounts of X and Y while taking the budget constraint into account. This is a problem in restricted maxima [17].

<u>Problem 3-6</u> [17]. A utility index is $U = e^{xy}$. Let $p_x = 1$, $p_y = 5$, and $I = 10$. Find the demand for X and Y.

<u>Solution:</u> Form the function

$$F(x,y,\lambda) = e^{xy} + \lambda(x + 5y - 10).$$

Take the first partial derivatives and set the same equal to zero, that is,

$$F_1 = ye^{xy} + \lambda = 0$$

$$F_2 = xe^{xy} + 5\lambda = 0$$

$$F_3 = x + 5y - 10 = 0, \quad \text{by Theorem 3-5.}$$

From the third equation, $x = 10 - 5y$. Substitute for x in the first two equations getting

$$ye^{10y - 5y^2} + \lambda = 0$$

$$(10 - 5y)e^{10y - 5y^2} + 5\lambda = 0$$

Divide the first equation by the second equation. One gets

$$y/(2 - y) = 1,$$

$$y = 2 - y$$

$$y = 1$$

$$x = 5$$

$$\lambda = -e^5.$$

The second partial derivatives are

$$F_{11} = U_{11} = y^2e^{xy}, \; U_{11}(5,1) = e^5$$

$$F_{22} = U_{22} = x^2e^{xy}, \; U_{22}(5,1) = 25e^5$$

$$F_{12} = U_{12} = xye^{xy}, \; U_{12}(5,1) = 5e^5.$$

Now $U_{11}(5,1) > 0$ and $[U_{11}(5,1)][U_{22}(5,1)] - [U_{12}(5,1)]^2$
$= 25e^{10} - 5e^5 > 0.$

Therefore, U has a maximum value at $(5,1)$, by Theorem 3-7.

CHAPTER IV

FUNDAMENTAL THEORY OF CALCULUS OF VARIATIONS

SIMPLEST VARIATIONAL PROBLEMS

WITH FIXED ENDPOINTS

Thus far in this report the writer has dealt with the problem of finding points at which differentiable functions of one or more variables possess maximum or minimum values. In the calculus of variations, one deals with the far more extensive problem of finding functional forms for which given integrals assume maximum or minimum values. Or, in the language of geometry, one may say that this calculus deals with the problem of finding paths of integration for which integrals admit extrema. In elementary calculus, the student learns how to find the length of a given curve, $y = f(x)$, between two fixed points, A and B, by means of the formula

$$s = \int_a^b (1 + y'^2)^{1/2} dx.$$

In analysis, the length of a curve is defined as the limiting length of a polygonal line inscribed in the curve (i.e., with vertices lying on the curve) as the maximum length of the chords forming the polygonal line goes to zero. If this limit exists and is finite, the curve is said to be rectifiable [19]. Suppose one generalizes this problem so that, instead of being given a curve, $y = f(x)$, and two fixed points on it, one is given only the two fixed points and is required to find the curve which minimizes the integral

$$s[y] = \int_{x_0}^{x_1} F(x,y,y')dx.$$

There are many curves connecting the two points A and B. See Figure 4-1.

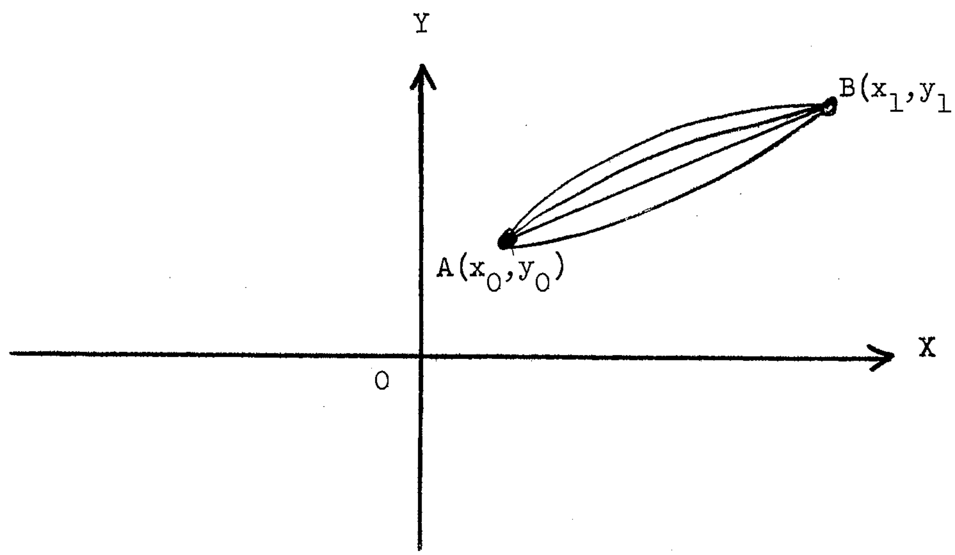

Figure 4-1.

The generalized problem is now one in the calculus of variations which reveals the answer as curves of the form $y = mx + b$, i.e., straight lines. There are other problems in elementary calculus which can be generalized in a similar fashion so as to become problems in the calculus of variations. For example, which curve revolved about the x-axis between two fixed points will generate a solid of revolution with the minimum surface area? The integrals mentioned above are called functionals, that is, functions from a function space to the reals. The domain of the functional is the set of all y, where $y = y(x)$ is a function defined on some closed interval $[a,b]$. Likewise, the set of all v, where $v = s[y]$ is a real number, comprises the range of the functional. The domain of a functional J is a function space, and the study of functionals J on various spaces Y is the material of courses in functional analysis. Variational theory is that section of functional analysis concerned with the existence and determination of $y_0 \in Y$ such that $J[y_0]$ is either an extremum of $J[y]$ or a stationary value. A stationary value requires solely the vanishing of the first differential, without any restriction on the second differential. In this chapter, the writer will discuss the basic properties of the functional such as the norm, continuity, first variation, second variation, and other properties which are needed to develop necessary and sufficient conditions for a functional to have an extremum. This treatment includes Euler's equation (a necessary condition), Legendre's necessary condition, Jacobi's necessary condition, and a set of sufficient conditions for a functional to have an extremum. A brief look is taken at the case of several independent variables. For the most part, though, our attention is directed to the simplest variational problem with fixed endpoints. Illustrative examples are included wherever they seem appropriate to clarify concepts.

Functionals and Function Spaces

In this section, the writer gives formal definitions of a functional and of a linear space whose elements may be functions and is then called a function space. The concept of a norm is introduced to explain the notion of closeness between two elements in the space, neighborhood of an element in the space, and the continuity of a functional in the space. Several examples of a normed linear space, in which the elements are functions, are given.

Let Y denote a given class of functions $y:[a,b] \to E_1$ and let $J:Y \to E_1$ be a given function on Y to the reals. Such a function J is called a functional to distinguish it from the more familiar functions, $f:E_1 \to E_1$. The latter are called point-functions because their values $f(x)$ depend on the choice of a point x in E_1. The value $J[y]$ depends on the choice of $y \in Y$.

The formal definition of a particular type of functional, i.e., $J[y]$ is now given.

Definition 4-1. If $[a,b]$ is a fixed interval, F a continuous function of three variables x,y,y', and Y the class of all continuously differentiable functions on $[a,b]$, then the statement that

$$J[y] = \int_a^b F(x,y(x),y'(x))dx$$

defines a functional $J:Y \to E_1$, i.e., a functional of a curve with fixed end points [18].

Since the domain Y of a functional J is usually a subset of some function space, it can be assumed that in some way or the other neighborhoods $U(y_0,\delta)$ of $y_0 \in Y$ can be defined which make it possible for us to deal with such concepts as convergence and continuity [18]. The concept of continuity plays an important role for functionals, just as it does for the ordinary functions considered in classical analysis. In order to formulate this concept for functionals, one must introduce a concept of "closeness" for elements in a function space. This is most conveniently done by introducing the concept of the norm of a function, analogous to the concept of the distance between a point in Euclidean space and the origin of coordinates. Although in what follows this paper will always be concerned with function spaces, it will be most convenient to introduce the concept of a norm in a more general and abstract form, by introducing the concept of a normed linear space [19].

Definition 4-2. A linear space is a set R of elements $x, y, z, \ldots$ of any kind, for which the operations of addition and multiplication by real numbers $a, b, \ldots$ are defined and obey the following axioms:

(1) $x + y = y + x$,

(2) $(x + y) + z = x + (y + z)$,

(3) There exists an element 0 (the zero element) such that $x + 0 = x$ for any $x \in R$,

(4) For each $x \in R$, there exists an element, $-x$, such that $x + (-x) = 0$,

(5) $1(x) = x$,

(6) $a(bx) = (ab)x$,

(7) $(a + b)x = ax + bx$,

(8) $a(x + y) = ax + ay$ [19].

Definition 4-3. A linear space R is said to be normed, if each element $x \in R$ is assigned a nonnegative number $\|x\|$, called the norm of x, such that

(1) $\|x\| = 0$ if and only if $x = 0$,

(2) $\|ax\| = |a| \; \|x\|$,

(3) $\|x + y\| \leq \|x\| + \|y\|$, [19].

On the basis of this concept of a normed linear space, one can talk about distances between elements, defining the distance between x and y to be the quantity $\|x - y\|$. The elements of a normed linear space can be objects of any kind, e.g., numbers, vectors, matrices, functions, etc. The writer now examines several normed linear spaces which are useful in our subsequent work.

(1) The space C, or more precisely C(a,b), consists of all continuous functions y(x) defined on the closed interval [a,b]. By addition of elements of C and multiplication of elements of C by numbers, one means ordinary addition of functions and multiplication of functions by numbers, while the norm is defined as the maximum of the absolute value, i.e.,

$$\|y\|_0 = \max_{a \leqslant x \leqslant b} |y(x)|.$$

Thus, in the space C, the distance between the functions $y_1(x)$ and $y_2(x)$ does not exceed ϵ if the graph of the function $y_2(x)$ lies inside a strip of width 2ϵ (in the vertical direction) "bordering" the graph of the function $y_1(x)$, as shown in Figure 4-2 [19].

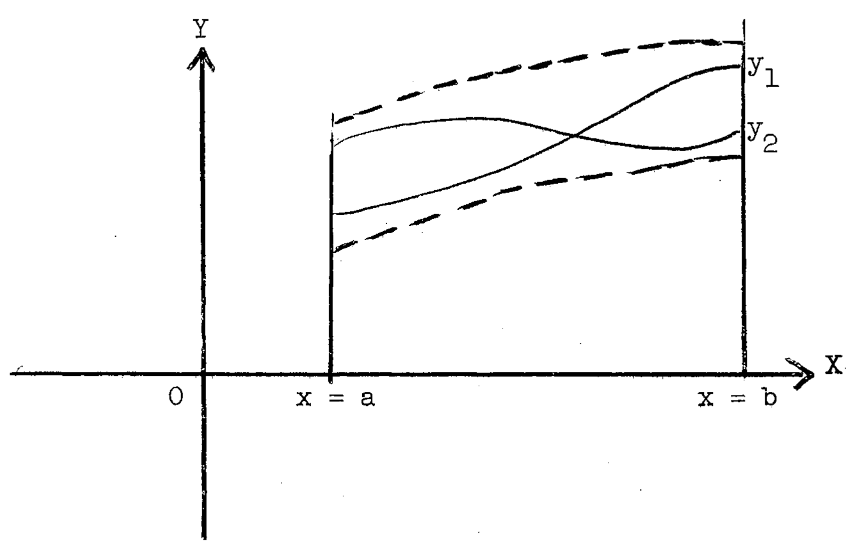

Figure 4-2.

(2) The space D_1, or more precisely $D_1(a,b)$, consists of all functions $y(x)$ defined on an interval $[a,b]$, which are continuous and have continuous first derivatives. The operations of addition and multiplication by numbers are the same as in the space C, but the norm is defined by the formula

$$\|y\|_1 = \max_{a \leqq x \leqq b} |y(x)| + \max_{a \leqq x \leqq b} |y'(x)|.$$

Thus, two functions in D_1 are regarded as close together if both the functions themselves and their first derivatives are close together, since

$$\|y - z\| < \epsilon$$

implies

$$|y(x) - z(x)| < \epsilon, \ |y'(x) - z'(x)| < \epsilon$$

for all $a \leqq x \leqq b$ [19].

