

ON THE EXISTENCE AND SMOOTHNESS PROBLEM OF THE  
MAGNETOHYDRODYNAMICS SYSTEM

By

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Fluid mechanics plays a pivotal role in engineering application to daily lives. The prominently famous fluid dynamics partial differential equations (PDE) due to its remarkable utility is the Navier-Stokes equations of which its mathematical and physical significance is so highly regarded that it has become one of the seven Millennium Prize problems declared by the Clay Research Institute. We study closely related systems of partial differential equations with focus on the magnetohydrodynamics system, of which its special case is the Navier-Stokes equations. Other systems of PDEs of our concern include the surface quasi-geostrophic equations, incompressible porous media equation governed by Darcy's law, Boussinesq system, Leray, Lans-alpha models, micropolar and magneto-micropolar fluid models. We discuss the properties of solutions to these systems such as the global regularity issue with fractional Laplacians, logarithmic supercriticality, component reduction results of regularity criteria.

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## CHAPTER 1

### INTRODUCTION

Fluid mechanics plays a pivotal role in engineering application to daily lives. The prominently famous fluid dynamics partial differential equations (PDE) due to its remarkable utility is the Navier-Stokes equations (NSE) (cf. [MB01]):

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla\pi = \nu\Delta u + f, \quad (1.1a)$$

$$\nabla \cdot u = 0, \quad u(x, 0) = u_0(x), \quad (1.1b)$$

where  $u : \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}^N$  represents the velocity vector field,  $\pi : \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}$  the pressure scalar field,  $f : \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}^N$  the external force (e.g. gravity) with  $N \geq 2, N \in \mathbb{N}$  and  $\nu \geq 0$  the viscosity of fluid. We note that

$$(u \cdot \nabla)u = \sum_{i=1}^N (u_i \frac{\partial}{\partial x_i})u, \quad \Delta u = \sum_{i=1}^N \frac{\partial^2 u}{(\partial x_i)^2}.$$

The dot product of a gradient vector and  $u$  represents the incompressibility (volume-preserving) property of fluid. Given  $u_0$ , the system (1.1a)-(1.1b) becomes an initial value problem of a non-linear evolutionary PDE. As important role as the system (1.1a)-(1.1b) plays in applications such as the design of aircraft and the study of blood flow, it remains unknown whether the solution preserves the smoothness of initial data globally in time. This problem has been so highly regarded that it has become one of the seven Millennium Prize problems declared by the Clay Research Institute (cf. [CJW06]).

My research interest concerns precisely the global regularity issue of the NSE

(1.1a)-(1.1b). Over decades since the study of the NSE problem was initiated by Professor Leray (cf. [L33]), various unique and creative ideas from countless number of great mathematicians flourished and their results inspired subsequent generations to come. My approach in research is to extend and improve previous results on not only the NSE but also its *related* PDE making careful observation on structure of each PDE.

In Chapter 2, we discuss the general reason why the existence and smoothness of the NSE has remained an outstanding open problem in mathematical analysis for more than 80 years. In particular, we discuss the bounded quantities and notions of criticality. We also introduce the system of our main concern, the magnetohydrodynamics (MHD) system of which its special case is the NSE.

In Chapter 3, we discuss a new research direction of the global regularity in the logarithmically supercritical case, initiated by Professor Tao in [T07]. Subsequently, we present our result on the logarithmically supercritical MHD system with zero diffusivity which was the end point case omitted in the work by Professor Wu in [W11] (cf. [Y14c]).

In Chapter 4, we discuss the global regularity issue of the two-dimensional generalized MHD system with fractional Laplacians. Subsequently, we elaborate on why the two-dimensional case is special and present our results from [Y14a] and [Y14e].

In Chapter 5, we discuss the component reduction results of the Serrin-type regularity criteria of the three-dimensional NSE and its related systems. Subsequently, we present our results on the three-dimensional MHD system from [Y14h] and [Y14i].

Thereafter, we conclude making comments on some open problems and work in progress toward these problems.

The style of this writing is rather a summary of the author's research work in the last four years. Mathematically we focus on the survey of previous results, motivation and difficulty and only sketch of the proofs of the actual author's new results. Most

of the main results in this thesis obtained by the author have already appeared in refereed journals: [Y14a], [Y14c], [Y14e], [Y14h], [Y14i].

## CHAPTER 2

### Navier-Stokes equations, bounded quantities and criticality

#### 2.1 Generalized Navier-Stokes equations

For the purpose of this chapter's discussion, we generalize the classical NSE (1.1a)-(1.1b) by replacing the dissipative term  $-\Delta u$  by a fractional Laplacian defined by

$$\widehat{\Lambda^{2\alpha}} f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi) \quad \forall \alpha \geq 0.$$

That is, we consider the generalized Navier-Stokes equations (g-NSE):

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi + \nu \Lambda^{2\alpha} u = f, \quad (2.1a)$$

$$\nabla \cdot u = \sum_{i=1}^N \frac{\partial u_i}{\partial x_i} = 0, \quad u(x, 0) = u_0(x). \quad (2.1b)$$

We remark that at  $\alpha = 1$ , the g-NSE reduces to the classical NSE (1.1a)-(1.1b). We also mention now that in case  $\nu = 0$ , the classical NSE is reduced furthermore to the Euler equations.

The integral formulation of the fractional Laplacian for a restricted range of  $\alpha \in (0, 1)$  has been studied intensively (e.g. [CC04] and [J05]). Hereafter let us write for brevity

$$\partial_t := \frac{\partial}{\partial t}, \quad \partial_i := \frac{\partial}{\partial x_i}.$$

Let us also denote a constant that depends on  $a, b$  by  $c(a, b)$  and  $A \lesssim B$  and  $A \approx B$  when there exists a constant  $c \geq 0$  of no significance such that  $A \leq cB$  and  $A = cB$

respectively.

We now recall the statement of the Navier-Stokes Existence and Smoothness Problem (cf. [CJW06]). We fix  $N = 3$  and assume that divergence-free  $u_0$  and  $f$  are both smooth and bounded; i.e.  $u_0, f \in C^\infty(\mathbb{R}^3)$ ,  $\nabla \cdot u_0 = 0$ ,  $\forall$  multi-index  $\gamma_1$  and a natural number  $\gamma_2$ ,  $\forall K > 0$ ,  $\exists C = C(\gamma_1, \gamma_2, K) > 0$  such that

1.

$$|\partial_x^{\gamma_1} u_0(x)| \leq \frac{C}{(1 + |x|)^K} \quad \forall x \in \mathbb{R}^3,$$

2.

$$|\partial_x^{\gamma_1} \partial_t^{\gamma_2} f(x, t)| \leq \frac{C}{(1 + |x| + t)^K} \quad \forall (x, t) \in \mathbb{R}^3 \times [0, \infty).$$

The problem is to show that

1. with  $f \equiv 0$ , for any  $u_0$  satisfying the condition above,  $\exists$  a solution to the NSE (1.1a)-(1.1b),  $u \in C^\infty(\mathbb{R}^3 \times [0, \infty))^3$  such that its kinetic energy  $\|u(t)\|_{L^2}^2$  is uniformly bounded  $\forall t \geq 0$ ,
2. or  $\exists u_0(x)$  and  $f(x, t)$  satisfying the conditions above and the smoothness breaks down in finite time.

Because all the discussion in this thesis will be toward the global regularity result rather than a blow-up, hereafter we assume  $f \equiv 0$  in both (1.1a) and (2.1a).

Despite much effort by great mathematicians in generations since the work of Professor Leray in [L33], the problem remains open. Yet, this problem continues to captivate the hearts of young aspiring mathematicians. In the author's opinion, it is due to so many beautiful results from creative ideas that continue to indicate that somehow the problem is "almost" solved. We briefly list some of these results in mind, most of which were obtained by Professor Leray in [L33] and we refer to [MB01] for a collective study on these directions of research:

1. The global existence of a weak solution in case  $N = 3$  has been shown, although its uniqueness remains open.
2. The local existence of the unique strong solution in case  $N = 3$  is also well-known but whether it may be extended globally in time remains open.
3. The global existence of the unique strong solution with  $N = 3$  is shown if the initial data is sufficiently small.
4. The global existence of the unique strong solution if  $N = 2$  is shown. In fact, even for the two-dimensional Euler equations, the global existence of the unique strong solution has been known (cf. [K67]).
5. Omitting  $\nabla\pi$  term, the NSE becomes the Burger's equation and the global existence of the unique strong solution in any dimension  $N \geq 2$  is known (e.g. [KNS08] in case  $N = 1$  and the domain is a torus).

## 2.2 Bounded quantities and criticality

The solution to the NSE, similarly to many other PDEs based on physics, possess conserved quantities. In this section we elaborate on a few and discuss their implications.

We take  $L^2$ -inner products of the g-NSE (2.1a) with  $u$  to obtain

$$\int \partial_t u \cdot u + \int (u \cdot \nabla) u \cdot u + \int \nabla \pi \cdot u + \nu \int \Lambda^{2\alpha} u \cdot u = 0.$$

We observe that the second and third terms immediately vanish after integration by parts due to the incompressibility condition (2.1b):

$$\int \sum_{i,j=1}^N u_i (\partial_i u_j) u_j = \int \sum_{i,j=1}^N \frac{1}{2} u_i \partial_i (u_j)^2 = - \int \sum_{j=1}^N \frac{1}{2} (\nabla \cdot u) (u_j)^2 = 0,$$

$$\int \nabla \pi \cdot u = - \int \pi (\nabla \cdot u) = 0.$$

Therefore, we obtain an identity of

$$\frac{1}{2}\partial_t\|u\|_{L^2}^2 + \nu\|\Lambda^\alpha u\|_{L^2}^2 = 0.$$

Integrating in time leads to

$$\sup_{t \in [0, T]} \|u(t)\|_{L^2}^2 + 2\nu \int_0^T \|\Lambda^\alpha u\|_{L^2}^2 dt \leq c(u_0). \quad (2.2)$$

This implies the conservation of kinetic energy  $\sup_{t \in [0, T]} \|u(t)\|_{L^2}^2$  and cumulative kinetic energy dissipation  $\nu \int_0^T \|\Lambda^\alpha u\|_{L^2}^2 dt$ . We note that more conserved quantities have been found (cf. [MB01]); however, these two quantities have proven to be most readily useful in utilizing for the global regularity proofs.

