A STUDY OF PERFECT NUMBERS AND UNITARY PERFECT NUMBERS

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CHAPTER I

HISTORY AND INTRODUCTION

This thesis is concerned with the positive integers and some of their special properties in terms of their divisors. An N will be used to denote this set of positive integers and only small letters will be used to denote positive integers unless otherwise specified.

From ancient times some positive integers have been considered to have magical properties. Some such positive integers are 7 and 40 from their use in the Bible, 6 which is the number of days in the creation of the world, and 28 which is the length of the lunar cycle [1;p. 94].[1]

One classification made by the ancient Greeks depended upon the sum of the <u>aliquot parts</u> of a positive integer n, that is, the divisors of n other than n itself. The positive integer n is <u>deficient</u>, <u>abundant</u>, or <u>perfect</u> if the sum of the aliquot parts is less than, greater than, or equal to n. For example, 8 is deficient since $1 + 2 + 4 = 7 < 8$, 12 is abundant since $1 + 2 + 3 + 4 + 6 = 16 > 12$, and 6 is perfect since $1 + 2 + 3 = 6$. The ancient Greeks knew five perfect numbers 6, 28, 496, 8128, and 33550336. Later mathematicians have extended this list to include twenty-four perfect numbers, the largest of which contains 12,003 digits [2].

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[1] Numbers in brackets refer to the bibliography at the end of the thesis.

In the third century B.C. Euclid proved that if

$$p = 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$$

is prime, then $2^k p$ is perfect. This is proposition 36 in Book IX of The Elements. In the eighteenth century Leonhard Euler proved that all even perfect numbers are of this type. Euler also determined some conditions necessary for odd numbers to be perfect [3]. However, no one has yet proven the existence or nonexistence of odd perfect numbers.

In 1965 M. V. Subbarao and L. J. Warren studied unitary perfect numbers. A number is unitary perfect if the sum of its unitary divisors, other than itself, is equal to the number where a divisor d of a number n is a unitary divisor if d and n/d are relatively prime numbers [4].

Chapter II will show a comparison between Euclid's method and the modern method of proving that a number of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a prime, is perfect. It will be shown that all even perfect numbers are of this form. The value of p for the twenty-four known perfect numbers and the numerical value of the first thirteen perfect numbers will be listed. Interesting, but not as well known, properties of even perfect numbers will be presented.

Chapter III will show that if an odd perfect number exists, it is of the form $n = p^a p_1^{2b_1} p_2^{2b_2} \cdots p_k^{2b_k}$, where $p \equiv a \equiv 1 \bmod 4$ and p, p_i, $i = 1, 2, \ldots, k$ are distinct odd primes. Other restrictions for p, a, p_i, and b_i, $i = 1, 2, \ldots, k$ are included. It will be shown that for n to be an odd perfect number, $n \equiv 1 \bmod 4$ and $n \equiv 1 \bmod 12$ or $n \equiv 9 \bmod 36$. It will be shown that there is at most a finite number of odd perfect numbers with a given number of distinct prime factors.

Included are some upper bounds for p_1, the smallest prime divisor of an odd perfect number, such as $p_1 \leq k$, $p_1 < \frac{2}{3}k + 2$, and $p_1 \leq \frac{3}{2}k + 1$ where k is the number of distinct prime divisors. In addition, if n is an odd perfect number, then

$$\frac{1}{2} < \sum_{p|n} \frac{1}{p} < 2 \ln \frac{\pi}{2} ,$$

where p is a prime. Improvements on these bounds will also be shown.

Chapter IV will show that all unitary perfect numbers are of the form $n = 2^t p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where the p_i's are distinct odd primes and $t > 0$. The five known unitary perfect numbers will be shown.

Parts of this thesis could be used in a seminar for high school students and for enrichment and supplementary material for an elementary number theory course. Also, this thesis could be used as a reference by others wishing to do work in the area of perfect numbers or unitary perfect numbers.

It is not necessary for a person to have an extensive knowledge of mathematics or number theory to read this thesis, but some background in selected topics of number theory such as congruences and number-theoretic functions would be helpful.

CHAPTER II

EVEN PERFECT NUMBERS

The theory of even perfect numbers is well developed. Euclid proved that if $p = 1 + 2 + \cdots + 2^k = 2^{k+1} - 1$ is prime, then $2^k p$ is perfect. Euler proved that all even perfect numbers are of this form. Many interesting facts about even perfect numbers are also known.

In his theorem, Proposition 36 in Book IX of The Elements, Euclid used Proposition 35 which states that if a set of numbers is in continued proportion (a geometric progression), and if the first number is subtracted from the second and last numbers, then the ratio of this first difference is to the first number as the second difference is to the sum of all the numbers before it [5;p. 420]. That is, stated in present day symbolic algebra, if $a, ar, ar^2, \ldots, ar^k$ is a geometric progression then

$$\frac{ar - a}{a} = \frac{ar^k - a}{a + ar + ar^2 + \cdots + ar^{k-1}}.$$

This is equivalent to

$$a + ar + ar^2 + \cdots + ar^{k-1} = \frac{a(r^k - 1)}{r - 1}$$

which is the well-known formula for the sum of a geometric progression.

The following is Euclid's proof of Proposition 36 as taken from the translation by Sir Thomas L. Heath [6;p. 421]. Although the proof is difficult to read, most of the terminology and symbolism of Heath's translation is retained in order to show, by comparison with a proof later in this chapter of the same proposition, the advantage of using the present day symbolic algebra and number theory techniques.

Let the numbers A, B, C, D (not necessarily four in number) beginning from a unit (the integer one) be set out in double proportion (A is double the unit and each of the others is double the preceding number) until the sum of all, including the unit, is a prime. Let E be equal to this sum. Let FG be the product of E and D. Then FG is perfect. For however many numbers there are in A, B, C, D, let the same amount E, HK, L, M be taken in double proportion beginning from E. Therefore, the product of E and D is equal to the product of A and M. But the product of E and D is FG. Therefore, the product of A and M is FG. Since A is the double of the unit, FG is the double of M. Then E, HK, L, M, FG are in double proportion. Subtract from HK and FG the numbers HN and FO each equal to E. Therefore, by Proposition 35, the ratio of the first difference NK is to E as the second difference OG is to the sum of E, HK, L, and M. But since HK is the double of E, NK is equal to E. Therefore, OG equals the sum of E, HK, L, and M. But FO is also equal to E which is the sum of the unit, A, B, C, and D. Therefore, the whole FG is equal to the sum of E, HK, L, M, A, B, C, D and the unit. Also, FG is measured by E, HK, L, M, A, B, C, D and the unit. That is, these are all factors of FG.

FG is not measured by any other number. For, let P measure FG and be different from E, HK, L, M, A, B, C, D, and the unit. Let Q be the number such that FG is the product of P and Q. Since the product of E and D is also FG, the ratio of E to Q is equal to the ratio of P to D. Since A, B, C, D are continuously proportional beginning from a unit, D is measured by no number other than A, B, or C. Since P is not A, B, or C, P does not measure D. Then E does not measure Q. Then, since E is prime, E and Q are prime to one another. Thus, the ratio of E to Q is a fraction reduced to lowest terms. Since the ratio of E to Q is equal to the ratio of P to D, E measures P the same number of times that Q measures D. Since D is measured only by A, B, and C, Q is either A, B, or C. Let D be equal to B. How many numbers there are in B, C, D, let the same amount E, HK, L be taken. Then the ratio of B to D is equal to the ratio of E to L. Therefore, the product of B and L is equal to the product of D and E. Since the product of D and E is equal to the product of Q and P, the product of Q and P is equal to the product of B and L. Therefore, the ratio of Q to B is equal to the ratio of L to P. Since Q is equal to B, L is equal to P. This is impossible since P is different from E, HK, L, M, A, B, C, D, and the unit. Therefore, no number other than A, B, C, D, E, HK, L, M, and the unit measures FG.

Since FG is the sum of A, B, C, D, E, HK, L, M, and the unit and is measured only by them, FG is perfect.

If $a,b \in N$, the greatest common divisor of a and b is denoted by (a,b). If a and b are relatively prime, $(a,b) = 1$. The notation $a|b$ indicates that a divides b.

A function f defined on the positive integers is said to be <u>multiplicative</u> if $f(mn) = f(m)f(n)$, whenever $(m,n) = 1$.

Any positive integer greater than 1 can be expressed uniquely in canonical form, that is, if $n \in N$, there exists primes $p_i \in N$ and numbers a_i, $i = 1, 2, \ldots, k$ such that

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} = \prod_{i=1}^{k} p_i^{a_i}.$$

If $\sigma(n)$ represents the sum of the divisors of n, including n itself, then

$$\sigma(n) = \sum_{d|n} d = \prod_{i=1}^{k} (1 + p_i + \cdots + p_i^{a_i})$$

$$= \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

This function is multiplicative [1;p. 95].

For $n \in N$, n is perfect if $\sigma(n) = 2n$; n is abundant if $\sigma(n) > n$; and n is deficient if $\sigma(n) < n$. These are equivalent to the definitions given in Chapter I.

<u>Theorem 2.1</u>. If n is a perfect number and $k \in N$ and $k > 1$, then kn is abundant.

PROOF: Let $d_1, d_2, \ldots, d_s$ be the divisors of n. Since n is perfect,

$$\sigma(n) = \sum_{i=1}^{s} d_i = 2n.$$

If $k \in N$ and $k > 1$, then some of the divisors of kn are 1, $kd_1, kd_2, \ldots, kd_s$. Therefore,

$$\sigma(kn) \geq 1 + \sum_{i=1}^{s} kd_i > k \sum_{i=1}^{s} d_i = 2kn,$$

and kn is abundant.

<u>Theorem 2.2</u>. If n is a perfect number and $k \neq n$ is a divisor of n, then k is deficient.

PROOF: If $d_1, d_2, \ldots, d_s$ are the divisors of k, then 1, $(n/k)d_1, (n/k)d_2, \ldots, (n/k)d_s$ are divisors of n. Then

$$2n = \sigma(n) \geq 1 + \sum_{i=1}^{s} \frac{n}{k} d_i > \frac{n}{k} \sum_{i=1}^{s} d_i = \frac{n}{k} \sigma(k).$$

Then, $\sigma(n) < 2n\left(\frac{k}{n}\right) = 2k$ and k is abundant.

Basic Theorems

Euclid's theorem can now be stated and proved in the following manner.

<u>Theorem 2.3</u>. If $2^k - 1$ is a prime, then $2^{k-1}(2^k - 1)$ is a perfect number.

PROOF: Since $2^k - 1$ is a prime, $(2^{k-1}, 2^k - 1) = 1$ and

$$\sigma[2^{k-1}(2^k - 1)] = \sigma(2^{k-1})\sigma(2^k - 1) = \frac{2^k - 1}{2 - 1}[1 + (2^k - 1)]$$

$$= (2^k - 1)2^k = 2[2^{k-1}(2^k - 1)].$$

Therefore, $2^{k-1}(2^k - 1)$ is perfect [1;p. 98].

The next theorem is the converse of Theorem 2.3 and was first proved by Euler [3;p. 19].

<u>Theorem 2.4</u>. If n is an even perfect number, then there exists a number k such that $n = 2^{k-1}(2^k - 1)$ where $(2^k - 1)$ is a prime number.

PROOF: Since n is even, $n = 2^t m$, where m is an odd integer and $t \in N$. Then $(2^t, m) = 1$ and

$$\sigma(n) = \sigma(2^t m) = \sigma(2^t)\sigma(m) = \frac{2^{t+1} - 1}{2 - 1}\sigma(m)$$

$$= (2^{t+1} - 1)\sigma(m).$$

But since n is perfect, $\sigma(n) = 2n = 2(2^t m) = 2^{t+1}m$. Thus, $(2^{t+1} - 1)\sigma(m) = 2^{t+1}m$ and then $(2^{t+1} - 1)|2^{t+1}m$. Since $2^{t+1} - 1$ is odd, $(2^{t+1} - 1)|m$. Then $m = (2^{t+1} - 1)u$, where u is an odd integer. Suppose $u > 1$. Then 1, u, and $(2^{t+1} - 1)u$ are distinct divisors of m, so that

$$\sigma(m) \geq 1 + u + (2^{t+1} - 1)u > 2^{t+1}u.$$

Therefore,

$$\sigma(n) = (2^{t+1} - 1)\sigma(m) > (2^{t+1} - 1)2^{t+1}u = 2n,$$

which contradicts n being perfect. Thus, $u = 1$ and $m = 2^{t+1} - 1$. Suppose that m is not a prime. Then,

$$\sigma(m) > 1 + (2^{t+1} - 1) = 2^{t+1}.$$

Then,

$$\sigma(n) = (2^{t+1} - 1)\sigma(m) > (2^{t+1} - 1)2^{t+1} = 2n.$$

This contradicts n being perfect. Therefore, m is a prime. Let $k = t + 1$ and then $n = 2^{k-1}(2^k - 1)$ with $(2^k - 1)$ a prime [1;p. 98].

Theorem 2.5. If $2^k - 1$ is prime, then k is prime.

PROOF: If $k = ab$, $a > 1$, $b > 1$, then

$$2^k - 1 = 2^{ab} - 1 = (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1)$$

and $2^k - 1$ is not prime. Thus, if $2^k - 1$ is prime, k must be prime also.

Numbers of the form $M_n = 2^n - 1$ are called Mersenne numbers after Marin Mersenne. The problem of finding even perfect numbers is, therefore, the problem of finding Mersenne primes of the form $M_p = 2^p - 1$. Mersenne in the seventeenth century stated that M_2, M_3, M_5, M_7, M_{13}, M_{17}, M_{19}, M_{31}, M_{67}, M_{127}, and M_{257} were prime. However, $M_{67} = 2^{67} - 1 = (193707721)(761838257287)$ and M_{257} is also composite [3;p. 29]. Twenty-three Mersenne primes, and hence, twenty-four perfect numbers are now known. They are M_p for $p = 2, 3, 5, 7, 13,$

17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, and 19,937. The twenty-first, twenty-second, and twenty-third of these were discovered by the use of Illiac II at the Digital Computer Laboratory of the University of Illinois. The times required by the computer were one hour and twenty-three minutes, one hour and thirty minutes, and two hours and fifteen minutes, respectively [6]. The last one was discovered by Bryant Tuckerman, a mathematician with the International Business Machines Corporation using a System/360 Model 91 computer, the largest IBM machine in common use today. The time required was nearly forty minutes [2].

The numerical values of the first thirteen perfect numbers have been listed by Uhler [7]. However, the fifth number listed is incorrect. It is listed as 33350336 but according to Dickson [3;p. 7] it should be 33550336. Uhler's list with this correction made is as follows:

$$2(2^2 - 1) = 6$$
$$2^2(2^3 - 1) = 28$$
$$2^4(2^5 - 1) = 496$$
$$2^6(2^7 - 1) = 8128$$
$$2^{12}(2^{13} - 1) = 3355\ 0336$$
$$2^{16}(2^{17} - 1) = 85898\ 69056$$
$$2^{18}(2^{19} - 1) = 13\ 74386\ 91328$$
$$2^{30}(2^{31} - 1) = 2305\ 84300\ 81399\ 52128$$
$$2^{60}(2^{61} - 1) = 26\ 58455\ 99156\ 98317\ 44654\ 69261\ 59538\ 42176$$
$$2^{88}(2^{89} - 1) = 1915\ 61942\ 60823\ 61072\ 94793\ 37808\ 43036$$
$$38130\ 99732\ 15480\ 69216$$

$2^{106}(2^{107} - 1)$ = 14 13164 03645 85696 48337 23975 34604 58722
91022 34723 18386 94311 77837 28128

$2^{126}(2^{127} - 1)$ = 47401 11546 64524 42794 63731 26085 98848
15736 77491 47483 58890 66354 34913 11991
52128

$2^{520}(2^{521} - 1)$ = 2356 27234 57267 34706 57895 48996 70990
49884 77547 85839 26007 10143 02759 75063
37283 17862 22397 30365 53960 26005 61360
25556 64625 03270 17505 28925 78043 21554
33824 98428 77715 24270 10394 49691 86640
28644 53412 80338 31439 79023 68386 24033
17143 59223 56643 21970 31017 20713 16352
74872 98747 40064 78019 39587 16593 64010
87419 37564 90579 18549 49216 05556 46976.

Some Congruence Relations

It was once thought that even perfect numbers ended alternately in 6 or 8. This was due to the belief that some Mersenne numbers were prime when they were actually composite, and consequently, did not give numbers that were perfect. However, even perfect numbers do end in a 6 or an 8. That they do not end alternately in a 6 or an 8 is seen from the fact that the fifth and sixth perfect numbers both end in 6.

Theorem 2.6. If $n = 2^{p-1}(2^p - 1)$ is a perfect number where p is a prime number, then n ends in 6 or 28.

PROOF: In $n \neq 6$, p is odd and of the form $p = 2k + 1$. Then,

$$n = 2^{2k}(2^{2k+1} - 1) = 4^k(2\cdot 4^k - 1).$$

It can be shown that $4^k \equiv 4$ or $6 \bmod 10$. If $4^k \equiv 6 \bmod 10$,

$$n = 4^k(2\cdot 4^k - 1) \equiv 6(12 - 1) \equiv 6 \bmod 10,$$

and n ends in a 6. If $4^k \equiv 4 \bmod 10$, there exists an integer m such that $4^k = 4 + 10m$. Since $4|4^k$ and $4|4$, then $4|10m$ which implies that $2|m$. Thus, $4^k = 4 + 20t$ for $m = 2t$. Then

$$\begin{aligned} n &= (20t + 4)(40t + 8 - 1) = (20t + 4)(40t + 7) \\ &= 800t^2 + 300t + 28 \equiv 28 \bmod 100, \end{aligned}$$

and n ends in a 28. Thus, n ends in a 6 or a 28.

If $n \in N$ is written in the usual base 10 notation with digits $a_1, a_2, \ldots, a_k$ where $0 \leq a_i \leq 9$ for $0 \leq i \leq k - 1$ and $0 < a_k \leq 9$, then

$$n = \sum_{i=0}^{k} a_i 10^i \equiv \sum_{i=0}^{k} a_i \bmod 9$$

and

$$n \geq \sum_{i=0}^{k} a_i.$$

Let n_1 be the sum of the digits of n, let n_2 be the sum of the digits of n_1, and continue this process until a one digit number n_k

is obtained. One obtains a finite sequence $n > n_1 > n_2 > \cdots > n_k$ such that $n \equiv n_1 \equiv n_2 \equiv \cdots \equiv n_k \bmod 9$.