(3) The space D_n, or more precisely $D_n(a,b)$, consists of all functions $y(x)$ defined on the interval $[a,b]$ which are continuous and have continuous derivatives up to order n inclusive, where n is a fixed integer. Addition of elements of D_n and multiplication of elements of D_n by numbers are defined just as they are in the spaces C and D_1 of the preceding cases, but the norm is now defined by the formula

$$\|y\|_n = \sum_{i=0}^{n} \max_{a \leqq x \leqq b} |y^{(i)}(x)|,$$

where $y^i(x) = (d/dx)^i y(x)$ and $y^{(0)}(x)$ denotes the function $y(x)$ itself. Thus, two functions in D_n are regarded as close together if the values of the functions themselves and of all their derivatives up to order n inclusive are close together. It is easily verified that all the axioms of a normed linear space are actually satisfied for each of the spaces C, D_1, and D_n [19].

In a similar manner, one can introduce spaces of functions of several variables, e.g., the space of continuous functions of n variables, the space of functions of n variables with continuous first partial derivatives, etc. After a norm has been introduced in the linear space R (which may be a function space), it is natural to talk about continuity of functionals defined on R.

Definition 4-4. The functional $J[y]$ is said to be continuous at the point $y_0 \in R$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|J[y] - J[y_0]| < \epsilon$$

provided that $\|y - y_0\| < \delta$ [19].

So far, the writer has talked about linear spaces and functionals defined on them. However, in many variational problems, one has to deal with functionals defined on sets of functions which do not form linear spaces. In fact, the set of functions (or curves) satisfying the constraints of a given variational problem, called the admissible functions (or admissible curves), is in general not a linear space. For example, the admissible curves for the "simplest" variational problem are the smooth plane curves passing through two fixed points, and the sum of two such curves does not pass through the two points. Nevertheless, the concept of a normed linear space and the related concepts of the distance between functions, continuity of functionals, etc., play an important role in the calculus of variations.

The First Variation of a Functional
A Necessary Condition
For an Extremum

In this section, the concept of the variation (or differential) of a functional, analogous to the concept of the differential of a function of n variables is introduced. The concept will then be used to find extrema of functionals. First, some preliminary facts and definitions are given.

Definition 4-5. Given a normed linear space R, let each element $h \in R$ be assigned a number $v[h]$, i.e., let $v[h]$ be a functional defined on R. Then $v[h]$ is said to be a (continuous) linear functional if

(1) $v[ah] = av[h]$ for any $h \in R$ and any real number a,

(2) $v[h_1 + h_2] = v[h_1] + v[h_2]$ for any $h_1, h_2 \in R$,

(3) $v[h]$ is continuous for all $h \in R$.

Illustrative Example 4-1. If one associates with each function $h(x) \in C(a,b)$ its value at a fixed point x_0 in $[a,b]$, i.e., if one defines the functional $v[h]$ by the formula

$$v[h] = h(x_0)$$

then $v[h]$ is a linear functional on $C(a,b)$.

Illustrative Example 4-2. The integral

$$v[h] = \int_a^b h(x)dx$$

defines a linear functional on $C(a,b)$.

The writer now considers a linear functional defined on $D_n(a,b)$

of the form

$$v[h] = \int_a^b [A_0(x)h(x) + A_1(x)h'(x) + \dots + A_n(x)h^{(n)}(x)]dx,$$

where the $A_i(x)$ are fixed functions in $C(a,b)$. Suppose $v[h]$ vanishes for all $h(x)$ belonging to some class. Then what can be said about the functions $A_i(x)$? Some typical results in this direction are given by the next four lemmas. The reader is referred to [19] for proofs.

Lemma 4-1. If $A(x)$ is continuous in $[a,b]$, and if

$$\int_a^b A(x)h(x)dx = 0$$

for every function $h(x) \in C(a,b)$ such that $h(a) = h(b) = 0$, then $A(x) = 0$ for all $x \in [a,b]$.

Lemma 4-2. If $A(x)$ is continuous in $[a,b]$, and if

$$\int_a^b A(x)h'(x)dx = 0$$

for every function $h(x) \in D_1(a,b)$ such that $h(a) = h(b) = 0$, then $A(x) = c$ for all x in $[a,b]$, where c is a constant.

Lemma 4-3. If $A(x)$ is continuous in $[a,b]$, and if

$$\int_a^b A(x)h''(x)dx = 0$$

for every function $h(x) \in D_2(a,b)$ such that $h(a) = h(b) = 0$ and $h'(a) = h'(b) = 0$, then $A(x) = c_0 + c_1 x$ for all x in $[a,b]$, where c_0 and c_1 are constants.

Lemma 4-4. If $A(x)$ and $B(x)$ are continuous in $[a,b]$, and if

$$\int_a^b [A(x)h(x) + B(x)h'(x)]dx = 0$$

for every function $h(x) \in D_1(a,b)$ such that $h(a) = h(b) = 0$, then $B(x)$ is differentiable, and $B'(x) = A(x)$ for all x in $[a,b]$.

The concept of the variation of a functional is now introduced. Let $J[y]$ be a functional defined on some normed linear space, and let

$$\Delta J[h] = J[y + h] - J[y]$$

be its increment, corresponding to the increment $h = h(x)$ of the "independent variable" $y = y(x)$. If y is fixed, $\Delta J[h]$ is a functional of h, in general a nonlinear functional. Suppose that

$$\Delta J[h] = v[h] + \epsilon \, \|h\|$$

where $v[h]$ is a linear functional and $\epsilon \to 0$ as $\|h\| \to 0$. Then the functional $J[y]$ is said to be differentiable, and the <u>principal linear part</u> of the increment $\Delta J[h]$, i.e., the linear functional $v[h]$ which differs from $\Delta J[h]$ by an infinitesimal of order higher than 1 relative to $\|h\|$, is called the variation of $J[h]$ and is denoted by $\delta J[h]$, [19].

<u>Illustrative Example 4-3</u>. Find the first variation of the functional

$$J[y] = \int_0^{\pi/2} (y'^2 - y^2)dx, \; y(0) = 0, \; y(\pi/2) = 1.$$

<u>Solution</u>: Assume that y is fixed and let h be an increment such that $h(0) = h(\pi/2) = 0$. The increment of $J[y]$ is determined as follows:

$$\Delta J[h] = \int_0^{\pi/2} [(y' + h')^2 - (y + h)^2]dx - \int_0^{\pi/2} (y'^2 - y^2)dx,$$

$$= \int_0^{\pi/2} [y'^2 + 2y'h' + h'^2 - y^2 - 2yh - h^2 - y'^2 + y^2]dx$$

$$= 2\int_0^{\pi/2} [-yh + y'h']dx + \int_0^{\pi/2} [-h^2 + h'^2]dx.$$

The principal linear part of the increment $\Delta J[h]$ is

$$2\int_0^{\pi/2} [-yh + y'h']dx.$$

By definition, this is the first variation of $J[y]$, i.e.,

$$\delta J[h] = 2\int_0^{\pi/2} [-yh + y'h']dx.$$

Theorem 4-1. The differential of a differentiable functional is unique [19].

Proof: Suppose that $v[h]$ is a linear functional and that

$$\frac{v[h]}{\|h\|} \to 0$$

as $\|h\| \to 0$. Then $v[h] \equiv 0$, i.e., $v[h] = 0$ for all h. In fact, suppose $v[h_0] \neq 0$ for some $h_0 \neq 0$. Then, setting

$$h_n = h_0/n, \quad \lambda = v[h_0]/\|h_0\|,$$

it is seen that $\|h_n\| \to 0$ as $n \to \infty$, but

$$\lim_{n \to \infty} \frac{v[h_n]}{\|h_n\|} = \lim_{n \to \infty} \frac{v[\frac{h_0}{n}]}{\frac{\|h_0\|}{n}} = \lim_{n \to \infty} \frac{\frac{1}{n} v[h_0]}{\|\frac{1}{n}\| \|h_0\|}$$

$$= \lim_{n \to \infty} \frac{\frac{1}{n} v[h_0]}{\frac{1}{n}\|h_0\|} = \lim_{n \to \infty} \frac{nv[h_0]}{n\|h_0\|}$$

$$= \lambda \neq 0$$

contrary to hypothesis.

Now, suppose the differential of the functional $J[y]$ is not uniquely defined, so that

$$\Delta J[h] = v_1[h] + \epsilon_1\|h\|,$$

$$\Delta J[h] = v_2[h] + \epsilon_2\|h\|,$$

where $v_1[h]$ and $v_2[h]$ are linear functionals, and ϵ_1, $\epsilon_2 \to 0$ as $\|h\| \to 0$. This implies

$$v_1[h] - v_2[h] + \epsilon_1\|h\| - \epsilon_2\|h\| = 0,$$

or

$$v_1[h] - v_2[h] = \epsilon_2\|h\| - \epsilon_1\|h\|,$$

and hence $v_1[h] - v_2[h]$ is an infinitesimal of order higher than 1 relative to $\|h\|$. But $v_1[h]$, $v_2[h]$ being linear functionals imply $v_1[h] - v_2[h]$ is a linear functional, and it follows from the first part of the proof that $v_1[h] - v_2[h]$ vanishes identically, as asserted. Therefore, $v_1[h] = v_2[h]$, [19].

It is now shown how the concept of variation can be used to establish a necessary condition for a functional $J[y]$ to have an extremum. To begin with, the corresponding concepts from differential calculus are reviewed. Let $F(x_1,\ldots,x_n)$ be a differentiable function of n variables. Then $F(x_1,\ldots,x_n)$ is said to have a (relative) extremum at the point $(a_1,\ldots,a_n)$ if

$$\Delta F = F(x_1,\ldots,x_n) - F(a_1,\ldots,a_n)$$

has the same sign for all points $(x_1,\ldots,x_n)$ belonging to some neighborhood of $(a_1,\ldots,a_n)$. The extremum $F(a_1,\ldots,a_n)$ is a minimum if $\Delta F \geq 0$ and a maximum if $\Delta F \leq 0$.

In the same manner, the functional $J[y]$ has a relative extremum for $y = \hat{y}$ if $J[y] - J[\hat{y}]$ does not change its sign in some neighborhood of the curve $y = \hat{y}(x)$. Subsequently, this paper shall be concerned with functionals defined on some set of continuously differentiable functions, and the functions themselves can be regarded either as elements of the space C or elements of the space D_1. In connection with these two spaces, two kinds of extrema are defined:

The functional $J[y]$ has a <u>weak extremum</u> for $y = \hat{y}$ if there exists an $\epsilon > 0$ such that $J[y] - J[\hat{y}]$ has the same sign for all y in the domain of definition of the functional which satisfy the condition $\|y - \hat{y}\|_1 < \epsilon$ where $\| \|_1$ denotes the norm in the space D_1. On the other hand, the functional $J[y]$ has a <u>strong extremum</u> for $y = \hat{y}$ if there exists an $\epsilon > 0$ such that $J[y] - J[\hat{y}]$ has the same sign for all y in the domain of definition of the functional which satisfy the condition $\|y - \hat{y}\|_0 < \epsilon$, where $\| \|_0$ denotes the norm in the space C. It is clear that every strong extremum is at the same time a weak extremum, since if $\|y - \hat{y}\|_1 < \epsilon$, then $\|y - \hat{y}\|_0 < \epsilon$, and hence, if $J[\hat{y}]$ is an extremum with respect to all y such that $\|y - \hat{y}\|_0 < \epsilon$, then $J[\hat{y}]$ is certainly an extremum with respect to all y such that $\|y - \hat{y}\|_1 < \epsilon$. However, a weak extremum may not be a strong extremum, and as a rule the weak extremum is easier to find than the strong extremum. The reason for this is that the functionals usually considered in the calculus of variations are continuous in the norm of the space D_1, and this continuity can be exploited in the theory of weak extrema. In general, however, functionals will not be continuous in the norm the space C.

