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NORMAL TORI IN $\sharp_n(S^2 \times S^1)$ AND THE DEHN TWIST AUTOMORPHISMS OF THE FREE GROUP

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NORMAL TORI IN $\sharp_n(S^2 \times S^1)$ AND THE DEHN TWIST AUTOMORPHISMS OF THE FREE GROUP

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BY

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Abstract

The 3–dimensional space which renders a $\mathrm{Out}(F_n)$ action is $M = \sharp_n(S^2 \times S^1)$. The relation between M and $\mathrm{Out}(F_n)$ is that the latter is isomorphic to the mapping class group of M up to rotations about 2-spheres in M.

Associated to M is a rich algebraic structure coming from the essential 2–spheres that M contains. Inspired by this and combining the work of Whitehead with that of Laudenbach, Hatcher in the paper [15] defined the notion of normal form with respect to a fixed sphere system and proved the existence of normal representatives of spheres in a given isotopy class of spheres in M. This is a local notion of minimal intersection of a sphere system with respect to a maximal sphere system in M.

In this work, a notion of being *normal* for tori in $\sharp_n(S^2 \times S^1)$ is defined. This notion is crucial to determine minimality of intersections between tori and between spheres and tori. We prove two theorems regarding existence and uniqueness of normal representatives in a given homotopy class of tori. Then we define criteria for minimal intersection in a local sense and prove that a normal representative from a given homotopy class of tori satisfies it.

Just as there is a 1-1 correspondence between the equivalence classes of free splittings of the free group and the isotopy classes of embedded essential spheres in M, we prove that there is a 1-1 correspondence between the equivalence classes of $\mathbb{Z}$– splittings of F_n and homotopy classes of embedded essential tori in M. This gives us the

opportunity to understand *Dehn twist automorphisms* of the free group, since they are defined with respect to $\mathbb{Z}$– splittings. To this end, we define *Dehn twist along a torus* in M using the mapping classes of M and describe these twists with respect to their actions on the universal cover of M.

In addition, we give the motivation behind this work by stating possible applications and reasons for the importance of studying tori in this manifold.

Chapter 1

Introduction

1.1 Study on $\mathrm{Out}(F_n)$

The study of the group of outer automorphisms $\mathrm{Out}(F_n)$ of the free group F_n on n letters is closely related to the study of the spaces on which it acts. One such space is Culler-Vogtmann space, or "Outer Space". It was first introduced by Culler and Vogtmann in [8] and it is based on regarding F_n as the fundamental group of a graph.

Another such space is obtained from a 3–manifold, $M = \sharp_n(S^2 \times S^1)$, the connected sum of n copies of $S^2 \times S^1$. The fundamental group of M is also F_n and $\mathrm{Out}(F_n)$ acts on the sphere complex, which is a simplicial complex whose simplices correspond to systems of 2–spheres in M. Hatcher and Vogtmann used the sphere complex to prove a homological stability property of $\mathrm{Aut}(F_n)$ in [16].

Hatcher in the paper [15] defined the notion of normal form for a sphere with respect to a fixed sphere system and proved the existence of normal representatives of spheres in a given isotopy class of spheres in M. This leads to arguments about intersection numbers and minimal intersection conditions of these spheres and a correspondence between the free splittings of the free group F_n and the embedded spheres in M, as in [12].

In [12] Gadgil and Pandit stated and proved a geometric and algebraic intersection number argument for the sphere complex of M, which again provides information about the nature of the splittings of F_n and hence of the complex of free factors related to these splittings. A relation between the simplicial automorphism group of the graph of free splittings of F_n and $\mathrm{Out}(F_n)$ given by using sphere complex of M can be found in [2].

All this previous work on M suggest that, due to the 3–dimensional topological nature of this manifold, it is expected that algebraic problems concerning $\mathrm{Out}(F_n)$ could be understood as topological-geometric problems and then could be solved using 3–dimensional topology techniques. The main concern of this work is contributing to this translation in the following ways.

- We will study embedded essential tori in M. This gives us a geometric interpretation of intersections of certain non-free group splittings and one might hope that it could be extended to other group splittings. Moreover, following Hatcher, we will define a notion of a normal form for a torus in M, and prove that such a representative exists fairly uniquely in a given homotopy class. We then show that this representative is the geodesic representative analogue of a curve on a surface, it provides minimal intersection with the spheres in the maximal sphere system, which is a pants decomposition analogue.

- As a part of our goal of understanding $\mathrm{Out}(F_n)$ topologically, we will define Dehn twists along a torus in M and relate them to Dehn twist automorphisms of $\mathrm{Out}(F_n)$.

- (Conjecture on an analogue of Thurston's theorem)

Thurston in [32] says that, in a group generated by two Dehn twists about two filling curves on a closed surface S with genus $g \geq 2$, the groups generated by twists with powers greater than a finite number $N = N(S)$ is free of rank 2. Adapting this theorem to $\mathrm{Out}(F_n)$ to find rank 2 free groups, Clay and Pettet in [7] used an algebraic definition of a Dehn twist automorphism relative to a $\mathbb{Z}$–splitting of the free group and obtained a number N for the minimum power of twists, yet this number N depended on the choice of the twists.

To find a number N which is independent of the choice of Dehn twists, it is necessary to leave the 1–dimensional model for $\mathrm{Out}(F_n)$ since the dependence was due to the necessity for picking a basis of F_n in the proof. The conjecture is that by working on M and looking at Dehn twists along normal tori instead of Dehn twist automorphisms, similar theorem to Thurston's might be stated and in this case a uniform value for N could be achieved.

1.2 The Outline of the Thesis

- We begin in Chapter 2 with the different descriptions of the manifold $M = \sharp_n(S^2 \times S^1)$ and the definition of $\mathrm{Out}(F_n)$. We then define the elements of the mapping class group $\mathrm{MCG}(M)$ of M explicitly. This is the first step in understanding why M is a model for $\mathrm{Out}(F_n)$.

- In Chapter 3, we continue translating the algebraic concepts related to the free group into objects in M on which we can work with 3–dimensional topological techniques. To this end, we give the correspondences between the splittings of the free group and homotopy classes of two basic surfaces in M: non-trivial spheres and non-trivial tori.

- In Chapter 4, the notion of normal form for a torus in M is defined. Then we give there theorem which guarantees the existence of a normal representative in a given homotopy class of tori in M.

- Chapter 5 gives the combinatorial description of a lift of a torus by corresponding it to a graph called *the decorated graph*. Such description of the lifts using the spheres of a given maximal sphere system and the transverse orientation is a crucial ingredient of the proof of the uniqueness theorem in Chapter 6.

- In Chapter 6 the notion of being unique for normal representatives is given. Then it is shown that a normal representative uniquely exists in a given homotopy class of tori. This chapter also contains the corollay of exictence and uniqueness theorems, which states that a normal representative is the one which intersects the spheres, in particular the spheres in any given maximal sphere system in M minimally.

- In Chapter 7, we investigate tori in M which are intersecting each other non-trivially. We define Dehn twist along a torus in M and describe the effect of a twist on intersection circles for different types of intersections between tori in M. We also look at the pictures in the universal cover of M. We focus on a particular type of intersection, which corresponds to a pair of hyperbolic-hyperbolic $\mathbb{Z}$– splittings of F_n. Dehn twist along a torus is used to interpret Dehn twist automorphisms of F_n and hence the groups generated by two such Dehn twist automorphisms. The conjecture here is that the group generated by two Dehn twist automorphisms given by a hyperbolic-hyperbolic pair of $\mathbb{Z}$–splittings of F_n gives a free group, after we apply the twist a uniform number of times.

Chapter 2

$\mathrm{Out}(F_n)$ and the Mapping Class Group of $M = \sharp_n(S^2 \times S^1)$

Let M be connected sum of n copies of $S^2 \times S^1$, denoted by $\sharp_n(S^2 \times S^1)$. In this chapter, we will give different descriptions of M and introduce its mapping class group $\mathrm{MCG}(M)$ and mapping classes. After this, we will give the connection between $\mathrm{MCG}(M)$ and $\mathrm{Out}(F_n)$.

2.1 Different Descriptions of M

One way to describe M is to remove the interiors of $2n$ disjoint 3-balls $B_1^+, B_1^-, \ldots, B_n^+, B_n^-$ from S^3 and identify the resulting 2-sphere boundary components in pairs by orientation-reversing diffeomorphisms. See Figure 2.1.

One can define M using the definition of connected sum as follows: We remove n 3–balls $B_1, B_2, \ldots, B_n$ from S^3, and one 3-ball from each of n corresponding $S^2 \times S^1$. Then we attach one $S^2 \times S^1 - \mathrm{int}(B)$ in place of each removed 3–ball B from M along the remaining 2–sphere boundaries with an orientation reversing homeomorphism.

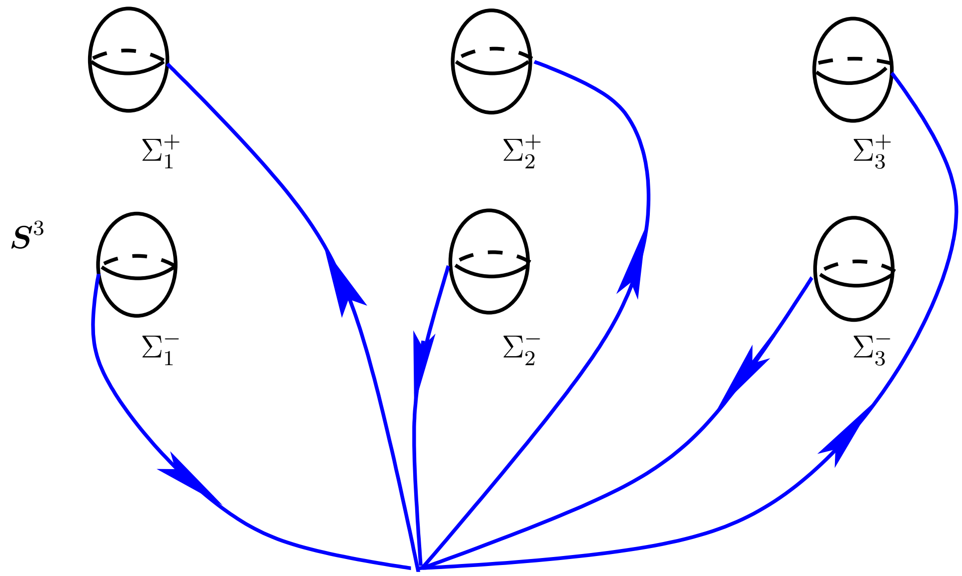

Figure 2.1: A schematic picture of $\sharp_3(S^2 \times S^1)$ with a generating set of F_3.

A *handlebody* is an orientable 3–manifold obtained from the 3–ball D^3 by attaching n copies of 1-handles, $D^2 \times [-1, 1]$. The gluing homeomorphisms match the $2n$ disks $D^2 \times \{-1, +1\}$ with $2n$ disjoint 2–disks in $\partial D^3 = S^2$ so that the resulting manifold is orientable. We will follow this definition of a handlebody to give another way of describing M as follows:

We take two genus n handlebodies and glue them by an identity homeomorphism along their boundary surfaces. For each nontrivial 2–disk D^2 on each of the handlebodies, we will call each regular neighborhood $D^2 \times I$ of D^2 a *1–handle*. After gluing the boundary surfaces, each $D^2 \times \{t\}$, $t \in I$ will be glued to an identical copy of itself, resulting in $S^2 \times I$ in the glued manifold. Since the disks are nontrivial, S^2 is a nontrivial 2–sphere in the resulting manifold. For each regular neighborhood of a nontrivial non-separating disk $D^2 \times \{t\}$ in each handlebody, we obtain a $S^2 \times S^1$ after gluing copies of $D^2 \times \{0\}$ and $D^2 \times \{1\}$. Hence M is a double handlebody where the gluing map is the identity homeomorphism.

2.2 The Free Group and its Group of Automorphisms

Definition 2.1. Let A be a subset of a group F. Then F is a *free group with basis* A if the following holds: If ϕ is any function from the set A into a group G, then there is a unique extension of ϕ to a homomorphism ϕ^* from F into G. Uniqueness of the extension is equivalent to requiring that A generate F. The cardinality of this generating set is the rank of the free group and the elements of the generating set are sometimes called *letters*.

Let F_n denote the free group with rank n. Every group G on n generators is a homomorphic image of F_n and as such the theory of free groups precedes the general theory of groups with generators and defining relators.

We denote by $\mathrm{Aut}(F_n)$ the group of automorphisms of the free group F_n.

Definition 2.2. Let F_n be a free group with basis $X = \{x_1, \ldots, x_n\}$. For any $x_i \in X$, let n_i be the automorphism satisfying $n(x_i) = x_i^{-1}$ and leaving other elements of X unchanged. For any x_i, x_j, $i \neq j$ let n_{ij} be the automorphism such that $n_{ij}(x_j) = x_i x_j$ and leaving other elements of X unchanged. The automorphisms n_i and n_{ij} are called *Nielsen automorphisms.*

The significance of these automorphisms is the following:

Theorem 2.3 ([21], [5])**.** *The group* $\mathrm{Aut}(F_n)$ *is generated by the set of all Nielsen automorphisms* n_i *and* n_{ij}.

Definition 2.4. Let G be a group. For any $g \in G$, we define the *inner automorphism* i_g of G by $i_g(x) = gxg^{-1}$ for each $x \in G$. The set $\{i_g : g \in G\}$ is a subgroup of $\mathrm{Aut}(G)$. It is called *subgroup of inner automorphisms of* G and denoted by $\mathrm{Inn}(G)$. Clearly $\mathrm{Inn}(G)$ is normal subgroup of $\mathrm{Aut}(G)$.

The group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$ is called the *outer automorphism group of* G.

2.3 The Mapping Class Group of M and $\mathrm{Out}(F_n)$

For the manifold $M = \sharp_n(S^2 \times S^1)$, the *Mapping Class Group of* M, denoted $\mathrm{MCG}(M)$ is defined by,

$$\mathrm{MCG}(M) = \mathrm{Homeo}^+(M)/\,\mathrm{Homeo}_0(M)$$

where $\mathrm{Homeo}^+(M)$ is the group of orientation preserving self homeomorphisms of M and $\mathrm{Homeo}_0(M)$ is the group of homeomorphism which are homotopic to the identity.

The action of $\mathrm{MCG}(M)$ on the fundamental group $\pi_1(M) = F_n$ of M gives a homomorphism $\mathrm{MCG}(M) \to \mathrm{Out}(F_n)$, sending a mapping class to its induced automorphism on the fundamental group. This is a map which gives a motivating connection between these two algebraic objects. Via this connection, automorphisms of the fundamental group can be described geometrically, in other words, in terms of isotopy classes of self homeomorphisms of the 3-manifold M.

The goal of this section is defining the mapping classes in $\mathrm{MCG}(M)$ and the kernel of the homomorphism $\mathrm{MCG}(M) \to \mathrm{Out}(F_n)$.

The Mapping Class Group $\mathrm{MCG}(M)$ of M is generated by mapping classes which are isotopy classes of three main types of self homeomorphisms of M: Rotations (twists) along 2-spheres, slide homeomorphisms and spin homeomorphisms.

Rotations about 2-spheres:

We take a product neighborhood $S^2 \times I$ of a non-trivial embedded sphere S^2 in M. Using the sphere $S = S^2 \times \{0\}$ one can define a homeomorphism as follows: Let $\tau : I \to SO(3,\mathbb{R})$ be a loop based at the identity rotation which generates $\pi_1(SO(3,\mathbb{R})) \cong \mathbb{Z}_2$. Define $\mathfrak{r} : M \to M$ by $\mathfrak{r}(x,t) = (\tau_t(x), t)$ for $(x,t) \in S^2 \times I$ and $\mathfrak{r}(m) = m$ for $m \notin S^2 \times I$. Since product neighborhoods are unique up to isotopy, the mapping class of this rotation is well defined. τ has order 2 in $\pi_1(SO(3,\mathbb{R}))$, $\mathfrak{r}^2$ is isotopic to the identity.

We have the following Lemma:

Lemma 2.5 ([25]). *Let $\mathfrak{r}_S$ be a rotation about the 2–sphere $S \subseteq M$ and let T be an embedded nontrivial 2–sphere in the interior of M. Then there is a product of rotations $\mathfrak{r}$ about 2–spheres disjoint from T so that $\langle \mathfrak{r}_S \rangle = \langle \mathfrak{r} \rangle$ in* $\mathrm{MCG}(M)$.

Let Σ' be the result of removing from M interiors of $2n$ disjoint 3–balls $B_1^+, B_1^-, \ldots, B_n^+, B_n^-$. We will denote the remaining boundary spheres by Σ_j^+, Σ_j^- where $j \in \{1, \ldots, n\}$. Let us also denote by Σ_j the 2–sphere obtained from gluing Σ_j^+ and Σ_j^-. Then M can be constructed from Σ' and n copies of $\Sigma_j \times I$ by identifying each $\Sigma_j \times \{0\}$ with Σ_j^+ and each $\Sigma_j \times \{1\}$ with Σ_j^-.