Theorem 2.7. If n is any even perfect number, except 6, then $n \equiv 1 \bmod 9$. Thus, if n_1 is the sum of the digits of n, n_2 the sum of the digits of n_1, ..., n_{i+1} the sum of the digits of n_i, then $n > n_1 > n_2 > \cdots > n_t > 1$ and $n \equiv n_1 \equiv n_2 \equiv \cdots \equiv n_t \equiv 1 \bmod 9$.

PROOF: If n is an even perfect number other than 6, there exists a positive integer k such that $n = 4^k(2 \cdot 4^k - 1)$ by the proof of Theorem 2.6. Since $4^k \equiv 1, 4,$ or $7 \bmod 9$, then

$$n \equiv 1(2 - 1) \equiv 1 \bmod 9,$$

$$n \equiv 4(8 - 1) \equiv 28 \equiv 1 \bmod 9,$$

or

$$n \equiv 7(14 - 1) \equiv 91 \equiv 1 \bmod 9.$$

Thus, $n > n_1 > n_2 > n_t > 1$ and $n \equiv n_1 \equiv n_2 \equiv \cdots \equiv n_t \equiv 1 \bmod 9$ by the remarks preceding the theorem.

As an illustration of Theorem 2.7 consider the sixth even perfect number $n = 8589869056$.

$$n_1 = 8 + 5 + 8 + 9 + 0 + 6 + 9 + 0 + 5 + 6 = 64$$

$$n_2 = 6 + 4 = 10$$

$$n_3 = 1 + 0 = 1$$

$$8589869056 \equiv 64 \equiv 10 \equiv 1 \bmod 9$$

Theorem 2.8. If n is any even perfect number other than 28, then $n \equiv \pm 1 \bmod 7$.

PROOF: Let $n = 2^{p-1}(2^p - 1)$. Then p is of the form $p = 3k$, $p = 3k + 1$, or $p = 3k + 2$. Since p is a prime, if $p = 3k$, $k = 1$ and $p = 3$. Then $n = 28$. If $p = 3k + 1$,

$$n = 2^{3k}(2^{3k+1} - 1) = 8^k(2\cdot 8^k - 1) \equiv 1(2 - 1) \equiv 1 \bmod 7.$$

If $p = 3k + 2$,

$$n = 2^{3k+1}(2^{3k+2} - 1) = 2\cdot 8^k(4\cdot 8^k - 1)$$
$$\equiv 2(4 - 1) \equiv 6 \equiv -1 \bmod 7.$$

Therefore, $n = 28$ or $n \equiv \pm 1 \bmod 7$.

<u>Theorem 2.9</u>. If n is an even perfect number, other than 6, then $n \equiv 1, 2, 3,$ or $8 \bmod 13$.

PROOF: Since n is perfect, $n = 2^{p-1}(2^p - 1)$ where p is a prime. If $p = 2$, $n = 6$. If $p = 3$, $n = 28 \equiv 2 \bmod 13$. If $p = 5$, $n = 496 \equiv 2 \bmod 13$. If $p = 7$, $n = 8128 \equiv 3 \bmod 13$. If $p \geq 13$, since p is prime, p is of the form $p = 12k + 1$, $p = 12k + 5$, $p = 12k + 7$, or $p = 12k + 11$. Now,

$$2^{12k} = 16^{3k} \equiv 3^{3k} \equiv 27^k \equiv 1^k \equiv 1 \bmod 13.$$

If $p = 12k + 1$,

$$n = 2^{12k}(2^{12k+1} - 1) = 2^{12k}(2\cdot 12^{12k} - 1)$$
$$\equiv 1(2 - 1) \equiv 1 \bmod 13.$$

If $p = 12k + 5$,

$$n = 2^{12k+4}(2^{12k+5} - 1) = 16\cdot 2^{12k}(32\cdot 2^{12k} - 1)$$

$$\equiv 3\cdot 1(6\cdot 1 - 1) \equiv 3(5) \equiv 15 \equiv 2 \bmod 13.$$

If $p = 12k + 7$,

$$n = 2^{12k+6}(2^{12k+7} - 1) = 16\cdot 4\cdot 2^{12k}(16\cdot 8\cdot 2^{12k} - 1)$$

$$\equiv 3\cdot 4(3\cdot 8 - 1) \equiv 12(23) \equiv -1(-3) \equiv 3 \bmod 13$$

If $p = 12k + 11$,

$$n = 2^{12k+10}(2^{12k+11} - 1) = 16^2 4\cdot 2^{12k}(16^2 8\cdot 2^{12k} - 1)$$

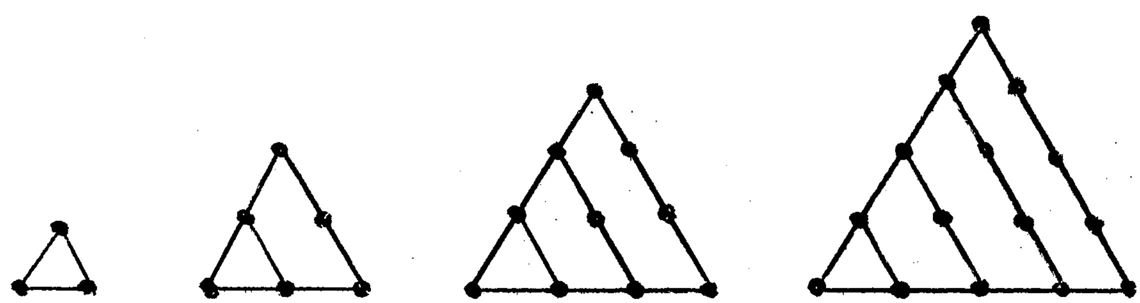
$$\equiv 3^2 4(3^2 8 - 1) \equiv 36(72 - 1) \equiv -3(71) \equiv -3(-7)$$

$$\equiv 21 \equiv 8 \bmod 13.$$

Therefore, $n = 6$ or $n \equiv 1, 2, 3,$ or $8 \bmod 13$.

Geometric Numbers

A number n is triangular if n points can be arranged in a triangular diagram by the following procedure. The diagram for the first triangular number is an equilateral triangle with sides of unit length and points at the three vertices. The first triangular number is then 3. Let one vertex be an origin. The diagram for the second triangular number is obtained by superimposing an equilateral triangle with sides of length 2 units on the diagram for the first triangular number so that a vertex and adjacent sides coincide with the origin and its adjacent sides. The third side of the superimposed triangle is then partitioned by points into two segments of unit length. The second triangular number is then the number of points that are now in the diagram. In general, the diagram for the $(k + 1)^{th}$ triangular number is constructed by superimposing an equilateral triangle with sides of length $k + 1$ units on the diagram for the k^{th} triangular number so that a vertex and adjacent sides coincide with the origin and its adjacent sides. The third side of the superimposed triangle is then partitioned by points into $k + 1$ segments of unit length. The $(k + 1)^{th}$ triangular number is then the number of points in the diagram. The first four triangular numbers are 3, 6, 10, and 15. Their diagrams are shown below.



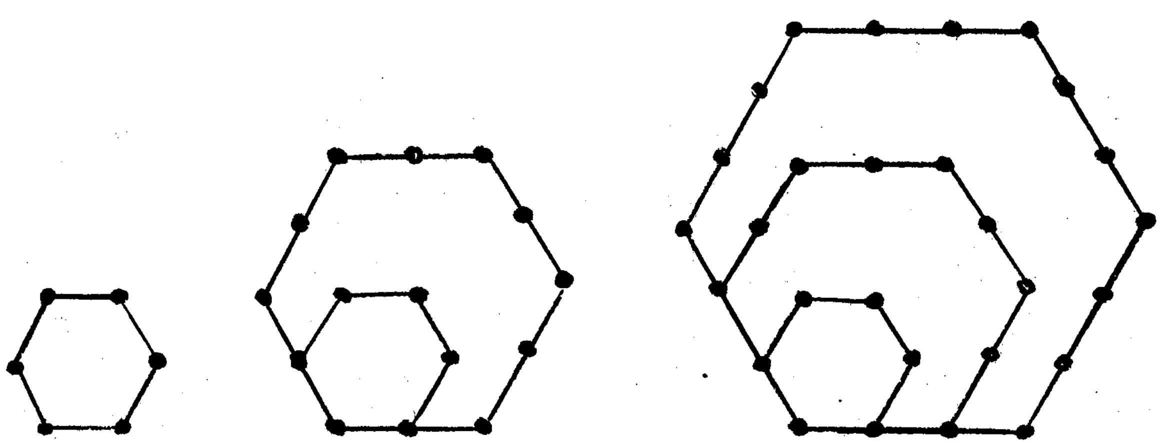
3 6 10 15

If n is the k^{th} triangular number,

$$n = 1 + 2 + \cdots + (k + 1) = \frac{(k + 1)(k + 2)}{2}.$$

Thus, a number is triangular if it is of this form.

A number n is hexagonal if n points can be arranged in a hexagonal diagram by the following procedure. The diagram for the first hexagonal number is a regular hexagon with sides of unit length and points at the 6 vertices. The first hexagonal number is then 6. Let one vertex be an origin. The diagram for the second hexagonal number is obtained by superimposing a regular hexagon with sides of length 2 units on the diagram for the first hexagonal number so that a vertex and adjacent sides coincide with the origin and its adjacent sides. The other four sides are then partitioned by points into two segments of unit length. The second hexagonal number is then the number of points that are now in the diagram. In general, the diagram for the $(k + 1)^{th}$ hexagonal number is constructed by superimposing a regular hexagon with sides of length $k + 1$ units on the diagram for the k^{th} hexagonal number so that a vertex and its adjacent sides coincide with the origin and its adjacent sides. The other four sides of the superimposed hexagon are then partitioned by points into $k + 1$ segments of unit length. The $(k + 1)^{th}$ hexagonal number is then the number of points in the diagram. The first three hexagonal numbers are 6, 15, and 28. Their diagrams are shown below.



6 15 28

If n is the k^{th} hexagonal number,

$$n = 1 + 5 + 9 + 13 + \cdots + (4k + 1)$$

$$= \frac{(k + 1)(4k + 2)}{2} = (k + 1)(2k + 1).$$

Thus, a number is hexagonal if it is of this form. Since,

$$n = (k + 1)(2k + 1) = \frac{(2k + 1)(2k + 2)}{2}$$

a hexagonal number is also a triangular number.

Theorem 2.10. If n is an even perfect number, n is a hexagonal number, and hence, also a triangular number.

PROOF: Since n is perfect,

$$n = 2^{p-1}(2^p - 1) = 2^{p-1}(2 \cdot 2^{p-1} - 2 + 1)$$

$$= (2^{p-1} - 1 + 1)[2(2^{p-1} - 1) + 1]$$

$$= (k + 1)(2k + 1)$$

for $k = 2^{p-1} - 1$. Therefore, n is hexagonal and, hence, also triangular.

Harmonic Mean of the Divisors

If $\tau(n)$ is the number of positive divisors of n, then

$$\tau(n) = \tau\left[\prod_{i=1}^{k} p_i^{a_i}\right] = \prod_{i=1}^{k} (a_i + 1).$$

The function $\tau(n)$ is multiplicative [1;p. 95].

Let $H(n)$ be the <u>harmonic mean</u> of the divisors of n, that is the reciprocal of the arithmetic mean of the reciprocals of the divisors of n. Then,

$$\frac{1}{H(n)} = \frac{1}{\tau(n)} \sum_{d|n} \frac{1}{d} = \frac{1}{n\tau(n)} \sum_{d|n} \frac{n}{d} = \frac{1}{n\tau(n)} \sum_{d'|n} d' = \frac{\sigma(n)}{n\tau(n)}$$

where $dd' = n$. Therefore,

$$H(n) = \frac{n\tau(n)}{\sigma(n)}$$

and $H(n)$ is a multiplicative function. Then

$$H(n) = H\left(\prod_{i=1}^{k} p_i^{a_i}\right) = \prod_{i=1}^{k} \frac{p_i^{a_i}(a_i + 1)}{1 + p_i + \cdots + p_i^{a_i}}$$

Laborde [8] proves that $H(n) > 1$ when $n > 1$ and $H(n) > 2$ except when n is a prime or when $n = 1, 4,$ or 6. $H(n) > 2$ for all odd composite numbers.

<u>Theorem 2.11</u>. If n is an even perfect number, then

$$n = 2^{H(n)-1}(2^{H(n)} - 1).$$

PROOF: Since n is perfect, $\sigma(n) = 2n$. Also, there exists a prime p such that $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is a prime so that

$$H(n) = \frac{n\tau(n)}{\sigma(n)} = \frac{n(p - 1 + 1)(1 + 1)}{2n} = p.$$

Therefore,

$$n = 2^{H(n)-1}(2^{H(n)} - 1).$$

<u>Theorem 2.12</u>. If n is even and has the form

$$n = 2^{H(n)-1}(2^{H(n)} - 1),$$

then n is perfect.

PROOF: It suffices to show that $P = 2^{H(n)} - 1$ is prime. Since n is even, $H(n) > 1$. Since P is odd $(2^{H(n)-1}, P) = 1$. Then,

$$H(n) = H(2^{H(n)-1})H(P) = \frac{2^{H(n)-1}\tau(2^{H(n)-1})}{\sigma(2^{H(n)-1})} H(P)$$

$$= \frac{2^{H(n)-1}[H(n) - 1 + 1]}{\frac{2^{H(n)} - 1}{2 - 1}} H(P) = \frac{2^{H(n)-1}H(n)}{2^{H(n)} - 1} H(P)$$

$$> \frac{2^{H(n)-1}H(n)H(P)}{2^{H(n)}} = \frac{H(n)H(P)}{2}.$$

This gives that $H(P) < 2$. Since P is odd and $P > 1$, P is a prime by the remarks preceding Theorem 2.11. Therefore, n is perfect [8].

Other Properties

<u>Lemma</u>:

$$\sum_{i=0}^{k} (2i + 1)^3 = (k + 1)^2(2k^2 + 4k + 1)$$

PROOF: The equation is satisfied for $k = 0$. If it is true for $k = m$, then

$$\sum_{i=0}^{m+1} (2i + 1)^3 = (m + 1)^2(2m^2 + 4m + 1) + (2m + 3)^3$$

$$= (m^2 + 2m + 1)(2m^2 + 4m + 1) + (2m + 3)^3$$

$$= 2m^4 + 8m^3 + 11m^2 + 6m + 1 + 8m^3 + 36m^2 + 54m + 27$$

$$= 2m^4 + 16m^3 + 47m^2 + 60m + 28 = (m + 2)^2(2m^2 + 8m + 7)$$

$$= (m + 2)^2[2(m + 1)^2 + 4(m + 1) + 1].$$

Thus by induction, the equation is true for any k.

Theorem 2.13. If n is an even perfect number, other than 6, then there exists an integer k such that

$$n = \sum_{i=0}^{k} (2i + 1)^3.$$

PROOF: Since n is perfect, $n \neq 6$, then $n = 2^{2s}(2^{2s+1} - 1)$. Then

$$n = 2^{2s}(2 \cdot 2^{2s} - 1) = (2^s)^2[2(2^s)^2 - 1].$$

Then if $k = 2^s - 1$,

$$n = (k + 1)^2[2(k + 1)^2 - 1] = (k + 1)^2(2k^2 + 4k + 1).$$

Thus, if $n \neq 6$,

$$n = \sum_{i=0}^{2^s-1} (2i + 1)^3.$$

<u>Theorem 2.14</u>. If $n = 2^{p-1}(2^p - 1)$ is perfect, then

$$\prod_{\substack{d|n \\ d<n}} d = n^{p-1}$$

PROOF: Since $2^p - 1$ is prime,

$$\begin{aligned}
\prod_{\substack{d|n \\ d<n}} d &= 1\cdot 2 \cdots 2^{p-1}\cdot(2^p - 1)\cdot 2(2^p - 1) \cdots 2^{p-2}(2^p - 1) \\
&= 2^{\left[\frac{(p-1)p}{2}\right]} 2^{\left[\frac{(p-2)(p-1)}{2}\right]} (2^p - 1)^{p-1} \\
&= 2^{\left[\frac{(p-1)(2p-2)}{2}\right]} (2^p - 1)^{p-1} \\
&= 2^{(p-1)^2}(2^p - 1)^{p-1} = [2^{p-1}(2^p - 1)]^{p-1} \\
&= n^{p-1}.
\end{aligned}$$

Binary Notation

If $n = 2^{p-1}(2^p - 1)$ is perfect and is expressed in binary notation, the binary notation will consist of p ones followed by $p - 1$ zeros because

$$n = 2^{p-1}(2^p - 1) = \frac{2^{p-1}(2^p - 1)}{2 - 1} = \sum_{i=p-1}^{2p-2} 2^i.$$

The binary representation of this is

$$\underbrace{111 \cdots 1}_{\substack{p \\ \text{ones}}}\underbrace{000 \cdots 0}_{\substack{p-1 \\ \text{zeros}}} \quad \text{(binary)}.$$

For example, $28 = 2^{3-1}(2^3 - 1)$, $p = 3$, and

$$28 = 2^4 + 2^3 + 2^2 = 11100 \quad \text{(binary)}.$$

If $1, d_1, d_2, \ldots, d_k$ are the divisors of an even perfect n, excluding n itself, then for each d_i there exists a d_j such that $n = d_i d_j$. Then,

$$n = 1 + \sum_{i=1}^{k} d_i$$

and

$$1 = \frac{1}{n} + \sum_{i=1}^{k} \frac{d_i}{n} = \frac{1}{n} + \sum_{i=1}^{k} \frac{1}{d_i}.$$

For the perfect number 28 this is

$$1 = \frac{1}{28} + \frac{1}{14} + \frac{1}{7} + \frac{1}{4} + \frac{1}{2}.$$

If these fractions are expressed in binary notation and added, the result is

$$\begin{aligned}
\frac{1}{28} &= .000010010010\cdots \quad \text{(binary)}\\
\frac{1}{14} &= .000100100100\cdots \quad \text{(binary)}\\
\frac{1}{7} &= .001001001001\cdots \quad \text{(binary)}\\
\frac{1}{4} &= .010000000000\cdots \quad \text{(binary)}\\
\frac{1}{2} &= .100000000000\cdots \quad \text{(binary)}\\
\hline
1 &= .111111111111\cdots \quad \text{(binary)}
\end{aligned}$$

The fractions add to 1 without a single carry. As Daniel Shanks [9;p. 25] has said, "Is this not perfection--of a sort?" This result is the same for any even perfect number.

Theorem 2.15. If n is an even perfect number and $d_1, d_2, \ldots, d_k$ are the divisors of n, other than 1, and if the reciprocals of $d_1, d_2, \ldots, d_k$ are expressed in binary notation their sum will be $1 = .11111\cdots$ without a single carry.