<u>Illustrative Example 4-4</u>. A functional continuous in the norm of D_1 but not continuous in the norm of C is now constructed. Consider the linear mapping $F : C \to R$ (reals). If F is bounded on

$\{f : f \in D_1,\ \|f\| + \|f'\| \leq 1\}$, then F is continuous in the norm of D_1. On the other hand if F is unbounded on $\{f : f \in C,\ \|f\| \leq 1\}$, then F is not continuous in the norm of C. Let $F_1 : f \to f'(0)$ for $f \in D_1$ and let F be an extension of F_1 to C. If $f \in D_1$ and $\|f\|_1 \leq 1$, then

$$\|f(x)\|_1 = \max_{0 \leq x \leq 1} |f(x)| + \max_{0 \leq x \leq 1} |f'(x)| \leq 1,$$

or $$|F(f)| \leq 1 \text{ if } \|f\|_1 \leq 1.$$

Hence, F is continuous in D_1. If $f \in \{p_n\}$ where $p_n(x) = e^{-nx}$ for $n = 0, 1, 2, \ldots,$ then $\|f\| = \max\limits_{0 \leq x \leq 1} e^{-nx} = 1$. Consequently, $|F(p_n)| = |p_n'(0)| = |-n| = n$ which implies F is unbounded on $\{f : \|f\| \leq 1\}$. Therefore, F is discontinuous in C.

The following theorem is a necessary condition for a functional $J[y]$ to have an extremum at $y = \hat{y}$.

Theorem 4-2. A necessary condition for the differentiable functional $J[y]$ to have an extremum for $y = \hat{y}$ is that its variation vanish for $y = \hat{y}$, i.e.,

$$\delta J[y] = 0$$

for $y = \hat{y}$ and all admissible h [19].

Proof: Suppose $J[y]$ has a minimum for $y = \hat{y}$. By the definition of the variation $\delta J[h]$,

$$\Delta J[h] = \delta J[h] + \epsilon\|h\|,$$

where $\epsilon \to 0$ as $\|h\| \to 0$. Thus, for sufficiently small $\|h\|$, the sign of $\Delta J[h]$ will be the same as the sign of $\delta J[h]$. Now suppose that $\delta J[h_0] \neq 0$ for some admissible h_0. Then for any $A > 0$, no matter how small,

$$\delta J[-Ah_0] = -\delta J[Ah_0],$$

since $\delta J[y]$ is linear. Hence, $\Delta J[h] = \delta J[h] + \epsilon\|h\|$ can be made to have either sign for arbitrarily small $\|h\|$. But this is impossible, since by hypothesis $J[y]$ has a minimum for $y = \hat{y}$, i.e.,

$$\Delta J[h] = J[\hat{y} + h] - J[\hat{y}] \geq 0$$

for all sufficiently small $\|h\|$. This contradiction proves the theorem [19].

The Simplest Variational Problem

The simplest variational problem can be formulated as follows:

Let $F(x,y,z)$ be a function with continuous first and second partial derivatives with respect to all its arguments. Then, among all functions $y(x)$ which are continuously differentiable for $a \leq x \leq b$ and satisfy the boundary conditions

$$y(a) = A, \; y(b) = B, \tag{1}$$

find the function for which the functional

$$J[y] = \int_a^b F(x,y,y')dx \tag{2}$$

has a weak extremum. To restate the problem simply, it is required to find a weak extremum of a functional of the form (2) where the class of admissible curves consists of all smooth curves joining two points (a smooth curve or function $f(x)$ being one continuous in $[a,b]$ and having a continuous derivative in $[a,b]$). To apply the necessary condition for an extremum (Theorem 4-2) to the problem just formulated, one needs to be able to calculate the variation of the functional of the type (2). The task now is to derive an appropriate formula for this variation.

Suppose $y(x)$ is given an increment $h(x)$, where, in order for the function $y(x) + h(x)$ to continue to satisfy the boundary conditions, $h(a)$ and $h(b)$ must vanish. Then the corresponding increment of the functional (2) equals

$$\Delta J = J[y + h] - J[y] = \int_a^b F(x,y + h,y' + h')dx - \int_a^b F(x,y,y')dx$$

$$\Delta J = \int_a^b [F(x,y + h,y' + h') - F(x,y,y')]dx,$$

but by Taylor's theorem (Theorem 3-2) it follows that

$$J = \int_a^b [F_y(x,y,y')h + F_{y'}(x,y,y')h']dx + (1/2)\int_a^b [F_{yy}(x,y,y')h^2 + 2F_{yy'}(x,y,y')hh' + F_{y'y'}(x,y,y')h'^2]dx + \ldots \tag{3}$$

where the subscripts denote partial derivatives with respect to the corresponding arguments, and the dots denote terms of order higher than 2 relative to h and h'. Thus the first integral in the right hand member of (3) is the principal linear part of ΔJ, and is by definition, the variation of $J[y]$, i.e.,

$$\delta J = \int_a^b [F_y(x,y,y')h + F_{y'}(x,y,y')h']dx.$$

According to Theorem 4-2, a necessary condition for $J[y]$ to have an extremum for $y = y(x)$ is that

$$\delta J = \int_a^b (F_y h + F_{y'} h')dx = 0 \tag{4}$$

for all admissible h. But by Lemma 4-4, (4) implies that

$$F_y - \frac{d}{dx} F_{y'} = 0, \tag{5}$$

a result known as Euler's equation. Thus, the following theorem is proved:

Theorem 4-3 [19]. Let $J[y]$ be a functional of the form

$$\int_a^b F(x,y,y')dx,$$

defined on the set of functions $y(x)$ which have continuous first derivatives in $[a,b]$ and satisfy the boundary conditions $y(a) = A$, $y(b) = B$. Then a necessary condition for $J[y]$ to have an extremum for a given function $y(x)$ is that $y(x)$ satisfy Euler's equation

$$F_y - \frac{d}{dx} y' = 0.$$

The integral curves of Euler's equation are called extremals. Since Euler's equation is a second-order differential equation, its solution will in general depend on two arbitrary constants, which are determined from the boundary conditions $y(a) = A$, $y(b) = B$. The problem usually considered in the theory of differential equations is that of finding a solution which is defined in the neighborhood of some point and satisfies given initial conditions (Cauchy's problem). However, in solving Euler's equation, a solution which is defined over all of some fixed region and satisfied given boundary conditions is sought. Therefore, the question of whether or not a certain variational problem has a solution does not just reduce to the usual existence theorems for differential equations. In connection with this, a theorem due to Bernstein is stated concerning the existence and uniqueness of solutions "in the large" of an equation of the form

$$y'' = F(x,y,y'). \tag{6}$$

Theorem 4-4 [19]. If the functions F, F_y, and $F_{y'}$ are continuous at every finite point (x,y) for any finite y', and if a constant $k > 0$ and functions

$$A = A(x,y) \geq 0, \ B = B(x,y) \geq 0$$

(which are bounded in every finite region of the plane) can be found such that

$$F_y(x,y,y') > k, \ |F(x,y,y')| \leq Ay'^2 + B,$$

then one and only one integral curve of equation (6) passes through any two points (a,A) and (b,B) with different abscissas $(a \neq b)$.

For a proof of this theorem the reader is referred to reference [22].

For a functional of the form

$$\int_a^b F(x,y,y')dx$$

Euler's equation is in general a second-order differential equation, but it may turn out that the curve for which the functional has its extremum is not twice differentiable. The following theorem gives conditions which guarantee that a solution of Euler's equation has a second derivative:

Theorem 4-5 [19]. Suppose $y = y(x)$ has a continuous first derivative and satisfies Euler's equation

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

Then, if the function $F(x,y,y')$ has continuous first and second derivatives with respect to all its arguments, $y(x)$ has a continuous second derivative at all points (x,y) where

$$F_{y'y'}[x,y(x),y'(x)] \neq 0.$$

The proof of this theorem is omitted.

Euler's equation, $F_y - \frac{d}{dx} F_{y'} = 0$, plays a fundamental role in the calculus of variations, and is in general a second-order differential equation. In some special cases, Euler's equation can be reduced to a first-order differential equation or its solution can be obtained entirely in terms of quadratures, i.e., by evaluating integrals. Now several such cases are examined.

Case 1. Suppose the integrand does not depend on y, i.e., let the functional under consideration have the form

$$\int_a^b F(x,y')dx$$

where F does not contain y explicitly. In this case, Euler's equation becomes

$$\frac{d}{dx} F_{y'} = 0$$

which obviously has the first integral

$$F_{y'} = C, \tag{7}$$

where C is a constant. This is a first-order differential equation which does not contain y. Solving (7) for y', one obtains an equation of the form

$$y' = f(x,C),$$

from which y can be found by a quadrature.

Case 2. If the integrand does not depend on X, i.e., if

$$J[y] = \int_a^b F(y,y')dx,$$

then

$$F_y - \frac{d}{dx} F_{y'} = F_y - F_{y'y} y' - F_{y'y'} y'' \tag{8}$$

by the chain rule. Multiplying (8) by y', one obtains

$$y'F_y - y' \frac{d}{dx} F_{y'} = F_y y' - F_{y'y} y'^2 - F_{y'y'} y' y''.$$

Here the right hand side is $\frac{d}{dx}(F - y'F_{y'})$ since

$$\frac{d}{dx}(F - y'F_{y'}) = F_y y' + F_{y'} y'' - y'F_{y'y} y' - y'F_{y'y'} y'' - y''F_{y'}$$

$$= F_y y' - F_{y'y} y'^2 - F_{y'y'} y' y''.$$

Thus Euler's equation becomes $\frac{d}{dx}(F - y'F_{y'}) = 0$ and has the first integral

$$F - y'F_{y'} = C,$$

where C is a constant.

Case 3. If F does not depend on y', Euler's equation takes the form

$$F_y(x,y) = 0,$$

and hence is not a differential equation, but a "finite" equation, whose solution consists of one or more curves $y = y(x)$.

Case 4. In a variety of problems, one encounters functionals of the form

$$\int_a^b f(x,y)\sqrt{(1 + y'^2)}\,dx,$$

representing the integral of a function $f(x,y)$ with respect to the arc length s, i.e., $ds = \sqrt{(1 + y'^2)}\,dx$. In this case, Euler's equation can be transformed into

$$F_y - \frac{d}{dx} F_{y'} = f_y(x,y)\sqrt{(1 + y'^2)} - \frac{d}{dx}\left[f(x,y)y'/\sqrt{(1 + y'^2)}\right]$$

$$= f_y\sqrt{(1 + y'^2)} - [f_x y'/\sqrt{(1 + y'^2)}] - [f_y y'^2/\sqrt{(1 + y'^2)}] -$$

$$[fy''/(1 + y'^2)^{3/2}]$$

$$= [1/\sqrt{(1 + y'^2)}][f_y - f_x y' - fy''/(1 + y'^2)] = 0$$

or

$$f_y - f_x y' - fy''/(1 + y'^2) = 0.$$

This section is closed by giving several examples illustrating several of the cases explained in the previous paragraph.

Illustrative Example 4-5. The functional

$$t[y] = \int_1^2 [\sqrt{(1 + y'^2)}/x]dx, \; y(1) = 0, \; y(2) = 1$$

is the time that passes when a particle is moving from one point to some other point along the curve $y = y(x)$ with the velocity $v = x$. Find the curve $y(x)$ which minimizes the time t [20].

Solution: Since the integrand does not contain y, case 1 is illustrated here, where Euler's equation assumes the form $F_{y'} = C$. Hence $F = \sqrt{(1 + y'^2)}/x$ and

$$F_{y'} = y'/x\sqrt{(1 + y'^2)} = C.$$

The above equation cleared of fractions becomes

$$y' = Cx\sqrt{(1 + y'^2)}. \qquad (9)$$

Further simplifications of (9) gives

$$y'^2(1 - C^2x^2) = C^2x^2, \quad \text{or}$$

$$y = Cx/\sqrt{(1 - C^2x^2)}. \qquad (10)$$

Equation (10) when solved and simplified becomes

$$x^2 + (y - C_1)^2 = 1/C^2.$$

This equation represents a two-parameter family of extremal curves (circles). The boundary conditions

$$y(1) = 0, \; y(2) = 1$$

are used to find the curve passing through the points (1,0) and (2,1). From these conditions, $C_1 = 2$ and $C = 1/5$.

