2D BOUSSINESQ EQUATIONS WITH LOGARITHMICALLY SUPER-CRITICAL CONDITIONS

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Abstract: This thesis focuses on the regularity problem of two generalized two dimensional Boussinesq equations. The first model contains the critical level of diffusion and a double logarithmically super-critical velocity. The second model contains logarithmically super-critical dissipation. The proof takes the advantage of the two equivalent definitions of the dissipative operator. We also extend the Besov spaces to better suit the new operator. In Chapter 5, we give a small data regularity result for super-critical Surface Quasi-Geostrophic equations. This is achieved by generalize the definition of Only Small Shock first introduced in [21]. The proof also use the modulus of continuity approach in [53]. The last chapter deal with an axisymmetric Navier-Stokes model by Hou and Li in n-dimensional setting. The local and global regularity result is achieved by requiring a strong enough fractional Laplacian dissipation.

TABLE OF CONTENTS

CHAPTER 1

Introduction

1.1 Navier-Stokes Equations

Fluid dynamics take a large proportion of the analysis of partial differential equations. They model phenomena in both micro and macro scales. We can find examples in geophysics, weather prediction and oceanic engineering, as well as plasma media and engine combustion analysis. Some of the models also find their analogues in economy, finance and social behavior contest. The most fundamental, also important, system among all the fluid dynamic models is the incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + f \\ \qquad u_0(x) = u(x, 0) \end{cases} \tag{1.1}$$

Here, $u = u(x,t) : \mathbb{R}^d \to \mathbb{R}^d$ is the velocity field usually defined in 2 dimensional or 3 dimensional space. p presents the pressure. This model considers the convolution, the term $u \cdot \nabla u$, and the dissipative effect, i.e. $\nu \Delta u$. The case $\nu = 0$ is called Euler equations. f on the right hand side is the external force. The incompressibility, or the preservation of the volume, is given by the divergence free condition, $\nabla \cdot u = 0$. The equations are derived based on the conservation of mass, momentum as well as energy.

The Navier-Stokes equations draw a great amount of attention among both mathematicians and physicists. After being found for over two hundred years, the problem regarding the existence and smoothness solutions of the 3D system remains open. In fact, it is one of the seven Millennium Prize Problems stated by the Clay Mathematics Institute [16]. A widely accepted understanding is that the potential loss of smoothness and differentiability is caused by the non-linear convection term while the dissipative term tends to compensate and stop the solution becoming singular. The major difficulty is that, the convolution term may be more singular than the smoothing term. This can be more explicitly shown in the vorticity form of the equations. We call the vector ω the vorticity related to the velocity field u, when

$$\omega = \nabla \times u \tag{1.2}$$

Taking the curl of the velocity equation, we have

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \nu \Delta \omega = \omega \cdot \nabla u + \tilde{f} \\ \qquad u_0(x) = u(x, 0) \end{cases} \tag{1.3}$$

Notice that the pressure will vanish under this operation. In the two dimensional setting, the vortex stretching term $\omega \cdot \nabla u$ on the right hand side will disappear since ω is always orthogonal to ∇u. We can prove the existence and uniqueness of the classic C^∞ solution by standard energy estimate. However, this term will not vanish and cause difficulty for regularity analysis when we have three or higher dimensional setting.

There are some conditional results for the well-posedness problem for Navier-Stokes equations. The existence of the Leray-Hopf type weak solutions [57] [46]. By Prodi, Serrin [67] [71] [70] and later Struwe [74], when $u \in L^p(0, T; L^q(\mathbb{R}^3)$ and $p, q \geq 1$ $\frac{2}{p} + \frac{3}{q} \leq 1$, the solution u is smooth in the spatial direction in the weak sense. Caffarilli, Kohn, Nirenberg [15] and Lin [58] show that, the 1-D Hausdorff measure of the singularity set to the weak solution is zero. The most frequently used *a priori* condition is given by Beale, Kato and Majda [5]:

$$\int_0^T \|\omega(\cdot,\tau)\|_{L^\infty} d\tau \leq \infty \qquad \forall T > 0 \tag{1.4}$$

if and only if the three dimensional system has a global in time solution in the function space $C([0,\infty], H^s) \cap C^1([0,\infty], H^{s-1})$, $s > 3$. We can find a large number of analog conclusions for the variants of the Navier-Stokes equations.

1.2 2D Boussinesq Equations

One variant of the Navier-Stokes equations is the 2D Boussinesq equations.

$$\begin{cases} \partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.5}$$

Here, $\kappa > 0$ is the diffusion coordinator. θ is a scalar function of $x \in \mathbb{R}^2$, which may stand for temperature. $\mathbf{e}_2$ is the unit vector $(0,1)$ in the vertical direction. The Boussinesq system is established to model geophysical flows such as atmospheric fronts and ocean circulations, where the gravity is taken into consideration [61]. It also plays a very important role in the study of Raleigh-Bernard convection. Interestingly, the 2D Boussinesq equations have some key feature of the 3D Euler and Navier-Stokes equations if we identify θ with ω. The troublesome vortex stretching term finds its counterpart as $\partial_{x_1}\theta$. Furthermore, the inviscid 2D Boussinesq system is identical to the 3 dimensional axi-symmetric swirling flows out side the symmetric axle. See Chapter 6 regarding this type of system.

Due to the similarity of the Boussinesq system and the Navier-Stokes system, the global regularity problem for (1.5) has drawn large attention in the past years and important progress has been made as well. When both the dissipation and thermal diffusion appear, the system acts exactly like the 2D Navier-Stokes. One can use the same energy method to jointly bound the norm of u and θ, see [11] for example. For the case $\nu > 0, \kappa = 0$ or $\nu = 0, \kappa > 0$, the global well-posedness is still manageable by the energy method except we need a logarithmic correction for bounding L^∞ norm. Chae [13] solves both cases while Hou and Li [39] solves the one with $\kappa = 0$. The needed inequality is the Brezis-Wainger inequality.

However, for the complete inviscid case, i.e. $\nu = \kappa = 0$, the global regularity problem remains open. The idea is to weaken either the dissipation or the diffusion to find a critical case where regularity still holds. One of the two ways is the anisotropic Boussinesq equations. Since the Laplacian operator is the sum of double partial differential of the x_1 and x_2 direction, the anisotropic operator only consider double differential in one direction. Danchin and Paicu [32] first study the case with horizontal dissipation or diffusion. Larios, Lunasin and Titi [59] re-established some results of Danchin and Paicu for the horizontal dissipation case under milder assumptions. Since the Boussinesq system is not uni-directional, the anisotropic system with vertical dissipation or diffusion is comparably harder. This type of system has been studied by Adhikari, Cao and Wu [1] [2] and was successively resolved by Cao and Wu in [24].

The second way to weaken the dissipation or diffusion is to change the Laplacian operator to a non-local operator. For example, we call $\Lambda = (\sqrt{-\Delta})^{\frac{1}{2}}$ the Zygmund operator [73] and define the fractional Laplacian as Λ^α for $\alpha \geq 0$. The meaning of this type of operator can be understood through Fourier multiplier or through a convolution type kernel, see section 2.2 for details. Hmidi, Keraani and Rousset, in [37] and [38], made a great progress by establishing the global regularity when the full Laplacian in dissipation or diffusion is replaced by the Zygmund operator while the other term is absent. Their method is to find a joint variable so that the vortex stretching term is transformed into a commutator structure (see section 2.4). By these two results, we may make the conjecture that the critical case for the Boussinesq is $\alpha + \beta = 1$, where the dissipation operator is Λ^α and the diffusion one is Λ^β (see Miao and Xue [64]). Recently, Jiu, Miao, Wu and Zhang [49] show the regularity for this critical condition when α is close enough to 1. In a preprint [21], Constantin and Vicol applied a nonlinear maximum principle for linear nonlocal operators to obtain their global regularity result when we have mixed fractional power of both dissipation and diffusion.

Our work in chapter 4 will focus on the slight super-critical dissipation. We can find the first attempt of super-critical context in Tao [75]. The principle behind this type result is a generalized Gronwall inequality, which can be further developed into Osgood condition (see section 2.5). We can write a generalized Boussinesq system by

$$\begin{cases} \partial_t u + u \cdot \nabla u + \mathcal{L} u = -\nabla p + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta + \mathcal{M} \theta = 0 \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.6}$$

Here, we assume $\nu = \kappa = 1$ for simplicity. Again, the operator $\mathcal{L}$ and $\mathcal{M}$ are defined by Fourier multiplier or by a nonlocal operator with a convolution kernel. The restriction of these operator can be found in [28]. Hmidi [45] shows the well-posedness for the logarithmically super-critical diffusion case, where $\mathcal{L} = 0$ and

$$\mathcal{M}(\Lambda) = \frac{\Lambda}{\log^\gamma(a + \Lambda)}$$

He develops the crucial estimate from the diffusion, which counter-act the convection effect, by showing the positive definite of the generator function with the help of Askey's theorem.

1.3 Active Scaler and Surface Quasi-Geostrophic Equations

The third way to find the super-critical Boussinesq equations is to alter the relation between ω and u. We start with the vorticity version of the Boussinesq equations

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \mathcal{L}\omega = \partial_{x_1}\theta, \\ \partial_t \theta + u \cdot \nabla \theta + \mathcal{M}\theta = 0, \\ u = \nabla^{\perp}\psi, \quad \Delta\psi = \omega, \\ \omega(x,0) = \omega_0(x), \ \theta(x,0) = \theta_0(x), \end{cases} \tag{1.7}$$

We introduce the stream function ψ to ensure the divergence free condition of u. We rewrite the third row to get the logarithmically generalized system

$$u = \nabla^{\perp}\psi, \quad \Delta\psi = \Lambda^{\sigma}\log^{\gamma}(I - \Delta)\omega, \tag{1.8}$$

Here,

$$\widehat{\Lambda^{\sigma} f}(\xi) = |\xi|^{\sigma}\widehat{f}(\xi) \quad \text{and} \quad \widehat{\log^{\gamma}(I-\Delta)f}(\xi) = \log^{\gamma}(I + |\xi|^2)\,\widehat{f}(\xi).$$

By the definition, u is more singular with respect to ω when compare to the case ω is the curl of the velocity. Chae and Wu [25] show the well-posedness for the system of which $\mathcal{L} = \Lambda$, $\mathcal{M} = 0$, $\sigma = 0$ and $\gamma \geq 0$. In Chapter 3, we give the regularity result when the diffusion is in the critical case while $\sigma = 0$ and $\gamma \in [0, \frac{1}{2})$.

These models can be regarded as examples of a large group called active scalar type. For x defined in $\mathbb{R}^2$ or the 2D torus $\mathbb{T}^2$ and $t \in [0, \infty)$, we consider the following system:

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \mathcal{L}\theta = 0 \\ u = \nabla^\perp \psi \,, \quad \Delta \psi = \Lambda^\beta \theta \end{cases} \tag{1.9}$$

The system is named in the way that, instead of being a given velocity field, u is determined by the variable θ. We can explicitly write

$$u = (-\mathcal{R}_2, \mathcal{R}_1)\theta \tag{1.10}$$

Here, $\mathcal{R}_i = (\partial_{x_i} \Lambda^{-1})$ is the non-local singular Riesz operator, of which the Fourier multiplier is $\frac{\xi_i}{|\xi|}$. It is not hard figure that, by identifying θ with ω, we have the 2D Navier Stokes equations when $\beta = 0$ and $\mathcal{L} = (-\Delta)$. When $\beta = 1$, we have the famous surface quasi-geostrophic equations. The SQG model is invented to describe the surface temperature of shallow water. Take the oceans as an example. The global regularity results for these two cases, as well as the case for $\beta \in (0, 1)$, have been established (see details in section 5.1). When $\beta \in (1, 2)$, it is in the super-critical regime, and the well-posedness problem remains wide open. However, interestingly, the system reduces to a trivial linear equation when $\beta = 2$.

$$\partial_t \theta + \nabla \theta \cdot \nabla^\perp \theta = 0 \quad or \qquad \partial_t \theta = 0 \,, \qquad \theta(x, t) = \theta_0(x)$$

CHAPTER 2

Preliminary

2.1 Besov Space

In this section, we introduce a new set of function spaces called Besov spaces. It provides more powerful tools when compared to L^p and Sobolev space $W^{s,p}$. We start with the idea of dyadic decomposition of unity.

Let

$$A = \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \right\}$$

$$B(0,1) = \{ \xi \in \mathbb{R}^n : |\xi| \leq 1 \}$$

Proposition 2.1.1 *Let $\mathcal{D}$ be an infinity smooth function. Then, there are two functions ψ and ϕ such that*

1.

$$\psi \in \mathcal{D}(B(0,1)) , \quad \phi \in \mathcal{D}(A)$$

2.

$$0 \leq \psi, \phi \leq 1 , \quad \psi(\xi) = \psi(|\xi|) , \phi(\xi) = \phi(|\xi|)$$

3.

$$\psi(\xi) + \sum_{j=0}^{\infty} \phi(\frac{\xi}{2^j}) \equiv 1 , \quad \xi \in \mathbb{R}^n$$

$$\sum_{j=-\infty}^{\infty} \phi(\frac{\xi}{2^j}) = 1\,, \quad \xi \neq 0$$

4.

$$\frac{1}{2} \leq \psi^2(\xi) + \sum_{j=0}^{\infty} \phi^2(\frac{\xi}{2^j}) \leq 1\,, \quad \xi \in \mathbb{R}^n$$

$$\frac{1}{2} \leq \sum_{j=-\infty}^{\infty} \phi^2(\frac{\xi}{2^j}) \leq 1\,, \quad \xi \neq 0$$

5. Let $\phi_j = \phi(\frac{\xi}{2^j})$

$$supp\, \phi_j \cap supp\, \phi_k = \emptyset \quad if \quad |j-k| \geq 2$$

$$supp\, \phi \cap supp\, \phi_j = \emptyset \quad if \quad j \geq 1$$

We try to find examples for ϕ and ψ. For a given annulus $\{\xi : 2^j \leq |\xi| \leq 2^{j+1}\}$, the summation in the third condition has non-zero contribution from the functions ϕ_j and ϕ_{j+1}. Due to the scaling property of ϕ_j, the third condition is equivalent to

$$\phi(x) + \phi(2x) = 1\,, \quad \text{for} \quad x \in (\frac{1}{2}, 1)$$

We can arbitrarily choose some infinitely smooth function ϕ defined on $(\frac{1}{2}, 1)$ with all of its derivatives being zero at $\frac{1}{2}$ and 1. Then. we extend its support to $x \in (1, 2)$ by calling $\phi(x) = 1 - \phi(\frac{3}{2} - \frac{1}{2}x)$ in this interval. Notice that, the forth condition would be satisfied automatically under this construction. We can then determine the function ψ by the third condition.

When given the set of functions $\{\phi_j\}$ and ψ, we can define the Fourier localization operator Δ_j

Definition 2.1.2 *We define the operator* Δ_j *for any integer* j *as* $(\phi_j \hat{f})^\vee = 2^{jn} h(2^j \cdot) \star f$*. Here* $h = \phi^\vee$.

Definition 2.1.3 *Let*

$$\mathcal{S}_0 = \left\{ \varphi \in \mathcal{S} \,:\, \int_{\mathbb{R}^d} \varphi(x) x^\gamma dx = 0, |\gamma| = 0, 1, 2, \ldots \right\}$$

and $\mathcal{S}_0'$ *be the dual space of* $\mathcal{S}_0$*. For* $s \in \mathbb{R}$*,* $1 \leq p, q \leq \infty$ *and a function* $f \in \mathcal{S}_0'$*, the homogeneous Besov norm is*

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js} \|\Delta_j f\|_{L^p}\|_{l^q} = \left\{ \sum_{j=-\infty}^{\infty} 2^{jqs} \left[\int_{\mathbb{R}^n} |\Delta_j f|^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

The homogeneous Besov space consists of f *with finite homogeneous Besov norm.*

Definition 2.1.4 *For the inhomogeneous Besov space, we redefine* $\Delta_j f = 0$ *for* $j = -2, -3, -4, \ldots$ *and* $\Delta_j f = (\psi \hat{f})^\vee$*.*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js} \|\Delta_j f\|_{L^p}\|_{l^q} = \left\{ \sum_{j=-1}^{\infty} 2^{jqs} \left[\int_{\mathbb{R}^n} |\Delta_j f|^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

Again, the inhomogeneous Besov space consists of f *with finite inhomogeneous Besov norm.*

The Besov norms can be defined in an alternative way [73]

$$\|f\|_{\mathring{B}^s_{p,q}} = \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^p}{|t|^{d+sq}} dt \right)^{\frac{1}{q}} \tag{2.1}$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^p}{|t|^{d+sq}} dt \right)^{\frac{1}{q}} \tag{2.2}$$

Proposition 2.1.5 *We list some frequently used embedding theorems regarding the Besov space*

- *For any* $s > 0$

$$B^s_{p,q} \subset \mathring{B}^s_{p,q}$$

- *If $s_1 \leq s_2$, for the inhomogeneous norm only*

$$B^{s_2}_{p,q} \subset B^{s_1}_{p,q}$$

- *If $1 \leq q_1 \leq q_2 \leq \infty$,*

$$\mathring{B}^{s}_{p,q_1} \subset \mathring{B}^{s}_{p,q_2}\,, \qquad B^{s}_{p,q_1} \subset B^{s}_{p,q_2}$$

- *If $1 \leq p_1 \leq p_2 \leq \infty$ and $s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}$*

$$\mathring{B}^{s_1}_{p_1,q} \subset \mathring{B}^{s_2}_{p_2,\infty}\,, \qquad B^{s_1}_{p_1,q} \subset B^{s_2}_{p_2,\infty}$$

- *The relation between the Besov space and the Hilbert space is, for $s \in \mathbb{R}$*

$$\mathring{H}^s \sim \mathring{B}^{s}_{2,2}\,, \quad H^s \sim B^{s}_{2,2}$$

- *Besov space is related to the general Sobolev space through*

$$\mathring{B}^{s}_{q,\min(q,2)} \hookrightarrow \mathring{W}^{s}_{q} \hookrightarrow \mathring{B}^{s}_{q,\max(q,2)}$$

Specially

$$\mathring{B}^{0}_{q,\min(q,2)} \hookrightarrow L^q \hookrightarrow \mathring{B}^{0}_{q,\max(q,2)}$$

Bernstein's inequalities are powerful tools in dealing with Fourier localized functions. These inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

Proposition 2.1.6 *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*

1) If f satisfies

$$supp\,\widehat{f} \subset \{\xi \in \mathbb{R}^d : \; |\xi| \le K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) If f satisfies

$$supp\,\widehat{f} \subset \{\xi \in \mathbb{R}^d : \; K_1 2^j \le |\xi| \le K_2 2^j\}$$

for some integer j and constants $0 < K_1 \le K_2$, then

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

For the last part of this section, we mention an often used technique in Besov space, which is called paraproduct. We first denote

$$S_j f = \sum_{k=-1}^{j-1} \Delta_k f\,, \qquad \tilde{\Delta}_j f = \Delta_{j-1} f + \Delta_j f + \Delta_{j+1} f\,, \quad j = 0, 1, 2, \ldots$$

Since for any function $f \in S_0'$, $f = \sum_{j=-1}^{\infty}$, the product of two functions f and g can be decomposed as

$$f \cdot g = \sum_{j=-1}^{\infty} S_j f \Delta_j g + \sum_{j=-1}^{\infty} \Delta_j f S_j g + \sum_{j=-1}^{\infty} \Delta_j f \tilde{\Delta}_j g$$

In the $\mathbb{Z}^2$ grid, these components represent the lower and the upper triangles and the center strip which includes the diagonal line and the two sub-diagonal lines. The reason for this type of decomposition is due to the fact that $supp\,\phi_j \cap supp\,\phi_k$ is non-empty if and only if $|j - k| \le 2$.

In the analysis of commutator (see the exact definition in section 2.4), the aforementioned three parts are often further split into five parts. For example,

$$[\Delta_j,\, u\cdot\nabla]\theta = J_1 + J_2 + J_3 + J_4 + J_5$$

with

$$\begin{aligned}
J_1 &= \sum_{|j-k|\le 2} [\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k\theta,\\
J_2 &= \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k\theta,\\
J_3 &= S_j u\cdot\nabla\Delta_j\theta,\\
J_4 &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta),\\
J_5 &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\theta),
\end{aligned}$$

When combined with the divergence free condition, any spacial integral over the third term will vanish.

2.2 Two Expressions of the Non-Local Operator

In the original Navier-Stokes equations, the dissipation part is given by $-\nu(-\Delta)$ with dissipation coefficient ν. It is defined locally. However, when we discuss the critical case through energy method, the operator has been modified to fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ or Λ^α. Here $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the Zygmund operator. The way we understand this operator is through the Fourier multiplier method, or for a function f in the Schwartz class

$$\Lambda^\alpha f = (|\xi|^\alpha \hat{f})^\vee$$

In [7], Cordoba and Cordoba gave an alternative way to define the fractional Laplacian operator by extending the work of Stein [73]. With the restriction $\alpha \in$

$(0, 2)$, for f in the Schwartz class defined on $\mathbb{R}^d$, we call the Reiz potential

$$\Lambda^{\alpha} f(x) = C_{\alpha}\, P.V. \int \frac{[f(x) - f(y)]}{|x - y|^{d+\alpha}} dy \tag{2.3}$$

Inspired by the work of Tao [75], researchers have introduced a large class of dissipation operators related to the slightly super-critical regime. In general, we may write them as $\mathcal{L}$. This operator may has been defined through the convolution way, like the one in [7]

$$\mathcal{L} f(x) = C_{\alpha}\, P.V. \int [f(x) - f(y)] \frac{m(|x - y|)}{|x - y|^{d}} dy \tag{2.4}$$

The function $m : (0, \infty) \to [0, \infty)$ is a non-increasing smooth function which is singular at the origin. To guarantee the convergence of the principal value integral, we need the sub-quadratic condition for m, i.e.

$$\int_0^1 r m(r) dr \leq \infty$$

This definition has its application in deriving the local maximum principal for the operator $\mathcal{L}$ ([7], [18]). More examples can be found in [50],[56].

On the other hand, We can define $\mathcal{L}$ by the Fourier multiplier method. Suppose $P(\xi)$ be a radially symmetric function defined on $\mathbb{R}^2$, which is smooth away from the origin, non-decreasing and $P(0) = 0$, $P(\xi) \to \infty$ as $|\xi| \to \infty$, we may define

$$\tilde{\mathcal{L}} f(x) = (P(\xi)\hat{f})^{\vee} \tag{2.5}$$

This definition is closely related to the general Besov space and Bernstein inequality. One important question is that, under what conditions for m and p, we have the equivalence of the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$. From the example of Λ^{α}, the most likelihood relation is $m(r) = P(\frac{1}{r})$. In [28], Dabkowski, Kiselev, Silvertre and Vicol give a rigorous proof that, when m and P satisfies the conditions listed below, $m(r) \sim P(\frac{1}{r})$ and L being equivalent to $\tilde{L}$. For $m(r)$

1. there exists $C_{m,1} > 0$ such that

$$rm(r) \leq C_{m,1} \qquad \text{for all } r \leq 1$$

2. there exists $C_{m,2} > 0$ such that

$$r|m'(r)| \leq C_{m,2}m(r) \qquad \text{for all } r > 0$$

3. there exists $\alpha > 0$ such that

$$r^\alpha m(r) \text{ is non-increasing.}$$

The first condition makes sure that $\mathcal{L}$ is in either critical and super-critical regime. The second one is a smoothness condition. The last one ensure a minimum level of regularity given from $\mathcal{L}$. Now, for $P(|\xi|)$

1. P satisfies the doubling condition: for any $\xi \in \mathbb{R}^2$,

$$P(2|\xi|) \leq c_D P(|\xi|)$$

with constant $c_D \geq 1$;

2. P satisfies the Hormander-Mikhlin condition (see [73]). With N be a positive integer only depending on C_D, for any $\xi \in \mathbb{R}^2$ and for all multi-indices $k \in \mathbb{Z}^n$,

$$|\xi|^{|k|} \, |\partial_\xi^k P(|\xi|)| \leq c_H P(|\xi|)$$

Here the constant $c_H \geq 1$.

3. P has sub-quadratic growth at ∞, i.e.

$$\int_0^1 P(|\xi|^{-1})|\xi|d|\xi| < \infty$$

4. P satisfies

$$(-\Delta)^{\frac{(d+2)}{2}} P(|\xi|) \geq c_H^{-1} P(\xi)|\xi|^{-d-2}$$

   for all $|\xi|$ sufficiently large.

We need to point out that the forth condition for P is a very strong one. However, $P(|\xi|) = \frac{|\xi|}{\log^a(1+|\xi|)}$ with $a \in (0,1)$ is an example satisfies all the conditions. The related $m(r)$ is $|r|^{-1}\log^a(1+\frac{1}{r})$. This sample of $\mathcal{L}$ is usually called logarithmically super-critical dissipation.

2.3 General Besov Space and Bernstein Inequality

Following the logarithmically super-critical dissipation operator in the last part of the previous section, we need to generalize the definition of the Besov Space. In particular, let us consider a class of $\mathcal{L}$, where $P|\xi| = \frac{|\xi|}{a(|\xi|)}$ for some positive increasing function a. This special form is due to the importance of the critical case. Another reason to generalize the Besov space comes from the active equation.