PROOF: Since n is perfect, there exists a prime p such that $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime. The divisors of n, other than 1, are $2, 2^2, \ldots, 2^{p-1}, (2^p - 1), 2(2^p - 1), 2^2(2^p - 1), \ldots, 2^{p-1}(2^p - 1)$.

$$\frac{1}{2^p - 1} = \frac{2^{-p}}{1 - 2^{-p}} = \sum_{i=1}^{\infty} 2^{-ip}$$

$$= .\underbrace{00\cdots0}_{\substack{p-1\\ \text{zeros}}}1\underbrace{00\cdots0}_{\substack{p-1\\ \text{zeros}}}1\underbrace{00\cdots0}_{\substack{p-1\\ \text{zeros}}}1\cdots \quad \text{(binary)}.$$

For $j = 1, 2, \ldots, p-1$,

$$\frac{1}{2^j(2^p - 1)} = \frac{2^{-j-p}}{1 - 2^{-p}} = \sum_{i=1}^{\infty} 2^{-j-ip}$$

$$= .\underbrace{00\cdots0}_{\substack{p+j-1\\ \text{zeros}}}1\underbrace{00\cdots0}_{\substack{p-1\\ \text{zeros}}}1\underbrace{00\cdots0}_{\substack{p-1\\ \text{zeros}}}1\cdots \quad \text{(binary)}$$

For $j = 1, 2, \ldots, p-1$,

$$\frac{1}{2^j} = 2^{-j} = .\underbrace{00 \cdots 0}_{\substack{j-1 \\ \text{zeros}}}100\cdots \quad \text{(binary)}.$$

$$\sum_{j=0}^{p-1} \frac{1}{2^j(2^p - 1)} = \sum_{j=0}^{p-1} \frac{1}{2^j} \sum_{i=1}^{\infty} \frac{1}{2^{ip}}$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^{ip}} + \sum_{i=1}^{\infty} \frac{1}{2^{1+ip}} + \sum_{i=1}^{\infty} \frac{1}{2^{2+ip}} + \cdots + \sum_{i=1}^{\infty} \frac{1}{2^{p-1+ip}}.$$

For any $m \geq p$, $m = kp + r$ where $0 \leq r \leq p - 1$ and $k \in N$. Hence, $1/2^m = 1/2^{r+kp}$ appears as an addend in only the sum

$$\sum_{i=1}^{\infty} \frac{1}{2^{r+ip}}.$$

Thus,

$$\sum_{j=0}^{p-1} \frac{1}{2^j(2^p - 1)} = \sum_{j=p}^{\infty} \frac{1}{2^j} = \sum_{j=p}^{\infty} 2^{-j}.$$

Then

$$\sum_{j=0}^{p-1} \frac{1}{2^j(2^p - 1)} + \sum_{j=1}^{p-1} \frac{1}{2^j} = \sum_{j=p}^{\infty} 2^{-j} + \sum_{j=1}^{p-1} 2^{-j} = \sum_{j=1}^{\infty} 2^{-j}$$

$$= .1111\cdots \quad \text{(binary)}.$$

CHAPTER III

ODD PERFECT NUMBERS

The theory of odd perfect numbers is not as well developed as the theory of even perfect numbers. No odd perfect numbers have been found, but no one has proven that they do not exist. However, many conditions that they must satisfy, if they do exist, are known.

Basic Structure

The first condition proven about odd perfect numbers is the following theorem which was first proven by Euler in the nineteenth century [3].

<u>Theorem 3.1.</u> If n is an odd perfect number, then

$$n = p^a \prod_{i=1}^{k} p_i^{2b_i}$$

where $p \equiv a \equiv 1 \bmod 4$ and $p, p_1, p_2, \ldots, p_k$ are distinct primes.

PROOF: Let $p_0, p_1, p_2, \ldots, p_k$ be distinct primes where

$$n = \prod_{i=0}^{k} p_i^{a_i}.$$

Since n is perfect,

$$\sigma(n) = \prod_{i=0}^{k} (1 + p_i + \cdots + p_i^{a_i}) = 2 \prod_{i=0}^{k} p_i^{a_i} = 2n.$$

For some i, say $i = 0$, $1 + p_0 + \cdots + p_0^{a_0}$ must be of the form $2m$, m odd. Thus, $a_0 = 2s + 1$ for some s. Let $p = p_0$ and $a = a_0$, then

$$1 + p_0 + \cdots + p_0^{a_0} = 1 + p + \cdots + p^a = \frac{p^{a+1} - 1}{p - 1}$$

$$= \frac{p^{2s+2} - 1}{p - 1} = \frac{(p^{s+1} + 1)(p^{s+1} - 1)}{p - 1}$$

$$= \frac{(p^{s+1} + 1)(p - 1)(p^s + p^{s-1} + \cdots + 1)}{p - 1}$$

$$= (p^{s+1} + 1)(p^s + p^{s-1} + \cdots + 1).$$

For any s, $p^{s+1} + 1$ is even. Therefore, $s = 2t$, for some t, in order that $p^s + p^{s-1} + \cdots + 1$ is odd. Thus, $a = 4t + 1$. Then $p^{s+1} + 1 = p^{2t+1} + 1 = 2w$, w odd. That is $p^{2t+1} + 1 \equiv 2 \bmod 4$. But $p^{2t+1} + 1 = p^{2t}p + 1 \equiv p + 1 \bmod 4$. Therefore, $p \equiv 1 \bmod 4$. Since $1 + p_i + \cdots + p_i^{a_i}$ is odd for $i = 1, 2, \ldots, k$ then $a_i = 2b_i$, $i = 1, 2, \ldots, k$. Therefore,

$$n = p^a \prod_{i=1}^{k} p_i^{2b_i}, \quad a \equiv p \equiv 1 \bmod 4.$$

<u>Corollary 3.2</u>. If n is an odd perfect number, $n \equiv 1 \bmod 4$.

PROOF: Since, for each i, p_i is odd, then $p_i^{2b_i} \equiv 1 \bmod 4$. Therefore, n represented as in Theorem 3.1 yields:

$$n \equiv p^a \equiv p^{4t+1} \equiv p^{4t}p \equiv p \equiv 1 \bmod 4.$$

Others have proven additional restrictions of this form. Steurwald proved that an odd number n is not perfect if $b_1 = b_2 = \cdots = b_k = 1$ [10;p. 44]. Kanold proved that n is not perfect if any of the following hold:

1. $b_1 = b_2 = \cdots = b_k = 2$,
2. 9, 15, 21, or 33 divide the greatest common divisor of $b_1 + 1, b_2 + 1, \ldots, b_k + 1$,
3. $b_1 = b_2 = 2$ and $b_3 = b_4 = \cdots = b_k = 1$,
4. $a = 5$ and $b_i = 1$ or 2 for $i = 1, 2, \ldots, k$,
5. 3 does not divide n, $b_2 = b_3 = \cdots = b_k = 1$, and $a = 1$ or 5, and
6. $b_2 = b_3 = \cdots = b_k = 1$ and $2b_1 < 10$ [10;p. 44].

The next theorem was proven by Brauer [11;p. 715]. In the proof Brauer used a theorem of J. J. Sylvester which states that if n is not divisible by 3, it contains at least 8 different prime factors. Also, Brauer used the following two lemmas [11;pp. 713-714].

<u>Lemma 3.3</u>. Let q be a positive prime. The Diophantine equation $q^2 + q + 1 = y^m$ has no solution for $m > 1$.

<u>Lemma 3.4</u>. Let r and s be different positive integers and p be a prime. The system of simultaneous Diophantine equations

$x^2 + x + 1 = 3p^r$, $y^2 + y + 1 = 3p^s$ has no solution in positive integers x and y.

Also, Brauer used the following lemma concerning cyclotomic polynomials. If e_m is an n^{th} root of 1 and all the numbers $e_m^0, e_m^1, e_m^2, \ldots, e_m^{n-1}$ are distinct, then e_m is a <u>primitive n^{th} root of unity</u>. The polynomial

$$F_n(x) = \prod_a (x - e_a),$$

the product extending over all primitive n^{th} roots of unity, is called the <u>cyclotomic polynomial</u> of index n or the n^{th} cyclotomic polynomial. The symbol $F_n(x)$ will be used for the n^{th} cyclotomic polynomial. The degree of $F_n(x)$ is $\phi(n)$ where $\phi(n)$ is the number of positive integers less than n which are relatively prime to n [12;p. 158].

<u>Lemma 3.5</u>. If p is a prime, the only divisors of $F_p(m)$, $m \in N$, is of the form $ph + 1$, $h \in N$ or p itself [11; p. 714].

<u>Theorem 3.6</u>. An odd number of the form $n = p^a q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$ where $p, q_1, q_2, \ldots, q_t$ are distinct primes and $p \equiv a \doteq 1 \bmod 4$ is not perfect.

PROOF: By changing notation, let n be written in the form $n = p^a q_1^2 q_2^2 \cdots q_k^2 r_1^2 r_2^2 \cdots r_m^2 s^4$, $k \geq 0$, $m \geq 0$, where the primes $q_i \equiv 1 \bmod 3$ and the primes $r_i \not\equiv 1 \bmod 3$. If n is perfect $\sigma(n) = 2n$. Then

$$2p^a q_1^2 q_2^2 \cdots q_k^2 r_1^2 r_2^2 \cdots r_m^2 s^4$$

$$= \sigma(p^a) \prod_{i=1}^{k} (1 + q_i + q_i^2) \prod_{i=1}^{m} (1 + r_i + r_i^2)(1 + s + s^2 + s^3 + s^4). \quad (1)$$

Since each q_i is of the form $3b + 1$, each

$$1 + q_i + q_i^2 = 1 + (3b + 1) + (3b + 1)^2$$
$$= 3 + 9b + 9b^2 = 3(1 + 3b + 3b^2)$$

is divisible by 3 but not by 9. If for some i, $r_i = 3$, then $1 + r_i + r_i^2 = 1 + 3 + 9 = 13$. For all other i, r_i is of the form $3b + 2$. Then each

$$1 + r_i + r_i^2 = 1 + (3b + 2) + (3b + 2)^2 = 9b^2 + 15b + 7.$$

Thus, 3 is not a factor of any $1 + r_i + r_i^2$. Since

$$x^2 + x + 1 = F_3(x),$$

by Lemma 3.5, all other prime factors of

$$\prod_{i=1}^{k} (1 + q_i + q_i^2) \prod_{i=1}^{m} (1 + r_i + r_i^2)$$

are of the form $3h + 1$, $h \in N$.

<u>Case I</u>: $n \not\equiv 0 \bmod 3$. This implies that $k = 0$. Since n is not divisible by 3, it follows from Sylvester's theorem that n contains at least 8 different primes. Hence, $m \geq 6$. Equation (1)

is then,

$$2p^a s^4 \prod_{i=1}^{m} r_i^2 = \sigma(p^a)(1 + s + s^2 + s^3 + s^4) \prod_{i=1}^{m} (1 + r_i + r_i^2).$$

Since the factors of each $1 + r_i + r_i^2$ are of the form $3h + 1$ and each $r_i \not\equiv 1 \bmod 3$,

$$\prod_{i=1}^{m} (1 + r_i + r_i^2)$$

is a divisor of $p^a s^4$. It could be that one of the m factors of this product equals p, but by Lemma 3.3 each of the remaining $m - 1$ factors cannot be a power of p. Hence, each of these $m - 1$ factors must be divisible by s and their product divisible by at least s^5. This is a contradiction.

The proof involves two more cases: $n \equiv 0 \bmod 3$ and $n \not\equiv 0 \bmod 27$; and $s = 3$. The second case involves nine subcases, and hence, both are referenced instead of being included for bulk [12].

The next three theorems concerning the form of an odd perfect number, if one exists, were proven by Paul J. McCarthy [13].

<u>Theorem 3.7</u>. If n is odd and

$$n = p^a \prod_{i=1}^{m} p_i^{2b_i},$$

$p \equiv a \equiv 1 \bmod 4$, r is a prime that does not divide n, and

$p^e \equiv 1 \bmod r$ for some $e \in N$, then n is not perfect if $a + 1 \equiv 0 \bmod (er)$ [13;p. 257].

PROOF: To prove that n is not perfect it is sufficient to show that $\sigma(p^a)$ has a factor which does not divide n. If $a + 1 \equiv 0 \bmod (er)$, there exists an integer k such that $a + 1 = erk$. Then

$$\sigma(p^a) = \frac{p^{a+1} - 1}{p - 1} = \frac{p^{erk} - 1}{p - 1}$$

$$= \frac{(p^{er} - 1)(p^{er(k-1)} + \cdots + p^{er} + 1)}{p - 1}$$

$$= \frac{(p^e - 1)(p^{e(r-1)} + \cdots + p^e + 1)(p^{er(k-1)} + \cdots + 1)}{p - 1}$$

with $p - 1$ a divisor of $p^e - 1$. Then since $p^e \equiv 1 \bmod r$,

$$p^{e(r-1)} + \cdots + p^e + 1 \equiv 1 + \cdots + 1 + 1 \equiv r \equiv 0 \bmod r.$$

Therefore, r divides $\sigma(p^a)$ but not n. Thus, n is not perfect.

The next theorem by McCarthy is an extension of one by Steuerwald.

Theorem 3.8. If n is odd, not divisible by 3, and

$$n = p^a \prod_{i=1}^{k} q_i^{2b_i},$$

with $p \equiv a \equiv 1 \bmod 4$, then n is not perfect if $b_i \equiv 1 \bmod 3$ for $i = 1, 2, \ldots, k$. [13;p. 258].

PROOF: If each $b_i \equiv 1 \bmod 3$, then b_i is of the form $3h + 1$. If, for any i, $q_i \equiv 1 \bmod 3$,

$$\sigma(q_i^{2b_i}) = \sigma(q_i^{2(3h+1)}) = 1 + q_i + \cdots + q_i^{6h+2}$$
$$\equiv 1 + 1 + \cdots + 1 \equiv 6h + 3 \equiv 0 \bmod 3.$$

Thus, $3|n$ which is impossible, and for each i, $q_i \equiv 2 \bmod 3$. Suppose that n is perfect. Let q be the smallest prime divisor of n. Since $a = 4t + 1$ for some nonnegative integer t,

$$\sigma(p^a) = \frac{p^{a+1} - 1}{p - 1} = \frac{p^{4t+2} - 1}{p - 1}$$
$$= \frac{(p^2 - 1)(p^{2t} + p^{2t-2} + \cdots + p^2 + 1)}{p - 1}$$
$$= (p + 1)(p^{2t} + p^{2t-2} + \cdots + p^2 + 1)$$
$$= 2\left(\frac{p + 1}{2}\right)(p^{2t} + p^{2t-2} + \cdots + p^2 + 1)$$

and $(p + 1)/2$ is a factor of n. If $p = q$, n is divisible by $(p + 1)/2 < q$. Hence, q is one of the q_i. Since

$$\sigma(q^{2(3h+1)}) = \frac{q^{6h+3} - 1}{q - 1} = \frac{q^{3(2h+1)} - 1}{q - 1}$$
$$= \frac{(q^3 - 1)(q^{3(2h)} + q^{3(2h-1)} + \cdots + q^3 + 1)}{q - 1}$$
$$= (q^2 + q + 1)(q^{3(2h)} + q^{3(2h-1)} + \cdots + q^3 + 1),$$

$q^2 + q + 1 = q'$ is a divisor of n. Since q does not divide q', if q' is composite and all prime divisors are larger than q, then $q' \geq (q + 1)^2 = q^2 + 2q + 1 > q'$ which is not possible. Thus, if q' is composite, it has a divisor less than q. Thus, n would have a prime factor less than q which is impossible. Thus, q' is a prime. Since

$$q' = q^2 + q + 1 \equiv 2^2 + 2 + 1 \equiv 1 \bmod 3,$$

$q' = p$. Then n is divisible by

$$q'' = \frac{q' + 1}{2} = \frac{q^2 + q + 2}{2} = \frac{q^2 + q}{2} + 1.$$

Since q does not divide q'' and $(q + 1)^2 > (q^2 + q)/2 + 1$, if q'' is composite, it contains a factor less than q. Therefore, q'' is a prime. Since

$$q'' = \frac{q' + 1}{2} \equiv \frac{1 + 1}{2} \equiv 1 \bmod 3,$$

then $q'' = p$. But,

$$q'' = \frac{q' + 1}{2} = \frac{p + 1}{2}.$$

This is a contradiction. Thus, n is not perfect.

From Theorem 3.8, if n is not divisible by 3, a necessary condition that

$$n = p^a q_1^{2b} q_2^2 \cdots q_k^2$$

is not perfect is the condition that $b \equiv 1 \bmod 3$. The next theorem shows that this requirement can be dropped if a condition is imposed on q_1.

Theorem 3.9. If 3 does not divide the odd n and

$$n = p^a q_1^{2b} q_2^2 \cdots q_k^2,$$

then n is not perfect if $q_1 \equiv 2 \bmod 3$ [13;p. 258].

PROOF: Suppose n is perfect and that $q_1 \equiv 2 \bmod 3$. If for any i, $2 \leq i \leq k$, $q_i \equiv 1 \bmod 3$, then

$$\sigma(q_i^2) = 1 + q_i + q_i^2 \equiv 1 + 1 + 1 \equiv 0 \bmod 3,$$

and $3|n$ which is impossible. Therefore, for $i = 2, 3, \ldots, k$, $q_i \equiv 2 \bmod 3$. Since $\sigma(q_2^2) = F_3(q)$, q_i cannot divide $\sigma(q_2^2)$ for $i = 1, 2, \ldots, k$ by Lemma 3.5. Thus, $\sigma(q_2^2) = p^m$. By Lemma 3.3, $m = 1$. The same is true for $i = 3, 4, \ldots, k$. Since $k > 3$ by the theorem of Sylvester, $\sigma(q_2^2) = \sigma(q_3^2) = p$, even though $q_2 \neq q_3$. This is impossible. Therefore, n is not perfect if $q_1 \equiv 2 \bmod 3$.

The next theorem was proven by G. Cuthbert Webber [14]. The proof, which is quite lengthy, has been omitted. The techniques and procedures used are very similar to those used in Theorem 3.6. Webber used Lemmas 3.3, 3.4, and 3.6. In addition, he used the following lemmas.

Lemma 3.10. If

$$f_k(x) = \sum_{i=0}^{k-1} x^i,$$

and m, q, and s are integers, t a prime, then

(a) $m|s$ implies $f_m(x)|f_s(x)$,

(b) If $q \equiv 1 \bmod t$, then $f_s(q) \equiv 0 \bmod t$, if and only if $t|s$, and

(c) If k is the smallest positive integer such that $q^k \equiv 1 \bmod t$, then $f_s(q) \equiv 0 \bmod t$ if and only if $k|s$.