Therefore, the solution is

$$(y - 2)^2 + x^2 = 5.$$

Illustrative Example 4-6 (The problem of minimum surface of revolution). Among all the curves joining two given points (x_0, y_0) and (x_1, y_1), find the one which generates the surface of minimum area when rotated about the x-axis [19].

Solution:

From elementary calculus, it is known that the area of a surface of revolution is

$$S[y(x)] = 2\pi \int_{x_0}^{x_1} y\sqrt{(1 + y'^2)}\,dx.$$

The integrand depends only on y and y' illustrating case 2. Thus Euler's equation has the first integral

$$F - y'F_{y'} = C_1.$$

Since $F = y\sqrt{(1 + y'^2)}$ and $F_{y'} = yy'/\sqrt{(1 + y'^2)}$, then

$$y\sqrt{(1 + y'^2)} - [yy'^2/\sqrt{(1 + y'^2)}] = C_1$$

$$y(1 + y'^2) - yy'^2 = C_1\sqrt{(1 + y'^2)}$$

$$y + yy'^2 - yy'^2 = C_1\sqrt{(1 + y'^2)}$$

$$y = C_1\sqrt{(1 + y'^2)}$$

Let $y' = \sinh(t)$. Then $y = C_1\sqrt{(1 + \sinh^2(t))}$ implying $y = C_1\cosh(t)$. Now $\frac{dy}{dx} = y'$ implying $dx = dy/y' = \frac{C_1\sinh(t)dt}{\sinh(t)}$

or

$$dx = C_1 dt, \; x = C_1 t + C_2.$$

Hence, the desired surface is obtained by revolving a curve with the parametric equations

$$x = C_1 t + C_2, \; y = C_1\cosh(t).$$

The parameter t is eliminated by substitution. Hence,

$$y = C_1 \cosh \frac{x - C_2}{C_1},$$

a family of catenaries. A surface that is generated by rotating catenary about the x-axis is called a <u>catenoid</u>. The constants C_1 and C_2 can be determined from the boundary conditions, i.e., $y(x_0) = A$, $y(x_1) = B$.

<u>Illustrative Example 4-7</u>. Find the general solution of the Euler's equation corresponding to the functional

$$J[y] = \int_a^b f(x)\sqrt{(1 + y'^2)}dx,$$

and investigate the special cases $f(x) = \sqrt{x}$ and $f(x) = x$.

(a) $F(x,y,y') = f(x)\sqrt{(1 + y'^2)}$

$$F_y - \frac{d}{dx}F_{y'} = -\frac{d}{dx}[f(x)y'/\sqrt{(1 + y'^2)}]$$

$$= [-f_x y'/\sqrt{(1 + y'^2)}] - [fy''/(1 + y'^2)^{3/2}] = 0$$

$\frac{d}{dx}[fy'/\sqrt{(1 + y'^2)}] = 0$, Euler's equation

$fy'/\sqrt{(1 + y'^2)} = C$, first integral to Euler's equation.

This equation is simplified as follows:

$$f(x)y' = C\sqrt{(1 + y'^2)},$$

$$f^2(x)y'^2 = C^2(1 + y'^2),$$

$$[f^2(x) - C^2]y'^2 = C^2,$$

$$y'^2 = C^2/[f^2(x) - C^2],$$

$$y' = C/\sqrt{(f^2(x) - C^2)},$$

$$\frac{dy}{dx} = C/\sqrt{[f^2(x) - C^2]},$$

$$dy = Cdx/[f^2(x) - C^2].$$

The general solution of the above differential equation is

$$y = C\int [dx/\sqrt{f^2(x) - C^2}] + C_1.$$

(b) Let $f(x) = \sqrt{x}$. Then

$$y = C [dx/\sqrt{(x - C^2)}] + C_1$$

$$y = 2C\sqrt{(x - C^2)} + C_1$$

$$y - C_1 = 2C\sqrt{(x - C^2)}$$

$$(y - C_1)^2 = 4C^2(x - C^2).$$

Therefore, the extremal curve is a family of parabolas each member with its vertex at (C^2, C_1) and having its axis of symmetry parallel to the x-axis.

(c) Let $f(x) = x$. Then

$$y = C\int [dx/\sqrt{(x^2 - C^2)}] + C_1$$

$$y = C\ln |x + \sqrt{(x^2 - C^2)}| + C_1.$$

Case of Several Variables

So far, only functionals depending on functions of one variable, i.e., on curves, have been considered. In many problems, however, one encounters functionals depending on functions of several independent variables, i.e., on surfaces. The case of two variables is utilized to illustrate how the formulation and solution of the simplest variational problem discussed in a previous section carries over to the case of functionals depending on surfaces. Let $F(x,y,z,p,q)$ be a function with continuous first and second partial derivatives with respect to all its arguments, and consider a functional of the form

$$J[z] = \int\int_R F(x,y,z,z_x,z_y)dxdy, \tag{11}$$

where z is defined as a function from a closed region R in E_2 to the reals, and z_x, z_y are the partial derivatives of $z = z(x,y)$. Suppose that a function with the following properties is desired:

1. $z(x,y)$ and its first and second derivatives are continuous in R,

2. $z(x,y)$ takes given values on the boundary T of R, and

3. the functional (11) has an extremum for $z = z(x,y)$.

Just as in the case of one variable, a necessary condition for the functional (11) to have an extremum is that its variation (i.e., the principal linear part of its increment) vanish. The proof of this statement is the same as it is for the case of one variable. But, in order, to find Euler's equation for the functional (11), the following lemma, analogous to Lemma 4-1, is needed.

Lemma 4-5 [19]. If $A(x,y)$ is a fixed function which is continuous in a closed region R, and if the integral

$$\int\int_R A(x,y)h(x,y)dxdy \tag{12}$$

vanishes for every function $h(x,y)$ which has continuous first and second derivatives in R and equals zero on the boundary T of R, then $A(x,y) = 0$ everywhere in R.

Proof: Suppose $A(x,y) > 0$ at some point in R. Then $A > 0$ in some circle (neighborhood)

$$(x - x_0)^2 + (y - y_0)^2 \leq \epsilon^2 \tag{13}$$

contained in R, with center (x_0,y_0) and radius ϵ. Let $h(x,y) = 0$ outside the circle (13) and

$$h(x,y) = [(x - x_0)^2 + (y - y_0)^2 - \epsilon^2]^3$$

inside the circle. Then $h(x,y)$ satisfies the conditions of the lemma. However, in this case (12) reduces to an integral over the circle (13) and is obviously positive. This contradicts the fact that the integral vanishes for every function $h(x,y)$. Therefore, $A(x,y)$ must be zero everywhere in R.

In order to apply the necessary condition for an extremum of the functional (11), i.e., $\delta J = 0$, there is a need to calculate the variation δJ. Let $h(x,y)$ be an arbitrary function which has continuous first and second partial derivatives in the region R and vanishes on the boundary T of R. Then if $z(x,y)$ belongs to the domain of definition of the functional (11), so does $z(x,y) + h(x,y)$. By definition

$$\Delta J = J[z + h] - J[z]$$

$$= \iint_R [F(x,y,z + h,\ z_x + h_x, z_y + h_y) - F(x,y,z,z_x,z_y)dxdy.$$

It follows from Taylor's Formula (Theorem 3-2) that

$$\Delta J = \iint_R [F_z h + F_{z_x} h_x + F_{z_y} h_y]dxdy + (1/2)\iint_R [F_{zz}h^2 + F_{z_x z_x} h_{xx}^2 +$$

$$F_{z_y z_y} h_{yy}^2 + 2F_{zz_x} hh_x + 2F_{zz_y} hh_y + 2F_{z_x z_y} h_x h_y]dxdy + \ldots,$$

where the dots denote terms of order higher than 2 relative to h, h_x, h_y. The first integral on the right represents the principal linear part of ΔJ and is therefore the variation of J, i.e.,

$$\delta J = \iint_R (F_z h + F_{z_x} h_x + F_{z_y} h_y)dxdy. \tag{14}$$

Consider the double integral of the second and third terms of the right member of the equation above, i.e.,

$$\iint_R (F_{z_x} h_x + F_{z_y} h_y)dxdy.$$

The integration by parts formula gives

$$\iint_R F_{z_x} h_x dxdy = \iint_R \frac{\partial}{\partial x} (F_{z_x} h)dxdy - \iint_R h \frac{\partial}{\partial x} F_{z_x} dxdy \tag{15}$$

and

$$\iint_R F_{z_y} h_y dxdy = \iint_R \frac{\partial}{\partial y} (F_{z_y} h)dxdy - \iint_R h \frac{\partial}{\partial y} F_{z_y} dxdy. \tag{16}$$

The sum of (15) and (16) is

$$\iint_R (F_{z_x} h_x + F_{z_y} h_y)dxdy$$

$$= \iint_R [\frac{\partial}{\partial x} (F_{z_x} h) + \frac{\partial}{\partial x} (F_{z_y} h)]dxdy - \iint_R [\frac{\partial}{\partial x} F_{z_x} + \frac{\partial}{\partial x} F_{z_x}]hdxdy$$

$$= \int_T (F_{z_x} hdy - F_{z_y} hdx) - \iint_R [\frac{\partial}{\partial x} F_{z_x} + \frac{\partial}{\partial x} F_{z_y}]hdxdy,$$

where in the last step Green's Theorem [8] is applied, i.e.,

$$\int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \int_T (Pdx + Qdy).$$

The line integral along T is zero, since $h(x,y)$ vanishes on T. Therefore,

$$\int\int_R (F_{z_x} h_x + F_{z_y} h_y)dxdy = - \int\int_R \left(\frac{\partial}{\partial x} F_{z_x} + \frac{\partial}{\partial y} F_{z_y}\right)hdxdy.$$

The substitution of the above result for $F_{z_x} h_x + F_{z_y} h_y$ in (14) gives

$$\delta J = \int\int_R \left(F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y}\right)h(x,y)dxdy. \tag{17}$$

Thus, the condition that $\delta J = 0$ implies that the double integral (17) vanishes for any $h(x,y)$ satisfying the stipulated conditions. Therefore, by Lemma 4-5,

$$F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} = 0, \tag{18}$$

which, again is known as Euler's equation. Consequently, y is found by solving the second-order partial differential equation (18).

Illustrative Example 4-8 [19]. Find the surface of least area spanned by a given contour (the boundary curve (in space) through which every admissible surface must pass).