Definition 2.3.1 *For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the generalized Besov spaces $\mathring{B}^{s,a}_{p,q}$ and $B^{s,a}_{p,q}$ are defined through the norms*

$$\|f\|_{\mathring{B}^{s,a}_{p,q}} \equiv \|2^{js}a(2^j)\,\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty,$$

$$\|f\|_{B^{s,a}_{p,q}} \equiv \|2^{js}a(2^j)\,\|\Delta_j f\|_{L^p}\|_{l^q} < \infty. \tag{2.6}$$

To best suit the logarithmically super-critical case, we let $a(x) = a(|x|) : (0,\infty) \rightarrow$

$(0,\infty)$ *be a non-decreasing function satisfying*

$$\lim_{|x|\to\infty} \frac{a(x)}{|x|^\sigma} = 0\,, \quad \forall\, \sigma > 0.$$

Similarly, we can define the space-time Besov spaces

Definition 2.3.2 *For $t > 0$, $s \in \mathbb{R}$ and $1 \le p, q, r \le \infty$, the space-time spaces $\widetilde{L}^r_t \mathring{B}^s_{p,q}$ and $\widetilde{L}^r_t B^s_{p,q}$ are defined through the norms*

$$\|f\|_{\widetilde{L}^r_t \mathring{B}^{s,a}_{p,q}} \equiv \|2^{js} a(2^j) \|\mathring{\Delta}_j f\|_{L^r_t L^p}\|_{l^q},$$

$$\|f\|_{\widetilde{L}^r_t B^{s,a}_{p,q}} \equiv \|2^{js} a(2^j) \|\Delta_j f\|_{L^r_t L^p}\|_{l^q}.$$

The factor 2^j is largely due to the support of each Fourier localization operator. These spaces are related to the classical space-time spaces $L^r_t \mathring{B}^s_{p,q}$, $L^r_t B^{s,\gamma}_{p,q}$, $L^r_t \mathring{B}^{s,a}_{p,q}$ and $L^r_t B^{s,a}_{p,q}$ via the Minkowski inequality.

Then, we focus on the generalized Bernstein inequality related to $\mathcal{L}$. The first lemma is given by Chae, Constantin and Wu in [10].

Lemma 2.1 *Assume that v and ω are related through*

$$v = \mathcal{R}\, Q\, \omega,$$

where $\mathcal{R}$ denotes the standard Riesz transform and Q a Fourier multiplier operator satisfying Condition 1.1 in [10][p.36]. Then, for any integer $j \ge 0$ and $N \ge 0$,

$$\begin{aligned} \|S_N v\|_{L^p} &\le C_p\, Q(C_0 2^N)\, \|S_N \omega\|_{L^p}, \quad 1 < p < \infty, \\ \|\Delta_j v\|_{L^q} &\le C\, Q(C_0 2^j)\, \|\Delta_j \omega\|_{L^q}, \quad 1 \le q \le \infty, \end{aligned}$$

where C_p is a constant depending on p only, C_0 and C are pure constants.

Notice the difference between the range of the indices. The inequality works for the L^1 and L^∞ norm only if the support of the inside function is away from zero. Then, we have two point-wise and Lebesgue-norm estimates associated with $\mathcal{L}$ in its convolution definition. The proof of the versions for Λ^α and the for $\frac{\Lambda}{\log^\alpha(1+\Lambda)}$ can be found in [7] and [18].

Lemma 2.2 *Let $\mathcal{L}$ be the operator defined by (4.2). Then, for $p > 1$,*

$$|f(x)|^{p-2}f(x)(\mathcal{L}f(x)) \geq \frac{1}{p}\mathcal{L}(|f|^p).$$

Proof. By (2.4),

$$\mathcal{L}f(x) = \text{ P.V. } \int \frac{f(x) - f(y)}{|x-y|^d} m(|x-y|)dy$$

and thus

$$|f(x)|^{p-2}f(x)\mathcal{L}f(x) = \text{ P.V. } \int \frac{|f(x)|^p - |f(x)|^{p-2}f(x)f(y)}{|x-y|^d} m(|x-y|)dy.$$

By Young's inequality,

$$|f(x)|^{p-2}f(x)f(y) \leq |f(x)|^{p-1}|f(y)| \leq \frac{p-1}{p}|f(x)|^p + \frac{1}{p}|f(y)|^p$$

Therefore,

$$\begin{aligned}
&|f(x)|^{p-2}f(x)\mathcal{L}f(x) \\
&\quad \geq \frac{1}{p} \text{ P.V. } \int \frac{p|f(x)|^p - (p-1)|f(x)|^p - |f(y)|^p}{|x-y|^d} m(|x-y|)dy \\
&\quad \geq \frac{1}{p}\mathcal{L}(|f|^p).
\end{aligned}$$

This completes the proof of Lemma 2.2. ∎

Lemma 2.3 *Let $\mathcal{L}$ be the operator defined by (2.4). Then, for $p \geq 2$,*

$$\int |f|^{p-2} f(\mathcal{L}f)dx \geq \frac{2}{p} \int \left| \mathcal{L}^{\frac{1}{2}}(|f|^{\frac{p}{2}}) \right|^2 dx.$$

Proof. The $p = 2$ case is trivial. For $p > 2$, let $\beta = \frac{p}{2} - 2$. By Lemma 2.2,

$$\begin{aligned} \int |f|^{p-2} f(\mathcal{L}f)dx &= \int |f|^{\frac{p}{2}} |f|^{\beta} f(\mathcal{L}f)dx \\ &\geq \int |f|^{\frac{p}{2}} \frac{2}{p} (\mathcal{L}(|f|^{\frac{p}{2}})dx \\ &= \frac{2}{p} \int \left| \mathcal{L}^{\frac{1}{2}}(|f|^{\frac{p}{2}}) \right|^2 dx. \end{aligned}$$

This completes the proof of Lemma 2.3. ∎

These two lemmas help us establish a lower bound for the contribution of the dissipation part when we estimate the L^p norm of the solution. Finally, the general Bernstein type inequality is stated as

Lemma 2.4 *Let $j \geq 0$ be an integer and $p \in [2, \infty)$. Let $\mathcal{L}$ be defined by (2.5) and (2.3.1). Then, for any $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$P(2^j) \|\Delta_j f\|^p_{L^p(\mathbb{R}^d)} \leq C \int_{\mathbb{R}^d} |\Delta_j f|^{p-2} \Delta_j f \mathcal{L} \Delta_j f \, dx, \tag{2.7}$$

where C is a constant depending on p and d only.

Proof. The case when $p = 2$ simply follows from Plancherel's theorem. Now we assume $p > 2$. The proof modifies the corresponding ones in [18, 45]. Let $N > 0$ be an integer to be specified later. Clearly,

$$\|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \leq \|S_N \Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} + \|(Id - S_N)\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \equiv I_1 + I_2.$$

By the standard Bernstein inequality 2.1.6, for $s > 0$,

$$I_2 \leq C2^{-Ns} \| |\Delta_j f|^{\frac{p}{2}} \|_{B^{1+s}_{2,2}}.$$

Applying Lemma 3.2 of [18], we have, for $s \in (0, \min(\frac{p}{2} - 1, 2))$,

$$\| |\Delta_j f|^{\frac{p}{2}} \|_{B^{1+s}_{2,2}} \leq C \|\Delta_j f\|^{\frac{p}{2}-1}_{B^0_{p,2}} \|\Delta_j f\|_{B^{1+s}_{p,2}} \leq C2^{j(1+s)} \|\Delta_j f\|^{\frac{p}{2}}_{L^p}.$$

Therefore,

$$I_2 \leq C2^{-Ns} 2^{j(1+s)} \|\Delta_j f\|^{\frac{p}{2}}_{L^p}.$$

By Lemma 2.1,

$$I_1 = \|S_N \Lambda \mathcal{L}^{-\frac{1}{2}} \mathcal{L}^{\frac{1}{2}} (|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \leq C\, 2^N\, (P(2^N))^{-\frac{1}{2}} \, \|\mathcal{L}^{\frac{1}{2}} (|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

Combining the estimates leads to

$$\|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \leq C2^{-Ns} 2^{j(1+s)} \|\Delta_j f\|^{\frac{p}{2}}_{L^p} \, + \, C\, 2^N (P(2^N))^{-\frac{1}{2}} \|\mathcal{L}^{\frac{1}{2}} (|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

By the second part of the Bernstein inequality, for Λ in proposition 2.1.6,

$$2^j \|\Delta_j f\|^{\frac{p}{2}}_{L^p} = 2^j \|(\Delta_j f)^{\frac{p}{2}}\|^{\frac{2}{p}\frac{p}{2}}_{L^2} \leq C\, \|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

Therefore,

$$2^j \|\Delta_j f\|^{\frac{p}{2}}_{L^p} \leq C2^{-Ns} 2^{j(1+s)} \|\Delta_j f\|^{\frac{p}{2}}_{L^p} \, + \, C\, 2^N (P(2^N))^{-\frac{1}{2}} \|\mathcal{L}^{\frac{1}{2}} (|\Delta_j f|^{\frac{p}{2}})\|_{L^2}. \qquad (2.8)$$

We now choose $j < N \leq j + N_0$ with N_0 independent of j such that

$$C\, 2^{-(N-j)s} \leq \frac{1}{2}.$$

(2.7) then follows from (2.8) and lemma 2.3. This completes the proof of Lemma 2.4. ■

2.4 Commutator Estimate

One of the contributions of the paper [37] from Hmidi, Keraani and Rousset is that, they transform the vortex stretching term $\partial_{x_1}\theta$ into a commutator estimate.

Definition 2.4.1 *The commutator is a binary operator. Let f and g be either functionals or a functions, h is a function,*

$$[f, g]h = f[g(h)] - g[f(h)]$$

For example, the product rule for the derivatives can be written in the commutator form $\frac{d}{dt}f \cdot g = \frac{d}{dt}(f \cdot g) - f \cdot \frac{d}{dt}g = [\frac{d}{dt}, f]g$. In [37], we will encounter the commutator $[\mathcal{R}, u \cdot \nabla]\theta$, which denotes $\mathcal{R}(u \cdot \nabla\theta) - u \cdot \nabla(\mathcal{R}\theta)$.

The estimate for some certain norm, e.g. the L^p norm and the Besov norm, needs the following important lemma

Lemma 2.5 *Consider two different cases: $\delta \in (0, 1)$ and $\delta = 1$.*

1. *Let $\delta \in (0, 1)$ and $q \in [1, \infty]$. If $|x|^\delta h \in L^1$, $f \in \mathring{B}^\delta_{q,\infty}$ and $g \in L^\infty$, then*

$$\|h * (fg) - f(h * g)\|_{L^q} \leq C \, \||x|^\delta \phi\|_{L^1} \, \|f\|_{\mathring{B}^\delta_{q,\infty}} \|g\|_{L^\infty}, \tag{2.9}$$

   where C is a constant independent of f, g and h.

2. *Let $\delta = 1$. Let $q \in [1, \infty]$. Let $r_1 \in [1, q]$ and $r_2 \in [1, \infty]$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then*

$$\|h * (fg) - f(h * g)\|_{L^q} \leq C \, \||x|h\|_{L^{r_1}} \, \|\nabla f\|_{L^q} \, \|g\|_{L^{r_2}}, \tag{2.10}$$

Proof. By Minkowski's inequality, for any $p \in [1, \infty]$

$$
\begin{aligned}
\|h*(fg) - f(h*g)\|_{L^p} &= \left[\int | \int h(z)(f(x) - f(x-z))g(x-z)dz|^p dx \right]^{\frac{1}{p}} \\
&\le \int \left[\int |h(z)(f(x) - f(x-z))g(x-z)|^p dx \right]^{\frac{1}{p}} dz \\
&\le \|g\|_{L^\infty} \int |h(z)| \|f(\cdot) - f(\cdot - z)\|_{L^p} dz \\
&\le \|g\|_{L^\infty} \sup_{|z|>0} \frac{\|f(\cdot) - f(\cdot - z)\|_{L^p}}{|z|^\delta} \||z|^\delta |h(z)|\|_{L^1}
\end{aligned}
$$

Notice that the second term in the last row is the norm of $\mathring{B}^{\delta}_{p,\infty}$ ■

The following property gives the bound for the Besov norm of the commutator involving $\mathcal{R}_\alpha$. The similar proof can be given to the cases for L^p norm or the Besov norm with a logarithmic factor.

Proposition 2.4.2 *Let a and $\mathcal{R}_a$ be defined as in (2.3.1). Assume*

$$p \in [2, \infty), \quad q \in [1, \infty], \quad 0 < s < \delta.$$

Let $[\mathcal{R}_a, u]F = \mathcal{R}_a(uF) - u\mathcal{R}_a F$ be a standard commutator. Then

$$\|[\mathcal{R}_a, u]F\|_{B^{s,a}_{p,q}} \le C\,(\|u\|_{\mathring{B}^{\delta}_{p,\infty}} \|F\|_{B^{s-\delta,\,a^2}_{\infty,q}} + \|u\|_{L^2}\,\|F\|_{L^2}),$$

where C denotes a constant independent of a and $\mathcal{R}_a$.

Proof. [Proof of Proposition 2.4.2] Let $j \ge -1$ be an integer. Using the notion of para-products on u and F, we decompose $\Delta_j[\mathcal{R}_a, u]F$ into three parts,

$$\Delta_j[\mathcal{R}_a, u]F \quad = \quad I_1 + I_2 + I_3,$$

where

$$
\begin{aligned}
I_1 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}_a(S_{k-1}u\cdot\Delta_k F) - S_{k-1}u\cdot\mathcal{R}_a\Delta_k F),\\
I_2 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}_a(\Delta_k u\cdot S_{k-1} F) - \Delta_k u\cdot\mathcal{R}_a S_{k-1} F),\\
I_3 &= \sum_{k\ge j-1} \Delta_j(\mathcal{R}_a(\Delta_k u\cdot\tilde{\Delta}_k F) - \Delta_k u\mathcal{R}_a\cdot\tilde{\Delta}_k F).
\end{aligned}
$$

When the operator $\mathcal{R}_a$ acts on a function whose Fourier transform is supported on an annulus, it can be represented as a convolution kernel. Since the Fourier transform of $S_{k-1}u\cdot\Delta_k F$ is supported on an annulus with radius 2^k, we can write

$$
h_k \star (S_{k-1}u\cdot\Delta_k F) - S_{k-1}u\cdot(h_k\star\Delta_k F),
$$

where h_k is given by the inverse Fourier transform of $i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi)$, namely

$$
h_k(x) = \left(i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi)\right)^{\vee}(x).
$$

Here $\widetilde{\Phi}_k(\xi)\in C_0^\infty(\mathbb{R}^2)$, $\widetilde{\Phi}_k(\xi)$ is also supported on an annulus around the radius of 2^k and is identically equal to 1 on the support of $S_{k-1}u\cdot\Delta_k F$. Therefore, recalling the definition of the Besov space, we can write

$$
i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi) = i\frac{\xi_1}{|\xi|}\tilde{\Phi}_0(2^{-k}\xi)\,a(|\xi|).
$$

Therefore,

$$
h_k(x) = 2^{2k}\,h_0(2^k x) * a^{\vee}(x), \quad h_0(x) = \left(\frac{\xi_1}{|\xi|}\tilde{\Phi}_0(\xi)\right)^{\vee}.
$$

By Lemma 2.5,

$$\begin{aligned}\|I_1\|_{L^p} &\le C\,\||x|^\delta h_j\|_{L^1}\|S_{j-1}u\|_{\mathring{B}^\delta_{p,\infty}}\,\|\Delta_j F\|_{L^\infty}\\ &\le C\,2^{-\delta j}\,a(2^j)\,\|S_{j-1}u\|_{\mathring{B}^\delta_{p,\infty}}\,\|\Delta_j F\|_{L^\infty}.\end{aligned}$$

I_2 in L^p can be estimated as follows.

$$\begin{aligned}\|I_2\|_{L^p} &\le C\,2^{-\delta j}\,a(2^j)\,\|S_{j-1}F\|_{L^\infty}\|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}}\\ &\le C\,2^{-\delta j}\,a(2^j)\sum_{m\le j-1}\|\Delta_m F\|_{L^\infty}\,\|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}}\\ &= C\,2^{-sj}a^{-1}(2^j)\sum_{m\le j-1}2^{(s-\delta)(j-m)}\frac{a^2(2^j)}{a^2(2^m)}\,2^{(s-\delta)m}a^2(2^m)\,\|\Delta_m F\|_{L^\infty}\|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}}.\end{aligned}$$

The estimate of $\|I_3\|_{L^p}$ is different. We need to distinguish between low frequency and high frequency terms. For $j=0,1$, the terms in I_3 with $k=-1,0,1$ have Fourier transforms containing the origin in their support and the lower bound part of Bernstein's inequality does not apply. To deal with these low frequency terms, we take advantage of the commutator structure and bound them by Lemma 2.5. The kernel h corresponding to $\mathcal{R}_a$ still satisfies, for any $r_1\in(1,\infty)$,

$$\||x|\,h\|_{L^{r_1}}\le C.$$

Therefore, by Lemma 2.5 and Bernstein's inequality, for $j=0,1$ and $k=-1,0,1$,

$$\begin{aligned}\|\Delta_j(\mathcal{R}_a(\Delta_k u\cdot\tilde{\Delta}_k F)-\Delta_k u\cdot\mathcal{R}_a\tilde{\Delta}_k F)\|_{L^p} &\le C\,\||x|h\|_{L^{r_1}}\,\|\nabla\Delta_k u\|_{L^p}\,\|\Delta_k F\|_{L^{r_2}}\\ &\le C\,\|u\|_{L^2}\,\|F\|_{L^2}.\end{aligned}$$

where $\frac{1}{r_1}+\frac{1}{r_2}=1$. For the high frequency terms, we do not need the commutator structure. By Lemma 2.1 and Hölder's inequality,

$$\begin{aligned}\|I_{31}\|_{L^p} \equiv \left\| \sum_{k\geq j-1} \Delta_j(\mathcal{R}_a(\Delta_k u \cdot \tilde{\Delta}_k F)) \right\|_{L^p} &\leq \sum_{k\geq j-1} C\, a(2^j)\, \|\Delta_k u\|_{L^p}\, \|\Delta_k F\|_{L^\infty} \\ &\leq C\ a(2^j) \sum_{k\geq j-1} 2^{-\delta k}\, 2^{\delta k} \|\Delta_k u\|_{L^p}\, \|\Delta_k F\|_{L^\infty} \\ &\leq C\, 2^{-sj}\, a^{-1}(2^j)\, \|u\|_{\mathring{B}^\delta_{p,\infty}} \sum_{k\geq j-1} 2^{s(j-k)} \frac{a^2(2^j)}{a^2(2^k)}\, 2^{(s-\delta)k} a^2(2^k)\ \|\Delta_k F\|_{L^\infty}.\end{aligned}$$

$I_{32} \equiv \sum_{k\geq j-1} \Delta_k u \cdot \mathcal{R}_a \tilde{\Delta}_k F$ admits the same bound. Therefore, by the definition of the norm in definition 2.3.1,

$$\begin{aligned}\|[\mathcal{R}_a, u]F\|_{B^{s,a}_{p,q}} \ \leq\ & \left[\sum_{j\geq -1} 2^{qsj} a^q(2^j) \|I_1\|^q_{L^p}\right]^{\frac{1}{q}} + \left[\sum_{j\geq -1} 2^{qsj} a^q(2^j) \|I_2\|^q_{L^p}\right]^{\frac{1}{q}} \\ & + \left[\sum_{j\geq -1} 2^{qsj} a^q(2^j) (\|I_{31}\|^q_{L^p} + \|I_{32}\|^q_{L^p})\right]^{\frac{1}{q}} + C\, \|u\|_{L^2}\, \|F\|_{L^2}.\end{aligned}$$

The first term on the right is clearly bounded by

$$C\, \|u\|_{\mathring{B}^\delta_{p,\infty}} \left[\sum_{j\geq -1} 2^{q(s-\delta)j} a^{2q}(2^j) \|\Delta_j F\|^q_{L^\infty}\right]^{\frac{1}{q}} = C\, \|u\|_{\mathring{B}^\delta_{p,\infty}}\, \|F\|_{B^{s-\delta,\, a^2}_{\infty,q}}.$$

Due to $s < \delta$, (2.3.1) and a convolution inequality for series,

$$\left[\sum_{j\geq -1} 2^{qsj} a^q(2^j) \|I_2\|^q_{L^p}\right]^{\frac{1}{q}} \leq C\, \|u\|_{\mathring{B}^\delta_{p,\infty}}\, \|F\|_{B^{s-\delta,\, a^2}_{\infty,q}}.$$

Thanks to $0 < s$, (2.3.1) and a convolution inequality for series,

$$\left[\sum_{j\geq -1} 2^{qsj} a^q(2^j) \|I_{31}\|^q_{L^p}\right]^{\frac{1}{q}} \leq C\, \|u\|_{\mathring{B}^\delta_{p,\infty}}\, \|F\|_{B^{s-\delta,\, a^2}_{\infty,q}}.$$

This completes the proof of Proposition 2.4.2. ■

2.5 Gronwall and Osgood Inequality

In the analysis of ordinary differential equations, we often use the Gronwall inequality to show the boundedness of a function or a certain norm.

Proposition 2.5.1 *Let I denote an interval of the real line. Let $f(t)$ and $g(t)$ be real valued continuous functions defined on I. If f ids differentiable in the interior of I and satisfies the differential inequality*

$$f'(t) \leq g(t)f(t)\,, \quad t \in I$$

then f is bounded by the solution of the corresponding differential equation $y'(t) = g(t)y(t)$

$$f(t) \leq f(a) \exp\left(\int_a^t f(s)dx\right)$$

It can be stated in the integral form also:

Proposition 2.5.2 *Let I be an interval. $f(t)$, $g(t)$ and $h(t)$ are real valued functions defined on I. Assume that $f(t)$ and $g(t)$ are continuous and that the negative part of $h(t)$ is integrable on every closed and bounded subinterval of I. If $g(t)$ is non-negative and if $f(t)$ satisfies the integral inequality*

$$f(t) \leq f(a) + \int_a^t g(s)f(s)ds\,, \quad \forall t \in I,$$

then

$$f(t) \leq h(t) + \int_a^t h(s)g(s) \exp\left(\int_s^t g(\tau)d\tau\right) ds\,, \quad t \in I$$

If, in addition, the function $h(t)$ is non-negative, then

$$f(t) \leq h(t) \exp\left(\int_a^t g(s)ds\right)\,, \quad t \in I$$

The following Osgood inequality can be regarded as an extension of the Gronwall inequality

Proposition 2.5.3 *Let $\alpha(t) > 0$ be a locally integrable function. Assume $\omega(t) \geq 0$ satisfies*

$$\int_0^\infty \frac{1}{\omega(r)} dr = \infty.$$

Suppose that $\rho(t) > 0$ satisfies

$$\rho(t) \leq a + \int_{t_0}^t \alpha(s)\omega(\rho(s))ds$$

for some constant $a \geq 0$. Then if $a = 0$, then $\rho \equiv 0$; if $a > 0$, then

$$-\Omega(\rho(t)) + \Omega(a) \leq \int_{t_0}^t \alpha(\tau)d\tau,$$

where

$$\Omega(x) = \int_x^1 \frac{dr}{\omega(r)}.$$

This inequality is also named Bihari's inequality [4]. When $\omega(r)$ is the identity function, we have the Gronwall inequality. Another example seen in the following chapter is for $\omega(r) = r\log(r)$.

2.6 Frequently Used Inequalities

Other than the Gronwall's and Osgood inequality mentioned in the previous section, we list some other often applied inequalities when estimate certain type of norms.

Hölder Inequality

For $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q} \tag{2.11}$$

Here

$$\|f\|_{L^p} = (\int |f|^p)^{\frac{1}{p}} \quad for \qquad 1 \leq p < \infty$$

is the standard Lebesgue space. It is equal to the essential supreme norm when $p = \infty$. The equation holds if and only if $f = Cg$ almost everywhere for a non-zero constant C. Notice that the coefficient on the right hand side is 1. One extension of the Hölder inequality is that, for $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$$

. The Hölder inequality is a generalization of the Cauchy-Schwarz inequality.

Young's Inequality

This inequality are mostly used to split multiplication between the norms. For $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, let a and b be two non-negative real numbers,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.12}$$

The equation holds if and only if $a^p = b^q$.