Lemma 3.11. If q and r are positive integers, then $f_{2r+1}(q)$ and $g_{2r+1}(q) = q^{2r} - q^{2r-1} + \cdots - q + 1$ do not have a common prime factor.

Lemma 3.12. If $2r + 1$ is a prime, and q an integer, then $f_{2r+1}(q)$ and $f_{2r+1}(q^{2r+1})$ do not have a common factor other than $2r + 1$; likewise, for $g_{2r+1}(q)$ and $g_{2r+1}(q^{2r+1})$.

Lemma 3.13. If $2r + 1$ is a prime and $p > 1$ is a positive integer, $f_{2r+1}(p)$, $g_{2r+1}(p)$, $f_{2r+1}(p^{2r+1})$ and $g_{2r+1}(p^{2r+1})$ are divisible by four distinct primes, that is, each of the functions is divisible by one of the four distinct primes and no two by a single one of the primes.

Lemma 3.14. If $3|f_{4r+2}(p)$ and, in case $p \equiv -1 \bmod 3$, $f_{4r+2}(p) \equiv 0 \bmod 3^j$ but $p + 1 \not\equiv 0 \bmod 3^j$, then $f_3(p)$ and $g_3(p)$ are factors of $f_{4r+2}(p)$.

The theorem that is then proved using these lemmas is the following one.

Theorem 3.15. The number

$$n = 3^{2b}p^a s_1^{2b_1} s_2^{2b_2} s_3^{2b_3},$$

where p, s_1, s_2, and s_3 are distinct odd primes $\neq 3$ and $p \equiv a \equiv 1 \bmod 4$, is not perfect.

The next theorem was proven by R. J. Levit [15]. The proof uses the following lemmas. In the first lemma and the theorem, the product notation is used with the convention that if $a > b$,

$$\prod_{i=a}^{b} x_i = 1.$$

The first lemma can easily be proven by induction.

<u>Lemma 3.16</u>. If $c_1, c_2, \ldots c_t$ are integers, $t \geq 2$, then

$$\sum_{j=1}^{t} \left[\prod_{i=1}^{j-1} (c_i - 1) \prod_{i=j+1}^{t} c_i \right] = \prod_{i=1}^{t} c_i - \prod_{i=1}^{t} (c_i - 1).$$

<u>Lemma 3.17</u>. If $a > 1$ is an integer and p a prime such that $a \equiv p \equiv 1 \bmod 4$, then $\sigma(p^a)$ is divisible by at least two distinct odd primes.

PROOF: It is sufficient to exhibit two odd nontrivial divisors of $\sigma(p^a)$ which are relatively prime. Since

$$\sigma(p^a) = 1 + p + \cdots + p^a \equiv 1 + 1 + \cdots + 1 \equiv a + 1 \equiv 2 \bmod 4,$$

$\sigma(p^a)$ has but one factor of 2. Then

$$\sigma(p^a) = \frac{p^{a+1} - 1}{p - 1} = 2\, \frac{p^{(a+1)/2} + 1}{2}\, \frac{p^{(a+1)/2} - 1}{p - 1}.$$

Then the required divisors are

$$d_1 = \frac{p^{(a+1)/2} + 1}{2} \quad \text{and} \quad d_2 = \frac{p^{(a+1)/2} - 1}{p - 1}.$$

They are relatively prime since $2d_1 - (p - 1)d_2 = 2$ so that if there were a common divisor of d_1 and d_2 it would have to divide 2. They are nontrivial since $d_1 > d_2 > 1$ for $a > 1$. Thus, $\sigma(p^a)$ has at least two distinct odd prime divisors.

<u>Theorem 3.18</u>. If

$$n = p^a \prod_{i=1}^{k} p_i^{a_i}$$

is odd with $p \equiv a \equiv 1 \bmod 4$ and $\sigma(p^a)/2$, $\sigma(p_1^{a_1})$, ..., $\sigma(p_k^{a_k})$ are all powers of primes, then n is not a perfect number.

PROOF: By Lemma 3.17, $\sigma(p^a)/2$ a power of a prime implies that $a = 1$. Suppose n is perfect. Then

$$2p \prod_{i=1}^{k} p_i^{a_i} = \sigma(p) \prod_{i=1}^{k} \sigma(p_i^{a_i}) = 2\,\frac{\sigma(p)}{2} \prod_{i=1}^{k} \sigma(p_i^{a_i}).$$

Without loss of generality the p_i's may be numbered recursively in the following manner. Let p_1 be that prime such that $p_1^{a_1} = \sigma(p)/2$, p_2 be that prime such that $p_2^{a_2} = \sigma(p_1^{a_1})$, and in general let p_m be that prime such that $p_m^{a_m} = \sigma(p_{m-1}^{a_{m-1}})$. This process can be continued until a prime p_t is reached such that $p = \sigma(p_t^{a_t})$. Suppose that $t < k$. Then numbering the remaining p_i in any order as p_{t+1}, p_{t+2}, ..., p_k, one obtains

$$\prod_{i=t+1}^{k} p_i^{a_i} = \prod_{i=t+1}^{k} \sigma(p_i^{a_i}).$$

But this is impossible since

$$\prod_{i=t+1}^{k} \sigma(p_i^{a_i}) = \prod_{i=t+1}^{k} (1 + p_i + \cdots + p_i^{a_i}) > \prod_{i=t+1}^{k} p_i^{a_i}.$$

Hence, $t = k$ and

$$p_1^{a_1} = \frac{\sigma(p)}{2} = \frac{p + 1}{2}, \quad p = \sigma(p_k^{a_k}) = \frac{p_k^{a_k+1} - 1}{p_k - 1}$$

$$p_i^{a_i} = \sigma(p_{i-1}^{a_{i-1}}) = \frac{p_{i-1}^{a_{i-1}+1} - 1}{p_{i-1} - 1}, \quad i = 2, 3, \ldots, k.$$

Let $c_i = p_i^{a_i}$ and $b_i = 1/(p_i - 1)$ for $i = 1, 2, \ldots, k$ then

$$c_1 = \frac{p + 1}{2}, \quad p = b_k p_k c_k - b_k \tag{1}$$

$$c_i = b_{i-1} p_{i-1} c_{i-1} - b_{i-1}, \quad i = 2, 3, \ldots, k. \tag{2}$$

Eliminating p from equation (1) gives

$$2c_1 - 1 = b_k p_k c_k - b_k. \tag{3}$$

The first two equations from (2) are

$$c_2 = b_1 p_1 c_1 - b_1, \quad c_3 = b_2 p_2 c_2 - b_2.$$

Together, these give

$$c_3 = b_2p_2(b_1p_1c_1 - b_1) - b_2$$
$$= (b_1p_1b_2p_2)c_1 - [b_1(b_2p_2) + b_2].$$

Then, using the third equation from (2) one obtains

$$c_4 = b_3p_3c_3 - b_3$$
$$= b_3p_3\{(b_1p_1b_2p_2)c_1 - [b_1(b_2p_2) + b_2]\} - b_3$$
$$= (b_1p_1b_2p_2b_3p_3)c_1 - [b_1(b_2p_2b_3p_3) + b_2(b_3p_3) + b_3].$$

Continuing inductively, one gets

$$c_k = \left(\prod_{i=1}^{k-1} b_ip_i\right)c_1 - \sum_{j=1}^{k-1} b_j \prod_{i=j+1}^{k-1} b_ip_i. \tag{4}$$

Combining equations (3) and (4) gives

$$2c_1 - 1 = b_kp_kc_k - b_k$$
$$= b_kp_k\left[\left(\prod_{i=1}^{k-1} b_ip_i\right)c_1 - \sum_{j=1}^{k-1} b_j \prod_{i=j+1}^{k-1} b_ip_i\right] - b_k$$
$$= \left(\prod_{i=1}^{k} b_ip_i\right)c_1 - \sum_{j=1}^{k} b_j \prod_{i=j+1}^{k} b_ip_i$$

or

$$\left(\prod_{i=1}^{k} b_ip_i - 2\right)c_1 - \sum_{j=1}^{k} b_j \prod_{i=j+1}^{k} b_ip_i + 1 = 0.$$

Multiplying both sides by

$$\prod_{i=1}^{k} (p_i - 1) \quad \text{or} \quad \prod_{i=1}^{k} \frac{1}{b_i}$$

which are equal, gives

$$\left(\prod_{i=1}^{k} p_i - 2\prod_{i=1}^{k} (p_i - 1)\right) c_1 - \sum_{j=1}^{k} \left(\prod_{i=1}^{j-1} (p_i - 1) \prod_{i=j+1}^{k} p_i\right) + \prod_{i=1}^{k} (p_i - 1) = 0.$$

Then by Lemma 3.16

$$\left(\prod_{i=1}^{k} p_i - 2\prod_{i=1}^{k} (p_i - 1)\right) c_1 - \prod_{i=1}^{k} p_i + \prod_{i=1}^{k} (p_i - 1) + \prod_{i=1}^{k} (p_i - 1) = 0$$

or

$$\left(\prod_{i=1}^{k} p_i - 2\prod_{i=1}^{k} (p_i - 1)\right) c_1 - \left(\prod_{i=1}^{k} p_i - 2\prod_{i=1}^{k} (p_i - 1)\right) = 0.$$

Then

$$(c_1 - 1)\left(\prod_{i=1}^{k} p_i - 2\prod_{i=1}^{k} (p_i - 1)\right) = 0.$$

This implies that either

$$p_1^{a_1} = c_1 = 1 \quad \text{or} \quad \prod_{i=1}^{k} p_i = 2\prod_{i=1}^{k} (p_i - 1).$$

The first of these is impossible since $p_1 \geq 3$. The second is impossible since the right member is even and the left member is odd. Hence, n cannot be perfect.

The results of this theorem can be restated in the following form.

Corollary 3.19. Let

$$n = p^a \prod_{i=1}^{k} p_i^{a_i}$$

with $p \equiv a \equiv 1 \bmod 4$. If n is an odd perfect number, then at least two of $\sigma(p^a)/2$, $\sigma(p_1^{a_1})$, ..., $\sigma(p_k^{a_k})$ must have a common factor greater than 1.

The next theorem, which was proven by Paul J. McCarthy [16], uses lemmas concerning cyclotomic polynomials.

Lemma 3.20.

$$\sum_{i=0}^{k} x^i = \prod_{\substack{d|(k+1) \\ d \neq 1}} F_d(x).$$

Lemma 3.21. If $r|F_n(q)$, q a prime, then either $r|n$ or $r \equiv 1 \bmod n$.

Theorem 3.23. If n is an odd integer and

$$n = p^a \prod_{i=1}^{k} q_i^{2b_i}$$

where $p \equiv a \equiv 1 \bmod 4$ and r is the smallest prime divisor of $\tau(n)/2$, then n is not perfect if it has a prime divisor r' such that $r > r'$ and $p + 1 \not\equiv 0 \bmod r'$. In particular, n is not perfect if $r > p$.

PROOF: Suppose n is perfect, and that r' is a divisor of n satisfying $r > r'$ and $p + 1 \not\equiv 0 \bmod r'$. Then

$$\sigma(p^a) \prod_{i=1}^{k} \sigma(q_i^{2b_i}) = 2n.$$

Therefore, $r' | \sigma(p^a)$ or $r' | \sigma(q_i^{2b_i})$ for some i. Since

$$\sigma(p^a) = \sum_{i=0}^{a} p^i \quad \text{and} \quad \sigma(q_i^{2b_i}) = \sum_{j=0}^{2b_i} q_i^j,$$

by Lemma 3.20 there is a divisor $d \neq 1$ of $a + 1$ such that $r' | F_d(p)$ or there is a divisor $d \neq 1$ of $2b_i + 1$, for some i, such that $r' | F_d(q_i)$. Then by Lemma 3.21, either $r' | d$ or $r' \equiv 1 \bmod d$. Since

$$\tau(n) = (a + 1) \prod_{i=1}^{k} (2b_i + 1),$$

$d | \tau(n)$. But, since $r' < r$ and r is the smallest prime divisor of $\sigma(n)/2$, it is impossible for the odd prime r' to divide d. Thus,

$r' \equiv 1 \bmod d$. Then $d|(r' - 1)$ which implies that $d \leq r' - 1$. But, since $d|\tau(n)$ and r is the smallest odd prime that divides $\tau(n)$, $d \geq r$ unless $d = 2$. Then, since $r' < r$, $d \geq r > r' > r' - 1$ unless $d = 2$. Thus, $d = 2$. Since $d = 2$, d cannot divide $2b_i + 1$ for any i and, therefore, d is a divisor of $a + 1$ and $r'|F_2(p)$. Thus, $r'|(p + 1)$ which is impossible since $p + 1 \not\equiv 0 \bmod r'$. Therefore, n is not perfect.

The following theorem has been proven by Jacques Touchard [17] and M. Raghavachari [18]. The following proof is the one by Raghavachari which is simpler than the one by Touchard.

Theorem 3.24. If

$$n = p^a \prod_{i=1}^{k} q_i^{2b_i}$$

is an odd perfect number with $p \equiv a \equiv 1 \bmod 4$, then n is of the form $12m + 1$ or $36m + 9$.

PROOF: By Corollary 3.2, $n \equiv 1 \bmod 4$.

Case I: $3|n$. This implies that n is of the form $12m + 3$ or $12m + 9$. Since $12m + 3 \equiv 3 \bmod 4$, n is not of the form $12m + 3$. Hence, n is of the form $12m + 9$. Since $p \equiv 1 \bmod 4$, $p \neq 3$. Hence, for some i, $q_i = 3$. Thus, $3^2|n$. Therefore, $3^2|(12m + 9)$ which implies that $3|m$. Therefore, n is of the form $36m + 9$.

Case II: 3 does not divide n. This implies that n is of the form $12m + 1$, $12m + 5$, $12m + 7$, or $12m + 11$. Since $12m + 7 \equiv 3 \bmod 4$ and $12m + 11 \equiv 3 \bmod 4$, n is not of the form

$12m + 7$ or $12m + 11$. Suppose n is of the form $12m + 5$. Since any odd prime, other than 3, is of the form $6t + 1$ or $6t + 5$, then for any i,

$$q_i^{2b_i} = (6t + 1)^{2b_i} = (36t^2 + 12t + 1)^{b_i} \equiv 1^{b_i} \equiv 1 \bmod 12$$

or

$$q_i^{2b_i} = (6t + 5)^{2b_i} = (36t^2 + 60t + 25)^{b_i} \equiv 1^{b_i} \equiv 1 \bmod 12.$$

Thus, for n to be of the form $12m + 5$, p^a must be of the form $12m + 5$. Since $a = 4s + 1$ for some s, $p^{4s}p$ is of the form $12m + 5$. As in the case of the q_i's, $p^{4s} \equiv 1 \bmod 12$ which implies that p is of the form $12m + 5$. Then since $a = 4s + 1$,

$$3 \mid (1 + p + \cdots + p^a) = \sigma(p^a).$$

Thus, $3 \mid n$ which is a contradiction. Therefore, n is of the form $12m + 1$.

<u>Corollary 3.25</u>. If

$$n = p^a \prod_{i=1}^{k} q_i^{2b_i}$$

is an odd even perfect number and 3 does not divide n, then $p \equiv 1 \bmod 12$ and $a \equiv 1$ or $9 \bmod 12$.

PROOF: From Case II of Theorem 3.24, $n \equiv 1 \bmod 12$ and

$$\prod_{i=1}^{k} q_i^{2b_i} \equiv 1 \bmod 12.$$

Thus, $p^a \equiv 1 \bmod 12$. Since $a = 4s + 1$ for some s and $p^{4s} \equiv 1 \bmod 12$,

$$p^a = p^{4s+1} = p^{4s}p \equiv p \bmod 12.$$

Hence, $p \equiv 1 \bmod 12$ which implies that $p \equiv 1 \bmod 3$. Since $a \equiv 1 \bmod 4$ then $a = 12t + 1$, $12t + 5$, or $12t + 9$ for some t. Suppose $a = 12t + 5$. Then

$$\begin{aligned}\sigma(p^a) = \sigma(p^{12t+5}) &= 1 + p + \cdots + p^{12t+5} \\ &\equiv 12t + 6 \equiv 0 \bmod 3.\end{aligned}$$

Thus, $3|\sigma(p^a)$ and, therefore, $3|n$ which is a contradiction. Therefore, $a \equiv 1$ or $9 \bmod 12$.

The Number of Prime Factors

Let n be an odd perfect number with k distinct prime factors. There seems to be disagreement among authors as to what has been proven about the value of k. Dickson [19] has stated that Sylvester has proven that $k \geq 5$ while Brauer [11] has stated that Sylvester has proven that $k \geq 4$. Also, according to Dickson [3], Sylvester proved that if 3 does not divide n, $k \geq 8$ while according to Brauer [11], Sylvester proved that $k \geq 7$. Dickson [3] also stated that Tepin proved that if $3 \cdot 7$ does not divide n, then $k \geq 11$, if $3 \cdot 5$ does not divide n then $k \geq 14$, and if $3 \cdot 5 \cdot 7$ does not divide n then $k \geq 19$. Also, Catalan proved that if $3 \cdot 5 \cdot 7$ does not divide n, $k \geq 26$. According to Karl K. Norton [20], Kühnel has proven that $k \geq 6$.

Norton [20] has also developed a formula for a lower bound on the value of k which is based on the value of the smallest prime factor of n. First, the following lemma is needed.

<u>Lemma 3.26</u>. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is perfect, then

$$2 < \prod_{i=1}^{k} \frac{p_i}{p_i - 1}.$$

PROOF: Since n is perfect

$$2 = \frac{\sigma(n)}{n} = \frac{\displaystyle\prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}}{\displaystyle\prod_{i=1}^{k} p_i^{a_i}} = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i^{a_i}(p_i - 1)}$$

$$= \prod_{i=1}^{k} \frac{p_i - \frac{1}{p_i^{a_i}}}{p_i - 1} < \prod_{i=1}^{k} \frac{p_i}{p_i - 1}.$$

If P_r represents the r^{th} prime, and P_m is the smallest prime divisor of the perfect number $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, then Lemma 3.26 implies that

$$2 < \prod_{i=m}^{m+k-1} \frac{P_i}{P_i - 1}.$$

Let the function $a(m)$ be defined for $m \geq 2$ by the following inequality:

$$\prod_{i=m}^{m+a(m)-2} \frac{P_i}{P_i - 1} < 2 < \prod_{i=m}^{m+a(m)-1} \frac{P_i}{P_i - 1}.$$

It follows that n must have a prime factor at least as large as P_s where $s = s(m) = m + a(m) - 1$. Norton provides a table of values for $a(m)$ and P_s for $2 \leq m \leq 100$. The values increase rapidly. For $m = 100$, $a(m) = 26308$ and $P_s = 304961$.