Solution: The formula for finding the surface area is given by the functional

$$J[z] = \int\int_R \sqrt{(1 + z_x^2 + z_y^2)}dxdy$$

where R is the area enclosed in the xy-plane by the projection of the given contour onto the xy-plane. In this case, Euler's equation is determined as follows:

$$F = \sqrt{(1 + z_x^2 + z_y^2)},\ F_z = 0,\ F_{z_x} = z_x/\sqrt{(1 + z_x^2 + z_y^2)}$$

$$F_{z_y} = z_y/\sqrt{(1 + z_x^2 + z_y^2)}.$$

Hence, from

$$F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} = 0,$$

one gets

$$\frac{\partial}{\partial x}[z_x/\sqrt{(1 + z_x^2 + z_y^2)}] + \frac{\partial}{\partial y}[z_y/\sqrt{(1 + z_x^2 + z_y^2)}] = 0$$

or

$$\frac{\sqrt{(1 + z_x^2 + z_y^2)}z_{xx} - z_x(1 + z_x^2 + z_y^2)^{-1/2}(z_x z_{xx} + z_y z_{xy})}{1 + z_x^2 + z_y^2}$$

$$+\frac{\sqrt{(1 + z_x^2 + z_y^2)}z_{yy} - z_y(1 + z_x^2 + z_y^2)^{-1/2}(z_x z_{xy} + z_y z_{yy})}{1 + z_x^2 + z_y^2} = 0$$

$$(1 + z_x^2 + z_y^2)z_{xx} - z_x(z_x z_{xx} + z_y z_{xy}) + (1 + z_x^2 + z_y^2)z_{yy}$$

$$- z_y(z_x z_{xy} + z_y z_{yy}) = 0,$$

$$z_{xx} + z_x^2 z_{xx} + z_y^2 z_{xx} - z_x^2 z_{xx} - z_x z_y z_{xy} + z_{yy} + z_x^2 z_{yy} - z_x z_y z_{xy} - z_y^2 z_{yy} = 0$$

$$z_{xx}(1 + z_y^2) - 2z_x z_y z_{xy} + z_{yy}(1 + z_x^2) = 0,$$

which can be written as

$$r(1 + q^2) - 2spq + t(1 + p^2) = 0,$$

where

$$p = z_x,\ q = z_y,\ r = z_{xx},\ s = z_{xy},\ t = z_{yy}.$$

The equation

$$r(1 + q^2) - 2spq + t(1 + p^2) = 0$$

has a simple geometric meaning, which can be explained by using the formula

$$M = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}$$

for the mean curvature of the surface, where E, F, G, e, f, g are the coefficients of the first and second fundamental quadratic forms of the surface [8]. If the surface is given by an explicit equation of the form $z = z(x,y)$, then

$$E = 1 + p^2,\ F = pq,\ G = 1 + q^2,\ e = r/\sqrt{(1 + p^2 + q^2)},$$
$$f = s/\sqrt{(1 + p^2 + q^2)},\ g = t/\sqrt{(1 + p^2 + q^2)}$$

and hence

$$M = \frac{(1 + p^2)t - 2spq + (1 + q^2)r}{\sqrt{(1 + p^2 + q^2)}}$$

Here, the numerator coincides with the left-hand side of Euler's equation. Thus, the equation

$$r(1 + q^2) - 2spq + t(1 + p^2) = 0$$

implies that the mean curvature of the required surface equals zero. Surfaces with zero mean curvature are called minimal surfaces.

Second Variation. Sufficient Conditions For A Weak Extremum

Until now, in studying extrema of functionals, only one necessary condition for a functional to have a weak (relative) extremum for a given curve $f(x)$ has been considered, i.e., the condition that the variation of the functional vanish for the curve $f(x)$. In this section, sufficient conditions for a functional to have a weak extremum are derived. In order to find some of these sufficient conditions, a new concept is needed, namely, the second variation of a functional. In studying the properties of the second variation, some new necessary conditions for an extremum are also derived.

A functional $B[x,y]$ depending on two elements x and y, belonging to some normed linear space R, is said to be bilinear if it is a linear functional of y for any fixed x and a linear functional of x for any fixed y. Thus,

$$B[x + y,z] = B[x,z] + B[y,z],$$
$$B[Ax,y] = AB[x,y],$$

and

$$B[x,y + z] = B[x,y] + B[x,z],$$
$$B[x,Ay] = AB[x,y]$$

for any $x,y,z \in R$ and any real number A.

If $y = x$ in a bilinear functional, the expression is called a quadratic functional.

A bilinear functional defined on a finite-dimensional space is called a bilinear form. Symbolically, every bilinear form can be expressed as follows:

$$B[x,y] = \sum_{i,k=1}^{n} b_{ik}x_i y_k$$

where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are the components of the "vectors" x and y relative to some basis. Of course, if $y = x$ in this expression, it becomes a quadratic form, i.e.,

$$A[x] = B[x,x] = \sum_{i,k=1}^{n} b_{ik}x_i x_k.$$

A quadratic functional $A[x] = B[x,x]$ is said to be <u>positive definite</u> if $A[x] > 0$ for every nonzero element x.

<u>Illustrative Example 4-9</u> [19]. The expression

$$B[x,y] = \int_a^b x(t)y(t)dt$$

is a bilinear functional defined on the space C of functions which are continuous in the interval $a \leq t \leq b$. The corresponding quadratic functional is

$$A[x] = \int_a^b x^2(t)dt.$$

<u>Illustrative Example 4-10</u> [19]. A more general bilinear functional defined on C is

$$B[x,y] = \int_a^b \alpha(t)x(t)y(t)dt,$$

where $\alpha(t)$ is a fixed function. If $\alpha(t) > 0$ for all t in $[a,b]$, then the corresponding quadratic functional

$$A[x] = \int_a^b \alpha(t)x^2(t)dt$$

is positive definite.

<u>Illustrative Example 4-11</u> [19]. The expression

$$A[x] = \int_a^b [\alpha(t)x^2(t) + \beta(t)x(t)x'(t) + \gamma(t)x'^2(t)]dt$$

is a quadratic functional defined on the space D_1 of all functions which are continuously differentiable in the interval $[a,b]$.

Let J[y] be a functional defined on some normed linear space R. Then J[y] is differentiable if its increment

$$\Delta J[h] = J[y + h] - J[y]$$

can be written in the form

$$\Delta J[h] = \phi_1(h) + \epsilon\|h\|,$$

where $\phi_1(h)$ is a linear functional and $\epsilon \to 0$ as $\|h\| \to 0$. The quantity $\phi[h]$ is the principal linear part of the increment $\Delta J[h]$, and is called the (first) variation (or first differential) of J[y], denoted by $\delta J[h]$.

Similarly, the functional J[h] is said to be twice differentiable if its increment can be written in the form

$$\Delta J[h] = \phi_1[h] + \phi_2[h] + \epsilon\|h\|^2,$$

where $\phi_1[h]$ is a linear functional, $\phi_2[h]$ is a quadratic functional, and $\epsilon \to 0$ as $\|h\| \to 0$. The quadratic functional $\phi_2[h]$ is called the second variation (or second differential) of the functional J[y], and is denoted by $\delta^2 J[h]$. The uniqueness of the second variation of a functional can be proved in the same way as it was proved for the first variation, see Theorem 4-1. For an example of the second variation of a functional, see the solution of Illustrative Example 4-3.

<u>Theorem 4-6</u> [19]. A necessary condition for the functional J[y] to have a minimum for $y = \hat{y}$ is that

$$\delta^2 J[y] \geq 0 \tag{19}$$

for $y = \hat{y}$ and all admissible h. For a maximum, the sign $\geq$ in (19) is replaced by $<$.

The proof is given in the same reference and is omitted here.

A quadratic functional $\phi_2[h]$ defined on some normed linear space R is strongly positive if there exists a constant $k > 0$ such that

$$\phi_2[h] \geq k \|h\|^2$$

for all h [19].

In a finite-dimensional space, strong positivity of a quadratic form is equivalent to positive definiteness of the quadratic form. Therefore, a function of a finite number of variables has a minimum at a point P where its first differential vanishes, if its second differential is positive at P. The latter statement is related to Theorem 3-12 (sufficient conditions for a function of n variables to have extrema).

Theorem 4-7 [19]. A sufficient condition for a functional $J[y]$ to have a minimum for $y = \hat{y}$, given that the first variation $\delta J[h]$ vanishes for $y = \hat{y}$ is that its second variation $\delta^2 J[h]$ be strongly positive for $y = \hat{y}$.

Proof: For $y = \hat{y}$, $\delta J[h] = 0$ for all admissible h by Theorem 4-2, and hence

$$\Delta J[h] = \delta^2 J[h] + \epsilon\|h\|^2,$$

where $\epsilon \to 0$ as $\|h\| \to 0$. Moreover, for $y = \hat{y}$,

$$\delta^2 J[h] \geq k\|h\|^2,$$

where k is a constant greater than 0. This follows from the definition of strong positivity. Thus, for sufficiently small ϵ_1, $|\epsilon| < (1/2)K$ if $\|h\| < \epsilon_1$. It follows that

$$\Delta J[h] = \delta^2 J[h + \epsilon\|h\|^2 > (1/2)k\|h\|^2 > 0$$

if $\|h\| < \epsilon_1$, i.e., $J[y]$ has a minimum for $y = \hat{y}$, as asserted [19].

The Formula for the Second Variation

Legendre's Condition

Let $F(x,y,z)$ be a function with continuous partial derivatives up to order three with respect to all its arguments. Smoothness requirements will be assumed to hold whenever needed. What is needed now is an expression for the second variation in the case of the simplest variational problem, i.e., for functionals of the form

$$J[y] = \int_a^b F(x,y,y')dx \tag{20}$$

defined for curves $y = y(x)$ with fixed endpoints

$$y(a) = A, \; y(b) = B.$$

To begin with, let the function $y(x)$ take on an increment $h(x)$ satisfying the boundary conditions

$$h(a) = 0, \; h(b) = 0. \tag{21}$$

Then by Taylor's theorem (Theorem 3-2) with remainder, the increment of the functional $J[y]$ can be written as

$$\Delta J[h] = J[y + h] - J[y]$$

$$= \int_a^b (F_y h + F_{y'} h')dx + \int_a^b (\bar{F}_{yy} h^2 + 2\bar{F}_{yy'} hh' + \bar{F}_{y'y'} h'^2)dx, \tag{22}$$

where the overbar indicates that the corresponding derivatives are evaluated along certain intermediate curves, i.e.,

$$\bar{F}_{yy} = F_{yy}(x,y + \theta h,y' + \theta h') \quad (0 < \theta < 1),$$

and similarly for $\bar{F}_{yy'}$ and $\bar{F}_{y'y'}$.

If $\bar{F}_{yy}$, $\bar{F}_{yy'}$, and $\bar{F}_{y'y'}$ are replaced by the derivatives F_{yy}, $F_{yy'}$, and $F_{y'y'}$ evaluated at the point $(x,y(x),y'(x))$ then (22) becomes

$$\Delta J[h] = \int_a^b (F_y h + F_{y'} h')dx + (1/2)\int_a^b (F_{yy} h^2 + 2F_{yy'} hh' + F_{y'y'} h'^2)dx + \epsilon \quad (23)$$

where ϵ can be written as

$$\int_a^b (\epsilon_1 h^2 + \epsilon_2 hh' + \epsilon_3 h'^2)dx. \quad (24)$$

The continuity of the derivatives F_{yy}, $F_{yy'}$, and $F_{y'y'}$ implies $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$ as $\|h\| \to 0$ from which is apparent that ϵ is an infinitesimal of order higher than 2 relative to $\|h\|^2$. Hence, the first term in the right-hand side of (23) is $\delta J[h]$, and the second term, which is quadratic in h, is the second variation $\delta^2 J[h]$. Consequently, for the functional (20),

$$\delta^2 J[h] = (1/2)\int_a^b (F_{yy} h^2 + 2F_{yy'} hh' + F_{y'y'} h'^2)dx. \quad (25)$$

Integration by parts and the boundary conditions imposed on h, i.e., $h(a) = h(b) = 0$, are used to transform (25) into a more convenient form.

The procedure is as follows:

Let $u = F_{yy'}$, $dv = 2hh'dx$.

Then

$$du = \frac{d}{dx}(F_{yy'})dx, \quad v = h^2.$$

Therefore, by the formula $\int udv = uv - \int vdu$,

$$\int_a^b 2F_{yy'} hh'dx = [h^2 F_{yy'}]_a^b - \int_a^b h^2 \frac{d}{dx}(F_{yy'})dx. \quad (26)$$

By the boundary conditions in (21), (26) reduces to

$$\int_a^b 2F_{yy'}hh'dx = -\int_a^b h^2 \frac{d}{dx}(F_{yy'})dx. \tag{27}$$

Substituting (27) in (25) gives

$$\delta^2 J[h] = (1/2)\int_a^b [F_{yy}h^2 - \frac{d}{dx}(F_{yy'})h^2 + F_{y'y'}h'^2]dx, \tag{28}$$

and (28) can be written as

$$\delta^2 J[h] = \int_a^b (Ph'^2 + Qh^2)dx, \tag{29}$$

where

$$P = P(x) = (1/2)F_{y'y'}, \quad Q = Q(x) = (1/2)(F_{yy} - \frac{d}{dx}F_{yy'}). \tag{30}$$

It was proved in Theorem 4-6 that a necessary condition for a functional $J[y]$ to have a minimum is that its second variation $\delta^2 J[h]$ be nonnegative. In the case of a functional of the form (20), formula (29) can be used to establish a necessary condition for the second variation to be nonnegative. The argument runs as follows: Consider the quadratic functional (29) for functions $h(x)$ satisfying the condition $h(a) = 0$. With this condition, the function $h(x)$ will be small in the interval $[a,b]$ if its derivative $h'(x)$ is small in $[a,b]$. The proof of this statement is as follows:

Assume $h(a) = 0$ and let $|h'(x)| < \epsilon$ in $[a,b]$ where ϵ represents an arbitrarily small positive real number. Then

$$h(x) = \int_a^x h'(t)dt$$

and

$$|h(x)| \leq \int_a^x |h'(t)|dt$$

$$\leq \epsilon(x - a)$$

$$\leq \epsilon(b - a).$$

However, the converse of the above statement is not true, i.e., a function $h(x)$ which is itself small but has a large derivative $h'(x)$ in $[a,b]$ can be constructed. For example, let $a = 0$, $b = \pi/2$, and $h(x) = \epsilon \sin(nx)$. Then $|h(x)| \leq \epsilon$, $h'(x) = n\,\epsilon \cos(nx)$, and $h'(0) = n\epsilon$. It is seen here that the derivative $h'(x)$ can be made larger than any preassigned value K by choosing n sufficiently large.