Young's Inequality for Convolution

For the convolution defined as

$$(f \star g)(x) = \int f(x-y)g(y)dy \tag{2.13}$$

We have the estimate of the L^r norm of the convolution function

$$\|(f \star g)\|_{L^r} = \|f\|_{L^p}\|g\|_{L^q} \tag{2.14}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

Sobolev Embedding Inequality

For the Sobolev spaces defined as, given $k \in \mathbb{N}$ and $1 \leq p \leq \infty$

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \; \forall \, |\alpha| \leq k\} \tag{2.15}$$

The definition can be developed into Bessel space, which allows non-integer k.

$$H^(k,p)(\mathbb{R}^n) = \left\{ f \in L^p : [(1+|\xi|^2)^{\frac{k}{2}} \hat{f}]^\vee \in L^p \right\}$$

The Sobolev embedding theorem shows that, we can gain some integrability by requiring more on derivatives based on these spaces. Suppose $k > l$ and $1 \leq p < q \leq \infty$ such that $(k-l)p < n$,

$$\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n} \tag{2.16}$$

We have $\|f\|_{W^{l,q}} \leq C\|f\|_{W^{k,p}}$

Brezis-Wainger inequality

The following Brezis-Wainger inequality is used to control the L^∞ norm through $\|\nabla f\|_{L^2}$ and a logarithmic factor. For $p > 2$

$$\|f\|_{L^\infty} \leq C(1 + \|\nabla f\|_{L^2}) \left[1 + \log^+(\|\nabla f\|_{L^p})\right]^{\frac{1}{2}} + C\|f\|_{L^2}$$

GagliardoCNirenberg Interpolation Inequality

This inequality is one of the most important from the interpolation theory. For three function space, if $A \subset B \subset C$, we have $\|\cdot\|_C \leq \|\cdot\|_B \leq \|\cdot\|_A$. The Gagliardo-Nirenberg inequality provides a refined bound for the norm of B with both $\|\cdot\|_A$ and $\|\cdot\|_C$ involved. We require the function spaces to be Sobolev type. For $u : \mathbb{R}^n \to \mathbb{R}$

and fixed $1 \leq q, r \leq \infty$, given two positive number j and m, if p and α satisfies

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\alpha + \frac{1-\alpha}{q} \qquad \frac{j}{m} \leq \alpha \leq 1 \tag{2.17}$$

$$\|D^j u\|_{L^p} \leq C\|D^m u\|_{L^r}^{\alpha} \|u\|_{L^q}^{1-\alpha} \tag{2.18}$$

We can find extend use of this inequality in the sixth chapter.

Calderón-Zygmund Inequality

For the Riesz transform $\mathcal{R}$, we have the relation

$$\|\mathcal{R}f\|_{L^p} \leq \|f\|_{L^p}, \quad for \qquad 1 \leq p < \infty$$

The limitation of this inequality is that it does not work for $p = \infty$. However, when the support of the Fourier transform of f does not contain the origin, the inequality works for the case $p = \infty$.

Minkowski Inequality

The Minkowski inequality can be regarded as a generalization of the triangle inequality. Suppose f is a function of two variable x and y, and $1 \leq p < \infty$

$$\left(\int \left|\int f(x,y)dy\right|^p dx\right)^{\frac{1}{p}} \leq \int \left(\int |f(x,y)dx|^p\right)^{\frac{1}{p}} dy \tag{2.19}$$

Tri-Functional Inequality

In [24] [22], Cao and Wu introduce the following inequality. We can find its application for anisotropic dissipation equations. Let $q \geq 2$. Assume that $f, g, g_y, h_x \in L^2(\mathbb{R}^2)$ and $h \in L^{2(q-1)}(\mathbb{R}^2)$. Then, for some constant C

$$\iint_{\mathbb{R}^2} |f\, g\, h| dxdy \leq C\|f\|_{L^2} \|g\|_{L^2}^{1-\frac{1}{q}} \|g_y\|_{L^2}^{\frac{1}{q}} \|h\|_{L^{2(q-1)}}^{1-\frac{1}{q}} \|h_x\|_{L^2}^{\frac{1}{q}} \tag{2.20}$$

The rest of this paper is organized as follows. Chapter 3 will focus on the Boussinesq system with an active scalar type logarithmically super-critical velocity. We will prove the existence and uniqueness of the solution in the function space $L^p(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ for ω and θ. The regularity result for the 2D Boussinesq system with slightly super-critical dissipation and no diffusion will be given in Chapter 4. The last two chapters will present my work on the other two interesting models. For the SQG equations, we found the small data wellposedness and an eventual regularity result by introducing the CRH and $OSSm$ condition. For a n-dimensional axi-symmetric Navier-Stokes model firstly introduced by Hou, Li and Lei, we prove that the solution will remain bounded for $u_1 \in H^1(\mathbb{R}^n)$, $\omega_1 \in L^2(\mathbb{R}^n)$ and $\psi_1 \in H^2(\mathbb{R}^n)$ in the case we have a strong enough fractional Laplacian dissipation operator.

CHAPTER 3

2D Boussinesq equations with supercritical velocity

3.1 Introduction

In this chapter, we will study the global in time regularity problem of the generalized Euler-Boussinesq equations:

$$\begin{cases} \partial_t v + u \cdot \nabla v - \sum_{j=1}^{2} u_j \nabla v_j = -\nabla p + \theta \mathbf{e}_2, \\ \nabla \cdot v = 0, \quad u = \Lambda^\sigma P(\Lambda) v, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, \end{cases} \tag{3.1}$$

Here, $P(\Lambda)$ is defined through the Fourier transform. When $\delta = 0$ and $P(\xi) = 1$, we have the critical Euler-Boussinesq system studied in Hmidi, Keraani and Russell [38]. When $\delta > 0$ or $P(\xi)$ is an unbounded function, we get into the super-critical regime. This can be more easily explained by the vorticity form of the equations

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_{x_1} \theta, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \Lambda^\sigma P(\Lambda) \omega, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, \end{cases} \tag{3.2}$$

where $\omega = \omega(x,t)$ is a scalar function and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. It is easy to find that, the major difference between the above system and the original Boussinesq equations

1.5 is that the ω is not defined as the curl of the velocity u. Rather than that, we start with ω and inversely define the velocity as

$$u = \nabla^{\perp}\Delta^{-1}\Lambda^{\sigma}P(\Lambda)\omega.$$

It explains the reason we have the vector field v instead of u in 3.1. At the same time, u is more singular than the original Boussinesq when compared with ω. One instant consequence of this change is that, when switching between the norm of ∇u and that of ω, it requires an extra factor, which will cause trouble in some cases. This will be handled by an general Bernstein's inequality as shown in the following sections. To achieve this approach, we may put the following conditions on $P(\xi)$

Condition 3.1.1 *The symbol $P(|\xi|)$ assumes the following properties:*

1. *P is continuous on $\mathbb{R}^2$ and $P \in C^\infty(\mathbb{R}^2 \setminus \{0\})$;*
2. *P is radially symmetric;*
3. *$P = P(|\xi|)$ is nondecreasing in $|\xi|$;*
4. *There exist two constants C and C_0 such that*

$$\sup_{2^{-1}\le|\eta|\le 2} \left|(I-\Delta_\eta)^n \, P(2^j|\eta|)\right| \le C\, P(C_0\, 2^j)$$

for any integer j and $n = 1, 2$.

We point out that the forth condition is a special example of the Hörmander-Mihlin condition, which is satisfied by a wide range of functions.

The goal of this chapter is to prove the following theorem

Theorem 3.1.2 *Let $\sigma = 0$. Assume the symbol $P(|\xi|)$ obeys Condition 3.1.1 and*

$$P(2^k) \le C\sqrt{k} \quad \text{for a constant } C \text{ and any large integer } k > 0, \tag{3.3}$$

$$\int_1^\infty \frac{1}{r\log(1+r)\,P(r)}dr = \infty. \tag{3.4}$$

Let $q > 2$ and let $s > 2$. Consider the IVP (3.2) and the initial data $\omega(x,0) = \omega_0(x)$, $\theta(x,0) = \theta_0(x)$ with $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Then the IVP (3.2) has a unique global solution (ω,θ) satisfying, for any $T > 0$ and $t \leq T$,

$$\omega \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2)), \quad \theta \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2) \cap L^1([0,T]; B^{s+1}_{q,\infty}(\mathbb{R}^2)). \tag{3.5}$$

An example of $P(|\xi|)$ that satisfies the 3.1.1 is the double logarithmic function

$$P(|\xi|) = (\log(1+\log(1+|\xi|)))^\gamma\,, \quad \gamma \in [0.1] \tag{3.6}$$

The major difficulty of the proof is the same as that mentioned in the first chapter, that is the vortex stretching term $\partial_{x_1}\theta$. The idea is to turn the control of the vortex stretching term into a commutator estimate. By defining

$$G = \omega + \mathcal{R}\theta \qquad \mathcal{R} = \Lambda^{-1}\partial_{x_1} \tag{3.7}$$

and taking $\mathcal{R}$ on the θ equation, we have

$$\partial_t G + u\cdot\ \nabla G = -[\mathcal{R}, u\cdot\nabla]\theta, \tag{3.8}$$

The estimate of the right hand side will use the technique in 2.4. For the proof of the theorem, we will first establish a bound on $\|\omega\|_{L^q}$, $\|\theta\|_{B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L^\infty}$. The bound of the $B^s_{q,\infty}$ norm is found in two steps. Firstly we consider the case $\frac{2}{q} < s < 1$. Then, we extend the range to $1 < s < 2 - \frac{2}{q}$. The case for an arbitrary value of s can be solved by iterating the second step.

3.2 Global *a priori* bounds for $\|\omega\|_{L^\infty_t L^q}$, $\|\theta\|_{L^1_t B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L^\infty_t L^\infty}$

This section establishes global bounds for $\|\omega\|_{L^\infty_t L^p}$, $\|\theta\|_{L^1_t B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L^\infty_t L^\infty}$. The first two can be bounded spontaneously. We first give two lemmas that showing the relation between these two quantities. Notice that, though the main theorem of this chapter 3.1.2 requires $\sigma = 0$, these two lemmas allow $\sigma \in [0,1)$.

Lemma 3.1 *Let $\sigma \in [0,1)$. Assume that the symbol P satisfies Condition 3.1.1 and (3.18). Let (ω, θ) be a smooth solution of (3.2). Then, for any $q \in [2,\infty)$ and for any $t > 0$,*

$$\|\omega(t)\|_{L^q} \le C\left(\|\omega_0\|_{L^q} + \|\theta_0\|_{L^q}\right) e^{C\,t\|\theta_0\|_{L^q}}\, e^{C\int_0^t \|\theta(\tau)\|_{B^{\sigma,P}_{\infty,2}}\, d\tau}, \tag{3.9}$$

where C's are pure constants.

Proof. [Proof of Lemma 3.1] We start with the equations satisfied by G and $\mathcal{R}\theta$,

$$\begin{aligned} &\partial_t G + u\cdot\ \nabla G = -[\mathcal{R}, u\cdot\nabla]\theta, \\ &\partial_t \mathcal{R}\theta + u\cdot\nabla\mathcal{R}\theta + \Lambda\mathcal{R}\theta = -[\mathcal{R}, u\cdot\nabla]\theta. \end{aligned} \tag{3.10}$$

By the embedding $B^0_{q,2} \hookrightarrow L^q$ for $q \ge 2$ and Lemma 2.4.2,

$$\begin{aligned} \|\omega(t)\|_{L^q} &\le \|G_0\|_{L^q} + \|\mathcal{R}\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}, u\cdot\nabla]\theta\|_{L^q} d\tau \\ &\le \|G_0\|_{L^q} + \|\mathcal{R}\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{q,2}} d\tau \\ &\le \|G_0\|_{L^q} + \|\theta_0\|_{L^q} + C\int_0^t \left[\|\omega(\tau)\|_{L^q}(\|\theta(\tau)\|_{B^{\sigma,P}_{\infty,2}} + \|\theta_0\|_{L^q})\right] d\tau, \end{aligned}$$

which implies (3.9), by Gronwall's inequality. ■

Lemma 3.2 *Let $\sigma \in [0,1)$. Assume that the symbol P satisfies Condition 3.1.1 and (3.18). Let $q \in (1,\infty)$. Then, any smooth solution (ω, θ) solving (3.12) satisfies, for* *each integer $j \ge 0$,*

$$2^{j(1-\sigma)}\|\Delta_j\theta\|_{L^1_tL^q} \le 2^{-j\sigma}\|\Delta_j\theta_0\|_{L^q} + C\,P(2^j)\,\|\theta_0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^q}d\tau, \qquad (3.11)$$

where C is a pure constant.

Proof. [Proof of Lemma 3.2] We will makes use of the dissipation in the θ-equation,

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0, \\ u = \nabla^\perp\psi, \quad \Delta\psi = \Lambda^\sigma P(\Lambda)\omega, \\ \theta(x,0) = \theta_0(x). \end{cases} \qquad (3.12)$$

Letting $j \ge 0$ and applying Δ_j to (3.12), multiplying by $\Delta_j\theta|\Delta_j\theta|^{q-2}$ and integrating over $\mathbb{R}^2$, we obtain, after integrating by parts,

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j\theta\|_{L^q}^q + \int \Delta_j\theta|\Delta_j\theta|^{q-2}\Lambda\Delta_j\theta\,dx = -\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Delta_j(u\cdot\nabla\theta)\,dx.$$

Due to the lower bound (see, e.g., [18, 80])

$$\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Lambda\Delta_j\theta\,dx \ge C2^j\|\Delta_j\theta\|_{L^q}^q$$

and the decomposition of $[\Delta_j, u\cdot\nabla]\theta$ into five parts,

$$\Delta_j(u\cdot\nabla\theta) = J_1 + J_2 + J_3 + J_4 + J_5$$

with

$$
\begin{aligned}
J_1 &= \sum_{|j-k|\le 2} [\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k\theta,\\
J_2 &= \sum_{|j-k|\le 2} (S_{k-1}u - S_ju)\cdot\nabla\Delta_j\Delta_k\theta,\\
J_3 &= S_ju\cdot\nabla\Delta_j\theta,\\
J_4 &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta),\\
J_5 &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\theta),
\end{aligned}
$$

we obtain, by Hölder's inequality,

$$
\frac{1}{q}\frac{d}{dt}\|\Delta_j\theta\|_{L^q}^q + C2^j\|\Delta_j\theta\|_{L^q}^q \le \|\Delta_j\theta\|_{L^q}^{q-1}\left(\|J_1\|_{L^q}+\|J_2\|_{L^q}+\|J_4\|_{L^q}+\|J_5\|_{L^q}\right).
$$

The integral involving J_3 becomes zero due to the divergence-free condition $\nabla\cdot S_ju = 0$. The terms on the right can be bounded as follows. To bound $\|J_1\|_{L^q}$, we write $[\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k\theta$ as an integral,

$$
[\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k\theta = \int \Phi_j(x-y)(S_{k-1}u(y)-S_{k-1}u(x))\cdot\nabla\Delta_k\theta(y)dy,
$$

where Φ_j is the kernel associated with the operator Δ_j (see the Appendix for more details). By the commutator estimate and the inequality

$$
\|\Phi_j(x)|x|^{1-\sigma}\|_{L^1} \le 2^{-j(1-\sigma)}\,\|\Phi_0(x)|x|^{1-\sigma}\|_{L^1} \le C\,2^{-j(1-\sigma)},
$$

we have

$$\begin{aligned}
\|J_1\|_{L^q} &\le \sum_{|j-k|\le 2} \|\Phi_j(x)|x|^{1-\sigma}\|_{L^1}\, \|S_{k-1}u\|_{B^{1-\sigma}_{q,\infty}}\, \|\nabla\Delta_k\theta\|_{L^\infty}\\
&\le C \sum_{|j-k|\le 2} 2^{-j(1-\sigma)}\, \|S_{k-1}u\|_{\mathring{B}^{1-\sigma}_{q,\infty}}\, 2^k \|\Delta_k\theta\|_{L^\infty}.
\end{aligned}$$

Recalling that $\Lambda^{1-\sigma}u = \nabla^\perp\Delta^{-1}\Lambda P(\Lambda)\omega$ and applying Lemma 2.1, we obtain

$$\|S_{k-1}u\|_{\mathring{B}^{1-\sigma}_{q,\infty}} \le C\,\|\Lambda^{1-\sigma}S_{k-1}u\|_{L^q} \le C\,P(2^j)\,\|S_{k-1}\omega\|_{L^q} \le C\,P(2^j)\,\|\omega\|_{L^q}.$$

Therefore,

$$\|J_1\|_{L^q} \le C2^{j\sigma}\,P(2^j)\,\|\omega\|_{L^q}\|\Delta_j\theta\|_{L^\infty}.$$

By Bernstein's inequality,

$$\begin{aligned}
\|J_2\|_{L^q} &\le \sum_{|j-k|\le 2} \|S_j u - S_{k-1}u\|_{L^q}\, \|\nabla\Delta_j\theta\|_{L^\infty} \le C\|\Delta_j u\|_{L^q}2^j\|\Delta_j\theta\|_{L^\infty}\\
&\le C\,\|\nabla\Delta_j u\|_{L^q}\|\Delta_j\theta\|_{L^\infty}\\
&\le C2^{j\sigma}\,P(2^j)\,\|\Delta_j\omega\|_{L^q}\|\Delta_j\theta\|_{L^\infty}.
\end{aligned}$$

We remark that we have applied the lower bound part of Bernstein's inequality in the second inequality above. This is valid for $j \ge 0$. Similarly,

$$\begin{aligned}
\|J_4\|_{L^q} &\le C\,\|\Delta_j u\|_{L^q}\|\nabla S_{j-1}\theta\|_{L^\infty} \le C\,\|\Delta_j u\|_{L^q}2^j\|S_j\theta\|_{L^\infty}\\
&\le C\,\|\nabla\Delta_j u\|_{L^q}\|\theta\|_{L^\infty} \le C\,2^{j\sigma}\,P(2^j)\,\|\Delta_j\omega\|_{L^q}\|\theta\|_{L^\infty}.
\end{aligned}$$

Thanks to $\sigma \in [0,1)$ and the condition on P in (3.18),

$$\begin{aligned}
\|J_5\|_{L^q} &\le C \sum_{k\ge j-1} 2^j \|\Delta_k u\|_{L^q} \|\widetilde{\Delta}_k \theta\|_{L^\infty} \\
&\le C \sum_{k\ge j-1} 2^{j-k} \|\nabla \Delta_k u\|_{L^q} \, \|\Delta_k \theta\|_{L^\infty} \\
&\le 2^{j\sigma} \sum_{k\ge j-1} 2^{(j-k)(1-\sigma)} \, P(2^k) \, \|\Delta_k \omega\|_{L^q} \|\Delta_k \theta\|_{L^\infty} \\
&\le C 2^{j\sigma} \, P(2^j) \, \|\omega\|_{L^q} \|\theta\|_{L^\infty}.
\end{aligned}$$

Collecting the estimates above, we obtain

$$\frac{d}{dt} \|\Delta_j \theta\|_{L^q} + C\, 2^j \|\Delta_j \theta\|_{L^q} \le C 2^{j\sigma} \, P(2^j) \, \|\omega\|_{L^q} \|\theta_0\|_{L^\infty}.$$

Integrating with respect to time yields

$$\|\Delta_j \theta(t)\|_{L^q} \le e^{-C 2^j t} \, \|\Delta_j \theta_0\|_{L^q} + C 2^{j\sigma} \, P(2^j) \, \|\theta_0\|_{L^\infty} \int_0^t e^{-C 2^j (t-\tau)} \, \|\omega(\tau)\|_{L^q} \, d\tau.$$

We further take the L^1-norm in time to obtain

$$2^j \|\Delta_j \theta\|_{L^1_t L^q} \le \|\Delta_j \theta_0\|_{L^q} + C 2^{j\sigma} \, P(2^j) \, \|\theta_0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^q} d\tau,$$

which is the desired result. This completes the proof of Lemma 3.2. ■

Now, with disposal of the above two lemmas, we turn to the main result of this section.

Proposition 3.2.1 *Let $\sigma = 0$. Assume the symbol P satisfies Condition 3.1.1 and (3.3). Let (ω, θ) be a smooth solution of (3.2) with $\omega_0 \in B^s_{q,\infty}$ and $\theta_0 \in B^s_{q,\infty}$. Then, for any $T > 0$ and $0 < t \le T$,*

$$\|\omega(t)\|_{L^q} \le C(T), \qquad \|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \le C(T), \qquad \|\omega(t)\|_{L^\infty} \le C(T)$$

for some constant C depending T and the initial norms of ω_0 and θ_0.

Proof. [Proof of Proposition 3.2.1] The proof uses the bounds in Lemmas 3.1 and 3.2 with $\sigma = 0$. By the definition of $B^{0,P}_{\infty,2}$ and the embedding $B^{0,P}_{\infty,1} \hookrightarrow B^{0,P}_{\infty,2}$,

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \leq \int_0^t \left[\sum_{j=-1}^{N-1} (P(2^j))^2 \|\Delta_j\theta\|^2_{L^\infty}\right]^{\frac{1}{2}} d\tau + \int_0^t \sum_{j=N}^{\infty} P(2^j)\|\Delta_j\theta\|_{L^\infty}\, d\tau.$$

Thanks to the condition on P in (3.3),

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \leq t\|\theta_0\|_{L^\infty} N + \sum_{j\geq N} P(2^j)\|\Delta_j\theta\|_{L^1_t L^\infty}. \tag{3.13}$$

Since $q \in (2,\infty)$ and P satisfies (3.3), we choose $\epsilon > 0$ such that

$$-1 + \epsilon + \frac{2}{q} < 0, \qquad (P(2^j))^2\, 2^{-j\epsilon} \leq 1.$$

By Bernstein's inequality and Lemma 3.2 with $\sigma = 0$,

$$\begin{aligned}
\sum_{j\geq N} P(2^j)\, \|\Delta_j\theta\|_{L^1_t L^\infty} &\leq \sum_{j\geq N} P(2^j)\, 2^{j\frac{2}{q}} \|\Delta_j\theta\|_{L^1_t L^q} \\
&\leq C \sum_{j\geq N} (P(2^j))^2\, 2^{j(\frac{2}{q}-1)} (\|\theta_0\|_{L^q} + \|\theta_0\|_{L^\infty}\|\omega\|_{L^1_t L^q}) \\
&\leq C \sum_{j\geq N} 2^{j(\frac{2}{q}+\epsilon-1)} (\|\theta_0\|_{L^q} + \|\theta_0\|_{L^\infty}\|\omega\|_{L^1_t L^q}) \\
&\leq C\, \|\theta_0\|_{L^q} + C\, 2^{N(-1+\epsilon+\frac{2}{q})} \|\theta_0\|_{L^\infty}\|\omega\|_{L^1_t L^q}.
\end{aligned}$$

Inserting the estimates above in (3.13) and choosing N to be the largest integer satisfying

$$N \leq \frac{\log(1 + \|\omega\|_{L^1_t L^q})}{(1 - \epsilon - \frac{2}{q})} + 1$$

leads to

$$\|\theta\|_{L_t^1 B^{0,P}_{\infty,2}} \le C\,\|\theta_0\|_{L^\infty\cap L^q} + C\,\|\theta_0\|_{L^\infty} t\;\log\left(1+\int_0^t \|\omega(\tau)\|_{L^q}\,d\tau\right).$$

It then follows from this estimate and (3.9) with $\sigma = 0$ that

$$\|\theta\|_{L_t^1 B^{0,P}_{\infty,2}} \le C\,t\log(1+C\,t) + C\,t\|\theta\|_{L_t^1 B^{0,P}_{\infty,2}}, \tag{3.14}$$

where C's are constants depending on $\|\theta_0\|_{L^q}$ and $\|\theta_0\|_{L^\infty}$. This inequality allows us to conclude that, for any $T > 0$ and $t \le T$,

$$\|\theta\|_{L_t^1 B^{0,P}_{\infty,2}} \le C(T, \|\omega_0\|_{L^q}, \|\theta_0\|_{L^q\cap L^\infty}). \tag{3.15}$$

In fact, (3.15) is first obtained on a finite-time interval and the global bound is then obtained through an iterative process. Finally we prove the global bound for $\|\omega\|_{L^\infty}$. By (3.11) with $\sigma = 0$ and (3.3), we have, for any integer $j \ge 0$ and any $\epsilon > 0$,

$$2^{j(1-\epsilon)}\|\Delta_j\theta\|_{L_t^1 L^q} \le \|\theta_0\|_{L^q} + C\|\theta_0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^q} d\tau \le C(T). \tag{3.16}$$

Since $q \in (2,\infty)$, we can choose $\epsilon > 0$ such that

$$2\epsilon + \frac{2}{q} - 1 < 0.$$

By Bernstein's inequality,

$$\|\theta\|_{B^{\epsilon}_{\infty,1}} \le \sum_{j\ge -1} 2^{(2\epsilon+\frac{2}{q}-1)j} 2^{(1-\epsilon)j}\|\Delta_j\theta\|_{L^q} \le C\,\sup_{j\ge -1} 2^{j(1-\epsilon)}\|\Delta_j\theta\|_{L^q}.$$

It then follows from (3.16) that, for any $t \le T$,

$$\|\theta\|_{L^1_t B^{\epsilon}_{\infty,1}} \le C(T). \tag{3.17}$$

Starting with the equations of G and $\mathcal{R}\theta$, namely (3.10), and applying Lemma 2.4.2, we have, for any $\epsilon > 0$,

$$\begin{aligned}
\|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty} &\le \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} + 2\int_0^t \|[\mathcal{R}, \ u\cdot\nabla]\theta\|_{B^0_{\infty,1}} d\tau \\
&\le \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} \\
&\quad + \int_0^t ((\|\omega\|_{L^q} + \|\omega\|_{L^\infty})\|\theta\|_{B^{\epsilon}_{\infty,1}} + \|\omega\|_{L^q}\|\theta\|_{L^q}) d\tau \\
&\le \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} + \int_0^t (\|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty})\|\theta\|_{B^{\epsilon}_{\infty,1}} d\tau \\
&\quad + \int_0^t (\|\omega\|_{L^q}\|\theta\|_{B^{\epsilon}_{\infty,1}} + \|\omega\|_{L^q}\|\theta\|_{L^q})\, d\tau.
\end{aligned}$$

By Gronwall's inequality, (3.17) and the global bound for $\|\omega\|_{L^q}$, we have

$$\|\omega\|_{L^\infty} \le \|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty} \le C(T).$$

This completes the proof of Proposition 3.2.1. ■

3.3 Global Bound for $\|(\omega, \theta)\|_{B^s_{q,\infty}}$

Before the proof of the bound, we state a logarithmic type interpolation inequality that bounds $\|\nabla u\|_{L^\infty}$.