If n is an abundant number and $d_1, d_2, \ldots, d_k$ are the divisors of n, then the divisors of mn include $1, md_1, md_2, \ldots, md_k$. Thus,

$$\sigma(mn) \geq 1 + \prod_{i=1}^{k} md_i > m \prod_{i=1}^{k} d_i > 2mn.$$

This with Theorem 2.1 gives that a multiple of a nondeficient number is nondeficient.

Definition: A nondeficient number is *primitive* if it is not the multiple of a smaller nondeficient number.

The set of all nondeficient numbers is equal to the set of all multiples of the primitive nondeficient numbers. Any perfect number is a primitive nondeficient number since by Theorem 2.2 a divisor of a perfect number is deficient.

There is an infinite number of nondeficient odd numbers having a given number, greater than two, of distinct prime factors. For example:

$$\sigma(945) = \sigma(3^3 \cdot 5 \cdot 7) = (1 + 3 + 9 + 27)(1 + 5)(1 + 7)$$
$$= 40(6)(8) = 1920 > 2(945).$$

Thus $3^3 \cdot 5 \cdot 7$ is abundant which implies that $3^3 \cdot 5 \cdot 7 \cdot p_1^{a_1} \cdots p_k^{a_k}$, where the p_i's are distinct primes greater than 7, is an abundant number. However, there are only a finite number of primitive non-deficient odd numbers having any given number of distinct prime factors, and hence, there cannot be an infinite number of odd perfect numbers with any given number of distinct prime factors. This has been proven by L. E. Dickson [19]. In order to prove this, the following lemmas are needed.

<u>Lemma 3.27</u>. If $p_1, p_2, \ldots, p_k$ are given prime numbers, then any set

$$S = \{p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \mid a_i\text{'s are integers} \geq 0\}$$

contains a finite number of integers, $n_1, n_2, \ldots, n_s$ such that every integer in S is a multiple of at least one n_i.

PROOF: For $k = 1$, every element of S is a multiple of p^{b_1} where b is the smallest a_i. To proceed by induction, let the lemma be true for $k - 1$ integers. Select at random $n_1 = p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}$ from the set S. Then any element of S is a multiple of n_1 if $a_i \geq c_i$ for all $i = 1, 2, \ldots, k$. If there are other elements in S, consider the elements of S for which $a_i = v$ for some i, $1 \leq i \leq k$ and v a fixed integer such that $0 \leq v < c_i$. After deleting the common factor p_i^v there is the set

$$S' = \{p_1^{a_1} \cdots p_{i-1}^{a_{i-1}} p_{i+1}^{a_{i+1}} \cdots p_k^{a_k} \mid a_j\text{'s are integers} \geq 0\}.$$

By the induction hypothesis, S' contains a finite number of integers $m_1, m_2, \ldots, m_t$ such that all elements of S' are multiples of at least one m_j. Thus, all elements of S for which $a_i = v$ are multiples of at least two of $p_i^v, m_1, m_2, \ldots, m_t$. The number of cases arising by varying i and v is finite. Therefore, there is a finite number of integers in S for which each integer in S is a multiple of at least one.

Lemma 3.28. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ where $p_1, p_2, \ldots, p_m$ are distinct primes and $0 < k < m$, $1 < p_i' \leq p_i$, p_i' a prime, for $i > k$. Let

$$P = \prod_{i=1}^{m} \frac{\sigma(p_i^{a_i})}{p_i^{a_i}} = \prod_{i=1}^{m} \frac{p_i - \frac{1}{p_i^{a_i}}}{p_i - 1}$$

$$P_0 = \prod_{i=1}^{m} \frac{p_i}{p_i - 1}, \quad P_k = \prod_{i=1}^{k} \frac{\sigma(p_i^{a_i})}{p_i^{a_i}} \prod_{i=k+1}^{m} \frac{p_i}{p_i - 1},$$

$$P_0' = \prod_{i=1}^{m} \frac{p_i'}{p_i' - 1}, \quad P_k' = \prod_{i=1}^{k} \frac{\sigma(p_i^{a_i})}{p_i^{a_i}} \prod_{i=k+1}^{m} \frac{p_i'}{p_i' - 1}.$$

Then n is deficient if $P_s \leq 2$ or $P_s' \leq 2$ where s is an integer such that $0 \leq s < m$. If n is odd and is deficient for all values of $a_{s+1}, \ldots, a_m$, then $P_s < 2$.

PROOF: By definition, n is deficient if $P < 2$ and nondeficient if $P \geq 2$. Since $P'_s \geq P_s > P$, $0 \leq s < m$, n is deficient if $P_s \leq 2$ and also if $P'_s \leq 2$. Since P_s is the limit of P for $a_i \to \infty$, $i = s + 1, \ldots, m$, and since $P < 2$ if n is deficient, then $P_s \leq 2$ if n is deficient for all values of $a_{s+1}, \ldots, a_m$. Suppose $P_s = 2$. Then $m = 1$ since if $m > 1$ and p_j is the greatest prime among $p_{s+1}, \ldots, p_m$, no number in the denominator of P_s is divisible by p_j. Thus,

$$P_s = P_0 = \frac{p_1}{p_1 - 1} = 2$$

which implies that $p_1 = 2$ and $n = 2^{a_1}$. Therefore, if n is odd and deficient for all values of $a_{s+1}, \ldots, a_m$, then $P_s < 2$.

<u>Theorem 3.29</u>. All primitive nondeficient odd numbers having a given number m of distinct prime factors are formed from a finite number of sets of m primes. Thus, there cannot be an infinite number of odd perfect numbers with any given number of distinct prime factors.

PROOF: Consider numbers of the form $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ where the p_i's are odd primes in ascending order of magnitude. Let $p'_i = p_1$ for $i = 1, 2, \ldots, m$ then n is deficient if

$$P'_0 = \prod_{i=1}^{m} \frac{p'_i}{p'_i - 1} = \left(\frac{p_1}{p_1 - 1}\right)^m \leq 2$$

which implies that

$$p_1 \geq \frac{\sqrt[m]{2}}{\sqrt[m]{2} - 1}.$$

Thus, if n is nondeficient

$$p_1 < \frac{\sqrt[m]{2}}{\sqrt[m]{2} - 1}.$$

Therefore, there is a finite number of distinct primes for p_1. Proceeding by induction, assume that $p_1, \ldots, p_v$, $v < m$ is a particular set of a finite number of sets of v distinct primes. Since n is to be a primitive nondeficient number $n_v = p_1^{a_1} p_2^{a_2} \cdots p_v^{a_v}$ must be deficient. Since each divisor of a deficient number is deficient, the deficient n_v's are the numbers in which certain exponents $a_{i_1}, \ldots, a_{i_k}$ are arbitrary, which each remaining exponent takes a limited number of values, and further numbers in which every exponent is limited. Consider one such type of n_v which is one of a finite number of analogous cases. After permuting $p_1, \ldots, p_v$, assume that u, $0 \leq u \leq v$ is an integer such that $a_1, \ldots, a_u$ are limited, while a_i, $i = u + 1, \ldots, v$ takes all values. By Lemma 3.28, the deficiency of n_v implies that

$$P_u = \prod_{i=1}^{u} \frac{\sigma(p_i^{a_i})}{p_i^{a_i}} \prod_{i=u+1}^{v} \frac{p_i}{p_i - 1} < 2$$

the second product being absent if $u = v$. Since there is a limited number of sets $a_1, \ldots, a_u$, each P_u is less than a constant $M < 2$. Then for P_u' use $p_i' = p_i$, $i = u + 1, \ldots, v$ and $p_i' = p_{v+1}$, $i = v + 1, \ldots, m$. Then n is deficient if

$$p'_u = \prod_{i=1}^{u} \frac{\sigma(p_i^{a_i})}{p_i^{a_i}} \prod_{i=u+1}^{m} \frac{p'_i}{p'_i - 1}$$

$$= \prod_{i=1}^{u} \frac{\sigma(p_i^{a_i})}{p_i^{a_i}} \prod_{i=u+1}^{v} \frac{p'_i}{p'_i - 1} \left(\frac{p_{v+1}}{p_{v+1} - 1}\right)^{m-v}$$

$$= M \left(\frac{p_{v+1}}{p_{v+1} - 1}\right)^{m-v} \leq 2.$$

Thus, if n is deficient

$$p_{v+1} \geq \frac{\left(\frac{2}{M}\right)^{\frac{1}{m-v}}}{\left(\frac{2}{M}\right)^{\frac{1}{m-v}} - 1} = M'.$$

Hence, if n is nondeficient, $p_{v+1} < M'$, and in a nondeficient n, p_{v+1} is less than the largest of the limits obtained in the various cases, finite in number. Consider the set S of primitive nondeficient numbers having as distinct prime factors $p_1, \ldots, p_m$ a particular one of the finite number of possible sets of m primes. Since any greater multiple of a nondeficient number is not primitive, the set S is finite by Lemma 3.27. Therefore, there can not be an infinite number of perfect numbers with any given number of distinct prime factors.

Bounds On the Prime Factors

If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is an odd perfect number with $p_1 < p_2 < \cdots < p_k$, then Cesàro proved that $p_1 \leq k\sqrt{2}$ and Desboves proved that $p_1 < 2^k$ [4]. Several such bounds have been proven for p_1. Servais proved the following theorem [4].

Theorem 3.30. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is an odd perfect number with $p_i < p_{i+1}$, $i = 1, 2, \ldots, k - 1$, then $p_1 \leq k$.

PROOF: Since the p_i's are odd, $p_1 + i < p_{i+1}$, $i = 1, 2, \ldots, k - 1$. Then for each i,

$$1 + \frac{1}{p_{i+1} - 1} < 1 + \frac{1}{p_1 + i - 1}$$

which gives

$$\frac{p_{i+1}}{p_{i+1} - 1} < \frac{p_1 + i}{p_1 + i - 1}.$$

This with Lemma 3.26 implies

$$2 < \prod_{i=1}^{k} \frac{p_i}{p_i - i} < \prod_{i=0}^{k-1} \frac{p_1 + i}{p_1 + i - 1} = \frac{p_1 + k - 1}{p_1 - 1}.$$

This implies that $p_1 < k + 1$. Therefore, $p_1 \leq k$.

Another theorem, similar to the last one, has been proven by M. Perisastri [21].

Theorem 3.31. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is an odd perfect number with $p_i < p_{i+1}$, $i = 1, 2, \ldots, k - 1$, then $p_1 < \frac{2}{3} k + 2$.

PROOF: Since the p_i's are odd primes, $p_1 + 2(i - 1) \leq p_i$ for $i = 1, 2, \ldots, k$. Then for each i,

$$1 + \frac{1}{p_i - 1} \leq 1 + \frac{1}{p_1 + 2i - 3}$$

which implies

$$\frac{p_i}{p_i - 1} \leq \frac{p_1 + 2i - 2}{p_1 + 2i - 3}.$$

This with Lemma 3.26 implies

$$2 < \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \leq \prod_{i=1}^{k} \frac{p_1 + 2i - 2}{p_1 + 2i - 3}.$$

But since

$$\frac{p_1 + 2i - 2}{p_1 + 2i - 3} < \frac{p_1 + 2i - 3}{p_1 + 2i - 4}$$

for $i = 1, 2, \ldots, k$, then

$$4 < \prod_{i=1}^{k} \left(\frac{p_1 + 2i - 2}{p_1 + 2i - 3}\right)^2 < \prod_{i=1}^{k} \frac{p_1 + 2i - 2}{p_1 + 2i - 3} \frac{p_1 + 2i - 3}{p_1 + 2i - 4}$$

$$= \prod_{i=1}^{k} \frac{p_1 + 2i - 2}{p_1 + 2i - 4} = \frac{p_1 + 2k - 2}{p_1 - 2}.$$

Then $p_1 < \frac{2}{3} k + 2$.

The next theorem has been proven by both T. M. Putnam [22] and M. Perisastri [23]. It uses a different technique for establishing a bound for the smallest prime divisor of an odd perfect number.

Theorem 3.32. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is an odd perfect number, there exists at least one p_i such that $p_i < \frac{k}{\ln 2} + 1$. That is, $p_i \leq \frac{3}{2} k + 1$.

PROOF: The proof of Lemma 3.26 implies

$$\frac{n}{\sigma(n)} > \prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right) = \prod_{i=1}^{k}\frac{1}{1 + \frac{1}{p_i - 1}}.$$

Suppose $p_i > \frac{k}{\ln 2} + 1$, $i = 1, 2, \ldots, k$. Since

$$\left(1 + \frac{1}{x}\right)^{x}$$

is an increasing function,

$$\left(1 + \frac{\ln 2}{x}\right)^{x} = \left[\left(1 + \frac{1}{\frac{x}{\ln 2}}\right)^{\frac{x}{\ln 2}}\right]^{\ln 2}$$

is an increasing function. Therefore, for $m > k$

$$\frac{n}{\sigma(n)} > \prod_{i=1}^{k}\frac{1}{1 + \frac{\ln 2}{k}} = \left(\frac{1}{1 + \frac{\ln 2}{k}}\right)^{k} > \left(\frac{1}{1 + \frac{\ln 2}{m}}\right)^{m}.$$

Hence,

$$\frac{n}{\sigma(n)} > \lim_{m\to\infty}\left(\frac{1}{1 + \frac{\ln 2}{m}}\right)^{n} = \frac{1}{e^{\ln 2}} = \frac{1}{2}.$$

Therefore, $2n > \sigma(n)$. Therefore, if n is perfect, at least one $p_i < \frac{k}{\ln 2} + 1 < 1.45k + 1$. Thus, there exists at least one p_i such that $p_i < \frac{3}{2}k + 1$.

By Theorem 3.24, if n is an odd perfect number, n is of the form $12m + 1$ or $36m + 9$. In the proof of the theorem it is seen that n is of the form $36m + 9$ only if $3|n$. Thus, if n is of the form

$36m + 9$, $p_1 = 3$ and no bound is needed. Thus, the bounds developed are for the case when n is of the form $12m + 1$.

The following theorem provides a bound for p_i other than p_1 [4].

Theorem 3.33. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $p_1 < p_2 < \cdots < p_k$, is an odd perfect number and L is defined by

$$\prod_{i=1}^{m-1} \frac{p_i}{p_i - 1} \leq L < 2$$

where $2 \leq m \leq k$, then

$$p_m < \frac{L(k - m) + 2}{2 - L}.$$

PROOF: For $i = m + 1, m + 2, \ldots, k$, $p_i < p_m + i$ implies that

$$\frac{p_i}{p_i - 1} < \frac{p_m + i}{p_m + i - 1} < \frac{p_m + i - m}{p_m + i - m - 1}.$$

This with Lemma 3.26 gives

$$2 < \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \leq L \prod_{i=m}^{k} \frac{p_i}{p_i - 1}$$

$$< L \prod_{i=m}^{k} \frac{p_m + i - m}{p_m + i - m - 1} = \frac{L(p_m + k - m)}{p_m - 1}.$$

This implies

$$p_m < \frac{L(k - m) + 2}{2 - L}.$$

Sum of the Reciprocals of the Prime Factors

In 1958, M. Perisastri [21] established both upper and lower bounds on the sum of the reciprocals of the prime factors of an odd perfect number, if one exists. In his proof he used the fact that

$$\prod_{p}\left(1-\frac{1}{p^2}\right)=\frac{6}{\pi^2}$$

where p runs through all primes. The following theorem is the one proven by Perisastri.

Theorem 3.34. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is an odd perfect number, then

$$\frac{1}{2} < \sum_{i=1}^{k} \frac{1}{p_i} < 2 \ln \frac{\pi}{2}.$$

PROOF: If n is perfect, then

$$2=\frac{\sigma(n)}{n}=\frac{\prod_{i=1}^{k}\frac{p_i^{a_i+1}-1}{p_i-1}}{\prod_{i=1}^{k}p_i^{a_i}}=\frac{\prod_{i=1}^{k}\left(1-\frac{1}{p_i^{a_i+1}}\right)}{\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)}.$$

Thus,

$$2\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)=\prod_{i=1}^{k}\left(1-\frac{1}{p_i^{a_i+1}}\right)<1.$$

Then,

$$\frac{1}{2} > \prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right) > 1 - \sum_{i=1}^{k}\frac{1}{p_i},$$

which implies

$$\frac{1}{2} < \sum_{i=1}^{k}\frac{1}{p_i}.$$

Since $p_1 \geq 3$, $p_2 \geq 5$, $\ldots$, $p_k \geq q_k$, where q_k is the k^{th} odd prime,

$$2\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right) = \prod_{i+1}^{k}\left(1 - \frac{1}{p_i^{a_i+1}}\right)$$

$$\geq \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots \left(1 - \frac{1}{q_k^2}\right)$$

$$= \frac{4}{3}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{q_k^2}\right)$$

$$> \frac{4}{3}\prod_{p}\left(1 - \frac{1}{p^2}\right) = \frac{4}{3}\frac{6}{\pi^2} = \frac{8}{\pi^2}.$$

Then, since $1 - x < e^{-x}$ for $0 < x < 1$,

$$\frac{8}{\pi^2} < 2\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right) < 2\prod_{i=1}^{k}\exp\left(-\frac{1}{p_i}\right) = 2\exp\left(-\sum_{i=1}^{k}\frac{1}{p_i}\right).$$

This gives

$$\sum_{i=1}^{k}\frac{1}{p_i} < 2\ln\frac{\pi}{2}.$$

Hence,

$$\frac{1}{2} < \sum_{i=1}^{k} \frac{1}{p_i} < 2 \ln \frac{\pi}{2}.$$

From this theorem it is seen that

$$.5 < \sum_{i=1}^{k} \frac{1}{p_i} < .903.$$

D. Suryanarayana and Venkateswara Rao [24] have improved on both the upper and lower bound in the following theorem.

Theorem 3.35. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is an odd perfect number, then

$$\frac{\ln 2}{5(\ln 5 - \ln 4)} < \sum_{i=1}^{k} \frac{1}{p_i} < \ln 2 + \frac{1}{338}$$

if n is of the form $12t + 1$ and

$$\frac{1}{3} + \frac{2 \ln 2 - \ln 3}{5(\ln 5 - \ln 4)} < \sum_{i=1}^{k} \frac{1}{p_i} < \ln \frac{18}{13} + \frac{53}{150}$$

if n is of the form $36t + 9$.