This implies that the term Ph'^2 plays the dominant role in the quadratic functional (29), in the sense that Ph'^2 can be much larger than the second term Qh^2 (assuming that $P \neq 0$). Therefore, one might expect that the coefficient $P(x)$ determines whether the functional (29) takes values with just one sign or values with both signs. The result of this argument is now stated as a lemma.

<u>Lemma</u> <u>4-6</u> [19]. A necessary condition for the quadratic functional

$$\delta^2 J[h] = \int_a^b (Ph'^2 + Qh^2)dx, \tag{30}$$

defined for all functions $h(x) \in D_1(a,b)$ such that $h(a) = h(b) = 0$ to be nonnegative is that

$$P(x) \geq 0 \qquad (a \leq x \leq b). \tag{31}$$

For the proof of this lemma, see [19].

Using this lemma and the necessary condition for a minimum proved in Theorem 4-6, one immediately gets Legendre's Condition.

<u>Theorem</u> <u>4-7</u> [LEGENDRE]. A necessary condition for the functional

$$J[y] = \int_a^b F(x,y,y')dx, \; y(a) = A, \; y(b) = B$$

to have a minimum for the curve $y = y(x)$ is that the inequality

$$F_{y'y'} \geq 0$$

(Legendre's condition) be satisfied at every point of the curve [19].

<u>Illustrative Example 4-12.</u> Is the Legendre condition satisfied by the extremal curve of the functional

$$J[y] = \int_0^1 (y'^2 + 12xy)dx,$$

which passes through the points $(0,0)$ and $(1,1)$?

<u>Solution:</u> Let $F = y'^2 + 12xy$. $F_y = 12x$, $F_{y'} = 2y'$. Therefore, the Euler equation which is obtained from $F_y - \frac{d}{dx}F_{y'} = 0$ is

$$12x - \frac{d}{dx}(2y') = 0, \quad \text{or}$$

$$y'' = 6x.$$

The solution of this differential equation is

$$y = x^3 + C_1x + C_2.$$

It follows from the boundary conditions that $C_1 = C_2 = 0$. Hence, Legendre condition (Theorem 4-7) is satisfied, i.e., $F_{y\,y} = 2 > 0$, at every point on the extremal $y = x^3$. In fact, the strengthened Legendre condition is satisfied here.

Analysis of the Quadratic Functional

$$\int_a^b (Ph'^2 + Qh^2)dx$$

In the previous section, it is indicated the condition

$$P(x) \geq 0 \qquad (a \leq x \leq b)$$

is necessary but not sufficient for the quadratic functional

$$\int_a^b (Ph'^2 + Qh^2)dx \tag{32}$$

to be ≥ 0 for all admissible $h(x)$. In this section, it is assumed that the condition

$P(x) > 0$ (strengthened Legendre's condition), $(a \leq x \leq b)$

holds. Conditions are stated which are both necessary and sufficient for the functional (32) to be greater than zero for all admissible $h(x) \neq 0$, i.e., for the functional (32) to be positive definite. These results will be used later to establish both necessary and sufficient conditions for a functional to have an extremum. To begin with, Euler's equation is written

$$-\frac{d}{dx}(Ph') + Qh = 0 \tag{33}$$

corresponding to the functional (32). This is a linear differential equation of the second order, which is satisfied, together with the boundary conditions

$$h(a) = 0,\ h(b) = 0 \tag{33}$$

or more generally, the boundary conditions

$$h(a) = 0,\ h(c) = 0,\ (a < c \leq b),$$

by the function $h(x) \equiv 0$. However, in general, (33) can have other non-trivial solutions satisfying the same boundary conditions. In this regard, the following new concept is introduced:

Definition 4-6. The point $\tilde{a}(\neq a)$ is said to be conjugate to the point a if the equation (33) has a solution which vanishes for $x = a$ and $x = \tilde{a}$ but is not identically zero [19].

Using this concept of a conjugate point and the fact $P(x) > 0$, a sufficient condition for (32) to be positive definite is now stated.

Theorem 4-8. If

$$P(x) > 0 \qquad (a \leq x \leq b)$$

and if the interval [a,b] contains no points conjugate to a, then the quadratic functional

$$\int_a^b (Ph'^2 + Qh^2)dx \tag{34}$$

is positive definite for all h(x) such that h(a) = h(b) = 0 [19].

The proof of this theorem is found in the same reference. The basic idea of the proof that (34) is positive definite hinges upon whether (34) can be reduced to the form

$$\int_a^b P(x)\phi^2(\ldots)dx$$

where $\phi^2(\ldots)$ is some expression which cannot be identically zero unless $h(x) \equiv 0$.

In the absence of points conjugate to a in the interval [a,b], it can be shown that this condition is not only sufficient but also necessary for the functional (34) to be positive definite. This result is summarized in the theorem which follows:

Theorem 4-9. If the quadratic functional

$$\int_a^b (Ph'^2 + Qh^2)dx \tag{35}$$

where

$$P(x) > 0 \qquad (a \leq x \leq b),$$

is positive definite for all h(x) such that h(a) = h(b) = 0, then the interval [a,b] contains no points conjugate to a, [19].

If the condition that the functional (35) be positive definite is replaced by the condition that it be nonnegative for all admissible $h(x)$, then the following result is obtained:

Theorem 4-10. If the quadratic functional

$$\int_a^b (Ph'^2 + Qh^2)dx \tag{36}$$

where

$$P(x) > 0 \qquad (a \leq x \leq b)$$

is nonnegative for all $h(x)$ such that $h(a) = h(b) = 0$, then the interval $[a,b]$ contains no interior points conjugate to a, [19].

The combination of Theorems 8 and 9 gives a necessary and sufficient condition for the quadratic functional (36) to be positive definite.

Theorem 4-11 (Necessary and Sufficient Condition). The quadratic functional

$$\int_a^b (Ph'^2 + Qh^2)dx$$

where

$$P(x) > 0 \qquad (a \leq x \leq b),$$

is positive definite for all $h(x)$ such that $h(a) = h(b) = 0$ if and only if the interval $[a,b]$ contains no points conjugate to a [19].

Jacobi's Necessary Condition

The results of the previous section are now applied to the simplest variational problem, i.e., to the functional

$$\int_a^b F(x,y,y')dx \tag{37}$$

with boundary conditions

$$y(a) = A, \; y(b) = B.$$

In a previous section, the second variation of the functional (37) in the neighborhood of some extremal $y = y(x)$ was expressed by

$$\int_a^b (Ph'^2 + Qh^2)dx \tag{38}$$

where

$$P = (1/2)F_{y'y'}, \; Q = (1/2)(F_{yy} - \frac{d}{dx} F_{yy'}).$$

Definition 4-7 [19]. The Euler equation

$$- \frac{d}{dx} (Ph') + Qh = 0 \tag{39}$$

of the quadratic functional (38) is called the Jacobi equation of the original functional (37).

Definition 4-8 [19]. The point $\tilde{a}$ is said to be conjugate to the point a with respect to the functional (37) if it is conjugate to a with respect to the quadratic functional (38) which is the second variation of (37), i.e., if it is conjugate to a in the sense of the definition 4-6.

Theorem 4-12 (Jacobi's Necessary Condition). If the extremal $y = \hat{y}(x)$ corresponds to a minimum of the functional

$$\int_a^b F(x,y,y')dx$$

and if

$$F_{y'y'} > 0$$

along this extremal, then the open interval (a,b) contains no points conjugate to a. This theorem remains true if the word "minimum" is replaced by "maximum" and the condition $F_{y'y'} > 0$ by $F_{y'y'} < 0$ [19].

Proof: By Theorem 4-6, the fact that $J[y]$ has a minimum at $y = \hat{y}(x)$ implies $\delta^2 J[y] \geq 0$ for all admissible h. Moreover, according to Theorem 4-10, if the quadratic functional (38) is nonnegative, the interval (a,b) can contain no points conjugate to a. The theorem follows at once from these two facts taken together.

Illustrative Example 4-13. Is the Jacobi condition satisfied by the extremal curve of the functional,

$$J[y] = \int_0^a (y'^2 - y^2)dx,$$

which passes through the points A(0,0) and B(a,0)?

Solution: Let $F = y'^2 - y^2$. Then $F_y = -2y$, $F_{yy} = -2$, $F_{y'} = 2y'$, $F_{y'y'} = 2$, $F_{yy'} = 0$. Hence,

$$P = (1/2)F_{y'y'} = 1,\ Q = (1/2)(F_{yy} - \frac{d}{dx}F_{yy'}) = -1,$$

and

$$\int_0^a (Ph'^2 + Qh^2)dx = \int_0^a (h'^2 - h^2)dx.$$

Now the Jacobi equation is defined by

$$-\frac{d}{dx}(Ph') + Qh = 0$$

and in this case is

$$-\frac{d}{dx}h' - h = 0, \quad \text{or} \quad h'' + h = 0.$$

Hence,

$$h = C_1 \sin(x - C_2).$$

Since $h(0) = 0$, it follows that $C_2 = 0$, and $h = C_1 \sin(x)$. The function h vanishes at the points $x = k\pi$, where k runs through all the integers. Therefore, if $0 < a < \pi$, there is only one point $x = 0$ in the interval $0 \leqq x \leqq a$ at which the function h vanishes, and so the Jacobi condition holds. If $a \geq \pi$, then there is at least one point more, $x = \pi$, in the interval $0 \leqq x \leqq a$, at which the function h vanishes, so that the Jacobi condition does not hold.

The necessary conditions for an extremum discussed in this paper are now summarized: If the functional

$$\int_a^b F(x,y,y')dx, \ y(a) = A, \ y(b) = B$$

has a weak extremum for the curve $y = y(x)$, then

1. The curve $y = y(x)$ is an extremal, i.e., satisfies Euler's equation

$$F_y - \frac{d}{dx} F_{y'} = 0,$$

2. Along the curve $y = y(x)$, $F_{y'y'} \geq 0$ for a minimum and $F_{y'y'} \leqq 0$ for a maximum,

3. The interval (a,b) contains no points conjugate to a.

Sufficient Conditions for a Weak Extremum

Finally, a set of conditions which is sufficient for a functional of the form

$$J[y] = \int_a^b F(x,y,y')dx, \ y(a) = A, \ y(b) = B \tag{40}$$

to have a weak extremum for the curve $y = y(x)$ is formulated. It will be noted that the sufficient conditions given below are very similar to the necessary conditions given at the end of the preceding section. The essential difference is that each necessary condition considered above is necessary by itself whereas the sufficient conditions have to be considered as a set, i.e., the existence of an extremum is assured only if all the conditions are satisifed simultaneously.

Theorem 4-13 (SUFFICIENT CONDITIONS).