Now a basic approach of energy method to try to prove the validity of the first possibility of the Millennium Prize Problem in  $N$  dimension would be to try to attain a bound of the form

$$\partial_t \|u(\cdot, t)\|_{H^s}^2 \leq A(t)B(t)\|u(\cdot, t)\|_{H^s}^2 \quad \forall t \in [0, T],$$

where

$$\int_0^T A(s)ds < \infty, \quad \sup_{t \in [0, T]} B(t) \leq c,$$

so that by Gronwall's inequality, this leads to

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^s}^2 \leq \|u_0(\cdot)\|_{H^s}^2 e^{\sup_{s \in [0, t]} B(s) \int_0^T A(s)ds} < \infty.$$

In case of the g-NSE, from (2.2) we see that the appropriate candidates for  $A$  and  $B$  are surely

$$A(s) = \|\Lambda^\alpha u(\cdot, s)\|_{L^2}^2, \quad B(t) = \|u(\cdot, t)\|_{L^2}^2.$$

We shall give a heuristic argument as to why the Existence and Smoothness problem of the Navier-Stokes equations remains difficult. Again, similarly to many other systems of PDE that arise from physics, the solution to the NSE and the g-NSE obey the scaling invariance; namely, if  $u : \mathbb{R}^N \times [0, T) \mapsto \mathbb{R}^N$  solves the g-NSE (2.1a)-(2.1b), then so does

$$u_\lambda : \mathbb{R}^N \times [0, \frac{T}{\lambda^{2\alpha}}) \mapsto \mathbb{R}^N, \quad u_\lambda(x, t) := \lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t).$$

This implies that upon scaling, the  $L^2$ -norm, on which we wish to rely on its uniform boundedness as we discussed, has the following issue:

$$\|u_\lambda(x, t)\|_{L^2}^2 = \int_{\mathbb{R}^N} |\lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t)|^2 dx = \lambda^{4\alpha-2-N} \|u(\cdot, \lambda^{2\alpha} t)\|_{L^2}^2.$$

Based on this observation, it is convention that we classify the g-NSE with various powers of  $\alpha$  as follows:

1. the g-NSE is energy-subcritical when  $\alpha > \frac{1}{2} + \frac{N}{4}$ ,
2. the g-NSE is energy-critical when  $\alpha = \frac{1}{2} + \frac{N}{4}$ ,
3. the g-NSE is energy-supercritical when  $\alpha < \frac{1}{2} + \frac{N}{4}$ .

Notice for  $N = 2$ , the g-NSE (2.1a)-(2.1b) is energy critical while for  $N = 3$ , g-NSE requires  $\alpha = \frac{5}{4}$  to be critical. The Millennium Prize Problem is set for the classical NSE (1.1a)-(1.1b) which is (2.1a)-(2.1b) at  $\alpha = 1, N = 3$  and hence in the “energy-supercritical” regime. The main point to which we come back hereafter repeatedly is that the higher power the fractional Laplacian of the dissipative (or diffusive) term, the easier the global regularity result becomes to show. We mention that on the other hand, making the non-linear term somehow *mild* also achieves the same goal (cf. [CIW08], [MX12], [K10] for the modified surface quasi-geostrophic equations and [Y11a] for the modified porous media equations).



### 2.3 Magnetohydrodynamics system

We now introduce the system of PDE which is very closely related to the NSE and is of our main concern, namely the generalized magnetohydrodynamics (g-MHD) system:

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi + \nu \Lambda^{2\alpha} u = (b \cdot \nabla)b, \quad (2.3a)$$

$$\partial_t b + (u \cdot \nabla)b - (b \cdot \nabla)u + \eta \Lambda^{2\beta} b = 0, \quad (2.3b)$$

$$u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x), \quad \nabla \cdot u = \nabla \cdot b = 0, \quad (2.3c)$$

where  $b : \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}^N$  is the magnetic field,  $\eta \geq 0$  the diffusivity constant and  $\beta \geq 0$ . The classical MHD system which is (2.3a)-(2.3c) at  $\alpha = \beta = 1$  describes the motion of electrically conducting fluids and plays a fundamental role in astrophysics, geophysics and plasma physics (cf. [ST83]). We observe that this system at zero magnetic field ( $b \equiv 0$ ) reduces to the g-NSE (2.1a)-(2.1b). The work by Serge and Temam in [ST83] shows that the analogous results to the g-NSE with  $u$  replaced by  $(u, b)$  can be obtained, e.g. the global existence of a weak solution pair  $(u, b)$  and the local existence of the unique strong solution pair  $(u, b)$ . However, the global regularity issue in three-dimensional case in particular remains open.

For this system of equations, following our discussion on the rescaling issue, we understand the following result by Professor Wu:

**Theorem 2.1** ([W03]) *Suppose  $N \geq 2$  and  $\alpha \geq \frac{1}{2} + \frac{N}{4}, \beta \geq \frac{1}{2} + \frac{N}{4}$ . Given  $(u_0, b_0) \in H^s(\mathbb{R}^N), N \geq 2, s > \max\{1 + 2\alpha, 1 + 2\beta\}, \exists (u, b)$  that solves the g-MHD system (2.3a)-(2.3c) and  $(u, b) \in H^s(\mathbb{R}^N)$  globally in time.*

In short, the global regularity result for the generalized MHD-system (2.3a)-(2.3c) was obtained above because at  $\alpha = \beta$ , the g-MDH system regains its rescaling property and  $\alpha, \beta \geq \frac{1}{2} + \frac{N}{4}$ , the system is in the sub-critical and critical regimes. Hence the dissipation and diffusion strength suffice to suppress the non-linear terms and close the a priori estimates, leading to the global regularity result.

## CHAPTER 3

### Logarithmic supercriticality

#### 3.1 Logarithmically supercritical PDEs

Although improving the lower bound on the power of the fractional Laplacian in the dissipative term of g-NSE (2.1a)-(2.1b) seems extremely difficult, Professor Tao introduced the notion of “logarithmic supercriticality” on the wave equations and the NSE (cf. [T07] and [T09]). The intuitive idea behind this phenomenon is as follows: as we discussed in Chapter 2, the inequality of

$$\partial_t \|u(\cdot, t)\|_{H^s}^2 \leq A(t)B(t)\|u(\cdot, t)\|_{H^s}^2$$

seems to be the threshold in a way that if the term  $\|u(\cdot, t)\|_{H^s}^2$  on the right hand side had any power bigger than 2, then we will not be able to show the finiteness as desired, namely

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 \exp \left( \sup_{t \in [0, T]} B(t) \int_0^T A(\tau) d\tau \right).$$

However, one can indeed obtain a “slightly worse” inequality and still show the finiteness of  $\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^s}^2$ , namely

$$\partial_t \|u(\cdot, t)\|_{H^s}^2 \leq A(t)B(t)\|u(\cdot, t)\|_{H^s}^2 \ln(e + \|u(\cdot, t)\|_{H^s}^2),$$

which by a separation of variables implies

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^s}^2 \lesssim \|u_0\|_{H^s}^2 \exp \left( \exp \left( \sup_{t \in [0, T]} B(t) \int_0^T A(\tau) d\tau \right) \right).$$

This is the heuristic idea behind the recent phenomenon of the global regularity result in the logarithmically supercritical regime, of which its proof naturally has the flavor of “Brezis-Wainger type inequality” argument (cf. [BW80]). This topic of breaking the threshold of criticality logarithmically has caught so much attention that by now there is an overwhelming amount of examples. We just wish to mention several prominent work:

1. Euler equations in [CCW11],
2. Boussinesq system in [H11],
3. Wave equations in [R09].

### 3.2 Logarithmically supercritical MHD system with zero diffusivity

Motivated by such results, Professor Wu improved his own result from those in [W03]: consider

$$\partial_t u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \pi + \nu \mathcal{L}_1^2 u = 0, \quad (3.1a)$$

$$\partial_t b + (u \cdot \nabla)b - (b \cdot \nabla)u + \eta \mathcal{L}_2^2 b = 0, \quad (3.1b)$$

where we denoted the operators  $\mathcal{L}_1, \mathcal{L}_2$  defined as a Fourier operator with a specific Fourier symbol e.g.

$$\widehat{\mathcal{L}_1 f}(\xi) = m_1(\xi) \hat{f}(\xi), \quad m_1(\xi) \geq \frac{|\xi|^\alpha}{g_1(\xi)}, \quad (3.2)$$

for all sufficiently large  $|\xi|$  and  $g_1 : \mathbb{R}^+ \mapsto \mathbb{R}^+, g_1 \geq 1$  is radially symmetric, non-decreasing function. The operator  $\mathcal{L}_2$  is defined analogously in terms of  $m_2, g_2$  and  $\beta$ . The following result was obtained in [W11]:

**Theorem 3.1** (cf. [W11]) *Suppose  $(u_0, b_0) \in H^s(\mathbb{R}^N), s > 1 + \frac{N}{2}$  and*

$$\alpha \geq \frac{1}{2} + \frac{N}{4}, \quad \beta > 0, \quad \alpha + \beta \geq 1 + \frac{N}{2} \text{ and } \int_1^\infty \frac{d\tau}{\tau(g_1^2(\tau) + g_2^2(\tau))^2} = \infty.$$

Then the unique global classical solution pair  $(u, b)$  to the system (3.1a)-(3.1b) exists.

(Cf. also [Y12a] for a general result on the Leray- $\alpha$  and Lans- $\alpha$  models.) For technical reasons, the endpoint case  $\alpha = 1 + \frac{N}{2}, \beta = 0$  was infeasible in Professor Wu's proof. We now elaborate on this issue.

In general, even without logarithmic worsening in the dissipative and diffusive terms, the end point case  $\alpha = 1 + \frac{N}{2}$  and zero diffusivity is slightly more subtle. Consider such a g-MHD system with zero diffusion,  $N \geq 3$ :

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi + \nu \Lambda^{2+N} u = (b \cdot \nabla)b, \quad (3.3a)$$

$$\partial_t b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0, \quad (3.3b)$$

for which the same procedure of taking  $L^2$ -inner products on (3.3a)-(3.3b) with  $(u, b)$  respectively as we did in Chapter 2 Section 2 on the g-NSE leads to the bounds of

$$\sup_{t \in [0, T]} (\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{1+\frac{N}{2}} u(\tau)\|_{L^2}^2 d\tau \lesssim_{u_0, b_0, T} 1.$$

Then, on the  $H^1$ -estimate, which is done by taking  $L^2$ -inner products on (3.3a)-(3.3b) with  $(-\Delta u, -\Delta b)$  respectively, we have

$$\begin{aligned} \partial_t (\|\nabla u\|_{L^2}^2 + \nu \|\nabla b\|_{L^2}^2) + \|\Lambda^{2+\frac{N}{2}} u\|_{L^2}^2 &\lesssim \int |\nabla u|^3 + |\nabla u| |\nabla b|^2 \\ &\lesssim \text{first term} + \|\nabla u\|_{L^\infty} \|\nabla b\|_{L^2}^2 \end{aligned}$$

where one faces the difficulty that  $H^{\frac{N}{2}}(\mathbb{R}^N)$  is not embedded in  $L^\infty(\mathbb{R}^N)$ . Fortunately the Brezis-Wainger type inequality (cf. [BW80]) can close the estimate in this case by possibly coupling with a higher-order estimate (cf. similar result on the Boussinesq system in [Y14g]); however, in case of the logarithmically supercritical case, same estimate will not work.