Proposition 3.3.1 *Assume that the symbol Q satisfies Condition 3.1.1 and (3.3). Let u and ω be related through*

$$u = \nabla^\perp \Delta^{-1} Q(\Lambda)\omega.$$

Then, for any $1 \le q \le \infty$, $\beta > 2/q$, and $1 < p < \infty$,

$$\|\nabla u\|_{L^\infty} \le C\,(1 + \|\omega\|_{L^p}) + C\,\|\omega\|_{L^\infty}\,\log(1 + \|\omega\|_{B^\beta_{q,\infty}})\,Q\left(\|\omega\|^{\frac{2q}{q\beta-2}}_{B^\beta_{q,\infty}}\right),$$

where C's are constants that depend on p, q and β only.

Proof. [Proof of Proposition 3.3.1] For any integer $N \ge 0$, we have

$$\|\nabla u\|_{L^\infty} \le \|\Delta_{-1}\nabla u\|_{L^\infty} + \sum_{k=0}^{N-1} \|\Delta_k \nabla u\|_{L^\infty} + \sum_{k=N}^{\infty} \|\Delta_k \nabla u\|_{L^\infty}.$$

By Bernstein's inequality and Lemma 2.1, we have

$$\|\nabla u\|_{L^\infty} \;\; \le \;\; C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + C\,\sum_{k=N}^{\infty} (2^k)^{\frac{2}{q}}\,\|\nabla \Delta_k u\|_{L^q}.$$

By Lemma 2.1,

$$\|\nabla u\|_{L^\infty} \;\; \le \;\; C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + \,C_d\,\sum_{k=N}^{\infty} (2^k)^{\frac{2}{q}}\,Q(2^k)\|\Delta_k \omega\|_{L^q}.$$

By the definition of Besov space $B^\beta_{q,\infty}$,

$$\|\Delta_k \omega\|_{L^q} \le 2^{-\beta\,k}\|\omega\|_{B^\beta_{q,\infty}}.$$

Therefore,

$$\|\nabla u\|_{L^\infty} \;\; \le \;\; C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + \,C\,\|\omega\|_{B^\beta_{q,\infty}}\,\sum_{k=N}^{\infty} (2^k)^{(\frac{2}{q}-\beta)}\,Q(2^k).$$

Due to $\frac{2}{q} - \beta < 0$ and (3.3), we can choose $\epsilon > 0$ such that

$$\epsilon + \frac{2}{q} - \beta < 0 \quad \text{and} \quad Q(2^N) \le 2^{\epsilon N}.$$

Especially, we take $\epsilon = \frac{1}{2}(\beta - \frac{2}{q})$ to get

$$\|\nabla u\|_{L^\infty} \leq C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + C\,\|\omega\|_{B^\beta_{q,\infty}}\,(2^N)^{(\frac{1}{q}-\frac{\beta}{2})}.$$

If we choose N to be the largest integer satisfying

$$N \leq \frac{1}{\frac{\beta}{2} - \frac{1}{q}} \log_2 \left(1 + \|\omega\|_{B^\beta_{q,\infty}}\right),$$

we then obtain the desired result in Proposition 3.3.1. ■

Also, we restate the commutator estimate proposition 2.4.2 with proper context of this chapter. The proof is similar to the one given in chapter 2.

Proposition 3.3.2 *Let $\mathcal{R} = \Lambda^{-1}\partial_{x_1}$ denote the Riesz transform. Assume that the symbol P satisfies Condition 3.1.1 and*

$$\text{for any } \epsilon > 0, \quad \lim_{|\xi|\to\infty} \frac{P(|\xi|)}{|\xi|^\epsilon} = 0. \tag{3.18}$$

Assume that u and ω are related by

$$u = \nabla^\perp \Delta^{-1} \Lambda^\sigma P(\Lambda)\omega$$

with $\sigma \in [0,1)$. Then, for any $p \in (1,\infty)$ and $r \in [1,\infty]$,

$$\|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{p,r}} \leq C\|\omega\|_{L^p}\|\theta\|_{B^{\sigma,P}_{\infty,r}} + C\,\|\omega\|_{L^p}\,\|\theta\|_{L^p} \tag{3.19}$$

and, for any $r \in [1,\infty]$, $q \in (1,\infty)$ and any $\epsilon > 0$,

$$\|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{\infty,r}} \leq C(\|\omega\|_{L^p} + \|\omega\|_{L^\infty})\|\theta\|_{B^{\sigma+\epsilon}_{\infty,r}} + C\,\|\omega\|_{L^q}\,\|\theta\|_{L^q} \tag{3.20}$$

for some constant C, where the generalized Besov space $B^{\sigma,P}_{\infty,r}$ with P being the symbol of the operator P is defined by

$$\|f\|_{\mathring{B}^{s,P}_{p,q}} \equiv \|2^{js}P(2^j)\,\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty,$$

$$\|f\|_{B^{s,P}_{p,q}} \equiv \|2^{js}P(2^j)\,\|\Delta_j f\|_{L^p}\|_{l^q} < \infty. \tag{3.21}$$

Proposition 3.3.3 *Assume that $\sigma = 0$ and the symbol $P(|\xi|)$ obeys Condition 3.1.1, (3.3) and (3.4). Let $q > 2$ and let $s > 2$. Consider the IVP (3.2) with $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Let (ω, θ) be a smooth solution of (3.2). Then (ω, θ) admits a global a priori bound. More precisely, for any $T > 0$ and $t \le T$,*

$$\|(\omega(t), \theta(t))\|_{B^s_{q,\infty}} \le C(s, q, T, \|(\omega_0, \theta_0)\|_{B^s_{q,\infty}}),$$

where C is a constant depending on s, q, T and the initial norm.

Proof. [Proof of Proposition 3.3.3] The proof is divided into two main steps. The first step provides bounds for $\|\omega\|_{B^\beta_{q,\infty}}$ and $\|\theta\||_{B^\beta_{q,\infty}}$ for β in the range $\frac{2}{q} < \beta < 1$ while the second step proves the global bounds for $\|\omega\|_{B^{\beta_1}_{q,\infty}}$ and $\|\theta\||_{B^{\beta_1}_{q,\infty}}$ for $1 \le \beta_1 < 2 - \frac{2}{q}$. The desired bounds in $B^s_{q,\infty}$ with $s > 2$ can be obtained by a repetition of the second step.

Let $j \ge -1$ be an integer. Applying Δ_j to the equation of G, namely (3.10), multiplying by $\Delta_j G|\Delta_j G|^{q-2}$ and integrating over $\mathbb{R}^2$, we obtain, after integrating by parts,

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\Delta_j G\|^q_{L^q} &= -\int \Delta_j G|\Delta_j G|^{q-2}\Delta_j(u\cdot\nabla G)\,dx \\ &\quad -\int \Delta_j[\mathcal{R}, u\cdot\nabla]\theta\,\Delta_j G|\Delta_j G|^{q-2}\,dx.\end{aligned}$$

Following the notion of paraproducts, we decompose $\Delta_j(u \cdot \nabla G)$ into five parts,

$$\Delta_j(u \cdot \nabla G) = J_1 + J_2 + J_3 + J_4 + J_5$$

with

$$\begin{aligned}
J_1 &= \sum_{|j-k|\le 2} [\Delta_j,\, S_{k-1}u \cdot \nabla]\Delta_k G, \\
J_2 &= \sum_{|j-k|\le 2} (S_{k-1}u - S_j u) \cdot \nabla \Delta_j \Delta_k G, \\
J_3 &= S_j u \cdot \nabla \Delta_j G, \\
J_4 &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u \cdot \nabla S_{k-1} G), \\
J_5 &= \sum_{k\ge j-1} \Delta_j(\Delta_k u \cdot \nabla \widetilde{\Delta}_k G).
\end{aligned}$$

By Hölder's inequality,

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j G\|_{L^q}^q \le \|\Delta_j G\|_{L^q}^{q-1} \left(\|J_1\|_{L^q} + \|J_2\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q} + \|J_6\|_{L^q}\right),$$

where $J_6 = \Delta_j[\mathcal{R}, u \cdot \nabla]\theta$. The integral involving J_3 becomes zero due to the divergence-free condition $\nabla \cdot S_j u = 0$. The terms on the right can be bounded as follows. To bound $\|J_1\|_{L^q}$, we write $[\Delta_j,\, S_{k-1}u \cdot \nabla]\Delta_k G$ as an integral,

$$[\Delta_j,\, S_{k-1}u \cdot \nabla]\Delta_k G = \int \Phi_j(x-y)(S_{k-1}u(y) - S_{k-1}u(x)) \cdot \nabla \Delta_k G(y)dy,$$

where Φ_j is the kernel associated with the operator Δ_j (see the Appendix for more details). By a standard commutator estimate (see, e.g., [19, p.39], [80, p.814-815]),

$$\|J_1\|_{L^q} \le C \sum_{|j-k|\le 2} \|\nabla S_{k-1}u\|_{L^\infty} \|\Delta_k G\|_{L^q}.$$

By Hölder's and Bernstein's inequalities,

$$\|J_2\|_{L^q} \leq C\,\|\nabla\widetilde{\Delta}_j u\|_{L^\infty}\,\|\Delta_j G\|_{L^q}.$$

We have especially applied the lower bound part in Bernstein's inequalities (see Proposition 2.1.6). The purpose is to shift the derivative ∇ from G to u. It is worth pointing out that the lower bound does not apply when $j=-1$. In the case when $j=-1$, J_2 involves only low modes and there is no need to shift the derivative from G to u. J_2 is bounded differently. When $j=-1$, J_2 becomes

$$J_2 = -S_0(u)\cdot\nabla\Delta_1\Delta_{-1}G = -\Delta_{-1}u\cdot\nabla\Delta_1\Delta_{-1}G,$$

whose L^q-norm can be bounded by

$$\|J_2\|_{L^q} \leq C\|\Delta_{-1}u\|_{L^\infty}\,\|\Delta_{-1}G\|_{L^q} \leq C\|\omega\|_{L^q}\,\|G\|_{L^q}.$$

For J_4 and J_5, we have, by Bernstein's inequality,

$$\begin{aligned}
\|J_4\|_{L^q} &\leq C\sum_{|j-k|\leq 2}\|\Delta_k u\|_{L^\infty}\,\|\nabla S_{k-1}G\|_{L^q} \\
&\leq C\sum_{|j-k|\leq 2}\|\nabla\Delta_k u\|_{L^\infty}\sum_{m\leq k-1}2^{m-k}\|\Delta_m G\|_{L^q}, \\
\|J_5\|_{L^q} &\leq C\sum_{k\geq j-1}2^j\,\|\Delta_k u\|_{L^\infty}\|\widetilde{\Delta}_k G\|_{L^q} \\
&\leq C\sum_{k\geq j-1}2^{j-k}\,\|\nabla\Delta_k u\|_{L^\infty}\,\|\widetilde{\Delta}_k G\|_{L^q}.
\end{aligned}$$

Furthermore, for any $\beta \in \mathbb{R}$,

$$\begin{aligned} \|J_1\|_{L^q} &\le C \sum_{|j-k|\le 2} \|\nabla u\|_{L^\infty} 2^{-\beta(k+1)}\, 2^{\beta(k+1)} \|\Delta_k G\|_{L^q} && (3.22)\\ &\le C\, 2^{-\beta(j+1)}\, \|G\|_{B^\beta_{q,\infty}}\, \|\nabla u\|_{L^\infty} \sum_{|j-k|\le 2} 2^{\beta(j-k)} && (3.23)\\ &\le C\, 2^{-\beta(j+1)}\, \|G\|_{B^\beta_{q,\infty}}\, \|\nabla u\|_{L^\infty}, && (3.24)\end{aligned}$$

where C is a constant depending on β only. It is clear that $\|J_2\|_{L^q}$ admits the same bound. For any $\beta < 1$, we have

$$\begin{aligned} \|J_4\|_{L^q} &\le C\, \|\nabla u\|_{L^\infty} \sum_{|j-k|\le 2} \sum_{m<k-1} 2^{m-k}\, 2^{-\beta(m+1)}\, 2^{\beta(m+1)}\, \|\Delta_m G\|_{L^q}\\ &\le C\, \|\nabla u\|_{L^\infty}\, \|G\|_{B^\beta_{q,\infty}} \sum_{|j-k|\le 2} \sum_{m<k-1} 2^{m-k}\, 2^{-\beta(m+1)}\\ &= C\, 2^{-\beta(j+1)}\, \|G\|_{B^\beta_{q,\infty}}\, \|\nabla u\|_{L^\infty} \sum_{|j-k|\le 2} 2^{\beta(j-k)} \sum_{m<k-1} 2^{(m-k)(1-\beta)}\\ &\le C\, 2^{-\beta(j+1)}\, \|G\|_{B^\beta_{q,\infty}}\, \|\nabla u\|_{L^\infty}.\end{aligned}$$

where C is a constant depending on β only and the condition $\beta < 1$ is used to guarantee that $(m-k)(1-\beta) < 0$. For any $\beta > -1$,

$$\begin{aligned} \|J_5\|_{L^q} &\le C\, \|\nabla u\|_{L^\infty}\, 2^{-\beta(j+1)} \sum_{k\ge j-1} 2^{(\beta+1)(j-k)}\, 2^{\beta(k+1)}\, \|\widetilde{\Delta}_k G\|_{L^q}\\ &\le C\, 2^{-\beta(j+1)}\, \|G\|_{B^\beta_{q,\infty}}\, \|\nabla u\|_{L^\infty}.\end{aligned}$$

$\|J_6\|_{L^q} = \|\Delta_j[\mathcal{R}, u\cdot\nabla]\theta\|_{L^q}$ can be estimated as in the proof of Proposition 2.4.2,

$$\|J_6\|_{L^q} \le C\, (\|\omega\|_{L^q} + \|\omega\|_{L^\infty}) 2^{\epsilon j} \|\Delta_j \theta\|_{L^q}$$

for any fixed $\epsilon > 0$, where C is a constant depending on ϵ. For the purpose to be specified later, we choose

$$\epsilon > 0, \quad \beta + \epsilon < 1.$$

Collecting these estimates and invoking the global bounds for $\|\omega\|_{L^q \cap L^\infty}$, we obtain, for any $-1 < \beta < 1$,

$$\frac{d}{dt}\|\Delta_j G\|_{L^q} \le C\, 2^{-\beta(j+1)}\, \|G\|_{B^{\beta}_{q,\infty}}\, \|\nabla u\|_{L^\infty} + C\, 2^{\epsilon j} \|\Delta_j \theta\|_{L^q} + C.$$

Let $\widetilde{\beta} = \beta + \epsilon < 1$. By applying the process above to the equation for θ and making use of the fact that

$$\int \Delta_j \theta |\Delta_j \theta|^{q-2} \Lambda \Delta_j \theta \, dx \ge 0,$$

we obtain

$$\frac{d}{dt}\|\Delta_j \theta\|_{L^q} \le C\, 2^{-\widetilde{\beta}(j+1)}\, \|\theta\|_{B^{\widetilde{\beta}}_{q,\infty}}\, \|\nabla u\|_{L^\infty}.$$

Integrating the inequalities in time and adding them up, we obtain

$$X(t) \le C + X(0) + \, C \int_0^t (1 + \|\nabla u(\tau)\|_{L^\infty}) X(\tau)\, d\tau. \tag{3.25}$$

where we have set

$$X(t) \equiv \|G(t)\|_{B^{\beta}_{q,\infty}} + \|\theta(t)\|_{B^{\widetilde{\beta}}_{q,\infty}}.$$

By Proposition 3.3.1, for any $\frac{2}{q} < \beta$,

$$\begin{aligned} \|\nabla u\|_{L^\infty} \;&\le\; C\,(1 + \|\omega\|_{L^p}) + C\,\|\omega\|_{L^\infty}\, Q\left(\|\omega\|_{B^{\beta}_{q,\infty}}^{\frac{2q}{q\beta - 2}} \right) \log(1 + \|\omega\|_{B^{\beta}_{q,\infty}}) \\ &\le\; C\,(1 + \|\omega\|_{L^p}) + C\,\|\omega\|_{L^\infty}\, P(X(t)^{\frac{2q}{q\beta-2}} \, \log(1 + X(t))\,). \end{aligned}$$

Inserting this inequality in (3.25) and applying Osgood's inequality, we obtain desired bound, for $t \leq T$,

$$\|\omega(t)\|_{B^{\beta}_{q,\infty}} \leq \|G(t)\|_{B^{\beta}_{q,\infty}} + \|\theta(t)\|_{B^{\tilde{\beta}}_{q,\infty}} = X(t) \leq C(T).$$

We now proceed to show that, for any $t \leq T$,

$$\|\omega(t)\|_{B^{\beta_1}_{q,\infty}} \leq C(T) \quad \text{for any } \beta_1 \text{ satisfying } 1 < \beta_1 < 2 - \tfrac{2}{q}.$$

The strategy is first to get the global bound for $\|\theta(t)\|_{B^{\beta_1}_{q,\infty}}$ from the equation for θ and then get the global bound for $\|G\|_{B^{\beta_1}_{q,\infty}}$. As we have seen from the previous part, J_4 is the only term that requires $\beta < 1$. In the process of estimating $\|\theta(t)\|_{B^{\beta_1}_{q,\infty}}$, the corresponding terms $\widetilde{J}_1$, $\widetilde{J}_2$, $\widetilde{J}_5$ can be bounded the same way as before, namely

$$\|\widetilde{J}_1\|_{L^q},\ \|\widetilde{J}_2\|_{L^q},\ \|\widetilde{J}_5\|_{L^q} \leq C\, 2^{-\beta_1(j+1)}\, \|\theta\|_{B^{\beta_1}_{q,\infty}}\, \|\nabla u\|_{L^\infty}. \tag{3.26}$$

$\|\widetilde{J}_4\|_{L^q}$ is estimated differently. We start with the basic bound

$$\|\widetilde{J}_4\|_{L^q} \leq C \sum_{|j-k|\leq 2} \|\nabla \Delta_k u\|_{L^\infty} \sum_{m<k-1} 2^{m-k} \|\Delta_m \theta\|_{L^q}.$$

Since $\beta_1 + \frac{2}{q} < 2$, we can choose $\frac{2}{q} < \beta < 1$ and $\epsilon > 0$ such that

$$\beta_1 + \frac{2}{q} + \epsilon < 2\beta. \tag{3.27}$$

By Berntsein's ienquality and Lemma 2.1,

$$\begin{aligned}
\|\nabla \Delta_k u\|_{L^\infty} &\leq C\, 2^{\frac{2k}{q}}\, \|\nabla \Delta_k u\|_{L^q} \leq C\, 2^{\frac{2k}{q}}\, P(2^k) \|\Delta_k \omega\|_{L^q} \\
&\leq C\, 2^{k(\frac{2}{q}+\epsilon)} \|\Delta_k \omega\|_{L^q} \leq C\, 2^{k(\frac{2}{q}+\epsilon-\beta)}\, \|\omega\|_{B^{\beta}_{q,\infty}}.
\end{aligned}$$

Clearly, for any $\beta < 1$,

$$\begin{aligned}\sum_{m<k-1} 2^{m-k}\,\|\Delta_m\theta\|_{L^q} &= 2^{-\beta k}\sum_{m<k-1} 2^{(m-k)(1-\beta)}2^{\beta m}\,\|\Delta_m\theta\|_{L^q}\\ &\le C\,2^{-\beta k}\|\theta\|_{B^\beta_{q,\infty}}.\end{aligned}$$

Therefore, according to (3.27) and the global bound in the first step,

$$\|\widetilde{J}_4\|_{L^q} \le C\,2^{-\beta_1(j+1)}\,\|\omega\|_{B^\beta_{q,\infty}}\,\|\theta\|_{B^\beta_{q,\infty}}\,2^{(\beta_1+\frac{2}{q}+\epsilon-2\beta)j} \le C\,2^{-\beta_1(j+1)}. \quad (3.28)$$

Collecting the estimates in (3.26) and (3.28), we have

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^q} \le C\,2^{-\beta_1(j+1)}\,\|\theta\|_{B^{\beta_1}_{q,\infty}}\,\|\nabla u\|_{L^\infty} + C\,2^{-\beta_1(j+1)}.$$

Bounding $\|\nabla u\|_{L^\infty}$ by the interpolation inequality in Proposition 3.3.1 and applying Osgood inequality lead to the desired global bound for $\|\theta\|_{B^{\beta_1}_{q,\infty}}$. With this bound at our disposal, we then obtain a global bound for $\|G\|_{B^{\beta_1}_{q,\infty}}$ by going through a similar process on the equation of G. Therefore, for any $t \le T$,

$$\|\omega\|_{B^{\beta_1}_{q,\infty}} \le \|\theta\|_{B^{\beta_1}_{q,\infty}} + \|G\|_{B^{\beta_1}_{q,\infty}} \le C(T).$$

If necessary, we can repeat the second step a few times to achieve the global bound for ω and θ in $B^s_{q,\infty}$ for any $s > 2$. This completes the proof of Proposition 3.3.3. ■

3.4 Uniqueness and Existence

We finish the proof of 3.1.2 in this section. Since we have shown that the solution $\omega \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2))$, $\theta \in C([]0,T); B^s_{q,\infty}(\mathbb{R}^2) \cap L^1([0,T]; B^{s+1}_{q,\infty}(\mathbb{R}^2))$ for all $s > 2$, the uniqueness of the solution is trival due to this high regularity. We focus on the existence of the solution. It stars with the construction of a local solution through themethod of successive approximation. That is, we consider a successive approximation sequence $\{(\omega^{(n)}, \theta^{(n)})\}$ solving

$$\begin{cases} \omega^{(1)} = S_2\omega_0, \quad \theta^{(1)} = S_2\theta_0, \\ u^{(n)} = \nabla^\perp \Delta^{-1} P(\Lambda)\, \omega^{(n)}, \\ \partial_t \omega^{(n+1)} + u^{(n)} \cdot \nabla \omega^{(n+1)} = \partial_{x_1} \theta^{(n+1)}, \\ \partial_t \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} + \Lambda \theta^{(n+1)} = 0, \\ \omega^{(n+1)}(x,0) = S_{n+2}\omega_0(x), \quad \theta^{(n+1)}(x,0) = S_{n+2}\theta_0(x). \end{cases} \tag{3.29}$$

In order to show that $\{(\omega^{(n)}, \theta^{(n)})\}$ converges to a solution of (3.1.2), it suffices to prove that $\{(\omega^{(n)}, \theta^{(n)})\}$ obeys the following properties:

(1) There exists a time interval $[0, T_1]$ over which $\{(\omega^{(n)}, \theta^{(n)})\}$ are bounded uniformly in terms of n. More precisely, we show that

$$\|(\omega^{(n)}, \theta^{(n)})\|_{B^s_{q,\infty}} \le C(T_1, \|(\omega_0, \theta_0)\|_{B^s_{q,\infty}}),$$

for a constant depending on T_1 and the initial norm only.

(2) There exists $T_2 > 0$ such that $\omega^{(n+1)} - \omega^{(n)}$ and $\theta^{(n+1)} - \theta^{(n)}$ are Cauchy in $B^{s-1}_{q,\infty}$, namely

$$\|\omega^{(n+1)} - \omega^{(n)}\|_{B^{s-1}_{q,\infty}} \le C(T_2)\, 2^{-n}, \quad \|\theta^{(n+1)} - \theta^{(n)}\|_{B^{s-1}_{q,\infty}} \le C(T_2)\, 2^{-n}$$

for any $t \in [0, T_2]$, where $C(T_2)$ is independent of n.

If the properties stated in (1) and (2) hold, then there exists (ω, θ) satisfying, for $T = \min\{T_1, T_2\}$,

$$\omega(\cdot, t) \in B^s_{q,\infty}, \quad \theta(\cdot, t) \in B^s_{q,\infty} \quad \text{for} \quad 0 \le t \le T,$$

$$\omega^{(n)}(\cdot, t) \to \omega(\cdot, t) \quad \text{in} \quad B^{s-1}_{q,\infty}, \qquad \theta^{(n)}(\cdot, t) \to \theta(\cdot, t) \quad \text{in} \quad B^{s-1}_{q,\infty}.$$

It is then easy to show that (ω, θ) solves (3.1.2) and we thus obtain a local solution and the global bounds in Sections 3.2 and 3.3 allow us to extend it into a global solution. It then remains to verify the properties stated in (1) and (2). Property (1) can be shown as in Sections 3.2 and 3.3. To verify Property (2), we consider the equations for the differences $\omega^{(n+1)} - \omega^{(n)}$ and $\theta^{(n+1)} - \theta^{(n)}$ and prove Property (2) inductively in n. The bounds can be achieved in a similar fashion in Sections 3.2 and 3.3. We thus omit further details. This completes the proof of Theorem 3.1.2.