Define $\mathfrak{R}(M)$ to be the subgroup of $\mathrm{MCG}(M)$ generated by rotations about embedded 2–spheres in M.

Lemma 2.6 ([25]). *$\mathfrak{R}(M)$ is generated by rotations about the 2–spheres $\Sigma_j, j \in \{1, \ldots, n\}$. It is a normal subgroup of $\mathrm{MCG}(M)$ isomorphic to $\oplus_n(\mathbb{Z}/2)$.*

Proof. If g is any homeomorphism of M and $\mathfrak{r}$ is a rotation about a 2–sphere S, then $g\mathfrak{r}g^{-1}$ is a rotation about the 2–sphere $g(S)$. Therefore, $\mathfrak{R}(M)$ is a normal subgroup of $\mathrm{MCG}(M)$. To prove the first part of the lemma, let us take T equal to the union of the images of Σ_j in M. Now, Lemma 2.5 shows that $\mathfrak{R}(M)$ is generated by rotation along spheres disjoint from T. Since $M - T$ is the interior of a punctured 3-cell, it has no non-trivial 2–spheres. Hence, a rotation about any 2–sphere disjoint from T is isotopic to a product of rotations about some subset of the 2–spheres in T. Therefore the rotations about the 2–spheres in T generate $\mathfrak{R}(M)$. Since these rotations commute, and each has order at most 2, we have proved the lemma. $\square$

Slide Homeomorphisms: Let M_j' be the result of replacing $\Sigma_j \times I$ by the balls B_j^+ and B_j^- for some fixed j. Let γ be an arc properly embedded in $M - \text{int}(\Sigma_j \times I)$, both of whose endpoints lie in the boundary S of B, where B is one of B_j^+ or B_j^-. Choose an isotopy J_t of M_j' satisfying:

1. $J_0 = 1_{M'}$,
2. J_1 is the identity on B,
3. There is a regular neighborhood of $B \cup \gamma$ such that each J_t is the identity outside this neighborhood,
4. The isotopy J_t moves B around γ, i.e, if e is the center of B, then the trace $J_t(e)$ is a loop representing the generator of the fundamental group of the regular neighborhood of $B \cup \gamma$ (which is a solid torus having infinite cyclic fundamental group) determined by the orientation of γ.

Define a homeomorphism h of M by taking J_1 on $M - (\Sigma \times I)$ and the identity on $\Sigma_j \times I$. We call h a *slide homeomorphism* which slides $\Sigma_j \times I$ around γ. Here, recall that B is one of B_j^+ and B_j^- for a fixed j while the isotopy fixes the other. If the isotopy moves B_j^+ (respectively B_j^-) around γ, we say h *slides the left end* (respectively, *the right end) of the j^{th} handle around γ.*

A change of the choice of γ in its homotopy class in $\pi_1(M_j', B)$ changes h by an isotopy and possibly a rotation about S. The possible rotation comes from rotating B around an axis as it is being moved along the loop. Consequently a choice of homotopy class of γ determines at most two isotopy classes of slide homeomorphism:

Lemma 2.7. *Isotopy class of a slide homeomorphism depends on the homotopy class of the sliding loop only.*

Proof. Let $\mathrm{Imb}(B, M')$ be the set of embeddings from B to M', where B is as defined before and $M' = M'_j$ for a fixed j. Since $\mathrm{Diff}(M') \to \mathrm{Imb}(B, M')$ is a fibration (a continuous mapping satisfying the homotopy lifting property with respect to any space), a loop in $\mathrm{Imb}(B, M')$ can be lifted to loops in $\mathrm{Diff}(M')$ starting at the identity. Hence, a slide homeomorphism can be defined using any loop, not just an embedded one.

Now consider two sliding loops, γ_0 and γ_1 in $\mathrm{Imb}(B, M')$. A homotopy between the loops extends γ_0 and γ_1 to a 1–parameter family γ_t of loops in $\mathrm{Imb}(B, M')$. Now, we have lifts Γ_0 and Γ_1 of the first and the last loops to $\mathrm{Diff}(M')$. Since $\mathrm{Diff}(M') \to \mathrm{Imb}(B, M')$ is a fibration, we can lift γ_t to a 1–parameter family of loops Γ_t agreeing with Γ_0 and Γ_1 at the endpoints. After removing B and regluing to get M, the restriction of $\Gamma_t(1)$ gives an isotopy between the two slide homeomorphisms. □

Spins:

Using a homeomorphism which interchanges B_j^+ and B_j^-, one constructs a homeomorphism which reverses the direction of an arc in M crossing $\Sigma_j \times I$ from Σ_j^+ to Σ_j^-. This is called a *spin of the* j^{th} *1-handle*.

The next result is a version of the well-known theorem for a compact orientable 3 manifold:

Proposition 2.8 ([25]). $\mathrm{MCG}(M)$ *is generated by the isotopy classes of rotations about the* 2*–spheres, slide homeomorphisms of the spheres* Σ_j*s and spins.*

The proof uses a result of M. Scharlemann and can be found in [25]. Next theorem is the core theorem of this section:

Theorem 2.9 ([25]). *The kernel of the homomorphism* $\mathrm{MCG}(M) \to \mathrm{Out}(F_n)$ *is generated by* $\mathfrak{R}(M)$.

To prove the theorem , we will use three results, the first of which are paraphrased from [19]:

Theorem 2.10 ([19]). *Let* $f : (M, x_0) \to (M, x_0)$ *be a map inducing the identity automorphism on* $\pi_1(M, x_0)$ *and having a local degree* $+1$ *at* x_0*. Then* f *is properly homotopic to a composite of rotations about* 2*–spheres.*

Lemma 2.11 ([20]). *Let* S *be an essential embedded sphere in a* 3*-manifold* M*. Let* $h : S^2 \times [0, 1] \to M$ *be a map such that*

1. $h(S^2 \times \{0, 1\}) \subset M - S$

2. $h \mid S^2 \times \{0\}$ *is an essential immersed image which is not homotopic to* S

Then, $h \mid S^2 \times \{0\}$ *and* $h \mid S^2 \times \{1\}$ *are homotopic in* $M - S$.

Lemma 2.12 ([25]). *Suppose that* $T_1, T_2, \ldots, T_n$ *is a collection of pairwise disjoint pairwise non-isotopic essential embedded* 2*–spheres in* M*. Let* h *be a homeomorphism of* M *such that* $h(T_i) = T_i$ *for* $1 \leq i \leq m$ *and* $h(T_i)$ *is homotopic to* T_i *for* $m + 1 \leq i \leq n$*. Then* h *is isotopic preserving* $T_1, \ldots, T_m$ *to a homeomorphism* h' *such that* $h'(T_j) = T_j$ *for* $1 \leq j \leq n$.

Proof. An extension of the Laudenbach Lemma 2.11 to collection of disjoint 2–spheres shows that $h(T_{m+1})$ is isotopic to T_{m+1} by a homotopy that avoids $\cup_{i=1}^{m} T_i$. Since by [20, Theorem III. 1.3] homotopic 2–spheres are isotopic, $h(T_{m+1})$ is isotopic to T_{m+1} in the complement of $\cup_{i=1}^{m} T_i$. Induction completes the proof. □

Proof of Theorem 2.9. Let $\langle h \rangle$ be an element of the kernel. Changing h by isotopy, we may assume that h preserves a basepoint of M and induces the identity automorphism on the fundamental group of M. Applying Theorem 2.10 and using Proposition 2.6 shows that after composition with a product of rotations about the spheres, we may assume that h is properly homotopic to the identity automorphism since rotations about the 2–spheres also induce the identity automorphism on $\pi_1(M)$. It follows that for any 2–sphere S embedded in M, the restriction of h to S is homotopic to the inclusion. Now we apply Lemma 2.12 to spheres $\Sigma_1 \times \{1/2\}, \ldots, \Sigma_n \times \{1/2\}$. Since h induces the identity automorphism, it cannot interchange the sides of any of these 2–spheres so h is isotopic to a map which is identity on Σ'. The rotations about 2–spheres can now be applied to make the homeomorphism equal to the identity. □

In our work, the main inspiration is the homomorphism $\mathrm{MCG}(M) \to \mathrm{Out}(F_n)$. It is very close to being an isomorphism, as its kernel is a finite elementary abelian 2–group. It is also surjective since the Nielsen automorphisms correspond to slide homeomorphisms and spins. From Lemma 2.6 and Theorem 2.9 we have the following exact sequence:

$$1 \to \mathfrak{R}(M) \to \mathrm{MCG}(M) \to \mathrm{Out}(F_n) \to 1$$

This connection with the mapping class group of a 3–manifold enables us to utilize a considerable body of three-dimensional techniques developed over many decades. Moreover, this exact sequence gives more opportunity to describe automorphisms geometrically since the kernel $\mathfrak{R}(M)$ is finite. This lets us to describe certain automorphisms, called *Dehn twist automorphisms*, geometrically. This is a significant difference between 3–dimensional and 2–dimensional settings, and a justification for working on a 3–manifold since Dehn twist automorphisms do not always correspond to Dehn twist mapping classes along curves on a surface. This will be investigated in detail in Chapter 7.

Chapter 3

The Structure of $M = \sharp_n(S^2 \times S^1)$ in Terms of the Splittings of the Free Group

In this chapter, we will give the correspondence between certain splittings of the free group and spheres and tori in M, which is the second step towards understanding $\mathrm{Out}(F_n)$ via M.

3.1 Spheres and Tori in M

We will be interested in *nontrivial* spheres and tori. To be precise on what we mean by nontrivial, we have the following definition:

Definition 3.1. An *essential sphere* in M is the one which does not bound a 3–cell in M.

A 3-manifold is said to be *irreducible* if all 2–spheres bound 3–balls. Hence, a 3–manifold is reducible when it contains essential spheres. The fact that M is reducible

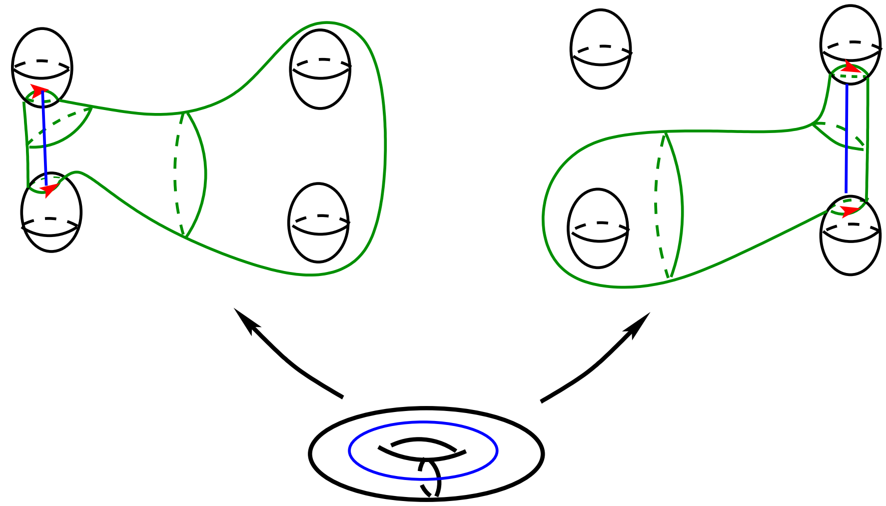

Figure 3.1: A separating (on the left) and a nonseparating (on the right) embedded torus in M.

can also be seen in Figure 3.2.

We have the following similar definition of *nontrivial* for a torus in M:

Definition 3.2. We will say that a torus α is *essential* in M if the image of $\pi_1(\alpha)$ under the homomorphism induced by the inclusion map $\iota : \alpha \to M$ is nontrivial in $\pi_1(M)$.

In Figure 3.1 we see two examples of essential embedded tori.

3.1.1 Spheres and Sphere Systems

A *sphere system* is a collection of isotopy classes of disjoint and essential 2–spheres in M no two of which are isotopic.

Throughout this work, we will call 3–punctured 3–spheres either 3–punctured spheres or twice-punctured 3–cells (balls). These are analogous to pairs of pants in dimension 2.

Recall that a *pair of pants* is a surface of genus zero with three boundary components. A *pants decomposition of a surface S* is a collection of disjointly embedded

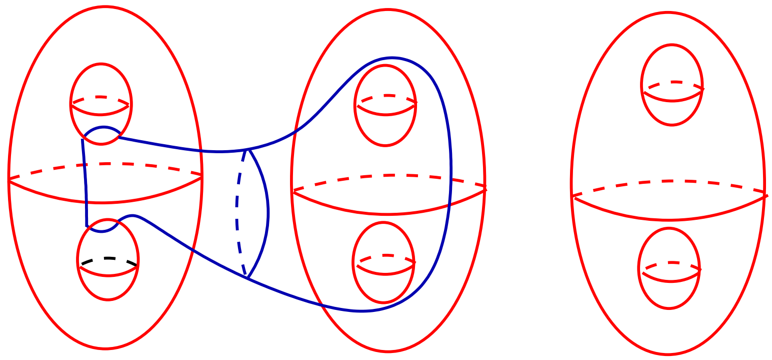

Figure 3.2: A maximal sphere system and a torus in $\sharp_3(S^2 \times S^1)$.

circles on S which decompose S into pairs of pants. This is also called a *maximal curve system*.

Using essential spheres in M, we can define 3–dimensional versions of pair of pants and of pants decompositions for M as follows:

Definition 3.3. We call a collection Σ of disjointly embedded essential, non-isotopic 2–spheres in M *a maximal sphere system* if every complementary component of Σ in M is a 3-punctured 3-sphere.

We see an example of a maximal sphere system in Figure 3.2. In this example $n = 3$ and there are 6 essential spheres in the maximal sphere system.

Hatcher in the paper [15] defined the notion of normal form with respect to a fixed sphere system and proved the existence of normal representatives of spheres in a given isotopy class of spheres in M. This leads to arguments about intersection numbers and minimal intersection conditions of these spheres and a correspondence between the free splittings of the free group F_n and the embedded spheres in M where $M = S^2 \times S^1$, as in [12].

3.1.2 The Sphere Complex Connecting Two Models for $\text{Out}(F_n)$

There is a simplicial complex associated to M called the *sphere complex* and denoted by $\mathbb{S}(M)$, having isotopy classes of non-trivial 2–spheres in M as vertices and sphere systems of $k+1$ spheres as k-dimensional simplices.

Definition 3.4. The *Outer Space* consists of the set of equivalence classes of triples $(G; f; \ell)$ where G is a graph, ℓ is a metric, and f a homotopy equivalence from the n rose to the graph, and $(G; f; \ell) \sim (G_0; f_0; \ell_0)$ if there is an isometry $\psi : (G; \ell) \to (G_0; \ell_0)$ so that $\psi \circ f$ is homotopic to f_0.

Outer space is a simplicial complex with some faces missing. The sphere complex has a subspace which is homeomorphic to Outer Space. This is the subspace which consists of sphere systems with all complementary components simply connected. This is the connection between two models for $\text{Out}(F_n)$.

3.2 Spheres and Tori As Splittings of $\pi_1(M)$

In this section, we will investigate the correspondences between two types of surfaces in M; tori and spheres, and we will give their algebraic interpretations with respect to the fundamental group of M.

3.2.1 G-graphs

In our work, a graph Γ is understood in the sense of Serre [27] : It consists of a set of vertices, $V(\Gamma)$, set of edges, $E(\Gamma)$, a function $\iota : E(\Gamma) \to V(\Gamma)$ which defines the terminal vertex τ by, $\tau(e) = \iota(\overline{e})$ where $e \to \overline{e}$ under the inversion function $- : E(\Gamma) \to E(\Gamma)$ and $e \neq \overline{e}$.

Now, a G–graph is a graph on which the group G acts (on the left) without inversions ($e \neq \overline{e}$). A *tree* can be described homologically: a graph Γ is a tree exactly when the following sequence is exact:

$$0 \to C_1(\Gamma) \to C_0(\Gamma) \to \mathbb{Z} \to 0$$

where $C_0(\Gamma)$ is the free abelian group on $V(\Gamma)$ and $C_1(\Gamma)$ is the free abelian group generated by $E(\Gamma)$.

3.2.2 Elementary Group Splittings

Definition 3.5. An *amalgamated free product* of two groups A and C amalgamated along a group B is the pushout of A and C when the maps $\alpha_1 \colon B \to A$ and $\alpha_2 \colon B \to C$ are group homomorphisms. It is denoted by $A *_B C$.

Similarly,

Definition 3.6. An amalgamated free product of A along B (or *HNN extension* of A along B) is defined to be the pushout of A where $\iota_{1,2} : B \to A$ are both homomorphisms. This universal group is denoted by $A*_B$.

By a $\mathbb{Z}$*–splitting* of a group G we mean an amalgamated free product or an HNN extension of the group G so that B is isomorphic to $\mathbb{Z}$.

Definition 3.7. A *free splitting* of a group G is the amalgamated free product of two groups A and C amalgamated along the trivial group.

The following theorem gives the structure of subgroups of free products. The similar result can be derived for amalgamated free products as well. The proof uses the theory of covering spaces.