We now briefly discuss the proof in [W11] to clarify why the endpoint case requires a different treatment. We refer to the Appendix for a review on Besov space

techniques.

Applying the Leray projection  $\mathbb{P}$ , which is a projection onto the divergence-free vector fields, on (3.1a) and then  $\Delta_j$  the Littlewood-Paley operator on (3.1a)-(3.1b) leads to

$$\begin{cases} \partial_t \Delta_j u + \nu \mathcal{L}_1^2 \Delta_j u = -\mathbb{P} \Delta_j ((u \cdot \nabla) u) + \mathbb{P} \Delta_j ((b \cdot \nabla) b), \\ \partial_t \Delta_j b + \eta \mathcal{L}_2^2 \Delta_j b = -\Delta_j ((u \cdot \nabla) b) + \Delta_j ((b \cdot \nabla) u). \end{cases}$$

Taking  $L^2$ -inner products with  $(\Delta_j u, \Delta_j b)$  lead to

$$\begin{cases} \partial_t \|\Delta_j u\|_{L^2}^2 + 2\nu \|\mathcal{L}_1 \Delta_j u\|_{L^2}^2 = -2 \int \Delta_j u \cdot \Delta_j ((u \cdot \nabla) u) + 2 \int \Delta_j u \cdot \Delta_j ((b \cdot \nabla) b), \\ \partial_t \|\Delta_j b\|_{L^2}^2 + 2\eta \|\mathcal{L}_2 \Delta_j b\|_{L^2}^2 = -2 \int \Delta_j b \cdot \Delta_j ((u \cdot \nabla) b) + 2 \int \Delta_j b \cdot \Delta_j ((b \cdot \nabla) u). \end{cases}$$

Multiplying by  $2^{2sj}$  and summing over all  $j \geq -1$  leads to

$$\begin{aligned} & \partial_t (\|u\|_{H^s}^2 + \|b\|_{H^s}^2) + 2\nu \sum_{j \geq -1} 2^{2sj} \|\mathcal{L}_1 \Delta_j u\|_{L^2}^2 + 2\eta \sum_{j \geq -1} 2^{2sj} \|\mathcal{L}_2 \Delta_j b\|_{L^2}^2 \\ &= 3 \text{ nonlinear terms} - 2 \sum_j 2^{2sj} \int \Delta_j b \cdot \Delta_j ((u \cdot \nabla) b). \end{aligned}$$

To handle the product terms in Besov space, Professor Wu used Bony paraproduct (cf. [C98]) and Besov commutator estimates. The idea of the proof is same as that of the logarithmically supercritical NSE in [T09]. We must split between the high and low frequency modes, for example by some  $M_1 > 0$ . However, to handle the logarithmic function  $g_i, i = 1, 2$  which is non-decreasing, at one point, it was crucial in the work of [W11] to obtain the following estimate

$$\begin{aligned} & \sum_{j > M_1} g_2^2(2^j) 2^{2sj} \|\Delta_j b\|_{L^2}^2 2^{-2\beta j} \left( \sum_{m \leq j-2} 2^{m(1+\frac{N}{2})} \|\Delta_m u\|_{L^2} \right)^2 \\ & \lesssim g_2^2(2^{M_1}) 2^{-2M_1(\beta-\delta)} \|b\|_{H^s}^2 \|u\|_{H^s}^2, \end{aligned}$$

where he chose  $\delta \in (0, \beta)$  so that

$$g_2^2(2^j) 2^{-2j(\beta-\delta)} \leq g_2^2(2^{M_1}) 2^{-2M_1(\beta-\delta)}$$

for  $j \geq M_1$  (provided  $M_1$  is sufficiently large) because  $g_2$  grows logarithmically. Hence, it is clear that  $\beta > 0$  indeed was important.

Therefore, it seems that perhaps the case  $\alpha = 1 + \frac{N}{2}$  with zero diffusivity requires a new type of proof. We mention that the conjecture based on numerical analysis results (e.g. [H85], [PPS95]) was that it should be possible to show the global regularity result in this case because the velocity vector field seems to play a more dominant role than the magnetic vector field in preserving their regularity. We now present our result:

**Theorem 3.2** (cf. [Y14c]) *Let  $\nu > 0, \eta = 0, \alpha \geq 1 + \frac{N}{2}$  and  $g_1$  satisfy*

$$\int_e^\infty \frac{d\tau}{g_1^2(\tau) \ln(\tau)\tau} = \infty. \quad (3.4)$$

*Then for all initial data pair  $(u_0, b_0) \in H^s, s \geq 3 + N$ , there exists a unique global classical solution pair  $(u, b)$  to (3.1a)-(3.1b).*

**Remark 3.1** 1. *There are various ways to obtain different initial regularity conditions. Indeed, improving the initial regularity is an important direction of research of much interest (cf. [Y11b], [Y14d]). We chose the statement above for simplicity of its proof.*

2. *This result completes all cases of global regularity of logarithmically supercritical MHD system with equal or more dependence on the dissipation than diffusion. It also improves the result of [W09] logarithmically. We mention that almost simultaneously and independently the authors in [TYZ13a] obtained a very similar but slightly different result (cf. Remark 1.1 (3) in [Y14c] for the detailed discussion of the similarities and differences between the result in [Y14c] and that of [TYZ13a]).*

3. *We mention that its proof was partially inspired by the work in [CMZ07], [Z10] and [Z05].*

We now give an idea of the proof, just  $H^1$ -estimate leaving interested readers to [Y14c] for details. We assume without loss of generality that  $\nu = 1$  in (3.1a). Taking  $L^2$ -inner products of (3.1a)-(3.1b) with  $u$  and  $b$  respectively, we obtain as before

$$\sup_{t \in [0, T]} (\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \int_0^T \|\mathcal{L}_1 u\|_{L^2}^2 d\tau \lesssim_{u_0, b_0, T} 1. \quad (3.5)$$

For simplicity let us denote by  $X(t) := \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2$ . We apply  $\nabla$  on (3.1a)-(3.1b), take  $L^2$ -inner products with  $\nabla u$  and  $\nabla b$  respectively and sum to obtain

$$\frac{1}{2} \partial_t X(t) + \|\mathcal{L}_1 \nabla u\|_{L^2}^2 \lesssim \int |\nabla u|^3 + |\nabla u| |\nabla b|^2.$$

We apply Hölder's inequalities to obtain

$$\frac{1}{2} \partial_t X(t) + \|\mathcal{L}_1 \nabla u\|_{L^2}^2 \lesssim \|\nabla u\|_{L^\infty} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2. \quad (3.6)$$

By Littlewood-Paley decomposition and Bernstein's inequality, for some  $M_1 > 0$  to be determined subsequently, we obtain

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\leq \sum_{j \geq -1} \|\Delta_j \nabla u\|_{L^\infty} \\ &\lesssim \sum_{2^j \leq M_1} \frac{2^{j(\frac{N}{2})}}{g_1(2^j)} \|\Delta_j \nabla u\|_{L^2} g_1(2^j) + \sum_{2^j > M_1} 2^{-j} \frac{2^{j(\frac{N}{2}+1)}}{g_1(2^j)} \|\Delta_j \nabla u\|_{L^2} g_1(2^j). \end{aligned}$$

Since  $g_1$  is increasing, we obtain

$$\|\nabla u\|_{L^\infty} \lesssim g_1(M_1) \sum_{2^j \leq M_1} \|\Delta_j \mathcal{L}_1 u\|_{L^2} + \sum_{2^j > M_1} 2^{-j} g_1(2^j) \|\Delta_j \mathcal{L}_1 \nabla u\|_{L^2}$$

by (3.2). We further bound using Hölder's inequalities to obtain

$$\|\nabla u\|_{L^\infty} \lesssim g_1(M_1) \sqrt{\ln(M_1)} \|\mathcal{L}_1 u\|_{L^2} + M_1^{-\frac{1}{2}} \|\mathcal{L}_1 \nabla u\|_{L^2}. \quad (3.7)$$

Now we set  $M_1 := e + X(t)$  so that by Young's inequality we have from (3.6) and (3.7)

$$\begin{aligned}
& \|\nabla u\|_{L^\infty} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \\
& \lesssim g_1(e + X(t)) \sqrt{\ln(e + X(t))} \|\mathcal{L}_1 u\|_{L^2} X(t) + \|\mathcal{L}_1 \nabla u\|_{L^2} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2}) \\
& \leq \frac{1}{2} \|\mathcal{L}_1 \nabla u\|_{L^2}^2 + c(g_1(e + X(t)) \sqrt{\ln(e + X(t))} \|\mathcal{L}_1 u\|_{L^2} X(t) + cX(t).
\end{aligned}$$

Absorbing the dissipative term, we have by Young's inequalities again from (3.6)

$$\partial_t X(t) + \|\mathcal{L}_1 \nabla u\|_{L^2}^2 \lesssim (g_1^2(e + X(t)) \ln(e + X(t))) (e + X(t)) (1 + \|\mathcal{L}_1 u\|_{L^2}^2). \quad (3.8)$$

Separation of variables immediately implies that for any  $t \in [0, T]$ , we have

$$\int_{e+X(0)}^{e+X(t)} \frac{d\tau}{g_1^2(\tau) \ln(\tau)\tau} \lesssim \int_0^T 1 + \|\mathcal{L}_1 u\|_{L^2}^2 d\tau \lesssim_{u_0, b_0, T} 1.$$

due to (3.5). By hypothesis (3.4) this implies

$$\sup_{t \in [0, T]} X(t) < \infty. \quad (3.9)$$

It now follows from integrating (3.8) in time and using (3.5) again that

$$\int_0^T \|\mathcal{L}_1 \nabla u\|_{L^2}^2 d\tau \lesssim_{u_0, b_0, g_1, T} 1. \quad (3.10)$$

Thus, we have shown

$$\sup_{t \in [0, T]} (\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2) + \int_0^T \|\mathcal{L}_1 \nabla u\|_{L^2}^2 d\tau \lesssim 1.$$

We omit further details.



## CHAPTER 4

### Two-dimensional generalized magnetohydrodynamics system

#### 4.1 Two-dimensional generalized MHD system

In this chapter we discuss the global regularity issue of the generalized MHD system in dimension two. From our discussion in Chapter 2 Section 2 concerning criticality, in two-dimensional case we should *require* that the g-NSE (2.1a)-(2.1b) has a full Laplacian in order to guarantee a global regularity result; no power less than one should be allowed. However, a remarkable fact in the two-dimensional case is that no Laplacian is necessary at all; i.e. the two-dimensional Euler equations still admits a global regularity result.