1. The curve $y = y(x)$ is an extremal, i.e., satisfies Euler's equation

$$F_y - \frac{d}{dx} F_{y'} = 0,$$

2. Along the curve $y = y(x)$,

$$P(x) = (1/2)F_{y'y'}[x,y(x),y(x)] > 0$$

(the strengthened Legendre condition),

3. The interval $[a,b]$ contains no points conjugate to the point a (the strengthened Jacobi condition).

Then the functional (40) has a weak minimum for $y = y(x)$, [19].

The proof of this theorem, given in the same reference, establishes sufficient conditions for a weak extremum in the case of the simplest variational problem.

Illustrative Example 4-14. Examine the extrema of the functional

$$J[y] = \int_0^2 (xy' + y'^2)dx, \; y(0) = 1, \; y(2) = 0.$$

Solution:

Here $F = xy' + y'^2$ which implies that the Euler equation is

$$-\frac{d}{dx} F_{y'}(x,y') = 0.$$

This has integral curves $F_{y'}(x,y') = C$, i.e.,

$$x + 2y' = C.$$

The solution of this equation is

$$y = -(1/4)x^2 \quad Cx + C_1,$$

a family of parabolas. From the boundary conditions, $C = 0$ and $C_1 = 1$. Hence

$$y(x) = (1/4)x^2 + 1.$$

Also, $F_{y'y'} = 2$. Therefore, along the curve $y(x) = (1/4)x^2 + 1$,

$$P(x) = (1/2)F_{y'y'}[x,y(x),y'(x)] = (1/2)(2) > 0.$$

Hence, Legendre (strengthened) condition is satisfied.

Finally, the following second-order partial derivatives are needed to determine the functions P and Q:

$$F_{y'y'} = 2, \; F_{yy} = 0, \quad \text{and} \quad F_{yy'} = 0.$$

This implies that

$$P(x) = (1/2)F_{y'y'}, \; Q(x) = (1/2)(F_{yy} - \frac{d}{dx}F_{y\,y}),$$

$$P(x) = 1 > 0, \qquad Q(x) = 0.$$

Therefore,

$$\int_0^2 P(x)h'^2 dx = \int_0^2 h'^2 dx.$$

Now, it needs to be shown that $\int_0^2 h'^2 dx > 0$.

Suppose that $\int_0^2 h'^2(x)dx = 0$. Then $h'(x) = 0$ implying $h(x) = c$ (a constant). The condition that $h(0) = h(2) = 0$ and the fact that h is continuous and constant imply that $h(\;) = 0$ for all x in $[0,2]$.

Hence,

$$\int_0^2 h'^2 dx > 0$$

which implies

$$\int_0^2 (Ph'^2 + Qh^2)dx$$

is positive definite. Therefore, [0,2] contains no points conjugate to 0, by Theorem 4-9.

Consequently, J[y] has a weak minimum for $y(x) = (1/4)x^2 + 1$, by Theorem 4-13.

Constrained Variational Problems.

Subsidiary Conditions

In concluding this chapter, a brief look is taken at two variational problems with subsidiary conditions and then basic techniques for handling such problems are stated.

In the simplest variational problem, the class of admissible curves was specified by conditions imposed on the endpoints of the curves. However, many applications of the calculus of variations lead to problems in which not only boundary conditions, but also conditions of quite a different type known as subsidiary conditions (side conditions or constraints) are imposed on the admissible curves.

In Chapter I, a brief history of the isoperimetric problem is given. Basically, the problem is one of finding the geometric figure with maximal area and given perimeter. The next example includes a technical statement of the isoperimetric problem.

Illustrative Example 4-16 (Isoperimetric Problem). Find the curve $y = y(x)$ for which the functional

$$J[y] = \int_a^b F(x,y,y')dx \tag{41}$$

has an extremum, where the admissible curves satisfy the boundary conditions

$$y(a) = A, \ y(b) = B,$$

and are such that another functional

$$K[y] = \int_a^b G(x,y,y')dx \tag{42}$$

takes on a fixed value L.

To solve this problem, it is assumed that the functions F, G defining the functionals (41) and (42) have continuous first and second partial derivatives in $[a,b]$ for arbitrary values of y and y'. From these assumptions, the following result can be established:

Theorem 4-14 [19]. Given the functional

$$J[y] = \int_a^b F(x,y,y')dx,$$

let the admissible curves satisfy conditions

$$y(a) = A, \ y(b) = B, \ K[y] = \int_a^b G(x,y,y')dx = \ell \tag{43}$$

where $K[y]$ is another functional, and let $J[y]$ have an extremum for $y = y(x)$. Then, if $y = y(x)$ is not an extremal of $K[y]$, there exists a constant λ such that $y = y(x)$ is an extremal of the functional

$$\int_a^b (F + \lambda G)dx,$$

i.e., $y = y(x)$ satisfies the differential equation

$$F_y - \frac{d}{dx} F_{y'} + \lambda(G_y - \frac{d}{dx} G_{y'}) = 0. \qquad (44)$$

The reader can easily recognize the analogy between the theorem and the familiar method of Lagrange Multipliers for finding extrema of functions of several variables, subject to subsidiary conditions. [See last part of Chapter III.]

To use Theorem 4-14 to solve a given isoperimetric problem, first the general solution of (44) which will contain two arbitrary constants in addition to the parameter λ is written. Then these arbitrary constants and parameter are determined from the boundary conditions $y(a) = A$, $y(b) = B$ and the subsidiary condition $K[y] = L$.

The results of Theorem 4-14 can be immediately extended to functionals depending on several functions $y_1, \ldots, y_n$ and subject to several subsidiary conditions of the form (42). For instance, suppose an extremum of the functional below is sought, i.e.,

$$J[y_1,\ldots,y_n] = \int_a^b F(x,y_1,\ldots,y_n,y_1',\ldots,y_n')dx, \qquad (45)$$

subject to the conditions

$$y_i(a) = A_i,\ y_i(b) = B_i \qquad (i = 1,\ldots,n) \qquad (46)$$

and

$$\int_a^b G_j(x,y_1,\ldots,y_n,y_1,\ldots,y_n)dx = \ell_j \qquad (j = 1,\ldots,k), \qquad (47)$$

where $k < n$. In this case a necessary condition for an extremum is that

$$\frac{\partial}{\partial y_i}(F + \sum_{j=1}^{k}\lambda_j G_j) - \frac{d}{dx}[\frac{\partial}{\partial y_i}(F + \sum_{j=1}^{k}\lambda_j G_j)] = 0 \qquad (48)$$

$$(i = 1,\dots,n)$$

The 2n arbitrary constants appearing in the solution of the system (48), and the values of the k parameters $\lambda_1,\dots,\lambda_k$, sometimes called Lagrange multipliers, are determined from the boundary conditions (46) and the subsidiary conditions (47).

In the isoperimetric problem, the subsidiary conditions which must be satisfied by the functions $y_1,\dots,y_n$ are specified by functionals. At this point, a different type of problem is stated as follows: Find the functions $y_i(x)$ for which the functional (45) has an extremum, where the admissible functions satisfy the boundary conditions

$$y_i(a) = A_i,\ y_i(b) = B_i \qquad (i = 1,\dots,n)$$

and k "finite" subsidiary conditions $(k < n)$

$$g_j(x,y_1,\dots,y_n) = 0 \qquad (j = 1,\dots,k). \tag{49}$$

In other words, the functional (45) is not considered for all curves satisfying the boundary conditions (46), but only for those which lie in the $(n - k)$-dimensional manifold defined by the system (49).

In the case where $n = 2$ and $k = 1$, the following theorem is stated:

<u>Theorem</u> <u>4-15</u> [19]. Given the functional

$$J[y,z] = \int_a^b F(x,y,z,y',z')dx, \tag{50}$$

let the admissible curves lie on the surface

$$g(x,y,z) = 0 \tag{51}$$

and satisfy the boundary conditions

$$y(a) = A_1,\ y(b) = B_1$$

$$z(a) = A_2, \ z(b) = B_2 \tag{52}$$

and moreover, let $J[y]$ have an extremum for the curve

$$y = y(x), \ z = z(x). \tag{53}$$

Then, if g_y and g_z do not vanish simultaneously at any point of the surface (51), there exists a function $\lambda(x)$ such that (53) is an extremal of the functional

$$\int_a^b [F + \lambda(x)g]dx,$$

i.e., satisfies the differential equations

$$F_y + g_y - \frac{d}{dx} F_{y'} = 0 \tag{54}$$

$$F_z + g_z - \frac{d}{dx} F_{z'} = 0$$

To use this theorem to solve a given variational problem of the type (50), first the general solution of the system (54), which will contain several arbitrary constants and a Lagrange multiplier, is obtained. These quantities can be determined by making use of the boundary conditions and the subsidiary condition.

Illustrative Example 4-17. Let P be the collection of all non-self-intersecting plane arcs in the upper half plane for which the total length has the given value L and whose endpoints lie on the x-axis. Among these arcs, find the one for which the area enclosed by it and the x-axis is a maximum [19].

Solution: The area enclosed by any member of P and the x-axis is given by

$$A[y] = \int_{-a}^{a} y dx, \tag{55}$$

and the total length, equal to the fixed value L for every member of P, is given by

$$J[y] = \int_{-a}^{a} \sqrt{(1 + y'^2)}dx. \tag{56}$$

The equation of the particular arc for which (55) is a maximum with respect to arcs $y = y(x)$ whose left-hand endpoints coincide at $(-a,0)$, whose right-hand endpoints coincide at $(a,0)$ and which give to (56) the prescribed value L is sought.

The function with Lagrange's multipliers is written as follows:

$$F^* = F + \lambda G = y + \lambda\sqrt{(1 + y'^2)}$$

of which the corresponding Euler equation is obtained from

$$F_y - \frac{d}{dx} F_{y'} + \lambda(G_y - \frac{d}{dx} G_{y'}) = 0, \quad \text{[Theorem 4-14]},$$

that is,

$$1 - \lambda \frac{d}{dx} [y/\sqrt{(1 + y'^2)}] = 0.$$

By direct integration, the following result is obtained:

$$\frac{\lambda y'}{\sqrt{(1 + y'^2)}} = x - C_1.$$

From this it follows that

$$dy = [\pm(x - C_1)dx/\sqrt{(\lambda^2 - (x - C_1)^2)}]$$

and therefore that

$$y = \mp \ (\lambda^2 - (x - C_1)^2) + C_2$$

or

$$(x - C_1)^2 + (y - C_2)^2 = \lambda^2,$$

a family of circles with center (C_1,C_2) and radius λ. The constants

C_1, C_2, λ are determined from the boundary conditions $y(-a) = 0$, $y(a) = 0$ and the subsidiary condition $J[y] = L$.

The solution of the system

$$(-a - C_1)^2 + C_2^2 = \lambda^2$$

$$(a - C_1)^2 + C_2^2 = \lambda^2$$

$$\pi\lambda = L$$

consists of $C_1 = 0$, $C_2 = -\ (L^2 - a^2\pi^2)/\pi$, and $\lambda = L/\pi$.

<u>Illustrative Example 4-18.</u> Find the shortest distance between two points $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$ lying on the surface of a sphere $x^2 + y^2 + z^2 = a^2$.