Theorem 3.8 (Kurosh Subgroup Theorem [26]). *If H is a subgroup of $G = G_1 * G_2$, then H is the free product of a free group with subgroups of conjugates of G_1 or G_2.*

3.2.3 Relating Graphs to Groups and to Elementary Splittings

Definition 3.9. A *graph of groups* consists of an abstract graph Γ (which will always assumed to be connected), together with a function $\mathfrak{G}$ assigning to each vertex v of Γ a group G_v and to each edge e a group G_e, with $G_e = G_{\overline{e}}$ and a homomorphism $f_e : G_e \to G_{\partial_0(e)}$ where $\partial_0(e)$ is $\iota(e)$ restricted to the initial vertex of e with respect to the fixed orientation on it.

Similarly, we define a graph $\mathfrak{X}$ of topological spaces, or of spaces with preferred basepoint. Given a graph $\mathfrak{X}$ of spaces, we can define a total space X_Γ as the quotient of $\bigcup\{X_v : v \in V(\Gamma)\} \cup \bigcup\{X_e \times I : e \in E(\Gamma)\}$ by the identifications,

$$X_e \times I \to X_{\overline{e}} \times I \quad \text{by } (x,t) \mapsto (x, t-1)$$

$$X_e \times \{0\} \to X_{\partial_0 e} \quad \text{by } (x,0) \mapsto f_e(x)$$

If $\mathfrak{X}$ is a graph of (connected) base spaces, then by taking fundamental groups we obtain a graph $\mathfrak{G}$ of groups with the same underlying abstract graph Γ. The fundamental group G_Γ of the graph $\mathfrak{G}$ of groups is defined to be the fundametal group of the total space X_Γ.

Observe that in the cases when Γ has only one pair $(e, \overline{e})$ of edges we obtain products $A *_B C$ and $A *_B$, as follows by van Kampen's theorem.

3.2.4 Spheres in M and Free Splittings of $\pi_1(M)$

The main theorem of this section is the following theorem, which corresponds each splitting of the fundamental group of a closed 3–manifold N as a free product corresponds to a sphere in the manifold. We will apply this theorem in our manifold, M.

Theorem 3.10 (Kneser's conjecture on free products [17])**.** *Let N be a compact 3–manifold such that each component of ∂N (possibly empty) is incompressible in N. If $\pi_1(M) \cong G_1 * G_2$, then $N = N_1 \sharp N_2$ where $\pi_1(N_i) \cong G_i$, $i = 1, 2$.*

Before we give the proof of this theorem, we will need some definitions of concepts common in 3–dimensional topology and some necessary lemmata for the proof.

Definition 3.11. Let N be a 3-manifold and S a surface which is either properly embedded in N or contained in the boundary ∂N. We say that S is *compressible in N* if one of the following conditions is satisfied:

1. S is an inessential 2–sphere in N,

2. S is a 2–cell which is either subset of the boundary ∂N or $S \cup \partial M$ contains the boundary of a 3–cell.

   or,

3. There is a 2–cell $D \subset N$ with $D \cap S = \partial D$ with ∂D not contractible in S.

Otherwise, S is said to be *incompressible*.

We have two well-known results concerning incompressible surfaces in a 3–manifold, the first one is also known as The Loop Theorem:

Lemma 3.12 ([17])**.** *If S is a 2–sided incompressible surface in a 3–manifold N, then* $\ker(\pi_1(S) \to \pi_1(M)) = 1$.

Definition 3.13. For a 3–manifold N and a space X we say that two maps $f, g : N \to X$ are *C-equivalent* if there are maps $f = f_0, f_1, \ldots, f_n = g$ such that either f_i homotopic to f_{i-1} or f_i agreeing with f_{i-1} on the complement of a 3-cell. C-equivalent maps induce the same homomorphism $\pi_1(N) \to \pi_1(X)$ up to choices of base points and inner automorphisms.

The following Lemma shows a way to find incompressible surfaces in a compact 3–manifold:

Lemma 3.14 ([17])**.** *Suppose N is a compact 3–manifold, X a k-manifold which contains a 2-sided $k-1$ submanifold Y with $\ker(\pi_1(Y) \to \pi_1(X)) = 1$ and $\pi_2(Y) = \pi_2(X - Y) = 0$. If $f : N \to X$ is any map, then there is a map $g : M \to X$ which is C-equivalent to f and so that each component of $g^{-1}(Y)$ is a properly embedded, 2–sided incompressible surface in N.*

We continue with the proof of the Theorem 3.10. The proof is based on the topological proof of Grushko's theorem, given by Stallings([29]).

Proof of Theorem 3.10. Choose complexes X_1 and X_2 with $\pi_1(X_i) \cong G_i$ and $\pi_2(X_i) = 0$. Join a point of X_1 to a point of X_2 by a simplex A to form a complex $X = X_1 \cup A \cup X_2$. Here, $\pi_1(X) \cong G_1 * G_2$ and $\pi_2(X_i) = 0$. So we can construct a map $f : N \to X$ such that $f_* : \pi_1(N) \to \pi_1(X)$ is an isomorphism. Choose $x_0 \in A$. By lemma 3.14 we can assume that each component of $f^{-1}(x_0)$ is a 2–sided incompressible surface properly embedded in N. Then if F is a component of $f^{-1}(x_0)$, by 3.12, $\pi_1(F) = 1$. If some component F is an incompressible 2-cell, then by hypothesis ∂F bounds a 2-cell D in ∂N. Then the 2–sphere $F \cup D$ can be pushed to ∂N where it gives an incompressible 2–sphere F'. Since $\pi_2(X_i) = 0$, f can be modified by a C-equivalance which replaces F by F'. Hence, we may assume that all components of $f^{-1}(x_0)$ are incompressible 2–spheres in interior of N. Now, we have two cases: If $f^{-1}(x_0)$ is connected, we are done. If not, there is a path $\beta : I \to N$ such that $\beta(0)$ and $\beta(1)$ connects two different components of $f^{-1}(x_0)$. Now, $f \circ \beta$ is a loop in X and since f_* is surjective, there is a loop γ based at $\beta(1)$ so that $f_*(\gamma) = [f \circ \beta]^{-1}$. Then $\alpha = \beta\gamma$ is a path satisfying:

1. $\alpha(0)$ and $\alpha(1)$ are in different components of $f^{-1}(x_0)$,
2. $[f \circ \alpha]$ is trivial in $\pi_1(X)$.

We may assume that α is a simple path which crosses $f^{-1}(x_0)$ transversely at each point of $\alpha(\text{int}I)$. We assume also that every path satisfying conditions above is chosen so that number of components of $\alpha^{-1}(f^{-1}(x_0))$ is minimal.

Now we claim that $\alpha(\text{int}I) \cup f^{-1}(x_0) = \varnothing$. If not, we can write $\alpha = \alpha_1\alpha_2 \cdots \alpha_k$ $(k \geq 2)$ where for each i, $\alpha_i(\text{int}I) \cup f^{-1}(x_0) = \varnothing$ and $\alpha_i(\partial I) \subset f^{-1}(x_0)$. Then, $[f \circ \alpha_1][f \circ \alpha_2] \cdots [f \circ \alpha_k]$ is a representation of the identity element as an alternating product in the free product $G_1 * G_2$. This means that $[f \circ \alpha_i] = 1$ for some i. If $\alpha_i(0)$ and $\alpha_i(1)$ are contained in the same component, then we reduce the number of components of $\alpha^{-1}(f^{-1}(x_0))$. If not, then since above conditions satisfied, minimality assumption is contradicted. Thus we have, $\alpha(\text{int}I) \cup f^{-1}(x_0) = \varnothing$.

Let F_j, $j = 1, 2$ be the component of $f^{-1}(x_0)$ containing $\alpha(j)$. Let also C be a small regular neighborhood of $\alpha(I)$ such that $C \cap F_j = D_j$ is a spanning 2-cell of C and $C \cap f^{-1}(x_0) = D_0 \cup D_1$. Let B be the annulus in ∂C bounded by $\partial D_0 \cup \partial D_1$. Push the interior of B slightly into the interior of C to obtain an annulus B' which has the same boundary as B and $B \cup B'$ is a boundary of a solid torus.

We define a map $f_1 : N \to X$ as follows: Let $f \mid N$ and $f_1 \mid N$ coincide on the complement of the interior of C and let $f_1(B') = x_0$. Since $[f \circ \alpha] = 0$, we can extend f_1 across a meridonal 2-cell of the torus. Now, it remains to extend it to two remaining two cells. This is possible since $\pi_2(X) = 0$ and also it is possible to make this extension so that $C \cap f^{-1}(x_0) = B'$. So f_1 is C-equivalent to f and $f_1^{-1}(x_0) = f^{-1}(x_0) - ((D_0 \cup D_1) \cup B')$ has one less component than $f^{-1}(x_0)$.

The proof is completed by induction. $\square$

All the above classical theorems in 3–manifold topology will be applied to our manifold, M. We would like to note here that M has no incompressible surfaces but the 2–sphere.

3.2.5 Tori in M and $\mathbb{Z}$–splittings of $\pi_1(M)$

In this section, we will relate $\mathbb{Z}$–splittings to tori. The main theorem of this section is the following:

Theorem 3.15 ([6]). *There is a bijection between the set of homotopy classes of essential embedded tori in M and equivalence classes of $\mathbb{Z}$–splittings of the free group.*

Throughout this section, we will drop the rank of the free group.

To prove the theorem 3.15, we will need the connection between free splittings and the amalgamated free products. This will be established by the following theorems:

Theorem 3.16 ([30]). *Let $H = A *_B C$ where $B \neq \{1\}$. Let F be a free group and let $\phi : F \to H$ be a surjective homomorphism. Then F has a free factorization $F = F_1 * F_2$ such that one of the following symmetric alternatives holds:*

1. *$\phi(F_1) \subset A$ and $\phi(F_1) \cap B \neq \{1\}$ or*

2. *$\phi(F_1) \subset C$ and $\phi(F_1) \cap B \neq \{1\}$*

A similar theorem holds for HNN extensions:

Theorem 3.17 ([30]). *Let $H = A *_B \alpha$, thus H is generated by its subgroup A and an additional element t, such that $B = A \cap tAt^{-1}$, with relations saying that for all $b \in B$, $b = t\alpha(b)t^{-1}$. Suppose that $B \neq 1$. Let F be a free group and let $\phi : F \to H$ be a surjective homomorphism. Then F has a free factorization $F = F_1 * F_2$ such that one of the following symmetric alternatives holds:*

1. *$F_1 \subset A$ and $F_1 \cap B \neq \{1\}$ or*

2. *$F_1 \subset tAt^{-1}$ and $F_1 \cap B \neq \{1\}$*

Definition 3.18. A subgroup S of a free group F is called *unsplittable in F* if for every free factorization $F = F_1 * F_2$ if $S \cup F_1 \neq 1$ then $S \subset F_1$.

Proposition 3.19. *Every cyclic subgroup of a free group F is unsplittable in F.*

Theorem 3.20 (Shenitzer [28]). *Suppose that a free group F is an amalgamated free product, $F = A *_B C$ in which the amalgamated subgroup B is cyclic. Then B is a free factor of A or a free factor of C.*

Proof. By theorem 3.16, taking the identity map from F to itself, we know that one of the two symmetric alternatives is true, let us suppose that $F = F_1 * F_2$ and $F_1 \subset A$ and that $F_1 \cup B \neq 1$. Now we apply the Kurosh Subgroup Theorem to C as a subgroup of $F = F_1 * F_2$; this implies that C has a free factor of the form $C \cap F_1$. Since $B \subset C$, that free factor of C contains $B \cap F_1$, which is nontrivial. Since B is cyclic, it is unsplittable in C by proposition 3.19 and therefore $B \subset C \cap F_1 \subset C \cap A$. Since $C \cap A = B$, the free factor $C \cap F_1$ is in fact B. □

Theorem 3.21 (Swarup [31]). *Suppose that a free group F is an HNN extension $F = A *_B \alpha$ in which the amalgamated subgroup B is cyclic. We express A in terms of A and an extra generator t such that $B = A \cap tAt^{-1}$. Then A has a free product structure $A = A_1 * A_2$ in such a way that one of the following symmetric alternatives hold:*

1. $B \subset A_1$*, and there exists* $a \in A$ *such that* $t^{-1}Bt = a^{-1}A_2a$ *or*

2. $t^{-1}Bt \subset A_1$ *and there exists* $a \in A$ *such that* $B = a^{-1}A_2a$

Definition 3.22. Let α be a torus and $\widetilde{\alpha}$ the set of lifts of α to the universal cover $\widetilde{M}$. There is a simplicial tree T_α whose vertices correspond to the complementary components of $\widetilde{M} - \widetilde{\alpha}$ and such that two vertices are adjacent if the closures of their corresponding regions intersect.

Definition 3.23. Two $\mathbb{Z}$–splittings of the free group F are said to be *equivalent* if there is a F-equivariant bijection between the Bass-Serre trees corresponding to the splittings.

By van Kampen's theorem, any essential embedded torus in M gives rise to a splitting of $\pi_1(M)$ over $\mathbb{Z}$. This is as an amalgamated free product if the torus is separating and as an HNN-extension if the torus is non-separating. Hence, the simplicial tree given in the definition is the Bass-Serre tree corresponding to the splitting given by the torus α.

We have the following lemma regarding this simplicial tree, which will be crucial for our main proof.

Lemma 3.24. *Let* α *and* β *be two homotopic tori. Then,* $T_\alpha = T_\beta$*. Hence, homotopic tori correspond to equivalent splittings.*

To prove this lemma, we will work on the *ends* of $\widetilde{M}$. An end of a topological space is a point of the so called Freudenthal compactification of the space. Namely,

Definition 3.25. Let X be a topological space. For a compact set K, let $C(K)$ denote the set of components of complement $X - K$. For L compact with $K \subset L$, we have a natural map $C(L) \to C(K)$. These compact sets define a *directed system* under inclusion. Let the set of ends $E(X)$ be the inverse limit of the sets $C(K)$.

The space $\widetilde{M}$ is non-compact and it has infinitely many ends. We denote the set of ends of $\widetilde{M}$ by $E(\widetilde{M})$. It is homeomorphic to a Cantor set, in particular, it is compact. The set $E(\Gamma)$ of ends of Γ where Γ is the tree dual to $\widetilde{M}$ is identified with the set $E(\widetilde{M})$.

Proof of Lemma 3.23. . The endpoint compactification of $\widetilde{M}$ is actually the 3–sphere, S^3, in which the ends form a Cantor set. The action of F on $\widetilde{M}$ extends to a highly non-free action of F on S^3.

Assuming that α does not bound a solid torus, each lift $L = S^1 \times R$ of α defines a decomposition of the set of ends into two sets $X(L)$ and $Y(L)$ where $X(L) \cap Y(L)$ consists of two endpoints, corresponding to the axis of a conjugate of the generator of the image of $\pi_1(M)$. Let us once and for all work on the complement of the countable set of these two endpoints which are connected by the axes of lifts and denote by the partition of the remaining set of ends $X(L)$ and $Y(L)$, given by a lift L. Since the lifts are disjoint, any two satisfy either $X(L_1) \subset X(L_2)$ or $X(L_1) \subset Y(L_2)$ for two lifts L_1 and L_2. Hence, for each lift, we will have a partition corresponding to it.

For each set of partitions we take a vertex and for each collection in a partition such that whenever $X \subset Y$ there is no collection of ends Z such that $X \subset Z \subset Y$, we take another vertex. We then connect the vertices corresponding to partitions to the vertices corresponding to its sets by edges. Since the partitions of ends do not intersect, we have a tree.

Now, since for each lift we have a partition of the ends, there is a 1-1 correspondence between the tree given by the partitions and the tree T_α as defined above. Namely, the "edge-midpoint" vertices of T_α correspond to the elements of this set of partitions (i. e. the components of $\widetilde{\alpha}$). The components of $\widetilde{M} - \widetilde{\alpha}$, i. e. the other vertices of the T_α, correspond to the collections of lifts (topologically, the frontier components of these components), having the property that if L_1 and L_2 are two of them, then (assuming that we select the notation so that $X(L_1) \subset X(L_2)$) there is no L_3 in the collection for which $X(L_1) \subset X(L_3) \subset X(L_2)$ or $X(L_1) \subset Y(L_3) \subset X(L_2)$. These latter vertices correspond to the ones given by the collections in a given partition. Hence making each such collection of partitions a vertex connected by an edge to each of its elements defines the corresponding simplicial tree T_α.

For a homotopy of embedded tori in M, the initial and final tori determine the same partition of the ends, and hence the same tree, in both senses. To see this:

Let α be homotopic to β. To show that we get the same partitions of the endpoints from the lifts of α and from the lifts of β, we need to show that if two endpoints are separated by a component L of $\widetilde{\alpha}$, then they are separated by the corresponding component L' of $\widetilde{\beta}$ (i. e. the one that L moves to).