In two-dimensional case with zero dissipation, we have

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi = 0.$$

Taking a curl leads to, with  $\Omega := \nabla \times u$ ,

$$\partial_t \Omega + (u \cdot \nabla)\Omega = 0. \tag{4.1}$$

This is a transport-diffusion equation and hence  $\Omega$  in  $L^\infty$ -norm becomes a conserved quantity: in detail, multiplying (4.1) by  $|\Omega|^{p-2}\Omega$ ,  $p \geq 2$  and integrating in space leads to

$$\frac{1}{p} \partial_t \|\Omega\|_{L^p}^p = 0, \tag{4.2}$$

where we used the fact that

$$\int (u \cdot \nabla \Omega) |\Omega|^{p-2} \Omega = \frac{1}{p} \int (u \cdot \nabla) |\Omega|^p = -\frac{1}{p} \int (\nabla \cdot u) |\Omega|^p = 0 \quad (4.3)$$

by integration by parts and the incompressibility condition (1.1b). Integrating in time now leads to

$$\sup_{t \in [0, T]} \|\Omega\|_{L^p} \leq c(u_0) \quad (4.4)$$

with the constant independent of  $p$ . Taking  $p \rightarrow \infty$  leads to

$$\sup_{t \in [0, T]} \|\Omega\|_{L^\infty} \leq c(u_0). \quad (4.5)$$

Although  $\Omega = \nabla \times u$  is not quite  $\nabla u$ , by the standard borderline techniques of Brezis-Wainger type argument (cf. [BW80]) and div-curl lemma ([CLMS93]), their difference can be overcome in some estimates. Hence, instead of the kinetic energy in the form of the  $L^2$ -norm of  $u$ , we have the  $L^\infty$  norm of the vorticity  $\Omega$ , more or less a derivative of  $u$ . Due to this *better* conserved quantity, it is well-known (e.g. [K67]) that the two-dimensional NSE with no dissipation, the Euler equations, allows a global regularity result. It also implies that the two-dimensional g-NSE with  $\nu > 0, \alpha \geq 0$  also admits the global regularity result. We remark that in the three-dimensional case, the right hand side of (4.1) has  $(\Omega \cdot \nabla)u$  and hence the argument above in (4.2)-(4.5) breaks down.

A natural question is whether or not the two-dimensional MHD system also admits such a special result. More precisely, we wish to answer the question of whether the two-dimensional inviscid MHD system, i.e. (2.3a)-(2.3c) with  $\nu = \eta = 0$ , admits a globally unique smooth solution pair  $(u, b)$  as in the two-dimensional Euler equations. This problem remains open. Indeed, a naive attempt at taking a curl on the (2.3a) and following the same  $L^p$ -estimate argument as we did above, one realizes that no better conserved quantity is obtained due to the presence of  $(b \cdot \nabla)b$ . Trying a similar idea on (2.3b) does not lead to any obvious progress either. However, due to the

effort by many mathematicians, we have recently seen much progress in this direction of research. We now give a heuristic reason behind recent developments and survey these results.

We denote by  $j := \nabla \times b$  which represents the current density. Taking a curl on (2.3a), we obtain

$$\partial_t \Omega + \nu \Lambda^{2\alpha} \Omega = -(u \cdot \nabla) \Omega + (b \cdot \nabla) j.$$

This implies that an identical procedure on the  $L^p$ -estimate,  $2 \leq p < \infty$  of  $\Omega$  as in (4.2)-(4.4) leads to

$$\partial_t \|\Omega\|_{L^p}^p + \nu \int \Lambda^{2\alpha} \Omega |\Omega|^{p-2} \Omega \leq \|(b \cdot \nabla) j\|_{L^p} \|\Omega\|_{L^p}^{p-1}$$

by Hölder's inequality. Then we discover an interesting relation, namely a sufficient bound on the magnetic field  $(b, j)$  leads to a certain bound on the velocity field  $(u, \Omega)$ . On the other hand, as we will discuss in the subsequent Chapter 5, it is well-known that a sufficient bound on the velocity field  $(u, \Omega)$  leads to a certain bound on the magnetic field  $(b, j)$ ; this is because for example, upon an  $H^1$ -estimate of  $(u, b)$  which starts by taking inner products of (2.3a) by  $-\Delta u$  and (2.3b) by  $-\Delta b$ , every one of four non-linear terms will have  $u$  but not necessarily  $b$ . The conclusion from this observation is that one can hope to complete some iteration argument, bounding  $u$  and  $b$  separately relying on previously attained bounds at each step, leading to higher regularity.

We emphasize that taking advantage of the vorticity  $\nabla \times u$  formulation rather than the gradient of the velocity  $\nabla u$  is extremely important. In [Y14f], the author obtained a global regularity result for the two-dimensional generalized magnetohydrodynamics- $\alpha$  system in a challenging case using the vorticity formulation.

Based on this observation, in less than one year so many results have appeared. Before we list them below, we firstly remind ourselves that prior to these results,

to the best of the author's knowledge, the global regularity issue of the generalized MHD system (2.3a)-(2.3c) at  $\alpha = 0$  was never investigated; moreover, if  $\beta = 1$ , except the logarithmic improvement in [W11], it was believed that  $\alpha \geq 1$  was necessary to guarantee the smoothness of the solution pair globally in time. Finally we emphasize that improving these lower bounds of the fractional Laplacians is extremely important as the classical three-dimensional NSE is energy-supercritical at  $\alpha = 1$  while for  $\alpha \geq \frac{5}{4}$ , the global regularity result is attainable. Now let us review recent results:

For fixed  $\alpha = 0$ ,

1. Professor Tran, Professor Yu and Professor Zhai in [TYZ13b] showed that  $\beta > 2$  suffices to obtain the global regularity result.
2. Independently from the result of [TYZ13b], the author in [Y14a] showed that in fact  $\beta > \frac{3}{2}$  suffices. Professor Jiu and Professor Zhao in [JZ14a] also obtained this result of  $\beta > \frac{3}{2}$ .
3. Professor Jiu and Professor Zhao in [JZ14b] and Professor Cao, Professor Wu and Professor Yuan in [CWY14] independently showed that  $\beta > 1$  suffices.

On the other hand, for fixed  $\beta = 1$ ,

1. Professor Tran, Professor Yu and Professor Zhai in [TYZ13b] showed that instead of  $\alpha = 1$ ,  $\alpha \geq \frac{1}{2}$  suffices. The author in [Y14a] also obtained this result independently.
2. Professor Yuan and Professor Bai in [YB14] and the author in [Y14e] independently showed that  $\alpha > \frac{1}{3}$  suffices.
3. Professor Ye and Professor Xu in [YX14] showed that  $\alpha \geq \frac{1}{4}$  suffices.

We also wish to mention that there is a pre-print, unpublished nor posted on arXiv, by Professor Fan, Professor Nakamura and Professor Zhou that claims that

in case  $\beta = 1$ ,  $\alpha > 0$  suffices. Moreover, we mention that numerical analysis results such as [TYB13] suggests the global regularity result at  $\alpha = 0, \beta = 1$ .

Because both our results in [Y14a] and [Y14e] are available in refereed journals published already, let us apply this idea and obtain a new result in the next section instead of repeat the proofs from [Y14a] and [Y14e].

## 4.2 Two-dimensional magneto-micropolar fluid system

In this section, we will apply the idea from the recent developments on the two-dimensional generalized MHD system to obtain a global regularity result of the following two-dimensional magneto-micropolar fluid (MMPF) system:

$$\partial_t u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \left( \pi + \frac{1}{2}|b|^2 \right) = (\mu + \chi)\Delta u + \chi(\nabla \times w), \quad (4.6a)$$

$$\partial_t w + (u \cdot \nabla)w = \gamma\Delta w - 2\chi w + \chi(\nabla \times u), \quad (4.6b)$$

$$\partial_t b + (u \cdot \nabla)b - (b \cdot \nabla)u = \nu\Delta b. \quad (4.6c)$$

where we denoted

1.  $w$  the micro-rotational velocity,
2.  $\chi$  the vortex viscosity,
3.  $\mu$  the kinematic viscosity,
4.  $\gamma$  the angular viscosities.

The MMPF system at  $b \equiv 0$  reduces to the micropolar fluid (MPF) system, introduced by Professor Eringen in [E64] and [E66]. The MPF system represents the fluids consisting of bar-like elements, e.g. anisotropic fluids, such as liquid crystals made up of dumbbell molecules and animal blood. The actual MMPF system was considered by the authors in [AS74]. It has been studied by many mathematicians

and engineers; in particular, Professor Galdi and Professor Rionero in [GR77] showed the results analogous to the classical MHD system such as

1. global existence of a weak solution  $(u, w, b)$  in dimensions  $N = 2, 3$ ,
2. global existence of the unique strong solution in case initial data is small if  $N = 3$ ,
3. the unique weak solution if  $N = 2$ .

We aim to obtain the global regularity result with zero angular viscosity; i.e.  $\gamma = 0$ , extending the result of [GR77]. Let us motivate our study. Firstly, similarly to the NSE and the MHD system, for the MMPF system (4.6a)-(4.6c) at  $\gamma = 0$ , we have the following conserved quantities:

$$\sup_{t \in [0, T]} (\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \int_0^T \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 d\tau \lesssim_{u_0, w_0, b_0} 1.$$

Taking a curl on (4.6a), we obtain

$$\partial_t \Omega - (\mu + \chi) \Delta \Omega = -(u \cdot \nabla) \Omega + (b \cdot \nabla) j - \chi \Delta w.$$

Upon an  $L^2$ -estimate of  $\Omega$ , due to the lack of angular viscosity, we have no obvious way to handle  $-\chi \Delta w$ . Natural idea is to couple this estimate with that of  $w$ , but taking  $L^2$ -inner products on (4.6b) with  $-\Delta w$  leads to

$$\frac{1}{2} \partial_t \|\nabla w\|_{L^2}^2 \leq \int \nabla u \cdot \nabla w \cdot \nabla w - \chi \int (\nabla \times u) \Delta w$$

and the first terms is too difficult to handle without angular viscosity.

However, this difficulty has been overcome. We claim the following result:

**Theorem 4.1** ([Y14j]) *Let  $\gamma = 0$ . For every  $(u_0, w_0, b_0) \in H^s(\mathbb{R}^2)$ ,  $s > 2$ , there exists a unique solution  $(u, w, b)$  to (4.6a)-(4.6c) such that*

$$u, b \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2([0, \infty); H^{s+1}(\mathbb{R}^2)), \quad w \in C([0, \infty); H^s(\mathbb{R}^2)).$$

The proof is an appropriate combination of the ideas from [DZ10] and [CWY14]. Let us sketch the outline of the proof.