<u>Solution</u>: The length of the curve $y = y(x)$, $z = z(x)$ is given by the functional

$$J[y,z] = \int_{x_0}^{x_1} \sqrt{(1 + y'^2 + z'^2)}\,dx.$$

Using theorem 4-15, the auxiliary functional is

$$J^*[y,z] = \int_{x_0}^{x_1} [\sqrt{(1 + y'^2 + z'^2)} + \lambda(x)(x^2 + y^2 + z^2 - a^2)]dx,$$

and the corresponding Euler equations are

$$2y\lambda(x) - \frac{d}{dx}[y'/\sqrt{(1 + y'^2 + z'^2)}] = 0$$

$$2z\lambda(x) - \frac{d}{dx}[z'/\sqrt{(1 + y'^2 + z'^2)}] = 0.$$

The solutions of these equations would be a family of curves depending on four constants, whose values would be determined by the boundary conditions:

$$y(x_0) = y_0, \qquad y(x_1) = y_1$$

$$z(x_0) = z_0, \qquad z(x_1) = z_1.$$

An Economic Model

This chapter is concluded with the discussion of an economic model which is related to the theory of the calculus of variations. The problem is concerned with determining the most profitable schedule of production of a commodity over a given period of time so as to meet certain requirements. Although the solution suggested by this model is a very profitable one, its realization involves what may appear to be a practical inconvenience (rate of output changing everyday). However, the objective here is to show how the calculus of variations can be applied to a problem in economics. Some new concepts are introduced and explained in the solution of the problem. A statement of the problem with solution now follows:

An Inventory Problem [23]. Find the most profitable schedule of production of a commodity over a given period of time $[0,T]$ so as to meet the following requirements:

a. The initial inventory, h_0, is given.

b. The sales function, $S(t)$ being the cumulative sales from 0 to t, is any piece-wise continuous function.

c. Inventory is nonnegative.

d. The terminal inventory is 0.

e. The cost of production is given: let $f(x)$ be the cost, per unit time, of producing x units of product per unit time, $f(x)/x$ is then the average cost and $f'(x)$ is the marginal cost.

f. Marginal cost is increasing $(f'' > 0)$.

g. The cost of storage is α per unit of product and per unit time.

The production schedule $X(t)$, which is unknown, is defined as

$X(t) = h_0$ + cumulative output up to time t.

This definition insures that X is continuous. However, it is not assumed here that the rate of production X is continuous.

<u>Solution:</u> The function $X(t)$ is subject to the following restrictions:

1. $X(0) = h_0$,
2. $X(T) = S(T)$,
3. $X(t) \geq S(t)$, for any $0 \leq t \leq T$.
4. $X(t)$ is continuous and nondecreasing and has a piecewise continuous derivative.

The cost of storage per unit time at time t is $\alpha(X - S)$, and the cost of production per unit time t is $f(X')$.

Therefore, the total cost for the period is

$$C[X] = \int_0^T [\alpha(X - S) + f(X')]dt.$$

It is required to find the function X which minimizes the functional above and is subject to conditions 1-4.

Step 1. Let $F(t,X,X') = \alpha(X - S) + f(X')$. Then $F_X = \alpha$, $F_{X'} = f'(X')$, $\frac{d}{dt} F_X = f''(X')X''$ which implies

$$F_X - \frac{d}{dt} F_{X'} = -X''f''(X') = 0, \quad \text{or} \quad X''f''(X') = \alpha.$$

The general solution of this differential equation gives a family of curves from which the desired one can be selected by means of the boundary conditions, i.e., $X = X(t)$.

To decide if the strengthened Legendre condition is satisfied, one has to find $F_{X'X'}$ and show that

$$P(t) = (1/2)F_{X'X'}[t,X(t),X'(t)] > 0.$$

But $F_{X'X'} = f''(X')$ which implies $P(t) = (1/2)f''(X') > 0$ by requirement f in the problem. Therefore, the strengthened Legendre condition is satisfied by Theorem 4-7.

Lastly, it has to be shown that the closed interval $[0,T]$ contains no points conjugate to 0, i.e., if the strengthened Jacobi's condition is satisfied. But $F_{XX} = 0$, $F_{XX'} = 0$ which implies $Q = (1/2)[F_{XX} - F_{XX'}] = 0$. Therefore,

$$Ph'^2 + Qh^2 = (1/2)f''(X')h'^2$$

and

$$\int_0^T (Ph'^2 + Qh^2)dt = (1/2)\int_0^T f''(X')h^2 dt > 0$$

since $f''(X') > 0$ and

$$\int_0^T f''(X')h'^2 dt = 0 \text{ if and only if } h' \equiv 0.$$

The reasoning here is the same as that used in Illustrative Example 4-14. Consequently, $[0,T]$ contains no points conjugate to the point 0, by Theorem 4-9.

Hence, Step 1 implies that the functional, $C[X]$, has a weak minimum for $X = X(t)$, by Theorem 4-13.

Step 2. At any corner point (t_0,X_0) [that is, a point at which the left-hand derivative is not equal to the right-hand derivative] of the solution curve, the following relation must hold between the two slopes X'_{-0} and X'_{+0} [25]:

$$F(t_0,X_0,X'_{+0}) - F(t_0,X_0,X'_{-0}) = (X'_{+0} - X'_{-0})F_{X'}(t_0,X_0,X'_{-0}).$$

In this problem the corner condition is satisfied as follows:

$$f(X'_{+0}) - f(X'_{-0}) = (X'_{+0} - X'_{-0})f'(X'_{-0})$$

where X'_{-0} and X'_{+0} designate the derivatives at the left and at the right of X_0, i.e., rates of production before and after t_0. This condition expresses that there is no advantage in changing the position of this boundary point [24].

Step 3. If the minimizing curve has an arc in common with a boundary $S(t)$ (above which it is bound to lie), then the arc of the boundary must satisfy [26]:

$$F_X(t,S,S') - \frac{d}{dt} F_{X'}(t,S,S') \geq 0.$$

In this problem there is a boundary solution if $S''f''(S') \geq \alpha$.

The general equation of the extremals contains two arbitrary constants since it is the solution of a differential equation of the second order. A particular extremal has, therefore, to be determined by two conditions.

Two examples are now given.

First, let $\alpha = 0$ (i.e., no storage cost). Then $X'' = 0$ and the solutions are formed of straight lines.

Secondly, let $\alpha \neq 0$, $f(X') = aX'^2 + bX' + c$ (marginal cost = $2aX' + b$ is a linear function of output). Then the equation of the extremals becomes

$$2aX'' = \alpha,$$

whose solution is:

$$X - \bar{X} = \frac{\alpha}{4a}(t - \bar{t})^2$$

with two arbitrary constants.

In conclusion, the optimal rates of production for inventory, given demand during a period of length T, are determined as a function of (continuous) time.

CHAPTER V

SUMMARY AND EDUCATIONAL IMPLICATIONS

In this paper the theory of extrema of differentiable functions is discussed and illustrated. This presentation clarifies the theory, making it more readable and more readily available to the beginning student of analysis. It also provides the necessary background material required to understand the procedures used in testing functions and functionals for extrema.

Summary

In Chapter I the nature and significance of the problem, the need for the study, and the scope and limitations of the paper are given. Chapter II includes the development of the theory of extrema of functions of one variable and applications to problems in business, economics, geometry and the physical sciences. In Chapter III the theory of extrema of functions of several variables is developed and illustrated. Also, applications to number theory, geometry, business, and economics are included. Chapter IV includes a presentation and discussion of some of the elementary concepts of the calculus of variations with illustrations. Except for a few basic theorems, most of the theorems are stated without proof. This chapter is concluded with an economic model which is related to the calculus of variations.

Educational Implications

A great deal of the theory of extrema of differentiable functions can be understood by the secondary school students and the beginning analysis students in college. It is important that some of these concepts be presented to these groups in a more systematic and rigorous manner. The usual textbook treatment of this subject is rather limited. For the most part, the student is left with the impression that he can always find the extrema of a function by taking the first derivative of the function and setting the same equal to zero. An analytic treatment of the subject such as this presents the necessary background needed for a thorough understanding of the problem.

As a result of reading this thesis, the student should gain a deeper insight into the theory of functions, as well as, deepen his understanding of the theory of extrema. It is also significant that the reader, who is a potential teacher of mathematics at the secondary or college level, may find motivation material for his class as well as material for the gifted student.

Certainly, the material discussed in this paper may serve as supplementary reading material and potential subject matter in the future undergraduate mathematics curriculum.

Undoubtedly the most important effect of this paper lies in the experience that the writer gained in its preparation.

A SELECTED BIBLIOGRAPHY

1. Courant and Robins, What is Mathematics?

2. Preston, Thomas, The Theory of Light, 5th Edition, London, 1928, pp. 102-103.

3. Kelvin, Lord, Popular Lectures and Essays, London, 1894, Vol. 2, pp. 571-572.

4. Newman, James R., The World of Mathematics, 1956.

5. Johnson and Kiokemeister, Calculus with Analytic Geometry, 3rd Edition, Allyn and Bacon, Inc., Boston, 1964.

6. Apostol, Tom M., Mathematical Analysis, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1957.

7. Walsh, J. L., A Rigorous Treatment of Maximum and Minimum Problems, D. C. Heath and Company, Boston, Massachusetts, 1962.

8. Widder, David V., Advanced Calculus, 2nd Edition, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1961.

9. Fleming, Wendell H., Functions of Several Variables, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1965.

10. Johnson, R. E. and F. L. Kiokemeister, Calculus, Allyn and Bacon, Inc., Boston, 1959.

11. Labarre, Jr., A. E., Elementary Mathematical Analysis, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1961.

12. Parker, William V. and James E. Eaves, Matrices, The Ronald Press Company, New York, 1960.

13. Perlis, S., Theory of Matrices, Reading, Massachusetts: Addison-Wesley, 1952.

14. Kaplan, Wilfred, Advanced Calculus, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1952.

15. Howell, James E. and Daniel Teichroew, Mathematical Analysis for Business Decisions, Richard D. Irwin, Inc., Homewood, Illinois, 1963.

16. Theodore, Chris A., Applied Mathematics: An Introduction, Richard D. Irwin, Inc., Homewood, Illinois, 1965.

17. Tintner, Gerhard, Mathematics and Statistics for Economists, Rinehart & Company, Inc., New York, 1953.

18. Ewing, George M., Calculus of Variations with Applications, University of Oklahoma, 1964.

19. Gelfand, I.M. and S. V. Fomin, Calculus of Variations, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963.

20. Elsgolc, L. E., Calculus of Variations, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962.

21. Browne, Edward T., Determinants and Matrices (The Theory of), The University of North Carolina Press, Chapel Hill, 1958.

22. Bernstein, S. N., Sur les equations du calcul des variations, Ann. Sci. Ecole Norm. Sup., 29, pp. 431-483, 1912.

23. Morin, Francois, "Note on an Inventory Problem," Econometrica, Vol. 23, No. 1 (January, 1955), pp. 447-450.

24. Lesourne, Jacques, Economic Analysis and Industrial Management, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, pp. 358-361.

25. Pars, L. A., An Introduction to the Calculus of Variations, John Wiley & Sons, Inc., New York, pp. 33-34.

26. Bolza, O., Lectures on the Calculus of Variations, reprinted by G. E. Stechert and Company, New York (1931), pp. 22, 35, and 43.

VITA

Marcus Harold Whitfield, Jr.

Candidate for the Degree of

Doctor of Education

Thesis: THEORY OF MAXIMA AND MINIMA AND APPLICATIONS

Major Field: Higher Education   Minor Field: Mathematics

Biographical:

Personal Data: Born in Jackson (Hinds County), Mississippi, May 8, 1928, the youngest of three children and the only son of the late Marcus Harold Whitfield, Sr., and Lula B. Whitfield Pennington.

Education: Attended Sally Reynolds Elementary School, Jim Hill Junior High School and Lanier Senior High School of Hinds County in the Jackson Municipal Separate School System. Graduated from high school in 1946; received the Bachelor of Arts Degree from Tougaloo College (Tougaloo, Mississippi) in Mathematics, Cum Laude, in 1950; received the Master of Arts for Teachers' Degree in Mathematics from Indiana University in 1957; attended Indiana University during the summer of 1958 for further study and the University of Illinois during the summer of 1962; completed requirements for the Doctor of Education Degree with an emphasis on Mathematics in July, 1967.

Professional Experience: Served in the United States Army as supply clerk and supply sergeant from 1950 to 1953; taught mathematics and served as school photographer at Utica Junior College (Utica, Mississippi) from 1954 to 1957; taught mathematics and sponsored the school yearbook at Jim Hill High School (Jackson, Mississippi) from 1957 to 1958; was assistant professor of mathematics at Grambling College (Grambling, Louisiana) from 1958 to 1960; was assistant professor of Mathematics at Tougaloo College (Tougaloo, Mississippi) from 1960 to 1963; was part-time instructor in Mathematics at Langston University (Langston, Oklahoma) from 1965 to 1967; has written several papers in religion.

Organizations: Member of the National Council of Teachers of Mathematics and the Mathematical Association of America.