Let p and q be two endpoints separated by L. Fix an arc between them that crosses $L = S^1 \times R$ in one point. During the homotopy, that component, although no longer embedded, moves in $\widetilde{M}$, i.e. it does not touch any endpoint. So assuming that the homotopy is transverse to the arc, its inverse image in $S^1 \times R \times I$ consists of circles and arcs properly imbedded in $S^1 \times R \times I$ (note that if the homotopy could cross an endpoint of the arc, then an arc of the inverse image could fail to be properly imbedded in $S^1 \times R \times I$). Since only one endpoint of the inverse image is in the end L, there is an odd number of endpoints in L' (i. e. the arc crosses L' an odd number of times) and therefore L' still separates p and q.

□

Using the theorems giving connections between splittings of free group, we can now deduce a correspondence between the homotopy classes of essential embedded tori in M and the equivalence classes of $\mathbb{Z}$–splittings of F. We end this section with the proof of the theorem 3.15, giving the proof of this correspondence:

Proof of Theorem 3.15. The first direction is given by Lemma 3.24 and the remarks given before that. In the other direction, we make use of the theorems 3.20 and 3.21 that relate a splitting of F over the trivial group to a $\mathbb{Z}$–splitting of F and then we use Kneser's Conjecture 3.10. We treat the amalgamated product and HNN-extension cases separately.

Case 1: We first consider the case of an amalgamated free product $F = A *_{\langle c \rangle} B$. By Shenitzer's Theorem 3.20, after possibly interchanging $A \leftrightarrow B$, there is a free splitting $F = A * B_0$ where $B = \langle c, B_0 \rangle$. Let $S \subset M$ be an embedded (separating) sphere representing this splitting. We fix a basepoint $* \in M$ and assume it lies on S. As $c \in A$, there is an embedded loop $\gamma \subset M$ that represents $c \in F$ and only intersects S at $*$. For small ϵ, boundary of the closed ϵ–neighborhood of $S \cup \gamma$ consists of two components: an embedded sphere isotopic to S and an embedded essential torus τ_γ.

It is clear from the construction, that the splitting of F associated to τ_γ by van Kampen's Theorem is the original splitting. However, there are some choices made in the construction of τ_γ and it must be shown that different choices result in homotopic tori. It is clear that changing S or γ by a homotopy results in a change of τ_γ by a homotopy.

Now since Shenitzer's theorem 3.20 gives many possible splittings, we need to consider two different complementary free factors B_0 and B_1 of A such that $\langle c, B_0 \rangle = \langle c, B_1 \rangle = B$ and show that the tori obtained after we add the loop to corresponding spheres are homotopic, even when the spheres themselves are not. For this, let S_0 and S_1 be the spheres representing the splittings $A * B_0$ and $A * B_1$ respectively and τ_0 and τ_1 be the tori as constructed above using these spheres. We assume that γ intersects S_0 only at the fix basepoint $* \in M$.

We first treat the special case that B_1 is obtained from B_0 by replacing a generator $b \in B_0$ by bc. Fix a basis for F consisting of a basis for A and a basis for B_0 where b is one of the generators for B_0. This corresponds to a sphere system in M which decomposes as $\Sigma_A \cup \Sigma_{B_0}$; the sphere S_0 separates the two sets Σ_A and Σ_{B_0}. In terms of these sphere systems, we can describe a homeomorphism that takes S_0 to (a sphere isotopic to) S_1.

Denote by Σ_γ the ordered set of spheres (all in Σ_A) pierced by γ starting from the basepoint. Cut M open along the sphere β corresponding to the generator b and via a homotopy push the boundary sphere β^- filled in with a 3–ball through the spheres in Σ_γ in order, dragging S_0 along. After removing the 3–ball and regluing β^+ and β^-, the image of S_0 is S_1 and the sphere β now corresponds to bc. By shrinking β^- and S_0, we can assume that homotopy is the identity on τ_0 and γ. Thus, we have a homeomorphism taking S_0 to S_1, $S_0 \cup \gamma$ to $S_1 \cup \gamma$ and is the identity on τ_0. As a homeomorphism takes a regular neighborhood to a regular neighborhood, τ_0 is homotopic to τ_1.

A similar argument works if we replace b by bc^{-1}.

The general case now follows as we can transform B_0 to B_1 by a sequence of the above transformations plus changes of basis that do not affect the associated spheres. For a proof of this argument we refer to the proof of Theorem 5 in [**?**].

Finally, we need to consider the possibility that $F = A_0 * \langle c \rangle * B_0$ where $A = \langle A_0, c \rangle$ and $B = \langle B_0, c \rangle$. Let S_A and S_B be the spheres representing the splittings $A * B_0$ and $A_0 * B$ respectively and τ_A and τ_B be the tori as constructed above using these spheres. In this case as $M - (S_A \cup S_B)$ is $S^1 \times S^2$ with two balls removed, it is easy to see that τ_A and τ_B are homotopic. Indeed, model $S^1 \times S^2$ are the region between the spheres of radius 1 and 2 in $\mathbb{R}^3$ after identifying the boundary spheres. Remove a ball of radius $\frac{1}{4}$ at each of the points $(0, 0, 3/2)$ and $(0, 0, -3/2)$. Then clearly the torus obtained from the intersection with the xy–plane is homotopic to both τ_A and τ_B.

Case 2: We now consider the case of an HNN-extension $F = A*_{\langle c\rangle}$. By Swarup's Theorem 3.21, there is a free factorization $A = A_0 * \langle t^{-1}ct\rangle$ for some $t \in F$, such that A_0 is a co-rank 1 free factor of F and such that $c \in A_0$. Let $S \subset M$ be an embedded (non-separating) sphere representing the trivial splitting $F = A_0*_{\{1\}}$. We fix a basepoint $* \in M$ and assume it lies on S. As $c \in A_0$, there is an embedded loop $\gamma \subset M$ that represents $c \in F$ and only intersects S at $*$. Further, both ends of γ are on the same side of S. For small ϵ, boundary of the closed ϵ–neighborhood of $S \cup \gamma$ consists of two components: an embedded sphere isotopic to S and an embedded essential torus τ_γ.

As above, it is clear from the construction, that the splitting of F associated to τ_γ by van Kampen's Theorem is the original splitting. Again, we must show that the choices along the way do not matter.

The idea here is really the same as above, different choices in the free factorization of A are understandable and can be thought of as applying a sequence of moves that either do not change the relevant spheres, or else lead to homotopic tori as above.

□

Chapter 4

Definition and Existence of Normal Form for Tori in $M = \sharp_n(S^2 \times S^1)$

In this chapter, we will define the notion of a normal form for a torus in M and state and prove an existence theorem for such a representative in a homotopy class. This work is included in [13] and most of it is inspired from Hatcher's work on normal form for sphere systems in [15].

4.1 Definition of Normal Form for a Torus in M

Given an imbedded torus in M and a maximal sphere system Σ, we can look at the number of intersections of the torus with the 2–spheres in each P, and define a notion of minimal intersection. In this work we are particularly interested in the existence of a torus in a homotopy class which intersects the 2–spheres of a maximal sphere system Σ minimally. There are certain pieces of a given torus in a P that are particularly important for minimal intersection. They are:

1. A disk piece, which is essential, in other words not parallel into any of the bound-

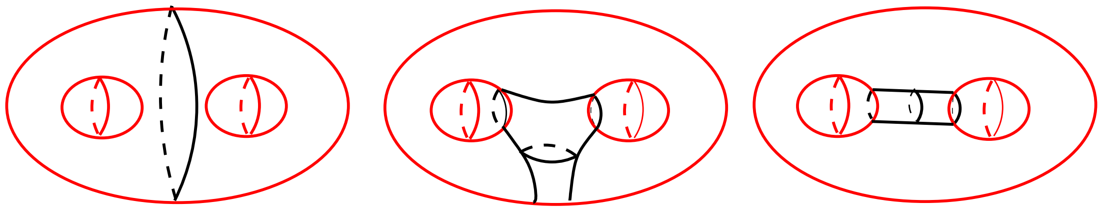

Figure 4.1: Disk, pants and cylinder pieces in M

ary 2–spheres and which has a single circle intersection with a single boundary sphere.

2. A cylinder piece, which is the topological boundary of a regular neighborhood of an arc connecting two different boundary 2–spheres.

3. A pants piece, which is the topological boundary of a regular neighborhood of a letter Y intersecting all three boundary components.

These pieces can be seen in Figure 4.1.

Definition 4.1. Given an embedded torus and a maximal sphere system Σ in M, we say that the torus is in *normal form with respect to* Σ if each intersection of the torus with each complementary 3-punctured 3-sphere is a disk, a cylinder or a pants piece (Figure 4.1).

4.2 Existence of Normal Form for Tori

We will show with the next theorem that any homotopy class of essential tori has a normal representative:

Theorem 4.2. *Every embedded essential torus in M is homotopic to a normal torus and the homotopy process does not increase the intersection number with any sphere of Σ.*

Our existence theorem and its proof is based on work of Hatcher in [15] and the following theorem:

Theorem 4.3 ([15])**.** *Given a maximal sphere system Σ, Every sphere system S can be isotoped into normal form with respect to Σ.*

Hatcher's work combines the works of Whitehead and Laudenbach, and the latter result is a core theorem for understanding spheres in M, hence we will state here:

Theorem 4.4 ([20])**.** *Let S and S' be two 2–spheres. Then, under the same conditions, if S and S' are homotopic, then they are isotopic.*

For tori, this latter Laudenbach result is not necessarily true and consequently we restrict ourselves to the homotopy classes instead of isotopy classes. Yet, our proof is similar to the proof of the first theorem above.

Proof of Theorem 4.2. Let us pick a representative α from a homotopy class of tori.

As the first step, in each P, we regard each piece of the torus as consisting of sphere pieces inside P and possibly concentric tubes connecting these sphere pieces to the boundary spheres of P. To do this, we first surger each piece of the torus along the intersection circles on the boundaries, starting from the innermost one, ending with the outermost one, resulting in a 2–sphere in P. On these sphere pieces, we reverse this surgery process by putting tubes between the sphere piece and the boundary spheres, in exactly the reverse order.

If α is not normal, there will be a piece F that meets a boundary sphere S of a thrice punctured sphere P in two intersection circles C_1 and C_2. Choose an arc ρ in F connecting C_1 to C_2. Let ρ' be an arc on $S \in \Sigma$ connecting C_1 to C_2. Reselecting ρ and F if necessary, we may assume that interior of ρ' does not meet α.

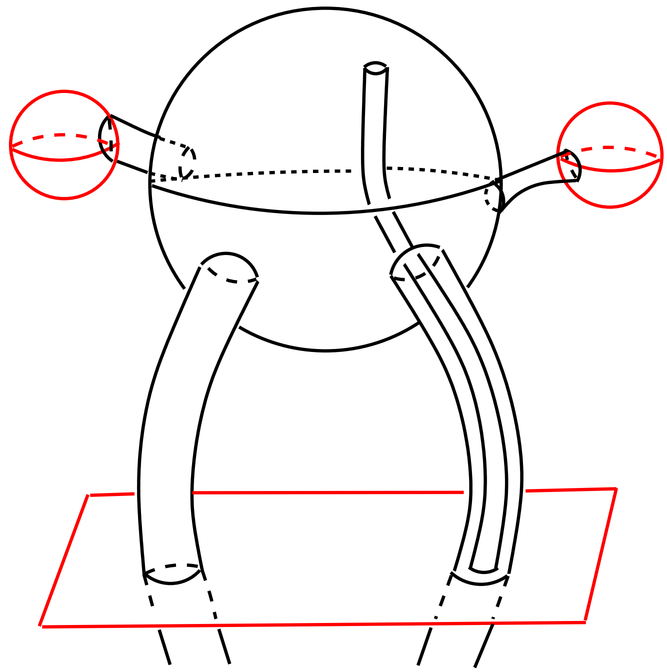

Figure 4.2: Realizing the pieces in 3–punctured 3–spheres

Let us call the tube portions of F which meet C_1 and C_2 T_1 and T_2, respectively. Let us assume that C_2 was surgered before C_1. Now, there is a homotopy of α which is an isotopy on F and whose effect is to slide the end of T_1 attached to the sphere part of F along ρ to T_2 and finally out of P. Any tubes of α inside of T_1 are slid along with it. If there are r such tubes, at a certain point of the homotopy, they create $2r + 1$ new intersection circles with S, $2r$ from the tubes inside T_1 and one from the intersection circle of the tube T_1 itself. (Here, we will redraw the picture in **??** as Figure 4.2).

The intersection of T_1 with P is now a cylinder. Since ρ is homotopic to ρ', this cylinder along with any tubes inside it are homotopic to an embedded position outside of P near ρ'. During the homotopy, self intersections of α may occur, but since the interior of ρ' does not meet α, the final position of α can be an embedding. This homotopy eliminates $2r + 2$ circles of intersection, giving a net decrease of 1 from the original position (Figure 4.2).

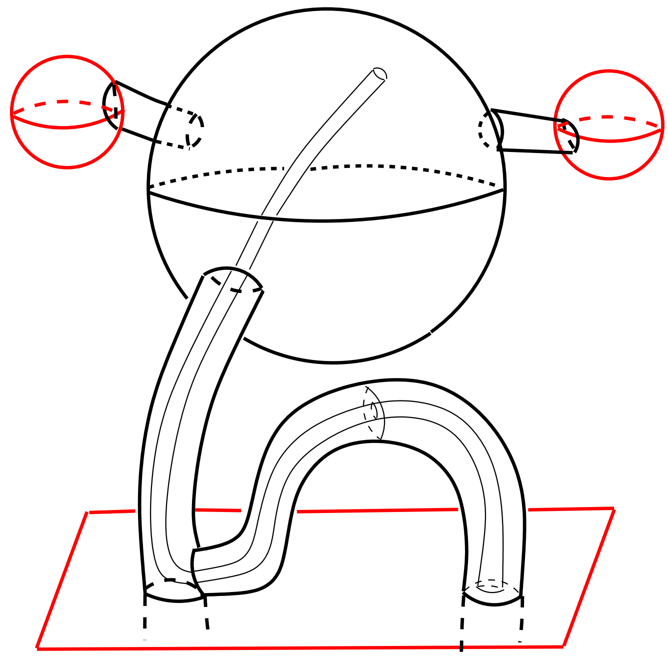

Figure 4.3: The homotopy eliminating intersection circles

A sequence of such homotopies in each P will give the desired homotopy in M. Since α is essential, its image under a composition of such homotopies will not be disjoint from Σ hence we will have a normal representative in the same homotopy class. $\square$

Chapter 5

Combinatorial Description of Lifts of Tori

In this chapter, we will describe each lift of a torus in M combinatorially in the universal cover $\widetilde{M}$. For this, we will start with describing $\widetilde{M}$ with respect to a maximal sphere system and model it with a tree, called *the dual tree*. After this, we will give the definition of the tree corresponding to a lift, and then "decorate" this tree with labels coming from the transverse orientation on the spheres in $\widetilde{M}$. A quotient of this latter tree with labels corresponding to a torus lift is called *the decorated graph*. Decorated graph will be crucial in Chapter 6 proving the "uniqueness" of normal representative.

5.1 The Universal Cover

To relate a torus to a tree, we will need to work on the universal cover of M.