PROOF: Since n is perfect

$$2 \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^{a_i+1}}\right) < 1. \qquad (1)$$

Therefore,

$$2 < \frac{1}{\prod_{k=1}^{k}\left(1 - \frac{1}{p_i}\right)}$$

which implies

$$\ln 2 < -\sum_{i=1}^{k} \ln\left(1 - \frac{1}{p_i}\right)$$

$$= -\sum_{i=1}^{k} \left(-\frac{1}{p_i} - \frac{1}{2}\frac{1}{p_i^2} - \frac{1}{3}\frac{1}{p_i^3} - \cdots\right)$$

$$= \sum_{i=1}^{k} \frac{1}{p_i} + \frac{1}{2}\sum_{i=1}^{k} \frac{1}{p_i^2} + \frac{1}{3}\sum_{i=1}^{k} \frac{1}{p_i^3} + \cdots . \tag{2}$$

Equation (1) also gives

$$2 = \frac{\prod_{i=1}^{k}\left(1 - \frac{1}{p_i^{a_i+1}}\right)}{\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right)}$$

which implies

$$\ln 2 = \sum_{i=1}^{k} \ln\left(1 - \frac{1}{p_i^{a_i+1}}\right) - \sum_{i=1}^{k} \ln\left(1 - \frac{1}{p_i}\right)$$

$$= \sum_{i=1}^{k}\left(-\sum_{j=1}^{\infty} \frac{1}{j}\frac{1}{p_i^{(a_i+1)j}}\right) - \sum_{i=1}^{k}\left(-\sum_{j=1}^{\infty} \frac{1}{j}\frac{1}{p_i^j}\right)$$

$$= \sum_{i=1}^{k} \frac{1}{p_i} + \sum_{i=1}^{k} \sum_{j=1}^{\infty} \frac{1}{(j+1)p_i^{j+1}} - \sum_{i=1}^{k} \sum_{j=1}^{\infty} \frac{1}{jp_i^{(a_i+1)j}} . \qquad (3)$$

Let p_1 be the prime divisor of n such that $p_1 \equiv 1 \bmod 4$ and $a_1 \equiv 1 \bmod 4$. Also, let the other primes be such that $p_2 < p_3 < \cdots < p_k$ with $2 | a_i$, $i = 2, 3, \ldots, k$.

Case 1: n is of the form $12t + 1$. By Corollary 3.25, 3 does not divide n and $p_1 \equiv 1 \bmod 12$. Hence, $p_i \geq 5$ for $i = 1, 3, \ldots, k$ and $p_1 \geq 13$. Then from (2)

$$\ln 2 < \sum_{i=1}^{k} \frac{1}{p_i} + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{5p_i} + \frac{1}{3} \sum_{i=1}^{k} \frac{1}{5^2 p_i} + \cdots$$

$$= 5 \sum_{i=1}^{k} \frac{1}{p_i} \left(\frac{1}{5} + \frac{1}{2}\frac{1}{5^2} + \frac{1}{3}\frac{1}{5^3} + \cdots \right)$$

$$= -5 \ln\left(1 - \frac{1}{5}\right) \sum_{i=1}^{k} \frac{1}{p_i} = 5 \ln \frac{5}{4} \sum_{i=1}^{k} \frac{1}{p_i}$$

which implies

$$\frac{\ln 2}{5(\ln 5 - \ln 4)} < \sum_{i=1}^{k} \frac{1}{p_i}.$$

From (3)

$$\ln 2 = \sum_{i=1}^{k} \frac{1}{p_i} + \sum_{i=2}^{k} \sum_{j=1}^{\infty} \left(\frac{1}{(j+1)p_i^{j+1}} - \frac{1}{jp_i^{(a_1+1)j}} \right)$$

$$+ \left(\frac{1}{2p_1^2} - \frac{1}{p_1^{a_1+1}} \right) + \sum_{j=2}^{\infty} \left(\frac{1}{(j+1)p_1^{j+1}} - \frac{1}{jp_1^{(a_1+1)j}} \right).$$

Since $a_i \geq 2$ for $i = 2, 3, \ldots, k$ each term in the second summation is positive, and hence, the sum is positive. Similarly, the fourth term is positive. Since $p_1 \geq 13$ and $a_1 \geq 1$,

$$\frac{1}{2p_1^2} - \frac{1}{p_1^{a_1+1}} \geq \frac{1}{2p_1^2} - \frac{1}{p_1^2} = -\frac{1}{2p_1^2} \geq -\frac{1}{2(13)^2} = -\frac{1}{338}.$$

Thus,

$$\ln 2 > \sum_{i=1}^{k} \frac{1}{p_i} - \frac{1}{338}$$

which gives

$$\sum_{i=1}^{k} \frac{1}{p_i} < \ln 2 + \frac{1}{338}.$$

Therefore, if n is of the form $12t + 1$

$$\frac{\ln 2}{5(\ln 5 - \ln 4)} < \sum_{i=1}^{k} \frac{1}{p_i} < \ln 2 + \frac{1}{338}.$$

<u>Case 2</u>: n is of the form $36t + 9$. Then clearly $3|n$ and $p_2 = 3$. Since $p_1 \equiv 1 \bmod 4$, $p_1 \geq 5$. Since $p_i \geq 5$ for $i = 3, 4, \ldots, k$ inequality (2) gives

$$\ln 2 < -\ln\left(1 - \frac{1}{3}\right) + \sum_{\substack{i=1 \\ i \neq 2}}^{k} \frac{1}{p_i} + \frac{1}{2}\sum_{\substack{i=1 \\ i \neq 2}}^{k} \frac{1}{p_i^2} + \frac{1}{3}\sum_{\substack{i=1 \\ i \neq 2}}^{k} \frac{1}{p_i^3} + \cdots$$

$$< \ln\frac{3}{2} + \sum_{\substack{i=1 \\ i \neq 2}}^{k} \frac{1}{p_i} + \frac{1}{2}\sum_{\substack{i=1 \\ i \neq 2}}^{k} \frac{1}{5p_i} + \frac{1}{3}\sum_{\substack{i=1 \\ i \neq 2}}^{k} \frac{1}{5^2 p_i} + \cdots$$

$$= \ln\frac{3}{2} + \frac{1}{p_1} + \frac{1}{2}\frac{1}{5}\frac{1}{p_1} + \frac{1}{3}\frac{1}{5^2}\frac{1}{p_1} + \cdots$$

$$+ \sum_{i=3}^{k} \frac{1}{p_i} + \frac{1}{2}\frac{1}{5}\sum_{i=3}^{k} \frac{1}{p_i} + \frac{1}{3}\frac{1}{5^2}\sum_{i=3}^{k} \frac{1}{p_i} + \cdots$$

$$= \ln\frac{3}{2} + 5\left(\ln\frac{5}{4}\right)\frac{1}{p_1} + 5\ln\frac{5}{4}\sum_{i=3}^{k} \frac{1}{p_i}$$

$$= \ln\frac{3}{2} + 5\ln\frac{5}{4}\sum_{i=1}^{k} \frac{1}{p_i} - \frac{5}{3}\ln\frac{5}{4}.$$

This implies

$$\frac{1}{3} + \frac{2\ln 2 - \ln 3}{5(\ln 5 - \ln 4)} < \sum_{i=1}^{k} \frac{1}{p_i}.$$

From (3)

$$\ln 2 = \sum_{i=1}^{k} \frac{1}{p_i} + \sum_{i=3}^{k}\sum_{j=1}^{\infty}\left(\frac{1}{(j+1)p_i^{j+1}} - \frac{1}{jp_i^{(a_i+1)j}}\right)$$

$$+ \left(\frac{1}{2p_1^2} - \frac{1}{p_1^{a_1+1}}\right) + \sum_{j=2}^{\infty}\left(\frac{1}{(j+1)p_1^{j+1}} - \frac{1}{jp_1^{(a_1+1)j}}\right)$$

$$+ \sum_{j=1}^{\infty} \left(\frac{1}{(j+1)p_2^{j+1}} - \frac{1}{jp_2^{(a_2+1)j}} \right).$$

Since $a_i \geq 2$ for $i = 2, 3, \ldots, k$ each term in the second summation is positive, and hence, the second summation is positive. The third summation is positive since every term is positive. Since $p_1 \geq 5$ and $a_1 \geq 1$,

$$\frac{1}{2p_1^2} - \frac{1}{p_1^{a_1+1}} \geq \frac{1}{2p_1^2} - \frac{1}{p_1^2} = - \frac{1}{2p_1^2} \geq - \frac{1}{2(5)^2} = - \frac{1}{50}.$$

Then since $p_2 = 3$ and $a_2 \geq 2$,

$$\begin{aligned}
\ln 2 &> \sum_{i=1}^{k} \frac{1}{p_i} - \frac{1}{50} + \sum_{j=1}^{\infty} \left(\frac{1}{(j+1)3^{j+1}} - \frac{1}{j(3^3)^j} \right) \\
&= \sum_{i=1}^{k} \frac{1}{p_i} - \frac{1}{50} + \sum_{j=1}^{\infty} \frac{1}{j3^j} - \frac{1}{3} - \sum_{j=1}^{\infty} \frac{1}{j(3^3)^j} \\
&= \sum_{i=1}^{k} \frac{1}{p_i} - \frac{1}{50} - \ln\left(1 - \frac{1}{3}\right) - \frac{1}{3} + \ln\left(1 - \frac{1}{3^3}\right) \\
&= \sum_{i=1}^{k} \frac{1}{p_i} - \ln \frac{2}{3} + \ln \frac{26}{3^3} - \frac{53}{150} \\
&= \sum_{i=1}^{k} \frac{1}{p_i} + \ln \frac{13}{9} - \frac{53}{150}.
\end{aligned}$$

This implies

$$\sum_{i=1}^{k} \frac{1}{p_i} < \ln \frac{18}{13} + \frac{53}{150}.$$

Therefore, if n is of the form $36t + 9$,

$$\frac{1}{3} + \frac{2 \ln 2 - \ln 3}{5(\ln 5 - \ln 4)} < \sum_{i=1}^{k} \frac{1}{p_i} < \ln \frac{18}{13} + \frac{53}{150}.$$

If the bounds in the last theorem are approximated to three decimal places the results are stronger bounds than those derived by Perisastri. In decimal form, this theorem says that if n is of the form $12t + 1$

$$.621 < \sum_{i=1}^{k} \frac{1}{p_i} < .696$$

and if n is of the form $36t + 9$

$$.591 < \sum_{i=1}^{k} \frac{1}{p_i} < .679.$$

D. Suryanarayana [25], using the same techniques as Suryanarayana and Rao, has improved these bounds even more. The proof of Suryanarayana's theorem which follows has been omitted. The proof is quite lengthy.

Theorem 3.36. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be an odd perfect number. If n is of the form $12t + 1$ and $5|n$, then

$$.644 < \frac{1}{5} + \frac{1}{7} + \frac{\ln \frac{48}{35}}{11 \ln \frac{11}{10}} < \sum_{i=1}^{k} \frac{1}{p_i} < \frac{1}{5} + \frac{1}{2738} + \ln \frac{50}{31} < .679.$$

If n is of the form $12t + 1$ and 5 does not divide n, then

$$.657 < \frac{1}{7} + \frac{\ln \frac{12}{7}}{11 \ln \frac{11}{10}} < \sum_{i=1}^{k} \frac{1}{p_i} < \ln 2 < .693.$$

If n is of the form $36t + 9$ and $5|n$, then

$$.596 < \frac{1}{3} + \frac{1}{5} + \frac{\ln \frac{16}{15}}{17 \ln \frac{17}{16}} < \sum_{i=1}^{k} \frac{1}{p_i}$$

$$< \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \ln \frac{65}{61} < .674.$$

If n is of the form $36t + 9$ and 5 does not divide n, then

$$.600 < \frac{1}{3} + \frac{\ln \frac{4}{3}}{7 \ln \frac{7}{6}} < \sum_{i=1}^{k} \frac{1}{p_i} < \frac{1}{3} + \frac{1}{338} + \ln \frac{18}{13}.$$

M. Perisastri [23] has used the Rieman Zeta function

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s}$$

to establish a lower bound on the sum of the primes. The following theorem states his results.

<u>Theorem 3.37</u>. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is an odd perfect number and s is the smallest a_i, $i = 1, 2, \ldots, k$, then

$$\sum_{i=1}^{k} \frac{1}{p_i} > \ln \frac{2^{s+1} - 1}{2^s} \zeta(s + 1).$$

Lower Bounds On n

Various lower bounds for odd perfect numbers have been obtained. In 1908, Turcaninov obtained $2(10)^6$ as a lower bound [11]. This was improved to 10^{10} by H. A. Bernhard [26] and to $1.4(10)^{14}$ by Kanold [11]. This bound was later improved to 10^{18} by J. B. Muskat and to 10^{20} by Kanold [27]. The best improvement which is 10^{36} has been made by Bryant Tuckerman [2].

CHAPTER IV

UNITARY PERFECT NUMBERS

Unitary perfect numbers are defined in terms of unitary divisors analogous to the way perfect numbers are defined in terms of divisors. For completeness, the following definitions which first appeared in Chapter I are restated. The positive integer d is a <u>unitary divisor</u> of the positive integer n, written $d \| n$, if $d | n$ and $(d, n/d) = 1$. If $n \in N$, then n is <u>unitary perfect</u> if

$$n = \sum_{\substack{d \| n \\ d \neq n}} d.$$

If $p^a \| n$, where p is a prime, then p^a is the largest power of p that divides n. For example, the unitary divisors of 28 are 1, 4, 7, and 28.

If $\sigma^*(n)$ represents the sum of the unitary divisors of n,

$$n = \prod_{i=1}^{k} p_i^{a_i} > 1,$$

then

$$\sigma^*(n) = \prod_{i=1}^{k} (1 + p_i^{a_i}).$$

Thus, $\sigma^*(n)$ is a multiplicative function [28;p. 37]. It is clear that n is unitary perfect if and only if $\sigma^*(n) = 2n$.

The first four unitary perfect numbers are 6, 60, 90, and 87,360 [4]. They are unitary perfect since

$$\sigma^*(6) = \sigma^*[2(3)] = (1 + 2)(1 + 3) = 3(4) = 12 = 2(6),$$

$$\begin{aligned}\sigma^*(60) = \sigma^*[2^2(3)(5)] &= (1 + 2^2)(1 + 3)(1 + 5)\\ &= 5(4)(6) = 120 = 2(60),\end{aligned}$$

$$\begin{aligned}\sigma^*(90) = \sigma^*[2(3)^2(5)] &= (1 + 2)(1 + 3^2)(1 + 5)\\ &= 3(10)(6) = 180 = 2(90),\end{aligned}$$

and

$$\begin{aligned}\sigma^*(87,360) &= \sigma^*[2^6(3)(5)(7)(13)]\\ &= (1 + 2^6)(1 + 3)(1 + 5)(1 + 7)(1 + 13)\\ &= 65(4)(6)(8)(14) = 174,720 = 2(87,360).\end{aligned}$$

While it is not known if there do or do not exist odd perfect numbers, it is quite easy to prove that there do not exist any odd unitary perfect numbers.

Theorem 4.1. There do not exist any odd unitary perfect numbers.

PROOF: Suppose $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is an odd unitary perfect number. Then

$$\sigma^*(n) = \prod_{i=1}^{k} (1 + p_i^{a_i}) = 2 \prod_{i=1}^{k} p_i^{a_i} = 2n.$$

Since $1 + p_i^{a_i}$ is even for $i = 1, 2, \ldots, k$ and $p_i^{a_i}$ is odd for $i = 1, 2, \ldots, k$, $k = 1$ and

$$1 + p_1^{a_1} = 2p_1^{a_1}$$

which implies that $p_1^{a_1} = 1$ which is a contradiction. Thus, there are no odd unitary perfect numbers.

<u>Theorem 4.2</u>. If $n = 2^t$, n is not unitary perfect.

PROOF: Suppose n is unitary perfect. Then

$$\sigma^*(n) = 1 + 2^t = 2 \cdot 2^t = 2n$$

which implies that $1 = 2^t$ which is impossible. Therefore, n is not unitary perfect.

Thus, any unitary perfect number is of the form

$$n = 2^t m = 2^t p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

where each p_i is an odd prime.

The following lemmas and theorem were proven by M. V. Subbarao and L. J. Warren [4]. However, the proofs presented here, in most cases, do not follow the pattern of Subbarao and Warren.

The following notation will be used throughout the remainder of the chapter. Unless otherwise specified, m represents an odd integer greater than 1 and n is an even integer given by $n = 2^t m$ with t a positive integer. If m is written in the form $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, then $p_1 < p_2 < \cdots < p_k$. If m is written in the form $m = m_1 m_2 m_3$, then $(m_1,m_2) = (m_1,m_3) = (m_2,m_3) = 1$; every prime divisor of m_1 is congruent to 1 modulo 4; every prime divisor of m_2 is congruent to 3 modulo 4 and occurs with an even exponent; and every prime divisor of m_3 is congruent to 3 modulo 4 and occurs with an odd exponent. For any fixed m, let a, b, and c denote the number of distinct prime factors of m_1, m_2, and m_3, respectively. For given nonnegative integers a, b, and c, not all zero, the set of all odd numbers $m = m_1m_2m_3$ associated with a, b, and c will be denoted by $K(a,b,c)$.

Lemma 4.3. If $n = p_1^{a_1}p_2^{a_2} \cdots p_3^{a_3}$, $m = q_1^{b_1}q_2^{b_2} \cdots q_k^{b_k}$ and $p_i^{a_i} \le q_i^{b_i}$, $i = 1, 2, \ldots, k$, then

$$\frac{\sigma^*(n)}{n} \ge \frac{\sigma^*(m)}{m}.$$

PROOF:

$$\frac{\sigma^*(n)}{n} = \prod_{i=1}^{k} \frac{1 + p_i^{a_i}}{p_i^{a_i}} = \prod_{i=1}^{k} \left(1 + \frac{1}{p_i^{a_i}}\right)$$

$$\ge \prod_{i=1}^{k} \left(1 + \frac{1}{q_i^{b_i}}\right) = \prod_{i=1}^{k} \frac{1 + q_i^{b_i}}{q_i^{b_i}} = \frac{\sigma^*(m)}{m}.$$

Lemma 4.4. If $n = 2^t m = 2^t p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$ is unitary perfect, then

(1) $p_k \mid (2^t + 1)$ if $a_1 = a_2 = \cdots = a_{k-1} = 1$;

(2) $a + b + 2c \le t + 1$ and equality holds when $c = 0$.