CHAPTER 4

2D Boussinesq equations with supercritical dissipation

4.1 Introduction

In this chapter, we will turn our focus onto the following general 2D Boussinesq system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \mathcal{L} u = -\nabla p + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{4.1}$$

The major generalization is the dissipation operator $\mathcal{L}$. As mentioned in the section 2.2, we can define the nonlocal dissipation operator in two ways:

$$\mathcal{L} f(x) = \text{ p.v. } \int_{\mathbb{R}^2} \frac{f(x) - f(y)}{|x-y|^2} m(|x-y|) dy \tag{4.2}$$

and $m\colon (0,\infty) \to (0,\infty)$ is a smooth, positive, non-increasing function, which obeys

(i) there exists $C_1 > 0$ such that

$$r m(r) \leq C_1 \qquad \text{for all } r \leq 1;$$

(ii) there exists $C_2 > 0$ such that

$$r|m'(r)| \leq C_2 m(r) \qquad \text{for all } r > 0;$$

(iii) there exists $\beta > 0$ such that

$$r^\beta m(r) \text{ is non-increasing.}$$

Alternatively, we have

$$\widehat{\mathcal{L}f}(\xi) = P(|\xi|)\widehat{f}(\xi) \tag{4.3}$$

When the Fourier multiplier satisfies the following conditions, $P(|\xi|) = C\, m(\frac{1}{|\xi|})$.

1. P satisfies the doubling condition: for any $\xi \in \mathbb{R}^2$,

$$P(2|\xi|) \leq c_D P(|\xi|)$$

   with constant $c_D \geq 1$;

2. P satisfies the Hormander-Mikhlin condition (see [73]): for any $\xi \in \mathbb{R}^2$,

$$|\xi|^{|k|}\, |\partial_\xi^k P(|\xi|)| \leq c_H P(|\xi|)$$

   for some constant $c_H \geq 1$, and for all multi-indices $k \in \mathbb{Z}^d$ with $|k| \leq N$, with N only depending on c_D;

3. P has sub-quadratic growth at ∞, i.e.

$$\int_0^1 P(|\xi|^{-1})|\xi|d|\xi| < \infty$$

4. P satisfies

$$(-\Delta)^2 P(|\xi|) \geq c_H^{-1} P(\xi)|\xi|^{-4}$$

for all $|\xi|$ sufficiently large.

Different from the system 3.2, the vorticity is defined conventionally as $\omega = \nabla \times u$. Then, the system can be reformulated as

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \mathcal{L}\omega = \partial_{x_1}\theta, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^{\perp}\psi, \quad \Delta\psi = \omega, \\ \omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{4.4}$$

The goal of this chapter is to prove the following theorem

Theorem 4.1.1 *Consider the IVP (4.1) and assume that $\mathcal{L}$ satisfies (4.2) and (4.3) with $P(|\xi|) = m(\frac{1}{|\xi|})$ obeying the aforementioned conditions. We further assume that $a(\xi) = a(|\xi|) \equiv |\xi|/P(|\xi|)$ is positive, non-decreasing and satisfies*

$$\lim_{|\xi|\to\infty} \frac{a(|\xi|)}{|\xi|^{\sigma}} = 0\,, \quad \forall\, \sigma > 0. \tag{4.5}$$

Let $q > 2$ and let the initial data (u_0, θ_0) be in the class

$$u_0 \in H^1(\mathbb{R}^2), \quad \omega_0 \in L^q(\mathbb{R}^2) \cap B^0_{\infty,1}(\mathbb{R}^2)\,, \quad \theta_0 \in L^2(\mathbb{R}^2) \cap B^{0,a^2}_{\infty,1}(\mathbb{R}^2),$$

where $\omega_0 = \nabla \times u_0$ is the initial vorticity. Then (4.1) has a unique global solution

(u, θ) *satisfying, for all* $t > 0$,

$$u \in L_t^\infty H^1, \quad \omega \in L_t^\infty L^q \cap L_t^1 B_{\infty,1}^0 \,, \quad \theta \in L_t^\infty L^2 \cap L_t^\infty B_{\infty,1}^{0,a^2} \cap L_t^1 B_{\infty,1}^{0,a}.$$

The major improvement over the result of Hmidi [45] is that, we are able to deal with a large group of $\mathcal{L}$ and avoid complicated calculation regarding the Askey theorem. This results in an easy to check condition and simple proof for the lower estimate of the dissipation term.

The section 4.2 and section 4.3 below directly control the L^p norm of ω and G. But, it is restricted to the range $2 < q < 4$. The Besov space technique will be used in section 4.4 to raise some differentiability for $q \in (2, 4)$. This gives the possibility to find the bound in $L_t^1 B_{\infty,1}^{0,a}$ in section 4.5, which gives the final proof on L^p, $q > 2$.

4.2 Global *a priori* Bound for $\|\omega\|_{B_{2,2}^{0,a^{-1}}}$

This section establishes a global *a priori* estimates for $\|G\|_{L^2}$. Due to the transport type equation for θ, we have the control over the L^p norm of θ. However, since $G = \omega - \mathcal{R}_a\theta$ and $\mathcal{R}_a = \mathcal{L}^{-1}\partial_{x_1}$, it is more likely to obtain the global bound for ω with a loss of an a factor, i.e. the $B_{2,2}^{0,a^{-1}}$ norm.

Proposition 4.2.1 *Assume that the initial data* (u_0, θ_0) *satisfies the conditions in Theorem 4.1. Let* (u, θ) *be the corresponding solution and let* $\omega = \nabla \times u$ *be the vorticity. Then, for any* $t \geq 0$,

$$\|G\|_{L^2}^2 + \int_0^t \|\mathcal{L}^{\frac{1}{2}} G(\tau)\|_{L^2}^2 \, d\tau \leq B(t)$$

and consequently

$$\|\omega(t)\|_{B_{2,2}^{0,\,a^{-1}}} \leq B(t),$$

where $B(t)$ *is integrable on any finite-time interval* $[0, T]$.

Proof. Trivially u and θ obey the following global *a priori* bounds

$$\|\theta(t)\|_{L^2\cap L^\infty} \le \|\theta_0\|_{L^2\cap L^\infty}, \quad \|u(t)\|_{L^2} \le \|u_0\|_{L^2} + t\|\theta_0\|_{L^2}. \tag{4.6}$$

It is easy to check that G satisfies

$$\partial_t G + u\cdot\nabla G + \mathcal{L}G = [\mathcal{R}_a, u\cdot\nabla]\theta. \tag{4.7}$$

Taking the inner product with G leads to

$$\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2 + \int G\,\mathcal{L}G\,dx = \int G\nabla\cdot[\mathcal{R}_a, u]\theta\,dx. \tag{4.8}$$

By the Hölder inequality and the boundedness of Riesz transforms on L^2,

$$\left|\int G\nabla\cdot[\mathcal{R}_a, u]\theta dx\right| \le \|\mathcal{L}^{\frac{1}{2}}G\|_{L^2}\,\|\mathcal{L}^{-\frac{1}{2}}\Lambda[\mathcal{R}_a, u]\theta\|_{L^2}.$$

Inserting this estimate in (4.8) and applying Young's inequality and 2.3, we obtain

$$\frac{d}{dt}\|G\|_{L^2}^2 + \|\mathcal{L}^{\frac{1}{2}}G\|_{L^2}^2 \le \|\mathcal{L}^{-\frac{1}{2}}\Lambda[\mathcal{R}_a, u]\theta\|_{L^2}^2. \tag{4.9}$$

By the definition of the norm in (2.6), $\|\mathcal{L}^{-\frac{1}{2}}\Lambda f\|_2 \le \|f\|_{B^{\frac{1}{2},\frac{a}{2}}_{2,2}}$. Applying Proposition 2.4.2 with $\frac{1}{2} < \delta < 1$ and $p = q = 2$, we obtain

$$\|[\mathcal{R}_a, u]\theta\|_{B^{\frac{1}{2},\frac{a}{2}}_{2,2}} \le C\|u\|_{B^\delta_{2,\infty}}\,\|\theta\|_{B^{\frac{1}{2}-\delta,\frac{a^2}{4}}_{\infty,2}} + C\,\|u\|_{L^2}\,\|\theta\|_{L^2}.$$

Since $u = \nabla^\perp\Delta^{-1}\omega$,

$$\begin{aligned}\|u\|_{B^\delta_{2,\infty}} &= \sup_{j\ge -1} 2^{\delta j}\,\|\Delta_j u\|_{L^2} \le \|\Delta_{-1}u\|_{L^2} + \sup_{j\ge 0} 2^{\delta j}\,\|\Delta_j\nabla^\perp\Delta^{-1}\omega\|_{L^2}\\ &\le \|u\|_{L^2} + \sup_{j\ge 0} 2^{(\delta-1)j}\|\Delta_j\omega\|_{L^2} \le \|u\|_{L^2} + \|\omega\|_{B^{0,a-1}_{2,2}}.\end{aligned}$$

For $\delta > \frac{1}{2}$, $\|\theta\|_{B^{\frac{1}{2}-\delta,\frac{a^2}{4}}_{\infty,2}} \le \|\theta\|_{L^\infty}$. Therefore,

$$\|\mathcal{L}^{-\frac{1}{2}}\Lambda[\mathcal{R}_a,u]\theta\|_{L^2} \le \|[\mathcal{R}_a,u]\theta\|_{B^{\frac{1}{2},\frac{a}{2}}_{2,2}} \le C\,\|u\|_{L^2}\,\|\theta\|_{L^2\cap L^\infty} + \|\omega\|_{B^{0,a-1}_{2,2}}\|\theta\|_{L^\infty}. \quad (4.10)$$

We can bound the $\|\omega\|_{B^{0,a-1}_{2,2}}$ by

$$\|\omega\|_{B^{0,a-1}_{2,2}} \le \|G\|_{B^{0,a-1}_{2,2}} + \|\mathcal{R}_a\theta\|_{B^{0,a-1}_{2,2}} \le \|G\|_2 + \|\theta\|_2. \quad (4.11)$$

$$\frac{d}{dt}\|G\|^2_{L^2} + \|\mathcal{L}^{\frac{1}{2}}\|^2_{L^2} \le C\|G\|^2_{L^2} + C \quad (4.12)$$

since $\|u\|_{L^2}$ and $\|\theta\|_{L^2\cap L^\infty}$ are bounded by (4.6). We combine (4.9), (4.10) and (4.11) to obtain the desired result. This completes the proof of Proposition 4.2.1. ■

4.3 Global *a priori* bound for $\|G\|_{L^q}$ with $q \in (2,4)$

This section establishes a global *a priori* bounds for $\|\omega\|_{L^q}$ with $q \in (2,4)$. We prepare the proof with the following lemma.

Lemma 4.1 *Let $q \in (2,\infty)$, $s \in (0,1)$, $0 < \epsilon(q-2) \le 2$ and $f \in L^{\frac{2q}{1+\epsilon}} \cap \mathring{H}^{s+(1-\frac{2}{q})(1+\epsilon)}$. Then*

$$\||f|^{q-2}f\|_{\mathring{H}^s} \le C\,\|f\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}}\,\|f\|_{\mathring{B}^s_{\frac{2q}{2-\epsilon(q-2)},2}} \le C\,\|f\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}}\,\|f\|_{\mathring{H}^{s+(1-\frac{2}{q})(1+\epsilon)}}. \quad (4.13)$$

Proof. This proof modifies the one in [38]. Identifying $\mathring{H}^s$ with $\mathring{B}^s_{2,2}$ and by the alternative definition of $\mathring{B}^s_{2,2}$ 2.1, we have

$$\||f|^{q-2}f\|^2_{\mathring{H}^s} = \int \frac{\||f|^{q-2}f(x+y) - |f|^{q-2}f(x)\|^2_{L^2}}{|y|^{2+2s}}\,dy.$$

Thanks to the inequality, for $q > 2$

$$\left||f|^{q-2}f(x+y) - |f|^{q-2}f(x)\right| \le C\,\left(|f|^{q-2}(x+y) + |f|^{q-2}(x)\right)|f(x+y) - f(x)|,$$

we have, by Hölder's inequality

$$\||f|^{q-2}f(x+y) - |f|^{q-2}f(x)\|_{L^2}^2 \le C\,\|f\|_{L^{\frac{2q}{1+\epsilon}}}^{2(q-2)}\,\|f(x+y) - f(x)\|_{L^\rho}^2,$$

where

$$\rho = \frac{2q}{2 - \epsilon(q-2)}.$$

Therefore,

$$\||f|^{q-2}f\|_{\mathring{H}^s}^2 \le C\,\|f\|_{L^{\frac{2q}{1+\epsilon}}}^{2(q-2)}\,\|f\|_{\mathring{B}^s_{\rho,2}}^2.$$

Further applying the Besov embedding inequality

$$\|f\|_{\mathring{B}^s_{\rho,2}} \le C\,\|f\|_{\mathring{H}^{s+1-\frac{2}{\rho}}},$$

we obtain (4.13) and this completes the proof of Lemma 4.1. ■

Now, we have the main conclusion for this section. Notice that, we still have to deduce a small factor for the norm of ω.

Proposition 4.3.1 *Assume that the initial data (u_0, θ_0) satisfies the conditions stated in theorem 4.1. Let (u, θ) be the corresponding solution and G be defined as in the previous section. Then, for any $q \in (2, 4)$, G obeys the global bound, for any $T > 0$ and $t \le T$,*

$$\|G(t)\|_{L^q}^q + C\int_0^t\int \left|\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\right|^2 dxdt + C\int_0^t \|G\|_{L^{\frac{2q}{1+\epsilon}}}^q d\tau \le B(t), \qquad (4.14)$$

where C is a constant depending on q only and $B(t)$ is integrable on any finite time *interval. A special consequence is that, for any small $\epsilon > 0$,*

$$\|\omega(t)\|_{B^{-\epsilon}_{q,\infty}} \le B(t). \tag{4.15}$$

Proof. Multiplying (4.7) by $G|G|^{q-2}$ and integrating with respect to x, we obtain

$$\frac{1}{q}\frac{d}{dt}\|G\|_{L^q}^q + \int G|G|^{q-2}\mathcal{L}G dx = -\int G|G|^{q-2}\nabla\cdot[\mathcal{R}_a, u]\theta\, dx.$$

By Lemma 2.3,

$$\int G|G|^{q-2}\mathcal{L}G dx \ge C\int |\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})|^2\, dx.$$

Set $\epsilon > 0$ to be small, say, for $q \in (2,4)$,

$$(1+\epsilon)\left(1-\frac{2}{q}\right) < \frac{1}{2}.$$

Thanks to the condition in (2.3.1) and by a Sobolev embedding,

$$\begin{aligned}
\|\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 &= \sum_{j\ge -1}\|\Delta_j\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&= \sum_{j\ge -1} 2^j a^{-1}(2^j)\|\Delta_j(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&\ge C\sum_{j\ge -1} 2^{(1-\epsilon)j}\|\Delta_j(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&= C\,\|\Lambda^{\frac{1}{2}-\frac{\epsilon}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&\ge C\,\|G\|^q_{L^{\frac{2q}{1+\epsilon}}}.
\end{aligned}$$

For $q \in (2,4)$, we choose $s > 0$ such that

$$s > \epsilon, \qquad s + (1+\epsilon)\left(1-\frac{2}{q}\right) = \frac{1}{2} - \epsilon.$$

By Hölder's inequality,

$$\left|\int G|G|^{q-2}\nabla\cdot[\mathcal{R}_a,u]\theta\right| \le \|G|G|^{q-2}\|_{\mathring{H}^s}\|[\mathcal{R}_a,u]\theta\|_{\mathring{H}^{1-s}}.$$

By Lemma 4.1 above,

$$\|G|G|^{q-2}\|_{\mathring{H}^s} \le C\,\|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}}\,\|G\|_{\mathring{H}^{s+(1+\epsilon)(1-\frac{2}{q})}} = C\,\|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}}\,\|G\|_{\mathring{H}^{\frac{1}{2}-\epsilon}}.$$

In addition, due to the condition in (2.3.1),

$$\|G\|^2_{\mathring{H}^{\frac{1}{2}-\epsilon}} = \sum_{j\ge -1} 2^{j-2\epsilon j}\|\Delta_j G\|^2_{L^2} \le \sum_{j\ge -1} 2^j a^{-2}(2^j)\|\Delta_j G\|^2_{L^2} \le \|\mathcal{L}^{\frac{1}{2}}(G)\|^2_{L^2}.$$

By Proposition 2.4.2, recalling $s>\epsilon$ and $u=\nabla^\perp\Delta^{-1}\omega$,

$$\begin{aligned}\|[\mathcal{R}_a,u]\theta\|_{\mathring{H}^{1-s}} &\le C\,\|u\|_{\mathring{B}^{1-s+\epsilon}_{2,\infty}}\,\|\theta\|_{B^{-\epsilon,1}_{\infty,2}} + C\,\|u\|_{L^2}\,\|\theta\|_{L^2}\\ &\le C\,\|\omega\|_{B^{0,a^{-1}}_{2,2}}\,\|\theta\|_{L^\infty} + C\,\|u\|_{L^2}\|\theta\|_{L^2}.\end{aligned}$$

Putting the estimates together, we obtain

$$\begin{aligned}&\frac{1}{q}\frac{d}{dt}\|G\|^q_{L^q} + C\int|\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})|^2\,dx + C\,\|G\|^q_{L^{\frac{2q}{1+\epsilon}}}\\ &\quad\le C\,\|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}}\,\|\mathcal{L}^{\frac{1}{2}}(G)\|_{L^2}\left(\|\omega\|_{B^{0,a^{-1}}_{2,2}}\,\|\theta\|_{L^\infty} + C\,\|u\|_{L^2}\|\theta\|_{L^2}\right).\end{aligned}$$

Applying Young's inequality to the right-hand side, noticing that $q\in(2,4)$ and resorting to the bounds in Proposition 4.2.1, we obtain (4.14). (4.15) follows from the inequality

$$\|\omega\|_{B^{-\epsilon}_{q,\infty}} \le \|G\|_{B^{-\epsilon}_{q,\infty}} + \|\mathcal{R}_a\theta\|_{B^{-\epsilon}_{q,\infty}} \le \|G\|_{L^q} + \|\theta\|_{L^q}.$$

This completes the proof of Proposition 4.3.1. ■

4.4 Global *a priori* bound for $\|G\|_{\widetilde{L}_t^r B_{q,1}^s}$ with $q \in [2,4)$

This section proves a global *a priori* bound for $\|G\|_{\widetilde{L}_t^r B_{q,1}^s}$ with $q \in (2,4)$. This bound serves as an important step towards a global bound for $\|\omega\|_{L^q}$ with general $q \in [2,\infty)$.

Proposition 4.4.1 *Assume that the initial data (u_0, θ_0) satisfies the conditions stated in Theorem 4.1. Let*

$$r \in [1,\infty], \quad s \in [0,1), \quad q \in (2,4).$$

Then, for any $t > 0$, G obeys the following global bound

$$\|G\|_{\tilde{L}_t^r B_{q,1}^s} \le B(t), \tag{4.16}$$

where B is integrable on any finite-time interval.

Proof. Let $j \ge -1$ be an integer. Applying Δ_j to (4.7) yields

$$\partial_t \Delta_j G + \mathcal{L}\Delta_j G = -\Delta_j(u\cdot\nabla G) - \Delta_j[\mathcal{R}_a, u\cdot\nabla]\theta.$$

Taking the inner product with $\Delta_j G|\Delta_j G|^{q-2}$, we have

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j G\|_{L^q}^q + \int \Delta_j G|\Delta_j G|^{q-2}\mathcal{L}\Delta_j G = J_1 + J_2, \tag{4.17}$$

where

$$J_1 = -\int \Delta_j(u\cdot\nabla G)\,\Delta_j G|\Delta_j G|^{q-2}, \tag{4.18}$$

$$J_2 = -\int \Delta_j[\mathcal{R}_a, u\cdot\nabla]\theta\ \Delta_j G|\Delta_j G|^{q-2}.$$

According to Lemma 2.4, for $j \geq 0$, the dissipation part can be bounded below by

$$\int \Delta_j G |\Delta_j G|^{q-2} \mathcal{L} \Delta_j G \geq C P(2^j) \|\Delta_j G\|_{L^q}^q. \tag{4.19}$$

By Lemma 4.2 below, J_1 can be bounded by

$$\begin{aligned} |J_1| \quad \leq \quad & C\, 2^{j(\epsilon+\frac{2}{q})} \, \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}} \Big[\|\Delta_j G\|_{L^q} + \sum_{m \leq j-2} 2^{(m-j)\frac{2}{q}} \|\Delta_m G\|_{L^q} \\ & + \sum_{k \geq j-1} 2^{(j-k)(1-\frac{2}{q})} \, \|\Delta_k G\|_{L^q} \Big] \, \|\Delta_j G\|_{L^q}^{q-1}, \end{aligned} \tag{4.20}$$

where we have taken ϵ to be small positive number, especially

$$s - 1 + 3\epsilon < 0.$$

To bound J_2, we first apply Hölder's inequality and then employ similar estimates as in the proof of Proposition 2.4.2 to obtain

$$\begin{aligned} |J_2| \quad & \leq \quad \|\Delta_j [\mathcal{R}_a, u \cdot \nabla] \theta\|_{L^q} \|\Delta_j G\|_{L^q}^{q-1} \\ & \leq \quad C \left(2^{j\epsilon} a(2^j) \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}} \|\theta\|_{L^\infty} + \|u\|_{L^2} \|\theta\|_{L^2} \right) \, \|\Delta_j G\|_{L^q}^{q-1}. \end{aligned} \tag{4.21}$$

Inserting (4.19), (4.20) and (4.21) in (4.17) and writing the bound for $\|\omega(t)\|_{B^{-\epsilon}_{q,\infty}}$ by $B(t)$, we obtain

$$\frac{d}{dt} \|\Delta_j G\|_{L^q} + C 2^j \, a^{-1}(2^j) \, \|\Delta_j G\|_{L^q} \leq C 2^{\epsilon j} \, a(2^j) \, B(t) \tag{4.22}$$

$$\begin{aligned} & + C 2^{j(\epsilon+\frac{2}{q})} B(t) \Big[\|\Delta_j G\|_{L^q} + \sum_{m \leq j-2} 2^{(m-j)\frac{2}{q}} \|\Delta_m G\|_{L^q} \\ & + \sum_{k \geq j-1} 2^{(j-k)(1-\frac{2}{q})} \, \|\Delta_k G\|_{L^q} \Big]. \end{aligned} \tag{4.23}$$

Due to (2.3.1), $a(2^j) \le 2^{\epsilon j}$. Integrating in time yields

$$\|\Delta_j G(t)\|_{L^q} \le e^{-C\,2^{(1-\epsilon)j}t}\|\Delta_j G(0)\|_{L^q} + C\,2^{-j(1-3\epsilon)}\,B(t) \tag{4.24}$$

$$+ C\,2^{j(\epsilon+\frac{2}{q})} B(t) \int_0^t e^{-C\,2^{(1-\epsilon)j}(t-\tau)} L(\tau)\,d\tau, \tag{4.25}$$

where, for notational convenience, we have written

$$L(t) = \Big[\|\Delta_j G\|_{L^q} + \sum_{m\le j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q} + \sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big].$$

Taking the L^r norm in time and applying Young's inequality for the time integral part lead to

$$\begin{aligned} \|\Delta_j G\|_{L^r_t L^q} \ &\le\ C\,2^{-\frac{1}{r}(1-\epsilon)j}\,\|\Delta_j G(0)\|_{L^q} + C\,2^{-j(1-3\epsilon)}\,\widetilde{B}(t) \\ &\quad + C\,2^{j(-1+2\epsilon+\frac{2}{q})}\widetilde{B}(t)\,\|L\|_{L^r}. \end{aligned}$$

Multiplying by 2^{js}, summing over $j \ge -1$ and noticing $s - 1 + 3\epsilon < 0$, we obtain

$$\|G\|_{\widetilde{L}^r_t B^s_{q,1}} \ \le\ C\,\|G(0)\|_{B^{s-1/r(1-\epsilon)}_{q,1}} + C\,\widetilde{B}(t) + K_1 + K_2 + K_3, \tag{4.26}$$

where

$$\begin{aligned} K_1 &= C\sum_{j\ge -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\,\widetilde{B}(t)\,2^{js}\|\Delta_j G\|_{L^r_t L^q}, \\ K_2 &= C\sum_{j\ge -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\,\widetilde{B}(t)\,2^{js}\sum_{m\le j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^r_t L^q}, \\ K_3 &= C\sum_{j\ge -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\,\widetilde{B}(t)\,2^{js}\sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^r_t L^q}. \end{aligned}$$

Since $-1 + 2\epsilon + \frac{2}{q} < 0$, we can choose an integer $N > 0$ such that

$$C\,2^{N(-1+2\epsilon+\frac{2}{q})}\widetilde{B}(t) \le \frac{1}{8}.$$

The sums in K_1, K_2 and K_3 can then be split into two parts: $j \le N$ and $j > N$. Since $\|G\|_{L^q}$ is bounded, the sum for the lower frequency part is bounded by $C\,\widetilde{B}(t)2^{sN}$. The sum over the high frequency section with $j > N$ is bounded by $\frac{1}{8}\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$. Therefore,

$$K_1, K_2, K_3 \le C\,\widetilde{B}(t)2^{sN} + \frac{3}{8}\|G\|_{\widetilde{L}^r_t B^s_{q,1}}.$$

Combining these bounds with (4.26) yields the desired estimates. This completes the proof of Proposition 4.4.1. ■

We now provide the details leading to (4.20). They bear some similarities as those in [25], but they are provided here for the sake of completeness.