The key observation by the authors in [DZ10] on the two-dimensional MPF system with zero angular viscosity was that by defining

$$Z := \Omega - \left( \frac{\chi}{\mu + \chi} \right) w, \quad (4.7)$$

we can take advantage of its evolution in time governed by

$$\partial_t Z + (u \cdot \nabla) Z = (\mu + \chi) \Delta Z - c_1 Z + c_2 w + (b \cdot \nabla) j \quad (4.8)$$

where

$$c_1 := \frac{\chi^2}{\mu + \chi} \geq 0, \quad c_2 := \frac{2\chi^2}{\mu + \chi} - \frac{\chi^3}{(\mu + \chi)^2}.$$

This transport-diffusion equation type structure with the forcing term of  $(b \cdot \nabla) j$  allows us to take advantage of the iteration scheme as follows:

1. sufficient bound on  $Z$  implies a certain bounds on  $w$  as (4.6b) is rather simple,
2. sufficient bound on  $Z$  and  $w$  implies the same bound on  $\Omega$  by definition of  $Z$  in (4.7),
3. sufficient bound on  $\Omega$  implies a certain bounds on  $b, j$  because both non-linear terms in (4.6c) has  $u$ ,
4. sufficient bound on  $b, j$  implies a certain bounds on  $Z$  due to (4.8).

Thus, we can try to iterate this procedure to obtain high regularity. Let us state these iterations in more detail; in contrast to the work in [CWY14], the iteration scheme runs only once. Firstly, following a similar argument in [TYZ13a], [Y14a] we can obtain

**Proposition 4.2.1** ([Y14j]) *Suppose  $(u, w, b)$  solves (4.6a)-(4.6c) in time interval  $[0, T]$ . Then*

$$\sup_{t \in [0, T]} (\|Z(t)\|_{L^2}^2 + \|\Omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2) + \int_0^T \|\nabla Z\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 d\tau \lesssim 1.$$

After this, we obtain slightly higher integrability:

**Proposition 4.2.2** ([Y14j]) *Suppose  $(u, w, b)$  solves (4.6a)-(4.6c) in time interval  $[0, T]$ . Then for any  $2 \leq q \leq 4$ ,*

$$\sup_{t \in [0, T]} (\|Z(t)\|_{L^q} + \|\Omega(t)\|_{L^q} + \|j(t)\|_{L^q}) \lesssim 1. \quad (4.9)$$

The following technical commutator estimate in Besov spaces is important part of the proof:

**Proposition 4.2.3** *Let  $s > -1$ ,  $f \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ ,  $\nabla g \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ ,  $\nabla \cdot g = 0$ . Then for any  $k \geq 3$ ,*

$$\|[\Delta_k, g^i \partial_i] f\|_{L^2} \lesssim c_k 2^{-ks} (\|\nabla g\|_{L^\infty} \|f\|_{H^s} + \|\nabla g\|_{H^s} \|f\|_{L^\infty}). \quad (4.10)$$

Moreover, for  $s > 0$  if additionally  $\nabla f \in L^\infty(\mathbb{R}^2)$ , then for any  $k \geq -1$ ,

$$\|[\Delta_k, g^i \partial_i] f\|_{L^2} \lesssim c_k 2^{-ks} (\|\nabla g\|_{L^\infty} \|f\|_{H^s} + \|g\|_{H^s} \|\nabla f\|_{L^\infty}) \quad (4.11)$$

where  $\{c_k\} \in l_{k \geq -1}^2$ .

The first inequality was done in [DZ10] and the second in [Y14j] for completeness after appropriate modifications. Using this commutator estimate, we can show the following blow-up criterion for the solution  $(u, w, b)$  of our concern:

**Proposition 4.2.4** *Suppose  $(u_0, w_0, b_0) \in H^s(\mathbb{R}^2)$ ,  $s > 2$  and its corresponding solution triple  $(u, w, b)$  to (4.6a)-(4.6c) in time interval  $[0, T]$  satisfies*



$$\int_0^T \|\Omega\|_{L^\infty} + \|w\|_{L^\infty}^2 + \|\nabla j\|_{L^2} d\tau \lesssim 1$$

where  $\Omega = \nabla \times u, j = \nabla \times b$ . Then  $(u, w, b) \in H^s(\mathbb{R}^2)$  for all time  $t \in [0, T]$ .

The crucial point here is that we do not need the bound on  $\nabla w$  but only on  $w$ . Using Propositions 4.2.2 and 4.2.3, this blow-up criterion can be shown to be satisfied. We omit further details.

## CHAPTER 5

### Sufficient condition for smoothness

#### 5.1 Serrin-type and Beale-Kato-Majda criteria

In this chapter we focus on the classical NSE and the MHD system and discuss their sufficient condition for smoothness, namely the Serrin-type and the Beale-Kato-Majda (BKM) criteria. Let us recall from Chapter 2 Section 2 that we have for the solution to the classical NSE (1.1a)-(1.1b),

$$\sup_{t \in [0, T]} \|u(t)\|_{L^2}^2 + 2\nu \int_0^T \|\nabla u\|_{L^2}^2 d\tau \leq c(u_0).$$

It is a standard practice to write  $u \in L_T^r L_x^p$  to imply that

$$\int_0^T \|u(\tau)\|_{L^p}^r d\tau < \infty.$$

Then, from above we know already that in the case of the classical NSE,

$$\sup_{t \in [0, T]} \|u(t)\|_{L^2} < \infty, \quad \int_0^T \|u(t)\|_{L^6}^2 d\tau \leq c \int_0^T \|\nabla u(t)\|_{L^2}^2 d\tau < \infty$$

due to the homogeneous Sobolev embedding of  $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ . Hence, by interpolation we see that  $u \in L_T^r L_x^p$  for  $\frac{3}{p} + \frac{2}{r} \geq \frac{3}{2}, \forall p \in [2, 6]$ . With this observation in mind, we recall two important results, namely the Serrin-type regularity criteria and the BKM criterion:

**Theorem 5.1** ([S62]) *If  $u$  is a weak solution to the three-dimensional NSE (1.1a)-(1.1b) in  $[0, T]$  and*

$$u \in L_T^r L_x^p \quad \text{for} \quad \frac{3}{p} + \frac{2}{r} < 1,$$

then  $u$  is unique.

**Theorem 5.2** ([BKM84]) *If  $u$  is a strong solution to the three-dimensional NSE (1.1a)-(1.1b) at  $\nu = 0$  in  $[0, L]$  and  $L < T$  is such that the solution  $u$  satisfies*

$$\int_0^T \|\Omega\|_{L^\infty} d\tau < \infty,$$

then there is no singularity up to  $T$ .

Despite the complication due to the additional three non-linear terms, we have analogous Serrin-type and BKM criteria for the MHD system; e.g.

**Theorem 5.3** ([CKS97]) *If  $(u, b)$  solves the MHD system (2.3a)-(2.3c) and*

$$\int_0^T \|\Omega\|_{L^\infty} + \|j\|_{L^\infty} d\tau < \infty,$$

where  $\Omega := \nabla \times u$  and  $j := \nabla \times b$ , then there is no singularity up to time  $T$ .

## 5.2 Component reduction results

We now discuss component reduction results. The idea of this relatively new direction of research is that because for example in dimension three, writing  $u = (u_1, u_2, u_3)$  and  $\nabla u$  as the three-by-three matrix of nine entries  $(\partial_i u_j)_{i,j=1}^3$ , it is clear that

$$\sum_{i=2}^3 \int_0^T \|u_i\|_{L^p}^r d\tau \lesssim \int_0^T \|u\|_{L^p}^r d\tau,$$

and

$$\int_0^T \|\partial_1 u_3\|_{L^p}^r d\tau \lesssim \int_0^T \|\nabla u\|_{L^p}^r d\tau,$$

where we chose  $\sum_{i=2}^3 u_i$  and  $\partial_1 u_3$  arbitrarily. Thus, component reduction results, if attained with same integrability condition on  $p, r$ , may be seen as a direct improvement of the Serrin-type and BKM criterions.

We now mention some of these results. For the NSE (1.1a)-(1.1b), Professor Chae and Professor Choe in [CC99] showed that the BKM-criteria which imposes conditions on all three components of  $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ , may be reduced to two:

**Theorem 5.4** ([CC99]) *Suppose  $(u, \pi)$  is a weak solution to the three-dimensional NSE (1.1a)-(1.1b) and  $(0, \Omega_2, \Omega_3) \in L_T^r L_x^p$  for  $\frac{3}{p} + \frac{2}{r} \leq 2, 1 < r < \infty, \frac{3}{2} < p < \infty$ , then there is no singularity up to time  $T$ .*

We remark that it remains open whether this BKM-criteria may be reduced to one component of  $\Omega$ , e.g. just  $\Omega_3$ .

For the MHD system, Professor Zhou in [Z05] and Professor He and Professor Xin in [HX05] obtained important results of the regularity criteria that involves only  $u$  eliminating conditions on  $b$  completely:

**Theorem 5.5** ([Z05] and [HX05]) *Suppose  $(u, b)$  solves the three-dimensional MHD system (2.3a)-(2.3b) at  $\alpha = \beta = 1$  in  $[0, T]$  and*

$$\int_0^T \|u\|_{L^p}^r d\tau < \infty, \quad \frac{3}{p} + \frac{2}{r} \leq 1, \quad 3 < p$$

$$\text{or } \int_0^T \|\nabla u\|_{L^p}^r d\tau < \infty, \quad \frac{3}{p} + \frac{2}{r} \leq 2, \quad 3 < p.$$

*Then there is no singularity up to time  $T$ .*

(cf. also [W08])

Thus, for the BKM criterion, according to the results in [CC99], we may drop the condition on the first component  $\Omega_1$  of  $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ . A natural question is whether any component reduction result attainable for the Serrin-type regularity

criteria which, according to the results in [S62] and [B95], it imposes conditions on all of

$$u = (u_1, u_2, u_3) \quad \text{or} \quad \nabla u = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix}.$$

The answer is positive:

**Theorem 5.6** ([KZ06]) *If  $u$  solves the three-dimensional NSE (1.1a)-(1.1b) in  $[0, T]$  and satisfies*

$$\begin{aligned} \nabla u_3 \in L_T^r L_x^p \text{ for } \frac{3}{p} + \frac{2}{r} \leq \frac{11}{6}, \quad \frac{54}{23} \leq p \leq \frac{18}{5} \\ \text{or } u_3 \in L_T^r L_x^p \text{ for } \frac{3}{p} + \frac{2}{r} \leq \frac{5}{8}, \quad \frac{24}{5} \leq p \leq \infty, \end{aligned}$$

*then there is no singularity up to time  $T$ .*

For example, the first part of the theorem implies that although previous results needed the bound on the entire Jacobian matrix of  $\nabla u$ , it turns out that we only need to bound the third row of the matrix.