PROOF: (1) If $a_1 = a_2 = \cdots = a_{k-1} = 1$ and n is unitary perfect,

$$\sigma^*(n) = (2^t + 1)(1 + p_k^{a_k}) \prod_{i=1}^{k-1} (1 + p_i)$$

$$= 2^{t+1} p_k^{a_k} \prod_{i=1}^{k-1} p_i = 2n.$$

Since $p_k > p_i$, $i = 1, 2, \ldots, k - 1$, p_k does not divide $1 + p_i$, $i = 1, 2, \ldots, k - 1$. Then, since p_k does not divide $1 + p_k^{a_k}$, $p_k | (2^t + 1)$.

(2) If $m = m_1 m_2 m_3$ and the prime $p | m_1$, then $p \equiv 1 \bmod 4$. Thus, $\sigma^*(p^s) = 1 + p^s \equiv 2 \bmod 4$ which implies that $2 \| (1 + p^s)$, where s is the exponent of p in n. If the prime $p | m_2$, then $p \equiv 3 \bmod 4$ and the power of p is even. Thus, $\sigma^*(p^{2s}) = 1 + p^{2s} \equiv 1 + (3^2)^s \equiv 2 \bmod 4$ which implies that $2 \| (1 + p^{2s})$, where $2s$ is the exponent of p in n. If the prime $p | m_3$, then $p \equiv 3 \bmod 4$ and the power of p is odd. Thus,

$$\sigma^*(p^{2s+1}) = 1 + p^{2s+1} \equiv 1 + 3(3^2)^s \equiv 4 \bmod 4$$

which implies that $4 | (1 + p^{2s+1})$ where $2s + 1$ is the exponent of p in n. Since n is unitary perfect,

$$2n = 2^{t+1} m_1 m_2 m_3 = (1 + 2^t) \prod_{i=1}^{k} (1 + p_i^{a_i}) = \sigma^*(n).$$

Thus, 2^{t+1} divides

$$\prod_{i=1}^{k} (1 + p_i^{a_i}).$$

But $a + b$ of the factors $(1 + p_i^{a_i})$ each contain exactly one factor of 2 while c of the factors $(1 + p_i^{a_i})$ contain at least two factors of 2. Thus, $a + b + 2c \leq t + 1$. If $c = 0$, then $a + b = t + 1$.

Lemma 4.5. If $n = 2^t m$ is unitary perfect and 3 does not divide n, then:

(1) t is an even integer;

(2) if $p^s \| m$, then $p^s \equiv 1 \bmod 6$;

(3) there is a prime p such that $p|m$, $p \equiv 5 \bmod 6$, and p occurs with an even exponent in m;

(4) m has an even number of distinct primes.

PROOF: (1) Since n is unitary perfect,

$$\sigma^*(n) = (1 + 2^t)\sigma^*(m) = 2^{t+1}m = 2n.$$

If $t = 2s + 1$ is odd, then

$$1 + 2^t = 1 + 2^{2s+1} \equiv 1 + 2(1)^s \equiv 0 \bmod 3.$$

Thus, $3|\sigma^*(n)$ which implies that $3|n$. Therefore, t cannot be odd and must be even.

(2) Since p is odd and $p \neq 3$, $p^s \equiv 1$ or $5 \bmod 6$. Suppose $p^s \equiv 5 \bmod 6$, then $1 + p^s \equiv 0 \bmod 6$ which implies that $3|\sigma^*(n)$, and hence, $3|n$ which is impossible. Therefore, $p^s \equiv 1 \bmod 6$.

(3) Since $t = 2s$ is even,

$$1 + 2^t = 1 + 2^{2s} = 1 + 4^s \equiv 5 \bmod 6.$$

Therefore, there exists a prime p such that $p|\sigma^*(n)$, and hence, $p|m$ with $p \equiv 5 \bmod 6$. Since $p \equiv 5 \bmod 6$,

$$p^{2s+1} = pp^{2s} \equiv 5(5^2)^s \equiv 5 \bmod 6.$$

Therefore, by (2) the power of p must be even.

(4) Let $n = 2^{2s}p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$. Since, for $i = 1, 2, \ldots, k$ $p_i^{a_i} \equiv 1 \bmod 6$, then $p_i^{a_i} \equiv 1 \bmod 3$. Then

$$2n = 2^{2s+1} \prod_{i=1}^{k} p_i^{a_i} \equiv 2\cdot 1 \equiv 2 \bmod 3$$

and

$$\sigma^*(n) = (2^{2s} + 1) \prod_{i=1}^{k} (1 + p_i^{a_i}) \equiv (1 + 1)(1 + 1)^k \equiv 2^{k+1} \bmod 3.$$

Therefore, $2^{k+1} \equiv 2 \bmod 3$ which implies k is even.

Although they have not been able to find any unitary perfect numbers not divisible by 3, Subbarao and Warren [5] have not been able to prove that there are none.

<u>Theorem 4.6</u>. Let $n = 2^t p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$ be unitary perfect.

(1) If $k = 1$, then $n = 6$.

(2) If $t = 1$, then $n = 6$ or 90.

(3) If $t = 2$, then $n = 60$.

(4) If $k = 2$, then $n = 60$ or 90.

(5) It is not possible for $k = 3$ or 5.

(6) It is not possible for $t = 3, 4, 5,$ or 7.

(7) If $t = 6$, then $n = 87{,}360$.

(8) If $k = 4$, then $n = 87{,}360$.

PROOF: (1) If $k = 1$, $n = 2^t p^a$. By Lemma 4.5 part (4), $3|n$. Therefore, $p = 3$. Then

$$\sigma^*(n) = (2^t + 1)(3^a + 1) = 2^{t+1}3^a = 2n.$$

Since 2 does not divide $2^t + 1$ and 3 does not divide $3^a + 1$,

$$2^t + 1 = 3^a \quad \text{and} \quad 3^a + 1 = 2^{t+1},$$

which implies

$$2^t + 1 + 1 = 2^{t+1}$$

or

$$2 = 2^t$$

which implies that $t = 1$, and hence, $a = 1$. Therefore, $n = 6$.

(2) If $t = 1$, by Lemma 4.5 part (1), $3|n$. Since 3 is a factor of m_2 or m_3, not both b and c can be 0. By Lemma 4.4 part (2) there are two cases. Either $a = b = 0$ and $c = 1$ or $a = b = 1$ and $c = 0$. In the first case, $k = 1$ and then by (1), $n = 6$. In the second case, $n = 2\cdot 3^{2r}p^s$. Then

$$\sigma^*(n) = 3(3^{2r} + 1)(p^s + 1) = 2^2 3^{2r} p^s = 2n$$

which implies that $3|(p^s + 1)$. This implies that $p \equiv 2 \bmod 3$ and s

is odd. Since p is one of the factors of m_1, $p \equiv 1 \bmod 4$. Therefore, $p \equiv 5 \bmod 12$. Suppose $p^s \geq 17$. Let $n' = 2\cdot 3^2 17$. Then by Lemma 4.3,

$$\frac{\sigma^*(2\cdot 3^{2r}p^s)}{2\cdot 3^{2r}p^s} \leq \frac{\sigma^*(2\cdot 3^2 17)}{2\cdot 3^2 17}.$$

Then, if n is unitary perfect,

$$2 \leq \frac{(2+1)(3^2+1)(17+1)}{2\cdot 3^2 17}$$

$$= \frac{3(10)(18)}{2(9)(17)} = \frac{30}{17}$$

which is a contradiction. Therefore, $p^s = 5$. Then $p = 5$ and $s = 1$. Then if n is unitary perfect

$$2n = 2^2 3^{2r} 5 = (2+1)(3^{2r}+1)(5+1) = \sigma^*(n).$$

This implies that

$$10\cdot 3^{2r} = 9\cdot 3^{2r} + 9$$

which shows that $3^{2r} = 9$. Thus, $r = 1$ and $n = 2\cdot 3^2 5 = 90$. Therefore, $n = 6$ or 90.

(3) Let $t = 2$. Then $2^2 \| n$ and $p = 2^2 + 1 = 5$ divides n. Then $a \geq 1$. By Lemma 4.4 part (2), $a + b + 2c \leq 3$. Thus, (i) $a = 1$, $b = 2$, and $c = 0$, (ii) $a = 2$, $b = 1$, and $c = 0$, or (iii) $a = c = 1$ and $b = 0$.

Case (i): $a = 1$, $b = 2$, and $c = 0$. Then $n = 2^2 3^{2r} 5^s p^{2u}$ with $p \geq 7$. Consider $n' = 2^2 3^2 5\cdot 7^2$. By Lemma 4.3,

$$\frac{\sigma^*(2^2 3^{2r} 5^s p^{2u})}{2^2 3^{2r} 5^s p^{2u}} \leq \frac{\sigma^*(2^2 3^2 5 \cdot 7^2)}{2^2 3^2 5 \cdot 7^2}.$$

Then if n is unitary perfect,

$$2 \leq \frac{(2^2+1)(3^2+1)(5+1)(7^2+1)}{2^2 3^2 5 \cdot 7^2}$$

$$= \frac{5(10)(6)(50)}{4(9)(5)(49)} = \frac{250}{147}$$

which is a contradiction. Therefore, n is not unitary perfect.

Case (ii): $a = 2$, $b = 1$, and $c = 0$. By Lemma 4.5 part (4), $3|n$. Then $n = 2^2 3^{2r} 5^s p^u$ with $p \geq 13$. Let $n' = 2^2 3^2 5 \cdot 13$. Then by Lemma 4.3

$$\frac{\sigma^*(n)}{n} \leq \frac{\sigma^*(2^2 3^2 5 \cdot 13)}{2^2 3^2 5 \cdot 13}.$$

Then if n is unitary perfect,

$$2 \leq \frac{(2^2+1)(3^2+1)(5+1)(13+1)}{2^2 3^2 5 \cdot 13}$$

$$= \frac{5(10)(6)(14)}{4(9)(5)(13)} = \frac{70}{39}$$

which is a contradiction. Therefore, n is not unitary perfect.

Case (iii): $a = c = 1$ and $b = 0$. Then $n = 2^2 5^r p^s$ with $p \equiv 3 \bmod 4$. Suppose $p \geq 7$. Let $n' = 2^2 5 \cdot 7$. Then by Lemma 4.3, if n is unitary perfect,

$$2 = \frac{\sigma^*(n)}{n} \leq \frac{\sigma^*(2^2 5 \cdot 7)}{2^2 5 \cdot 7} = \frac{(2^2+1)(5+1)(7+1)}{2^2 5 \cdot 7} = \frac{12}{7}$$

which is a contradiction. Therefore, $p = 3$ if n is to be unitary perfect and $n = 2^2 3^s 5^r$. Suppose $s > 1$. Let $n'' = 2^2 3^2 5$. Then by Lemma 4.3

$$2 = \frac{\sigma^*(n)}{5} \leq \frac{{}^*(2^2 3^2 5)}{2^2 3^2 5} = \frac{(2^2 + 1)(3^2 + 1)(5 + 1)}{2^2 3^2 5} = \frac{5}{3}$$

which is a contradiction. Therefore, $s = 1$ and $n = 2^2 3 \cdot 5^r$. Then since n is unitary perfect,

$$2 \cdot 2^2 3 \cdot 5^r = (2^2 + 1)(3 + 1)(5^r + 1)$$

or

$$24 \cdot 5^r = 20(5^r + 1).$$

Thus, $5^r = 5$ which implies that $r = 1$. Therefore, $n = 2^2 3 \cdot 5 = 60$.

(4) Let $k = 2$. Then $n = 2^t p_1^r p_2^s$. Suppose $t \geq 3$. Since $p_1^r \geq 3$ and $p_2^s \geq 5$, Lemma 4.3 shows that if n is unitary perfect, then

$$n = \frac{\sigma^*(n)}{n} \leq \frac{\sigma^*(2^3 3 \cdot 5)}{2^3 3 \cdot 5} = \frac{(2^3 + 1)(3 + 1)(5 + 1)}{2^3 3 \cdot 5}$$

$$= \frac{9(4)(6)}{8(3)(5)} = \frac{9}{5}$$

which is a contradiction. Therefore, for n to be unitary perfect, $t = 1$ or 2. But (2) states that if $t = 1$, then $n = 90$ and (3) states that if $t = 2$, then $n = 60$. Therefore, if $k = 2$, then $n = 60$ or 90.

(5) Case (1): $k = 3$. By Lemma 4.6 part (4), if n is to be unitary perfect, $3|n$. Let $n = 2^t 3^r p_1^{a_1} p_2^{a_2}$ be unitary perfect.

Then by Lemma 4.3,

$$2 = \frac{\sigma^*(n)}{n} \leq \frac{\sigma^*(2^t 3\cdot 5\cdot 7)}{2^t 3\cdot 5\cdot 7}$$

$$= \frac{(2^t + 1)(3 + 1)(5 + 1)(7 + 1)}{2^t 3\cdot 5\cdot 7}$$

$$= \frac{(2^t + 1)(4)(6)(8)}{2^t 3\cdot 5\cdot 7}.$$

This implies that

$$35(2^t) \leq 32(2^t) + 32$$

or

$$3(2^t) \leq 32,$$

which implies that $t = 1, 2,$ or 3. But by (2) and (3), t cannot be 1 or 2. Thus, $t = 3$. Since $2^3 + 1 = 9$, $r \geq 2$. Then by Lemma 4.3,

$$2 = \frac{\sigma^*(n)}{n} \leq \frac{\sigma^*(2^3 3^2\cdot 5\cdot 7)}{2^3 3^2\cdot 5\cdot 7}$$

$$= \frac{(2^3 + 1)(3^2 + 1)(5 + 1)(7 + 1)}{2^3 3^2\cdot 5\cdot 7}$$

$$= \frac{9(10)(6)(8)}{8(9)(5)(7)} = \frac{12}{7}$$

which is a contradiction. Therefore, n is not unitary perfect for $k = 3$.

Case (2): $k = 5$. Then $n = 2\, p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4} p_5^{a_5}$. Suppose n is unitary perfect. Then by Lemma 4.3

$$2 = \frac{\sigma^*(n)}{n} \le \frac{\sigma^*(2^t 3\cdot 5\cdot 7\cdot 11\cdot 13)}{2^t 3\cdot 5\cdot 7\cdot 11\cdot 13}$$

$$= \frac{(2^t + 1)(3 + 1)(5 + 1)(7 + 1)(11 + 1)(13 + 1)}{2^t 3\cdot 5\cdot 7\cdot 11\cdot 13}$$

which gives

$$715(2^t) \le 768(2^t + 1)$$

which is not true for any $t \geq 1$. Thus, n is not unitary perfect for $k = 5$.

(6) Case (1): $t = 3$. Since $2^3 + 1 = 9$, $3^2 | n$ if n is unitary perfect. Since $a + b + 2c \le 4$, there are at most 4 odd prime factors of n. From (1) and (4) there are at least 3 odd prime factors of n. Since

$$\frac{(2^3 + 1)(3^2 + 1)(5 + 1)(7 + 1)}{2^3 3^2 5\cdot 7} < \frac{(2^3 + 1)(3^2 + 1)(5 + 1)(7 + 1)(11 + 1)}{2^3 3^2 5\cdot 7\cdot 11}$$

$$= \frac{9(10)(6)(8)(12)}{8(9)(5)(7)(11)} = \frac{144}{77} < 2$$

by Lemma 4.3 no such n can be unitary perfect.

Case (2): $t = 4$. Since $2^4 + 1 = 17$, $17 | n$. Since $a + b + 2c \le 5$, $k \le 5$. But (1), (4), and (5) imply that $k = 4$ if n is to be a unitary perfect number. Suppose there exists a p such that $p | m_2$. Then $p \geq 3$ and has an exponent greater than or equal to 2. Then by Lemma 4.3, if n is unitary perfect,

$$2 = \frac{\sigma^*(n)}{n} \le \frac{\sigma^*(2^4 3^2 5\cdot 7\cdot 17)}{2^4 3^2 5\cdot 7\cdot 17}$$

$$= \frac{(2^4 + 1)(3^2 + 1)(5 + 1)(7 + 1)(17 + 1)}{2^4 3^2 5 \cdot 7 \cdot 17}$$

$$= \frac{17(10)(6)(8)(18)}{16(9)(5)(7)(17)} = \frac{12}{7}$$

which is a contradiction. Therefore, for n to be unitary perfect, $b = 0$. Thus, the only possible case is $a = 3$ and $c = 1$. Suppose $17^2 | n$, then if n is unitary perfect, by Lemma 4.3

$$2 = \frac{\sigma^*(n)}{n} \leq \frac{\sigma^*(2^4 3 \cdot 5 \cdot 13 \cdot 17^2)}{2^4 3 \cdot 5 \cdot 13 \cdot 17^2}$$

$$= \frac{(2^4 + 1)(3 + 1)(5 + 1)(13 + 1)(17^2 + 1)}{2^4 3 \cdot 5 \cdot 13 \cdot 17^2}$$

$$= \frac{17(4)(6)(14)(290)}{16(3)(5)(13)(289)} = \frac{406}{221}$$

which is a contradiction. Thus, $17 \| n$ if n is unitary perfect. Then $17 + 1 = 18$ implies that $3^2 | n$. Then by Lemma 4.3, if n is unitary perfect,

$$2 = \frac{\sigma^*(n)}{n} \leq \frac{\sigma^*(2^4 3^2 5 \cdot 13 \cdot 17)}{2^4 3^2 5 \cdot 13 \cdot 17}$$

$$= \frac{(2^4 + 1)(3^2 + 1)(5 + 1)(13 + 1)(17 + 1)}{2^4 3^2 5 \cdot 13 \cdot 17}$$

$$= \frac{17(10)(6)(14)(18)}{16(9)(5)(13)(17)} = \frac{21}{13}$$

which is a contradiction. Therefore, n is not unitary perfect for $t = 4$.