Lemma 4.2 *Let J_1 be defined as in (4.18). Then we have the following bound*

$$\begin{aligned}\|J_1\|_{L^q} \;\; \le \;\; & C\,2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\Big[\|\Delta_j G\|_{L^q} + \sum_{m\le j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q} \\ & + \sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big]\,\|\Delta_j G\|^{q-1}_{L^q}.\end{aligned}$$

Proof. Using the notation of paraproducts, we write

$$\Delta_j(u\cdot\nabla G) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15},$$

where

$$\begin{aligned}
J_{11} &= \sum_{|j-k|\le 2} [\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k G,\\
J_{12} &= \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k G,\\
J_{13} &= S_j u\cdot\nabla\Delta_j G,\\
J_{14} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}G),\\
J_{15} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k G).
\end{aligned}$$

Since $\nabla\cdot u = 0$, we have

$$\int J_{13}|\Delta_j G|^{q-2}\Delta_j G\,dx = 0.$$

By Hölder's inequality,

$$\left|\int J_{11}|\Delta_j G|^{q-2}\Delta_j G\right| \le \|J_{11}\|_{L^q}\|\Delta_j G\|_{L^q}^{q-1}.$$

We write the commutator in terms of the integral,

$$J_{11} = \int \Phi_j(x-y)\,(S_{k-1}u(y) - S_{k-1}u(x))\cdot\nabla\Delta_k G(y)\,dy,$$

where Φ_j is the kernel of the operator Δ_j found in section 2.1. As in the proof of Lemma 3.3, we have, for any $0<\epsilon<1$,

$$\|J_{11}\|_{L^q} \le \||x|^{1-\epsilon}\Psi_j(x)\|_{L^1}\,\|S_{j-1}u\|_{\mathring{B}^{1-\epsilon}_{q,\infty}}\,\|\nabla\Delta_j G\|_{L^\infty}.$$

By the definition of Φ_j and Bernstein's inequality, we have

$$\begin{aligned}\|J_{11}\|_{L^q} &\le C\, 2^{j(\epsilon+\frac{2}{q})}\, \||x|^{1-\epsilon}\Psi_0(x)\|_{L^1}\, \|S_{j-1}\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}} \|\Delta_j G\|_{L^\infty} \\ &\le C\, 2^{j(\epsilon+\frac{2}{q})} \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\, \|\Delta_j G\|_{L^q}.\end{aligned}$$

Again, by Bernstein's inequality,

$$\begin{aligned}\|J_{12}\|_{L^q} &\le C\|\Delta_j u\|_{L^q} \|\nabla \Delta_j G\|_{L^\infty} \\ &\le C 2^{j(\epsilon+\frac{2}{q})}\, \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\, \|\Delta_j G\|_{L^q};\end{aligned}$$

$$\begin{aligned}\|J_{14}\|_{L^q} &\le C\|\Delta_j u\|_{L^q} \|\nabla S_{j-1} G\|_{L^\infty} \\ &\le C 2^{j(\epsilon+\frac{2}{q})}\, \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}} \sum_{m\le j-2} 2^{(m-j)\frac{2}{q}} \|\Delta_m G\|_{L^q};\end{aligned}$$

$$\begin{aligned}\|J_{15}\|_{L^q} &\le C 2^{j(\epsilon+\frac{2}{q})} \sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\, \|\Lambda^{1-\epsilon}\Delta_k u\|_{L^q} \|\Delta_k G\|_{L^q} \\ &\le C 2^{j(\epsilon+\frac{2}{q})}\, \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}} \sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\, \|\Delta_k G\|_{L^q}.\end{aligned}$$

Combining the estimates above yields

$$\begin{aligned}\|J_1\|_{L^q} \le\ & C\, 2^{j(\epsilon+\frac{2}{q})}\, \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}} \Big[\|\Delta_j G\|_{L^q} + \sum_{m\le j-2} 2^{(m-j)\frac{2}{q}} \|\Delta_m G\|_{L^q} \\ & + \sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\, \|\Delta_k G\|_{L^q}\Big]\, \|\Delta_j G\|_{L^q}^{q-1}.\end{aligned}$$

This completes the proof of Lemma 4.2. ∎

4.5 Global *a priori* bounds for $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}$ and $\|\omega\|_{L^q}$ for any $q \geq 2$

This section shows that, if the initial data ω_0 is in L^q, then the solution ω is also *a priori* in L^q at any time. This is established by first proving the time integrability $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}$. More precisely, we have the following theorem.

Proposition 4.5.1 *Assume that the initial data (u_0, θ_0) satisfies the conditions as stated in Theorem 4.1.1. Then we have the following global a priori bounds. For any $T > 0$ and $t \leq T$,*

$$\|\omega(t)\|_{L^1_t B^{0,a}_{\infty,1}} \leq C(T), \quad \|\theta(t)\|_{B^{0,a^2}_{\infty,1}} \leq C(T), \quad \|\omega(t)\|_{L^q} \leq C(T),$$

where $C(T)$ are constants depending on T and the initial norms only.

Proof. [Proof of Proposition 4.5.1] We first explains that (4.16) in Proposition 4.4.1 implies that, for $t \leq T$,

$$\|G\|_{L^1_t B^{0,a}_{\infty,1}} \leq C(T).$$

In fact, if we choose $s \in [0, 1)$ satisfying $s > \frac{2}{q}$ for $q \in (2, 4)$ and set $\epsilon > 0$ satisfying $\epsilon + \frac{2}{q} - s < 0$, then

$$\begin{aligned} \|G\|_{B^{0,a}_{\infty,1}} &\equiv \sum_{j \geq -1} a(2^j) \|\Delta_j G\|_{L^\infty} \leq \sum_{j \geq -1} a(2^j) 2^{\frac{2}{q} j} \|\Delta_j G\|_{L^q} \\ &\leq \sum_{j \geq -1} a(2^j) 2^{-\epsilon j}\, 2^{j(\epsilon + \frac{2}{q} - s)} 2^{js} \|\Delta_j G\|_{L^q} \leq C \|G\|_{B^s_{q,1}}, \end{aligned}$$

where we have used the fact that $a(2^j)2^{-\epsilon j} \leq C$ for a constant C independent of j. Furthermore,

$$\|\omega\|_{L^1_t B^{0,a}_{\infty,1}} \leq \|G\|_{L^1_t B^{0,a}_{\infty,1}} + \|\mathcal{R}_a \theta\|_{L^1_t B^{0,a}_{\infty,1}}.$$

By the definition of the norm in $B^{0,a}_{\infty,1}$ and recalling that $\mathcal{R}_a \theta$ is defined by the multiplier $a(|\xi|)\frac{i\xi_1}{|\xi|}$, we have

$$\begin{aligned}
\|\mathcal{R}_a\theta\|_{B^{0,a}_{\infty,1}} &= a(2^{-1})\,\|\Delta_{-1}\mathcal{R}_a\theta\|_{L^\infty} + \sum_{j\geq 0} a(2^j)\,\|\Delta_j\mathcal{R}_a\theta\|_{L^\infty} \\
&\leq C\,\|\theta_0\|_{L^2} + \sum_{j\geq 0} a^2(2^j)\,\|\Delta_j\theta\|_{L^\infty} \\
&\leq C\,\|\theta_0\|_{L^2} + \|\theta\|_{B^{0,a^2}_{\infty,1}}.
\end{aligned}$$

By a lemma in [25],

$$\begin{aligned}
\|\theta\|_{B^{0,a^2}_{\infty,1}} &\leq C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}}\left(1+\int_0^t \|\nabla u\|_{L^\infty} dt\right) \\
&\leq C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}}\left(1+\|u\|_{L^1_t L^2} + \|\omega\|_{L^1_t B^0_{\infty,1}}\right) \\
&\leq C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}}\left(1+\|u\|_{L^1_t L^2} + \|\omega\|_{L^1_t B^{0,a}_{\infty,1}}\right). \qquad (4.27)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\omega\|_{L^1_t B^{0,a}_{\infty,1}} \leq\ & \|G\|_{L^1_t B^{0,a}_{\infty,1}} + C\left(\|\theta_0\|_{L^2} + \|\theta_0\|_{B^{0,a^2}_{\infty,1}}\right) t \\
& + C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \int_0^t \|u\|_{L^1_\tau L^2}\, d\tau + C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \int_0^t \|\omega\|_{L^1_\tau B^{0,a}_{\infty,1}}\, d\tau.
\end{aligned}$$

By Gronwall's inequality, $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}$ is bounded by C(T), which, in turn, implies that, by (4.27),

$$\|\theta(t)\|_{B^{0,a^2}_{\infty,1}} \leq C(T).$$

We finish this section by finding the bound for $\|\omega\|_{L^q}$. From the equations of G and $\mathcal{R}_a\theta$,

$$\begin{aligned}
\|\omega\|_{L^q} &\leq \|G\|_{L^q} + \|\mathcal{R}_a\theta\|_{L^q} \\
&\leq \|G_0\|_{L^q} + \|\mathcal{R}_a\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{L^q}\, d\tau \\
&\leq \|G_0\|_{L^q} + \|\mathcal{R}_a\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{B^0_{q,1}}\, d\tau.
\end{aligned}$$

Following the steps as in the proof of Proposition 2.4.2, we can show that

$$\|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{B^0_{q,1}} \le C\|\omega\|_{L^q}\,\|\theta\|_{B^{0,a}_{\infty,1}} + C\,\|\theta_0\|_{L^2}\,\|u\|_{L^2}.$$

Gronwall's inequality and the bound $\|\theta\|_{L^1_t B^{0,a}_{\infty,1}} \le C(T)$ then imply the bound for $\|\omega\|_{L^q}$. This completes the proof of Proposition 4.5.1. ■

4.6 Proof of the Main Theorem

In this section, we complete the proof of the theorem of 4.1 by showing the uniqueness and uniqueness of the solutions in the stated class.

Theorem 4.6.1 *Assume that the initial data (u_0, θ_0) satisfies the conditions stated in Theorem 4.1. Then, the solutions (u, θ) in the class*

$$u \in L^\infty([0,T];H^1), \quad \omega \in L^\infty([0,T];L^q)\cap L^1_T B^{0,a}_{\infty,1}, \quad \theta \in L^\infty([0,T], L^2\cap B^{0,a}_{\infty,1}) \tag{4.28}$$

must be unique.

Proof. Assume that $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ are two solutions in the class (4.28). Let $p^{(1)}$ and $p^{(2)}$ be the associated pressure. The differences

$$u = u^{(2)} - u^{(1)}, \qquad p = p^{(2)} - p^{(1)}, \qquad \theta = \theta^{(2)} - \theta^{(1)}$$

satisfy

$$\begin{cases} \partial_t u + u^{(1)}\cdot\nabla u + u\cdot\nabla u^{(2)} + \mathcal{L}u = -\nabla p + \theta \mathbf{e}_2, \\ \partial_t \theta + u^{(1)}\cdot\nabla\theta + u\cdot\nabla\theta^{(2)} = 0. \end{cases}$$

By Lemmas 4.3 and 4.4 below, we have the following estimates

$$\begin{aligned}\|u(t)\|_{B^0_{2,\infty}} &\le \|u(0)\|_{B^0_{2,\infty}} + C\,\|\theta\|_{L^\infty_t B^{-1,a}_{2,\infty}} \\ &\quad + C\int_0^t \|u(\tau)\|_{L^2}\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}})\,d\tau\end{aligned}$$

and

$$\begin{aligned}\|\theta(t)\|_{B^{-1,a}_{2,\infty}} &\le \|\theta(0)\|_{B^{-1,a}_{2,\infty}} + C\int_0^t \|\theta(\tau)\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}})\,d\tau \\ &\quad + C\int_0^t \|u(\tau)\|_{L^2}\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}\,d\tau.\end{aligned}$$

In addition, we bound $\|u\|_{L^2}$ by the following interpolation inequality, which can be found in [37]

$$\|u\|_{L^2} \le C\,\|u\|_{B^0_{2,\infty}}\,\log\left(1 + \frac{\|u\|_{H^1}}{\|u\|_{B^0_{2,\infty}}}\right)$$

together with $\|u\|_{H^1} \le \|u^{(1)}\|_{H^1} + \|u^{(2)}\|_{H^1}$. These inequalities allow us to conclude that

$$Y(t) \equiv \|u(t)\|_{B^0_{2,\infty}} + \|\theta(t)\|_{B^{-1,a}_{2,\infty}}$$

obeys

$$Y(t) \le 2\,Y(0) + C\int_0^t D_1(\tau)Y(\tau)\,\log\left(1 + D_2(\tau)/Y(\tau)\right)\,d\tau, \tag{4.29}$$

where

$$\begin{aligned}D_1 &= \|\theta^{(2)}\|_{B^{0,a}_{\infty,1}} + \|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}, \\ D_2 &= \|u^{(1)}\|_{H^1} + \|u^{(2)}\|_{H^1}.\end{aligned}$$

Applying Osgood's inequality to (4.29) and noticing that $Y(0) = 0$, we conclude that $Y(t) = 0$. This completes the proof of Theorem 4.6.1. ■

We now state and prove two estimates used in the proof of Theorem 4.6.1.

Lemma 4.3 *Assume that $u^{(1)}$, $u^{(2)}$, u, p and θ are defined as in the proof of Theorem 4.6.1 and satisfy*

$$\partial_t u + u^{(1)} \cdot \nabla u + u \cdot \nabla u^{(2)} + \mathcal{L} u = -\nabla p + \theta \mathbf{e}_2. \tag{4.30}$$

Then we have the a priori bound

$$\begin{aligned} \|u(t)\|_{B^0_{2,\infty}} &\le \|u(0)\|_{B^0_{2,\infty}} + C\,\|\theta\|_{L^\infty_t B^{-1,a}_{2,\infty}} \\ &+ C \int_0^t \|u(\tau)\|_{L^2} \left(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}\right) d\tau. \end{aligned} \tag{4.31}$$

Proof. [Proof of Lemma 4.3] Let $j \ge -1$ be an integer. Applying Δ_j to (4.30) and taking the inner product with $\Delta_j u$, we obtain, after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|^2_{L^2} + \|\mathcal{L}^{\frac{1}{2}}\Delta_j u\|^2_{L^2} = J_1 + J_2 + J_3, \tag{4.32}$$

where

$$\begin{aligned} J_1 &= -\int \Delta_j u\, \Delta_j(u^{(1)} \cdot \nabla u)\, dx, \\ J_2 &= -\int \Delta_j u\, \Delta_j(u \cdot \nabla u^{(2)})\, dx, \\ J_3 &= \int \Delta_j u\, \Delta_j(\theta \mathbf{e}_2)\, dx. \end{aligned}$$

By Plancherel's theorem,

$$\|\mathcal{L}^{\frac{1}{2}}\Delta_j u\|^2_{L^2} \ge C\, 2^j a^{-1}(2^j)\, \|\Delta_j u\|^2_{L^2},$$

where $C = 0$ in the case of $j = -1$ and $C > 0$ for $j \ge 0$. The estimate for J_3 is easy and we have, by Hölder's inequality,

$$|J_3| \le \|\Delta_j u\|_{L^2} \, \|\Delta_j \theta\|_{L^2} \le 2^j a^{-1}(2^j) \, \|\Delta_j u\|_{L^2} \, \|\theta\|_{B^{-1,a}_{2,\infty}}.$$

To estimate J_1, we need to use a commutator structure to shift one derivative to $u^{(1)}$. For this purpose, we write

$$\Delta_j(u^{(1)} \cdot \nabla u) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15}, \tag{4.33}$$

where

$$\begin{aligned}
J_{11} &= \sum_{|j-k|\le 2} [\Delta_j, S_{k-1}u^{(1)} \cdot \nabla]\Delta_k u, \\
J_{12} &= \sum_{|j-k|\le 2} (S_{k-1}u^{(1)} - S_j u^{(1)}) \cdot \nabla \Delta_j \Delta_k u, \\
J_{13} &= S_j u^{(1)} \cdot \nabla \Delta_j u, \\
J_{14} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u^{(1)} \cdot \nabla S_{k-1} u), \\
J_{15} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u^{(1)} \cdot \nabla \widetilde{\Delta}_k u).
\end{aligned}$$

Since $\nabla \cdot u^{(1)} = 0$, we have

$$\int J_{13} \, \Delta_j u \, dx = 0.$$

J_{11}, J_{12}, J_{14} and J_{15} can be bounded in a similar fashion as in the proof of Lemma 4.2 and we have

$$\begin{aligned}
&\|J_{11}\|_{L^2}, \, \|J_{12}\|_{L^2} \le C \, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \|\Delta_j u\|_{L^2}, \\
&\|J_{14}\|_{L^2} \le C \, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{m\le j-1} 2^{m-j} \|\Delta_m u\|_{L^2}, \\
&\|J_{15}\|_{L^2} \le C \, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k} \|\Delta_k u\|_{L^2}.
\end{aligned}$$

To estimate J_2, we write

$$\Delta_j(u \cdot \nabla u^{(2)}) = J_{21} + J_{22} + J_{23}, \tag{4.34}$$

where

$$\begin{aligned}
J_{21} &= \sum_{|j-k|\le 2} \Delta_j(S_{k-1}u \cdot \nabla \Delta_k u^{(2)}),\\
J_{22} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u \cdot \nabla S_{k-1} u^{(2)}),\\
J_{23} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u \cdot \nabla \widetilde{\Delta}_k u^{(2)}).
\end{aligned}$$

Therefore, by Hölder's inequality,

$$\begin{aligned}
&\|J_{21}\|_{L^2} \le C\,\|u\|_{L^2}\,\|\nabla\Delta_j u^{(2)}\|_{L^\infty},\\
&\|J_{22}\|_{L^2} \le C\,\|\Delta_j u\|_{L^2}(\|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}),\\
&\|J_{23}\|_{L^2} \le C\,(\|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k}\|\Delta_k u\|_{L^2}.
\end{aligned}$$

Inserting the estimates above in (4.32), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|_{L^2} + C\,2^j a^{-1}(2^j)\,\|\Delta_j u\|_{L^2} \le C\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}} + K(t), \tag{4.35}$$

where

$$\begin{aligned}
K(t) \;=\;& C\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}})\|\Delta_j u\|_{L^2}\\
&+ C\,\|u\|_{L^2}\,\|\nabla\Delta_j u^{(2)}\|_{L^\infty} + (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{m\le j-1} 2^{m-j}\|\Delta_m u\|_{L^2}\\
&+ C\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k}\|\Delta_k u\|_{L^2}.
\end{aligned}$$

Integrating (4.35) in time and taking supreme over j, we obtain (4.31). This completes the proof of Lemma 4.3. ■

Lemma 4.4 *Assume that θ, $u^{(1)}$, u and $\theta^{(2)}$ are defined as in the proof of Theorem 4.6.1 and satisfy*

$$\partial_t\theta + u^{(1)}\cdot\nabla\theta + u\cdot\nabla\theta^{(2)} = 0. \tag{4.36}$$

Then we have the a priori bound

$$\begin{aligned}\|\theta(t)\|_{B^{-1,a}_{2,\infty}} \;\le\; & \|\theta(0)\|_{B^{-1,a}_{2,\infty}} + C\int_0^t \|\theta(\tau)\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}})\,d\tau \\ & + C\int_0^t \|u(\tau)\|_{L^2}\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}\,d\tau. \end{aligned} \tag{4.37}$$

Proof. [Proof of Lemma 4.4] Let $j \ge -1$ be an integer. Applying Δ_j to (4.36) and taking the inner product with $\Delta_j\theta$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta\|_{L^2}^2 = K_1 + K_2, \tag{4.38}$$

where

$$\begin{aligned} K_1 &= -\int \Delta_j\theta\,\Delta_j(u^{(1)}\cdot\nabla\theta)\,dx, \\ K_2 &= -\int \Delta_j\theta\,\Delta_j(u\cdot\nabla\theta^{(2)})\,dx. \end{aligned}$$

To estimate K_1, we decompose $\Delta_j(u^{(1)}\cdot\nabla\theta)$ as in (4.33) and estimate each component in a similar fashion to obtain

$$\begin{aligned}|K_1| \;\le\; & C\,\|\Delta_j\theta\|_{L^2}^2\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \\ & + C\,\|\Delta_j\theta\|_{L^2}\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}). \end{aligned}$$

To estimate K_2, we decompose $\Delta_j(u\cdot\nabla\theta^{(2)})$ as in (4.34) and bound the components in a similar fashion to have

$$|K_2| \;\le\; C\,\|\Delta_j\theta\|_{L^2}\,\|u\|_{L^2} 2^j a^{-1}(2^j)\,\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}.$$

Combining these estimates, we find

$$\begin{aligned}\frac{d}{dt}\|\Delta_j\theta\|_{L^2} \;&\le\; C\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2}+\|\omega^{(1)}\|_{B^0_{\infty,1}})\\ &\quad + C\,\|u\|_{L^2} 2^j a^{-1}(2^j)\,\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}.\end{aligned}$$

Integrating in time, multiplying by $2^{-j}a(2^j)$ and taking $\sup_{j\ge -1}$, we obtain (4.37). This completes the proof of Lemma 4.4. ■

We now sketch the proof of Theorem 4.1.

Proof. [Proof:] Thanks to Theorem 4.6.1, it suffices to establish the existence of solutions. The first step is to obtain a local (in time) solution and then extend it into a global solution through the global *a priori* bounds obtained in the previous section. The local solution can be constructed through the method of successive approximation. That is, we consider a successive approximation sequence $\{(\omega^{(n)},\theta^{(n)})\}$ solving

$$\begin{cases}\omega^{(1)} = S_2\omega_0, \quad \theta^{(1)} = S_2\theta_0,\\ \partial_t\omega^{(n+1)} + u^{(n)}\cdot\nabla\omega^{(n+1)} + \mathcal{L}\omega^{(n+1)} = \partial_{x_1}\theta^{(n+1)},\\ \partial_t\theta^{(n+1)} + u^{(n)}\cdot\nabla\theta^{(n+1)} = 0,\\ \omega^{(n+1)}(x,0) = S_{n+2}\omega_0(x), \quad \theta^{(n+1)}(x,0) = S_{n+2}\theta_0(x).\end{cases} \tag{4.39}$$

To show that $\{(\omega^{(n)},\theta^{(n)})\}$ converges to a solution of (4.4), it suffices to prove that $\{(\omega^{(n)},\theta^{(n)})\}$ obeys the following properties:

(1) There exists a time interval $[0, T_1]$ over which $\{(\omega^{(n)}, \theta^{(n)})\}$ are bounded uniformly in terms of n. More precisely, we show that

$$\|\omega^{(n)}\|_{L_t^\infty(L^2\cap L^q)\cap L_t^1 B^{0,a}_{\infty,1}} \le C(T_1), \quad \|\theta^{(n)}\|_{L_t^\infty(L^2\cap B^{0,a^2}_{\infty,1})\cap L_t^1 B^{0,a}_{\infty,1}} \le C(T_1),$$

where $C(T_1)$ is a constant independent of n.

(2) There exists $T_2 > 0$ such that $\omega^{(n+1)} - \omega^{(n)}$ is a Cauchy sequence in $L_t^\infty B^{-1}_{\infty,1}$ and $\theta^{(n+1)} - \theta^{(n)}$ is Cauchy in $L_t^1 B^{-1,a}_{\infty,1}$, namely

$$\|\omega^{(n+1)} - \omega^{(n)}\|_{L_t^\infty B^{-1}_{\infty,1}} \le C(T_2)\, 2^{-n}, \quad \|\theta^{(n+1)} - \theta^{(n)}\|_{L_t^1 B^{-1,a}_{\infty,1}} \le C(T_2)\, 2^{-n}$$

for any $t \in [0, T_2]$, where $C(T_2)$ is independent of n.

If the properties stated in (1) and (2) hold, then there exists (ω, θ) satisfying

$$\omega \in L_t^\infty(L^2 \cap L^q) \cap L_t^1 B^{0,a}_{\infty,1}, \quad \theta \in L_t^\infty(L^2 \cap B^{0,a^2}_{\infty,1}) \cap L_t^1 B^{0,a}_{\infty,1},$$

$$\omega^{(n)} \to \omega \quad \text{in} \quad L_t^\infty B^{-1}_{\infty,1}, \qquad \theta^{(n)} \to \theta \quad \text{in} \quad L_t^1 B^{-1,a}_{\infty,1}$$

for any $t \le \min\{T_1, T_2\}$. It is then easy to show that (ω, θ) solves (4.39) and we thus obtain a local solution and the global bounds in the previous sections allow us to extend it into a global solution. It then remains to verify the properties stated in (1) and (2). Property (1) can be shown as in the previous sections (Section 4.2 through Section 4.5) while Property (2) can be checked as in the proof of Theorem 4.6.1. We thus omit further details. This completes the proof of Theorem 4.1. ∎

CHAPTER 5

Sufrace Quasi-Geostrophic equations

5.1 Introduction

In this chapter, we will pay our attention to the 2D Surface Quasi-Geostrophic equations. As mentioned in the first chapter, the equations read

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \kappa\Lambda^\alpha\theta = 0 \\ u = \nabla^\perp\phi\,, \quad \Lambda\phi = \theta \end{cases}$$

Very similar to the 2D Boussinesq equations, we have three different regimes depending on the choice of α

- The sub-critical case for $\alpha > 1$
- The critical case for $\alpha = 1$
- The super-critical case for $0 < \alpha < 1$.