A fixed maximal sphere system in M gives a description of the universal cover $\widetilde{M}$ of M as follows. Let $\mathbb{P}$ be the set of twice punctured 3-balls in M given by a maximal sphere system Σ and regard M as obtained from copies of P in $\mathbb{P}$ by identifying pairs

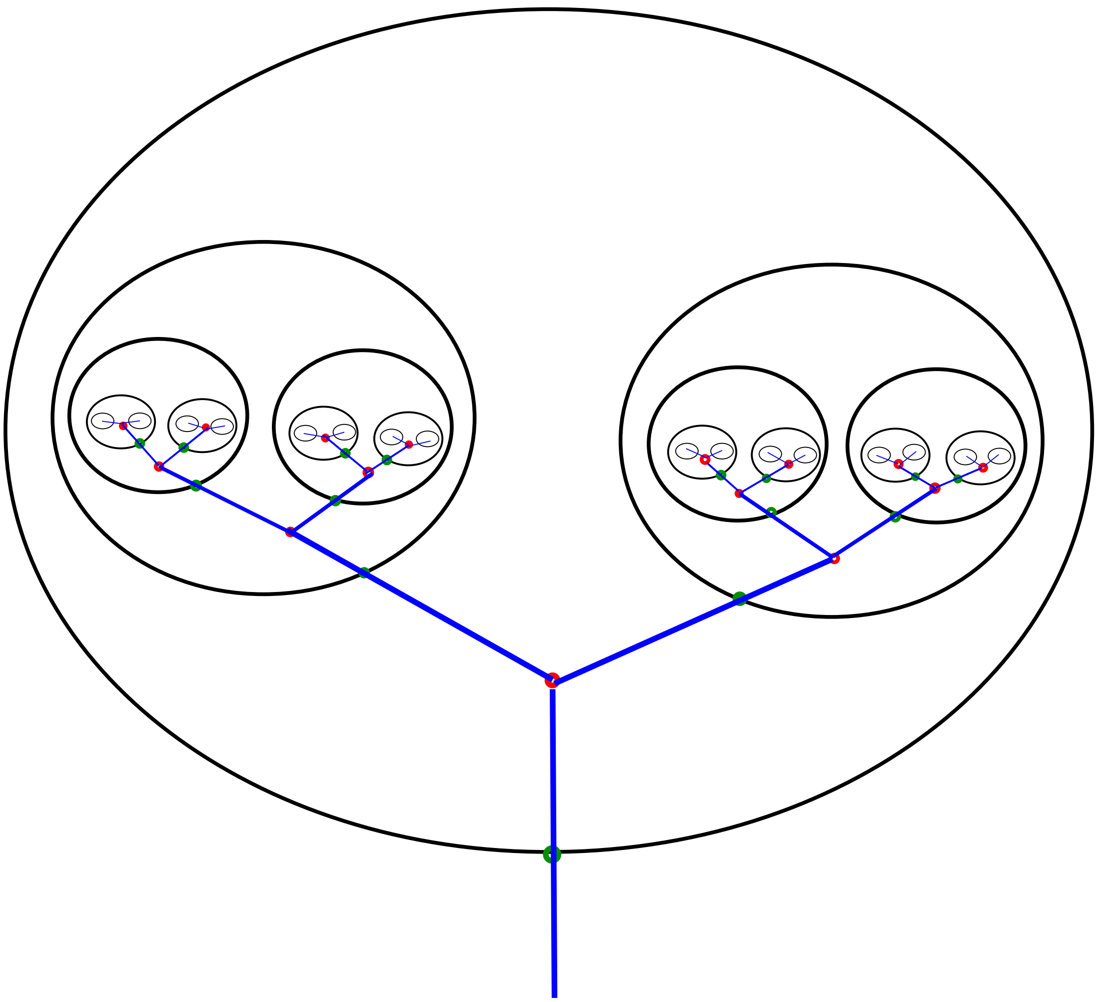

Figure 5.1: The universal cover $\widetilde{M}$ and the dual tree $\widetilde{\Gamma}$.

of boundary spheres. Note that a pair might both be contained in a single P, in which case the image of P in M is a once-punctured $S^2 \times S^1$. To construct $\widetilde{M}$, begin with a single copy of P and attach copies of the P in $\mathbb{P}$ inductively along boundary spheres, as determined by unique path lifting. Repeating this process gives a description of $\widetilde{M}$ as a treelike union of copies of the P. We remark that $\widetilde{M}$ is homeomorphic to the complement of a Cantor set in S^3.

The universal cover $\widetilde{M}$ is modeled by a tree $\widetilde{\Gamma}$, called the *dual tree,* as follows. For each copy P in $\mathbb{P}$ there is a vertex corresponding to the interior of P and a vertex for each of the three boundary spheres, and there are three edges connecting the interior vertex to the boundary sphere vertices. Hence there are two types of vertices: the valence-3 vertices indicating the 3-punctured spheres and valence-2 vertices indicating boundary spheres. To obtain $\widetilde{\Gamma}$, identify the boundary vertices according to how the corresponding sphere boundary components of the copies of the P are identified to form $\widetilde{M}$.

We will call the union of the three edges for a copy of a P a "Y", since it is homeomorphic to a letter Y. We also write $\widetilde{\Sigma}$ for the union in $\widetilde{M}$ of the inverse images of the spheres in the fixed sphere system.

Given a lift $\widetilde{\alpha}$ of an imbedded torus in normal form, there is a corresponding dual subgraph of $\widetilde{\Gamma}$ obtained by taking the union of the Y's for the copies of the P that meet $\widetilde{\alpha}$. We call this graph $T(\widetilde{\alpha})$. The inclusion of $\widetilde{\alpha}$ into $\widetilde{M}$ is modeled by the inclusion of $T(\widetilde{\alpha})$ into $\widetilde{\Gamma}$, which is injective, hence we will have at most one component of $\widetilde{\alpha}$ in each P. Note that

1. An extremal Y of $T(\widetilde{\alpha})$, that is, a Y that meets the rest of $T(\widetilde{\alpha})$ in a single vertex, occurs exactly when an intersection of $\widetilde{\alpha}$ with a copy of a P is a disk. We will call such Y's type-1.

2. A Y meeting the rest of $T(\widetilde{\alpha})$ in exactly two vertices occurs exactly when an intersection of $\widetilde{\alpha}$ with a copy of a P is a cylinder. These Y's will be called type-2.

3. A Y meeting the rest of $T(\widetilde{\alpha})$ in its three boundary vertices occurs exactly when an intersection of $\widetilde{\alpha}$ with a copy of a P is a pair of pants. These are type-3.

Since $\widetilde{\alpha}$ is connected, $T(\widetilde{\alpha})$ is also connected and hence is a tree. See Figure 5.1 for a picture of universal cover and the dual tree.

5.2 The Decorated Graph

To prove the uniqueness theorem in the next chapter, we will provide a combinatorial description of the lift of a torus in terms of a tree defined in the universal cover of M. Such a lift equipped with a transverse orientation will be associated to a *decorated graph*.

Let α be an embedded essential torus in M. The image of $\pi_1(\alpha)$ under the homomorphism induced by the inclusion $i\colon \alpha \to M$ is an infinite cyclic subgroup of $\pi_1(M)$, defined up to conjugacy. Fixing a specific lift of the inclusion to the universal cover, with image $\widetilde{\alpha}$, determines a specific subgroup in this conjugacy class, and a generator γ of this subgroup acts as a covering transformation of $\widetilde{M}$ that preserves $\widetilde{\alpha}$. Note that γ does not interchange the sides of $\widetilde{\alpha}$ since the image of α is two-sided in M. There is a corresponding action of $\pi_1(M)$ on $\widetilde{\Gamma}$ as simplicial isomorphisms. The generator γ has an invariant axis which is topologically a line and $T(\widetilde{\alpha})$ consists of this axis and finite trees meeting the axis. The action of γ on $T(\widetilde{\alpha})$ takes vertices to vertices and edges to edges. A fundamental domain for the action of γ on $T(\widetilde{\alpha})$ could be described

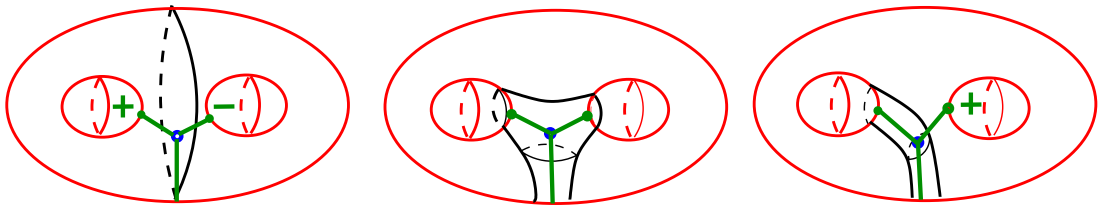

Figure 5.2: Labeling of the vertices induced from the labels on the spheres

as an arc on the invariant axis of γ whose endpoints are translates by γ, together with some finite trees attached to this arc. Translates of these finite trees are all of the finite trees meeting the invariant axis of γ.

Motivated by Hatcher [15], we will obtain the decorated graph with respect to the transverse orientation chosen as follows:

We pick a transverse orientation of the lift $\widetilde{\alpha}$ and label the sides; one with $+$ and the other $-$. This induces a corresponding orientation on $\widetilde{\alpha}/\gamma$. Split $\widetilde{M}/\gamma$ along $(\widetilde{\alpha}/\gamma) \cup (\bigcup \widetilde{S}_i)$ where $\widetilde{S}_i$ are the spheres corresponding to the γ-orbits of the spheres in $\widetilde{M}$ which are disjoint from $\widetilde{\alpha}$. Now, let $\widetilde{X_+}$ and $\widetilde{X_-}$ be the two components that contain copies of $\widetilde{\alpha}/\gamma$ and define $S_+ = \partial\widetilde{X_+} - \widetilde{\alpha}/\gamma$ and $S_- = \partial\widetilde{X_-} - \widetilde{\alpha}/\gamma$ where $\partial\widetilde{X_+}$ denotes the boundary of $\widetilde{X_+}$, etc. Note that $\widetilde{X_+}$,$\widetilde{X_-}$ and $\widetilde{\alpha}/\gamma$ are compact submanifolds. See Figure 5.3.

We label the spheres S_+ with $+$ and the spheres S_- with $-$. This gives a labeling of the vertices representing these spheres, which are extremal vertices of $T(\widetilde{\alpha})/\gamma$. For a disk piece of $\widetilde{\alpha}/\gamma$, the corresponding two extremal vertices of $T(\widetilde{\alpha})/\gamma$ will have the opposite signs. There will be no signs on a Y corresponding to a pants piece, since all spheres are intersected. For a cylinder piece, one of the boundary spheres will not be intersected hence will be on one side of the torus and will correspond to a labeled extremal vertex on a type-2 Y. This is illustrated in Figure 5.2.

In particular if we have a torus with one of S_+ and S_- empty, the corresponding

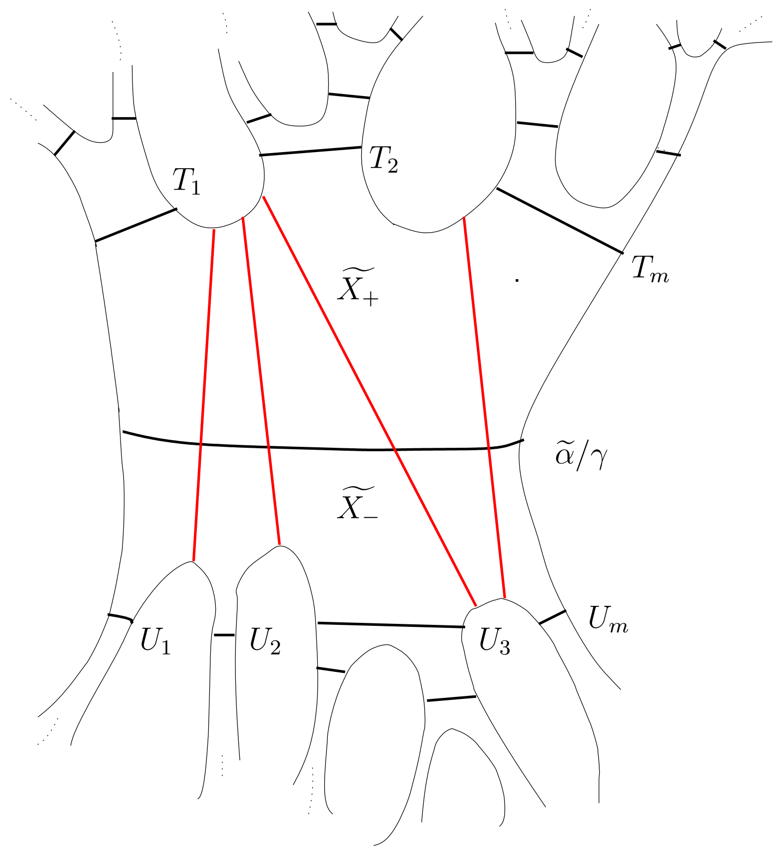

Figure 5.3: A schematic picture of $\widetilde{\alpha}/\gamma$, the submanifolds $\widetilde{X_+}$, $\widetilde{X_-}$ and spheres $T_i \in S_+$ and $U_j \in S_-$.

graph will be some union of type-2 Y's, in other words finitely many extremal edges attached to the axis of γ with one label on each of them, all labels the same. In this case, α bounds a solid torus in M, $\widetilde{\alpha}/\gamma$ bounds a solid torus $\widetilde{X_+}$ or $\widetilde{X_-}$ in $\widetilde{M}/\gamma$, and $\widetilde{\alpha}/\gamma$ represents the trivial element in $H_2(\widetilde{M}/\gamma)$.

The above construction will give the graph $T(\widetilde{\alpha})/\gamma$ in $\widetilde{\Gamma}/\gamma$ a "decoration" of signs on the ending vertices resulting from the transverse orientation on the torus $\widetilde{\alpha}/\gamma$. We will call this decorated graph g_α since it will be shown that it is uniquely determined by the normal homotopy class of the normal torus α. See Figure 5.4.

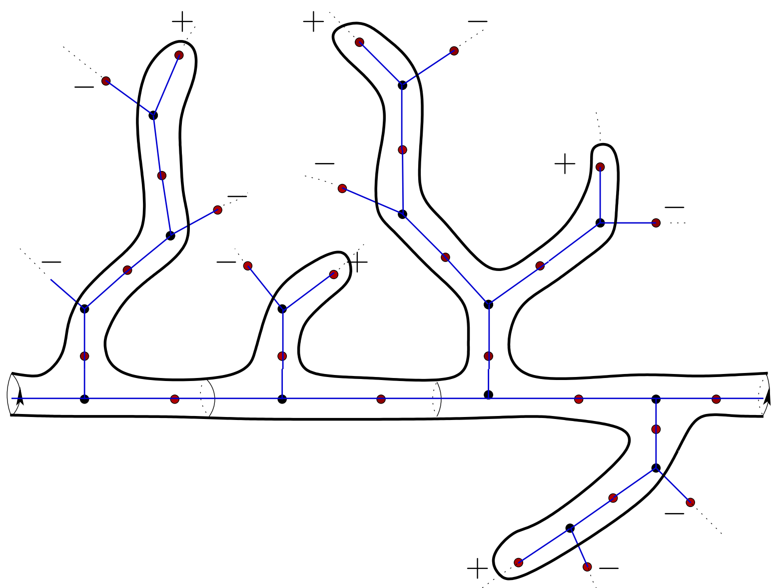

Figure 5.4: The relation between $\widetilde{\alpha}/\gamma$ and g_α.

Chapter 6

Uniqueness of Normal Form for Tori in $M = \sharp_n(S^2 \times S^1)$

In this chapter we will prove that given a homotopy class of tori, a normal representative exists and it is fairly "unique". We then prove that a normal representative gives a notion of local minimal intersection in M. Most of this work was included in [13].

6.1 Uniqueness of Normal Form for Tori

Let Σ be a maximal sphere system and α and α' be two essential tori in M. The theorem we would like to prove in this section is the following:

Theorem 6.1. *If α and α' are two homotopic tori in M, both in normal form with respect to a maximal sphere system Σ, then they are normally homotopic.*

The proof of this theorem will be divided into two Lemmata. To begin, we first assume that the images of $\pi_1(\alpha)$ and $\pi_1(\alpha')$ in $\pi_1(M)$ are conjugate, say to the subgroup generated by γ.

Lemma 6.2. *Let α and α' be two homotopic tori, both in normal form with respect to Σ. Then their transverse orientations may be chosen so that the corresponding decorated graphs with respect to the axis of γ are equal.*

Proof. Let α and α' be as given. Define, as before, $S_+ = \partial\widetilde{X_+} - \widetilde{\alpha}/\gamma$, $S_- = \partial\widetilde{X_-} - \widetilde{\alpha}/\gamma$, and $S'_+ = \partial\widetilde{X_+} - \widetilde{\alpha'}/\gamma$, $S'_- = \partial\widetilde{X_-} - \widetilde{\alpha'}/\gamma$. We pick transverse orientations on $\widetilde{\alpha}/\gamma$ and $\widetilde{\alpha'}/\gamma$. These transverse orientations determine transverse orientations on S_+, S_-, S'_+ and S'_- in $\widetilde{M}/\gamma$, and hence $+$ and $-$ labeling of them so that $\widetilde{\alpha}/\gamma$ is homologous to both S_+, S_-, and $\widetilde{\alpha'}/\gamma$ is homologous to both S'_+ and S'_-.

Now, any homotopy from α to α' lifts to a homotopy from $\widetilde{\alpha}/\gamma$ to $\widetilde{\alpha'}/\gamma$. Therefore we may fix transverse orientations on α and α' so that $\widetilde{\alpha}/\gamma$ and $\widetilde{\alpha'}/\gamma$ represent the same element of $H_2(\widetilde{M}/\gamma; \mathbb{Z})$. Then, S_+, S_-, S'_+ and S'_- all represent the same homology class.

Assume that $g_\alpha \neq g_{\alpha'}$. Suppose first that $T(\widetilde{\alpha})/\gamma \neq T(\widetilde{\alpha'})/\gamma$. Then one of them, say $T(\widetilde{\alpha})/\gamma$ contains an extremal Y, say Y_0, not in $T(\widetilde{\alpha'})/\gamma$.

Consider the valence-2 vertex of Y_0 which connects it to the rest of the graph. Let us call it v. Now, v represents a 2-sphere, which is a component of the boundary of the 3-punctured sphere $\widetilde{P}$ associated to the middle valence-3 vertex of the Y_0. This 2-sphere separates $\widetilde{M}$ into two parts. One part contains exactly one of the spheres in S_+ and one sphere in S_- and the other part contains all of the spheres of S'_+ and S'_- and all but the one of the spheres of S_+ and S_-. We will call this latter part $\widetilde{M_0}$. But then, S'_+ and S'_- represent zero in $H_2(\widetilde{M}/\gamma, \widetilde{M_0}/\gamma)$ and S_+ and S_- do not. This contradicts the fact that S_+, S_-, S'_+ and S'_- represent the same homology class in $H_2(\widetilde{M}/\gamma)$.