Case (3): $t = 5$. Since $2^5 + 1 = 33 = 3(11)$, $3|n$ and $11|n$. Since $a + b + 2c \leq 6$, $k < 6$. By (1), (4), and (5) the only possibilities are $k = 4$ or 6 if n is to be unitary perfect. Suppose there exists a p such that $p|m_2$. Then $p \geq 3$ and has an even exponent. Since

$$\frac{(2^5+1)(3^2+1)(5+1)(7+1)(11+1)}{2^5 3^2 5\cdot 7\cdot 11}$$

$$< \frac{(2^5+1)(3^2+1)(5+1)(7+1)(11+1)(13+1)(17+1)}{2^5 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17}$$

$$= \frac{33(10)(6)(8)(12)(14)(18)}{32(9)(5)(7)(11)(13)(17)} = \frac{432}{221} < 2$$

by Lemma 4.3, n is not unitary perfect for either $k = 4$ or $k = 6$. Thus, for n to be unitary perfect $b = 0$. Then $3|m_3$ and $11|m_3$ and $c \geq 2$. This leaves no possibility for n to be unitary perfect which is $a = 2$ and $c = 2$. If n is unitary perfect, by Lemma 4.3

$$2 = \frac{\sigma^*(n)}{n} \leq \frac{\sigma^*(2^5 3\cdot 5\cdot 11\cdot 13)}{2^5 3\cdot 5\cdot 11\cdot 13}$$

$$= \frac{(2^5+1)(3+1)(5+1)(11+1)(13+1)}{2^5 3\cdot 5\cdot 11\cdot 13}$$

$$= \frac{33(4)(6)(12)(14)}{32(3)(5)(11)(13)} = \frac{126}{65}$$

which is a contradiction. Thus, n cannot be unitary perfect for $k = 5$.

Case (4): $t = 7$. Since $2^7 + 1 = 129 = 3(43)$, $3|n$ and $43|n$ if n is unitary perfect. Suppose n is unitary perfect. Since $a + b + 2c \leq 8$, $k \leq 8$. By (1), (3), and (5), $k = 4, 6, 7,$ or 8.

Since

$$\frac{\sigma^*(2^7 3\cdot5\cdot7\cdot43)}{2^7 3\cdot5\cdot7\cdot43} = \frac{(2^7+1)(3+1)(5+1)(7+1)(43+1)}{2^7 3\cdot5\cdot7\cdot43}$$

$$= \frac{129(4)(6)(8)(44)}{128(3)(5)(7)(43)} = \frac{66}{35} < 2,$$

by Lemma 4.3, n is not unitary perfect for $k = 4$. Thus, $k = 6, 7,$ or 8. Since

$$\frac{\sigma^*(2^7 3^2 5\cdot7^2 11\cdot13\cdot43)}{2^7 3^2 5\cdot7^2 11\cdot13\cdot43} < \frac{\sigma^*(2^7 3^2 5\cdot7^2 11\cdot13\cdot17\cdot43)}{2^7 3^2 5\cdot7^2 11\cdot13\cdot17\cdot43}$$

$$< \frac{\sigma^*(2^7 3^2 5\cdot7^2 11\cdot13\cdot17\cdot19\cdot43)}{2^7 3^2 5\cdot7^2 11\cdot13\cdot17\cdot19\cdot43}$$

$$= \frac{(2^7+1)(3^2+1)(6)(7^2+1)(12)(14)(18)(20)(44)}{2^7(3^2)(5)(7^2)(11)(13)(17)(19)(43)}$$

$$= \frac{54000}{29393} < 2$$

by Lemma 4.3, n is not unitary perfect if $b > 1$. Thus, since 3 and 43 divide either m_2 or m_3, either (i) $a = 5$ and $b = c = 1$, (ii) $a = 4$ and $b = c = 1$, (iii) $a = 3$, $b = 1$, and $c = 2$, or (iv) $a = 4$, $b = 0$, and $c = 2$. Since

$$\frac{\sigma^*(2^7 3^2 5\cdot7\cdot13\cdot17\cdot43)}{2^7 3^2 5\cdot7\cdot13\cdot17\cdot43} < \frac{\sigma^*(2^7 3^2 5\cdot13\cdot17\cdot29\cdot43)}{2^7 3^2 5\cdot13\cdot17\cdot29\cdot43}$$

$$< \frac{\sigma^*(2^7 3^2 5\cdot13\cdot17\cdot29\cdot37\cdot43)}{2^7 3^2 5\cdot13\cdot17\cdot29\cdot37\cdot43}$$

$$= \frac{(2^7+1)(3^2+1)(6)(14)(18)(30)(38)(44)}{2^7(3^2)(5)(13)(17)(29)(37)(43)}$$

$$= \frac{395010}{237133} < 2$$

by Lemma 4.3, (i), (ii), and (iii) are not possible. Since

$$\frac{\sigma^*(2^7 3\cdot 5\cdot 13\cdot 17\cdot 29\cdot 43)}{2^7 3\cdot 5\cdot 13\cdot 17\cdot 29\cdot 43} = \frac{(2^7+1)(4)(6)(14)(18)(30)(44)}{2^7(3)(5)(13)(17)(29)(43)}$$

$$= \frac{12474}{6409} < 2$$

by Lemma 4.3, (iv) is not possible and n is not unitary perfect. Therefore, n is not unitary perfect for $t = 7$.

(7) Let $t = 6$. Since $2^6 + 1 = 65 = 5(13)$, if n is unitary perfect $5|n$, $13|n$ and $a \geq 2$. Since $a + b + 2c \leq 7$, $k \leq 7$. Then by (1), (4), and (5), $k = 4$, 6, or 7. Since

$$\frac{\sigma^*(2^6 3^2 5\cdot 7^2 13)}{2^6 3^2 5\cdot 7^2 13} < \frac{\sigma^*(2^6 3^2 5\cdot 7^2 11\cdot 13\cdot 17)}{2^6 3^2 5\cdot 7^2 11\cdot 13\cdot 17}$$

$$< \frac{\sigma^*(2^6 3^2 5\cdot 7^2 11\cdot 13\cdot 17\cdot 19)}{2^6 3^2 5\cdot 7^2 11\cdot 13\cdot 17\cdot 19}$$

$$= \frac{(2^6+1)(3^2+1)(6)(7^2+1)(12)(14)(18)(20)}{2^6 3^2 5\cdot 7^2 11\cdot 13\cdot 17\cdot 19}$$

$$= \frac{45000}{24871} < 2,$$

then $b < 2$. Thus, $b = 1$ or 0. Then either (i) $a = 7$ and $b = c = 0$, (ii) $a = 6$, $b = 1$ and $c = 0$, (iii) $a = 2$ and $b = c = 1$, (iv) $a = 3$, $b = 0$, and $c = 1$, (v) $a = 4$ and $b = c = 1$, (vi) $a = 5$, $b = 0$, and $c = 1$, or (vii) $a = c = 2$ and $b = 0$. Since 3 does not divide n and k is odd, (i) does not give a unitary perfect number. Consider

$$n_2 = 2^6 m_2' = 2^6 3^2 5\cdot 13\cdot 17\cdot 29\cdot 37\cdot 41, \quad m_2' \in K(6,1,0),$$

$$n_3 = 2^6 m_3' = 2^6 3^2 5\cdot 7\cdot 13, \quad m_3' \in K(2,1,1),$$

$$n_4 = 2^6 m_4' = 2^6 3\cdot 5\cdot 13\cdot 17, \quad m_4' \in K(3,0,1),$$

$$n_5 = 2^6 m_5' = 2^6 3^2 5\cdot 7\cdot 13\cdot 17\cdot 29, \quad m_5' \in K(4,1,1),$$

and

$$n_6 = 2^6 m_6' = 2^6 3\cdot 5\cdot 13\cdot 17\cdot 29\cdot 37, \quad m_6' \in K(5,0,1).$$

Then for $i = 2, 3, 4, 5, 6$, $\sigma^*(n_i)/n_i \leq \sigma^*(n)/n$ for any $n = 2^6 m$ with m in the appropriate $K(a,b,c)$. Also,

$$\frac{\sigma^*(n_3)}{n_3} < \frac{\sigma^*(n_5)}{n_5} < \frac{\sigma^*(n_2)}{n_2}$$

and

$$\frac{\sigma^*(n_4)}{n_4} < \frac{\sigma^*(n_6)}{n_6} < \frac{\sigma^*(n_2)}{n_2}.$$

Then since

$$\frac{\sigma^*(n_2)}{n_2} = \frac{\sigma^*(2^6 3^2 5\cdot 13\cdot 17\cdot 29\cdot 37\cdot 41)}{2^6 3^2 5\cdot 13\cdot 17\cdot 29\cdot 37\cdot 41}$$

$$= \frac{(2^6+1)(3^2+1)(6)(14)(18)(30)(38)(42)}{2^6(3^2)(5)(13)(17)(29)(37)(41)}$$

$$= \frac{65(10)(6)(14)(18)(30)(38)(42)}{64(9)(5)(13)(17)(29)(37)(41)}$$

$$= \frac{1256850}{747881} < 2,$$

there can be no unitary perfect number in cases (ii), (iii), (iv), (v) and (vi). This leaves only the case with $a = c = 2$ and $b = 0$. Then $2^6 5^r 13^s p^u q^v$. Since

$$\frac{\sigma^*(2^6 5\cdot 7\cdot 11\cdot 13)}{2^6 5\cdot 7\cdot 11\cdot 13} = \frac{5(13)(6)(8)(12)(14)}{2^6(5)(7)(11)(13)} = \frac{18}{11} < 2$$

and

$$\frac{\sigma^*(2^6 3^2 5\cdot 7\cdot 13)}{2^6 3^2 5\cdot 7\cdot 13} = \frac{5(13)(10)(6)(8)(14)}{2^6(3^2)(5)(7)(13)} = \frac{5}{3} < 2,$$

by Lemma 4.3, $3\|n$. Then $n = 2^6 3\cdot 5^r 13^s p^u$. Since

$$\frac{\sigma^*(2^6 3\cdot 5^2 7\cdot 13)}{2^6 3\cdot 5^2 7\cdot 13} = \frac{5(13)(4)(26)(8)(14)}{2^6(3)(5^2)(7)(13)} = \frac{26}{15} < 2,$$

and

$$\frac{\sigma^*(2^6 3\cdot 5\cdot 7\cdot 13^2)}{2^6 3\cdot 5\cdot 7\cdot 13^2} = \frac{5(13)(4)(6)(8)(170)}{2^6(3)(5)(7)(13^2)} = \frac{170}{91} < 2$$

by Lemma 4.3, $r = s = 1$. Then $n = 2^6 3\cdot 5\cdot 13 p^u$. Then if n is unitary perfect

$$2n = 2^7 3\cdot 5\cdot 13 p^u = 5(13)(4)(6)(14)(p^u + 1) = \sigma^*(n)$$

or

$$8p^u = 7(p^u + 1)$$

which gives $p^u = 7$. Then $n = 2^6 3\cdot 5\cdot 7\cdot 13 = 87{,}360$.

(8) $k = 4$. Let n be unitary perfect. Suppose

$$2 = \frac{\sigma^*(n)}{n} \le \frac{\sigma^*(2^t 5\cdot 7\cdot 11\cdot 13)}{2^t 5\cdot 7\cdot 11\cdot 13} = \frac{(2^t + 1)(6)(8)(12)(14)}{2^t(5)(7)(11)(13)}.$$

Then

$$5005(2^t) \leq 4032(2^t + 1)$$

or

$$973(2^t) \leq 4032,$$

which implies that $t = 1$ or 2 which leads to a contradiction. Therefore, $3|n$. Suppose

$$2 \leq \frac{\sigma^*(2^t 3^2 5\cdot 7\cdot 11)}{2^t 3^2 5\cdot 7\cdot 11} = \frac{(2^t + 1)(10)(6)(8)(12)}{2^2(3^2)(5)(7)(11)}.$$

Then

$$77(2^t) \leq 64(2^t + 1)$$

or

$$13(2^t) \leq 64,$$

which implies that $t = 1$ or 2 which is a contradiction. Therefore, $3\|n$. Suppose

$$2 \leq \frac{\sigma^*(2^t 3\cdot 7\cdot 11\cdot 13)}{2^t 3\cdot 7\cdot 11\cdot 13} = \frac{(2^t + 1)(4)(8)(12)(14)}{2^t(3)(7)(11)(13)}.$$

Then

$$1001(2^t) \leq 896(2^t + 1)$$

or

$$105(2^t) \leq 896,$$

which implies that $t = 1, 2,$ or 3 which is a contradiction. Thus, $5|n$. Suppose

$$2 \leq \frac{\sigma^*(2^t 3\cdot 5^2 7\cdot 11)}{2^t 3\cdot 5^2 7\cdot 11} = \frac{(2^t + 1)(4)(26)(8)(12)}{2^t(3)(5^2)(7)(11)}$$

Then

$$1925(2^t) \leq 1664(2^t + 1)$$

or

$$261(2^t) \leq 1664,$$

which implies that $t = 1$ or 2 which is a contradiction. Thus, $5 \| n$. Suppose

$$2 \leq \frac{\sigma^*(2^t 3 \cdot 5 \cdot 11 \cdot 13)}{2^t 3 \cdot 5 \cdot 11 \cdot 13} = \frac{(2^t + 1)(4)(6)(12)(14)}{2^t(3)(5)(11)(13)}.$$

Then

$$715(2^t) \leq 672(2^t + 1)$$

or

$$43(2^t) \leq 672,$$

which implies that $t = 1, 2,$ or 3 which is a contradiction. Then $7|n$. Suppose

$$2 \leq \frac{\sigma^*(2^t 3 \cdot 5 \cdot 7^2 11)}{2^t 3 \cdot 5 \cdot 7^2 11} = \frac{(2^t + 1)(4)(6)(50)(12)}{2^t(3)(5)(7^2)(11)}.$$

Then

$$539(2^t) \leq 480(2^t + 1)$$

or

$$59(2^t) \leq 480,$$

which implies that $t = 1, 2,$ or 3 which is a contradiction. Therefore, $7 \| n$. Thus, $n = 2^t 3 \cdot 5 \cdot 7 p^r$. Suppose $p \geq 17$. Then

$$2 \le \frac{\sigma^*(2^t 3\cdot 5\cdot 7\cdot 17)}{2^t 3\cdot 5\cdot 7\cdot 17} = \frac{(2^t + 1)(4)(6)(8)(18)}{2^t(3)(5)(7)(17)}$$

which implies that

$$595(2^t) \le 570(2^t + 1)$$

or

$$19(2^t) \le 576.$$

Then $t = 1, 2, 3,$ or 4 which is a contradiction. Therefore, $p = 11$ or 13. Since

$$\sigma^*(3\cdot 5\cdot 7p^r) = 4(6)(8)(p^r + 1) = 2^6(3)(p^r + 1),$$

$t \ge 6$. Then

$$2 \le \frac{\sigma^*(2^6 3\cdot 5\cdot 7p^r)}{2^6 3\cdot 5\cdot 7p^r} = \frac{65(4)(6)(8)(p^r + 1)}{2^6(3)(5)(7)p^r}$$

which implies that

$$14p^r \le 13(p^r + 1)$$

or

$$p^r \le 13.$$

Thus, $r = 1$ for $p = 11$ or 13. Suppose $p = 11$, then

$$2 = \frac{\sigma^*(2^t 3\cdot 5\cdot 7\cdot 11)}{2^t 3\cdot 5\cdot 7\cdot 11} = \frac{(2^t + 1)(4)(6)(8)(12)}{2^t(3)(5)(7)(11)}$$

which implies that

$$385(2^t) = 384(2^t + 1)$$

or

$$2^t = 384$$

which is a contradiction. Thus, p can only be 13. If $p = 13$,

$$2 = \frac{\sigma^*(2^t 3\cdot5\cdot7\cdot13)}{2^t 3\cdot5\cdot7\cdot13} = \frac{(2^t + 1)(4)(6)(8)(14)}{2^t(3)(5)(7)(13)}$$

which implies that

$$65(2^t) = 64(2^t + 1)$$

or

$$2^t = 64$$

which implies that $t = 6$. Then $n = 2^6 3\cdot5\cdot7\cdot13 = 87{,}360$.

Subbarao [29] has stated that he has proven the following theorem with "extensive and exhausting calculations using a desk calculator."

Theorem 4.7. If $n = 2^t m$ is a unitary perfect number with the same notation as in Theorem 4.6,

(1) it is not possible for $t = 8, 9,$ or $10,$ and

(2) it is not possible for $k = 6$.

These theorems can be used to show that after 87,360 there exist no unitary perfect number with less than 20 digits. Charles R. Wall of the University of Tennessee has discovered one with 24 digits [29].

It is

$$2^{18}\cdot 3\cdot 5^{4}\cdot 7\cdot 11\cdot 13\cdot 19\cdot 37\cdot 79\cdot 109\cdot 157\cdot 313.$$

Subbarao conjectures that there is only a finite number of unitary perfect numbers [29].

CHAPTER V

SUMMARY

The study of perfect numbers has fascinated mathematicians for centuries. Perhaps this collection of known facts about perfect numbers can aid others in working in this interesting area of mathematics.

The theory of even perfect numbers seems well established, and the form is well known (See Theorem 2.3, page 8 and Theorem 2.4, page 9). Other even perfect numbers can and, undoubtably, will be found by finding new Mersenne primes. This will need to be done by the use of computers. It will take considerable time, even with computers, to check Mersenne numbers until a prime is found.

It still is not known whether or not there exists an infinite number of even perfect numbers. This fact depends, of course, upon whether or not there are an infinite number of Mersenne primes. Perhaps some day someone will be able to prove that there are either an infinite number or a finite number of Mersenne primes.

The situation with odd perfect numbers is much different. The existence of odd perfect numbers is still an open question. Many mathematicians are still working on this problem today. With all the restrictions that have been proven, it looks doubtful that there do exist any odd perfect numbers.

As was pointed out in Chapter III, authors do not agree on what has been proven, especially about the number of distinct prime factors that an odd perfect number, if it exists, must have. Perhaps it would be worthwhile for someone to research the original works of some, such as J. J. Sylvester, to determine what has been proven.

The basic form of an odd perfect number, if it exists, is well known (See Theorem 3.1, page 27). More restrictions on this form can be made. However, it appears that unless other techniques are developed, such proofs will be quite lengthy. Perhaps better bounds on the prime divisors or the sum of the reciprocals of the prime divisors is a better area for investigation.

The study of unitary perfect numbers, since it is a much newer topic, presents a topic for much more investigation. However, it appears that to continue the search for unitary perfect numbers would involve quite lengthy proofs unless other techniques are developed. The procedures that have been used involve considerable numerical calculations.

Subaarao's conjecture that there is only a finite number of unitary perfect numbers is interesting. This presents a challenge for someone to prove or disprove. If it could be shown that there are only a finite number, it would then become an interesting problem to discover all of them. If there are an infinite number of unitary perfect numbers, perhaps more about them can be studied. Something analogous to what has been done with perfect numbers could be done.

There are still many questions that remain unanswered. Is 3 always a factor of a unitary perfect number? Except for 6, all of the known unitary perfect numbers contain the factor of 5. Do all of the unitary perfect numbers greater than 6 contain 5 as a factor?

There remain many areas of investigation in the study of unitary perfect numbers.

It is hoped that the work done in this dissertation will be helpful to someone desiring to investigate further the subject of perfect or unitary perfect numbers.

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