The main idea of the proof in [KZ06] is that firstly, the authors estimated not  $\|\nabla u\|_{L^2}^2$  of  $u$  that solves the NSE, but  $\|\nabla_h u\|_{L^2}^2$  where  $\nabla_h = (\partial_1, \partial_2, 0)$ . This requires taking  $L^2$ -inner products of (1.1a) with  $-\Delta_h u$  where  $\Delta_h u = (\partial_{11}^2 + \partial_{22}^2)u$  is a horizontal Laplacian. Secondly, through a key lemma, the authors succeeded on separating  $u_3$  as follows:

$$\left| \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta_h u \right| \leq c \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla \nabla_h u| dx.$$

The important fact used to obtain this key inequality was that by incompressibility, e.g.,

$$\partial_1 u_1 + \partial_2 u_2 = -\partial_3 u_3.$$

This inequality implies

$$\begin{aligned} \partial_t \|\nabla_h u\|_{L^2}^2 + \|\nabla \nabla_h u\|_{L^2}^2 &= \int (u \cdot \nabla) u \cdot \Delta_h u \\ &\leq c \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla \nabla_h u| dx. \end{aligned}$$

The heuristic hereafter is to manipulate inequalities to *put all the responsibility of global regularity* to just  $u_3$  alone.

**Remark 5.1** *As we elaborated, the crucial property of the solution that must be used in this type of result is the incompressibility of the solution. However, the incompressibility within the non-linear term even if the solution itself is not even a vector may still lead to the component reduction result. In [Y12b] and [Y13b], the author was able to obtain this type of component reduction results for the regularity criteria of the solution to the surface quasi-geostrophic equations and the incompressible porous media equation governed by Darcy's law, both of which the solution is a scalar.*

Extensions and improvements of such component reduction results by many mathematicians followed (e.g. [KZ07], [PP04], [PP11], [CT08], [FQ13a], [FQ13b]). In particular, Professor Cao and Professor Titi in [CT11] obtained a regularity criteria of either one of

$$\begin{cases} \int_0^T \|\partial_3 u_3\|_{L^p}^r d\tau < \infty, & \frac{3}{p} + \frac{2}{r} \leq \frac{3(p+2)}{4p}, \quad 2 < p, \\ \int_0^T \|\partial_i u_3\|_{L^p}^r d\tau < \infty, & \frac{3}{p} + \frac{2}{r} \leq \frac{p+3}{2p}, \quad 3 < p, \quad i = 1 \text{ or } 2, \end{cases}$$

for the three-dimensional NSE. The natural question is whether we can obtain similar component reduction results for the three-dimensional classical MHD system. Specifically, we wish to know whether or not we can reduce the regularity criteria in terms of  $u$  or  $\nabla u$  by Professor Zhou, Professor He and Professor Xin to just  $u_3$  or  $\partial_i u_j$  for any  $i, j = 1, 2, 3$ , as done in the case of the three-dimensional NSE. This question

remains open; however, due to a lot of effort by many mathematicians, we have seen significant progress recently.

Professor Cao and Professor Wu in [CW10] showed in particular that its regularity criteria can be reduced to  $\partial_3 u$ , not necessary  $\nabla u$ . Its integrability conditions were improved in [JZ12b]. Subsequently, Professor Lin and Professor Du in [LD13] and the author in [Y13a] obtained as slightly more general result. It was shown that its regularity criteria may be depend on any three partial derivatives to any direction of  $u_1, u_2$  and  $u_3$ . The integrability condition in [LD13] and [Y13a] were recently improved in [JZ14].

Let us describe the idea of the proof in [Y13a] (and [LD13]). The key was that the following decomposition is attainable:

$$\begin{aligned} & \int (u \cdot \nabla) u \cdot \Delta_h u - \int (b \cdot \nabla) b \cdot \Delta_h u + \int (u \cdot \nabla) b \cdot \Delta_h b - \int (b \cdot \nabla) u \cdot \Delta_h b \quad (5.1) \\ & \lesssim \int (|u_1| + |u_2| + |u_3|)(|\nabla u| + |\nabla b|)(|\nabla \nabla_h u| + |\nabla \nabla_h b|). \end{aligned}$$

We also mention the following inequality provided in [CT11] which has seen much applications:

**Lemma 5.1** (*[CT11]*) *For any  $f, g, h \in C_c^\infty(\mathbb{R}^3)$  and  $i, j, k$  any distinct choices from  $\{1, 2, 3\}$ ,*

$$\left| \int fgh dx \right| \lesssim \|f\|_{L^2}^{\frac{\gamma-1}{\gamma}} \|\partial_i f\|_{L^{\frac{2}{3-\gamma}}}^{\frac{1}{\gamma}} \|g\|_{L^2}^{\frac{\gamma-2}{\gamma}} \|\partial_j g\|_{L^2}^{\frac{1}{\gamma}} \|\partial_k g\|_{L^2}^{\frac{1}{\gamma}} \|h\|_{L^2}, \quad 2 < \gamma < 3.$$

With this lemma applied to our decomposition (5.1), we could place any partial derivative on  $u_1, u_2, u_3$  and obtain the criteria of our claim.

To the best of the author's knowledge, to this day, a regularity criteria of the classical three-dimensional MHD system, i.e. (2.3a)-(2.3c) at  $\alpha = \beta = 1$ , in terms of any two entries of  $\nabla u$  has not been obtained. However, the results of [LD13] and [Y13a] suggested the following:

**Theorem 5.7** ([LD13], [Y13a]) *Suppose the solution  $(u, b)$  to the three-dimensional MHD system (2.3a)-(2.3c) at  $\alpha = \beta = 1$  in  $[0, T]$  satisfies*

$$\int_0^T \|\partial_2 u_2\|_{L^p}^r + \|\partial_3 u_3\|_{L^p}^r d\tau < \infty, \quad 3 < p < \infty, \quad \frac{3}{p} + \frac{2}{r} \leq \frac{3}{2p} + \frac{1}{2}.$$

*Then there is no singularity up to time  $T$ .*

This was an immediate corollary to our result and the divergence-free property of  $u$ ; i.e.  $\partial_1 u_1 = -\partial_2 u_2 - \partial_3 u_3$ . This result led to an interesting observation, related to the BKM-criteria. Firstly, we note that to the best of the author's knowledge, a regularity criteria of the three-dimensional MHD system in terms of two vorticity components has not been shown, despite the fact that it was achieved by Chae and Choe in [CC99]. In [PP11], the following lemma was provided:

**Lemma 5.2** ([PP11]) *Let  $u$  be a divergence-free sufficiently smooth vector field in  $\mathbb{R}^3$ . Then there exists a constant  $C = C(q)$  such that for  $1 < q < \infty, i, j = 1, 2$*

$$\|\partial_i u_j\|_{L^q} \leq C(\|\Omega_3\|_{L^q} + \|\partial_3 u_3\|_{L^q}).$$

Applying this Lemma 5.2 to Theorem 5.7, we obtain the following result:

**Corollary 5.1** ([Y13a]) *Suppose the solution  $(u, b)$  to the three-dimensional MHD system (2.3a)-(2.3c) at  $\alpha = \beta = 1$  in  $[0, T]$  satisfies*

$$\int_0^T \|\Omega_3\|_{L^p}^r + \|\partial_3 u_3\|_{L^p}^r d\tau < \infty, \quad 3 < p < \infty, \quad \frac{3}{p} + \frac{2}{r} \leq \frac{3}{2p} + \frac{1}{2}.$$

*Then there is no singularity up to time  $T$ .*

The results in [LD13] and [Y13a] were based on several observations. However, a new result of deeper significance required a completely new decomposition of the four non-linear terms as Professor Kukavica and Professor Ziane did in [KZ06] for the three-dimensional NSE. Unfortunately, the difference between the NSE and the MHD system is quite significant. Observe that upon the  $\|\nabla_h u\|_{L^2}$ -estimate of the solution to the NSE (1.1a)-(1.1b), one only must estimate



$$\int (u \cdot \nabla) \Delta_h u = \sum_{i,j=1}^3 \sum_{k=1}^2 \int u_i \partial_i u_j \partial_{kk}^2 u_j,$$

which consists of 18 terms. However, in the case of the MHD system (2.3a)-(2.3b),

$$\int (u \cdot \nabla) u \cdot \Delta_h u - \int (b \cdot \nabla) b \cdot \Delta_h u + \int (u \cdot \nabla) b \cdot \Delta_h b - \int (b \cdot \nabla) u \cdot \Delta_h b$$

consists of 72. Moreover, due to the mixture of  $b$ , the cancellation of terms that used to occur for the NSE case in [KZ06] is no longer valid. However, some progress has been made. Firstly, Professor Jia and Professor Zhou in [JZ12a] obtained a decomposition of

$$\begin{aligned} & \int (u \cdot \nabla) u \cdot \Delta_h u - \int (b \cdot \nabla) b \cdot \Delta_h u + \int (u \cdot \nabla) b \cdot \Delta_h b - \int (b \cdot \nabla) u \cdot \Delta_h b \\ & \lesssim \int (|u_3| + |b|)(|\nabla u| + |\nabla b|)(|\nabla \nabla_h u| + |\nabla \nabla_h b|) \end{aligned}$$

with which they obtained a regularity criteria of

$$\int_0^T \|u_3\|_{L^p}^r + \|b\|_{L^p}^r d\tau < \infty, \quad \frac{3}{p} + \frac{2}{r} \leq \frac{3}{4} + \frac{1}{2p}, \quad \frac{10}{3} < p$$

(cf. [JZ12a]). Subsequently, in [Y14k] the dependence on  $b_3$  of  $b = (b_1, b_2, b_3)$  was completely eliminated only in the endpoint case. Yet, a decomposition of the four non-linear terms of the MHD system such as

$$\begin{aligned} & \int (u \cdot \nabla) u \cdot \Delta_h u - \int (b \cdot \nabla) b \cdot \Delta_h u + \int (u \cdot \nabla) b \cdot \Delta_h b - \int (b \cdot \nabla) u \cdot \Delta_h b \\ & \leq c \int |u_3| (|\nabla u| + |\nabla b|) (|\nabla \nabla_h u| + |\nabla \nabla_h b|) \end{aligned}$$

or even

$$\begin{aligned} & \int (u \cdot \nabla) u \cdot \Delta_h u - \int (b \cdot \nabla) b \cdot \Delta_h u + \int (u \cdot \nabla) b \cdot \Delta_h b - \int (b \cdot \nabla) u \cdot \Delta_h b \quad (5.2) \\ & \leq c \int (|u_2| + |u_3|) (|\nabla u| + |\nabla b|) (|\nabla \nabla_h u| + |\nabla \nabla_h b|) \end{aligned}$$

has never been seen, to the author's best knowledge. Until very recently, there was no result that has even reduced the criteria of the three-dimensional MHD system to two components of  $u$  or any two entries of the Jacobian matrix of  $u$ .