Now we are reduced to the case that $\widetilde{\alpha}/\gamma$ and $\widetilde{\alpha'}/\gamma$ have the same topological graphs. We must prove that their orientations may be selected so that the decorations are equal.

Suppose first that $[\widetilde{\alpha}/\gamma]$ and hence $[\widetilde{\alpha'}/\gamma]$ are 0 in $H_2(\widetilde{M}/\gamma)$. Then each bounds a compact submanifold of $\widetilde{M}/\gamma$, so the decorations each have either all plus signs or all minus signs. If they agree, there is nothing to prove. If they are opposite, we may reverse the orientation on one of them (not changing its homology class, since the class is 0) to make the signs all agree, and again we are finished. So we may assume that $[\widetilde{\alpha}/\gamma]$ is nonzero in $H_2(\widetilde{M}/\gamma)$.

We have $\partial\widetilde{X_+} = T_1 + \cdots + T_m - \widetilde{\alpha}/\gamma$ and $\partial\widetilde{X_-} = \widetilde{\alpha}/\gamma - U_1 - \cdots - U_n$ where $T_i \in S_+$, $U_j \in S_-$ and $m, n \in \mathbb{Z}$. Since $[\widetilde{\alpha}/\gamma]$ is nonzero, m and n are both at least 1.

We have $H_2(\widetilde{X_+}, \widetilde{\alpha}/\gamma) \cong \mathbb{Z}^{m-1} = \langle (T_1\rangle \oplus \cdots \oplus \langle T_m\rangle)/(T_1 + \cdots + T_m = 0)$. This is a subgroup of $H_2(\overline{\widetilde{M}/\gamma - \widetilde{M}_1}, \widetilde{\alpha}/\gamma)$, where $\widetilde{M}_1$ is the component of $\widetilde{M}/\gamma$ cut along $\widetilde{\alpha}/\gamma$ that contains $\widetilde{X_-}$.

In fact, we have

$$H_2(\widetilde{X_+}, \widetilde{\alpha}/\gamma) \subset H_2(\overline{\widetilde{M}/\gamma - \widetilde{M}_1}, \widetilde{\alpha}/\gamma) \cong H_2(\widetilde{M}/\gamma, \widetilde{M}_1)\ ,$$

the latter isomorphism by excision, and under

$$H_2(\widetilde{M}/\gamma) \to H_2(\widetilde{M}/\gamma, \widetilde{M}_1)\ ,$$

the homology class $[\widetilde{\alpha}/\gamma]$ goes into the subgroup $H_2(\widetilde{X_+}, \widetilde{\alpha}/\gamma)$ and equals $T_1 + T_2 + \cdots + T_m = 0$.

Now $[\widetilde{\alpha'}/\gamma] = [T_{i_1} + \cdots + T_{i_r} + U_{j_1} + \cdots + U_{j_s}]$ corresponding to the extremal vertices of the graph that are decorated with plus signs for $\widetilde{\alpha'}/\gamma$. Under

$$H_2(\widetilde{M}/\gamma) \to H_2(\widetilde{M}/\gamma, \widetilde{M}_1)\ ,$$

$[\widetilde{\alpha'}/\gamma]$ goes to $[T_{i_1} + \cdots + T_{i_r}]$, and must equal 0 since it equals $[\widetilde{\alpha}/\gamma]$ in $H_2(\widetilde{M}/\gamma)$.

Therefore it contains either all or none of the T_i. That is, in the decoration for the graph obtained from $\widetilde{\alpha'}/\gamma$, either all the T_i have plus signs or all have minus signs.

Applying the same argument to the minus side (with $\widetilde{X_-}$ in the role of $\widetilde{X_+}$), we conclude that for the decoration obtained from $\widetilde{\alpha'}/\gamma$, either all the U_i have plus signs or all have minus signs. That is, we have

$$[\widetilde{\alpha'}/\gamma] = [\epsilon T_1 \cdots + \epsilon T_m + \delta U_1 + \cdots + \delta U_n] \ ,$$

where $\epsilon, \delta \in \{0, 1\}$.

Suppose that $\epsilon = \delta = 0$ or $\epsilon = \delta = 1$, that is, in the decoration for $\widetilde{\alpha'}/\gamma$ all extremal vertices have either plus signs or minus signs. Then $\widetilde{\alpha'}/\gamma$ bounds a compact submanifold of $\widetilde{M}$, contradicting the fact that $[\widetilde{\alpha}/\gamma]$ is nonzero.

If $\epsilon = 1$ and $\delta = 0$, then the decorations are the same and there is nothing to prove.

In the remaining case, when $\epsilon = 0$ and $\delta = 1$, we may reverse the orientation on $\widetilde{\alpha'}/\gamma$ to make the decorations equal, and the proof is complete.

Therefore, since the decorations agree also, we have $g_\alpha = g_{\alpha'}$. □

Here we define the notion of being *normally homotopic*, which is our uniqueness criterion, as follows:

Definition 6.3. Two tori are said to be normally homotopic if there is a homotopy of M changing one of the tori to the other one without introducing new intersections on the sphere crossings, hence through normal, but possibly immersed tori at each level.

Lemma 6.4. *For two tori α and α' normal with respect to Σ, suppose that the corresponding decorated graphs are the same. Then, α and α' are normally homotopic.*

Proof. We will construct a normal homotopy of $\widetilde{\alpha}'/\gamma$ in $\widetilde{M}/\gamma$, moving it onto $\widetilde{\alpha}/\gamma$. This projects to a normal homotopy of α' onto α in M.

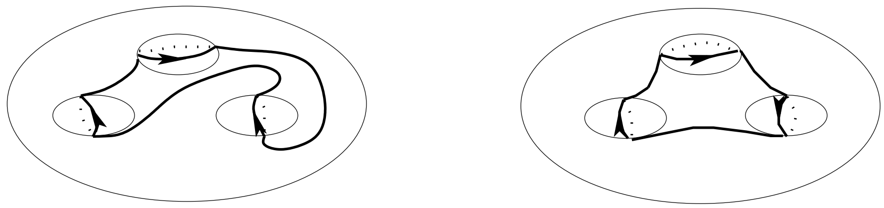

Figure 6.1: A twist of one of the pieces on a boundary sphere.

Since the decorated graphs are the same, both tori will have the same type of piece in each $\widetilde{P}$. To describe the normal homotopy, we start at one of the endpoints of the arc giving the axis of γ.

Let us call the first P on the axis of γ, P_1. By normal isotopy, we may move $\widetilde{\alpha}'/\gamma \cap P_1$ onto $\widetilde{\alpha}/\gamma \cap P_1$. On the next P along the axis, say P_2, we may move $\widetilde{\alpha}'/\gamma \cap P_2$ onto $\alpha/\gamma \cap \widetilde{P_2}$ without moving $\widetilde{\alpha}'/\gamma \cap (P_1 \cap P_2)$. It may be necessary to move $\widetilde{\alpha}'/\gamma$ on the other components of P_2 using a "twist". Figure 6.1 illustrates such a twist.

We continue along the axis of γ in this way, until we reach P_n that meets P_1. The isotopy moving $\widetilde{\alpha}'/\gamma \cap P_n$ onto $\widetilde{\alpha}/\gamma \cap P_n$ can be accomplished without moving $\widetilde{\alpha}'/\gamma \cap (P_n \cap P_1)$, since if not, $\widetilde{\alpha}'/\gamma$ would be a Klein bottle.

Now, we move to the finite tree branches on the axis of γ and continue moving the pieces of $\widetilde{\alpha}'/\gamma$ in each P corresponding to the Y's on branches, one after the other, fixing the already coinciding intersection circles we start with. Again, we might have the situation in Figure 6.1, so we might need to twist one intersection circle to make the pieces coincide. After a sequence of such homotopies we eventually reach an extremal Y, which must have one disk piece from each torus, one of the pieces with a twist, perhaps as in Figure 6.2. Now, if we fill in the boundary spheres in this P with 3-cells, we will obtain a 3-ball, and by an isotopy we will be able to move one disk piece to the other one without moving boundary. Regarding the 3-cells as points, this determines an element of the braid group of two points in the 3-ball. This group is of order 2. But

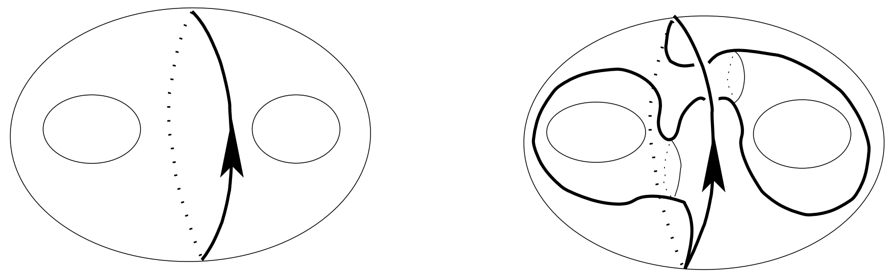

Figure 6.2: A disk piece and a twisted disk piece.

since the decorations are the same, the braid actually lies in the pure braid group of the 3-ball, which is trivial. So the disk pieces are isotopic relative to the boundary of P.

We observe that at each stage of each of these isotopies, we have a normal torus because each piece only moves within a single P. As the last step, we take the composition of these isotopies and project it into M to see that the two tori are normally homotopic. Since self intersections are possible, at some levels we might have immersed normal tori during this final homotopy. □

Now the proof of the Theorem 6.1 will be clear:

Proof of Theorem 6.1. Let α and α' be two homotopic normal tori. We start with Lemma 6.2 to see that two tori have the same decorated graphs and continue with Lemma 6.4 to conclude that they are normally homotopic. □

6.2 The Minimal Intersection

By the intersection number of a torus α we will mean the number $i(\alpha, \Sigma)$ of components of intersection of α with spheres of Σ when the intersection is transversal. From now we assume that the intersections with spheres of Σ are all transversal. As a result of the two main theorems in the previous chapters, we obtain the following:

Corollary 6.5. *If a torus α is in normal form with respect to a maximal sphere system Σ, then the intersection number of α with any S in Σ is minimal among the representatives of the homotopy class $[\alpha]$ in each P.*

Proof. Let $i(\alpha, S)$ denote the number of components of $\alpha \cap S$. Suppose α is normal but there is a torus α_1 which is homotopic to α with $i(\alpha_1, S) < i(\alpha, S)$. Then, by Theorem 4.2, α_1 is homotopic to a normal torus α_2 with $i(\alpha_2, \Sigma) \leq i(\alpha_1, \Sigma)$. Now, by Theorem 6.1, any two homotopic normal tori are normally homotopic, which implies $i(\alpha, S)=$ $i(\alpha_2, S) \leq i(\alpha_1, S)< i(\alpha, S)$, a contradiction. Therefore $i(\alpha, S)$ was minimal among the tori in $[\alpha]$. □

Chapter 7

Understanding Dehn Twist Automorphisms of the Free Group via Normal Tori in $M = \sharp_n(S^2 \times S^1)$

7.1 Intersecting Tori in M

We now consider two intersecting tori in M and from now on always use normal representatives whenever a maximal sphere system has been chosen. We also require the essential tori not to bound 3–balls in this chapter. An example of an intersecting essential pair of tori is illustrated in Figure 7.1.

We know that each homotopy class of tori in M gives an equivalence class of a $\mathbb{Z}$–splitting of F_n. The dual tree in $\widetilde{M}$ corresponding to such a splitting is called *Bass-Serre tree* and hence we will have a Bass-Serre tree corresponding to each homotopy class of tori. Given an essential embedded torus α in M, the image of $\pi_1(\alpha)$ under the homomorphism induced by the inclusion $i\colon \alpha \to M$ is an infinite cyclic subgroup of $\pi_1(M)$, defined up to conjugacy. These are $\mathbb{Z}$–subgroups of $\pi_1(M)$. Two $\mathbb{Z}$–splittings

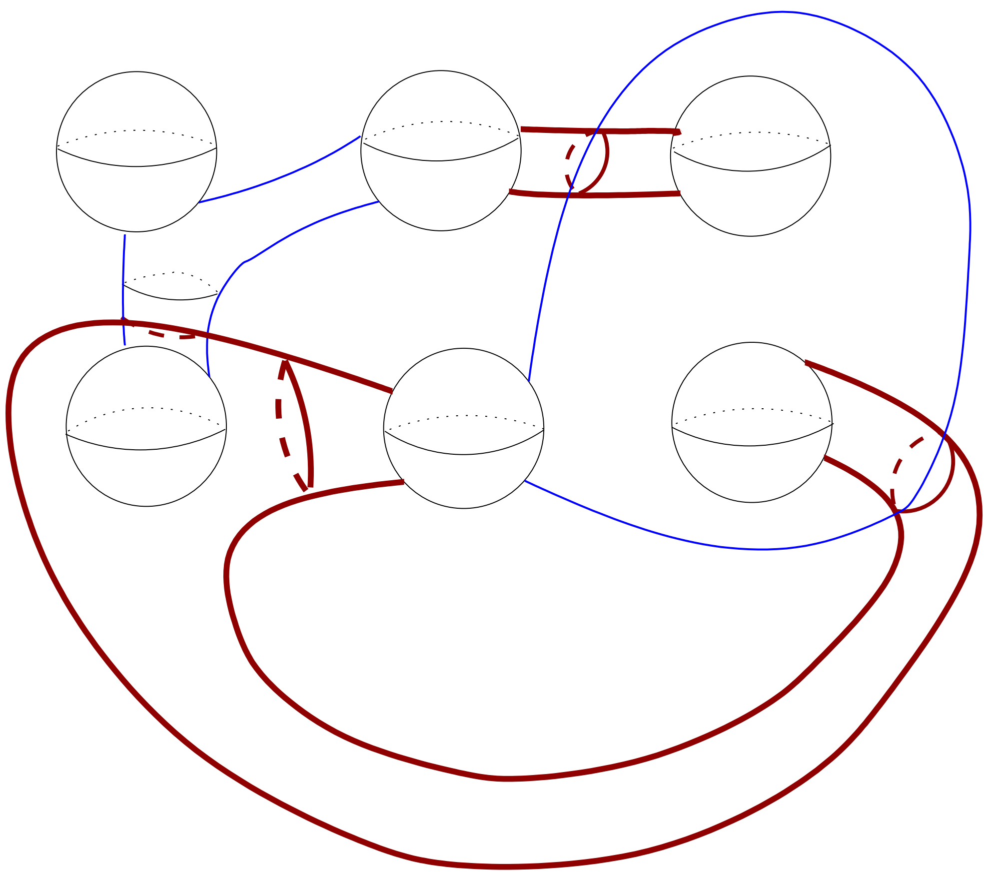

Figure 7.1: Two thrice-intersecting tori α (in black) and β (in red) in $\sharp_3(S^2 \times S^1)$. Here β intersects α twice nontrivially and once trivially whereas α intersects β once nontrivially and twice trivially.

correspond to two tori and the $\mathbb{Z}$–subgroups of $\pi_1(M)$ corresponding to these tori act on the Bass Serre trees of each other as elliptic or hyperbolic automorphisms. This action is by multiplication from the right. Recall also that given two elementary $\mathbb{Z}$–splittings $A_1 *_C B_1$ (or $A_1 *_C$) and $A_2 *_{C_2} B_2$ (or $A_2 *_{C_2}$) where $C_1 = \langle c_1 \rangle$ and $C_2 = \langle c_2 \rangle$, the element c_2 is said to be *elliptic* in the Bass-Serre tree of the first splitting if it is contained in a conjugate of A_1 or B_1 and called *hyperbolic* otherwise. These definitions also match with the way these automorphisms act on Bass-Serre trees:

Definition 7.1. Let $A_1 *_\alpha B_1$ (or $A_1 *_\alpha$) and $A_2 *_\beta B_2$ (or $A_2 *_\beta$) be two $\mathbb{Z}$–splittings of F_n corresponding to tori α and β. The *translation length* of α in the Bass-Serre tree T_β of the splitting corresponding to β is defined as

$$\min \left\{ d(\alpha(x), x) : x \in T_\beta \right\}.$$

We will denote this length by $\ell_\beta(\alpha)$.

It is clear that $\ell_\beta(\alpha) > 0$ when α is hyperbolic in T_β and zero if it is elliptic.

Depending on the action of the generator of the $\mathbb{Z}$ subgroups of $\pi_1(M)$ corresponding to each torus, we have three types of splittings: hyperbolic-hyperbolic, hyperbolic elliptic and elliptic-elliptic.

Definition 7.2. We will say that an intersection between two tori is *trivial* in one torus if it bounds a disk in that torus.