The global regularity problem for sub-critical case has been solved (as given in [26] [68]). However, the conventional energy method can not be applied to the critical case. In fact, in the recent years, a huge amount of effort has been dedicated to this problem. In 2001, Constantin, Cordoba and Wu [8] proved the existence and uniqueness problem under the condition that the initial data has a L^∞ -norm comparable to or less than the diffusion coefficient κ. i.e. the small data condition. For large initial data cases, we have four distinguish proofs at this point of time. Caffarelli and Vasseur [20] proved the global regularity of the Leray-Hopf weak solutions to the critical SQG equations in the whole space setting. Their method used the Di-Giorgi iterative estimates. One crucial fact they use in the second step is that the operator $\Lambda = (-\Delta)^{\frac{1}{2}}$ is equivalent to the normal derivative of a harmonic function which is obtained by extending the solution into the $3D$ half space. Kiselev and Nazarov in [52] proved the Holder continuity, i.e. the second step of Caffarelli and Vasseur, by investigating the revolution of the solutions in the $\mathcal{U}_r(\mathbb{T}^d)$ class. Here we pay more attention to the following two methods.

In [KNV07], Kiselev, Nazarov and Volberg introduce the idea of Modulus of Continuous (MOC).

Definition 5.1.1 *We call a function $\omega : [0, \infty) \to [0, \infty)$ a modulus of continuity if it is an increasing continuous concave function such that $\omega(0) = 0$. In addition, we may assume $\omega'(0) < \infty$ and $\lim_{\xi \to 0+} \omega"(\xi) = -\infty$. We say that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ has modulus of continuity ω if*

$$|f(x,t) - f(y,t)| \leq \omega(|x-y|) \qquad \textit{for all} \quad x, y \in \mathbb{R}^n$$

The procedure of the proof is to show that the solution θ follows the same MOC in the time perio $[0,T]$ as it does at the initial time. Then, the regularity will follow as $\|\nabla\theta\|_\infty$ is bounded by $\omega'(0)$.

In [[21]], Constantin and Vicol achieved the newest approach through the idea of Only Small Shock (OSS).

Definition 5.1.2 *Let $\delta > 0$ and $t > 0$. We say $\theta(x,t)$ has the OSS_δ property, if there exists an L such that*

$$\sup_{\{(x,y):|x-y|<L\}} |\theta(x,t) - \theta(y,t)| \leq \delta$$

Moreover, for $T > 0$, we say $\theta(x,t)$ has the uniform OSS_δ property on $[0,T]$, if there exists an $L > 0$ such that

$$\sup_{\{(x,y,t):|x-y|<L, 0\le t\le T\}} |\theta(x,t) - \theta(y,t)| \le \delta$$

The entire proof can be divided into two parts. The solution θ follows a uniform OSS property over time with respect to δ and L. And then the OSS implies the regularity of the solution. The new idea behind these two proofs is that we focus ourself on point-wise property of the solutions instead of a space norm over the entire domain.

The regularity problem for the supercritical case still remains open. Constantin and Wu [23] reached an conditional result which assumes that the solution remains in $C^{1-\alpha}$. Dabkowski [27] shows an eventual regularity results with $\alpha \in (0,1)$ by using the same $\mathcal{U}_r$ class as Kiselev and Nazarov did. The result is an alternation to the result of Silvestre [72], which uses the Caffarelli and Vasseur's approach. Kiselev, by investigating a time dependent MOC condition, reached the similar eventual regularity in [51]. It worth mention that Dabkowski [27], as well as Kiselev [51], also solves the finite time regularity theorem for $\alpha \in (0, \frac{1}{2})$, which is an extension to the work of Silvestre [72].

In the rest of this chapter, we attempt to extend the idea of OSS in to the super critical regime. The second step is done by assuming the solution being in a Holder continuity class uniformly over time, which is more regular than the OSS condition. This is consistent with the result of Constantin and Wu. Applying the method of MOC for the first step, we can reach a small data global regularity with requirement only on $\|\cdot\|_{L^\infty}$, which is weaker when compared with Yu [81]. Another corollary is that, with the decaying of the $\|\theta\|_\infty$ over time, we can reach a eventual regularity by the small data result.

5.2 Hólder to Regularity

For the super-critical case, i.e. $\alpha < 1$ in 5.1, the solution ought to be more regular than the OSS condition. One may find a hint from Constantin and Wu [23], which show the Hólder continuity $C^{1-\alpha}$ implies global regularity. Though the OSS condition only regularize the solution for $|x-y| < L$, its requirement can be represented as

$$|\theta(x,t) - \theta(y,t)| \leq \delta |x-y|^0$$

which is consist with the Hólder condition. One explanation is that the dissipation is weakened when α getting smaller. The regularity contribution from the dissipation operator will not counter the singularity from the nonlinear term. So, we should seek a certain amount of differentiability from the solution itself. We will have a rigorous proof below. First, we extend the idea of OSS condition.

Definition 5.2.1 (Close Range Holder) *Let $\delta > 0$ and $t > 0$. We say the solution $\theta(x,t)$ has the Close Rang Holder ($CRH_{\delta,s}$) property with parameter s at time t, if there exists an $L > 0$ such that*

$$\sup_{(x,y):|x-y|\leq L} |\theta(x,t) - \theta(y,t)| \leq \delta |x-y|^s$$

Moreover, we say $\theta(x,t)$ has the uniform $CRH_{\delta,s}$ property on $[0,T]$, if the same L works for all $0 \leq t \leq T$.

Theorem 5.1 *There exist a $\delta_0 > 0$, depending on $\|\theta_0\|_\infty$, so that if θ is a bounded weak solution of the super critical SQG equations with the uniform $CRH_{\delta_0,s}$ property on $[0,T]$, then it is a smooth solution on $[0,T]$. Here we require that $\frac{\alpha^2+\alpha-2}{\alpha-2} \leq s < 1$. Also*

$$\sup_{t\in[0,T]} \|\nabla\theta\|_\infty \leq C(\|\theta_0\|_\infty, \|\nabla\theta_0\|_\infty, L)$$

Proof. Using the standard procedure, we have the decaying of L^p norm of θ for $1 \le p \le \infty$. The goal here is to show that $\|\nabla\theta\|_\infty$ is bounded uniformly over time. Thus, by a BKM type criteria [5], i.e. the smooth solution may be continued pass through T if

$$\lim_{t\to T}\int_0^t \|\nabla\theta\|_{L^\infty} < \infty,$$

we can reach the regularity conclusion as expected.

Apply ∇ operator to the θ equation in 5.1 and then take the inner product with $\nabla\theta$

$$\frac{1}{2}(\partial_t + u\cdot\nabla)|\nabla\theta|^2 + \nabla\theta\cdot\Lambda^\alpha\nabla\theta + \nabla u : \nabla\theta\cdot\nabla\theta = 0$$

For the dissipation part, Constantin and Vicol [21] have improved the work of Cordoba and Cordoba [7]

$$\begin{aligned}
\nabla\theta\cdot\Lambda^\alpha\nabla\theta &= C_0 P.V.\int_{\mathbb{R}^d}\frac{\nabla\theta(x)(\nabla\theta(x)-\nabla\theta(y))}{|x-y|^{d+\alpha}}dy \\
&= \frac{1}{2}\Delta^\alpha|\nabla\theta|^2 + \frac{1}{2}D_\alpha \\
where \quad D_\alpha &= C_0 P.V.\int_{\mathbb{R}^d}\frac{|\nabla\theta(x,t)-\nabla\theta(y,t)|^2}{|x-y|^{d+\alpha}}
\end{aligned}$$

By the Theorem 2.5 of [21], we have the lower bound

$$\frac{1}{4}D_\alpha(x,t) \ge c_1\frac{|\nabla\theta|^{2+\alpha}}{\|\theta_0\|_\infty^\alpha}$$

Then we have

$$\frac{1}{2}(\partial_t + u\cdot\nabla)|\nabla\theta|^2 + c_1\frac{|\nabla\theta|^{2+\alpha}}{\|\theta_0\|_\infty} + \frac{D_\alpha(x,t)}{4} \le |\nabla u||\nabla\theta|^2$$

To bound the term $|\nabla u|$, we depends on the relation between u and θ

$$\nabla u = \mathcal{R}^{\perp}\nabla\theta = P.V. \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^{d+1}}(\nabla\theta(x,t) - \nabla\theta(y,t))dy$$

The greatest trouble occurs when $|x-y|$ is close to zero. We have the estimate, for some $\rho > 0$

$$\begin{aligned}
& \int_{|x-y|<\rho} \frac{(x-y)^{\perp}}{|x-y|^{d}+1}(\nabla\theta(x,t) - \nabla\theta(y,t))dy \\
= & \int_{\mathbb{R}^2} \frac{\nabla\theta(x,t) - \nabla\theta(y,t)}{|x-y|^{\frac{d+\alpha}{2}}} \frac{1}{|x-y|^{\frac{d-\alpha}{2}}} dy \\
\leq & \; c_2\sqrt{D_\alpha}\rho^{\frac{\alpha}{2}}
\end{aligned}$$

We expect

$$c_2\sqrt{D_\alpha}\rho^{\frac{\alpha}{2}}|\nabla\theta|^2 \leq \frac{D_\alpha}{8} + 2c_2^2\rho^{\alpha}|\nabla\theta|^4 \leq \frac{D_\alpha}{4}$$

which might be achieved by letting

$$\rho = (\frac{c_1}{4c_2^2|\nabla\theta|^{-2+\alpha}})^{\frac{1}{\alpha}}\|\theta_0\|_\infty^{-1}$$

When $|x-y|$ is large, i.e. $|x-y| > L$ in the theorem, we have

$$\int_{|x-y|\geq L} \frac{(x-y)^{\perp}}{|x-y|^{d}+1}(\nabla\theta(x,t) - \nabla\theta(y,t))dy \leq 2c_4\frac{\|\theta_0\|_\infty}{L}$$

If we $\rho = L$, the right hand side of 5.2 will be too large. This is exactly the reason we introduce the OSS or CRH condition. For the section $\rho \leq |x-y| < L$, we would do an integral by parts and use the fact that

$$\int_{|x-y|=r} (x-y)^{\perp}|x-y|^{-d-1}\nabla\theta(x)dy = 0$$

$$
\begin{aligned}
|\nabla u_{med}(x,t)| &= \int_{\rho\le|x-y|<L} \frac{(x-y)^{\perp}}{|x-y|^{d}+1}(\nabla\theta(x,t)-\nabla\theta(y,t))dy \\
&\le c_3 \int_{\rho\le|x-y|\le L} \frac{|\theta(x,t)-\theta(y,t)|}{|x-y|^{d+1}} \\
&\le c_3\delta_0 \int |x-y|^{-d-1+s} dy
\end{aligned}
$$

Here we use the $CRH_{\delta_0,s}$ property of θ. The integral is non-singular since we assume that $s<1$.

$$|\nabla u_{med}(x,t)| \le c_3\delta_0\rho^{-1+s}$$

applying the definition of ρ mentioned before, we have

$$|\nabla u_{med}(x,t)||\nabla\theta(x,t)|^2 \le c_4\|\theta_0\|_{\infty}^{1-s}|\nabla\theta|^{\frac{(1-s)(2-\alpha)}{\alpha}+2}$$

In order to hide this term in the left hand side, we need $\frac{(s-1)(\alpha-2)}{\alpha}+2\le 2+\alpha$, which is equivalent to $s\ge\frac{\alpha^2+\alpha-2}{\alpha-2}$, and

$$\delta_0 = c_5\|\theta_0\|_{\infty}^{-\alpha+\frac{s-1}{\alpha}}$$

Then, we reach a point-wise inequality

$$\frac{1}{2}(\partial_t+u\cdot\nabla+\Lambda^{\alpha})|\nabla\theta(x,t)|^2+\frac{c_1|\nabla\theta(x,t)|^{2+\alpha}}{2\|\theta_0\|_{\infty}^{\alpha}} \le 2c_4\frac{\|\theta_0\|_{\infty}|\nabla\theta(x,t)|^2}{L}$$

when $|\nabla\theta(x,t)|\ge(\frac{4c_4\|\theta_0\|_{\infty}^{1+\alpha}}{c_1L})^{\frac{1}{\alpha}}$, we have

$$(\partial_t+u\cdot\nabla+\Lambda^{\alpha})|\nabla\theta|^2\le 0$$

which means, $|\nabla\theta|$ would not exceed the threshold mentioned above. This bound serves for arbitrary x, which is equivalent to say $\|\nabla\theta\|_{\infty}<\infty$ uniformly for $t\in[0,T]$.

■

5.3 From $OSSm$ to Regularity

This section is a joint work with Constantin, Vicol and Wu. Inspired by the work of Dabkowski, Kiselev and Vicol, we may consider a more general dissipative operator for the SQG equations

$$\partial_t \theta + u \cdot \nabla \theta + \mathcal{L}\theta = 0, \qquad u = \mathcal{R}^{\perp}\theta$$

where the dissipative nonlocal operator $\mathcal{L}$ is defined as

$$\mathcal{L}\theta(x) = P.V. \int_{\mathbb{R}^2} (\theta(x) - \theta(x+y)) \frac{m(|y|)}{|y|^2} dy$$

The requirements for the function $m(r)$ are similar to those mentioned for the Boussinesq equations.

1. there exists $C_{m,1} > 0$ such that

$$r m(r) \leq C_{m,1} \qquad \text{for all } r \leq 1$$

2. there exists $C_{m,2} > 0$ such that

$$r|m'(r)| \leq C_{m,2} m(r) \qquad \text{for all } r > 0$$

3. there exists $\alpha > 0$ such that

$$r^{\alpha} m(r) \text{ is non-increasing.}$$

Examples of such functions $m(r)$ are

$$m(r) = \tfrac{1}{r^\gamma}, \qquad \text{for } r > 0 \text{ and } \gamma \in (0,1), \text{ which yields } \mathcal{L} = \Lambda^\gamma \tag{5.1}$$

$$m(r) = \tfrac{1}{r\log(2/r)}, \qquad \text{for } r \leq 1 \text{ and extend suitably for } r > 1 \tag{5.2}$$

$$m(r) = \tfrac{1}{r\log\log(2/r)}, \qquad \text{for } r \leq 1 \text{ and extend suitably for } r > 1 \tag{5.3}$$

$$m(r) = \tfrac{1}{r}, \qquad \text{for } r > 0, \text{ which yields the critical dissipation } \mathcal{L} = \Lambda. \tag{5.4}$$

For this new type of operator $\mathcal{L}$, we define the property $OSS_{m,\delta}$

Definition 5.3.1 (Only Small m Shocks) *We say the function $\theta(x)$ has the $OSS_{m,\delta}$* property *if there exits $L > 0$ such that*

$$\frac{|\theta(x) - \theta(y)|}{|x-y|m(|x-y|)} \leq \delta \qquad \textit{whenever} \quad |x-y| \leq L.$$

We say the function $\theta(x,t)$ has the uniform $OSS_{m,\delta}$ property on $[0,T]$ *if there exits $L > 0$, independent of $t \in [0,T]$ such that $\theta(\cdot,t)$ obeys* (5.3.1) *for all $t \in [0,T]$.*

Theorem 5.2 (OSS implies regularity) *Let θ_0 be smooth, decaying sufficiently fast at infinity, and assume that the operator $\mathcal{L}$ is such that m obeys* (1)–(3)*. There exits an $\delta_0 = \delta_0(\|\theta_0\|_{L^\infty})$ such if a solution $\theta(x,t)$ of* (5.3) *has the uniform OSS_{m,δ_0} property on $[0,T]$, for some $T > 0$, then $\theta(x,t)$ is Lipschitz continuous (in x) for $t \in [0,T]$.*

Proof. We need to show that $\sup_x |\nabla\theta(x,t)|^2$ remains uniformly bounded on $[0,T]$. For $(x,t) \in \mathbb{R}^2 \times [0,T]$ we have

$$(\partial_t + u\cdot\nabla + \mathcal{L})\,|\nabla\theta(x,t)|^2 + D_m[\nabla\theta](x,t) = -2(\nabla u \colon \nabla\theta\cdot\nabla\theta)(x,t)$$

where we denote $D_m[f] = 2f\mathcal{L}f - \mathcal{L}(f^2) \geq 0$, which is in turn given explicitly as

$$D_m[f](x) = P.V. \int_{\mathbb{R}^2} (f(x) - f(y))^2 \frac{m(|x-y|)}{|x-y|^2} dy.$$

Let L be the constant from the uniform OSS_{m,δ_0} property, and let $\rho \in (0, L)$ to be chosen later. We split the integral expression for ∇u, i.e.

$$\nabla u(x) = P.V. \int_{\mathbb{R}^2} (\nabla\theta(x) - \nabla\theta(y)) \frac{(x-y)^\perp}{|x-y|^3}$$

into an inner piece ($0 < |x-y| \leq \rho$), a medium piece ($\rho < |x-y| \leq L$), and an outer piece ($L < |x-y|$). For the inner piece, using the Cauchy-Schwartz inequality we have

$$|\nabla u_{in}(x)| \leq C \left(D_m[\nabla\theta](x)\right)^{1/2} \left(\int_0^\rho \frac{1}{rm(r)} dr\right)^{1/2} \leq \frac{D_m[\nabla\theta](x)}{4|\nabla\theta(x)|^2} + C|\nabla\theta(x)|^2 \int_0^\rho \frac{1}{rm(r)} dr.$$

For the medium piece we integrate by parts, use the uniform OSS_{m,δ_0} property and (2) to obtain

$$|\nabla u_{med}(x)| \leq C \int_{\rho<|x-y|\leq L} |\theta(x) - \theta(y)| \frac{1}{|x-y|^3} dy \leq C\delta_0 \int_\rho^L \frac{m(r)}{r} dr.$$

At last, the outer piece is direct via integration by parts

$$|\nabla u_{out}(x)| \leq C\|\theta_0\|_{L^\infty} \int_L^\infty \frac{1}{r^2} dr \leq C\frac{\|\theta_0\|_{L^\infty}}{L}.$$

For the positive term $D_m[\nabla\theta]$, using that θ has the uniform OSS_{m,δ_0} property on $[0, T]$, we have the lower bound

$$\begin{aligned} D_m[\nabla\theta](x) &\geq |\nabla\theta(x)|^2 \int_{|x-y|\geq\rho} \frac{m(|x-y|)}{|x-y|^2} dy - 2|\nabla\theta(x)| \left(\int_{|x-y|\geq\rho} |\theta(x) - \theta(y)| \left|\nabla_y \frac{m(|x-y|)}{|x-y|^2}\right| dy\right) \\ &\geq 2\pi|\nabla\theta(x)|^2 \int_\rho^\infty \frac{m(r)}{r} dr - C\delta_0|\nabla\theta(x)| \int_\rho^L \frac{m(r)^2}{r} dr - C|\nabla\theta(x)|\|\theta_0\|_{L^\infty} \int_L^\infty \frac{m(r)}{r^2} dr \end{aligned} \tag{5.5}$$

Here we also used (2). We now combine (5.3) with (5.3)–(5.5) and obtain

$$
\begin{aligned}
&(\partial_t + u\cdot\nabla + \mathcal{L})\,|\nabla\theta|^2 + \pi|\nabla\theta|^2 \textstyle\int_\rho^\infty \frac{m(r)}{r}dr \\
\le\; & C_0|\nabla\theta|^4 \textstyle\int_0^\rho \frac{1}{rm(r)}dr + C_0\delta_0|\nabla\theta| \int_\rho^L \frac{m(r)^2}{r}dr + C_0\delta_0|\nabla\theta|^2 \int_\rho^L \frac{m(r)}{r}dr \\
&+ C_0|\nabla\theta|\|\theta_0\|_{L^\infty} \textstyle\int_L^\infty \frac{m(r)}{r^2}dr + C_0\frac{\|\theta_0\|_{L^\infty}}{L}|\nabla\theta|^2
\end{aligned}
\tag{5.6}
$$

for $(x,t) \in \mathbb{R}^2 \times [0,T]$, where the constant C_0 may depend on m through (2). We rewrite (5.6) in compact form as

$$
(\partial_t + u\cdot\nabla + \mathcal{L})\,|\nabla\theta|^2 + T_0 \le T_1 + T_2 + T_3 + T_4 + T_5
$$

where the meaning of the T_i's is given in (5.6).

First we choose ρ so that $T_1 \le T_0/2$. Using (3), we have

$$
\int_0^\rho \frac{1}{rm(r)}dr \le \frac{1}{\rho^\alpha m(\rho)} \int_0^\rho \frac{1}{r^{1-\alpha}}dr \le \frac{1}{\alpha m(\rho)}, \tag{5.7}
$$

and using (2) and the fact that $m(\infty) = 0$ (due to (3)), we have

$$
\int_\rho^\infty \frac{m(r)}{r}dr \ge \frac{1}{C_{m,2}} \int_\rho^\infty -m'(r)dr = \frac{m(\rho)}{C_{m,2}}, \tag{5.8}
$$

so that we need to choose ρ sufficiently small to satisfy

$$
m(\rho) = C_1|\nabla\theta| \tag{5.9}
$$

for some sufficiently large constant C_1 that depends on $C_0, C_{m,2}$, and α. Note that since m is decreasing, this means ρ will be small, and by possibly increasing C_1 we can make sure that $\rho < L$.

From (2) we have

$$\int_\rho^L \frac{m(r)}{r}dr \le \rho^\alpha m(\rho)\int_\rho^\infty \frac{1}{r^{1+\alpha}}dr = \frac{m(\rho)}{\alpha},$$

and by the monotonicity of m and (5.8) we obtain

$$T_2 + T_3 \le \frac{C_0\delta_0}{\alpha}\left(|\nabla\theta| m(\rho)^2 + |\nabla\theta|^2 m(\rho)\right) = \frac{C_0(1+C_1)\delta_0}{\alpha}|\nabla\theta|^2 m(\rho) \le \frac{T_0}{4}$$

once we let δ_0 be sufficiently small (depending only on C_0, C_1, and α).

At last, using (3), we have

$$T_4 \le C_0\|\theta_0\|_{L^\infty}|\nabla\theta|\int_L^\infty \frac{L^\alpha m(L)}{r^{2+\alpha}}dr \le C_0\|\theta_0\|_{L^\infty}|\nabla\theta|\frac{m(L)}{L}$$

and therefore, inserting the above bounds into (5.6) we arrive at

$$(\partial_t + u\cdot\nabla + \mathcal{L})\,|\nabla\theta|^2 \le \frac{C_0\|\theta_0\|_{L^\infty}}{L}\left(m(L)|\nabla\theta| + |\nabla\theta|^2\right) \tag{5.10}$$

which concludes the proof of the Theorem. ■

The is theorem is an improvement of the theorem in th previous section, since we allows $s = 1-\alpha$. It can be regarded as an alternative proof of the theorem 3.1 in [23] when $m(r) = \frac{1}{r^\alpha}$.

5.4 Regularity with Small Data

In this section, we try to conclude that, when the initial data satisfies the $CRH_{\delta_0,s}$ and some smallness conditions, the solution remains the same property uniformly for $t \in [0,T]$ for $T > 0$.

Theorem 5.3 *Assume the initial value θ_0 for the equations satisfies*

$$|\theta_0(x) - \theta_0(y)| \leq \delta_0 |x-y|^s \quad \textit{when}\, |x-y| < L$$

for a constant L, then the solution for the super critical SQG equation have the uniform $CRH_{\delta_0,s}$ property with the parameter L.

Proof. We prove the theorem by a method close to the Modules of Continuity method invented by Kiselev, Nazarov and Volberg. For the simplicity of writing, we denote $\omega(\xi) = \delta_0 \xi^s$. At the beginning, we assume this formula holds for all $\xi \geq 0$.

We will show that, if the property breaks down at points x and y), with $|x-y| \leq L$, at time T, we would have $|\theta(x,T) - \theta(y,T)| = \omega|x-y|$. In Kiselev, Nazarov and Volberg paper, this is called breaking through scenario. This can be explained as the time continuity of the function $\theta(x,t) - \theta(y,t)$ for a fixed pair of points (x,y). However, we will also show that $\partial_t(|\theta(x,t) - \theta(y,t)|) < 0$, which would contradict the choice of x, y and t.