We claim a step forward in this direction via a new approach. In [Y14h] we indeed were able to reduce the criteria of 3-D MHD to any two components of  $u$ :

**Theorem 5.8** ([Y14h]) *Suppose  $(u, b)$  is the solution pair to the three-dimensional MHD system (2.3a)-(2.3c) at  $\alpha = \beta = 1$  in  $[0, T]$  and it satisfies*

$$\int_0^T \|u_2\|_{L^{p_1}}^{r_1} + \|u_3\|_{L^{p_2}}^{r_2} d\tau < \infty, \quad \frac{3}{p_i} + \frac{2}{r_i} \leq \frac{1}{3} + \frac{1}{2p_i}, \quad \frac{15}{2} < p_i, \quad i = 1, 2.$$

*Then there is no singularity up to time  $T$ .*

The proof required two new ideas. Firstly, the author to this day has not found a decomposition of the four non-linear terms in which two velocity components such as  $u_2, u_3$  are separated as in (5.2). In fact, the decomposition that was discovered was the following:

**Proposition 5.2.1** *Suppose  $(u, b)$  is the solution pair to the three-dimensional MHD system (2.3a)-(2.3b) at  $\alpha = \beta = 1$ . Then*

$$\begin{aligned} & \int (u \cdot \nabla) u \cdot \Delta_h u - \int (b \cdot \nabla) b \cdot \Delta_h u + \int (u \cdot \nabla) b \cdot \Delta_h b - \int (b \cdot \nabla) u \cdot \Delta_h b \\ & \lesssim \int (|u_2| + |u_3| + |b_2| + |b_3|)(|\nabla u| + |\nabla b|)(|\nabla \nabla_h u| + |\nabla \nabla_h b|). \end{aligned} \quad (5.3)$$

This proposition alone leads only to a criteria in terms of  $u_2, u_3, b_2$  and  $b_3$  which has been obtained before (e.g. [BGR13], [JL10]). The result in [Y14h] required the following proposition:

**Proposition 5.2.2** (cf. [Y14h], [Y14i]) *Let  $(u, b)$  be the solution pair to the three-dimensional MHD system (2.3a)-(2.3b) with  $\alpha = \beta = 1$  in time interval  $[0, T]$  and  $j \in \{1, 2, 3\}$ . Then for for any  $p \in (2, 6)$ , there exists a constant  $c(p) \geq 0$  such that*

$$\sup_{\tau \in [0, t]} \|b_j(\tau)\|_{L^p}^2 \leq \|b_j(0)\|_{L^p}^2 + c(p) \int_0^t \|\nabla b(\lambda)\|_{L^2}^2 \|u_j(\lambda)\|_{L^{\frac{6p}{6-p}}}^2 d\lambda$$

for any  $t \in [0, T]$ . In case  $p = 6$ , there exists a constant  $c \geq 0$  such that

$$\sup_{\tau \in [0, t]} \|b_j(\tau)\|_{L^6}^2 \leq \|b_j(0)\|_{L^6}^2 + c \int_0^t \|\nabla b(\lambda)\|_{L^2}^2 \|u_j(\lambda)\|_{L^\infty}^2 d\tau$$

for any  $t \in [0, T]$ .

**Remark 5.2** 1. Let us first point out that the main utility of this proposition is that we can “shift” the dependence on  $b_j$  to  $u_j$  so that applying this proposition twice appropriately to  $b_2, b_3$ , we may obtain the regularity criteria in terms of  $u_2, u_3$  alone.

2. The idea of the proof of this proposition was to observe that the  $j$ -th component of the magnetic field  $b$  is governed in time by the equation

$$\partial_t b_j + (u \cdot \nabla) b_j - (b \cdot \nabla) u_j = \eta \Delta b_j$$

so that the  $L^p$ -estimates of  $b_j$  leads to

$$\frac{1}{p} \partial_t \|b_j\|_{L^p}^p + \eta(p-1) \int \left| |\nabla b_j| |b_j|^{\frac{p-2}{2}} \right|^2 = \int (b \cdot \nabla) u_j |b_j|^{p-2} b_j$$

where the first non-linear term  $(u \cdot \nabla) b_j$  vanished due to the incompressibility condition (2.3c). At this stage, we observe that  $u_j$  is already separated in the right hand side. Appropriate sequence of energy estimates leads to the desired result in Proposition 5.2.2.

3. We note that without a new idea, we cannot shift the dependence on  $u_j$  to  $b_j$  due to the presence of  $\nabla \pi$ .

We present another component reduction result which was quite surprising due to the reasons on which we shall now elaborate. As we stated, the BKM-criterion for the

three-dimensional NSE (1.1a)-(1.1b) in terms of  $\Omega = \nabla \times u$  has been reduced by one component in [CC99] to  $\Omega_2, \Omega_3$  of  $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ . However, it remains unknown whether we can do the same for the three-dimensional MHD system although in [Y13a], we saw that a regularity criteria may be reduced to  $\partial_3 u_3$  and  $\Omega_3$ .

We remark that because  $\Omega_3 = \partial_1 u_2 - \partial_2 u_1$ , the regularity criteria in terms of  $\partial_3 u_3$  and  $\Omega_3$  still requires some dependence on the behavior of all of  $u_1, u_2, u_3$ . We also recall the numerical analysis results in [H85] and [PPS95] that indicated that the velocity field  $u$  plays more dominant role than the magnetic field  $b$ . Finally, when Professor Jia and Professor Zhou in [JZ12a] tried to reduce the dependence on  $u$  to  $u_3$  alone, they had to include all three components  $b_1, b_2, b_3$ .

Hence, the question is, when we put the dependence on  $u_3$  alone and no other component of  $u$ , can we reduce the dependence on  $b_1, b_2, b_3$  down to less components. The answer has been shown to be positive. With  $j_3 = \partial_1 b_2 - \partial_2 b_1$  the third component of the current density  $j = (j_1, j_2, j_3)$ , we present our result:

**Theorem 5.9** ([Y14i]) *Suppose  $(u, b)$  is the solution pair to the three-dimensional MHD system (2.3a)-(2.3b) at  $\alpha = \beta = 1$  in  $[0, T]$  and it satisfies*

$$\int_0^T \|u_3\|_{L^{p_1}}^{r_1} + \|j_3\|_{L^{p_2}}^{r_2} d\tau < \infty, \quad \begin{cases} \frac{3}{p_1} + \frac{2}{r_1} \leq \frac{1}{3} + \frac{1}{2p_1}, & \frac{15}{2} < p_1, \\ \frac{3}{p_2} + \frac{2}{r_2} \leq 2, & \frac{3}{2} < p_2. \end{cases} \quad (5.4)$$

*Then there is no singularity up to time  $T$ .*

We wish to point out a remarkable achievement of this theorem that may be of interest to many working in the research direction of the Serrin-type regularity criteria. The condition on  $j_3$  above is Serrin-class. That is, while if  $(u, b)(x, t)$  solves the MHD system, then so does  $(u_\lambda, b_\lambda)(x, t) = \lambda(u(\lambda x, \lambda^2 t), b(\lambda x, \lambda^2 t))$ ,  $\lambda > 0$  and for any  $i = 1, 2, 3$ ,  $\frac{3}{p} + \frac{2}{r} = 2$  if and only if

$$\int_0^T \|\partial_i u\|_{L^p}^r d\tau = \int_0^T \|\partial_i u_\lambda\|_{L^p}^r d\tau, \quad \int_0^T \|\partial_i b\|_{L^p}^r d\tau = \int_0^T \|\partial_i b_\lambda\|_{L^p}^r d\tau.$$

The proof required a new decomposition of the four non-linear terms that was not obvious at first sight:

**Proposition 5.2.3** *Let  $(u, b)$  solve (2.3a)-(2.3b) at  $\alpha = \beta = 1, N = 3$ . Then*

$$\begin{aligned} & \int (u \cdot \nabla) u \cdot \Delta_h u - (b \cdot \nabla) b \cdot \Delta_h u + (u \cdot \nabla) b \cdot \Delta_h b - (b \cdot \nabla) u \cdot \Delta_h b \quad (5.5) \\ & \lesssim \int |u_3| (|\nabla u| |\nabla \nabla_h u| + |\nabla b| |\nabla \nabla_h b|) \\ & \quad + |b_3| (|\nabla u| |\nabla \nabla_h b| + |\nabla b| |\nabla \nabla_h u|) + |\nabla_h u| |\nabla_h b| |j_3|. \end{aligned}$$

Let us give an idea of the complexity of this decomposition. Firstly, using integration parts and the incompressibility conditions (2.3c) we obtain

$$\begin{aligned} & \int (u \cdot \nabla) u \cdot \Delta_h u - (b \cdot \nabla) b \cdot \Delta_h u + (u \cdot \nabla) b \cdot \Delta_h b - (b \cdot \nabla) u \cdot \Delta_h b \\ & = \sum_{i,j=1}^3 \sum_{k=1}^2 \int u_i \partial_i u_j \partial_{kk}^2 u_j - b_i \partial_i b_j \partial_{kk}^2 u_j + u_i \partial_i b_j \partial_{kk}^2 b_j - b_i \partial_i u_j \partial_{kk}^2 b_j \\ & = \sum_{i,j=1}^3 \sum_{k=1}^2 \int -\partial_k u_i \partial_i u_j \partial_k u_j + \partial_k b_i \partial_i b_j \partial_k u_j - \partial_k u_i \partial_i b_j \partial_k b_j + \partial_k b_i \partial_i u_j \partial_k b_j \\ & := I + II + III + IV. \end{aligned}$$