7.2 Understanding $\mathrm{Out}(F_n)$ via Intersecting Tori

7.2.1 Dehn Twist on a Surface as a Motivation

The following definitions are explained in detail in [10]

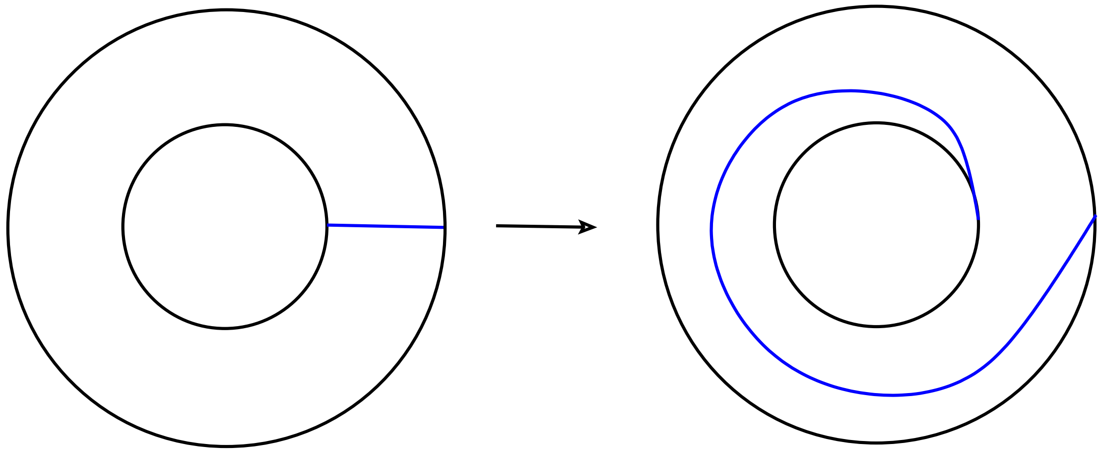

Figure 7.2: A view of a Dehn twist.

Definition 7.3. Consider the annulus $A = S^1 \times [0,1]$. To orient A we embed it in the (θ, r). plane via the map $(\theta, t) \mapsto (\theta, t+1)$, and take the orientation induced by the standard orientation of the plane Let $D : A \mapsto A$ be the *twist* mapof A given by the formula $D(\theta, t) = (\theta + 2\pi t, t)$.

The map D is an orientation-preserving homeomorphism that fixes ∂A pointwise. Note that instead of using $\theta + 2\pi t$ we could have used $\theta - 2\pi t$. Our choice is a left twist,while the other is a right twist. Figure 7.2 gives a pictorial description of the twist map D.

Now let S be an arbitrary (oriented) surface and let α be a simple closed curve in S. Let N be a regular neighborhood of α, and choose an orientation preserving homeomorphism $\phi : A \mapsto N$. We obtain a homeomorphism $D_\alpha : S \mapsto S$, called a *Dehn twist about* α, as follows:

$$D_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x), & \text{if } x \in N \\ x, & \text{if } x \in S - N. \end{cases}$$

In other words, the instructions for D_α are: perform the twist map D on the annulus

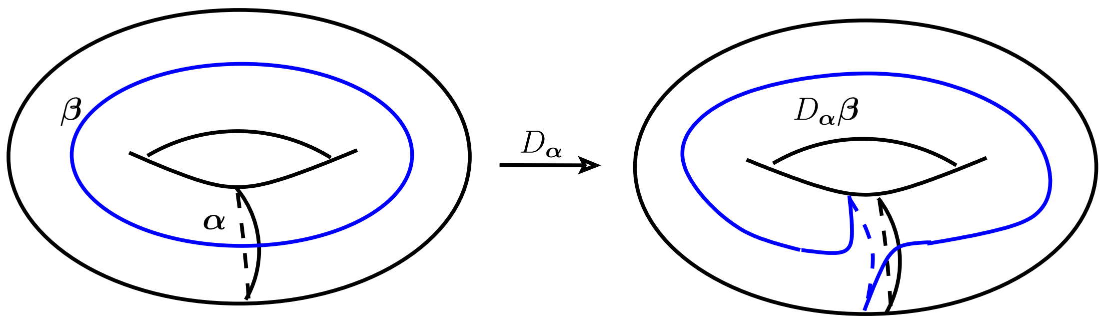

Figure 7.3: Another view of a Dehn twist.

N, and fix every point outside of N. The Dehn twist D_α depends on the choice of N and the homeomorphism ϕ. However, by the uniqueness of regular neighborhoods the isotopy class of D_α does not depend on either of these choices. What is more, D_α does not depend on the choice of the simple closed curve α within its isotopy class. Thus, if $\boldsymbol{\alpha}$ denotes the isotopy class of α, then D_α is well-defined as an element of $\mathrm{Mod}(S)$, called the Dehn twist about $\boldsymbol{\alpha}$. Check the Figure 7.3 for another view of a Dehn twist.

7.2.2 Dehn Twist Automorphisms of $\mathrm{Out}(F_n)$

Given a $\mathbb{Z}$–splitting of F_n as an amalgamated free product $F_n = A *_{\langle c \rangle} B$ or as an HNN extension $F_n = A *_{\langle c \rangle}$ of the free group, A *Dehn twist automorphism* is an element of $\mathrm{Out}(F_n)$ induced by the following automorphisms:

$$A *_{\langle c \rangle} B : a \mapsto a \text{ for } a \in A \qquad\qquad A *_{\langle tct^{-1} = c' \rangle} : a \mapsto a \text{ for } a \in A$$

$$b \mapsto cbc^{-1} \text{ for } b \in B \qquad\qquad t \mapsto tc$$

7.2.3 Dehn Twist Along a Torus in M

We would like to start by defining what a *Dehn twist along a surface in a* 3*–manifold* is, which is similar to the definition of a *rotation along a* 2*–sphere*:

Definition 7.4. Let S be a two sided surface in M, and γ a loop in $\pi_0(\mathrm{Homeo}(M))$. Then the *twist about S* is a function D where:

$$D(z,s) = (\gamma_s(z), s), \text{whenever } (z,s) \in S \times I$$
$$D(x) = x, \text{otherwise} .$$

Twists about tori in M in the meridinal direction correspond to composition of rotations along 2–spheres, which are homeomorphisms of M isotopic to the identity. Twists about tori in the longitude direction on the other hand, correspond to *slide homeomorphisms*. We can describe these homeomorphisms using tori as follows: we split a component of $M - \alpha$ along its sphere boundary and fill inside the sphere with a 3–cell. Then we slide this new 3-ball along a curve which connects the ball to itself in the longitude direction. We slide the 3-ball until we make one complete loop and replace the 3–cell with the component of $M - \alpha$ back.

For our purposes, we will describe the image of homotopy class of a torus β under a Dehn twist along α denoted by $D_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$, in the universal cover $\widetilde{M}$ when two tori are intersecting. This description will change depending on the type of the intersection. First we take two normal representatives α and β from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively, where bold letters denote the homotopy classes from now on.

Trivial-nontrivial intersections: For each trivial intersection of β with α in M, the intersection circle bounds a disk in β. To describe the image of such intersection disk under a twist about α, we use surgery in $\widetilde{M}$. If this intersection circle is nontrivial in

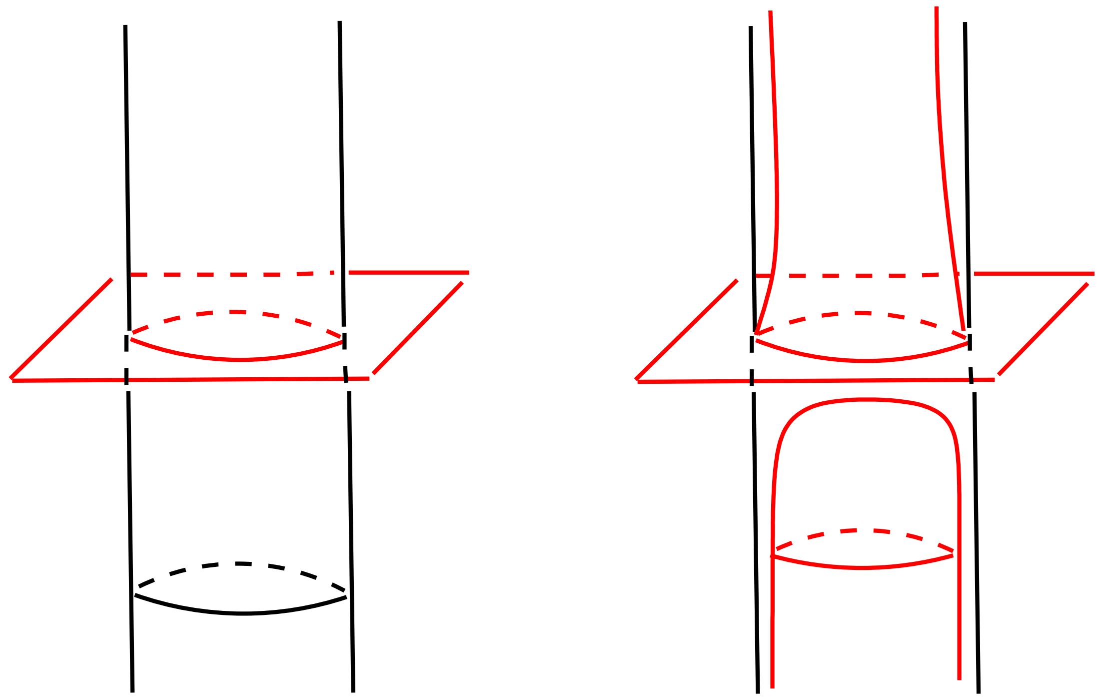

Figure 7.4: Dehn twist when one intersection is trivial in one torus (red) and nontrivial in the other one.

α, we take a lift of the intersection disk in $\widetilde{M}$ in a lift $\widetilde{\alpha}$ of α, cut it off and glue another disk to its boundary which follows $\widetilde{\alpha}$. An example for a lift of this type of intersection is the first intersection given in Figure 7.5 where the black torus is a lift of α and the red one is a lift of β. Images in $\widetilde{M}$ after twisting once are given in Figure 7.6.

Trivial-Trivial intersections: For each trivial intersection circle of β which is also trivial in α, we will follow a similar procedure, given again by a surgery in $\widetilde{M}$. We first fix lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ of α and β, respectively. A twist about α will lift to a twist about the chosen lift of α. To follow the image of the lift of intersection disk under a twist about $\widetilde{\alpha}$, we first take an arc in $\widetilde{M}$ connecting the lift of the intersection disk to another representative of itself located in the next fundamental domain of $\widetilde{\alpha}$. Then take two copies of the intersection circle in $\widetilde{\beta}$, cut $\widetilde{\beta}$ along these. We cap off the one whose image in M bounds a disk in β with another disk, and attach to the second

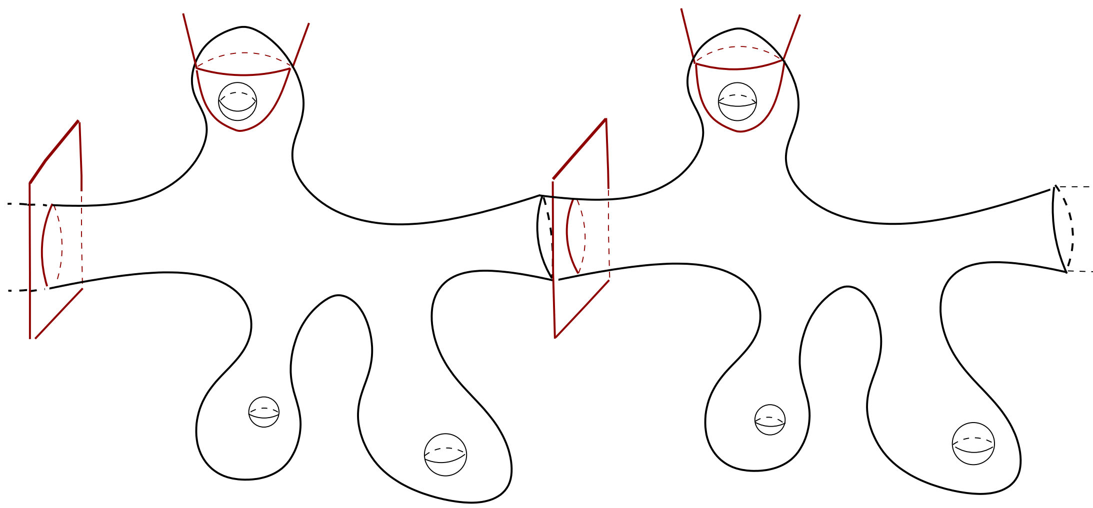

Figure 7.5: Lifts of two intersecting tori, shown in two copies of the fundamental domain of a lift of a black torus in $\widetilde{M}$.

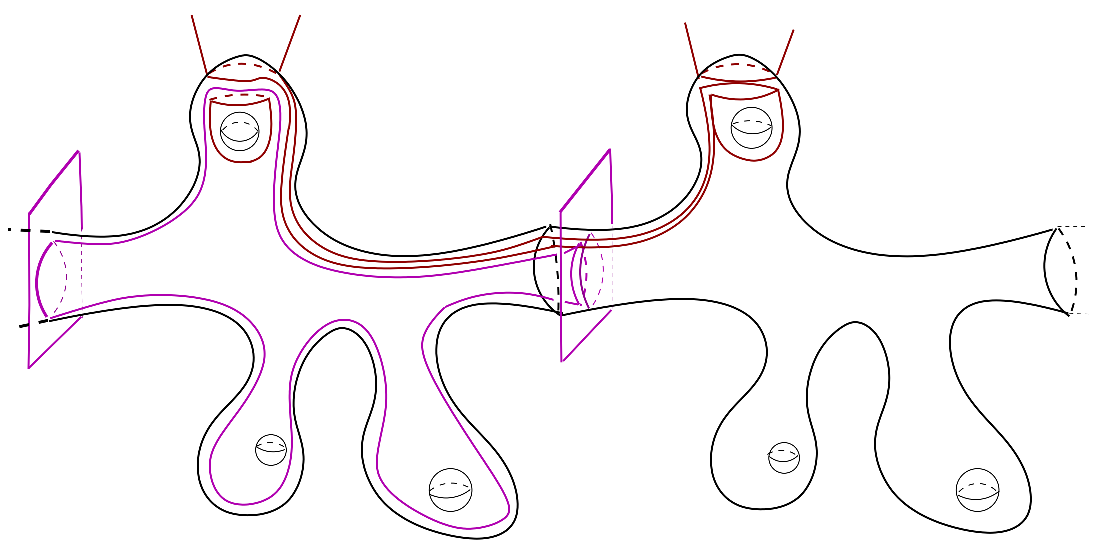

Figure 7.6: Image of the intersections given in Figure 7.5 under the twist about the black torus once.

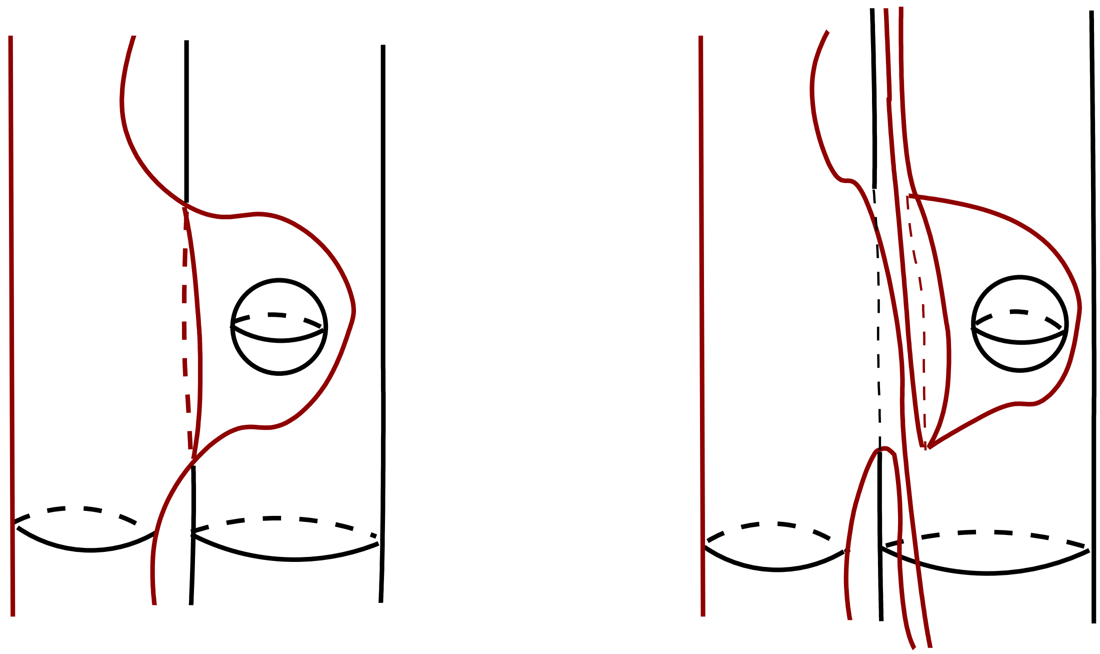

Figure 7.7: A schematic picture of the image of the red torus under a twist of the black torus in $\widetilde{M}$ (the upper tube comes from a different lift if considered in $\widetilde{M}$).

one an annulus which follows the arc and is glued to the capped off part of the next representative of the intersection circle. Observe that such annuli might intersect lifts of some spheres when they are following the arc in fundamental domain of $\widetilde{\alpha}$ but since twist about α has a support in a neighborhood of α only, so has a twist about a lift of α. Since such annuli occupy only a part of a neighborhood of $\widetilde{\alpha}$, they may not cross all sphere intersections $\widetilde{\alpha}$ makes. An example for such a (trivial-trivial) intersection circle is given in Figure 7.7. Another example of a same type of intersection is represented by the second intersection given in Figure 7.5. In this latter example, the intersection circle is trivial in both tori and $\widetilde{\alpha}$ is black and $\widetilde{\beta}$ is red. Describing its image in a fundamental domain under a twist about $\widetilde{\alpha}$ once requires the same type of surgery and this image is given in Figure 7.6 in purple. Sphere intersections were not depicted in these pictures.