We will use the equations proven in [53]. Denote $\Omega(|x-y|)$ the modulus of continuity which is followed by $u(x)$:

$$\partial_t(\theta(x,t) - \theta(y,t)) = \Omega(|x-y|)\omega'(|x-y|) + \mathcal{D}$$

$$\begin{aligned}
\Omega(\xi) &= A[\int_0^{\frac{\xi}{2}} \frac{\omega(\eta)}{\eta} d\eta + \xi \int_{\frac{\xi}{2}}^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta] \\
&= A\delta_0[\frac{1}{s}(\xi^2) + \frac{-1}{s-1}\xi^{s-1}\xi] = \frac{A\delta_0}{s(1-s)}\xi^s \\
\omega'(\xi) &= \delta_0 s \xi^{s-1} \\
\mathcal{D} &= C[\int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta) + \omega(\xi-2\eta) - 2\omega(\xi)}{\eta^{1+\alpha}} d\eta \\
&\quad + \int_{\frac{\xi}{2}}^{\infty} \frac{\omega(2\eta+\xi) - \omega(2\eta-\xi) - 2\omega(\xi)}{\eta^{1+\alpha}} d\eta]
\end{aligned}$$

Both of the integrals for $\mathcal{D}$ return a negative value, but we will take the advantage of the first one only.

$$\begin{aligned}\mathcal{D} &\leq \int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+\alpha}} d\eta \\ &\leq C\xi^{2-\alpha}\omega''(\xi) = C\delta_0 s(s-1)\xi^{2-\alpha+s-2}\end{aligned}$$

We would expect

$$\frac{A}{1-s}\delta_0^2\xi^{2s-1} - C\delta_0 s(1-s)\xi^{s-\alpha} < 0$$

$$\xi^{s+\alpha-1} < \frac{Cs(1-s)^2}{A\delta_0}$$

This requires

$$L = (\frac{Cs(1-s)^2}{A\delta_0})^{\frac{1}{s+\alpha-1}} = C\|\theta_0\|_\infty^{\frac{\alpha+1-s}{\alpha-1+s}}$$

The gap of the above proof is that, we assume the MOC property for the region $|x-y| > L$, which is not covered by the CRH condition. To fix this, we will use the fact that $|\theta(x)-\theta(y)|$ is bounded by $2\|\theta(\cdot,t)\|_\infty$. This is further bounded by $2\|\theta_0\|_{L^\infty}$. So, we are able to go through the above steps if, for $\xi = L$, $\delta_0\xi^s > 2\|\theta_0\|_\infty$. We will draw our conclusion. Notice that, we have the requirement that $s+\alpha-1 > 0$, i.e. $s > 1-\alpha$. But this condition is weaker than $s \geq \frac{\alpha^2+\alpha-2}{\alpha-2}$ when $\alpha \in (0,1]$. Using the formulas for L and δ_0 in 5.2 and 5.4, we have the inequality

$$\|\theta_0\|_\infty^{-\alpha+\frac{s-1}{\alpha}+\left(\frac{\alpha+1-s}{\alpha-1+s}\right)s-1} \geq \frac{2}{C}$$

The constant C is the one defined in L equation. In the region $\frac{\alpha^2+\alpha-2}{\alpha-2} \leq s < 1$, the power on the left hand side is negative, which implies the smallness condition.

■

5.5 Eventual Regularity

One of the corollary of the small data regularity theorem is the so called eventually regularity. Is is usually stated as follows,

Theorem 5.4 *For the SQG equations with* $0 < \alpha < 1$*, there exist* $T > 0$ *such that, if* θ *is a local solution on* $[0, T]$*, it is Lipschitz continuous for* $t \in [0, \infty)$

One wildly used idea for proving this type of theorem is that, if we have the decay of one certain norm and a small data regularity theorem related to the same norm, the solution will become regular when the norm drops below the small data threshold. The decay of the $\|theta\|_\infty$ is due to the work of Cordoba and Cordoba [7]

Theorem 5.5 *If* θ *and* u *are smooth solutions to the SQG equation with* $0 < \alpha \leq 1$ *on* $[0, \bar{T}]$*, then*

$$\|\theta(\cdot, t)\|_\infty \leq \frac{\|\theta_0\|_\infty}{(1 + \alpha C t \|\theta_0\|_\infty^\alpha)^{\frac{1}{\alpha}}} \qquad 0 \leq t < \bar{T}$$

Now, let $T > 0$, such that

$$\|\theta(T)\|_\infty^{-\alpha + \frac{s-1}{\alpha} + \left(\frac{\alpha+1-s}{\alpha-1+s}\right)s-1} \geq \frac{2}{C}$$

as claimed in the previous section, we can prove the theorem.

One remark is that the eventual regularity result does not imply the global regularity. The solution can possibly becomes singular before T.

CHAPTER 6

Axi-symmetric Navier-Stokes equations in $\mathbb{R}^n$

6.1 n-dimensional Axi-symmetric Navier-Stokes Model

In this chapter, we focus on a model for axisymmetric Navier-Stokes and Euler equations introduced by Hou and his collaborators. In [42],[43],[40],[41], Hou and Li, Hou and Lei proposed two systems of equations for study in order to understand the stabilizing effects of the nonlinear terms in the 3D axisymmetric Navier-Stokes and Euler equations. The following is a briefly summary of the derivation of these model equations. The incompressible 3D axisymmetric Navier-Stokes equations can be written as

$$\begin{cases} \frac{\widetilde{D}}{Dt}\, u^r - \frac{(u^\theta)^2}{r} = -p_r + \nu \left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz} - \frac{1}{r^2}\right) u^r, \\ \frac{\widetilde{D}}{Dt}\, u^\theta + \frac{u^r\, u^\theta}{r} = \nu \left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz} - \frac{1}{r^2}\right) u^\theta, \\ \frac{\widetilde{D}}{Dt}\, u^z = -p_z + \nu \left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz}\right) u^z, \\ \partial_r u^r + \frac{1}{r}\, u^r + \partial_z u^z = 0, \end{cases} \tag{6.1}$$

where u^r, u^θ and u^z are the cylindrical coordinates of the velocity field $\mathbf{u}$, and

$$\frac{\widetilde{D}}{Dt} = \partial_t + u^r \partial_r + u^z \partial_z.$$

When $\partial_\theta(\cdot) = 0$, these equations reduce to the axisymmetric Euler equations. The corresponding vorticity $\omega = \nabla \times \mathbf{u}$ obey

$$\begin{cases} \frac{\widetilde{D}}{Dt}\omega^r = \nu\left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz} - \frac{1}{r^2}\right)\omega^r + (\omega^r\partial_r + \omega^z\partial_z)u^r, \\ \frac{\widetilde{D}}{Dt}\omega^\theta + \frac{u^\theta\omega^r}{r} = \nu\left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz} - \frac{1}{r^2}\right)\omega^\theta + (\omega^r\partial_r + \omega^z\partial_z)u^\theta + \frac{u^r\omega^\theta}{r}, \\ \frac{\widetilde{D}}{Dt}\omega^z = \nu\left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz}\right)\omega^z + (\omega^r\partial_r + \omega^z\partial_z)u^z. \end{cases} \tag{6.2}$$

Noticing that u^r and u^z can be represented by ψ^θ, ω^r and ω^z by u^θ and the equation relating ω^θ and ψ^θ

$$-\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2}\right)\psi^\theta = \omega^\theta, \tag{6.3}$$

the axisymmetric Navier-Stokes equations reduce to a system of equations for the swirl components ψ^θ, u^θ and ω^θ. By substituting the new variables

$$u_1 = \frac{u^\theta}{r}, \quad \omega_1 = \frac{\omega^\theta}{r}, \quad \psi_1 = \frac{\psi^\theta}{r}$$

in the swirl component equations of (6.1), (6.2) and in (6.3), and dropping the convection terms, Hou and Lei [40] obtained the following system of model equations

$$\begin{cases} \partial_t u_1 = \nu\left(\partial_{rr} + \frac{3}{r}\partial_r + \partial_{zz}\right)u_1 + 2\partial_z\psi_1\, u_1, \\ \partial_t\omega_1 = \nu\left(\partial_{rr} + \frac{3}{r}\partial_r + \partial_{zz}\right)u_1 + \partial_z(u_1^2), \\ -\left(\partial_{rr} + \frac{3}{r}\partial_r + \partial_{zz}\right)\psi_1 = \omega_1. \end{cases} \tag{6.4}$$

Clearly this system of equations is self-contained. When the convection terms are added back to this system of equations, the 3D axisymmetric Navier-Stokes equations can be recovered. Even without the convection terms, these equations possess many similarities as the 3D axisymmetric Navier-Stokes equations. As demonstrated in [40] and [41], regularity criteria of the Prodi-Serrin type and of the Beal-Kato-Majda type [5] still hold for this system of equations.

Our attention is focused on the open problem of whether classical solutions of (6.4) are global in time. The issue is investigated here from two different perspectives. First, we generalize this model to include dissipation given by a fractional Laplacian. For this purpose, we need to interpret these equations as a system of equations in 5-dimensional space. To be more precise, we set $y = (y_1, y_2, y_3, y_4, z) \in \mathbf{R}^5$ and write Δ_y for the 5D Laplacian, namely

$$\Delta_y = \sum_{j=1}^{4} \partial_{y_j y_j} + \partial_{zz}.$$

If a function $f = f(y)$ is axisymmetric about the z-axis, then

$$\Delta_y f = \left(\partial_{rr} + \frac{3}{r}\partial_r + \partial_{zz}\right) f.$$

Identifying u_1, ω_1 and ψ_1 as 5D axisymmetric functions and replacing Δ_y by the fractional Laplacian $-(-\Delta_y)^\alpha$ for a parameter $\alpha > 0$, we obtain the generalized Hou-Lei model

$$\begin{cases} \partial_t u_1 = -\nu(-\Delta_y)^\alpha u_1 + 2\partial_z \psi_1 \, u_1, \\ \partial_t \omega_1 = -\nu(-\Delta_y)^\alpha \omega_1 + \partial_z(u_1^2), \\ (-\Delta_y)\psi_1 = \omega_1. \end{cases} \tag{6.5}$$

6.2 Global Regularity with $\alpha \geq \frac{1}{2} + \frac{n}{4}$

In this section, We study the initial-value problems of these generalized Hou-Lei equations with the initial data

$$u_1(x, 0) = u_{10}(x), \quad \omega_1(x, 0) = \omega_{10}(x), \quad \psi_1(x, 0) = \psi_{10}(x). \tag{6.6}$$

Theorem 6.2.1 *Consider the generalized 3D model (6.5). Assume that the initial data $(u_{10}, \omega_{10}, \psi_{10})$ in (6.6) satisfies*

$$u_{10} \in H^1(\mathbf{R}^5), \quad \psi_{10} \in H^2(\mathbf{R}^5) \quad \text{and} \quad \omega_{10} = -\Delta_y \psi_{10}.$$

When $\alpha \geq \frac{5}{4}$, the solution (u_1, ω_1, ψ_1) emanating from $(u_{10}, \omega_{10}, \psi_{10})$ remains bounded in $H^1(\mathbf{R}^5) \times L^2(\mathbf{R}^5) \times H^2(\mathbf{R}^5)$ for all time. More precisely, we have, for any $0 \leq t < \infty$,

$$\left(\|u_1\|_{H^1(\mathbf{R}^5)} + 2\|\omega_1\|_2^2\right) + \nu \int_0^t \left(\|\Lambda_y^\alpha u_1\|_2^2 + \|\Lambda_y^{1+\alpha}(u_1, \psi_1)\|_2^2 + 2\|\Lambda_y^\alpha \omega_1\|_2^2\right) dt \leq C,$$

where $\Lambda_y = (-\Delta_y)^{1/2}$ and C is a constant depending on $\|u_{10}\|_{H^1}$, $\|\omega_1\|_2$ and $\|\psi_{10}\|_{H^2}$ only.

Proof. [Proof of Theorem 6.2.1] Multiplying the first equation in (6.5) by u_1, the second by $2\psi_1$, integrating over $y \in \mathbf{R}^5$ and performing several integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^5} (u_1^2 + 2|\nabla_y \psi_1|^2)\, dy + \nu \int_{\mathbf{R}^5} (|\Lambda_y^\alpha u_1|^2 + 2|\Lambda_y^{1+\alpha}\psi_1|^2)\, dy = 0,$$

where $\Lambda_y = (-\Delta_y)^{\frac{1}{2}}$. Integrating in time yields

$$\begin{aligned} \int_{\mathbf{R}^5} (u_1^2 + 2|\nabla_y \psi_1|^2)\, dy + 2\nu \int_0^t \int_{\mathbf{R}^5} (|\Lambda_y^\alpha u_1|^2 + 2|\Lambda_y^{1+\alpha}\psi_1|^2)\, dy\, dt \\ = \int_{\mathbf{R}^5} (u_{10}^2 + 2|\nabla_y \psi_{10}|^2)\, dy. \end{aligned} \tag{6.7}$$

To obtain further bounds, we multiply the first equation in (6.5) by $\Delta_y u_1$, the second by $2\,\omega_1$, integrate over $y \in \mathbf{R}^5$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int \left(|\nabla_y u_1|^2 + 2|\omega_1|^2\right)\, dy + \nu \int \left(|\Lambda_y^{1+\alpha} u_1|^2 + 2|\Lambda_y^\alpha \omega_1|^2\right)\, dy = J_1 + J_2, \tag{6.8}$$

where

$$J_1 = \int 2\partial_z\psi_1\, u_1\Delta_y u_1\, dy, \qquad J_2 = \int 2\omega_1\, \partial_z u_1^2\, dy.$$

We estimate J_1 and J_2. By Hölder's inequality,

$$|J_1| \le C\, \|\Delta_y u_1\|_2\, \|\partial_z\psi_1\|_4\, \|u_1\|_4. \tag{6.9}$$

By the Gagliardo-Nirenberg type inequality, for $\alpha \ge 1$,

$$\|\Delta_y u_1\|_2 \le C\, \|u_1\|_2^{\frac{\alpha-1}{\alpha+1}}\, \|\Lambda_y^{1+\alpha} u_1\|_2^{\frac{2}{1+\alpha}}, \tag{6.10}$$

we obtain

$$\|u_1\|_4 \le C\, \|u_1\|_2^a\, \|\nabla_y u_1\|_2^b\, \|\Lambda^\alpha u_1\|_2^c\, \|\Lambda_y^{1+\alpha} u_1\|_2^d, \tag{6.11}$$

where the indices $a, b, c, d \in [0,1]$ and satisfy

$$a+b+c+d = 1, \qquad \frac{1}{4} = \frac{a}{2} + b\left(\frac{1}{2} - \frac{1}{5}\right) + c\left(\frac{1}{2} - \frac{\alpha}{5}\right) + d\left(\frac{1}{2} - \frac{1+\alpha}{5}\right). \tag{6.12}$$

Writing a and b in terms of c and d, we have

$$a = -\frac{1}{4} + (\alpha-1)c + \alpha d, \qquad b = \frac{5}{4} - \alpha c - (1+\alpha)d. \tag{6.13}$$

Similarly,

$$\begin{aligned} \|\partial_z\psi_1\|_4 &\le C\|\partial_z\psi_1\|_2^e\, \|\nabla_y\partial_z\psi_1\|_2^f\, \|\Lambda_y^\alpha\partial_z\psi_1\|_2^g\, \|\Lambda_y^{1+\alpha}\partial_z\psi_1\|_2^h \\ &\le C\, \|\nabla_y\psi_1\|_2^e\, \|\omega_1\|_2^f\, \|\Lambda_y^{1+\alpha}\psi_1\|_2^g\, \|\Lambda_y^\alpha\omega_1\|_2^h, \end{aligned} \tag{6.14}$$

where the indices $e, f, g, h \in [0,1]$ and satisfy

$$e+f+g+h = 1, \qquad \frac{1}{4} = \frac{e}{2} + f\left(\frac{1}{2} - \frac{1}{5}\right) + g\left(\frac{1}{2} - \frac{\alpha}{5}\right) + h\left(\frac{1}{2} - \frac{1+\alpha}{5}\right) \tag{6.15}$$

Or

$$e=(\alpha-1)g+\alpha h-\frac{1}{4},\qquad f=\frac{5}{4}-\alpha g-(1+\alpha)h. \tag{6.16}$$

Inserting (6.10), (6.11) and (6.14) in (6.9), we obtain

$$\begin{aligned}|J_1| \;\le\;& C\,\|u_1\|_2^{\frac{\alpha-1}{\alpha+1}+a}\,\|\nabla_y\psi_1\|_2^e\,\|\nabla_y u_1\|_2^b\,\|\omega_1\|_2^f\,\|\Lambda^\alpha u_1\|_2^c\,\|\Lambda_y^{1+\alpha}\psi_1\|_2^g\\ &\times\|\Lambda_y^{1+\alpha}u_1\|_2^{\frac{2}{1+\alpha}+d}\,\|\Lambda_y^\alpha\omega_1\|_2^h.\end{aligned} \tag{6.17}$$

When

$$\frac{2}{1+\alpha}+d+h\le 2,$$

we apply Young's inequality with

$$\frac{h}{2}+\frac{1}{1+\alpha}+\frac{d}{2}+\frac{1}{p}=1\quad\text{or}\quad p=\frac{2(\alpha+1)}{2\alpha-(\alpha+1)(h+d)} \tag{6.18}$$

to obtain

$$\begin{aligned}|J_1| \;\le\;& \frac{\nu}{2}\|\Lambda_y^\alpha\omega_1\|_2^2+\frac{\nu}{2}\,\|\Lambda_y^{1+\alpha}u_1\|_2^2\\ &+C(\nu)\,\|u_1\|_2^{\gamma_1}\,\|\nabla_y\psi_1\|_2^{\gamma_2}\|\nabla_y u_1\|_2^{\gamma_3}\,\|\omega_1\|_2^{\gamma_4}\,\|\Lambda^\alpha u_1\|_2^{\gamma_5}\,\|\Lambda_y^{1+\alpha}\psi_1\|_2^{\gamma_6},\end{aligned}$$

where

$$\gamma_1=p\left(\frac{\alpha-1}{\alpha+1}+a\right),\quad \gamma_2=p\,e,\quad \gamma_3=p\,b,\quad \gamma_4=p\,f,\quad \gamma_5=p\,c,\quad \gamma_6=p\,g.$$

When $\gamma_3+\gamma_4\le 2$ and $\gamma_5+\gamma_6\le 2$, namely

$$p(b+f)\le 2\quad\text{and}\quad p(c+g)\le 2, \tag{6.19}$$

we can apply Young's inequality again to further bound J_1 by

$$
\begin{aligned}
|J_1| \;\leq\; & \frac{\nu}{2}\|\Lambda_y^{\alpha}\omega_1\|_2^2 + \frac{\nu}{2}\,\|\Lambda_y^{1+\alpha}u_1\|_2^2 \,+ C(\nu)\,\|u_1\|_2^{\gamma_1}\,\|\nabla_y\psi_1\|_2^{\gamma_2} \\
& \times \left(\|\nabla_y u_1\|_2^2 + \,\|\omega_1\|_2^2\right)\,\left(\|\Lambda_y^{\alpha}u_1\|^2 + \,\|\Lambda_y^{1+\alpha}\psi_1\|_2^2\right),
\end{aligned}
\tag{6.20}
$$

Invoking (6.13), (6.16) and (6.18), the conditions in (6.19) can be rewritten as

$$
\frac{2(\alpha+1)}{2\alpha-(\alpha+1)(d+h)}\cdot\left(\frac{5}{2}-\alpha(c+g)-(1+\alpha)(d+h)\right)\leq 2, \tag{6.21}
$$

$$
\frac{2(\alpha+1)}{2\alpha-(\alpha+1)(d+h)}\,(c+g)\leq 2. \tag{6.22}
$$

Equivalently,

$$
\frac{\alpha+5}{2\alpha(\alpha+1)}\leq (c+g)+(d+h)\leq \frac{2\alpha}{\alpha+1}. \tag{6.23}
$$

When $\alpha \geq \frac{5}{4}$,

$$
\frac{\alpha+5}{2\alpha(\alpha+1)}\leq \frac{2\alpha}{\alpha+1}
$$

and we can select suitable c, g, d and h so that (6.23) holds and thus (6.19) holds. Some special choices of the indices a, b, c, d and e, f, g, h are

$$
a=0,\quad b=\frac{4}{9},\quad c=\frac{4}{9},\quad d=\frac{1}{9},\quad e=0,\quad f=\frac{4}{9},\quad g=\frac{4}{9},\quad h=\frac{1}{9}
$$

in the case $\alpha = \frac{5}{4}$, and

$$
a=e=0,\quad b=f=\frac{4\alpha^2+3\alpha-5}{4\alpha(\alpha+1)},\quad c=g=\frac{1}{\alpha+1},\quad d=h=\frac{5-3\alpha}{4\alpha(\alpha+1)}
$$

in the case of $\alpha \geq \frac{5}{4}$.

We now bound J_2. By the third equation in (6.5), J_2 can be written as

$$
J_2 = -4\int u_1\,\partial_z u_1\,\Delta_y\psi_1\,dy.
$$

For any $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, we have, by Hölder's inequality,

$$|J_2| \le \|u_1\|_p \, \|\partial_z u_1\|_q \|\omega_1\|_2. \tag{6.24}$$

Furthermore, by the Gagliardo-Nirenberg type inequalities

$$\begin{aligned} \|u_1\|_p &\le C \, \|u_1\|_2^{a_1} \, \|\nabla_y u_1\|_2^{b_1} \, \|\Lambda^\alpha u_1\|_2^{c_1} \, \|\Lambda^{1+\alpha} u_1\|_2^{d_1}, \\ \|\partial_z u_1\|_q &\le C \, \|\nabla_y u_1\|_2^{b_2} \, \|\Lambda^\alpha u_1\|_2^{c_2} \, \|\Lambda^{1+\alpha} u_1\|_2^{d_2} \end{aligned} \tag{6.25}$$

with the indices satisfying

$$a_1 + b_1 + c_1 + d_1 = 1, \quad b_2 + c_2 + d_2 = 1,$$

$$\frac{1}{p} = \frac{a_1}{2} + b_1 \left(\frac{1}{2} - \frac{1}{5}\right) + c_1 \left(\frac{1}{2} - \frac{\alpha}{5}\right) + d_1 \left(\frac{1}{2} - \frac{1+\alpha}{5}\right),$$
$$\frac{1}{q} - \frac{1}{5} = b_2 \left(\frac{1}{2} - \frac{1}{5}\right) + c_2 \left(\frac{1}{2} - \frac{\alpha}{5}\right) + d_2 \left(\frac{1}{2} - \frac{1+\alpha}{5}\right),$$

we obtain

$$\|u_1\|_p \, \|\partial_z u_1\|_q \le C \, \|u_1\|_2^{a_1} \, \|\nabla_y u_1\|_2^{b_3} \, \|\Lambda^\alpha u_1\|_2^{c_3} \, \|\Lambda^{1+\alpha} u_1\|_2^{d_3}, \tag{6.26}$$

where $b_3 = b_1 + b_2$, $c_3 = c_1 + c_2$ and $d_3 = d_1 + d_2$. Clearly

$$a_1 + b_3 + c_3 + d_3 = 2, \tag{6.27}$$
$$\frac{a_1}{2} + b_3 \frac{3}{10} + c_3 \frac{5-2\alpha}{10} + d_3 \frac{3-2\alpha}{10} = \frac{3}{10}. \tag{6.28}$$

Inserting (6.26) in (6.24) and applying Young's inequality, we obtain

$$|J_2| \le \frac{\nu}{2} \|\Lambda^{1+\alpha} u_1\|_2^2 + C(\nu) \, \|u_1\|_2^{\frac{2a_1}{2-d_3}} \, \|\nabla_y u_1\|_2^{\frac{2b_3}{2-d_3}} \, \|\Lambda^\alpha u_1\|_2^{\frac{2c_3}{2-d_3}} \, \|\omega_1\|_2^{\frac{2}{2-d_3}}.$$

If

$$\frac{2c_3}{2-d_3} \leq 2, \qquad \frac{2b_3}{2-d_3} + \frac{2}{2-d_3} \leq 2, \tag{6.29}$$

a further application of Young's inequality implies

$$|J_2| \leq \frac{\nu}{2}\|\Lambda^{1+\alpha}u_1\|_2^2 + C(\nu)\,\|u_1\|_2^{\frac{2a_1}{2-d_3}}\,\|\Lambda^{\alpha}u_1\|_2^2\left(\|\nabla_y u_1\|_2^2 + \|\omega_1\|_2^2\right). \tag{6.30}$$

When $\alpha \geq \frac{5}{4}$, we can choose suitable a_1, b_2, c_3 and d_3 so that they satisfy (6.27), (6.28) and (6.29). In fact, these conditions are equivalent to

$$\begin{aligned} &a_1 + c_3 = 2 - (b_3 + d_3), \\ &(b_3 + d_3) + \alpha(c_3 + d_3) = \frac{7}{2}, \\ &c_3 + d_3 \leq 2, \quad b_3 + d_3 \leq 1 \end{aligned}$$

and all of them are obviously satisfied if we set

$$a_1 = 0, \quad b_3 = 2 - \frac{5}{2\alpha}, \quad c_3 = 1 \quad \text{and} \quad d_3 = \frac{5}{2\alpha} - 1.$$

Combining (6.8), (6.20) and (6.30), we find that

$$\begin{aligned} &\frac{d}{dt}\left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2\right) + \nu\left(\|\Lambda_y^{1+\alpha}u_1\|_2^2 + 2\|\Lambda_y^{\alpha}\omega_1\|_2^2\right) \\ &\quad \leq C(\nu)\,\|u_1\|_2^{\gamma_1}\,\|\nabla_y\psi_1\|_2^{\gamma_2}\left(\|\Lambda_y^{\alpha}u_1\|^2 + \|\Lambda_y^{1+\alpha}\psi_1\|_2^2\right)\left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2\right) \\ &\qquad + C(\nu)\,\|u_1\|_2^{\frac{2a_1}{2-d_3}}\,\|\Lambda^{\alpha}u_1\|_2^2\left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2\right). \end{aligned}$$

It then follows from Gronwall's inequality and (6.7) that

$$\left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2\right) + \nu\int_0^t \left(\|\Lambda_y^{1+\alpha}u_1\|_2^2 + 2\|\Lambda_y^{\alpha}\omega_1\|_2^2\right)\,dt \leq C.$$

where C is a constant depending on the norms of the initial data, namely $\|u_{10}\|_2 +$

$\|\nabla_y u_{10}\|_2$, $\|\nabla_y \psi_{10}\|_2$ and $\|\omega_{10}\|_2$. When the initial data are more regular, the solution of (6.5) can be shown to be more regular. In particular, smooth data yield smooth solutions. This completes the proof of Theorem 6.2.1. ■

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