At this point, we have 72 terms many of which do not contain  $u_3$  or  $b_3$  and certainly not  $j_3$  in itself. The goal is to show that each and every one of them can be manipulated to contain one of  $u_3, b_3$  or  $j_3$ . We further decompose I, II, III and IV as follows:

$$\begin{aligned} I & = - \sum_{i=1}^3 \sum_{j,k=1}^2 \int \partial_k u_i \partial_i u_j \partial_k u_j - \sum_{i=1}^3 \sum_{k=1}^2 \int \partial_k u_i \partial_i u_3 \partial_k u_3 \\ & = - \sum_{i,j,k=1}^2 \int \partial_k u_i \partial_i u_j \partial_k u_j - \sum_{j,k=1}^2 \int \partial_k u_3 \partial_3 u_j \partial_k u_j - \sum_{i=1}^3 \sum_{k=1}^2 \int \partial_k u_i \partial_i u_3 \partial_k u_3, \end{aligned}$$

$$\begin{aligned}
II &= \sum_{i=1}^3 \sum_{j,k=1}^2 \int \partial_k b_i \partial_i b_j \partial_k u_j + \sum_{i=1}^3 \sum_{k=1}^2 \int \partial_k b_i \partial_i b_3 \partial_k u_3 \\
&= \sum_{i,j,k=1}^2 \int \partial_k b_i \partial_i b_j \partial_k u_j + \sum_{j,k=1}^2 \int \partial_k b_3 \partial_3 b_j \partial_k u_j + \sum_{i=1}^3 \sum_{k=1}^2 \int \partial_k b_i \partial_i b_3 \partial_k u_3, \\
III &= - \sum_{i=1}^3 \sum_{j,k=1}^2 \int \partial_k u_i \partial_i b_j \partial_k b_j - \sum_{i=1}^3 \sum_{k=1}^2 \int \partial_k u_i \partial_i b_3 \partial_k b_3 \\
&= - \sum_{i,j,k=1}^2 \int \partial_k u_i \partial_i b_j \partial_k b_j - \sum_{j,k=1}^2 \int \partial_k u_3 \partial_3 b_j \partial_k b_j - \sum_{i=1}^3 \sum_{k=1}^2 \int \partial_k u_i \partial_i b_3 \partial_k b_3,
\end{aligned}$$

and

$$\begin{aligned}
IV &= \sum_{i=1}^3 \sum_{j,k=1}^2 \int \partial_k b_i \partial_i u_j \partial_k b_j + \sum_{i=1}^3 \sum_{k=1}^2 \int \partial_k b_i \partial_i u_3 \partial_k b_3 \\
&= \sum_{i,j,k=1}^2 \int \partial_k b_i \partial_i u_j \partial_k b_j + \sum_{j,k=1}^2 \int \partial_k b_3 \partial_3 u_j \partial_k b_j + \sum_{i=1}^3 \sum_{k=1}^2 \int \partial_k b_i \partial_i u_3 \partial_k b_3.
\end{aligned}$$

We now see that it is only the first sum in each decomposition of  $I, II, III$  and  $IV$  that we do not have any  $u_3, b_3$  or  $j_3$  as every other sum has either  $u_3$  or  $b_3$ . We give one example of how to handle the problematic sums leaving detail to [Y14i]. Consider

$$- \int \partial_1 u_1 (\partial_1 b_2)^2$$

from the first integral of III,

$$\int (\partial_1 b_2)^2 \partial_2 u_2$$

from the first integral of IV, neither of which contains  $u_3$  or  $b_3$  and certainly not  $j_3$ .

We combine them to obtain

$$\begin{aligned}
&\int -\partial_1 u_1 (\partial_1 b_2)^2 + (\partial_1 b_2)^2 \partial_2 u_2 \tag{5.6} \\
&= \int \partial_3 u_3 (\partial_1 b_2)^2 + 2\partial_2 u_2 (\partial_1 b_2)^2 = - \int u_3 \partial_3 (\partial_1 b_2)^2 + \int 2\partial_2 u_2 (\partial_1 b_2)^2,
\end{aligned}$$

where we used the incompressibility conditions on  $u$  and integration by parts. We have now obtained  $u_3$  in the first integral; however, the second integral remains problematic without  $u_3, b_3$  or  $j_3$ . We now consider

$$- \int \partial_2 u_2 (\partial_2 b_1)^2$$

from the first integral of III and

$$\int (\partial_2 b_1)^2 \partial_1 u_1$$

from the first integral of IV, again neither of which contains  $u_3$  or  $b_3$ . We combine them to obtain

$$\begin{aligned} & \int -\partial_2 u_2 (\partial_2 b_1)^2 + (\partial_2 b_1)^2 \partial_1 u_1 & (5.7) \\ & = \int \partial_3 u_3 (\partial_2 b_1)^2 + 2\partial_1 u_1 (\partial_2 b_1)^2 = - \int u_3 \partial_3 (\partial_2 b_1)^2 + \int 2\partial_1 u_1 (\partial_2 b_1)^2. \end{aligned}$$

Now we combine the second integrals from (5.6) and (5.7) to obtain remarkably that

$$\begin{aligned} & 2 \int \partial_2 u_2 (\partial_1 b_2)^2 + \partial_1 u_1 (\partial_2 b_1)^2 \\ & = 2 \int \partial_2 u_2 (\partial_1 b_2)^2 + (-\partial_2 u_2 - \partial_3 u_3) (\partial_2 b_1)^2 \\ & = 2 \int \partial_2 u_2 ((\partial_1 b_2)^2 - (\partial_2 b_1)^2) - \partial_3 u_3 (\partial_2 b_1)^2 \\ & = 2 \int \partial_2 u_2 (\partial_1 b_2 - \partial_2 b_1) (\partial_1 b_2 + \partial_2 b_1) + 2 \int u_3 \partial_3 (\partial_2 b_1)^2 \\ & = 2 \int \partial_2 u_2 j_3 (\partial_1 b_2 + \partial_2 b_1) + 2 \int u_3 \partial_3 (\partial_2 b_1)^2 \\ & \lesssim \int |\nabla_h u| |j_3| |\nabla_h b| + |u_3| |\nabla b| |\nabla \nabla_h b|. \end{aligned}$$

Hence, we accomplished in separating  $u_3, b_3$  or  $j_3$  in these four terms. After repeating similar procedure, some of which are easier, we can finish the proof of Proposition 5.2.3.

With this decomposition, we only need to apply Proposition 5.2.2 as before to *shift* the dependence on  $b_3$  to  $u_3$  to obtain the desired regularity criteria that depends only on  $u_3$  and  $j_3$ . We omit further details.



## CHAPTER 6

### CONCLUSIONS

We list here several interesting problems that are work in progress, all for which proofs seem non-trivial:

1. As discussed in Chapter 5, for the three-dimensional NSE, there is a regularity criteria of the solution in terms of any one entry of the Jacobian matrix of the velocity vector field  $u$  (cf. [CT11]). Due to the addition of three more non-linear terms that involve the magnetic vector field  $b$ , it seems to be a very challenging open problem whether we can obtain a regularity criteria of the solution pair to the MHD system in terms of any two entries of the Jacobian matrix of the velocity vector field  $u$  (cf. [Y13a] for partial progress). Moreover, it remains unknown whether we can reduce the two-velocity component regularity criteria in [Y14h] to one velocity component.
2. Results of reducing the regularity criteria in terms of a gradient of the solution to less number of partial derivatives may be seen as a type of anisotropic study of the fluid mechanics PDEs. In this regard, in [Y14b] the author obtained the global regularity result for the  $N$ -dimensional MHD system in anistoropic spaces using anisotropic Littlewood-Paley theory. In short, while usually an initial data in  $H^s(\mathbb{R}^N)$  being small implies that every partial derivative being small in this norm, in [Y14b] the author showed that these “smallness” may be under different regularity indices  $s_1, s_2, \dots, s_N$  under some additional conditions. The proof followed the work of [I99]. To the best of the author’s knowledge some of these results are difficult to extend to the active scalars such as the surface quasi-

geostrophic or the incompressible porous media equation governed by Darcy's law.

3. Natural phenomenon such as fluid motion is in general difficult to model completely due to its turbulent behavior unless one adds some stochastic term (expressed in the form of Brownian motion). Moreover, the study of stochastic NSE has increasingly attracted attention from many mathematicians recently (e.g. [F08] and references found therein). This is another direction of research of much interest and this project is already in progress.

## APPENDIX A

### Besov Spaces

We denote by  $\mathcal{S}(\mathbb{R}^N)$  the Schwartz space,  $\mathcal{S}'(\mathbb{R}^N)$ , its dual and  $\mathcal{S}_0$  to be the subspace of  $\mathcal{S}$  in the following sense:

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \int_{\mathbb{R}^N} \phi(x) x^\gamma dx = 0, |\gamma| = 0, 1, 2, \dots \right\}.$$

Its dual  $\mathcal{S}'_0$  is given by  $\mathcal{S}'_0 = \mathcal{S}/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P}$  where  $\mathcal{P}$  is the space of polynomials. For  $j \in \mathbb{Z}$  we define

$$A_j = \{ \xi \in \mathbb{R}^N : 2^{j-1} < |\xi| < 2^{j+1} \}.$$

It is well-known that there exists a sequence  $\{\Phi_j\} \in \mathcal{S}(\mathbb{R}^N)$  such that

$$\text{supp } \hat{\Phi}_j \subset A_j, \quad \hat{\Phi}_j(\xi) = \hat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jN} \Phi_0(2^j x)$$

and

$$\sum_{j=-\infty}^{\infty} \hat{\Phi}_j(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^N \setminus \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Consequently, for any  $f \in \mathcal{S}'_0$ ,

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f.$$

To define the inhomogeneous Besov space, we let  $\Psi \in C_0^\infty(\mathbb{R}^N)$  be such that

$$\hat{\Psi}(\xi) + \sum_{j=0}^{\infty} \hat{\Phi}_j(\xi) = 1, \quad \Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f,$$

for any  $f \in \mathcal{S}'$ . With that, we set  $\Delta_j f = 0$  for  $j \leq -2$  and otherwise

$$\Delta_j f = \begin{cases} \Psi * f & \text{if } j = -1, \\ \Phi_j * f & \text{if } j = 0, 1, 2, \dots, \end{cases}$$

and define for any  $s \in \mathbb{R}, p, q \in [1, \infty]$ , the inhomogeneous Besov space

$$B_{p,q}^s = \{f \in \mathcal{S}' : \|f\|_{B_{p,q}^s} < \infty\},$$

where

$$\|f\|_{B_{p,q}^s} = \begin{cases} (\sum_{j=-1}^{\infty} (2^{js} \|\Delta_j f\|_{L^p})^q)^{\frac{1}{q}}, & \text{if } q < \infty, \\ \sup_{-1 \leq j < \infty} 2^{js} \|\Delta_j f\|_{L^p} & \text{if } q = \infty. \end{cases}$$

In particular  $B_{2,2}^s = H^s$ . Finally, we have the following lemma:

**Lemma A.1** (cf. [C98]) *Bernstein's Inequality: Let  $f \in L^p(\mathbb{R}^N)$  with  $1 \leq p \leq q \leq \infty$  and  $0 < r < R$ . Then for all  $k \in \mathbb{R}^+ \cup \{0\}$ , and  $\lambda > 0$ , there exists a constant  $C_k > 0$  such that*

$$\|\Lambda^k f\|_{L^q} \leq C_k \lambda^{k+N(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p} \quad \text{if } \text{supp } \hat{f} \subset \{\xi : |\xi| \leq \lambda r\}.$$

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