Nontrivial-Nontrivial intersections: These are described the similar way as before,

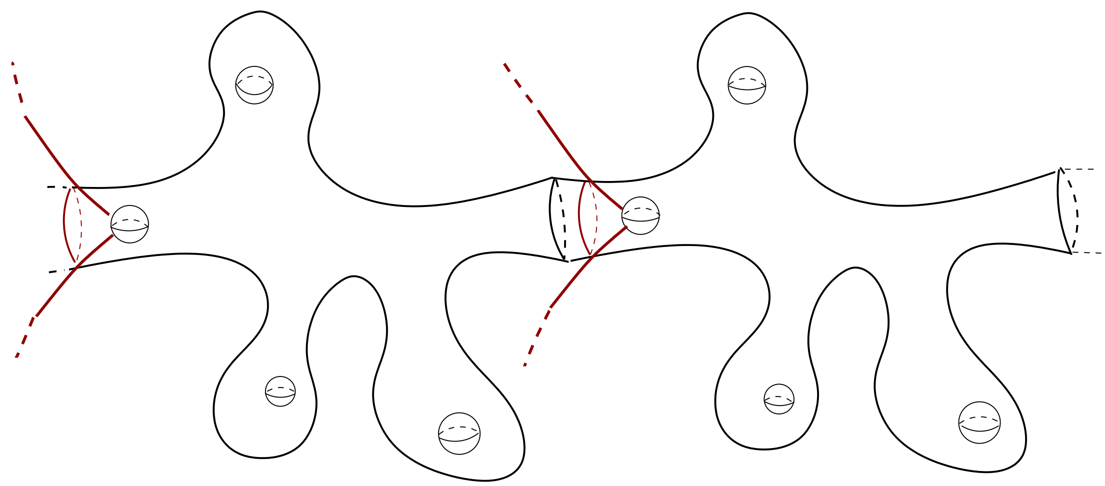

Figure 7.8: Two copies of a nontrivial-nontrivial intersection in $\widetilde{M}$

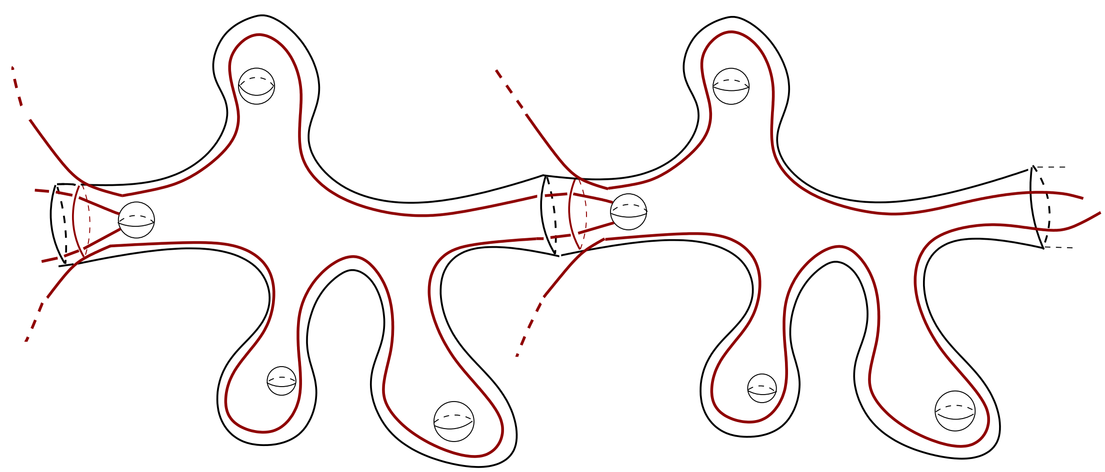

Figure 7.9: The image of the intersection in Figure 7.8 after twisting once.

but to see an example of such intersections, we refer to the Figures 7.8 and 7.9

Lemma 7.5. *A slide homeomorphism is a Dehn twist along a torus when the sliding loop is embedded.*

Proof. Let M' be the manifold obtained from M after cutting out a sphere and filling it with a ball, B. As explained before, a slide homeomorphism is an isotopy of M' which moves B along a loop connecting B to itself. In B, take a disk D_1 with radius 1, which is perpendicular to the sliding loop. ∂D_1 sweeps out a torus α, bounding a solid torus X, during the isotopy. Now, take another disk, D_2 of radius 2, containing D_1 as its disk of radius 1. Call the torus swept out by α' and the solid torus swept out by X'. So X is a concentric solid torus to X' and the closure of $X - X'$ is $\alpha \times I$. The isotopy of M' that moves B rotates all of X around the core circle, so at time 1 we have identity on X. We can extend the isotopy so that nothing outside X' moves. The homeomorphism we have is one which is identity outside $\alpha \times I$, hence a Dehn twist. The trace of this Dehn twist is some longitude of the solid torus X.

When we fix coordinates on X as $D_1 \times S^1$, and make the isotopy of B rotate X in the S^1 factor, then the trace of the Dehn twist is the loop that a point on ∂D_1 traces out. This is the longitude ℓ. But a different choice of coordinates on X will change the trace to $\ell + k \cdot m$, where $m = \partial D_1$ is a meridian of X, and k is an integer. This differs from the original slide by k rotations in ∂B. Since the rotation has order 2, when k is even we have an isotopic homeomorphism of M, while when k is odd, the result differs by a rotation in ∂B, so it may or may not be isotopic to the other slide.

□

Lemma 7.6. *Dehn twists about homotopic tori are isotopic.*

Proof. This is clear from the Lemma 7.5, since Dehn twist along a torus corresponds to a slide homeomorphism. When two tori are homotopic, the longitude curves of them are homotopic, and by the lemma mentioned, slide homeomorphisms along these curves, and hence the corresponding Dehn twists are isotopic. □

Now, the following lemma is crucial to describe the Dehn twists along a homotopy class of a torus:

Lemma 7.7 ([6]). *Let α and α' be two homotopic tori. Then we have $D_\alpha = D_{\alpha'}$ as the corresponding Dehn twist automorphisms.*

Proof. We need to show that the map D_α does not depend on the choice of the non-trivial (longitudinal) direction. This is true since any two choices for the non-trivial direction differ by a Dehn twist in the meridonal direction, which is in the kernel of the homomorphism $\mathrm{MCG}(M) \to \mathrm{Out}(F_n)$ as remarked before. Hence the induced element of $\mathrm{Out}(F_n)$ is well-defined.

If α and α' are isotopic, so are D_α and $D_{\alpha'}$. However, homotopic tori are not necessarily isotopic but this will not be a problem since the homotopy is described by passing one nested family of tubes through the other (from existence theorem 4.2), and hence it is supported inside a 3-ball in M. Since the homotopy between two tori extends to a homotopy equivalence of the 3-ball, the action of D_α and $D_{\alpha'}$ on loops in M is the same and therefore the lemma holds. □

7.2.4 Possible Applications

We have the following conjecture:

Theorem 7.8 (Conjecture). *Given a pair of hyperbolic-hyperbolic $\mathbb{Z}$–splittings $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and integers $k, l \geq N$, where N is a finite integer, the group $\langle D^k_{\boldsymbol{\alpha}}, D^l_{\boldsymbol{\beta}} \rangle$ is a free group of rank 2.*

The existence of free groups gives a dynamical property of the subgroup structure, in our case, of $\mathrm{Out}(F_n)$ the following way: Given a generating set of a finitely generated group, the growth rate of a group tells us the number of elements that can be written as a product of a given number of elements from the generating set, and much about the geometry and dynamics of a group and its elements can be learned from the growth rate. The exponential growth rate $\omega(G, S)$ of such a group G with a generating set S is given by:

$$\omega(G, S) = \lim_{n\to\infty} \sqrt[n]{|B_S(n)|},$$

where

$$B_S(n) = \{g \in G : \ell_S(g) \leq n\}.$$

Here, the *length* $\ell_S(g)$ is the least integer k so that the g can be expressed as a product of k elements from S.

If $\omega(G, S) > 1$ then G said to have *exponential growth*. In particular, in a free semi group generated by two elements, the number of elements of length n is the same as the number of ways to form an n-letter word using the generating set. As a result, any finitely generated group which contains a free semi group on two generators has exponential growth. It is possible to take the infimum over all generating sets in the above formula, which is denoted by $\omega(G)$. Now, if $\omega(G) > 1$, G is said to have *uniform exponential growth*. Finitely generated subgroups of the general linear group have this property, which in that setting is equivalent to being not virtually nilpotent [9].

It is also known that homotopy classes of homeomorphisms of surfaces (mapping class groups) and analogous groups of automorphisms of free groups have uniform exponential growth [1]. In the mapping class group setting, the question of whether finitely generated subgroups of mapping class groups have uniform exponential growth rate was answered positively by Mangahas in [22]. The main theorem of Mangahas in [22] states that the subgroups which are not abelian have uniform exponential growth and minimal growth rate is bounded below by a constant depending only on the surface. The Tits Alternative for the mapping class groups proven by Ivanov in [18] combined with the result of Birman, Lubotzky and McCarthy in [4] saying that any solvable subgroup of mapping class group is virtually abelian gives an idea of where to look for free groups inside all finitely generated subgroups of mapping class groups. Mangahas uses the classification of subgroups of mapping class groups due to Ivanov [18] along with concepts and techniques such as subsurface projection in the curve complex [23], minimal translation of pseudo Anosovs [24] and results of Fujiwara [11], and Hamidi-Tehrani [14] (completing her arguments in the details when finding a uniform number for the exponential growth of free subgroups of rank 2). Unfortunately, some of these crucial concepts are not fully developed in the $\mathrm{Out}(F_n)$ setting, and some others are far more complicated, so further techniques need to be developed and more cases need to be investigated.

Since $\mathrm{Out}(F_n)$ satisfies the Tits Alternative [3] and virtually nilpotent groups have polynomial growth, it will be sufficient to look for the free groups of rank 2 in non virtually abelian subgroups.

Thurston in [32] says that, in a group generated by two Dehn twists about two filling curves on a closed surface with genus $g \geq 2$, the groups generated by twists with powers greater than a finite number N is free of rank 2 and the elements from these groups which are not conjugate to powers of Dehn twists themselves are pseudo Anosov. Adapting this theorem to $\mathrm{Out}(F_n)$ to generate fully irreducible elements and to find rank 2 free groups, Clay and Pettet in [7] used an algebraic definition of a Dehn twist automorphism relative to a $\mathbb{Z}$–splitting of the free group and obtained a number N for the minimum power of twists, yet this number N depended on the choice of the twists:

Theorem 7.9 (Clay-Pettet [7])**.** *Let* D_1 *and* D_2 *be two Dehn twist automorphisms corresponding to two hyperbolic-hyperbolic* $\mathbb{Z}$*–splittings of* F_n*. Then, there exists* $N = N(D_1, D_2)$ *such that, for* $k, \ell \geq N$*,* $\langle D_1^k, D_2^\ell \rangle \cong F_2$*.*

To find a number N which is independent of the choice of Dehn twists, it is necessary to leave the 1-dimensional model for $\mathrm{Out}(F_n)$ since the dependence in [7] was due to the necessity for picking a basis of F_n in the proof. To prove the conjecture, we would like to understand the hyperbolic-hyperbolic $\mathbb{Z}$–splittings of F_n as two hyperbolic-hyperbolic intersecting tori in M and instead of working with Dehn twist automorphisms, we would like to use the topological Dehn twists along corresponding tori in M. All these correspondences are up to homotopy classes of tori and equivalences of splittings, and hence we would like to work with a normal representative, which gives a minimal intersection with the spheres in M. We need to use the normal representative to bound the intersection number of a twisted torus with the spheres, so that a classic Ping-Pong argument would yield a free group.

Bibliography

[1] J. W. Anderson, J. Aramayona, and K. J. Shackleton. Uniformly exponential growth and mapping class groups of surfaces. In *In the tradition of Ahlfors-Bers. IV*, volume 432 of *Contemp. Math.*, pages 1–6. Amer. Math. Soc., Providence, RI, 2007.

[2] Javier Aramayona and Juan Souto. Automorphisms of the graph of free splittings. arXiv:0909.3660v1 [math.GR], 2009.

[3] Mladen Bestvina, Mark Feighn, and Michael Handel. The Tits alternative for $\mathrm{Out}(F_n)$. I. Dynamics of exponentially-growing automorphisms. *Ann. of Math. (2)*, 151(2):517–623, 2000.

[4] Joan S. Birman, Alex Lubotzky, and John McCarthy. Abelian and solvable subgroups of the mapping class groups. *Duke Math. J.*, 50(4):1107–1120, 1983.

[5] Oleg Bogopolski. *Introduction to group theory*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. Translated, revised and expanded from the 2002 Russian original.

[6] Matt Clay, Funda Gültepe, and Kasra Rafi. Essential tori and Dehn twists of the free group. In preparation.

[7] Matt Clay and Alexandra Pettet. Twisting out fully irreducible automorphisms. *Geom. Funct. Anal.*, 20(3):657–689, 2010.

[8] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.

[9] Alex Eskin, Shahar Mozes, and Hee Oh. Uniform exponential growth for linear groups. *Int. Math. Res. Not.*, (31):1675–1683, 2002.

[10] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[11] Koji Fujiwara. Subgroups generated by two pseudo-Anosov elements in a mapping class group. I. Uniform exponential growth. In *Groups of diffeomorphisms*,

volume 52 of *Adv. Stud. Pure Math.*, pages 283–296. Math. Soc. Japan, Tokyo, 2008.

[12] Siddhartha Gadgil and Suhas Pandit. Algebraic and geometric intersection numbers for free groups. arXiv:0809.3109v1 [math.GT], 2008.

[13] Funda Gültepe. Normal tori in $\sharp_n(S^2 \times S^1)$. *Topology Appl.*, 160(8):953–959, 2013.

[14] Hessam Hamidi-Tehrani. Groups generated by positive multi-twists and the fake lantern problem. *Algebr. Geom. Topol.*, 2:1155–1178 (electronic), 2002.

[15] Allen Hatcher. Homological stability for automorphism groups of free groups. *Comment. Math. Helv.*, 70(1):39–62, 1995.

[16] Allen Hatcher and Karen Vogtmann. Homology stability for outer automorphism groups of free groups. *Algebr. Geom. Topol.*, 4:1253–1272, 2004.

[17] John Hempel. *3-manifolds*. AMS Chelsea Publishing, Providence, RI, 2004. Reprint of the 1976 original.

[18] Nikolai V. Ivanov. *Subgroups of Teichmüller modular groups*, volume 115 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by E. J. F. Primrose and revised by the author.

[19] John Kalliongis and Darryl McCullough. π_1-injective mappings of compact 3-manifolds. *Proc. London Math. Soc. (3)*, 52(1):173–192, 1986.

[20] François Laudenbach. *Topologie de la dimension trois: homotopie et isotopie*. Société Mathématique de France, Paris, 1974. With an English summary and table of contents, Astérisque, No. 12.

[21] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[22] Johanna Mangahas. Uniform uniform exponential growth of subgroups of the mapping class group. *Geom. Funct. Anal.*, 19(5):1468–1480, 2010.

[23] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000.

[24] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.

[25] Darryl McCullough. Topological and algebraic automorphisms of 3-manifolds. In *Groups of self-equivalences and related topics (Montreal, PQ, 1988)*, volume 1425 of *Lecture Notes in Math.*, pages 102–113. Springer, Berlin, 1990.

[26] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge, 1979.

[27] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[28] Abe Shenitzer. Decomposition of a group with a single defining relation into a free product. *Proc. Amer. Math. Soc.*, 6:273–279, 1955.

[29] John R. Stallings. A topological proof of Grushko's theorem on free products. *Math. Z.*, 90:1–8, 1965.

[30] John R. Stallings. Foldings of G-trees. In *Arboreal group theory (Berkeley, CA, 1988)*, volume 19 of *Math. Sci. Res. Inst. Publ.*, pages 355–368. Springer, New York, 1991.

[31] G. A. Swarup. Decompositions of free groups. *J. Pure Appl. Algebra*, 40(1):99–102, 1986.

[32] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.